DEFORMATION OF GENERIC SUBMANIFOLDS IN A COMPLEX MANIFOLD

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Abstract. This paper shows that an arbitrary generic submanifold in a complex manifold can be deformed into a 1-parameter family of generic submanifolds satisfying strong nondegeneracy conditions. The proofs use a careful analysis of the jet spaces of embeddings satisfying certain nondegeneracy properties, and also make use of the Thom transversality theorem, as well as the stratification of real-algebraic sets. Optimal results on the order of nondegeneracy are given.

1. Introduction

Many interesting properties of real submanifolds in complex space or, more generally, in a complex manifold, require the imposition of some nondegeneracy conditions. For hypersurfaces, and in particular for boundaries of domains in $\mathbb{C}^N$, the most studied condition is that of Levi-nondegeneracy. Although it is well-known that the boundary of any weakly pseudoconvex domain can be deformed into a strictly pseudoconvex one, it is not true that an arbitrary smooth boundary can be deformed into a (smooth) boundary that is everywhere Levi-nondegenerate. In contrast, in this paper we show that any (not totally real) generic submanifold of a complex manifold can be deformed into another generic submanifold satisfying a higher order nondegeneracy condition at every point of the deformed submanifold. The higher order nondegeneracy conditions considered here have been used to establish many properties that were previously only known to hold for Levi-nondegenerate hypersurfaces.

The first nondegeneracy condition we consider here is that of finite nondegeneracy or, more precisely, k-nondegeneracy for some integer $k \geq 1$, generalizing Levi nondegeneracy. In particular, a hypersurface $M \subset \mathbb{C}^N$ is 1-nondegenerate at a point p if and only if it is Levi-nondegenerate at p. The second nondegeneracy condition we consider is that of finite type in the sense of Kohn [K72] and Bloom-Graham [BG77]. For hypersurfaces, finite nondegeneracy implies the finite type property, but the two notions are independent for generic submanifolds of higher codimension. The precise definitions of these two conditions will be given in Section 2. The importance of these conditions is illustrated, for instance, by the fact that, taken together, they are sufficient to guarantee that the full CR automorphism group of a generic submanifold, equipped with its

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natural topology, is a finite-dimensional Lie group (see [BRWZ04]). Also, finite nondegeneracy and finite type at a given point p of a real-analytic generic submanifold $M \subset \mathbb{C}^N$ imply that the stability group of M at p (i.e. the group of germs of biholomorphisms fixing p and preserving M) is a Lie group, see [CM74, BS75, Be02, BER97, Z97, BER99a].

The present paper is devoted to approximation of general generic submanifolds of a complex manifold (with respect to the strong or Whitney topology, see e.g. [GG73, H94]) by those satisfying the two nondegeneracy conditions mentioned above. More precisely, our first result is the following.

Theorem 1.1. *Let M be a smooth (resp. real-analytic) connected manifold and X be a complex manifold such that $\dim_{\mathbb{C}} X < \dim_{\mathbb{R}} M < 2\dim_{\mathbb{C}} X$. Let $\tau\colon M \to X$ be a smooth (resp. real-analytic) embedding such that $\tau(M)$ is a generic submanifold of X. Then any open neighborhood of τ in $C^\infty(M, X)$, equipped with the Whitney topology, contains a smooth (resp. real-analytic) embedding whose image is a generic submanifold that is both finitely nondegenerate and of finite type.*

For the case of a smooth, generic embedding, we shall also prove a stronger version of Theorem 1.1 in which the approximation is accomplished via a 1-parameter family of embeddings. We have the following.

Theorem 1.2. *Let M be a smooth connected manifold, X a complex manifold with*

$$\dim_{\mathbb{C}} X < \dim_{\mathbb{R}} M < 2\dim_{\mathbb{C}} X$$

and $\tau\colon M \to X$ a smooth embedding such that $\tau(M)$ is a generic submanifold of X. Then for any open neighborhood U of τ in $C^\infty(M, X)$, there exists a smooth map $\widetilde{\tau}\colon M \times (-1, 1) \to X$ such that $\widetilde{\tau}(p, 0) = \tau(p)$ for all $p \in M$, and, for each $t \neq 0$, the map $p \mapsto \widetilde{\tau}(p, t)$ is in U and is an embedding of M into X whose image is a generic manifold that is finitely nondegenerate and of finite type.

In the course of the proof of Theorems 1.1 and 1.2, we shall give explicit estimates on the order of the nondegeneracy and the type of the approximating family of embeddings. This is made more precise in Theorems 2.2 and 2.4 below. In fact, the nondegeneracy and the type of the approximating embeddings can be estimated in terms of the dimensions of M and X alone. We will also show that some of these estimates are sharp, as is illustrated by Example 8.1; see also Remark 8.2.

A number of authors have studied deformations of Levi-nondegenerate real hypersurfaces and their embeddings into a complex manifold. Of the work in this direction, we mention here, in particular, the papers of Burns-Shnider-Wells [BSW78], Catlin-Lempert [CL92], Bland-Epstein [BE96], Huisken-Klingenberg [HK99], and Huang-Luk-Yau [HLY06].

Two of our main tools in the proof of our results are the Thom transversality theorem and the stratification of semi-algebraic sets (see e.g. [BR90]). The paper is organized as follows. In Section 2 we give various definitions used throughout the paper and we state Theorems 2.2 and 2.4, which are the more precise versions of Theorems 1.1 and 1.2, giving in particular the order of nondegeneracy of the approximating manifolds. In Sections 3 through 5 we describe a precise stratification of jets of degenerate embeddings and of those not of strong type (see Definition 2.1

below) and calculate their codimension in the space of all jets. The proof of Theorems 2.2 is given in Section 6 and that of Theorem 2.4 is given in Section 7. In Section 8, we conclude with some remarks and examples.

2. Definitions and precise results

Let $\mathcal{M}$ be a real submanifold in a complex manifold X. Recall that $\mathcal{M}$ is called *generic* if, for every $p \in \mathcal{M}$,

$$T_p\mathcal{M} + JT_p\mathcal{M} = T_pX,$$

where J denotes the complex structure of X, and $T_p\mathcal{M}$ and T_pX are the real tangent spaces of $\mathcal{M}$ and X respectively. In what follows, $\mathcal{M}$ is assumed to be generic. We let

$$N := \dim_{\mathbb{C}} X, \quad m := \dim_{\mathbb{R}} \mathcal{M}, \quad d := \operatorname{codim}_{\mathbb{R}}\mathcal{M} = 2N - m,$$

and denote by $T^{(0,1)}\mathcal{M}$ the bundle of $(0,1)$ vector fields on $\mathcal{M}$. Then for any $p \in \mathcal{M}$ we have $\dim_{\mathbb{C}} T^{(0,1)} = N - d =: n$. We note here that the condition $N < m < 2N$ imposed in the hypotheses of Theorems 1.1 and 1.2 for $\mathcal{M} = \tau(M)$ is equivalent to the conditions $d > 0$ and $n > 0$. We shall assume these conditions in the remainder of this paper.

We shall need the following definitions. Given $p \in \mathcal{M}$ and a system of local holomorphic coordinates $Z = (Z_1, \ldots, Z_N)$ in X near p, let $\rho = (\rho^1, \ldots, \rho^d)$ be a system of smooth, real defining functions of $\mathcal{M}$ near p (i.e. $\partial\rho^1(p,\overline{p}) \wedge \cdots \wedge \partial\rho^d(p,\overline{p}) \neq 0$). For an integer $k \geq 1$, define the *k-degeneracy* of $\mathcal{M}$ at p as the integer

$$r_p^1(k) := N - \dim \mathsf{span}_{\mathbb{C}} \left\{ (L_1 \ldots L_s \rho_Z^j)(p,\overline{p}) : 1 \leq j \leq d; 0 \leq s \leq k \right\}, \tag{2.1}$$

where $L_1, \ldots, L_s$ runs through arbitrary systems of $(0,1)$ vector fields on $\mathcal{M}$, i.e. sections of $T^{0,1}\mathcal{M}$. Here $\rho_Z^j = (\rho_{Z_1}^j, \ldots, \rho_{Z_N}^j)$ denotes the complex gradient of ρ^j (with respect to the coordinates Z) and is regarded as a vector in $\mathbb{C}^N$. Observe that $0 \leq r_p^1(k) \leq N - d$. It can be shown, see e.g. [BER99b, Chapter 11], that the right-hand side in (2.1) is an invariant, i.e. independent of the choice of the defining function ρ and the complex coordinates Z.

Recall that the generic submanifold $\mathcal{M} \subset X$ is called *k-nondegenerate at $p \in \mathcal{M}$* if and only if $r_p^1(k) = 0$ and $r_p^1(j) > 0$ for $1 \leq j < k$. Furthermore, $\mathcal{M}$ is said to be *finitely nondegenerate at p* if it is k-nondegenerate at p for some $k \geq 1$. Finally, we say that $\mathcal{M}$ is *k-degenerate at p* if $r_p^1(k) > 0$. With this definition, a hypersurface $\mathcal{M} \subset X$ is Levi nondegenerate at p if and only if it is 1-nondegenerate at p.

Recall also that the generic real submanifold $\mathcal{M} \subset X$ is said to be of *finite type k* at $p \in \mathcal{M}$ (in the sense of Bloom-Graham-Kohn) if $\mathbb{C}T_p\mathcal{M}$ is spanned by the commutators of the form

$$[\mathcal{L}_1, [\mathcal{L}_2, \ldots, [\mathcal{L}_{l-1}, \mathcal{L}_l] \ldots]](p), \quad 1 \leq l \leq k, \tag{2.2}$$

where each $\mathcal{L}_r$, $1 \leq r \leq l$, is either a $(1,0)$ or a $(0,1)$ vector field on $\mathcal{M}$, and k is minimal with this property. Here, for $l = 1$, the quantity (2.2) is simply $\mathcal{L}_1(p)$. Furthermore, $\mathcal{M}$ is said to be of *finite type at p* if it is of finite type k for some $k \geq 1$.

We shall now introduce a stronger version of the above mentioned finite type condition, which turns out to be more easily computable in terms of the defining functions of the manifold $\mathcal{M}$ (see Lemma 5.1 below).

Definition 2.1. Let $\mathcal{M} \subset X$ be a generic submanifold and $1 \leq k < \infty$ be an integer. We say that $\mathcal{M}$ is of *strong type* k at p if $\mathbb{C}T_p\mathcal{M}$ is spanned by the commutators of the form

$$[L_1, [L_2, \ldots, [L_{l-1}, \overline{L}_l] \ldots]](p), \quad 1 \leq l \leq k, \tag{2.3}$$

and their complex conjugates, where L_r, $1 \leq r \leq l$, are any $(1,0)$ vector fields on $\mathcal{M}$, and k is minimal with this property. Here, for $l = 1$, the quantity (2.3) is simply $\overline{L}_1(p)$.

It is an immediate consequence of the definition that, if M is of strong type k at p, then it is of finite type $\leq k$ at p. Moreover, for $k = 2, 3$, the notions of finite type k and strong type k coincide. However, for $k \geq 4$, finite type k may not imply strong type l for any l, even for hypersurfaces. For example, the hypersurface $\mathsf{Im}\, w = |z|^4$ in $\mathbb{C}^2$ is of finite type 4 at 0 but not of strong type l for any $l \geq 1$.

Note that in the case of hypersurfaces in X, k-nondegeneracy implies strong type $\leq k+1$. On the other hand, in higher codimension, the conditions of being of strong type $\leq k$ and l-nondegeneracy are independent for any k and l (see Lemma 5.1 below).

For an arbitrary generic manifold $\mathcal{M} \subset X$ and $p \in \mathcal{M}$ and an integer $k \geq 1$, we define the *k-defect of $\mathcal{M}$ at p* as the nonnegative integer $r_p^2(k)$ given by

$$r_p^2(k) := \dim_{\mathbb{R}} \mathcal{M} - \dim_{\mathbb{C}} V, \tag{2.4}$$

where V is the span of the commutators in (2.3) and their complex conjugates. Note that $\mathcal{M}$ is of strong type k at p if and only if $r_p^2(k) = 0$ and $r_p^2(j) > 0$ for $1 \leq j < k$.

Before stating a more precise version of Theorem 1.1, from which the latter follows, we introduce the following notation. For any pair of positive integers (m, N), with $2 \leq N < m < 2N$, we let $k_1(m, N)$ be the positive integer defined by

$$k_1(m,N) = \begin{cases} 1 & \text{if } \ N+2 \leq m \leq 2N-3, (m,N) \neq (7,5), \\ 3 & \text{if } (m,N) = (3,2), \\ 2 & \text{otherwise}, \end{cases} \tag{2.5}$$

and $k_2(m, N)$ be the smallest positive integer k for which the following inequality holds:

$$2(m-N)\binom{k+m-N-1}{k-1} \geq (m-N)^2 + 2m. \tag{2.6}$$

Observe in particular that if $m = 2N - 1$ (which will correspond to the case of a hypersurface in Theorem 2.2 below) then the integer $k_2(m, N)$ is 4 for $N = 2$, it is 3 for $N = 3$, and it is 2 for $N > 3$.

Theorem 2.2. *Let M, X and $\tau \colon M \to X$ be as in Theorem 1.1 and set $m := \dim_{\mathbb{R}} M$, $N := \dim_{\mathbb{C}} X$. Let $k_1 = k_1(m, N)$ and $k_2 = k_2(m, N)$ be defined by* (2.5) *and* (2.6) *respectively. Then any neighborhood of τ in $C^\infty(M, X)$, equipped with the Whitney topology, contains a smooth (resp.*

real-analytic) embedding whose image is a generic submanifold that, at every point, is both ℓ_1-nondegenerate, for some $\ell_1 \leq k_1$, and of strong type ℓ_2, for some $\ell_2 \leq k_2$.

Remark 2.3. In contrast to k_1, the integer k_2 in Theorem 2.2 has no uniform bound for all m and N. Indeed, one can check by using Lemma 5.1 below that if $\mathcal{M} \subset \mathbb{C}^N$ is a real $(N+1)$-dimensional generic submanifold of strong type k at some point, then necessarily $N < 2k$.

In analogy with Theorem 2.2, we have the following more precise version of Theorem 1.2, from which the latter follows.

Theorem 2.4. *Let M, X and $\tau\colon M \to X$ be as in Theorem 1.2. Then there exist positive integers k_1' and k_2', depending only on the dimensions $m := \dim_{\mathbb{R}} M$ and $N := \dim_{\mathbb{C}} X$, such that any neighborhood U of τ in $C^\infty(M,X)$ contains a smooth 1-parameter deformation $\widetilde{\tau}$ of τ as in Theorem 1.2 such that for each $t \neq 0$, the image of the embedding $p \mapsto \widetilde{\tau}(p,t)$ is, at each point p, both ℓ_1-nondegenerate, for some $\ell_1 \leq k_1'$, and of strong type ℓ_2, for some $\ell_2 \leq k_2'$. Moreover, when $X = \mathbb{C}^N$, k_1' can be chosen to be the same as the integer $k_1(m,N)$ given by (2.5) and k_2' to be the smallest positive integer k, for which the following holds:*

$$2(m-N)\binom{k+m-N-1}{k-1} \geq (m-N)^2 + 2m + 1. \tag{2.7}$$

In particular, if $\tau(M)$ is a real hypersurface in $\mathbb{C}^N$, i.e. $m = 2N-1$, k_2' can be chosen to be 4 for $N=2$, to be 3 for $N=3$, and to be 2 for $N>3$.

The following is an immediate consequence of Theorem 2.4.

Corollary 2.5. *Let $M \subset \mathbb{C}^N$ be a real hypersurface with $N \geq 3$. Then M can be smoothly approximated by a 1-parameter family of real hypersurfaces that are 2-nondegenerate and of strong type 3.*

In contrast, for $N=2$, there are real-analytic hypersurfaces in $\mathbb{C}^2$ that cannot be approximated by 2-nondegenerate ones, as Example 8.1 below shows.

3. Decomposition of jet spaces

In what follows, if A and B are smooth manifolds and k a positive integer, we use the standard notation $J^k(A,B)$ for the manifold of all k-jets of smooth mappings from A to B. For $p \in A$ and $q \in B$, we denote by $J^k_p(A,B)$ the submanifold of all k-jets in $J^k(A,B)$ with source p and by $J^k_{p,q}(A,B)$ — of those with source p and target q. (See e.g. [GG73] for properties of these manifolds.) If $f : A \to B$ is a map of class C^k and $p \in A$, we denote by $j^k_p f \in J^k_p(A,B)$ the jet of f at p.

Recall that if $f : A \to B$ is a smooth mapping, we say that f is an embedding if f is an immersion at every point of A and is homeomorphism onto its image $f(A)$. If M and X are as in Theorem 1.1, an embedding $f : M \to X$ is called *generic* if $f(M)$ is a generic submanifold of X.

If $p \in M$ and $g : (M, p) \to X$ is a germ of a smooth mapping at p, then g is a *germ of a generic embedding* if $g'(p) : T_pM \to T_{g(p)}X$ is injective and

$$T_{g(p)}X = g'(p)T_pM + Jg'(p)T_pM.$$

For $k \geq 1$ and $p \in M$, we denote by $W_p^0 \subset J_p^k(M, X)$ the k-jets of germs of generic embeddings as defined above.

If U is a local coordinate chart on M, then it is standard (see e.g. [GG73]) that $J^k(U, \mathbb{C}^N)$ can be naturally identified with $U \times \mathbb{R}^K$ for $K := \dim_{\mathbb{R}} J_p^k(U, \mathbb{C}^N)$ for $p \in U$.

Before stating the main result of this section, Lemma 3.4, we need the following.

Lemma 3.1. *Let $Z = (Z_1, \ldots, Z_N)$ be fixed coordinates in $\mathbb{C}^N$ and $g : (\mathbb{R}^{2n+d}, p) \to \mathbb{C}^N$ a germ of a generic embedding. Then after reordering the coordinates Z_j and multiplying some by $\sqrt{-1}$ if necessary, we can write $Z = (z, w) = (z_1, \ldots, z_n, w_1, \ldots, w_d)$ such that the following holds. After identifying $\mathbb{C}^N$ with $\mathbb{R}_x^n \times \mathbb{R}_y^n \times \mathbb{R}_s^d \times \mathbb{R}_t^d$ by $(x, y, s, t) = (\mathsf{Re}\, z, \mathsf{Im}\, z, \mathsf{Re}\, w, \mathsf{Im}\, w)$ and writing $g = (g_1, g_2)$ with $g_1 : (\mathbb{R}^{2n+d}, p) \to \mathbb{R}_x^n \times \mathbb{R}_y^n \times \mathbb{R}_s^d$ and $g_2 : (\mathbb{R}^{2n+d}, p) \to \mathbb{R}_t^d$, one has that g_1 is a germ of a diffeomorphism at p, and for U a sufficiently small neighborhood of p in $\mathbb{R}^{2n+d}$, $g(U)$ is given near $g(p)$ by*

$$\mathsf{Im}\, w = \varphi(\mathsf{Re}\, z, \mathsf{Im}\, z, \mathsf{Re}\, w), \tag{3.1}$$

where φ is the germ at $g_1(p)$ given by $\varphi = g_2 \circ g_1^{-1}$. Moreover, the $d \times d$ matrix $(\mathsf{id} + i\partial\varphi/\partial s)$ is invertible at $g_1(p)$.

Proof. For a sufficiently small neighborhood U of p in $\mathbb{R}^{2n+d}$, let $\rho = (\rho_1, \ldots, \rho_d)$ be some real-valued local defining functions for $g(U)$ near $g(p)$. Since the $N \times d$ matrix $(\partial\rho/\partial Z)$ has rank d at $g(p)$, after reordering the coordinates $(Z_1, \ldots, Z_N)$ and multiplying some of the Z_j by $\sqrt{-1}$ if necessary, we may assume that $Z = (z, w) = (z_1, \ldots, z_n, w_1, \ldots, w_d)$ with the $d \times d$ matrices $(\partial\rho/\partial w)$ and $(\partial\rho/\partial t)$ invertible at $g(p)$, where $t = \mathsf{Im}\, w$. Hence, by the implicit function theorem, $g(U)$ is given near $g(p)$ by (3.1) with $\varphi = g_2 \circ g_1^{-1}$. We observe that since $\rho = t - \varphi(x, y, s)$ is also a vector-valued defining function of $g(U)$ near $g(p)$, the invertibility of the matrix $(\mathsf{id} + i\partial\varphi/\partial s)$ follows from that of $(\partial\rho/\partial w)$. The proof of the lemma is complete. □

Remark 3.2. For a germ of a generic embedding g, as in the statement of Lemma 3.1, even if $g(U)$ is given by an equation of the form (3.1) for some coordinates (z, w) in $\mathbb{C}^N$, it does not necessarily follow that the matrix $(\mathsf{id} + i\partial\varphi/\partial s)$ is invertible. For example, for the generic embedding $g : \mathbb{R}^4 \to \mathbb{C}^3$ with

$$g(x, y, s_1, s_2) = \big(x + iy, s_1 + i(x + s_2), s_2 + i(y - s_1)\big),$$

$g(\mathbb{R}^4)$ is given by (3.1) with $\varphi(x, y, s) = (x + s_2, y - s_1)$, but the 2×2 matrix $(\mathsf{id} + i\partial\varphi/\partial s)$ is not invertible.

The following definition will be needed for Lemma 3.4 below.

Definition 3.3. Let $(\mathcal{M}, p)$ and $(\mathcal{M}', p')$ be two germs of smooth submanifolds of $\mathbb{C}^N$ and k a positive integer. We shall say that $(\mathcal{M}, p)$ and $(\mathcal{M}', p')$ are *k-equivalent* if there exists a germ of a local biholomorphism $H : (\mathbb{C}^N, p) \to (\mathbb{C}^N, p')$ such that if $x \mapsto Z(x)$ is a local parametrization of $\mathcal{M}$ defined in a neighborhood of 0 in $\mathbb{R}^{\dim \mathcal{M}}$ with $Z(0) = 0$ and ρ' is a vector-valued defining function for $\mathcal{M}'$, then

$$\rho'(H(Z(x)), \overline{H(Z(x))}) = O(|x|^{k+1}).$$

It is clear from the definitions that if $(\mathcal{M}, p)$ and $(\mathcal{M}', p')$ are generic submanifolds of $\mathbb{C}^N$ which are k-equivalent, then $(\mathcal{M}', p')$ is $(k-1)$-nondegenerate (resp. of strong type k) if and only if the same holds for $(\mathcal{M}, p)$.

For a positive integer k, we split the jet space $J^k_{0,0}(\mathbb{C}^n \times \mathbb{R}^d, \mathbb{R}^d)$ into its harmonic and nonharmonic (free from harmonic terms) parts, i.e.

$$J^k_{0,0}(\mathbb{C}^n \times \mathbb{R}^d, \mathbb{R}^d) = (J^k)_h \oplus (J^k)_{nh}, \tag{3.2}$$

where

$$(J^k)_h := \{ j^k_0 \psi : \psi = \mathsf{Re}\,(\sum_{0 \neq |\alpha|+|\gamma| \leq k} \psi_{\alpha\gamma} z^\alpha s^\gamma),\ \psi_{\alpha\gamma} \in \mathbb{C}^d,\ z \in \mathbb{C}^n,\ s \in \mathbb{R}^d \} \tag{3.3}$$

and

$$(J^k)_{nh} := \{ j^k_0 \psi : \psi = \mathsf{Re}\,(\sum_{\substack{\alpha \neq 0, \beta \neq 0 \\ |\alpha|+|\beta|+|\gamma| \leq k}} \psi_{\alpha\beta\gamma} z^\alpha \bar{z}^\beta s^\gamma),\ \psi_{\alpha\beta\gamma} \in \mathbb{C}^d,\ z \in \mathbb{C}^n,\ s \in \mathbb{R}^d \}. \tag{3.4}$$

We now state the main result of this section, a splitting of the jet spaces of generic embeddings, which will be used in Sections 4 and 5.

Lemma 3.4. *Let (z, w) be linear coordinates in $\mathbb{C}^N$ $(N = n + d, m = 2n + d)$, $p \in \mathbb{R}^m$, and let $\mathcal{O} \subset J^k_p := J^k_p(\mathbb{R}^m, \mathbb{C}^N)$ be the Zariski open subset of k-jets of the form $j^k_p g$, where $g : (\mathbb{R}^m, p) \to \mathbb{C}^N$ is a germ at p of a generic embedding whose image is of the form* (3.1) *near $g(p)$ with $(\mathsf{id} + i\partial\varphi/\partial s)$ invertible (as in Lemma 3.1). Then there exists a birational map*

$$\Psi = (\Psi_1, \Psi_2, \Psi_3, \Psi_4) \colon J^k_p \to \mathbb{C}^N \times J^k_{0,0}(\mathbb{R}^m, \mathbb{R}^m) \times (J^k)_h \times (J^k)_{nh} \tag{3.5}$$

smooth on $\mathcal{O}$, such that, if $y = j^k_p g \in \mathcal{O}$ and $\Psi_4(y) = j^k_0 \psi$, then the germ at $g(p)$ of the image of g in $\mathbb{C}^N$ is k-equivalent to the germ at 0 of the submanifold $\mathsf{Im}\, w = \psi(z, \bar{z}, \mathsf{Re}\, w)$. Moreover, Ψ is a diffeomorphism between $\mathcal{O}$ and a Zariski open subset in the target space in (3.5).

Proof. We use the natural identification $J^k_p \cong \mathbb{C}^N \times J^k_{0,0}(\mathbb{R}^m, \mathbb{C}^N)$ and the notation $y = (y_\alpha)_{0 \leq |\alpha| \leq k} \in J^k_p$ with $y_\alpha \in \mathbb{R}^{2N}$, $\alpha \in \mathbb{Z}^m_+$, $y_\alpha = (y^j_\alpha)_{1 \leq j \leq 2N}$. We set

$$\Psi_1(y) := y_0 \in \mathbb{R}^{2N} \cong \mathbb{C}^N, \quad \Psi_2(y) := (y^j_\alpha)_{1 \leq |\alpha| \leq k, 1 \leq j \leq m} \in J^k_{0,0}(\mathbb{R}^m, \mathbb{R}^m).$$

Next, we complete the pair (Ψ_1, Ψ_2) to a birational map

$$(\Psi_1, \Psi_2, \Phi) \colon \mathcal{O} \to \mathbb{C}^N \times J^k_{0,0}(\mathbb{R}^m, \mathbb{R}^m) \times J^k_{0,0}(\mathbb{C}^n \times \mathbb{R}^d, \mathbb{R}^d), \tag{3.6}$$

as a diffeomorphism onto its image, by constructing Φ as follows. Let $(y_\alpha) = (\partial^\alpha g(p))$ for $1 \leq |\alpha| \leq k$ and some germ $g = (g_1, g_2) \colon (\mathbb{R}^m, p) \to (\mathbb{R}^{2n+d} \times \mathbb{R}^d, 0)$ as in Lemma 3.1. Recall that g_1 is an invertible germ at p and $\varphi := g_2 \circ g_1^{-1}$ so that for U a sufficiently small neighborhood of p in $\mathbb{R}^{2n+d}$,

$g(U)$ is given near $g(p)$ by (3.1). We then set $\Phi(y) := j_0^k\varphi$, with that is, $\Phi(y) = (\partial^\alpha\varphi(0))_{1\le|\alpha|\le k}$. It is clear that $\Phi(y)$ depends only on $y = j_p^k g$ and not on the representative g, and that $\Phi(y)$ is rational in view of the chain rule. Moreover, since $j_p^k g_2 = j_p^k(\varphi\circ g_1)$, the inverse of the map (3.6) is a polynomial map. Thus the map (3.6) is a birational diffeomorphism onto its image. In fact, the image of (3.6) is the Zariski open set given by (Z,μ,ν) with $Z\in\mathbb{C}^N$, μ invertible in $J_{0,0}^k(\mathbb{R}^m,\mathbb{R}^m)$ and $\nu = j_0^k\varphi \in J_{0,0}^k(\mathbb{C}^n\times\mathbb{R}^d,\mathbb{R}^d)$ with $(\mathsf{id}+i\varphi_s(0))$ (with $s\in\mathbb{R}^d$) being an invertible $d\times d$ matrix.

We now set $\Psi_3(y)$ to be the component of $\Phi(y)$ in $(J^k)_h$ according to the decomposition (3.2). We may assume that $\Phi(y) = j_0^k\varphi$ with φ being a polynomial. In order to define $\Psi_4(y)$, we need to eliminate the harmonic components in φ by an appropriate biholomorphic change of coordinates near 0 in $\mathbb{C}^N$. We look for new coordinates $(z',w')\in\mathbb{C}^n\times\mathbb{C}^d$ given by $z=z'$, $w=h(z',w')$ with $h(0)=0$, $h_{w'}(0)\neq 0$. In these coordinates $g(U)$ is given by

$$h(z',w') - \bar h(\bar z',\bar w') = 2i\varphi\big(z',\bar z',\frac{h(z',w')+\bar h(\bar z',\bar w')}{2}\big). \tag{3.7}$$

We may consider $z',\bar z',w',\bar w'$ as independent variables. In order that $g(U)$ be given in the (z',w') coordinates by an equation of the form (3.1), where the defining function has no nonzero harmonic terms, it is necessary and sufficient that the equality (3.7) holds identically when $\bar z'=0$ and $\bar w'=w'$ (see e.g. [BER99b]). That is, we must have

$$h(z',w') - \bar h(0,w') \equiv 2i\varphi\big(z',0,\frac{h(z',w')+\bar h(0,w')}{2}\big). \tag{3.8}$$

To solve (3.8) for h we first set $z'=0$ and $w'=s\in\mathbb{R}^d$ to obtain

$$\mathsf{Im}\, h(0,s) \equiv \varphi(0,0,\mathsf{Re}\, h(0,s)). \tag{3.9}$$

It suffices to take

$$h(0,s) := s + i\varphi(0,0,s). \tag{3.10}$$

With this choice, $h(z',w')$ is uniquely determined from (3.8) by an immediate application of the implicit function theorem (since the $d\times d$ matrix $(\mathsf{id}+i\varphi_s(0))$ is assumed invertible). In the new coordinates (z',w') we can write a defining equation of $g(M)$ in the form $\mathsf{Im}\, w' = \psi(z',\bar z',\mathsf{Re}\, w')$, with $j_0^k\psi$ in $(J^k)_{nh}$ (as given by (3.4)). Conversely, knowing $j_0^k\psi$ and the harmonic part $\Psi_3(y)$ of $j_0^k\varphi$, we can recover $j_0^k h$ from (3.10) and (3.8) and hence the entire jet $j_0^k\varphi$. We may now define $\Psi_4(y) := j_0^k\psi$, completing the proof of the lemma. □

4. Jet spaces of finitely nondegenerate embeddings

We need to identify those jets of germs $g:(M,p)\to X$ of generic embeddings whose images are finitely nondegenerate at $g(p)$. For this it suffices to consider the case $M=\mathbb{R}^m$, with $m=\dim M = 2n+d$ and $X=\mathbb{C}^N$. We fix an integer $k\ge 2$, and, as in the previous section, we let $J_p^k := J_p^k(\mathbb{R}^m,\mathbb{C}^N)$. The following stratification result will be an important ingredient in the proofs of our main results.

Proposition 4.1. *Let $W_p^0 \subset J_p^k$ be the set of all k-jets at p of germs of generic embeddings $g\colon (\mathbb{R}^m, p) \to \mathbb{C}^N$ and $W_p' \subset W_p^0$ be the subset consisting of all those k-jets of generic embeddings that are $(k-1)$-degenerate. Then W_p^0 is a Zariski open subset of J_p^k and W_p' is a real-algebraic subset of W_p^0 admitting the stratification into real-analytic submanifolds*

$$W_p' = \bigcup_{1\le e\le n} W_{p,e}',$$

where $W_{p,e}'$ is the set of all k-jets of generic embeddings whose $(k-1)$-degeneracy at $g(p)$ (as defined by (2.1)*) is precisely e. In addition, for each e, $1 \le e \le n$, the closure of $W_{p,e}'$ in W_p^0 admits the stratification*

$$\overline{W_{p,e}'} = \bigcup_{e\le c\le n} W_{p,c}'.$$

Furthermore, for $1 \le e \le n$, the (real) codimension of the real-analytic submanifold $W_{p,e}'$ in J_p^k is at least

$$2d\binom{k+n-1}{k-1} - 2n - 3d + 2, \tag{4.1}$$

and this bound is achieved for $e = 1$.

Proof. Recall that, if g is defined in an open neighborhood U of 0 in $\mathbb{R}^m$ and $g(U)$ is given by (3.1) with $j_0^k\varphi$ in $(J^k)_{nh}$ (as defined by (3.4)), then $g(U)$ is $(k-1)$-degenerate at 0 if and only if the rank of the matrix A whose rows are

$$\varphi^j_{z,\bar z^\alpha}(0) = (\varphi^j_{z_1,\bar z^\alpha}(0), \ldots, \varphi^j_{z_n,\bar z^\alpha}(0)), \quad 1 \le j \le d, \quad 1 \le |\alpha| \le k-1, \tag{4.2}$$

is less than or equal $n-1$ (see e.g. [BER99b]). We should observe here that the rank of the matrix A is independent of the choice of complex coordinates in $\mathbb{C}^n$, as well as of the representative φ of the jet $j_0^k\varphi$. Since the number of multiindices $\alpha \in \mathbb{Z}_+^n$ with $0 \le |\alpha| \le k-1$ is $\binom{k+n-1}{k-1}$, it follows that the number of rows of A is $d\binom{k+n-1}{k-1} - d$. We will need the following lemma.

Lemma 4.2. *Let $(J^k)_{nh}$ be defined by* (3.4) *and, for $0 \le r \le n$, let $\mathcal{A}_r \subset (J^k)_{nh}$ be the (semialgebraic) subset defined by*

$$\mathcal{A}_r := \{ j_0^k\varphi \in (J^k)_{nh} : \operatorname{rank} A = r \}, \tag{4.3}$$

where A is the matrix associated to $j_0^k\varphi$ whose rows are given by (4.2)*. Then $\mathcal{A}_r$ is a real-analytic submanifold of $(J^k)_{nh}$ with*

$$\operatorname{codim} \mathcal{A}_r = 2d(n-r)\binom{k+n-1}{k-1} - (n-r)(2d + d(n-r) + 2r). \tag{4.4}$$

Moreover, if $\overline{\mathcal{A}}_r$ denotes the closure of $\mathcal{A}_r$ in $(J^k)_{nh}$, then we have the stratification

$$\overline{\mathcal{A}}_r = \bigcup_{0\le c\le r} \mathcal{A}_c. \tag{4.5}$$

Proof. We fix $j_0^k\widehat{\varphi} \in \mathcal{A}_r$ and denote by A_0 the corresponding matrix. Since $\operatorname{rank} A_0 = r$, after a complex-linear change of variables in $\mathbb{C}^n$, we may assume that the kernel of A_0 is spanned by e^a with $a = r+1, \ldots, n$, where $e^1, \ldots, e^n$ are the standard basis vectors in $\mathbb{C}^n$. Furthermore, we can choose r linearly independent rows $\widehat{\varphi}^{j_1}_{z,\bar{z}^{\alpha_1}}(0), \ldots, \widehat{\varphi}^{j_r}_{z,\bar{z}^{\alpha_r}}(0)$ of A_0. Moreover, since e^a are in the kernel of A_0 for $a = r+1, \ldots, n$, we have $\widehat{\varphi}^j_{z_l,\bar{z}_a} = \overline{\widehat{\varphi}^j_{z_a,\bar{z}_l}} = 0$, for $1 \le l \le n$. Hence, in the above choice of α_l, $1 \le l \le r$, we must have $\alpha_l \neq \varepsilon^b$ for any $b = r+1, \ldots, n$, where $\varepsilon^b := (0, \ldots, 1, \ldots, 0)$ with 1 at the b-th position.

Then, for $j_0^k\varphi$ near $j_0^k\widehat{\varphi}$ in $(J^k)_{nh}$, the system of linear equations in the unknowns $(v_1, \ldots, v_n)$,

$$\sum_{q=1}^{n} \varphi^{j_l}_{z_q,\bar{z}^{\alpha_l}}(0) v_q = 0, \quad 1 \le l \le r, \tag{4.6}$$

has an $(n-r)$-dimensional space of solutions, whose basis can be chosen to be of the form

$$v^i = (v^i_1, \ldots, v^i_r, 0, \ldots, 1, \ldots, 0) \in \mathbb{C}^n \tag{4.7}$$

with 1 at the $(r+i)$th position, where $i = 1, \ldots, n-r$. Note that v^i_q, $1 \le i \le n-r$, $1 \le q \le r$, are uniquely determined by (4.6) and depend only on the chosen rows of A, the matrix associated to the jet $j_0^k\varphi$.

To prove that $\mathcal{A}_r$ is a manifold of codimension K_r, where K_r is the number given by the right-hand side of (4.4), we shall show that $\mathcal{A}_r$ is given near $j_0^k\widehat{\varphi}$ by the vanishing of K_r real-analytic functions, whose differentials are independent at every point near $j_0^k\widehat{\varphi}$.

We now observe that a jet $j_0^k\varphi \in (J^k)_{nh}$ near $j_0^k\widehat{\varphi}$ belongs to $\mathcal{A}_r$ if and only if $Av^i = 0$ for $1 \le i \le n-r$, where A is the matrix corresponding to $j_0^k\varphi$ as usual. The latter condition, in view of (4.6), is equivalent to the system,

$$\sum_{q=1}^{n} \varphi^{j}_{z_q,\bar{z}^{\alpha}}(0) v^i_q = 0, \tag{4.8}$$

for $1 \le j \le d$, $1 \le i \le n-r$, $1 \le |\alpha| \le k-1$, where $(j,\alpha) \neq (j_l, \alpha_l)$ for $1 \le l \le r$. Note that the coefficients v^i_q, $1 \le i \le n-r$, $1 \le q \le r$, depend only on the variables $\varphi^{j_l}_{z_q,\bar{z}^{\alpha_l}}(0)$, $1 \le l \le r$, $1 \le q \le n$. Hence (4.8) can be considered as a linear system of equations in the jet variables $\varphi^j_{z_q,\bar{z}^\alpha}(0)$ for j, q, α as in (4.8).

Recall that we have $\alpha_l \neq \varepsilon^b$ for any $b = r+1, \ldots, n$, hence the system (4.8) contains, in particular, each equation

$$\sum_{q} \varphi^{j}_{z_q,\bar{z}_{a+r}}(0) v^i_q = 0, \quad 1 \le a, i \le n-r, \ 1 \le j \le d. \tag{4.9}$$

We now consider a new system of equations obtained from (4.8) by replacing each equation in (4.9) by the new equation

$$(\bar v_1^a,\dots,\bar v_n^a)\begin{pmatrix}\varphi^j_{z_1,\bar z_1}(0) & \dots & \varphi^j_{z_n,\bar z_1}(0)\\ \vdots & \ddots & \vdots\\ \varphi^j_{z_1,\bar z_n}(0) & \dots & \varphi^j_{z_n,\bar z_n}(0)\end{pmatrix}\begin{pmatrix}v_1^i\\ \vdots\\ v_n^i\end{pmatrix}=0 \tag{4.10}$$

for the given a,i,j. Note that the equations in the new system obtained in this fashion are linear combinations of equations in (4.8). Moreover, it is easy to see from the normalization (4.7) that the two systems are actually equivalent.

Recall that, in view of (4.7), $v^i_q=1$ for $q=r+i$. Hence each equation in (4.8) with fixed j,i,α, is linear in the variable $\varphi^j_{z_{i+r},\bar z^\alpha}(0)$ with coefficient 1. Moreover, the latter variable is an arbitrary complex number if either $|\alpha|\ge 2$ or $\alpha=\varepsilon^a$ with $1\le a\le r$. These correspond precisely to the equations of the old system (4.8) that were not replaced. On the other hand, each of the new equations (4.10) is linear in $\varphi^j_{z_{i+r},\bar z_{a+r}}(0)$ with coefficient 1. Moreover, for $i=a$, both the variable $\varphi^j_{z_{i+r},\bar z_{a+r}}(0)$ and the equation (4.10) are real, whereas for $i\ne a$, the corresponding equation (4.10) is conjugate of that obtained by exchanging i and a. In total, we obtain K_r independent real equations, proving the first part of the lemma.

To prove (4.5), observe first that the rank of a matrix is semi-continuous and hence the left-hand side of (4.5) is contained in the right-hand side. For the converse inclusion, let $j_0^k\varphi\in\mathcal{A}_c$ for $0\le c\le r$. As in the beginning of the proof of this lemma, we may assume that the kernel of the corresponding matrix A is spanned by the standard basis vectors e^a, $c+1\le a\le n$. Then, for any sufficiently small real $\varepsilon\ne 0$, it is easy to see that $j_0^k(\varphi+\varepsilon\widetilde\varphi)\in\mathcal{A}_r$ with $\widetilde\varphi(z,\bar z,s):=(\sum_{l=1}^r z_l\bar z_l,0,\dots,0)\in\mathbb{R}^d$. This proves (4.5) completing the proof of the lemma. □

We now return to the proof of Proposition 4.1. We shall first show that we can cover the set W_p^0 by finitely many open subsets $\mathcal{O}_l\subset J_p^k$ such that for every l, there exist linear coordinates $(z^l,w^l)\in\mathbb{C}^N$ with $j_p^k g\in\mathcal{O}_l$ if and only if $g\colon(\mathbb{R}^m,p)\to\mathbb{C}^N$ is a germ of a generic embedding whose image is of the form (3.1). Indeed, we start with a fixed set $(Z_1,\dots,Z_N)$ of linear coordinates in $\mathbb{C}^N$ and take finitely many new sets of linear coordinates $(z^l,w^l)\in\mathbb{C}^n\times\mathbb{C}^d$, obtained by permutations of the Z_j's and multiplication of some of the Z_j's by $i=\sqrt{-1}$, and take $\mathcal{O}_l$ to be the set of all k-jets of germs of generic embeddings g whose images can be graphed as in (3.1) with respect to the coordinates (z^l,w^l). Then it is easy to see that $\mathcal{O}_l$ is Zariski open and $\cup_l\mathcal{O}_l=W_p^0$. The rest of the proof follows from Lemmas 3.4 and 4.2. Indeed, for each choice of linear coordinates (z^l,w^l) as above, we may apply Lemma 3.4 to obtain the map Ψ^l as in (3.5), which is a diffeomorphism on $\mathcal{O}_l$. Then it easily follows that $W'_{p,e}=\cup_l(\Psi_4^l|_{\mathcal{O}_l})^{-1}(\mathcal{A}_{n-e})$ for each e, $1\le e\le n$, where $\mathcal{A}_r$ is given by (4.3). This completes the proof of Proposition 4.1, in view of Lemma 4.2. □

Remark 4.3. Note that the proof of Lemma 4.2 actually shows that $\mathcal{A}_r$ contains the open dense subset

$$\widehat{\mathcal{A}}_r:=\{j_0^k\varphi\in\mathcal{A}_r:\exists(b_1,\dots,b_d)\in\mathbb{R}^d\setminus\{0\},\ \mathrm{rank}\,(b_1\varphi^1_{z,\bar z}(0)+\dots+b_d\varphi^d_{z,\bar z}(0))=r\},$$

where $\varphi^j_{z,\bar z}(0)$ denotes the corresponding Hermitian $n \times n$ matrix. Note also that in general, $\widehat{\mathcal{A}}_r$ is a proper subset of $\mathcal{A}_r$ even for $k = 2$. Indeed, for $z \in \mathbb{C}^3$, $s \in \mathbb{R}$, consider the function

$$\varphi(z, \bar z, s) := (|z_1|^2 - |z_2|^2, 2\mathsf{Re}\,(z_1\bar z_3 + z_2\bar z_3)).$$

Then $j^2_0\varphi \in \mathcal{A}_2$ but $j^2_0\varphi \notin \widehat{\mathcal{A}}_2$, i.e. the joint kernel of the Hermitian matrices $\varphi^1_{z,\bar z}(0)$ and $\varphi^2_{z,\bar z}(0)$ is zero but any (real) linear combination of them has a nontrivial kernel. Such an example is closely related to the so-called "null-quadrics", i.e. quadrics of the form $\mathsf{Im}\, w = H(z, \bar z)$, $(z, w) \in \mathbb{C}^n \times \mathbb{C}^d$, where $H = (H_1, \ldots, H_d)$ is a vector-valued Hermitian form with trivial common kernel but such that any real linear combination of the H_l's is a degenerate Hermitian form. See [Be02] for further details.

5. Jet spaces of embeddings of finite strong type

In this section we fix an integer $k \geq 2$ and identify those k-jets of germs $g : (M, p) \to X$ of generic embeddings whose images are not of strong type l at $g(p)$ for any $l \leq k$. In this context we shall prove a stratification similar to that given by Proposition 4.1 (see Proposition 5.2 below). We start with the following lemma.

Lemma 5.1. *Assume that $(\mathcal{M}, 0) \subset \mathbb{C}^N$ is a germ of a generic submanifold given by* (3.1) *near the origin with $j^k_0\varphi \in (J^k)_{nh}$ (as defined by* (3.4)*). Then the codimension of the span of the vectors* (2.3) *and their conjugates at $p = 0$ in $\mathbb{C}T_0\mathcal{M}$ coincides with that of the span of all vectors $\varphi_{z_r,\bar z^\alpha}(0), \varphi_{\bar z_r,z^\alpha}(0)$ in $\mathbb{C}^d$ for $1 \leq r \leq n$, $1 \leq |\alpha| \leq k - 1$.*

Proof. If $\mathcal{M}$ is given by (3.1), a local basis of the $(1, 0)$ vector fields can be chosen to be

$$L_j = \frac{\partial}{\partial z_j} + i\varphi_{z_j}(z, \bar z, s)(\mathsf{id} - i\varphi_s(z, \bar z, s))^{-1}\frac{\partial}{\partial s}, \quad j = 1, \ldots, n.$$

Here we view φ_{z_j} as a row vector in $\mathbb{C}^d$, φ_s as a $d \times d$ matrix with $s = \mathsf{Re}\, w \in \mathbb{R}^d$ and $\frac{\partial}{\partial s}$ as a column vector with d components. We shall prove, for $j_1, \ldots, j_r, l \in \{1, \ldots, n\}$ and $r \leq k - 1$, the following identity, which will imply the conclusion of the lemma:

$$[L_{j_1}, [L_{j_2}, \ldots, [L_{j_r}, \overline{L}_l] \ldots]](0) = -2i\varphi_{z_{j_1},z_{j_2},\ldots,z_{j_r},\bar z_l}(0)\frac{\partial}{\partial s}. \tag{5.1}$$

Indeed, using induction on r, it follows from the assumption $j^k_0\varphi \in (J^k)_{nh}$, that

$$[L_{j_1}, [L_{j_2}, \ldots, [L_{j_r}, \overline{L}_l] \ldots]](z, \bar z, s) = \big(- 2i\varphi_{z_{j_1},z_{j_2},\ldots,z_{j_r},\bar z_l}(z, \bar z, s) + \langle \bar z\rangle \; + O(k - r)\big)\frac{\partial}{\partial s}, \tag{5.2}$$

where $\langle \bar z\rangle$ denotes a $\mathbb{C}^d$-valued row of polynomials of the form $\sum \bar z_j f_j(z, \bar z, s)$ and $O(k - r)$ a vector-valued function vanishing at the origin of order at least $k - r$. This proves the lemma. ☐

Proposition 5.2. *As in Proposition 4.1, let $W^0_p \subset J^k_p$ be the set of all k-jets at p of germs of generic embeddings $g\colon (\mathbb{R}^m, p) \to \mathbb{C}^N$. Denote by $W''_p \subset W^0_p$ the subset consisting of all k-jets of* *generic embeddings that are not of strong type l for any $l \le k$. Then W_p'' is a real-algebraic subset of W_p^0 admitting the stratification into real-analytic submanifolds*

$$W_p'' = \bigcup_{1 \le e \le d} W_{p,e}'',$$

where $W_{p,e}''$ is the set of all k-jets of generic embeddings whose k-defect (as defined by (2.4)) is precisely e at $g(p)$. Furthermore the closure of $W_{p,e}''$ in J_p^k admits the stratification

$$\overline{W_{p,e}''} = \bigcup_{e \le c \le d} W_{p,c}''$$

and the (real) codimension of $W_{p,e}''$ for $1 \le e \le d$, in J_p^k is at least

$$\Big[2n\binom{k+n-1}{k-1} - n^2 - 2n - d + 1\Big]^+ \tag{5.3}$$

with the notation $[r]^+ := \max(r, 0)$ for $r \in \mathbb{Z}$, and this bound is achieved for $e = 1$.

Proof. If g is defined in open neighborhood U of $\mathbb{R}^m$, and $g(U)$ is given by (3.1) with φ in $(J^k)_{nh}$, then, by Lemma 5.1, $g(U)$ is not of strong type k at 0 if and only if the rank of the matrix B whose rows are

$$\varphi_{z^\alpha, \bar z_r}(0) = (\varphi^1_{z^\alpha, \bar z_r}(0), \dots, \varphi^d_{z^\alpha, \bar z_r}(0)), \quad 1 \le r \le n, \quad 1 \le |\alpha| \le k-1, \tag{5.4}$$

and their conjugates, is less than d. Since the number of multiindices $\alpha \in \mathbb{Z}_+^n$ with $0 \le |\alpha| \le k-1$ is $\binom{k+n-1}{k-1}$, it follows that the number of rows of B with $|\alpha| > 1$ is $2n\left(\binom{k+n-1}{k-1} - n - 1\right)$. Since each row appears together with its conjugate, we don't change the rank of B by replacing the pair of each row and its conjugate by the real and imaginary parts of that row. Hence we obtain $2n\left(\binom{k+n-1}{k-1} - n - 1\right)$ real rows for $|\alpha| > 1$. Furthermore, each row in (5.4) with $|\alpha| = 1$ appears twice in the matrix B. In fact, the rows $(\varphi^1_{z_r, \bar z_r}(0), \dots, \varphi^d_{z_r, \bar z_r}(0))$, $1 \le r \le n$, are real and hence coincide with their conjugates, whereas the conjugate of each row $(\varphi^1_{z_r, \bar z_l}(0), \dots, \varphi^d_{z_r, \bar z_l}(0))$ with $r \ne l$, appears again as the row $(\varphi^1_{z_l, \bar z_r}(0), \dots, \varphi^d_{z_l, \bar z_r}(0))$. Hence the rank of B does not change if we remove these n^2 repeated rows. We thus end up with a real matrix B' having the same rank as B with $K'' := 2n\left(\binom{k+n-1}{k-1} - n - 1\right) + n^2$ real rows, whose entries can be seen as part of independent coordinates of $j_0^k\varphi$ in $(J^k)_{nh}$.

We now proceed as in the proof of Proposition 4.1 and Lemma 4.2. We make use of the following lemma.

Lemma 5.3. *Let $(J^k)_{nh}$ be defined by (3.4) and, for $0 \le r \le d$, let $\mathcal{B}_r \subset (J^k)_{nh}$ be the (semialgebraic) subset defined by*

$$\mathcal{B}_r := \{j_0^k\varphi \in (J^k)_{nh} : \operatorname{rank} B = r\}, \tag{5.5}$$

where B is the matrix associated to $j_0^k\varphi$ whose rows are given by (5.4)*. Then $\mathcal{B}_r$ is a real-analytic submanifold of $(J^k)_{nh}$ with*

$$\operatorname{codim}\mathcal{B}_r = (d-r)\Big[2n\binom{k+n-1}{k-1} - n^2 - 2n - r\Big]^+ \tag{5.6}$$

with the notation $[r]^+ := \max(r,0)$ for $r \in \mathbb{Z}$. Moreover, if $\overline{\mathcal{B}}_r$ denotes the closure of $\mathcal{B}_r$ in $(J^k)_{nh}$, then we have the stratification

$$\overline{\mathcal{B}}_r = \bigcup_{0\le c\le r} \mathcal{B}_c. \tag{5.7}$$

Proof. It follows from the discussion preceeding the lemma that we can replace the matrix B by the real matrix B' in (5.5). Then each $\mathcal{B}_r$ can be regarded as an orbit in the space of all real $K''\times d$ matrices under the action of $\mathbf{GL}(K'',\mathbb{R})\times\mathbf{GL}(d,\mathbb{R})$ by right and left multiplication. Hence each $\mathcal{B}_r$ is a manifold of codimension $(d-r)[K''-r]^+$ which yields (5.6). We leave the remaining details of the proof to the reader. □

The rest of the proof of the proof of Proposition 5.2 follows closely that of Proposition 4.1. □

6. Proof of Theorems 1.1 and 2.2

Proof. We start with the proof in the case where M is a smooth manifold and $\tau : M \to X$ is a smooth map. We shall make use of Thom transversality theorem (see e.g. [GG73] or [H94]). We first choose $k_1 = k_1(m,N)$ as defined by (2.5) and fix $k := k_1 + 1$. We shall take $W' \subset J^k(M,X)$ to be the subset of all k-jets of the form $j_p^k g$, $p \in M$, with $g\colon (M,p) \to X$ a germ of a generic embedding whose image is $(k-1)$-degenerate at $g(p)$. Observe that the latter condition imposed on g depends only on $j_p^k g$ and not on the representative g of this jet.

We now introduce a natural stratification of the set W'. For each integer $1 \le c \le n$, let $W'(c) \subset J^k(M,X)$ be the set of all k-jets of germs of generic embeddings $g\colon (M,p) \to X$ whose image has $(k-1)$-degeneracy at $g(p)$ equal to c. Note that

$$W' = \bigcup_{1\le c\le n} W'(c). \tag{6.1}$$

We claim that each $W'(c)$ is a smooth submanifold of $J^k(M,X)$ of codimension greater than or equal to the integer given by (4.1). Moreover, the bound given by (4.1) is achieved for $c = 1$. Indeed, since the $(k-1)$-degeneracy is invariant under translations in the target space $\mathbb{C}^N$, in the notation of Proposition 4.1 we have

$$W'_{p,c} = W'_{(p,0),c} \times \mathbb{C}^N \subset J^k_{p,0}(M,\mathbb{C}^N)\times\mathbb{C}^N,$$

with $W'_{(p,0),c} \subset J^k_{p,0}(M,\mathbb{C}^N)$ being a real-analytic submanifold of the same codimension as $W'_{p,c}$ in $J^k_p(M,\mathbb{C}^N)$. If U and V are local real and complex coordinate charts on M and X respectively, then it is standard (see e.g. [GG73]) that $J^k(U,V)$ can be naturally identified with $U\times V\times J^k_{0,0}(\mathbb{R}^m,\mathbb{C}^N)$, where $m := \dim M = 2n + d$ and $N := \dim_{\mathbb{C}} X = n + d$. Then the claim for $W'(c)$ follows from Proposition 4.1.

For $g \in C^\infty(M, X)$, let $j^k g := \{(p, j^k_p g) \in J^k(M, X) : p \in M\}$. Recall that it follows from Thom's transversality theorem that, if M and X are as in Theorem 1.1 and $W \subset J^k(M, X)$ a real smooth submanifold whose codimension is greater than $\dim M$, then

$$\{g \in C^\infty(M, X) : j^k g \cap W = \emptyset\}$$

is a *residual* subset in $C^\infty(M, X)$, i.e. a countable intersection of open dense subsets. Here $C^\infty(M, X)$ is equipped with the Whitney topology. For more details, see e.g. [H94, Chapter 2, Theorem 1.2], [GG73, Chapter 2, Theorem 4.9].

We now apply Thom's transversality theorem to each submanifold $W'(c)$, $1 \leq c \leq n$, which are the strata of W' given by (6.1). With the choice $k = k_1 + 1$, it follows that $j^k_p g \in W'$ if and only if $g(M)$ is k_1-degenerate at $g(p)$. One can check that if k_1 is the integer given by (2.5) and $k = k_1 + 1$, then the integer given by (4.1) is strictly greater than $\dim M = 2n + d$. Hence, by Thom's transversality theorem, the set

$$\{g \in C^\infty(M, X) : j^k g \cap W' = \emptyset\}$$

is residual in $C^\infty(M, X)$.

To approximate by embeddings whose images are of strong type, we now let $k := k_2(m, N)$, the integer given by (2.6). Observe that for this choice, the number given by (5.3) is greater than $\dim M = 2n + d$. We let $W'' \subset J^k(M, X)$ be the subset of all jets $j^k_p g$ with $g : (M, p) \to X$ a germ of a generic embedding whose image is not of strong type $\leq k_2$ at $g(p)$. As we did for W', we may stratify W'' and write

$$W'' = \bigcup_{1 \leq e \leq d} W''(e), \tag{6.2}$$

where $W''(e) \subset J^k(M, X)$ is the set of all k-jets of germs of generic embeddings $g : (M, p) \to X$ whose image has k-defect at $g(p)$ equal to e. It easily follows from Proposition 5.2 that each stratum $W''(e)$ is a smooth submanifold of $J^k(M, X)$ whose codimension is at least the number given by (5.3) and that this bound is achieved for $e = 1$. We again apply Thom's transversality theorem to each stratum $W''(e)$ of W'' to conclude that

$$\{g \in C^\infty(M, X) : j^k g \cap W'' = \emptyset\}$$

is also residual in $C^\infty(M, X)$.

Since the intersection of two residual sets is again residual and, in particular, dense, and since the generic embeddings from M to X form an open subset in $C^\infty(M, X)$ we obtain the approximation property as stated in Theorem 2.2 for the smooth case. Since strong type ℓ for some positive ℓ implies finite type, we have also proved Theorem 1.1 for the smooth case.

For the real-analytic case, i.e. when both M and τ are real-analytic, let $U \subset C^\infty(M, X)$ be any open neighborhood of τ (in the Whitney topology). Note that the set of all generic embeddings σ in U that are ℓ_1-nondegenerate, for some $\ell_1 \leq k_1$, and of strong type ℓ_2, for some $\ell_2 \leq k_2$, is an open subset of U in the Whitney topology, which is nonempty, by the proof above for the smooth case. Since the set of all real-analytic maps from M to X is dense in $C^\infty(M,X)$ (see e.g. [H94]), the desired generic embedding σ can also be chosen to be real-analytic. This completes the proofs of Theorems 1.1 and 2.2.

□

7. Proof of Theorems 1.2 and 2.4

Proof. The proof will follow in several steps. In Step 1, we consider the case $X=\mathbb{C}^N$ and M an open set in $\mathbb{R}^m$. In Step 2, we still take $X=\mathbb{C}^N$, but allow M to be a general abstract manifold. Step 3 deals with the general case.

7.1. **Step 1.** Assume first that $X=\mathbb{C}^N$ and M is an open set in $\mathbb{R}^m$ containing the origin. For any integer $k>1$, we use the shorthand notation and the identifications $J^k_p = J^k_p(M,\mathbb{C}^N)\cong J^k_0(\mathbb{R}^m,\mathbb{C}^N)$ and $J^k = J^k(M,\mathbb{C}^N)\cong M\times J^k_0(\mathbb{R}^m,\mathbb{C}^N)$. We look for $\widetilde{\tau}(p,t)$ of the form

$$\widetilde{\tau}(p,t) := \tau(p) + tf(p), \quad t\in(-1,1),\ f\in C^\infty(M,\mathbb{C}^N). \tag{7.1}$$

As in the proof of Theorem 1.1, we shall use Thom's transversality theorem to select f appropriately. For this, we first choose $k=k_1+1$, with $k_1=k_1(m,N)$ given by (2.5), and let $W'_p\subset J^k_p$ be the semialgebraic subset defined in Proposition 4.1. We shall show that the set

$$\{f\in C^\infty(M,\mathbb{C}^N) : j^k_p\tau + tj^k_pf \notin W'_p,\ \forall p\in M,\ \forall t\in(-1,1)\setminus\{0\}\} \text{ with } k=k_1+1, \tag{7.2}$$

is residual.

Similarly, we take $k=k'_2$, where k'_2 is the smallest integer k for which inequality (2.7) holds, and let $W''_p\subset J^k_p$ be the semialgebraic subset defined in Proposition 5.2. We shall again show that the set

$$\{f\in C^\infty(M,\mathbb{C}^N) : j^k_p\tau + tj^k_pf \notin W''_p,\ \forall p\in M,\ \forall t\in(-1,1)\setminus\{0\}\} \text{ with } k=k'_2, \tag{7.3}$$

is also residual.

We shall now complete the proof of Theorem 2.4 in the case where M is an open subset of $\mathbb{R}^m$ and $X=\mathbb{C}^N$. As in the statement of the theorem, let U be an open neighborhood of τ in $C^\infty(M,X)$. Since τ is a generic embedding, and the set of all generic embeddings of M into X is open in $C^\infty(M,X)$, we may assume, by shrinking U, if necessary, that all elements in U are also generic embeddings. It follows from the definition of the Whitney topology that there exists a continuous positive function $\delta(p)$ on M and a positive integer r such that the set

$$U^r_\delta := \{g\in C^\infty(M,\mathbb{C}^N) : |j^r_pg - j^r_p\tau| < \delta(p),\ \forall p\in M\}$$

is contained in U. Here $|\cdot|$ is any norm on $J^r_0(M,\mathbb{C}^N)$, with the identification given above. Choose $f\in C^\infty(M,\mathbb{C}^N)$ such that $\tau+f\in U^r_\delta$ and f in each of the residual sets defined by (7.2) and (7.3), which is possible since the intersection of two residual sets is residual, and hence dense. It follows from the definition of U^r_δ that $\tau+tf$ is also in $U^r_\delta\subset U$ for any $t\in[-1,1]$. It is now clear that $\widetilde{\tau}(p,t)=\tau(p)+tf(p)$, with f as chosen above, satisfies the conclusion of Theorem 2.4.

To complete Step 1 of the proof, it remains to show that the sets (7.2) and (7.3) are residual. Note that with the identification above, we have

$$W'_p = W'_0, \quad W''_p = W''_0, \quad p \in M. \tag{7.4}$$

Recall that a semialgebraic set A admits a finite (semialgebraic) stratification into a disjoint union of real-analytic submanifolds, and the maximum stratum dimension (resp. minimum stratum codimension) is independent of the stratification and is said to be the dimension (resp. codimension) of A (see e.g [BR90]). We consider first the set given by (7.2), for which $k := k_1 + 1$.

For $p \in M$, let $S_p \subset (J^k_p)^2 \times ((-1,1) \setminus \{0\})$ be given by

$$S_p := \{(\Lambda_0, \Lambda, t) : \Lambda_0, \Lambda \in J^k_p,\, t \in (-1,1) \setminus \{0\},\, \Lambda_0 + t\Lambda \in W'_p\}.$$

By Proposition 4.1, for any p, S_p is semialgebraic with $\operatorname{codim} S_p = \operatorname{codim} W'_p$. Consider the following natural projections

$$\pi_p \colon (J^k_p)^2 \times ((-1,1) \setminus \{0\}) \to (J^k_p)^2, \quad \rho_p \colon (J^k_p)^2 \times ((-1,1) \setminus \{0\}) \to J^k_p, \quad \sigma_p \colon (J^k_p)^2 \to J^k_p$$

given by

$$\pi_p(\Lambda_0, \Lambda, t) = (\Lambda_0, \Lambda), \quad \rho_p(\Lambda_0, \Lambda, t) = \Lambda_0, \quad \sigma_p(\Lambda_0, \Lambda) = \Lambda_0.$$

Then we have

$$\operatorname{codim}_{\rho_p^{-1}(\Lambda_0)}(\rho_p^{-1}(\Lambda_0) \cap S_p) = \operatorname{codim}_{J^k_p} W'_p.$$

Furthermore, $B_p := \pi_p(S_p) \subset (J^k_p)^2$ is also semialgebraic by a theorem of Tarski-Seidenberg (see e.g. [BR90]). Since the dimension of a semialgebraic set cannot increase after projection (see e.g. [BR90]), we have

$$\operatorname{codim} B_p \geq \operatorname{codim} S_p - 1 = \operatorname{codim} W'_p - 1.$$

We also have for every $\Lambda_0 \in J^k_p$,

$$\pi_p(\rho_p^{-1}(\Lambda_0) \cap S_p) = \sigma_p^{-1}(\Lambda_0) \cap B_p,$$

implying

$$\dim(\sigma_p^{-1}(\Lambda_0) \cap B_p) \leq \dim(\rho_p^{-1}(\Lambda_0) \cap S_p),$$

and hence

$$\operatorname{codim}_{\sigma_p^{-1}(\Lambda_0)}(\sigma_p^{-1}(\Lambda_0) \cap B_p) \geq \operatorname{codim}_{\rho_p^{-1}(\Lambda_0)}(\rho_p^{-1}(\Lambda_0) \cap S_p) - 1 = \operatorname{codim} W'_p - 1. \tag{7.5}$$

It follows from (7.4) that $S_p = S_0$ and $B_p = B_0$. Since B_0 is a semialgebraic set, we may consider its stratification into a finite union of real-analytic disjoint submanifolds $B_0 = \cup_j B^j_0$. For clarity and motivation of the proof, we begin with the simplifying assumption that B_0 consists of a single stratum, i.e. B_0 is a real-analytic submanifold of $(J^k_0)^2$, and that $\sigma_p|_{B_0}$ is a submersion onto an open subset $A_0 \subset J^k_0$ (i.e. $\sigma_p|_{B_0}$ is of maximal rank equal to $\dim_{\mathbb{R}} J^k_0$ at every point of B_0). We set

$$B := \cup_{p\in M} B_p = M \times B_0 \subset \cup_{p\in M} (J^k_p)^2, \quad A := M \times A_0 \subset J^k.$$

Then the map

$$\sigma \colon M \times (J^k_0)^2 \to M \times J^k_0 = J^k, \quad \sigma(p, \Lambda_0, \Lambda) := (p, \Lambda_0),$$

when restricted to B, is a submersion onto the open subset $A \subset J^k$. Define the map $\lambda_0 \colon M \to J_0^k$ by $\lambda_0(p) := j_p^k \tau \in J_0^k \cong J_p^k$, where $\tau \colon M \to \mathbb{C}^N$ is the given generic embedding.

Recall that k_1 is chosen to be $k-1$, where k is the minimum integer for which $\mathrm{codim} W_p'$ (given by (4.1)) is greater than $\dim_{\mathbb{R}} M (= 2n+d)$. The reader can also check that the same k is also the minimum integer, for which $\mathrm{codim} W_p' - 1 > \dim_{\mathbb{R}} M$.

We now consider the subset $V \subset J^k \cong M \times J_0^k$ defined by

$$V := \{(p, \Lambda) \in J^k : (\lambda_0(p), \Lambda) \in B_0\}, \tag{7.6}$$

which is a submanifold of J^k of codimension $\mathrm{codim}_{\sigma_0^{-1}(\Lambda_0)} B_0$. Hence by (7.5) and the choice of k, we have $\mathrm{codim} V \geq \mathrm{codim} W_p - 1 > \dim_{\mathbb{R}} M$. Thus we can apply Thom's transversality theorem to V to obtain a residual set of functions $f \in C^\infty(M, \mathbb{C}^N)$ with $j^k f \cap V = \emptyset$, i.e. $j_p^k \tau + t j_p^k f \notin W_p'$ for all $t \in (-1,1) \setminus \{0\}$ and all $p \in M$. This completes the proof in the simplifying case where $B_0 \subset (J_0^k)^2$ is a real-analytic submanifold and $\sigma_0|_{B_0}$ is a submersion onto an open subset $A_0 \subset J_0^k$.

We now consider the less restrictive assumption that B_0 is still a real-analytic submanifold of J_0^k (i.e. a single stratum) and $\sigma_0|_{B_0}$ is a submersion onto a submanifold $A_0 \subset J_0^k$ which is not necessarily open. It should be noted that in this case, the set V defined by (7.6) may not necessarily be a submanifold of J^k. To remedy this, we will enlarge both A_0 and B_0 as follows. Let $\Omega_0 \subset J_0^k$ be an open neighborhood of the submanifold $A_0 \subset J_0^k$ with a retraction $r_0 \colon \Omega_0 \to A_0$, i.e. $r_0 \in C^\infty(\Omega_0, A_0)$ and $r_0(\Lambda_0) = \Lambda_0$ for $\Lambda_0 \in A_0$. The existence of such Ω_0 and r_0 is well-known (see e.g. [GG73, Chapter 2, §7]). We now define the enlargement of B_0, denoted by $\widetilde{B}_0 \subset (J_0^k)^2$, to be

$$\widetilde{B}_0 := \{(\Lambda_0, \Lambda) : \Lambda_0 \in \Omega_0,\ (r_0(\Lambda_0), \Lambda) \in B_0\}.$$

Then it is easy to check that $\widetilde{B}_0$ is a submanifold of $(J_0^k)^2$ containing B_0 and $\sigma_0|_{\widetilde{B}_0}$ is a submersion onto the open subset Ω_0. Moreover, it is also easy to check that

$$\mathrm{codim}_{\sigma_0^{-1}(\Lambda_0)}(\sigma_0^{-1}(\Lambda_0) \cap \widetilde{B}_0) = \mathrm{codim}_{\sigma_0^{-1}(\Lambda_0')}(\sigma_0^{-1}(\Lambda_0') \cap B_0) \geq \mathrm{codim} W_0 - 1$$

for any $\Lambda_0 \in \Omega_0$ and $\Lambda_0' \in A_0$, where the latter inequality follows from (7.5). (The reader should note that if A_0 is not open in J_0^k, then $\dim \widetilde{B}_0 > \dim B_0$.) The rest of the proof in this case can be reduced to the previous case above (with A_0 open in J_0^k) by replacing B_0 with $\widetilde{B}_0$.

Finally, we consider the general case, i.e. when no restrictions are imposed on B_0; it is merely semialgebraic and hence is a finite union of real-analytic strata. Then the proof is reduced to the previously considered two cases by applying the following general result about semialgebraic sets that follows by induction from the stratification of semialgebraic sets and the theorem of Tarski-Seidenberg mentioned above.

Lemma 7.1. *Let $C \subset \mathbb{R}^{K_1} \times \mathbb{R}^{K_2}$ be a semialgebraic subset and $\sigma \colon \mathbb{R}^{K_1} \times \mathbb{R}^{K_2} \to \mathbb{R}^{K_1}$ be the canonical projection. Then C can be decomposed into a finite union of disjoint real-analytic submanifolds C_j such the restriction of σ to each C_j is a submersion onto a real-analytic submanifold of $\mathbb{R}^{K_1}$.*

Sketch of the proof of Lemma 7.1. The proof is by induction on the dimension of C. Since C can be stratified as a finite union of real-analytic connected submanifolds, we may take a stratum C_0 of maximal dimension and assume, after removing a semialgebraic subset of lower dimension, that this stratum is globally defined by the vanishing of finitely many polynomials with independent differentials. Furthermore, after removing another lower-dimensional semialgebraic subset, we may assume that the restriction of σ to C_0 is of constant rank. By a theorem of Tarski-Seidenberg, $\sigma(C_0)$ is again semialgebraic and hence can be stratified. Let $D_0 \subset \sigma(C_0)$ be the union of strata of maximum dimension. Then $\widetilde{C}_0 := \sigma^{-1}(D_0)$ is an open submanifold of C_0, whose complement is semialgebraic of lower dimension and such that the restriction of σ to $\widetilde{C}_0$ is a submersion onto a real-analytic submanifold. □

The proof that the set given by (7.2) is residual is complete under the hypotheses of Step 1. The proof that the set given by (7.3) is also residual follows by repeating verbatim the proof for (7.2) by replacing W'_p by W''_p and using Proposition 5.2 in place of Proposition 4.1. The proofs of Theorems 1.2 and 2.4 in the case when M is an open set in $\mathbb{R}^m$ and $X = \mathbb{C}^N$ are now complete.

7.2. **Step 2.** We now consider the case when M is a general smooth manifold but still assume $X = \mathbb{C}^N$. We again look for a suitable deformation $\widetilde{\tau}$ of τ, of the form (7.1), where we choose $f \in C^\infty(M, \mathbb{C}^N)$ appropriately. As in Step 1, we must show that the sets given by (7.2) and (7.3) are residual. For this, we cover M by countably many coordinate charts M_j. Then the arguments of Step 1, applied first to $k = k_1 + 1$ (as in Step 1), to each M_j, yield subsets $V_j \subset J^k(M_j, \mathbb{C}^N)$ such that each V_j is a finite union of smooth manifolds and, for $f \in C^\infty(M, \mathbb{C}^N)$, $(p, j^k_p f) \notin V_j$ for $p \in M_j$ implies that $\widetilde{\tau}(\cdot, t)$ (given by (7.1)), restricted to M_j is a generic embedding, whose image is k_1-nondegenerate. Then we regard each V_j as a finite union of smooth submanifolds of $J^k(M, \mathbb{C}^N)$ and apply Thom's transversality theorem to the union of the V_j's in $J^k(M, \mathbb{C}^N)$ to obtain that (7.2) is residual. An identical argument with $k = k'_2$ shows that (7.3) is residual. The proof is then completed as in Step 1. At this point we have proved Theorems 1.2 and 2.4 in the case where $X = \mathbb{C}^N$, including the estimates for k'_1 and k'_2.

7.3. **Step 3.** We now treat the general case where X is a complex manifold and we are given a neighborhood U of τ in $C^\infty(M, X)$. By possibly shrinking U, we may assume that all maps in U are generic embeddings of M in X. We begin with locally finite coverings of M and X by countably many relatively compact coordinate charts

$$M = \bigcup_j M_j = \bigcup_j \widetilde{M_j}, \quad X = \bigcup_j X_j, \quad j = 1, 2, \dots,$$

with $M_j \Subset \widetilde{M_j} \Subset M$ and $\tau(\widetilde{M_j}) \Subset X_j \Subset X$. We now choose k such that both integers (4.1) and (5.3) are greater than $\dim M + 2$. For every $p \in M$, we consider the semialgebraic sets $W_p \subset J^k_p(M, X)$ consisting of k-jets of germs of smooth generic embeddings $g\colon (M, p) \to X$ whose images are either $(k-1)$-degenerate or not of strong type $\leq k$. With this choice, we have

$$\operatorname{codim} W_p > \dim_{\mathbb{R}} M + 2 \tag{7.7}$$

as consequence of Propositions 4.1 and 5.2.

We also set $M_0 = \widetilde{M}_0 := \emptyset$, $X_0 := \emptyset$ and shall construct inductively a sequence of deformations $\widetilde{\tau}_j \colon M \times (-1,1) \to X$, $j = 0, 1, \ldots$, with the following properties:

(1) $\widetilde{\tau}_j(p,0) \equiv \tau(p)$ for all $p \in M$;
(2) $j_p^k \widetilde{\tau}_j(\cdot,t) \notin W_p$ for $p \in \bigcup_{l \le j} \overline{M}_l$ and $t \in (-1,1) \setminus \{0\}$;
(3) $\widetilde{\tau}_j(\widetilde{M}_l \times (-1,1)) \Subset X_l$ for all l;
(4) $\widetilde{\tau}_j(\cdot,t) \in U$ for all $t \in (-1,1)$;
(5) $\widetilde{\tau}_j(p,t) \equiv \widetilde{\tau}_{j-1}(p,t)$ for $p \notin \widetilde{M}_j$, $t \in (-1,1)$, and $j \ge 1$.

Then it is clear that $\widetilde{\tau}_0(p,t) := \tau(p)$ satisfies the hypotheses (1)–(5) for $j = 0$.

We now assume that the $\widetilde{\tau}_j$ have been chosen for $0 \le j \le j_0$, and construct $\widetilde{\tau}_{j_0+1}$. For this, we slightly modify the end of the proof in Step 1 by introducing an additional real parameter. In that proof we take M to be $\widetilde{M}_{j_0+1}$, regarded as a open set in $\mathbb{R}^m$, and choose the same sets A, B, A_0, B_0. We first look for $\widehat{\tau}$ of the form

$$\widehat{\tau}(p,t,t') = \widetilde{\tau}_{j_0}(p,t') + t f(p,t'), \quad p \in \widetilde{M}_{j_0+1}, \tag{7.8}$$

where $f \in C^\infty(\widetilde{M}_{j_0+1} \times (-1,1), \mathbb{C}^N)$, with the addition and multiplication by t understood to be with respect to the chosen coordinates in X_{j_0+1}. This is possible by the assumption (3) for $\widetilde{\tau}_{j_0}$.

We now define $\lambda_0 \colon \widetilde{M}_{j_0+1} \times (-1,1) \to J_0^k$ by $\lambda_0(p,t') := j_p^k \widetilde{\tau}_{j_0}(\cdot,t')$ and follow the arguments of Step 1 as follows. We fix a finite stratification $(B_0^l)_l$ of B_0 as in Lemma 7.1 such that $\sigma|_{B_0^l}$ is a submersion onto a real-analytic submanifold A_0^l together with corresponding retractions $r^l \colon \Omega_0^l \to A_0^l$ as in Step 1. Then each set

$$\widetilde{V}^l := \{(p,\Lambda,t') \in \widetilde{M}_{j_0+1} \times J_0^k \times (-1,1) : \lambda_0(p,t') \in \Omega_0^l,\ (r^l(\lambda_0(p,t')),\Lambda) \in B_0^l\},$$

is a submanifold of $\widetilde{M}_{j_0+1} \times J_0^k \times (-1,1)$. As in the proof of Step 1 we see that $\operatorname{codim} \widetilde{V}^l \ge \operatorname{codim} W_0 - 1$ and hence $\operatorname{codim} \widetilde{V}^l > \dim_{\mathbb{R}} M + 1$ in view of (7.7). We finally set

$$V^l := \Pi^{-1}(\widetilde{V}^l) \subset J^k(\widetilde{M}_{j_0+1} \times (-1,1), \mathbb{C}^N),$$

where $\Pi \colon J^k(\widetilde{M}_{j_0+1} \times (-1,1), \mathbb{C}^N) \to \widetilde{M}_{j_0+1} \times J_0^k \times (-1,1)$ is the natural projection. Note that each V^l is also a submanifold of $J^k(\widetilde{M}_{j_0+1} \times (-1,1), \mathbb{C}^N)$ of the same codimension as V^l. By Thom's transversality theorem applied to V^l, we obtain a residual set of maps $f \in C^\infty(\widetilde{M}_{j_0+1} \times (-1,1), \mathbb{C}^N)$ with $j^k f \cap V^l = \emptyset$ for all l, i.e. $j_p^k \widetilde{\tau}_{j_0}(\cdot,t') + t j_p^k f(\cdot,t') \notin W_p$ for all $t' \in (-1,1)$, $t \in (-1,1) \setminus \{0\}$, and $p \in \widetilde{M}_{j_0+1}$.

We next choose a smooth function $\varphi \in C^\infty(\widetilde{M}_{j_0+1}, \mathbb{R})$ with compact support with $\varphi(p) = 1$ for p in a neighborhood of $\overline{M}_{j_0+1}$. Furthermore, as a consequence of Property (2) for j_0, the compactness of $\cup_{j \le j_0} \overline{M}_j$ and the fact that W_p is closed in the open set in J_p^k of all k-jets of germs of generic embeddings, we have the positive continuous distance function

$$d(t) := \inf_{p \in \cup_{j \le j_0} \overline{M}_j} \operatorname{dist}(j_p^k \widetilde{\tau}_{j_0}(\cdot,t), W_p) > 0,\ t \ne 0,$$

where the distance is taken with respect to any fixed metric on $J^k(M,X)$. We now define

$$\widetilde{\tau}_{j_0+1}(p,t) := \begin{cases} \widetilde{\tau}_{j_0}(p,t) + \delta(t)\varphi(p)f(p,t), & p \in \widetilde{M}_{j_0+1}, \\ \widetilde{\tau}_{j_0}(p,t), & p \in M \setminus \widetilde{M}_{j_0+1}, \end{cases}$$

where $\delta(t)$ is a smooth real valued function on $(-1,1)$ with $\delta(0)=0$ and $\delta(t)>0$ for $t \neq 0$, chosen sufficiently small such that

$$\operatorname{dist}\big(j_p^k\widetilde{\tau}_{j_0+1}(\cdot,t), j_p^k\widetilde{\tau}_{j_0}(\cdot,t)\big) < d(t), \quad t \in (-1,1)\setminus\{0\}, \quad p \in M,$$

and such that Properties (3) and (4) hold for $j = j_0+1$. Here we use the standard fact that, for a given continuous positive function on $(-1,1)\setminus\{0\}$, there exists a smooth function on $(-1,1)$, which is smaller than the given function and positive away from 0. Then it follows that $\widetilde{\tau}_{j_0+1}(p,t)$ satisfies Property (2) for $j = j_0+1$. The remaining Properties (1) and (5) follow directly from the construction.

We have now constructed the sequence $(\widetilde{\tau}_j)_j$ satisfying Properties (1)–(5) above. Then we define the desired deformation $\widetilde{\tau}(p,t)$ as follows. Since the covering $(\widetilde{M}_j)_j$ of M is locally finite, there exists an integer function $\nu(p) \in \mathbb{N}$, $p \in M$, such that every $p \in M$ has a neighborhood that is disjoint from the union $\bigcup_{l \geq \nu(p)} \widetilde{M}_l$. We set

$$\widetilde{\tau}(p,t) := \widetilde{\tau}_{\nu(p)}(p,t).$$

Then it is an immediate consequence of Property (5) that $\widetilde{\tau}(p,t)$ is a smooth map from $M \times (-1,1)$ to X and Properties (1), (2) and (4) imply the desired conclusions of Theorems 1.2 and 2.4. □

8. Remarks and examples

It should be mentioned that even a hypersurface in $\mathbb{C}^2$, cannot, in general, be approximated by, or deformed into, a 1-nondegenerate (i.e. Levi-nondegenerate) hypersurface. Indeed, this is not possible if the (scalar-valued) Levi form of the hypersurface changes sign. For instance, the hypersurface $M \subset \mathbb{C}^2$ given by $\{(z,w) \in \mathbb{C}^2 : \mathsf{Im}\, w = (\mathsf{Re}\, z)^3\}$ cannot be approximated by a Levi nondegenerate hypersurface, since a Levi nondegenerate hypersurface would have either a strictly positive or a strictly negative scalar Levi form (while the Levi form of M takes both signs, depending on the sign of $\mathsf{Re}\, z$). However, it follows from Theorems 2.2 and 2.4 that any real hypersurface in $\mathbb{C}^2$ can be approximated by a 3-nondegenerate hypersurface. The following example shows that in $\mathbb{C}^2$ approximation by a 2-nondegenerate hypersurface is impossible in general, so that these theorems are sharp for $N=2$ and $m=3$.

Example 8.1. Consider the real-analytic hypersurface $M \subset \mathbb{C}^2$ given by

$$\mathsf{Im}\, w = 2\mathsf{Re}\,(z^3\bar{z}) + z\bar{z}\mathsf{Re}\, w. \tag{8.1}$$

It is easy to check that M is 3-nondegenerate at 0. Let $W' \subset J^3(\mathbb{R}^3,\mathbb{C}^2)$ be the set of all 3-jets of 2-degenerate embeddings of $\mathbb{R}^3$ into $\mathbb{C}^2$. As in (3.2), we write the decomposition

$$J^3_{0,0}(\mathbb{R}^3,\mathbb{R}) = (J^3)_h \oplus (J^3)_{nh}.$$

We observe that $(J^3)_{nh}$ is a 4-dimensional vector space and that W' is a submanifold of $J^3(\mathbb{R}^3, \mathbb{C}^2)$ of real codimension 3. Indeed, using Lemma 3.4 with $k = 3$, $N = 2$, and $m = 3$ (as well as the definition and invariance of 2-degeneracy), we see that, in the notation of the lemma, W' is given by the vanishing of the 3 real components of Ψ_4 corresponding to $\psi_{z\bar{z}}(0)$ and the real and imaginary parts of $\psi_{z^2\bar{z}}(0)$. We write Ψ_4' for the mapping corresponding to these three real components of Ψ_4. We now claim that if τ is the embedding from $\mathbb{R}^3$ to $\mathbb{C}^2$ whose image is M (given by (8.1)), the 3-dimensional submanifold $j^3\tau \subset J^3(\mathbb{R}^3, \mathbb{C}^2)$ intersects W' transversally. The fact that M cannot be approximated by a 2-nondegenerate hypersurface will then follow from this transversality. Indeed, if τ' is an embedding close to τ, then $j^3\tau'$ must also intersect W' at some point, and hence $\tau'(\mathbb{R}^3)$ cannot be everywhere 2-nondegenerate.

To prove the claimed transversality, we define the mapping

$$\mathbb{R}^3 \ni (x, y, s) \mapsto \mu(x, y, s) := \Psi_4'\big(j^3_{(x,y,s)}\tau\big).$$

Since W' is defined by the equation $\Psi_4' = 0$, the transversality of $j^3\tau$ to W' is equivalent to the fact that μ is of rank 3 at the origin of $\mathbb{R}^3$. The latter fact can be proved by a direct calculation using the definition of Ψ_4 given in Lemma 3.4, and is left to the reader.

Remark 8.2. Though a hypersurface $M \subset \mathbb{C}^2$ cannot be approximated in general by 2-nondegenerate ones, we claim that such a hypersurface can be approximated by a 3-nondegenerate one that is 2-nondegenerate outside a discrete subset. To see this, let M be a 3-dimensional manifold, $W_3' \subset J^4(M, \mathbb{C}^2)$ the set of all 4-jets of 3-degenerate embeddings of M into $\mathbb{C}^2$, and $W_2' \subset J^3(M, \mathbb{C}^2)$ the set of all 3-jets of 2-degenerate embeddings of M into $\mathbb{C}^2$. Note that $\operatorname{codim} W_3' = 6$ and $\operatorname{codim} W_2' = 3$. By Thom transversality theorem, the set

$$\{f \in C^\infty(M, \mathbb{C}^2) : j^3 f \pitchfork W_2',\ j^4 f \cap W_3' = \emptyset\} \tag{8.2}$$

is residual, and hence any embedding $\tau : M \to \mathbb{C}^2$ can be approximated by an embedding in (8.2). The rest of the claim follows from the fact that, by dimension, if $j^3 f$ is transversal to W_2', then the intersection of the two sets is necessarily discrete (or empty).

In particular, it follows from Remark 8.2 that any compact hypersurface in $\mathbb{C}^2$ can be approximated by another one which is 3-nondegenerate everywhere and 2-nondegenerate except at finitely many points. Our final example shows that for any positive integer ℓ, there is a real-analytic boundary of a bounded domain in $\mathbb{C}^2$ that cannot be approximated by a hypersurface that is 2-nondegenerate except at $\ell - 1$ points.

Example 8.3. Let ℓ be a positive integer, and let $(r, \omega) \in \mathbb{R}_+ \times S^3$ be the usual polar coordinates in $\mathbb{C}^2 \cong \mathbb{R}^4$. We shall show that there a real polynomial P defined in $\mathbb{R}^4$ such that the boundary of the domain

$$D := \{(r, \omega) \in \mathbb{C}^2 : 0 \le r < e^{P(\omega)}\} \tag{8.3}$$

satisfies the above property. For this, we choose any ℓ distinct points $\{p_1, \ldots, p_\ell\}$ on the unit sphere $S^3 \subset \mathbb{C}^2$ and let M be the hypersurface given in Example 8.1. The reader can easily check that there is a real polynomial P defined in $\mathbb{C}^2$ vanishing at $p_1, \ldots, p_\ell$ such that the following hold:

(1) If D is given by (8.3) then its boundary, ∂D, is tangent to the (real) tangent hyperplane to the sphere at p_k, for $k = 1, \ldots, \ell$.

(2) For each k, $k = 1, \ldots, \ell$, the 4-th jet at p_k of the hypersurface ∂D is the same as that of M at 0, after an appropriate translation and complex linear transformation in $\mathbb{C}^2$.

Note that the conditions above involve only the derivatives of P at p_k, $k = 1, \ldots, \ell$, of order ≤ 4, and that P is obviously not unique.

We now observe that, by the transversality argument used in Remark 8.2, any real hypersurface $\widetilde{M} \subset \mathbb{C}^2$ that is tangent to M (given in Example 8.1) at the origin of order at least 4, has the same property as M, i.e. $\widetilde{M}$ cannot be approximated by smooth 2-nondegenerate hypersurfaces. We conclude that any smooth compact hypersurface sufficiently close to ∂D must have at least ℓ 2-degenerate points. Recall that the approximating hypersurface can always be chosen to be everywhere 3-nondegenerate as a consequence of Theorem 2.2.

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