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University of Dublin

Trinity College

Developing Mathematics Teachers' Pedagogical Content Knowledge through Lesson Study:

A Multiple Case Study at a Time of Curriculum Change

A thesis written in fulfilment of the requirements for the degree of

Doctor in Philosophy (Ph.D.)

2015

Aoibhinn Ní Shúilleabháin

Supervisor: Dr Aidan Seery

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SUMMARY

This research was interested in investigating a model of in-school professional development that would support teachers in their practices and in their learning, particularly within a backdrop of curriculum reform. Lesson study (Lewis, Perry, & Murata, 2006) was identified as a model of professional development which would provide structure for collaboration within a teacher community where teachers could be provided with opportunity to develop their knowledge through participation. In contributing to advancing knowledge, this research offers a detailed analysis of: how teachers' pedagogical content knowledge was enhanced both in terms of detailing how teachers employed and enhanced their knowledge, and in detailing the mechanisms within lesson study which contributed to teacher learning by mapping that knowledge to an existing framework as suggested by Ball, Thames & Phelps, (2008). The research also suggests a new dimension of teacher knowledge - teacher knowledge of student - which may be added to the existing framework of mathematical knowledge of teaching.

The literature used to guide the methods of this research was based on the study's research question:

1. How is teacher knowledge, particularly pedagogical content knowledge, enhanced through lesson study?

The literature brought together three strands of literature: curriculum reform, professional development, and lesson study. Within these strands, literature on mathematical knowledge for teaching, professional noticing of students' mathematical thinking, and literature available on the Project Maths reform were also explored.

With regards to the methodology employed in this research, qualitative data was generated as a multi-site case study based in two post-primary schools and involving 12 mathematics teachers. These two groups of teachers participated in iterative cycles of lesson study throughout one academic year and data was generated through audio recordings of the teacher meetings conducted around lesson study and through interviews with participating teachers. As researcher, I participated in the investigation as a participant observer, facilitating lesson study and participating in teacher meetings as a former mathematics teacher and lecturer of mathematics pedagogy.

The data was analysed in four stages. The first two stages of analysis were performed as thematic analysis, and a researcher audio log and field notes assisted in building a holistic interpretation of the data. Stages three and four of the analysis were a refinement of the themes. Codes were identified based around teacher learning and the new perspectives teachers were developing on their practice due to their participation in lesson study. The data, analysis and discussion are presented in three chapters with the first analysis (Chapter 4) presenting a refined framework of analysis of teacher knowledge and the following two chapters (Chapters 5 and 6) dedicated to a temporal analysis of teachers' developing knowledge over iterative lesson study cycles.

Preliminary findings from the data revealed that viewing teaching and learning practices from a student's perspective, or employing a critical student lens (Fernandez Canon, & Chokshi, 2003), was a major theme within teachers' discussions around lesson study. This employ of a student lens by teachers who had not regularly incorporated this perspective on their teaching, enhanced their pedagogical content knowledge and impacted on their teaching and learning practices. Using a framework of teacher knowledge suggested by Ball et al. (2008) how teachers used and enhanced their KCS and KCT was analysed and detailed. A new dimension of teacher knowledge, that of teachers' knowledge of their own students or TKS also emerged as an important finding within this research.

The main contributions of this research are to the literature on lesson study in detailing how teachers' pedagogical knowledge, KCS and KCT, can be developed through participating in iterative cycles of lesson study. This research also contributes to theory in identifying a third dimension of teacher knowledge, that of teacher knowledge of students, which is situated alongside teachers' specialised content knowledge and pedagogical content knowledge. Furthermore, this study contributes to practice in its recommendation for the inclusion of structured, collaborative, in-school professional development, led by the teacher, as part of teaching and learning practices in Irish post-primary schools.

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LIST OF ABBREVIATIONS

CCK	Common Content Knowledge
CPD	Continuous Professional Development
DES	Department of Education & Skills
KCS	Knowledge of Content and Students
KCT	Knowledge of Content and Teaching
MKT	Mathematical Knowledge for Teaching
NCCA	National Council of Curriculum & Assessment
NCTM	National Council of Teachers of Mathematics
PCK	Pedagogical Content Knowledge
PMDT	Project Maths Development Team
PMISG	Project Maths Implementation Support Group
SCK	Specialised Content Knowledge
SMK	Subject Matter Knowledge
TKS	Teacher Knowledge of Students
TY	Transition Year

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Chapter One

1 Introduction

1.1 The Research

This research is conducted during the most dramatic reform of post-primary mathematics in the history of the Irish education system. The purpose of this study is to investigate how participating in a school-based model of professional development may benefit teachers, specifically in their learning within a backdrop of curriculum reform. As in all educational reform, teachers play a key role and appropriate professional development for teachers is paramount to ensuring the implementation and sustainability of introducing a new curriculum (Cohen & Barnes, 1993; Remillard, 2000, 2005; Remillard & Bryans, 2004). In addition, context specific school improvements can positively impact both teacher learning and student learning (Hopkins & Reynolds, 2001; Putnam & Borko, 2000).

Similar to the 'Math Wars' in the USA (Boaler, 2009; Oldham & Ní Shúilleabháin, 2014), the introduction of Project Maths has brought with it its fair share of controversy. The introduction of Project Maths has changed the mathematical content taught to post-primary students, the layout of students' assessments and, most dramatically, has called for a change in how teachers approach their teaching and learning of mathematics (Cosgrove, Perkins, Shiel, Fish, & McGuinness, 2012; Jeffes et al., 2012; Oldham & Close, 2009). Project Maths encourages a more constructivist approach to learning where students' communication of their mathematical understanding is encouraged and where there is an explicit focus on developing higher-order, problem solving skills (Lynch, 2011; National Council for Curriculum and Assessment, 2005, 2012a, 2012b). Without offering opinion on the merits of the curriculum or on the reform processes, the teaching and learning practices envisaged by the curriculum are relevant to this research since they are directly impacted by teacher knowledge (D. Ball, Hill, & Bass, 2005; Ball & Hyman, 2003; Ball, Thames, & Phelps, 2008; H. Hill, Sleep, Lewis, & Ball, 2007; H. C. Hill, Rowan, & Ball, 2005; Rowland, 2007; Rowland, Huckstep, & Thwaites, 2005).

Teacher knowledge and what encompasses all that teachers need to know in order to teach has been debated for decades. Shulman (1986) and Grossman (1995) have both suggested elements of teacher knowledge that are required for a teacher to not only have

an in-depth knowledge of the subject they are teaching, but to also know how to teach that subject. In more recent years teacher knowledge has been separated into two not necessarily distinct forms of knowledge: subject matter knowledge (SMK) and pedagogical content knowledge (PCK) (D. Ball et al., 2005; Ball et al., 2008; Baumert et al., 2010; Rowland, 2007; Rowland et al., 2005). This research focuses on pedagogical content knowledge since a prominent feature of the new Irish post-primary curriculum is not just that it will change *what* mathematics teachers will introduce to their students but *how* they will approach the teaching and learning of mathematics (Jeffes et al., 2012). This research will therefore focus on how teachers can be supported in strengthening and building their pedagogical content knowledge through lesson study for the purposes of the teaching of mathematics.

Specifically, the research questions:

1. ***How is teacher knowledge, particularly pedagogical content knowledge, enhanced through lesson study?***
    a. How are knowledge of content and students (KCS) and knowledge of content and teaching (KCT) enhanced within lesson study?
    b. Are other forms of teacher knowledge, external to subject matter knowledge, employed within lesson study?

In focusing on teachers' pedagogical content knowledge it is necessary to align this research with a mathematical framework of teacher knowledge. While there are many detailed and relevant frameworks of mathematical knowledge (Baumert et al., 2010; Delaney, 2012; Rowland et al., 2005), Ball et al.'s (2008) model provides a framework of mathematical knowledge for teaching which aligns with the data generated within this research. This framework was used in analysing how teachers employed and enhanced their PCK due to their participation in iterative cycles of lesson study.

In attempting to answer this question, this study will identify and investigate the literature surrounding lesson study and mathematical knowledge for teaching. It will then attempt to build on the literature by engaging teachers in two schools in a model of lesson study and by analysing the knowledge employed by and enhanced through their participation.

1.2 Teaching Mathematics

Before highlighting the contents of this thesis, I would like to take some time in outlining my own perspective of mathematics and of mathematics teaching. Mathematics is not solely about memorising rules, facts or equations or rhyming off numerical operations, but instead is about finding patterns, coming up with inventive solutions, and putting forward suggestions that might match an evolving body of knowledge (Boaler, 2009; Feachem, 2014; Polya, 1945; Schoenfeld, 1992, 1994). Maths is powerful in its ability to reduce and represent information in a way that is concise, accurate and universally understood (Ball & Hyman, 2003). It should be seen as a creative, developing activity and not as a rigid, historically defined, static body of knowledge as it is often represented to those learning mathematics (Greer, 1996; Mason, 1988).

Mathematical learning and teaching are linked by classroom interactions (Voigt, 1996). Mathematics teachers are in a privileged and unique position to not only make their students aware of the skills, facts, and rules of mathematics but to also open the door to the creativity, power, and universality of what mathematics can represent. To represent this with an analogy, if mathematics were represented by a meticulously detailed and fragile piece of lace in building students' mathematical understanding teachers have to carefully un-pick portions of it in order to make visible the various structures, patterns and relationships contained inside. Students could then themselves rebuild the design where the teacher has left room for them to embellish or accent certain aspects of the pattern as they see relevant to them. A teacher must be aware not to unravel too much too quickly lest the original composition get lost or too little in case the student does not understand the construct.

Teaching mathematics may be unusual in the fact that it is a subject configured by set rules and systems and yet is composed through amalgamating creative and new ways of tackling a problem. In reducing mathematics to portions of knowledge that are workable for students, teachers create a bridge between what a student does not yet know and the body of knowledge that is mathematics. In order to do this, teachers must have a solid knowledge of the subject, have knowledge of how to teach that subject and have an understanding of how their students might learn that subject (Ball et al., 2005; Ball et al., 2008; Grossman, 1995; Krauss et al., 2008; Shulman, 1987).

This research is about mathematics teachers and is based on how teachers can be supported in their learning and therefore be supported in their classroom practices which will positively impact on how students learn mathematics.

1.3 Researcher Note

As a former post-primary Mathematics teacher, I have personal experience of the post-primary school structure and environment and enter this research as a former teacher in one of the 24 national Project Maths phase 1 schools (see Section 2.2.1). As member of a productive, pro-active, well organised, and communicative Mathematics Department, I was very lucky as an early career teacher to be encouraged to collaborate with more experienced colleagues and to be exposed to good practice. However, as a Mathematics Department engaged in the initial curriculum reform, my colleagues and I experienced the need for opportunities to meet with one another and discuss teaching methodologies in implementing this new, unfamiliar, and evolving mathematics curriculum. As teachers who were positive about participating in this initial stage of reform, we struggled to find suitable times in which we could collaborate outside of administrative duties and found it difficult to establish a useful, productive structure for meetings where we were able to talk about our pedagogy.

As a facilitator of the new curriculum providing external professional development courses for teachers participating in the second phase of the reform (Section 2.2.1), I thought that a school-based model of professional development would enhance and support Irish post-primary teachers' engagement with the new curriculum alongside those in-service workshops and modular courses provided by the Project Maths Development Team (PMDT). Furthermore, as a lecturer of mathematics pedagogy to pre-service post-primary teachers, I have felt that collaboration and communication between teachers would have benefits for colleagues of all experiences and backgrounds in teaching mathematics.

My initial quest in beginning this research was to investigate how collaboration would assist teachers in their practice. In researching literature on teacher communities I came across lesson study as a model of professional development which could provide structure for teachers to collaborate while still focusing on how their students' learnt mathematics. This seemed like a useful model to investigate and, although I was unfamiliar with this form of professional development, I wanted to investigate how

lesson study might be incorporated within an Irish post-primary structure and study how teachers would benefit and learn from their participation. From a further investigation of the literature the research question emerged. As has been referenced in the summary, teachers engaged positively in this model of professional development and the generated data provided evidence of teacher learning, specifically a development of pedagogical content knowledge.

1.4 Thesis Contents

This research is divided into seven chapters. This introductory chapter has served to establish the motivation and interest in the research, as well as the aims and philosophical grounding which guides the investigation.

Chapter Two presents key literature around the research question focusing on: the Project Maths curriculum and curriculum reform, teacher professional development, mathematical knowledge for teaching, and lesson study. Within teacher professional development, a specific focus is given to teacher community as a structured form of engaging teachers in their own learning (P. Grossman, Wineburg, & Woolworth, 2001; Little, 2002; M. W. McLaughlin & Talbert, 2001, 2006). In examining the literature on mathematical knowledge for teaching, particular attention is directed towards Ball et al.'s (2008) proposed model, previously referenced in Tepylo and Moss' (2011) investigation into changing teacher mathematical knowledge through lesson study. Within the section on lesson study, literature on professional noticing of students' mathematical thinking is incorporated as part of the lesson study process.

Chapter Three presents the methods of this research. In considering the research question the approach of the research is first explored as a multiple case study and the qualitative nature of the data generated is examined. The researcher role as a participant-observer (Bogdan & Biklen, 2007) is also explored since I acted as a facilitator of lesson study but also as the sole researcher in this investigation. Information on contacting the schools who participated in the research and on the schools themselves is then shared. Before presenting the practical methodology, research boundaries and ethics are discussed. The research instruments are explored in detailing how the qualitative data was generated through audio recordings of teachers' lesson study meetings and of interviews with participating teachers. While the analysis was conducted using the aforementioned qualitative data, other data generated within the study is also discussed including the use

of field notes and a researcher audio log. As the sole researcher, it was important to include perspectives on researcher reflexivity and these are detailed within this chapter. Finally, an outline of the analysis of data is articulated.

Chapter Four is the first of three data analysis chapters. This chapter outlines a refinement and enhancement of a model of mathematical knowledge for teaching as presented by Ball et al. (2008). The chapter argues that the results of the research provide a greater understanding and detail of teachers' knowledge of content and students (KCS) and of teachers' knowledge of content and teaching (KCT) in detailing how teachers employed their knowledge through their participation in lesson study and how this knowledge was enhanced over the course of the research. This enriched understanding is achieved by referencing an earlier finding by Fernandez et al. (2003) who suggested that in participating in lesson study, teachers employed a 'critical student lens' on their practice and began incorporating how they thought their own students might view any teaching and learning approaches and strategies. In building on the perspective of the student and in incorporating this perspective in their planning of and reflection on research lessons, teachers enhanced their KCS and KCT. In this chapter a further dimension of teacher knowledge is also postulated from the data, that of teacher knowledge of student (TKS). As a case study on two sites, the teacher communities at Doone and Crannóg are analysed separately in this research.

Chapter Five is the second of three data analysis chapters but is the first chapter to analyse changes to teachers' PCK in detail. This chapter is dedicated to chronologically analysing how teachers' KCS was employed in both schools and how changes to teachers' KCS are evident in the planning and reflection of research lessons. Proposed features of KCS emergent from the data and which align with the literature are outlined. The analysis begins by detailing how teachers' KCS in both schools was utilised in the first cycle of lesson study and proceeds by tracing changes in how features of KCS were incorporated and built on in subsequent lesson study cycles. Features of KCS emerging from the data and aligning with the literature are outlined.

Chapter Six is the last of the three data analysis chapter and the second chapter to analyse changes to teachers' PCK in detail. This chapter is dedicated to chronologically analysing how teachers' KCT was employed in both schools and how changes to particular proposed features of KCT are evident in the planning and reflection of research

lessons. These suggested features of KCT emergent from the data, some of which align with the literature, are outlined. Similar to the previous chapter, the employment of teachers' KCT is chronologically analysed through iterative lesson study cycles. This chapter additionally analyses evidence of TKS throughout teachers' planning and reflection of research lessons. Two proposed features of TKS which emerged from the data are detailed.

Chapter Seven takes the study's research question and connects it to both the data and the literature to present the study's findings and contributions. The limitations of the research are outlined, as are recommendations for policy and practice.

1.5 Summary

Mathematics is a deep and abstract intellectual achievement, but while there is a place for rigor, organisation and logic within the subject, mathematics can also be enjoyed as a social endeavour, particularly for students encountering mathematics topics for the first time. Students actively construct their mathematical ways of knowing and learn through social interactions and engagement with the subject (Cobb, 2013) and this approach to learning is reflected within the Project Maths curriculum (National Council for Curriculum and Assessment, 2012a; 2012b). To this end, teachers can also engage in professional social interactions in planning detailed, creative trajectories of learning (Suh, 2014) for their students which also encourage communication of their understanding of the topic. In this research, lesson study reflects the same social interactions as put forward within the new Project Maths syllabus where teachers communicate, collaborate, and reflect on their teaching while putting forward and developing new and creative ideas on teaching mathematics. Lesson study provides opportunity for teachers to experience the benefit of discussing and developing teaching and learning content and materials with their colleagues.

With regards to performing this research in an Irish context over the duration of one academic year, the nature and extent of the investigations carried out with mathematics teachers at post-primary level have not been recorded previously in the literature.

Chapter Two

2 Literature Review

2.1 Introduction to Literature Review

In this chapter, four separate literatures are reviewed and brought together. In the first part, literature on the new Irish post-primary mathematics curriculum in Ireland, Project Maths, and curriculum reform will be explored. Leading directly from the findings of curriculum reform research, there is a defined need for professional development in order to support teachers in a time of curriculum change (Nicol & Crespo, 2006; Remillard, 2000) and, as the second part of the chapter, literature on teacher professional development will be explored in the form of teacher community as site for teacher learning (Arbaugh, 2003; P. Grossman et al., 2001). In the third section, mathematical knowledge for teaching will be examined based on a framework described by Ball et al. (2008) and stemming from an introduction of pedagogical content knowledge as suggested by Shulman (1986). Finally, the literature of lesson study (Lewis, Perry, & Murata, 2006) will be explored as a model of professional development and teacher community which develops teacher knowledge.

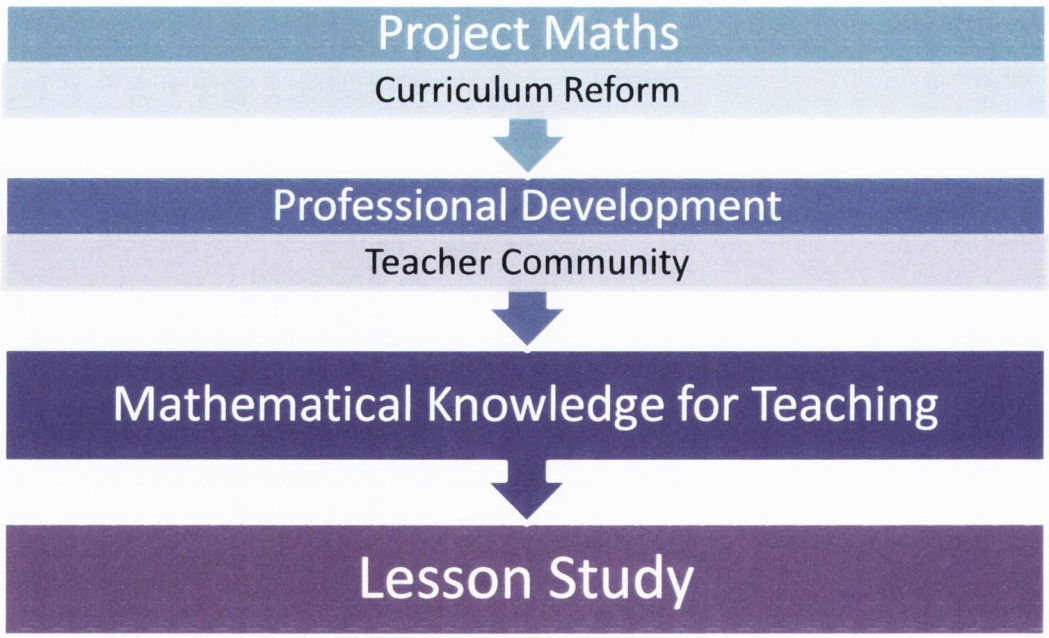


Figure 1 Outline of the Literature Review Chapter

2.2 Post-primary Curriculum Reform in Ireland: Five Features of Project Maths

As a broad definition, curriculum refers to the substance or content of teaching and learning (M. K. Stein, Remillard, & Smith, 2007). It refers to the expectation for instruction which is laid out in policy documents and is sometimes referred to as the 'intended' curriculum signifying that there are, in general, differences between an intended, implemented and enacted curriculum (Kilpatrick, 2009; van den Akker, 2003). In more recent years, the term 'curriculum' has been broadened to incorporate the material resources that have been designed to accompany a syllabus (M. K. Stein et al., 2007) including: official publications, special websites being developed, printed copies distributed to schools and central education authorities providing notes and implementation rules (Eurydice, 2011).

In Ireland, as in the vast majority of European countries (Eurydice, 2011), the curriculum is constructed outside the school and approved by a central education authority the National Council for Curriculum & Assessment (NCCA). This central initiative enables broad and uniform education and the engagement of experts in the curriculum development process while also providing teachers with constructed concrete materials which can be used in class. These curricula define the objectives, learning outcomes and content of teaching and it is assumed that the curriculum documents enable teachers to implement the curriculum faithfully.

In the past decade there has been an international shift of curricula emphases towards critical thinking, technological modernity, and mathematical problem solving and away from those which mainly emphasise skill-level instruction (Galbraith, 2013; Lesh & Zawojewski, 2007; Pampaka et al., 2012; Singer & Voica, 2013; Verschaffel et al., 2014). In recent years, the issue of competence in mathematics has become increasingly important and many new national mathematics curricula have emphasised competencies and problem solving skills, an increase in cross-curricular links, and a greater emphasis on the application of mathematics in everyday life (Eurydice, 2011).

In Ireland it has been argued that there is much room for improvement in mathematics education and concern has primarily lain with students' failure to make a successful transition to third level mathematically and scientifically oriented undergraduate courses (EGFSN, 2008; Project Maths Implementation Support Group, 2010). These concerns

are coupled with a growing concern over school-leavers literacy and numeracy levels and with post-primary students' underperformance in international mathematics assessments such as PISA and TIMSS (Cosgrove et al., 2012; Department of Education and Skills, 2011; Ríordáin & Hannigan, 2011; Shiel, Perkins, Close, & Oldham, 2007).

The move towards a curriculum in Ireland which had an increasing emphasis on problem solving began in the 1990s with a revision of the primary school curriculum published in 1999 (National Council for Curriculum and Assessment, 1999) and with the mathematics element formally introduced to classrooms in 2002. At the turn of the millennia, plans to construct a new post-primary curriculum that would better align with the primary curriculum and incorporate more problem solving skills began (Conway & Sloane, 2005; National Council for Curriculum and Assessment, 2003, 2005, 2008, 2009).

In 2010, the national phased roll-out of a new centralised mathematics post-primary curriculum entitled Project Maths began (National Council for Curriculum and Assessment, 2012c; Ní Shúilleabháin, 2014). Due to the high stakes nature of the Irish Leaving Certificate Examination, which governs students' entrance into third-level education, teaching and learning approaches at post-primary level over the past decades have developed to focus on rote learning of techniques and strategies (Lyons, Lynch, Close, Sheerin, & Boland, 2003; National Council for Curriculum and Assessment, 2005; Oldham, 2001). In the prior mathematics curriculum many teachers avoided teaching of mathematics topics that were deemed more difficult (State Examinations Commission, 2005) and many students, while proficient in instrumental skills, may not have had a relational understanding (Skemp, 1976) of their mathematics.

The Project Maths curriculum was designed not only to change *what* students learn about mathematics, but *how* they learn and how they are assessed (Jeffes et al., 2012). The curriculum is based on an approach which amalgamates elements of constructivism (Ernest, 1996) and realistic mathematics education (RME) where learners' active participation and sense-making is an important basis in designing the teaching and learning approaches experienced within the classroom (Lynch, 2011). Through experiencing this intended curriculum, students are expected to improve their higher-order skills such as the ability to justify and reason logically. This curriculum places greater emphasis on problem solving, on an investigative approach for students, and on students developing skills in communicating their mathematics (Cosgrove et al., 2012;

Jeffes et al., 2013; Perkins, Shiel, Merriman, Cosgrove, & Moran, 2013). Project Maths thus represents a philosophical shift in Irish post-primary classrooms from a highly didactic approach with relatively little emphasis on problem solving (Lyons, Lynch, Close, Sheerin, & Boland, 2003; Oldham, 2001) towards a dialogic, investigative, problem-focused approach to teaching and learning mathematics. Five features of the Project Maths reform, gleaned from available literature, will now be explored:

1. Mechanisms of the reform
2. Sociocultural underpinning of the curriculum
3. Curriculum encompassing more than content
4. Available curriculum materials
5. Teaching approaches encouraged by the curriculum

2.2.1 Mechanisms of the Project Maths Reform

In September 2008, 24 pilot schools began phase 1 of Project Maths introducing revised strands of the curriculum to specific year groups (National Council for Curriculum and Assessment, 2012c). In 2010 the national Phase 2 roll-out began with the rest of the approximately 750 post-primary schools around the country introducing the new curriculum. Roll-out at both phases was accompanied by amended state examinations which deviated greatly from their previous layout and assessment format. By 2015 the roll out of the new common curriculum will be completed and all students will sit a common state examination amalgamating both phases of the reform.

Phase	Syllabus strands	Pilot Schools Commence	Pilot Schools Changed Exams	All Schools Commence	All Schools Changed Exams
1	1 + 2	Sept. 2008	JC 2011 LC 2010	Sept. 2010	JC 2013 LC 2012
2	3 + 4	Sept. 2009	JC 2012 LC 2011	Sept. 2011	JC 2014 LC 2013
3[1]	5	Sept. 2010	JC 2013 LC 2012	Sept. 2012	JC 2015 LC 2014

Figure 2 Timeline of roll-out of Project Maths (NCCA, 2008)

A Project Maths Development Team (PMDT), through the Department of Education and Skills (DES), has provided professional development workshops and modular courses,

taught by teacher facilitators, outside of school for mathematics teachers throughout the country as well as providing curriculum materials available on-line for teachers (Project Maths Implementation Support Group, 2010). Regional Development Officers (RDOs) have been made available to visit schools for particular training and curriculum content updates during the introduction of this new curriculum (Project Maths Implementation Support Group, 2010). While the Project Maths approach is innovative from an Irish perspective of curriculum reform, it still takes the form of the workshop style training (Corcoran, 2011b) and the curriculum reform may benefit from more school-based, teacher-led professional development models (Charalambos & Philippou, 2010; OECD, 2009; Yoshida, 2012). As the phased curriculum introduction comes to an end, and in order to continue to provide professional development and support for teachers, it is now timely that a sustainable school-based model for mathematics teachers be investigated.

2.2.2 Underpinning Project Maths: A Sociocultural View of Learning

Learning and performance in schools has traditionally focused on the individual, whereas out-of-school activities are much more often carried out in groups. This form of learning de-emphasises the social and cultural characteristics of the student, and the classroom becomes out of context with the real world (Greer, 1996; Johnson & Johnson, 1999). Learning has more recently been noted as a culturally and socially situated activity and where knowledge is constructed by individuals as a result of their cultural and social experiences (Cobb & Bowers, 1999; Cobb, Jaworski, & Presmeg, 1996; Forman, 1996; E. Smith, 1995). This view of learning is one based on social interactions and was a paradigm shift from a view of learning which was based on instrumental and behavioural cognitive theories (Cobb et al., 1996; Hughes, 2007; Wertsch, 1985). In this sociocultural view, learning is seen as a type of transformation (Gadamer, 1989; Wenger, 1998) and places the lived experience of the learner at the centre of its theory.

A constructivist view of learning acknowledges that teachers should reflect on their students' strategies of doing mathematics instead of the previous behaviourist view of learning which was of passive reception (Ernest, 1996). Within this research, constructivism describes a sociocultural epistemology where the learner is seen as an active agent in constructing their own knowledge through interactions with the world around them. In the case of mathematics, students need to be actively involved in

practising mathematics, in working in small groups, in negotiating ways to define and organise their idea, and in explaining their ideas to each other (Boaler, 1999; Forman, 1996; Goos, 2004; Schoenfeld, 1992). In this context students should see themselves and their peers as sources of and constructors of knowledge instead of relying on the teacher or textbook as the fountain of all knowledge (Forman, 1996). Mathematics should be culturally meaningful and engaging to students (van Oers, 1996) but should also see the subject as having the potential for challenge and creativity.

Vygotsky's belief in the social foundations of learning (Bruner, 1985; Stone & Wertsch, 1985; Wertsch, 1985), Rousseau's certainty that children need to be active learners and Dewey's declaration that true education only occurs in the demands of social situations lend roots to the notion of learning as a social event and these roots are most certainly embedded in the activity of dialogue. One of the most important tools of mediating learning is thus language (Adler, 2000; Cobb et al., 1996; Gerald, 2005) and hence within this constructivist pedagogy, communication within the classroom should be to the fore. Equally within teacher professional development, structured dialogue between colleagues may provide valuable opportunities for learning.

2.2.3 Project Maths Curriculum

Project Maths not only implies changes to curriculum content, but also changes to teaching and learning approaches within the classroom (Jeffes et al., 2012). Teachers are encouraged to incorporate a more constructivist approach to learning where students are actively engaged in their learning through investigative and problem solving approaches and within the syllabi there is a strong emphasis on developing students' relational understanding and incorporating more communication within the classroom (National Council for Curriculum and Assessment, 2008, 2012a, 2012b).

The curriculum is influenced by other similar curriculum reforms (National Council of Teachers of Mathematics, 2000; Oldham, 2005) and represents a philosophical shift in Irish post-primary classrooms from a highly didactic approach (Oldham, 2001) towards a dialogic, investigative, problem-focused approach to teaching and learning mathematics. This curriculum bases its objectives on the National Research Council's five strands of mathematical proficiency: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition, which replace Skemp's (1976) terminology which was employed in all previous syllabi since 1990 (Oldham &

Ní Shúilleabháin, 2014). The revised Project Maths curriculum's teaching and learning approaches may be summarised as the following:

(1) A greater focus on learners' relational understanding of mathematical concepts, building from the concrete to the abstract and from the informal to the formal (Lynch, 2011)
(2) An approach to teaching and learning which gives prominence to developing learners' skills in communicating their mathematical understanding with others with teachers as facilitators of discussions (National Council for Curriculum and Assessment, 2008, 2012a, 2012b)
(3) An approach to teaching and learning mathematics which encourages an investigative, problem-solving approach with emphasis on application in real-life settings and contexts where students become active participants in developing their mathematical knowledge and skills (Lynch, 2011; Oldham & Close, 2009)

An additional feature of the curriculum encourages teachers to use supplementary curriculum materials instead of a traditional over-reliance on textbooks as a curriculum source (Cosgrove et al., 2012; O'Keefe & O'Donoghue, 2012).

The content of the Project Maths curriculum is defined through a 'strand' format detailed with specific student learning outcomes (National Council for Curriculum and Assessment, 2012a). This deviates greatly from previous mathematics curricula where content was listed without reference to students' learning outcomes (National Council for Curriculum and Assessment, 2000). The five strands are laid out as follows:

1. Statistics and Probability
2. Geometry and Trigonometry
3. Number
4. Algebra
5. Functions

The syllabus explicitly states that the strands should not be studied in isolation but, where appropriate "connections should be made within and across the strands and with other areas of learning" (National Council for Curriculum and Assessment, 2012a, p.8). A further distinction between this new curriculum and its predecessor is that within Project Maths, the various levels of the curriculum (Foundation, Ordinary, and Higher) are subsets of one another making it easier for students to transition between levels where

necessary. This strand format is very similar to that of the primary mathematics curriculum (An Roinn Oideachas, 1971) and can also be aligned with the Junior Cycle Framework which is due to be implemented in mathematics in 2019 (National Council for Curriculum and Assessment, 2011).

2.2.4 Curriculum Materials

2.2.4.1 Curriculum Resources

The majority of teachers rely on curriculum materials for their teaching and if those materials do not exist, it is likely that topic will not be taught (M. K. Stein et al., 2007). Remillard & Bryan's (2004) investigation into how primary teachers engaged with a new mathematics curriculum showed that teachers' orientations toward using curriculum materials influenced the way they used them, regardless of whether they agreed with the mathematical vision within the materials. Therefore, how teachers are introduced to any supplied curriculum materials leads to varying curricular experiences for both student and teacher. Curriculum reform efforts should thus include assisting teachers in examining unfamiliar curriculum resources and developing new approaches to using these materials (Hanley & Torrance, 2011; Remillard & Bryans, 2004).

2.2.4.2 Textbook as a Curriculum Source

Textbooks have been commonly viewed as sources of explanations and exercises for students but often represent mathematics in terms of formulae without providing a relational understanding of the content (D. Ball, Lubienski, & Mewborn, 2001; M. K. Stein et al., 2007). Textbooks can provide a framework for teachers in teaching certain content at specific stages (Nicol & Crespo, 2006), but also group together topics in a way that is often not inter-linked and does not cater for all students. In a review of the role of textbooks in curriculum reform, teachers have been criticised for their over-dependence on textbooks as the only curriculum source (Christou, Eliophotou-Menon, & Philippou, 2004; Teddlie, Creemers, Kyriakides, Muijs, & Fen, 2006).

Teachers can become reliant on centralised curricula and unused to critically reviewing them (Handelzalts, 2009). Remillard (2000) found that teacher learning was enhanced when teachers engaged in using and evaluating innovative curriculum materials as part of reform instead of relying on textbooks to script and govern the content of a lesson. It is therefore important in a time of curriculum change that teachers are supplied with

curriculum materials that support texts but where teachers are encouraged to critically analyse and modify these materials to suit their own students (Hanley, 2010).

2.2.4.3 Project Maths Curriculum Materials

In Ireland, teachers have traditionally referenced textbooks as syllabus documents and it was hoped that with the new curriculum teachers would instead more frequently reference the syllabus (Cosgrove et al., 2012). An extensive range of curriculum materials have been developed by the PMDT and teachers are encouraged to access these through professional development workshops and web based support (Project Maths Development Team, 2011). These materials include teaching and learning plans based on many elements within the curriculum, particularly strands which were not commonly taught on the previous curriculum. The materials also include interactive games for students, planning documents for teachers and animated on-line resources all available from the dedicated curriculum website. Some prior resources from the previous curriculum are still relevant as they were only available at Junior Cycle level (National Council for Curriculum and Assessment, 2002).

2.2.5 Teaching Approaches Encouraged within Project Maths

2.2.5.1 Problem Solving

As has been previously noted, problem solving has been a major part of many curriculum reforms in the past two decades (Burkhardt & Bell, 2007; Conway & Sloane, 2005; Lesh & Zawojewski, 2007; National Council of Teachers of Mathematics, 2000; Oldham, 2005; Pampaka et al., 2012; Schoenfeld, 2007). Ireland has joined in this trend with the introduction of Project Maths which includes as a specific objective to develop students' "ability to apply their mathematical knowledge and skill to solve problems in familiar and in unfamiliar contexts" (National Council for Curriculum and Assessment, 2010, p. 6). It is worth noting that while the previous Irish mathematics curricula did include the objectives of problem solving and developing students' relational understanding, these higher level thinking skills were not necessarily assessed within the corresponding state examination (Perkins, Shiel, Close, & Oldham, 2007) and were therefore traditionally not taught in the classroom (Lyons et al., 2003)

'Problem solving' refers to students' developing higher level thinking in order for them to greet a previously unseen problem (Conway & Sloane, 2005; Oldham, 2005). From

the perspective of solving a problem the crucial episodes are: analysing the problem, selecting appropriate mathematical knowledge, making a plan, carrying it out, and checking the answer against the question asked (Polya, 1945; Schoenfeld, 1992). However, developing a problem solving approach in teaching involves more than simply utilising such a tool in class and instead refers to: creating a classroom culture where emphasis is on students' understanding of processes rather than achieving the 'correct answer' (Lesh & Zawojewski, 2007; Schoenfeld, 1994); encouraging communication within the classroom, supported both verbally and in written form, where students work in independent groups (Forman, Larreamendy-Joerns, Stein, & Brown, 1998; Goos, 2004; Goos & Galbraith, 1996; Schoenfeld, 1992; J. D. Stein, 2008; Verschaffel et al., 2014), and utilising contextualised problems in employing interesting and engaging tasks through which mathematical concepts or procedures are illuminated and offers a rationale to students for learning mathematical topics (Galbraith, 2013; Hiebert et al., 1996; Lave, 1988; Mason, 1988; Prusak, Hershkowitz, & Schwarz, 2013).

Higher order thinking, promoted by encouraging problem solving skills, is best supported by adopting a conjecturing attitude where students are not afraid of being wrong, but rather are encouraged to be creative in suggesting solutions (Mason, 1988). Providing students with relevant, contextual questions motivates students to engage with the question, to make discoveries, to attempt different approaches, and offers a rationale to students for learning mathematical topics (Hiebert et al., 1996; Prusak et al., 2013). These are all features of the Project Maths curriculum which have been highlighted to teachers within the curriculum documents and through teachers' professional development courses (Project Maths Development Team, 2011). This approach to mathematics, highlighting communication and context, led to the qualitative nature of the data in this research focusing on teacher learning through conversations around the teaching and learning mathematics.

2.2.5.2 Engaging Students in their Learning of Mathematics

2.2.5.2.1 Providing Context

A constructivist teacher does not think of their students as empty vessels but as thinking, personal beings (Holten & Thomas, 2001). A broad social goal for some teachers is that they want their students engaged (Schoenfeld, 2011) and it is important that teachers anticipate what students will find interesting and motivating since affective factors play a

critical role in the teaching and learning of mathematics (Ball et al., 2008; Ma, 1999). As noted in the previous section, in encouraging students to solve unseen problems they need to be engaged with the mathematics in question. One of the explicit aims of the Project Maths curriculum is to engage students in solving problems that are based in a real-life context that are pertinent to learners. The literature would suggest that students want to feel that what they are studying is relevant and of practical use (Galbraith, 2013; Noyes, 2007) in order to illuminate the importance of mathematics to students and also engage them in their own learning.

It is important to take into account students' opinions of mathematics since a more negative opinion correlates with lower grades in mathematics (M. Brown, Brown, & Bibby, 2008). Students who disengage with mathematics attribute permanent characteristics either to themselves in thinking they are not interested in the subject, or to the subject itself which they see as 'boring'. On the other hand Brown et al. (2008) found a strong and significant correlation between the extent to which students enjoyed mathematics and their participation in the subject. Engaging students in their learning is an important aspect of the Project Maths curriculum which includes as an explicit aim to "foster a positive attitude to mathematics in the learner" (National Council for Curriculum and Assessment 2012, p. 6). Interest, curiosity, engagement and intrinsic motivation all carry the implication that the student has become actively involved with the content, or some aspect of, the learning activity (Ainley, 2001; Blackhouse, Haggarty, Pirie, & Stratton, 1992). Unfortunately, mathematics is one of the more dreaded school subjects for students (M. Brown et al., 2008) and of all the school subjects mathematics is most likely to hinder a students' profession towards higher education and employment opportunities (ATTRACT Project, 2012; Noyes, 2007).

2.2.5.2.2 Collaborating

Students' mathematics anxiety can overwhelm their ability to learn (Blackhouse et al., 1992). In order to overcome this anxiety, teachers can engage students in their own learning by encouraging students to collaborate with one another in their mathematical learning thus developing students relationships and communication with one another and developing the class as a community of learners (Goos, 2004; Johnson & Johnson, 1999; Solomon, Schaps, Watson, & Battistich, 1993). Within the Project Maths (2012) curriculum, communication is emphasised as important to incorporate within the mathematics classroom. Such engagement can often mean a loud and noisy classroom

(since students who are really involved in their mathematical activity will be talking with each other about it) (Boaler, 2009; Schoenfeld, 2011) and this constitutes a very different classroom from a traditional approach to learning that has traditionally been of the 'chalk and talk' model where students are solitary and quiet in their learning (Lyons et al., 2003; Oldham, 2001). However, it is important that teachers incorporate these approaches within their practices in order to engage students in the topic at hand and to develop students' skills in managing their own learning (National Council for Curriculum and Assessment, 2011).

2.2.5.3 Communication within the Classroom

2.2.5.3.1 Students Communicating their Mathematical Thinking

Leading on from engaging students in their learning through collaboration, communication within the mathematics classroom has been promoted in the last two decades as an important aspect of building students' understanding of mathematics (Boaler, 1999; Goos, 2004; National Council of Teachers of Mathematics, 2000; Stigler & Hiebert, 1999). Learning mathematics can be supported by providing opportunities for students to do and talk about mathematics through which they can get a sense of mathematical concepts and facts (Boaler, 1998, 1999; Goos, 2004; Mason, 2002). In encouraging students to communicate their mathematical understanding in class, teachers have more access to student talk which can allow them to interpret how students are thinking about their mathematics (Breen, McCluskey, Meehan, O'Donovan, & O'Shea, 2014; Mason, 2002). Classrooms can provide opportunities for students to feel competent both academically and socially (Solomon et al., 1993) and encouraging students to communicate their understanding develops their mathematical thinking as well as their inter-personal skills.

2.2.5.3.2 Teacher as a Facilitator of Learning

Rather than simply giving students an answer (which can sometimes be detrimental to students' learning) (Chiu, 2004; Gillies, 2004), teachers can enhance the explaining ability of their students by pressing their students for explanations of their ideas (Ermeling, 2010; Schoenfeld, 2011; Webb et al., 2009). In this sense the teacher is more of a coach or facilitator of student learning and actively engages with students while they work on problems by probing their thinking, by asking them questions, or by utilising reflective tosses (Schoenfeld, 2011). In this context, the teachers' role shifts from asking

questions to ensure that students arrive at a correct answer, to creating prompts that encourage students' exploratory talk (Schoenfeld, 2011; Watkins, 2005). Within this role, it is important that teachers anticipate what students' responses may be in order to plan how a lesson might be sequenced or how to respond to students depending on their questions, misconceptions or difficulties (Schoenfeld, 2011; Stein et al., 2008).

The Project Maths classroom, which places a strong emphasis on incorporating more communication within the classroom (Lynch, 2011; Oldham, 2005), thus looks and sounds different when compared to the more traditional approach to teaching that was common with the prior post-primary mathematics curriculum (Oldham, 2001). For teachers used to a traditional style of teaching this approach can be daunting and learning to facilitate classroom discussions can be considered as teachers enhancing their pedagogical content knowledge (Krauss et al., 2008).

2.3 Introducing a New Curriculum Reform

Introducing change in education, such as the introduction of Project Maths, is a complex process which requires careful planning, sufficient time for implementation and adequate funding (Eurydice, 2011). "Curricular reforms may be fragile and transient" (Senger, 1999, p. 201) and educational planners should develop professional strategies that continuously support teachers in understanding the philosophy behind a curriculum, engaging with that curriculum, and attempting to implement it in a meaningful way (Remillard & Bryans, 2004). Reform which occurs with the involvement and acknowledgement of teachers leads to sustainable, significant and coherent reform.

2.3.1 The Role of the Teacher in Curriculum Reform

Educational reforms typically impose new demands on the already complex and over-loaded work of teaching and the literature suggests that teachers will respond to imposed curriculum change by either embracing change, resisting and ignoring the change, or modifying the curriculum, with the latter two often the norm (Christou et al., 2004; M. K. Fetters, C. M. Czerniak, L. Fish, & J. Shawberry, 2002; Jenkins, 2000). Teachers' attitudes towards curriculum reform is often negative since teachers can feel that their content knowledge is inadequate, their teaching experience does not qualify them for this particular reform, and that the reform imposes on their disposition towards, approaches to and philosophy of teaching their particular subject (M. Fetters, C. Czerniak, L. Fish, & J. Shawberry, 2002; Handal & Herrington, 2003). Teachers typically feel pressure to cover

a curriculum and there is consistently echoes of insufficient time and resources when new curricula and reforms are introduced (van den Berg & Ros, 1999) where teachers are preoccupied with the structural aspects of the reform and the curriculum and their attention is drawn away from teaching and learning approaches (Imants, 2002).

Reform requires a change at the individual teacher level focusing on how they perceive their subject, their approaches to teaching and learning, their classroom practices (Charalambos & Philippou, 2010; Harris, 2003; Hopkins & Reynolds, 2001; van Driel, Bulte, & Verloop, 2007), and whether they believe their knowledge can be developed and enhanced (Cohen & Barnes, 1993; Nicol & Crespo, 2006; Remillard, 2000). It is thus important in attempting to translate a curriculum into a particular school's day-to-day practice that teachers are afforded the professional development supports and opportunities to engage meaningfully with the curriculum in a way that provides them with a knowledge and ownership of the syllabus they are being asked to implement (Charalambos & Philippou, 2010; Hanley & Torrance, 2011; Remillard & Bryans, 2004).

Another challenge in achieving sustainable curriculum reform is that most teachers teach alone in isolated classrooms without the opportunity to reflect on their practices or observe other teachers' practices (Remillard, 2005; Remillard & Bryans, 2004). Teachers lack the opportunity to discuss new ideas with their colleagues, give one another feedback or collaborate in developing initiatives. It is thus important in attempting to translate the intended to the enacted curriculum that teachers are afforded opportunities to engage meaningfully with the curriculum (Handelzalts, 2009) and the most effective place to do this is in the context of the classroom (Putnam & Borko, 2000).

2.3.2 School-specific Interventions within Curriculum Reform

The premise of introducing a new curriculum initiative is that it will bring about improvement in classroom practice as well as changes to students' experiences. National educational policies and curriculum implementation tend toward a universal approach to treating all schools the same, despite the considerable differences between schools (Hopkins & Reynolds, 2001). Context specific school improvements are required in order to ensure an intervention positively impacts student learning and that an associated model of professional development is sustainable (Hopkins & Reynolds, 2001; Putnam & Borko, 2000). This suggestion is of key importance to this study where any school intervention could be specifically tailored, within the defined structure of lesson study

(see Section 2.7) according to the resources, culture, teachers, and students within that school.

The implementation of new curriculum must take into account the nature of classrooms and schools and teachers should be encouraged to collaborate with one another on curriculum (D. Ball et al., 2001). Introducing a school based approach in introducing new curriculum strengthens reform by making it a shared practice within and across a school (P. Grossman et al., 2001; Handelzalts, 2009; Hord, 2004). Teachers are then more likely to be professionally and individually engaged with the reform and instead of keeping their ideas to themselves, share their practices and ideas with colleagues.

2.3.3 Professional Development and Curriculum Reform

In discussing the role of the teacher in reform and in discussing the need for school specific interventions, there is an obvious need for an holistic approach to curriculum reform and professional development of teachers in order for any change to be meaningful and sustainable (Fullan, 2003; Hopkins & Reynolds, 2001). Cohen and Barnes (1993) and others argue that real pedagogical change could not occur without significant teacher learning (Remillard, 2000; Remillard & Bryans, 2004). In order to teach differently, teachers need opportunities to learn mathematics in new ways and to consider new ideas about teaching and learning (Remillard, 2005; Remillard & Bryans, 2004). Teachers need concrete images and structures to influence pedagogy and require opportunities to see the curriculum being taught in classrooms (Christou et al., 2004; Remillard & Bryans, 2004). Teachers should also be encouraged to utilise and modify curriculum materials in order to best suit teachers' own teaching and learning approaches and practices (Hanley, 2010; Hanley & Torrance, 2011).

The literature of curriculum reform would suggest that a community approach to professional development, where teachers collaborate with their colleagues sharing ideas on implementing and experimenting with a new curriculum, would seem to be a positive way to support teacher learning and thus provide a strategy to closer align the intended, implemented and experienced curricula (Little, 2002; Little, Gearhart, Curry, & Kafka, 2003; Louis, 2006; Louis, Marks, & Sharon, 1996; M. W. McLaughlin & Talbert, 2001). This will be further discussed in Section 2.6 of this chapter.

2.3.4 Impact of Project Maths Reform thus far

A recent report on the impact on Project Maths on student learning concluded that the expected pedagogical change towards active learning and a more student-centred approach was not yet occurring and that there were no discernible differences between phase 1 and phase 2 schools (Jeffes et al., 2013, 2014). Although there is anecdotal evidence of specific positive developments in terms of implementing the teaching approaches of the new curriculum (Department of Education and Skills, 2014b), it is obvious from a cursory sampling of subject-specific department inspection reports that schools and teachers are encountering difficulties in engaging in curriculum reform processes (Department of Education and Skills, 2014a, 2014d). Concurrent with the literature (Imants, 2002; van den Berg & Ros, 1999) most noted difficulties are: a lack of time and resources for work on the reforms, a negative attitude of teachers towards the reform, and teachers' perceived lack of knowledge of the philosophies behind the form and therefore difficulties in participation (Irish Mathematics Teachers Association, 2013). There is also an issue of teachers not having the relevant subject matter knowledge to teach certain topics within the new Project Maths content (Ríordáin & Hannigan, 2011). All of these factors hinder the implementation of a revised curriculum and which align with the policy behind the syllabus.

2.4 Conclusion of Literature on Curriculum Reform

From the literature it is obvious that teacher learning is at the heart of curriculum reform (Cohen & Barnes, 1993; Nicol & Crespo, 2006; Remillard, 2000). It is particularly important that within a curriculum reform, teachers' knowledge is enhanced through engaging with and modifying innovative curriculum materials, through observing the curriculum being enacted, through reflecting on their own practice (Hanley, 2010; Remillard, 2000; Remillard & Bryans, 2004), and that these activities occur in the setting of their own classrooms and schools (Putnam & Borko, 2000).

Teachers can be supported in introducing a reform by engaging with their colleagues as part of a community (Remillard, 2005). In this context, teachers can be supported in developing new ideas and through sharing their teaching experiences with one another. These communities should be school based to understand the various policy, structural and nuanced differences between schools (D. Ball et al., 2001; Hopkins & Reynolds, 2001; Putnam & Borko, 2000). Introducing a reform as a school based approach can

considerably strengthen that reform by making it a shared practice across the school (P. Grossman et al., 2001; Handelzalts, 2009; Hord, 2004).

In Ireland, while some teachers are incorporating the teaching and learning approaches of the Project Maths curriculum, this is often happening in isolation. Through participating in this professional development model (see Section 2.8), teachers will be supported in their learning through:

1. modifying and developing teaching and learning resources,
2. enacting and observing the teaching of the new curriculum, and
3. reflecting on students' engagement in those lessons.

And as participants in a lesson study model of in-school professional development which invokes a teacher community approach, these teacher activities should, in turn, should positively impact the introduction of the new curriculum (Nicol & Crespo, 2006; Remillard, 2000).

2.5 Professional Development at a Time of Curriculum Reform

2.5.1 Introduction to Professional Development

An OECD Talis report (2009) defines professional development as the "activities that develop an individual's skills, knowledge, expertise and other characteristics as a teacher" (p. 49) and such learning comes about through participating in the practices of teaching (Adler, 2000). Professional development encompasses very many settings from the formal to the informal and should be integrated within teachers' entire career (OECD, 2005) and school life without being viewed as an unnecessary addendum to the school day (Guskey, 2002).

Opportunities for teachers to engage in continuing professional development can have a substantial impact on their work, achievement, skill and attitudes as well as on their performance and job satisfaction (Villegas-Reimers, 2003). Moreover, the changes to teachers' knowledge and behaviour in the classroom brought about by professional development also have an impact on student learning (Eurydice, 2011). In order to change a teaching approach, there is a need to move from the more traditional form of professional development to one that provides opportunity for reflection on practice and hence impacts on teachers' beliefs (M. K. Fetters et al., 2002).

Teachers' classrooms are powerful contexts for their learning (Ball & Cohen, 1996; Putnam & Borko, 2000) and professional development opportunities should also offer subject-specific options in order to integrate new materials, methodologies and technologies for specialised and experienced teachers (A. Smith, 2004; Yow, 2008). Programs which engage teachers in solving mathematical problems and in focusing on subject matter can help teachers develop an understanding of their practice and of how students think about that particular subject (Borko, 2004).

2.5.2 Professional Development as Teacher Learning

The demands of professional development of in-career teachers are different to those of the needs of pre-service or initial teachers (Jacobs, Lamb, & Philipp, 2010; Mooney Simmie, 2007) and teachers of varying experiences are also affected differently through their participation in professional development programmes (Jacobs et al., 2010). Experienced teachers often find that they gain a more creative approach to teaching in leaving the isolation of the classroom to collaborate with their colleagues (Mooney Simmie, 2007). On the other hand, newly qualified teachers are often initially pre-occupied with issues of classroom management and later reflect on the content in question within their lessons (Jacobs et al., 2010). Professional development interventions should therefore be undertaken with two assumptions in mind (Zawojewski, Chamberlin, Hjalmarson, & Lewis, 2008):

1) Teachers will grow in ways that cannot be always predicted,
2) Teachers' growth will take place in complex situations.

Some teachers' practices and knowledge are more easily affected than others (Fennema et al., 1996; Franke, Carpenter, Levi, & Fennema, 2001) and teachers' development is often unpredictable in engaging with professional development because they use their own interpretive systems to function in their daily instructional practice. Professional development should be relevant to local needs (Borko, 2004; Lieberman, 2009), should engage teachers in work directly related to their classroom practice (Darling-Hammond, Chung Wei, Andree, Richardson, & Orphanos, 2009; Delaney, Ball, Hill, Schilling, & Zopf, 2008), and should record how teachers are developing through their participation. Models of professional development should also motivate teachers to share their ways of thinking about their own practice with their peers (Guskey, 2002; M. W. McLaughlin &

Talbert, 2006; Zawojewski et al., 2008). Amalgamating each of these demands within one model demonstrates the complexity of such an undertaking.

2.5.3 Professional Development and Educational Reform

As highlighted in Section 2.3, changes to classroom practices that are demanded by reform require a great deal of learning on the part of teachers (Nicol & Crespo, 2006; Remillard, 2000) and would be difficult to make without appropriate support and guidance (Borko, 2004). Professional development is hence a cornerstone for the implementation of reform (Fishman, Marx, Best, & Tal, 2003).

In order to improve teaching, the most effective place to do so is in the context of a classroom lesson. If lessons become a focus point of professional development, any problems of applying research findings in the classroom disappear since the improvements are devised within the classroom where teachers' attention is focused on identifying what will improve their students' learning and they can then share this knowledge with their colleagues (Stigler & Hiebert, 1999). The power of a classroom-level based approach to generate high quality school improvement is shown by the experience of Japan (Lewis, 2000; Sloane, 2005; Stigler & Hiebert, 1999) where teachers adopt a problem-solving orientation to their teaching through lesson study. In lesson study, groups of teachers meet regularly over long periods of time (ranging from several months to a year) to work on the design, implementation, testing and improvement of one or several 'research lessons' (Hopkins & Reynolds, 2001). As a cyclical process of planning, conducting, observing, reflecting on, and revising, lesson study embeds professional development within the classroom (Sims & Walsh, 2009) and will be further explored in Section 2.8.6.

2.5.4 Professional development and Teacher Community

The literature suggests that strong professional learning communities are the best vehicles for inducing change in teachers' knowledge and instructional practices (P. Grossman et al., 2001; Little, 2002) and many studies have shown the positive teaching and learning outcomes of teachers engaging in such in-school professional development (Arbaugh, 2003; Fernandez, 2005; Hord, 2004; Louis, 2006). A key rationale for teacher community is that it provides an ongoing venue for teacher learning (Cochran-Smith & Lytle, 1999; Darling-Hammond et al., 2009; T. H. McLaughlin, 2004) because they are

situated between educational policies and the realities of schools (P. Grossman et al., 2001; Guskey, 2002; M. W. McLaughlin & Talbert, 2006).

Little (2002), weighing both options on the advantages and disadvantages of basing teachers' work in subject specific groups, eventually concluded that combining whole school reform with specific support for teachers' work in subject-related groups is a viable option since it potentially yields the best results in terms of teacher learning. Clandinin et al. (1998) and Grossman et al. (2001) also advocate centring developments on groups of teachers within schools. Thus, in a time of curriculum reform, engaging school based subject specific communities of teachers in reflecting on teaching and learning within the school may provide a model through which teachers can successfully engage with and learn from (Krainer, 2003) and this idea is taken up in the design of this research.

2.5.5 Professional Development Practices for the Introduction of Project Maths

Teachers in Ireland currently engage with formal professional development in the form of courses and workshops, in-service sessions that are usually held outside of the school (OECD, 2009). While teachers here engage in professional development activities that include informal dialogue and reading professional literature, teachers have felt that the majority of professional development in which they participated in were not suitable to their professional needs and conflicted with their work schedules (Gilleece, Shiel, Perkins, & Proctor, 2009). Releasing of teachers from school in the current culture of in-service training may also contribute to the perception that professional development is an "add on" rather than a continuous process for all teachers. In this current model, the absence of support at a classroom level leads to un-sustained learning for teachers without appropriate support and context sensitive feedback (Guskey, 2002; Sugrue, 2006).

Irish teachers engage with mentoring and peer observations less than the European average, particularly at post-primary level (Corcoran, 2011b; Gilleece et al., 2009; OECD, 2009). At post-primary level, in-service training days have been taken voluntarily, attracting individuals passionate about their own learning (Fishman et al., 2003). With the introduction of the Project Maths syllabus, 10 days of in-service workshops (in which I participated as a teacher in a phase 1 school) have been offered to

all post-primary mathematics teachers in the country with the majority (approximately an average of 70% of teachers over the 10 workshops) participating in these out-of-school training days where substitution cover was provided to schools (Project Maths Implementation Support Group, 2013). Approximately 63% of mathematics teachers around the country also volunteered to participate in modular courses (taught by teacher facilitators) held at the weekends or in the evenings during term (Project Maths Implementation Support Group, 2013). While feedback has been positive on these courses, and while all of the in-services have been made readily available on-line (Project Maths Development Team, 2011), an obvious drawback rests on the assumption that it is possible to take individuals out of their workplace, transform them in an unfamiliar setting, and then return them to an unchanged workplace where they may want to introduce ideas different to the status quo (P. Grossman et al., 2001). Corcoran (2011b) suggests that further shifts are needed in the provision of continuous professional development (CPD) in Ireland which are school-based, teacher-centred and focused on the range and nuances of student learning.

Both the NCCA and Teaching Council have now suggested that teachers should collaborate together in extending their professional boundaries as has been exemplified by the Junior Cycle Reform Network Schools (National Council for Curriculum and Assessment, 2011; The Teaching Council, 2010). The PMDT piloted small scale professional development interventions with phase 1 schools that were loosely based on lesson study in the academic year 2012/2013 concurrent with this research (Project Maths Development Team, 2013), further exemplifying a policy to develop professional development interventions embedded in teacher communities.

2.5.6 Conclusion of Literature on Professional Development

Professional development is based in the practices of teaching and teachers' subject matter and pedagogical content knowledge can be enhanced through engaging with colleagues in reflecting on teaching and learning approaches and by doing specific mathematics activities themselves. Professional development should be a continuous endeavour throughout the career of the teacher and while there is merit in providing external in-service workshops for teachers, they should also be provided with the support and structure of engaging with in-school models of teacher community where teachers of varying knowledge and experiences are supported in their teaching.

Combining the literature on curriculum reform and professional development, engaging a teacher community with teaching and learning practices such as those encompassed within lesson study would seem to be a well-placed model of professional development for mathematics teachers and will be investigated in this research.

2.6 Teacher Community

2.6.1 Introduction to Teacher Community

The idea of community is established on the premise that we are inherently social beings and that our learning, actions, and knowledge of ourselves are enhanced through social interactions (Bruner, 1985; Jovchelovitch, 2007; Wenger, 1998; Wertsch, 1985). Communities are socially based on relationships and on the interdependencies that nurture them and are defined by their centres of values and shared idea structures that comes about through participants' sharing of concepts, images, values and beliefs (de Lima, 2001).

The term 'community' has been used with great prevalence in recent educational research. Terms such as 'communities of learners', 'classroom communities', 'discourse communities', 'professional learning communities' and 'communities of practice' have been added as almost an obligatory appendage to many educational innovations (P. Grossman et al., 2001; O'Sullivan, 2010). What is it that all of these have in common?

The literature identifies a number of common features which are necessary in order for a community to evolve and develop and these have been further detailed for communities of teachers (P. Grossman et al., 2001; P. L. Grossman, Wineburg, & Woolworth, 2000; Louis & Marks, 1998; Louis et al., 1996; Stoll, Bolam, McMahon, Wallace, & Thomas, 2006; Wenger, 1998):

- Shared vision based on shared values
- Shared leadership
- Collective responsibility and interdependence
- Mutual trust and respect with inclusive membership
- Individual and collective professional learning
- Supportive conditions

In the context of a school, a community of teachers will also have a collective focus on student learning (Cochran-Smith & Lytle, 1999; Darling-Hammond et al., 2009). These features develop as members of the community engage with one another in various shared practices and through dialogue where a shared meaning is constructed (Bohm, 1996; Burbules, 1993; Wenger, 1998). A number of research interventions based on communities of teachers have likely failed due to the absence of cultivating these features. In Blue Mountain school (Giles & Hargreaves, 2006) there was an absence of shared leadership and mutual trust, and in Fernandez' (2002) attempt to build a lesson study community teachers did not have a shared vision of what a research lesson entailed.

The shared practices of a community provide a constructive medium through which members of that community can engage with one another. The concept of 'practice' connotes doing, but it is not just the activity of doing alone. It includes what is said and what is left unsaid. What is represented and what is assumed. It includes the language, tools, documents, images, symbols, defined roles, specified criteria, and codified procedures that are explicit. Simultaneously it includes all of the implicit relations, tacit conversations, subtle cues, unspoken rules of thumb, unconscious systems of principles, recognizable intuitions, specific perceptions, well-tuned sensitivities, embodied understandings, underlying assumptions, and shared cultural views (Ng & Tan, 2009; Wenger, 1998). These generated customs may not ever be articulated, but are embedded subtly in the culture of community and are unmistakable signs of membership in a community.

In this research, the term 'community' will refer to the group of teachers who are working together on the common enterprise of lesson study. Other terms have been avoided since, for example, a 'professional learning community' implies that teachers must feel they are learning within all endeavours of the community and 'discourse communities' may draw attention away from student learning. Thus teachers will be neutrally referred to as a 'teacher community' within this research due to their shared common endeavour in engaging in a model of professional development, that of lesson study. In addition, the features outlined above are expected to emerge during the research, thus qualifying this group of teachers to be defined as a community.

2.6.2 Why Teacher Community as Professional Development?

Engaging in a teacher community provides structure for professional dialogue to be taken out of the privacy of the classroom and into a forum that allows for open and critical discussion (de Lima, 2001; P. Grossman et al., 2001). A challenge in creating or developing a community of teachers as a form of professional development, however, is to maintain awareness of two aims (P. Grossman et al., 2001):

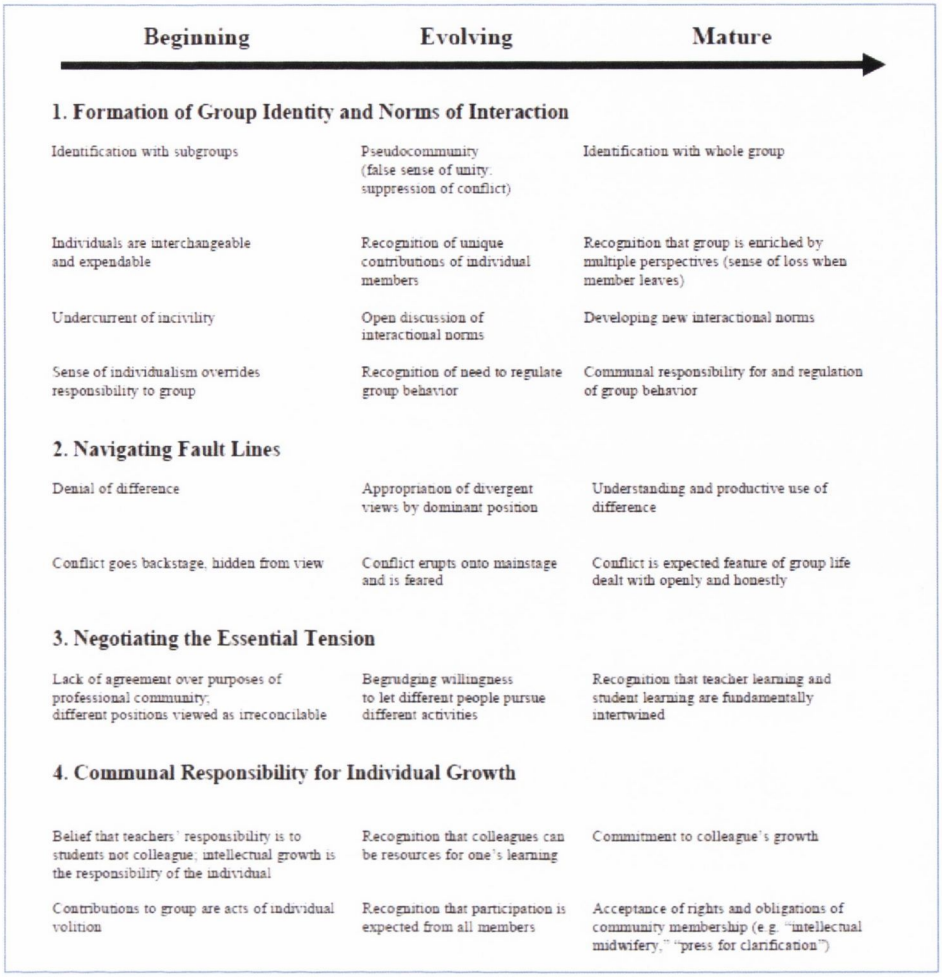
1. Maintaining a focus on student learning, while
2. Creating structures for teachers to engage as learners themselves with the subject matters they teach.

A teacher community must be concerned with teacher learning with an equal or subsequent focus on student learning. It is this dual focus which forms a basis of this research where teachers are explicitly focused on student learning but where the explicit focus of the research, shared with participants, is their own learning. Arbaugh (2003) saw success in his research since the focus of the study group was on teachers who were in control of their own professional growth. Borko et al. (2008) also saw the necessity of a strong community, where trust was prevalent, when teachers shared moments within their own teaching and observing each other's classroom practices.

Teacher collaboration is beneficial to student learning (Louis & Marks, 1998; M. W. McLaughlin & Talbert, 2001) and collaborative teams have the most impact when their focus is directed toward student learning and when the initiatives taken are related to curriculum and instruction (Sackney, Walker, & Mitchell, 2005; Vescio, Ross, & Adams, 2008). With an approach to mathematics where teachers are encouraged to incorporate more communication into their classroom and where students engage in dialogue about mathematics thus engaging in rich learning tasks (Boaler, 1998, 1999; National Council for Curriculum and Assessment, 2012a, 2012b), it seems logical that the teachers themselves should also engage and participate in a community of learning. Professional development that provides opportunity for teachers to build on their subject matter and pedagogical content knowledge should be embedded within classroom practices, situated within subject-specific groups, be focused on student learning, and be supported by strong teacher community.

2.6.3 The Evolution of Community

Communities often evolve naturally as individuals engage with others on norms and practices (Wenger, 1998; Wenger, McDermott, & Snyder, 2002), but it is also worth noting that a community is not instantly formed by virtue of having the right structures in place. When a group of professionals are working together, it does not imply that they are working as a community until there is a mutual engagement of all participants in the activities of the community, until all participants are engaging in a common joint enterprise and are developing a shared repertoire of the community (Lave, 1988; Lave & Wenger, 1991; Wenger, 1998; Wenger et al., 2002). Participants develop shared interpretations and meaning as they engage in the features of community and the community evolves, grows and develops over time as the relationships between members of the community and the practices within the community become more established (Cassidy et al., 2008; P. Grossman et al., 2001). Grossman et al. (2001) identified three stages of evolution of community in their research into school-based teacher communities engaged in professional development (Figure 3):



Beginning	Evolving	Mature
1. Formation of Group Identity and Norms of Interaction		
Identification with subgroups	Pseudocommunity (false sense of unity: suppression of conflict)	Identification with whole group
Individuals are interchangeable and expendable	Recognition of unique contributions of individual members	Recognition that group is enriched by multiple perspectives (sense of loss when member leaves)
Undercurrent of incivility	Open discussion of interactional norms	Developing new interactional norms
Sense of individualism overrides responsibility to group	Recognition of need to regulate group behavior	Communal responsibility for and regulation of group behavior
2. Navigating Fault Lines		
Denial of difference	Appropriation of divergent views by dominant position	Understanding and productive use of difference
Conflict goes backstage, hidden from view	Conflict erupts onto mainstage and is feared	Conflict is expected feature of group life dealt with openly and honestly
3. Negotiating the Essential Tension		
Lack of agreement over purposes of professional community; different positions viewed as irreconcilable	Begrudging willingness to let different people pursue different activities	Recognition that teacher learning and student learning are fundamentally intertwined
4. Communal Responsibility for Individual Growth		
Belief that teachers' responsibility is to students not colleague; intellectual growth is the responsibility of the individual	Recognition that colleagues can be resources for one's learning	Commitment to colleague's growth
Contributions to group are acts of individual volition	Recognition that participation is expected from all members	Acceptance of rights and obligations of community membership (e.g. "intellectual midwifery," "press for clarification")

Figure 3 Model of the formation of Teacher Community (Grossman et al., 2001)

As has already been noted, a community is not established by grouping individuals together but progresses through the relationships that are developed between participants. A community evolves through the development and adoption of practices, meaning, and engagement that occurs in the participants as individuals and as a collective. It is an informal aggregation defined not only by its members but also by the shared manner in which they do things and interpret events (Jovchelovitch, 2007; Ng & Tan, 2009). In order to work together, teachers need to feel a kind of commonality with the people with whom they collaborate (Hargreaves, 2001) and teachers motivation to work with one another stems from their awareness of their interdependence in terms of the likeliness of achieving successful performance (Little, 2003). While strong personal bonds may not be necessary to form a community of professionals (de Lima, 2001), it is imperative that teachers share a deep commitment to students' learning, development and well-being while engaging in community. These indications were acknowledged within this project.

Although they may not define it as such, it is likely that in the Irish post-primary context teachers are engaging in evolving and mature communities within their everyday teaching and learning practices, based within their schools and subject departments.

2.6.4 Tension within Community

As identified in Grossman et al.'s (2001) model, disagreement and competition, as well as mutual support and interpersonal allegiances, are all forms of engagement in community and cooperation and conflict tend to go hand in hand (Johnson & Johnson, 1999). Within communities teachers will experience disagreement between individual ideas and ideas from colleagues, research, students, the curriculum, or other sources (Lesh, Kelly, & Yoon, 2008), but rebellion from the norm and events of conflict may in fact reveal a greater commitment to the community than does passive conformity and should not necessarily be viewed as negative interactions within a community (Wenger, 1998; Wineburg & Grossman, 1998). In establishing community it is important to acknowledge and allow for tensions to develop.

Teachers often deal with professional conflict by avoiding interactions and engaging in a superficial politeness (Hargreaves, 2001; Lewis & Hurd, 2011) and thus communication and small-scale social interactions should be encouraged in order to balance any tensions of power and conflict within the group (Tusting, 2005). Carter (1998, cited in Grossman et al., 2001) spoke about an 'etiquette of democracy' where individuals were mindful about how they expressed dissent and negotiated disagreement, and such norms can gradually be established within an evolving community. Such norms are explicitly defined within the practices of lesson study (Lewis & Hurd, 2011) and will be later explored in this chapter (Section 2.8).

2.6.5 Teacher Learning in Community

While teachers generally welcome the opportunity to talk about ideas and curriculum resources with their colleagues, research has found that their conversations rarely include critical examination of teaching (M. W. McLaughlin & Talbert, 2001). These conversations are necessary however in order for teachers to explore ways of improving teaching and learning in an environment where trust and respect are the norm (Tschannen-Moran & Hoy, 2000). Being in a community that collaboratively articulates and recognises a problem in daily education practice enhances teachers' motivation to re-

design that practice (Handelzalts, 2009) and provides teachers with opportunities to enhance both their subject matter and pedagogical content knowledge.

The composition of the teacher community which includes teachers of varying years of experience will affect how that community may develop and what teachers may gain from their engagement. Experienced teachers who have long-standing norms of privacy and isolation may both lose and gain from their participation. Some experienced teachers may have more difficulty with giving up their autonomy in favour of collaboration (Handelzalts, 2009), however, when they buy into the idea of community and reform their participation this may impact more on their practice (Guy Thomas, Wineburg, Grossman, Myhre, & Woolworth, 1998). New teachers are less socialised in the tradition of isolation and privacy in their practice and therefore have fewer problems with collaboration. However, newly-qualified teachers begin in their practice with more concerns about classroom management than about the teaching task and student learning. Thus, within a teacher community, it is important to include members and participants of varying teaching experiences and to recognise that teachers will benefit in different ways from their participation (Jacobs et al. 2010).

Similar to the introduction of teacher design teams where teachers collaborate in developing curriculum and curriculum resources (Handelzalts, 2009; Mooney Simmie, 2007), a community of teachers participating in lesson study aims to provide a secure space where teachers' creativity can flourish and where teaching and learning practices can be considered and reflected upon. Teachers can critically analyse their practices in an environment which is situated within the familiar environment of the school with colleagues and where relations have a strongly pedagogical element in which participants seek to teach and learn from one another (Burbules, 1993).

2.6.6 Supporting Teacher Community within the School

It is crucial to the success and sustainability of communities within schools that management agree with and support the shared aim of the community and support members in active participation through organisational structures within the school (Fullan, 2003; M. W. McLaughlin & Talbert, 2006; Morrissey & D'Ette, 2004). With a historical legacy of top-down administration of professional development and fragmented subject-based departments, teacher communities can be particularly difficult to develop (Giles & Hargreaves, 2006; Handelzalts, 2009; M. W. McLaughlin & Talbert,

2006; Morrissey & D'Ette, 2004). In establishing any school-based teacher community it is thus important that principals are acknowledged and informed of the structural workings of the community as was addressed in this research (see Section 3.6.1).

2.6.7 Conclusion of Discussion of Teacher Community

Engaging teachers in a community of their colleagues provides a structure within which teachers can engage in dialogue based on sharing their teaching and learning experiences, reflecting on their practices, and building on their teacher knowledge. Teacher community should be focused on student learning but also on building and sharing teachers' knowledge with one another. Teacher community goes through various stages of evolution and these are strengthened through teachers sharing leadership of the community and engaging in the community with a shared goal. Embedding professional development within teacher community provides opportunity for a sustainable model in which teachers can develop their own practices, enhance their knowledge for teaching, and be supported in engaging with educational reform.

2.7 Mathematical Knowledge for Teaching

In referencing teacher learning as an important aspect of curriculum reform, it is necessary to explore the knowledge of mathematics teachers. Mathematical knowledge for teaching is multidimensional and research literature would suggest that teachers not only require knowledge of their subject (content knowledge) but also require a knowledge of how to teach that subject (pedagogical content knowledge), (Ball et al., 2001; Ball et al., 2008; Baumert et al., 2010; Rowland et al., 2005; Rowland, 2014; L. Shulman, 1987; L. S. Shulman, 1986).

Shulman first drew attention to a subject-specific knowledge of teaching and learning that was necessary and separate to a content knowledge of the subject at his presidential address for the American Educational Research Association (AERA) (Baumert et al., 2010; H. C. Hill, Ball, & Schilling, 2008). He referred to this as 'pedagogical content knowledge' and later described it as knowledge which included how best to represent and formulate a topic and to make it understandable to others, incorporating a familiarity with students' conceptions and misconceptions of the material (L. S. Shulman, 1986). In his estimation, teachers' knowledge encompassed not only *that* something was so but also understood *why* it was so, and further teachers were able to represent this information in a number of ways which were particular and relevant to their students.

Shulman (1987, p. 8) identified categories of teacher knowledge within which he paid a particular focus to PCK (Figure 4).

- content knowledge;
- general pedagogical knowledge, with special reference to those broad principles and strategies of classroom management and organization that appear to transcend subject matter;
- curriculum knowledge, with particular grasp of the materials and programs that serve as "tools of the trade" for teachers;
- pedagogical content knowledge, that special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding;
- knowledge of learners and their characteristics;
- knowledge of educational contexts, ranging from the workings of the group or classroom, the governance and financing of school districts, to the character of communities and cultures; and
- knowledge of educational ends, purposes, and values, and their philosophical and historical grounds.

Figure 4 Categories of Knowledge Base (Shulman, 1987)

Similarly, Grossman (1995) also identified the multi-dimensional nature of teacher knowledge and detailed six domains of teacher knowledge:

a) knowledge of content,
b) knowledge of learners and learning,
c) knowledge of general pedagogy,
d) knowledge of curriculum,
e) knowledge of context, and
f) knowledge of self.

Shulman's identification of PCK led to a plethora of educational research into knowledge for teaching (Delaney, 2012) and there is now a shared understanding that domain-specific and general pedagogical knowledge are both important factors in the teaching and learning of any subject (D. Ball et al., 2001; Ball et al., 2008; Baumert et al., 2010; H. C. Hill et al., 2005; Rowland et al., 2005; van Driel, Verloop, & de Vos, 1998). In the following sections teachers' pedagogical content knowledge and subject content knowledge will be discussed.

2.7.1 Pedagogical Content Knowledge

Mathematics education research has found that teachers' subject content knowledge can impact on student learning (H. C. Hill et al., 2005; Krauss et al., 2008) but also that their pedagogical content knowledge (PCK) plays a vital role in students' learning experiences (Ball et al., 2008; Baumert et al., 2010; H. C. Hill et al., 2008). PCK combines knowledge of mathematics, students, and pedagogy (D. Ball et al., 2001) and was defined by Krauss et al. (2008) as having a number of important features: knowledge of mathematical tasks as instructional tools, knowledge of students' thinking and assessment of understanding, and knowledge of multiple representations and explanations of mathematical problems.

Pedagogical content knowledge is an amalgam of both subject matter and pedagogy employed in teachers' interpretations, decision-making, and practices in facilitating student learning (Ball & Hyman, 2003; Mishra & Koehler, 2006; L. Shulman, 1987; van Driel et al., 1998). It refers to both the tacit and unique practical knowledge, built up through experience and through exchange with others, on how to teach a particular subject with an understanding of common learning difficulties, preconceptions of students and various representations of a particular topic. This knowledge also requires that teachers know what topics students find interesting or difficult and also recognise the representations which are most useful for teaching a specific content idea (Ball & Hyman, 2003; Kinach, 2002). Teachers gain a certain amount of knowledge from reading research based in practice, but also enhance their knowledge through experience and through observation and reflection of classroom practices. PCK also includes teachers' knowledge and development of resources used in the teaching and learning of their subject. From drawing on their PCK teachers decide on what activities should be chosen and undertaken during a lesson, how a classroom discussion should be facilitated, how to interpret students' responses, how to assess students' understanding, and how to build and develop students' thinking (Ball et al., 2008; Baumert et al., 2010; Krauss et al., 2008).

2.7.2 Subject Matter Knowledge

While PCK may be necessary in creating instructional moments, it is inconceivable without subject matter knowledge (as referred to by Grossman, 1995) or content knowledge (as referred to by Shulman 1987) (Baumert et al., 2010; Rowland et al.,

2005). Shulman (1987) defines this category of knowledge as the "understanding, skill, and disposition that are to be learned by school children" (p. 8). The content knowledge required of teachers is far more than the finished product of what students need to learn, but rather encompasses an in-depth and relational knowledge of a subject where teachers recognise why something is so as well as how it is so. Content knowledge extends to knowledge of one's subject which allows teachers to draw on a number of representations or examples in order to build students' mathematical thinking.

The subject matter required for mathematics teaching is not a watered down version of formal mathematics, but rather a serious area incorporating content and pedagogical knowledge (Ball et al., 2008; Davis & Simmt, 2006; Rowland et al., 2005). Teachers' repertoire of strategies, alternative mathematical representations and explanations are dependent on their conceptual understanding of the subject (D. Ball et al., 2001; Baumert et al., 2010; P. L. Grossman & Stodolsky, 1995) and the greater their knowledge, the more options available to the teacher in allowing students to make sense of a topic (D. Ball et al., 2001; Borko et al., 1992; Ma, 1999; Rowland et al., 2005; Schoenfeld, 2011). Experienced teachers have knowledge of and can quickly access a wide range of content which enable them to deal easily with issues that arise within their interactions with students (Schoenfeld, 2011; L. Shulman, 1987; L. S. Shulman, 1986). Teachers must have that particular knowledge (specialised content knowledge) which, when combined with PCK, will allow them to represent mathematical ideas and provide mathematical explanations for common procedures in a way that is meaningful for students and examine and understand students' solutions to problems (H. C. Hill et al., 2008; L. Shulman, 1987).

Teachers' subject matter knowledge is also relevant since teachers are likely to emphasise the areas which they feel most confident and knowledgeable in and avoid or de-emphasise areas in which they have relatively less content knowledge (P. L. Grossman, 1995) and as anecdotal evidence in Ireland has shown based on the percentage attempt on particular questions in the Leaving Certificate exams in the former curriculum in Ireland (SEC, 2005).

2.7.3 A Framework for Teacher Knowledge

In England, Rowland and his colleagues used Shulman's categories of knowledge to develop the Knowledge Quartet, focusing on how initial teacher knowledge is utilised in

the classroom and combining SMK and PCK (Rowland et al., 2005). The COACTIVE (Cognitive Activation in the Classroom) used Shulman's categories of knowledge to conceptualise teacher knowledge used in post-primary teaching (Baumert et al., 2010). Ball, Thames and Phelps (2008) proposed another model based on Shulman's categories developed by analysing the work of teachers and detailing sub-sections of SMK and PCK. Each of these frameworks find the separate categories of knowledge to be related (Delaney, 2012) and to be multidimensional (Ball et al., 2008).

Ball et al. (2008) define MKT as "the mathematical knowledge needed to carry out the work of teaching mathematics" (p. 395) which is unique to teachers. Drawing on this (2008) model, both subject matter knowledge and pedagogical content knowledge are subsets of this domain and while their categories have not been statistically verified (McCrory, Floden, Ferrini-Mundy, Reckase, & Sharon, 2012), conceptually the ideas are compelling and useful in understanding the multidimensional and complex nature of knowledge for teaching mathematics. This framework of MKT (Figure 5) as suggested by Ball et al. (2008) is further detailed and adapted here in this research and is explored in the following sections.

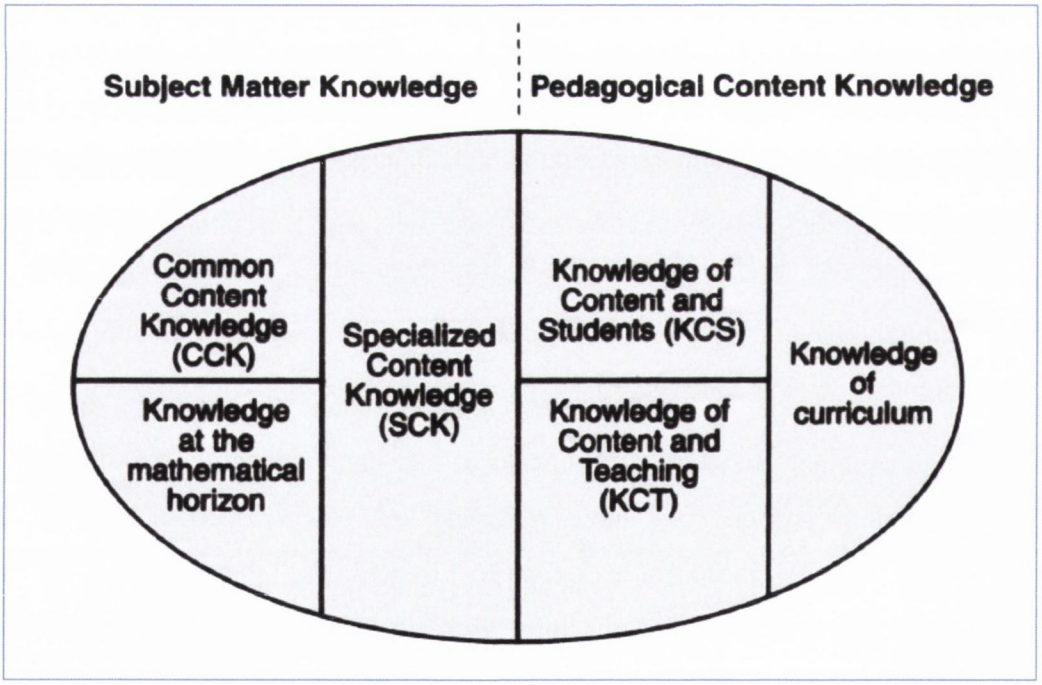


Figure 5 Framework of Mathematical Knowledge for Teaching, Ball et al. (2008)

2.7.4 Subject Matter Knowledge

Subject Matter Knowledge refers to the mathematical content knowledge which is required of teachers but lies outside the domain of pedagogy, i.e. the mathematics that teachers should be familiar with without ever thinking about teaching this mathematics to a learner. SMK includes three sub-sections which are now explored.

2.7.4.1 Common Content Knowledge

In Ball et al.'s (2008) model common content knowledge (CCK) includes calculating a shopping bill or correctly figuring a sales discount and represents knowledge that is common to many that have experienced mathematics education. CCK is that which is used in a variety of settings in everyday life in many occupations and is also essential to teachers but is not sufficient on its own for teaching (Ball et al., 2008; H. C. Hill et al., 2008). It is likely that CCK represents Shulman's (1986) supposition of subject matter knowledge and also includes Lave's (1991) hypothesis of knowledge of mathematics which is situated in real-world experiences. Common content knowledge, as a subset of SMK, includes that of simply calculating an answer or correctly solving a mathematics problem. Teachers require CCK to recognise incorrect answers or to realise when a textbook may give an incorrect decision (Ball et al., 2008) but also to use correct symbols and notation in their exposition. Similar to SCK (detailed below), no knowledge of student learning or of building mathematical understanding is required within CCK.

2.7.4.2 Specialised Content Knowledge

Specialised content knowledge was defined as "the mathematical knowledge and skill unique to teaching" (Hill et al., 2008, p. 400) and is not necessarily required in other professions (Ball et al., 2008). For example (to use an example regularly employed within the research literature) outside of teaching one may need to know how to divide two fractions but would not need to know why a fraction is commonly inverted and then multiplied in order to perform that division. This item of SCK is, however, required of teachers in their teaching of division of fractions with various demonstrations of models and examples of why it is so (D. Ball et al., 2005; D. Ball et al., 2001). Schoenfeld (2011) also highlights this form of knowledge (although not identified as SCK) in demonstrating the decisions teachers must make in deconstructing content in order to build students' mathematical understanding of a topic.

SCK is required of teachers in order to represent mathematical ideas in various different ways, in order to provide explanations for common procedures and to interpret and understand solutions provided by students to mathematical problems (D. Ball et al., 2005; H. C. Hill et al., 2008). SCK represents teachers' relational understanding of mathematics and while it is true that it is an objective for students to also develop a relational understanding of their mathematics, it is not required that all students should be able to demonstrate multiple representations of the mathematics topic under scrutiny while this is required of the teacher (Ball et al., 2008).

2.7.4.3 Knowledge at the Mathematical Horizon

Horizon knowledge, according to Ball et al. (2008), is an awareness of how mathematics topics are related over the span of mathematics, including also how mathematics topics are related within a curriculum. It is a little unclear how this type of knowledge is separate from that of knowledge of content and curriculum within PCK in Figure 5.

It is interesting, however, to note that within the Project Maths curriculum (Section 2.2), there is an explicit aim that teachers outline and find curriculum links within the syllabus and teach mathematics as a cohesive topic as opposed to discrete topics held together within a textbook. This knowledge is likely closely linked with knowledge of curriculum as denoted in Ball et al.'s (2008) framework of mathematical knowledge for teaching but will not be further investigated in this research.

2.7.5 Pedagogical Content Knowledge

As described above, Schulman's description of pedagogical content knowledge (PCK) includes knowledge of what explanations work best for particular students and knowing how students will engage or interpret certain activities (L. Shulman, 1987; L. S. Shulman, 1986). Ball et al. (2005, 2008) extend this definition of PCK as knowledge of content and student (KCS) as well as knowledge of content and teaching (KCT), and knowledge of curriculum. Knowing what aspects of a topic a student will find difficult, knowing what particular parts of a question or activity to highlight to students for different purposes, knowing how to communicate a certain point in order to purposefully confuse or enlighten a student are all elements of PCK.

All three elements of this knowledge are required of teachers in constructing mathematics questions, in identifying student strategies, in determining the validity of a

mathematical argument, in choosing or designing a model, analogy or representation to teach a particular item, in pre-empting student responses, in knowing common student conceptions and misconceptions and purposefully utilising this knowledge within the teaching and learning of mathematics. This knowledge is unique to mathematics teachers and especially relevant in employing teaching and learning approaches that encourage student discussion and student inquiry (Ball et al., 2008; Baumert et al., 2010; Rowland et al., 2005).

2.7.5.1 Knowledge of Content and Students

As detailed above, specialized content knowledge is the mathematical knowledge and skill unique to teaching. It requires that teachers are quick in identifying errors, can identify patterns and are flexible in their thinking about mathematical meaning (Ball et al., 2008). In contrast, knowledge of content and students (KCS) is focused on teachers' understanding of how students learn particular content and encompasses teachers' knowledge of common student errors and conceptions or misconceptions about mathematical topics (Ball et al., 2008; H. C. Hill et al., 2008). Hill et al. (2008) build on Shulman's (1986) definition of PCK in detailing KCS as an element of PCK, in particular "as content knowledge intertwined with knowledge of how students think about, know, or learn this particular content" (p. 375). Hill et al. (2008) put forward empirical evidence for KCS from research investigating improvements to student learning due to teachers' enhanced knowledge of how students learned. This research is supported by other professional development interventions where teachers' classroom practices changed due to an increased awareness of how students' thought about mathematics (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Cobb et al., 1991; Franke et al., 2001). Within Ball et al.'s (2008) framework of teacher knowledge, KCS links mathematical content with how students think about that content (Baumert et al., 2010).

Under PCK, Krauss et al. (2008) simultaneously defined knowledge of students' thinking and assessment of their understanding as a feature since teachers had to work with students' existing prior knowledge, misconceptions, typical errors and frequently used strategies. They acknowledged that errors and mistakes provide valuable insights into students' thinking and thus provide teachers with knowledge of how to build on students' understanding.

From their own research Hill et al. (2008) and Ball et al. (2008) defined a number of items within KCS which align with some of Krauss et al.'s (2008) definition and are highlighted here:

- Students' understanding of content: working with students' existing beliefs and prior knowledge (Baumert et al., 2010; Krauss et al., 2008) and interpreting students' productions as sufficient to show understanding and deciding which student productions indicate better understanding (Ball et al., 2008)
- Student developmental sequences: identifying the problem types, topics, or mathematical activities that are easier/more difficult at particular ages, knowing what students typically learn "first", having a sense for what third graders might be able to do etc. (knowledge of the potential of tasks for learning (Krauss et al., 2008)) (Ball et al., 2008; Hiebert et al., 2005)
- Common student errors: identifying and providing explanation for errors, having a sense for what errors arise with what content (Ball et al., 2008; Krauss et al., 2008)
- Anticipating what students are likely to think or find confusing (Ball et al., 2008)
- Common student computational strategies: being familiar with landmark numbers, fact families, etc. (Ball et al., 2008)

As Hill et al. (2008) found, these items of KCS often require teachers to use their knowledge of students' thinking in parallel with their mathematical content knowledge. While their findings were not statistically corroborated, they still provide a useful frame through which to view how teachers employ and can develop their KCS.

2.7.5.2 Knowledge of Content and Teaching

Ball et al. (2008) define KCT as that combining teaching and mathematics. Decisions that teachers make incorporate their knowledge of content but also their knowledge of instruction. As part of their KCT, teachers decide on which strategy can best illuminate a concept for a student (Ball et al., 2008) and can call on a number of strategies depending on what might best suit particular students. Teachers also decide on which questions will best provide learning opportunities for particular students.

Teachers sequence how content should be constructed and Krauss et al. (2008), in their alternative construction of PCK, identified the knowledge of and choreographing of

mathematical tasks as instructional tools in sequencing and building on learning opportunities as an important element of PCK.

As well as sequencing mathematical tasks, a teacher must also use their knowledge in identifying and choosing which student answers to incorporate within a lesson and which should be used as a point of clarification for further learning (Ball et al., 2008; Schoenfeld, 2011; M. K. Stein et al., 2008). This facilitation of student learning incorporates teachers' KCT in directing the group of students' learning about particular content.

2.7.5.3 Knowledge of Curriculum

Shulman (1987) and Grossman (1995) referred to curriculum knowledge as an element of the knowledge base of teachers encompassing teachers' knowledge of the syllabus and programmes designed for teaching a particular topic at a given level and the curriculum materials available to teachers in relation to that programme. Shulman also referred to a lateral knowledge of the content which students would be taught in other subjects at that given level and a vertical knowledge of content which students had already experienced and would experience in the future (L. Shulman, 1987). Directly relevant to this research it is particularly important that in a time of curriculum reform, teachers become familiar with the materials available to them as part of that reform (Hanley, 2010; Hanley & Torrance, 2011; Remillard, 2000). As explored in (Section 2.3), it is important that teachers are familiar with the intended curriculum and that their knowledge of the curriculum includes the various methods of teaching elements of the curriculum as well as a familiarity with its content.

2.7.6 Knowledge of Students

In addition to the framework identified by Ball et al. (2008), both Shulman (1987) and Grossman (1995) identified knowledge of learners and learning as domains or categories of teacher knowledge. Grossman defines this domain as "knowledge of learning theories, the physical, social, psychological, and cognitive development of students, motivational theory and practice, and ethnic, socioeconomic, and gender diversity among students" (p. 20). Grossman also refers to knowledge of context where teachers are familiar with their students and their families, as well as the local community. While KCS and KCT may incorporate some of this knowledge as defined by Ball et al. (2008), they do not incorporate familiarity and knowledge of students which Grossman (1995) directly

references the social and cultural aspects of learners. This knowledge, which is developed over time through relationships teachers have with their own students, will be addressed in this research.

2.7.7 Knowledge of Self

Both Shulman (1987) and Grossman (1987) identify teachers' knowledge of self as an important component of teacher knowledge. This knowledge of self encompasses teachers' education philosophies, educational beliefs, personal values and goals. Again, this form of knowledge is not explicitly contained within Ball et al.'s (2008) framework of mathematical knowledge for teaching but the literature demonstrates that in the context of reform, teachers' educational beliefs are important since they impact on how successful a reform may be in its implementation and follow-through from policy to the classroom (Charalambos & Philippou, 2010; M. Fetters et al., 2002; Remillard & Bryans, 2004) (Section 2.3.1).

2.7.8 MKT and Teachers in Ireland

In Ireland, it has been found that almost 50% of the teachers teaching mathematics do not have the recognised qualification to do so as designated by the Teaching Council (Ríordáin & Hannigan, 2011). This suggests that while teachers may be qualified as post-primary teachers, these out-of-field teachers do not have the requisite modules of mathematics completed in their undergraduate studies and hence do not have a sufficient level of subject matter knowledge required to teach post-primary mathematics. Since 2012 free post-graduate courses have been offered (by lottery) to these out-of-field teachers in order to enhance teachers' MKT (Project Maths Implementation Support Group, 2013). It is thus timely that an in-school model of professional development also be investigated as a sustainable form of CPD for mathematics teachers.

Ball et al. (2001) suggest that the New Math reform in the USA failed since it did not incorporate the classroom settings and cultures of schools and teachers had little opportunities to access professional development or collaborate on the curriculum. At a time of curriculum reform, where teachers feel unsure of their own content knowledge (Charalambos & Philippou, 2010; Wallace & Priestley, 2011), it is important that teachers are provided with opportunity to build their confidence in their own MKT while also familiarising themselves with the new curriculum (Charalambos & Philippou, 2010; Christou et al., 2004; Hanley & Torrance, 2011).

2.7.9 Conclusions from Mathematical Knowledge for Teaching

A person may have a very strong content knowledge of mathematics, but this is of little benefit if they have a poor knowledge of how students learn. Similarly, a person may have an in-depth knowledge of how students learn but without a relational understanding of mathematics, this knowledge is of little benefit to them (H. C. Hill et al., 2008; Rowland et al., 2005). Mathematical knowledge for teaching encompasses both subject matter knowledge and pedagogical content knowledge as mutual requirements of teaching mathematics (Ball et al., 2008; P. L. Grossman, 1995; Krauss et al., 2008; L. S. Shulman, 1986).

Ball et al. (2008) have put forward a model of mathematical knowledge for teaching that is regularly referenced within mathematics education research (Howell, 2014). In this model subject content knowledge and pedagogical content knowledge are taken as overlapping entities, both of which are necessary for teaching.

2.8 Lesson Study

This section will attempt to lay out the current landscape of research on lesson study since this was chosen as a model for in-school professional development in this research. Firstly, this section will sketch the history and outline the main premise of lesson study. Then the main avenues of research that have been pursued will be outlined in terms of lesson study as a basis for professional development and teacher learning.

2.8.1 The Origins of Lesson Study

The expression *lesson study* is a literal translation from the Japanese word *jugyokenky* where the separated phrase *jugyo* means *lesson* and *kenkyu* means *study* or *research* (Lewis, 2000; Lewis, 2002) and has been practiced in Japan for over a century (Lewis, 2000). This translation can be misleading in the sense that lesson study is not a simplistic study of lessons, but rather is a systematic inquiry into teaching and learning which is carried out by teachers collaboratively planning, examining, conducting, observing, and reflecting on lessons taught (Fernandez, 2002; Lewis, 2000; Lewis, 2002; Lewis & Hurd, 2011; Lewis et al., 2006; Murata, Bofferding, Pothen, Taylor, & Wischnia, 2012; Perry & Lewis, 2008; Takahashi & Yoshida, 2004; Yoshida, 2012).

2.8.2 Introduction to Lesson Study

Lesson study is a teacher-oriented and teacher-directed practice where members of a lesson study community determine a particular objective and establish how to explore their chosen goal through their examination of practice. Teachers collaborate in planning, conducting, observing, and reflecting on a research lesson (Fernandez, 2002; Fernandez & Chokshi, 2002; Lewis et al., 2006; Murata et al. 2002; Murata et al. 2012). While lesson study is generally adopted as an in-school practice involving teachers within the same subject (Fernandez, Cannon, & Chokshi, 2003; Lewis & Hurd, 2011; Lewis, Perry, & Hurd, 2009; Stigler & Hiebert, 1999), it can also be implemented as a practice for teachers in different schools or indeed for pre-service teachers (Cajkler, Wood, Norton, & Peddar, 2013; Corcoran, 2007). External experts or additional guests may also be invited from other educational institutions to participate in the planning and reflecting of a lesson (Lewis, 2002). A facilitator may also be involved in guiding the initial activities of the group participating in lesson study (Lewis & Hurd, 2011).

The aim of lesson study is not to construct the perfect lesson and it is important to note that the emphasis on the lesson plan which is in-built in this model should not be viewed as an endorsement or recommendation of the use of highly prescriptive lessons (Lewis & Hurd, 2011; Lewis et al., 2006). To the contrary, the purpose of lesson study is instead to engage teachers in dialogue around their practices where the subtleties and nuances within teaching can come to the surface through the sharing of teachers' experiences, ideas, comments and questions (Sims & Walsh, 2009). Through lesson study, collaborative planning can illuminate the possibilities of how students' learning experiences can be enhanced.

Lesson study provides teachers with a structure to learn how their students engage in mathematics, how students' learning can be improved, and provides opportunities for teachers to introduce new teaching approaches and to learn from their colleagues. Teachers examine student learning by anticipating their responses and by collecting data on students' engagement in activities during research lessons (C. Fernandez et al., 2003; Maria Lorelei Fernandez, 2010; Lewis & Hurd, 2011; Lewis et al., 2009; Lewis & Tsuchida, 1998). Participating in lesson study can open up discussions about the invisible aspects of teaching such as how students are engaging in activities and how classroom discussions can best be facilitated (Sims & Walsh, 2009).

Lesson study also encourages teachers to take risks in their teaching (Dudley, 2012), to try new teaching strategies and to learn from what does work as well as from what does not work as well in reaching a learning objective. This model of professional development helps teachers experience good teaching practice in live research lessons, particularly when new educational ideas may not appeal to them theoretically (Burney, 2004; Hanley, 2010; Murata et al., 2012;Yoshida, 2012). Seeing new ideas demonstrated within research lessons and participating in reflective discussions around practice helps teachers understand the benefit of different and unfamiliar practices (Lewis, 2000; Murata et al., 2012; Yoshida, 2012).

2.8.3 School Mechanisms of Lesson Study

Lesson study is designed to be incorporated within the school day and within the school structure. It is thus important that teachers participating in lesson study and those promoting it within a school structure are aware of the mechanisms within the model that will impact on school structures and many of these are amalgamated from literature in both lesson study and teacher community. In Japan, lesson study is often conducted as part of a year-long school-wide study with a set research theme and scheduled time for implementation of lesson study (Chichibu & Kihara, 2013). Communities need to be facilitated with time to meet and work together within the school setting and within the school day (Hord, 2004; Louis et al., 1996).

Work on a typical lesson study cycle generally entails 6 to 10 hours of group meetings spread over a 3 to 4 week period, with the two teachings (if a lesson is to be re-taught) separated by only a few days (Fernandez, 2002; Lewis & Hurd, 2011; Murata et al., 2012). It is usual however, that the community focus on the same agreed upon goal for the duration of the academic year (Lewis, 2001; Lewis & Hurd, 2011). A number of cycles can be completed during the year focusing on different areas of the curriculum and on different year groups under the umbrella of one over-arching goal.

In Billet's (2007) case study of a community of practice, the situational factors and practices of those familiar with those particular work settings were an important feature in the development of that community. Thousand and Villa (1992) recommend regular meeting times be scheduled for the community within the school schedule and adhered to respectfully. Meetings should also be limited in time and this should allow teachers to focus on their specified goals within a specific meeting (Lewis & Hurd, 2011; Thousand

& Villa, 1992). Neglecting to incorporate the norms and structures of the work environment can result in a disintegration of the community, as occurred in Blue Mountain high school (Giles & Hargreaves, 2006) where teachers felt too constrained to engage in work that felt relevant to their school and their students, and it was important to adhere to these norms in this research.

In Japan, teachers often produce a report at the end of their lesson study detailing the various aspects of their lesson (Fernandez, 2002; Sloane, 2005). Whether or not teachers provide a report at the end of the cycle, it is important that teachers within the community keep a track of the work of their group through minutes taking, task lists and noting ideas held within planning and reflection meetings (Lewis & Hurd, 2011; Thousand & Villa, 1992). While also helping to structure the work of the group, it also makes the work of individual teachers more visible and therefore encourages accountability within the team. Monitoring and assessing of the tasks of participants should happen on a regular basis (Thousand & Villa, 1992) but this should preferably occur in an informal, non-confrontational manner (Lewis & Hurd, 2011). It is also preferable that all members of the community are acknowledged as equal in terms of their input and the leadership of the lesson study cycles should be mutual and interchangeable in as far as is possible as will be referenced in Section 2.8.8 as part of the role of the facilitator in lesson study (Lewis & Hurd, 2011; Wenger, 1998).

2.8.4 Lesson Study as Professional Development

Many teachers often glean information in fleeting moments of conversations with their colleagues along the school corridors (P. Grossman et al., 2001) and would sometimes themselves suggest that these brief conversations with their colleagues are of more benefit to them than formalised, external in-service sessions that are broad in content and view schools in a general sense rather than specific environments with their own cultures and structures (Borko, 2004). Lesson study provides teachers with a structure within which they can engage in informal 'photocopier' conversations (sharing experiences, ideas and materials) while planning collaborative research lessons for students within their own school. Lesson study focuses on *how* to teach, not *what* to teach (Lewis, 2001), and as such is a valuable approach to professional development which is particularly relevant to the Project Maths reform. In Japan, teachers see lesson study as an integral part of their professional development which changes their beliefs and practices of

teaching (Yoshida, 2012) – a vital component of professional development in a time of curriculum change (Section 2.3) (M. Fetters et al., 2002).

As previously noted, teachers react differently to professional development (Jacobs et al., 2010) and lesson study is not intended to change all teachers in identical ways (Lewis & Hurd, 2011). Kamina and Tinto (2011) attended to the different learning trajectories of participating teachers in lesson study, noting that one newly qualified teacher was not at all changed through their participation. Despite this finding, the literature does suggest that in general, lesson study provides teachers with opportunities to develop their teacher knowledge in different ways through different elements of lesson study (Figure 6) (Fernandez, 2005; Lewis et al., 2009; Murata, 2011; Murata et al., 2012; Tepylo & Moss, 2011).

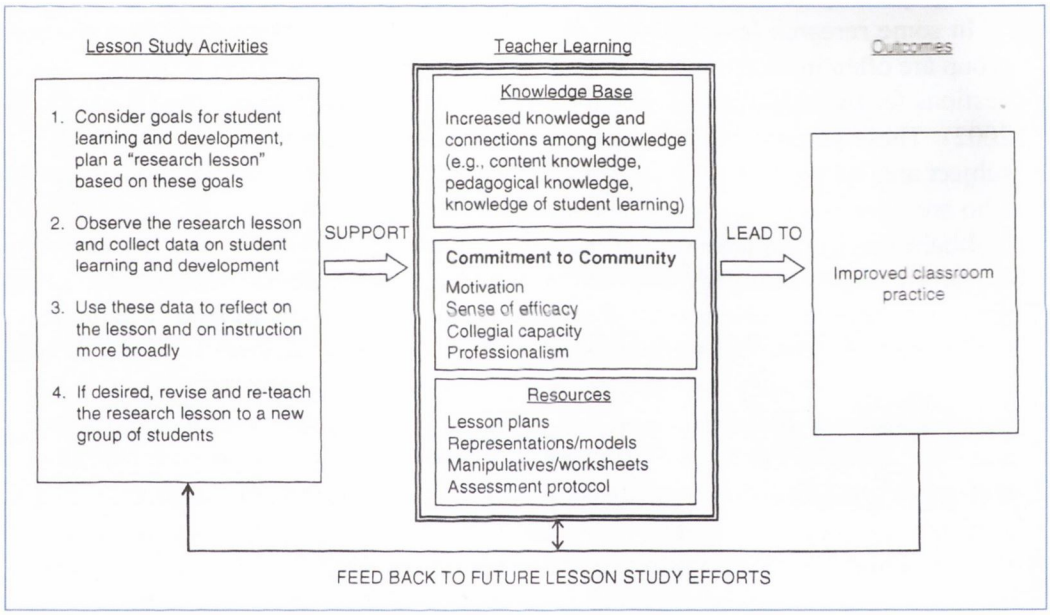


Figure 6 Lesson Study Activities, Teacher Learning, and Outcomes (Murata, 2011)

Lewis et al. (2009) present case study evidence that participating in lesson study builds teacher subject matter knowledge and focuses teachers on student thinking. Murata et al. (2012) report that novice teachers felt their discussions with more experienced colleagues helped them understand the importance of new ideas and implementing them practically in class. Tepylo and Moss (2011) in their research concluded that the four teachers involved in research in a rural school increased their content knowledge, KCS, and KCT, through their engagement in lesson study cycles over the course of an academic year. During their third lesson study cycle these teachers began anticipating student responses

and noticing how students were attempting to answer questions (KCS) within the research lesson (Tepylo & Moss, 2011). The teachers also began to create new strategies for teaching a topic and began to realise the importance of sequencing content in building students' mathematical thinking (KCT). Similar to the teachers in Fernandez et al.'s (2003) study, these teachers did not build their initial research lesson around students' thinking nor did they focus on student thinking within the research lesson, but rather around the activities of the teacher. However, by the third research lesson Tepylo and Moss (2011) observed teacher learning in how teachers were able to provide examples of student thinking in their post-research test. Engaging teachers in the collaborative development of educational artefacts, particularly in iterative cycles of testing, revising, and reflecting on those artefacts in reference to teaching and learning in lesson study, leads them to grow professionally.

2.8.5 Lesson Study as a Vehicle for Teacher Community

The teacher communities created and evolved through lesson study support the development of teacher knowledge and also support the connections between those types of knowledge (Murata, 2011). Lesson study brings teachers out of the isolated practice noted by Remillard & Bryans (2004) and into an environment where goals, ideas and teaching experiences are shared. Teachers' engagement in such a community can strengthen their commitment to teaching and motivate them in improving on their practice (P. Grossman et al., 2001). These features provide a strong rationale for the design of the present project.

The features of lesson study align with many of the recommendations and features of teacher community as outlined in Section 2.6: teachers construct a shared goal for their lesson study, there is no one leader and the role of teacher facilitator is rotated, teachers collaborate within lesson study meetings to build the curriculum materials, teachers enhance their knowledge both as a group and as individuals and they engage in a collective focus on student learning (Lewis & Hurd, 2011; Lewis et al., 2009; Lewis et al., 2006; Murata, 2011; Murata & Takahashi, 2002). Mutual trust and respect are requirements within a lesson study community and these may develop over time for a newly evolving community (P. Grossman et al., 2001). It is also necessary that a lesson study community can operate within supportive conditions within the school, required by

any in-school professional development intervention (Kruse, Louis, & Bryk, 1994; Lewis & Hurd, 2011).

Teachers need time to conduct lesson study on an ongoing basis (Willis, 2002) as distinct from engaging in continuous professional development as a series of once-off events. The community can then also become a 'community of memory' (P. Grossman et al., 2001) where a common history is developed and a common narrative arises based on the community's common and shared practices as teachers are involved in iterative lesson study cycles with their colleagues. The value of such a community of memory has been touched on by Tepylo and Moss (2011) in realising changes to teachers' knowledge through participating in iterative cycles of lesson study.

2.8.6 The Lesson Study Cycle

A lesson study cycle consists of a number of steps (Lewis et al., 2006; Zawojewski et al., 2008):

1. Study curriculum and formulate goals
2. Plan research lesson
3. Conduct or observe the research lesson
4. Reflect on research lesson and planning process

4a. Revise and re-teach research lesson

These four steps are adapted from Lewis et al.'s (2006) four features of investigation, planning, research lesson, reflection and are included within this cycle which can be repeated (revising certain elements of a research lesson) or can be iterative in focusing on different topics or aspects of teaching.

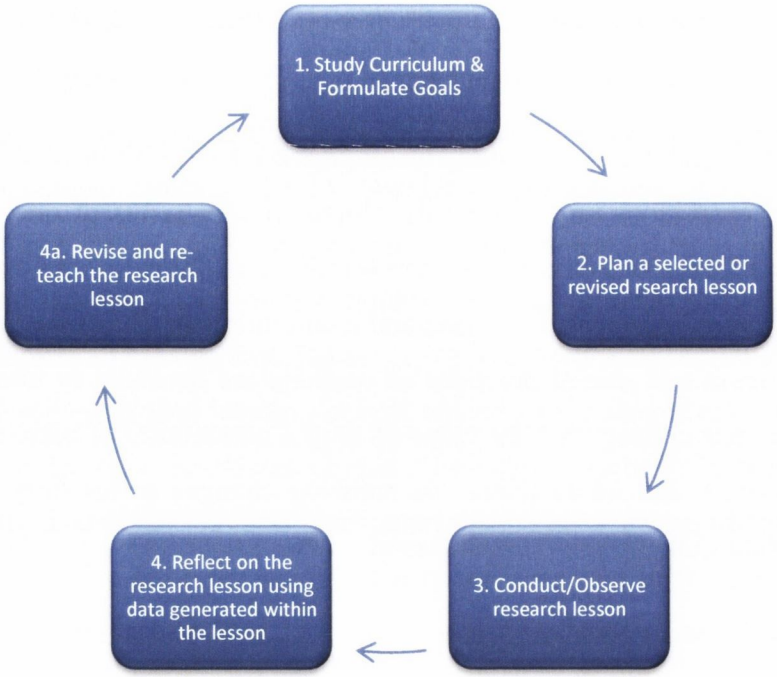


Figure 7 Lesson Study Cycle modified from Lewis et al. (2006)

Although the premise of lesson study can seem very simplistic, Doig et al. (2011) describe the lesson study process as an iceberg where the visible entity of the research lesson is in fact due to the far deeper, hidden work of, planning, of task design and selection, anticipation of students' thinking and reflection of the lesson that acts as a foundation of that research lesson. Each of the lesson study steps will be detailed in the following sections.

2.8.6.1 Study the Curriculum and Formulate Goals

At the initial stage of lesson study, teachers agree on a common objective for the duration of their involvement, be that one cycle of lesson study or a full academic term (Lewis et al., 2006; Meyer & Wilkerson, 2011). The focus of the community needs to be relevant to the participants as individuals and as a whole, but also provide a focus for observation of student engagement and learning during the research lesson (Lewis & Hurd, 2011), and this goal should be repeatedly examined and negotiated within the group (Thousand & Villa, 1992).

It is not expected that each member of a teacher community holds the exact same opinions or philosophies (Weick, 1969), but failure to establish a common objective can lead to a failure of outcomes in both the observation and reflection of the research lesson, as documented in a longitudinal case study of mathematics teachers in the USA (Perry &

Lewis, 2008) and also in the study of professional learning communities (Giles & Hargreaves, 2006). Additionally, lesson study may have failed in Fernandez's research (2002) since the research goal was incredibly broad in attempting to investigate "the effect that visuals have on student questions" (p. 402) and teachers were unable to address this research goal within lesson study. It is thus important that teachers agree on an overall aim which is specific and relevant to their practice and that teachers feel is possible to achieve.

It is at this beginning stage of the lesson study cycle that teachers can directly engage with their syllabus documents in detailing which area of the curriculum they would like to focus on for one or more of their lesson study cycles (Lewis & Hurd, 2011). This is likely an important factor in engaging teachers in a curriculum reform as was demonstrated in Kamina and Tinto's (2011) research into teachers' engagement with the *Investigations* fifth grade mathematics curriculum. Once teachers have agreed on which element of the curriculum they would like to focus on (because they find it difficult to teach or because they feel students have difficulty with it (Takahashi & Yoshida, 2004)), they can continue to the next step of planning the research lesson.

2.8.6.2 Planning or Revising a Research Lesson

Following the devising of a shared goal, teachers meticulously plan a lesson as a group based on their chosen area of curriculum and addressing their agreed goal as agreed by the group. Teachers enjoy planning the learning processes of their students in their own subject matter (Black & Atkin, 1996; P. L. Grossman & Stodolsky, 1995) and the product of this collaborative planning is a lesson plan that describes in detail the design of a research lesson. Anticipating student misconceptions, sequencing student activities and focusing on how students may approach a problem is an important part of planning the research lesson (C. Fernandez et al., 2003; Lewis et al., 2009; Lewis et al., 2006; Meyer & Wilkerson, 2011) and teachers can develop their PCK through these conversations (Fernandez & Cannon, 2005; Lewis et al., 2009; Tepylo & Moss, 2011).

Lesson study, as taken for this research, involves collaborative building of a lesson plan by *all* teachers participating in the community. The lesson is written during the research meetings with all teachers contributing ideas for content and approaches to be included within the lesson. This varies from a model practiced in Japanese high schools where one

teacher volunteers to 'take charge' of a lesson including writing the lesson plan which is then critiqued by colleagues before the lesson is taught (Chichibu & Kihara, 2013).

The construction of content for a mathematics lesson is a valuable skill. Lesson study shifts the focus of teachers from the exposition of a subject, to choosing and assembling content in a way that will be engaging and comprehensible to students and will build on their mathematical thinking (Fernandez, 2002; Fernandez & Chokshi, 2004). During this planning phase, teachers are encouraged to complete or solve the tasks they plan to give students in order to anticipate student thinking. By working on the task themselves, teachers surface their own strategies and misunderstandings, creating opportunities to correct their misunderstanding and jointly develop responses to students' expected thinking discussed in developing teacher knowledge (Section 2.7) (Mason, 1998, 2009; Perry & Lewis, 2008; M. K. Stein et al., 2008). Within this stage of the cycle, it is important that teachers attend to the presentation of a stimulus activity (referred to as *hatsumo*n in Japan) (Doig, Groves, & Fujii, 2011) and pay particular attention to the problems that they select for students to work on (Takahashi & Yoshida, 2004). The planning process can also be enhanced through teachers utilising available curriculum materials (Kamina & Tinto, 2011), developing curriculum materials, or utilising research literature (Tepylo & Moss, 2011) and is investigated in the following two sections.

A. Curriculum Materials

In constructing research lessons during lesson study, teachers can develop views and perspectives pertinent to their own students and their own teaching through creating materials themselves or by utilising and modifying readily available curriculum materials (Hanley, 2010). Ball & Cohen (1996) advocate teachers using curriculum materials in their planning since it supports teachers in focusing on and interpreting students' responses and in anticipating what learners may think when meeting particular mathematical activities.

When teachers produce educational objects for use in their own practice they have an opportunity to examine their own thinking since these created artefacts are an external representation of their own interpretive systems (Zawojewski et al., 2008). Teacher knowledge is developed when testing and revising take place through teachers negotiating the development of the curriculum resource, field-testing it in the classroom,

and reflecting on the usefulness of the material. This stage of planning is thus very important in viewing lesson study as a vehicle of professional development for teachers.

B. Research Literature in Planning Research Lessons

Research literature can assist in developing teachers' perspectives on their practices and contribute to teacher learning (PCK) and within lesson study teachers are encouraged to engage with research literature on teaching a particular mathematics topic (Lewis & Hurd, 2011). The practice of anticipating typical student responses to tasks can be supported by reading research literature or by using mathematics curricula that include typical student responses to problems, as done in many Japanese curricula (Fernandez & Chokshi, 2004; Stigler & Hiebert, 1999).

Tepylo and Moss (2011) in their lesson study research provided teachers with specific research literature based on the curricular content of fractions, a topic which those elementary teachers had decided as their chosen content for their participation in lesson study. From reading this research literature, Tepylo and Moss (2011) defined a particular instance where a teacher realised for the first time that fraction multiplication, where the resulting answer is smaller, would provide a difficulty for students. The teacher, who had been teaching for eight years, realised for the first time how students might find this concept difficult to grasp. Incorporating research literature within the planning phase of lesson study can thus be an important element in benefitting teachers' practices (Lewis & Hurd, 2011; Tepylo & Moss, 2011).

2.8.6.3 Conducting and Observing the Research Lesson

Part of creating a community where teachers can learn and share from each other is opening up classrooms for others to visit. Not only can these observations benefit the conducting teacher and students by discussing the strengths and possible areas of improvement within the lesson, but also the observer gains ideas for other classes (Briscoe & Wells, 2002; Ginns, Heirdsfield, Atweh, & Watters, 2001; Yow, 2008). Since the community will observe and reflect on learning within the classroom during a research lesson, teachers are provided with opportunity to positively impact students' engagement and learning of mathematics.

In lesson study one teacher conducts the lesson to his or her students while the other group members observe the lesson and take careful notes. As part of their planning,

teachers decide on what to observe during the research lesson (C. Fernandez et al., 2003; Lewis & Hurd, 2011; Meyer & Wilkerson, 2011). Lewis & Hurd (2011) suggest that the most beneficial approach to observation of a research lesson is to follow one student or group of students for the duration of the lesson while avoiding engaging with the students as instructing teachers.

Observing research lessons provides a very new and unique opportunity for teachers to look in on a lesson which they have participated in planning. Since only one teacher is conducting the lesson it provides the observing teachers with opportunity to notice how students are engaging with and responding to the mathematics within the lesson (Breen et al., 2014; Mason, 2002). This differs from observations in, for example, video studies where teachers are asked to view edited classroom clips and respond on significant events since teachers are viewing only one segment of a lesson (Jacobs et al., 2010; Sherin & Han, 2004a; van Es & Sherin, 2008). It also differs to teachers observing within their own classroom (Breen et al., 2014; Mason, 2002) since that depends on them having opportunity to note particular moments in their students' learning while simultaneously instructing. In lesson study, since all of the observing teachers are familiar with the learning objectives of the lesson and the sequence of activities, their attention can be focused on how individual students are engaging with the mathematical ideas within the lesson. Teachers can then interpret student events and use that information to inform their pedagogical decisions (Schoenfeld, 2011; van Es & Sherin, 2008). This theme is further detailed in the following section.

2.8.6.3.1 Noticing of Students' Mathematical Thinking

'Noticing' is defined as "making sense of how individuals process complex situations" (Jacobs, 2010, p111) and while experienced teachers may often have 'check points' in mind during instruction that they use to assess the progress of a lesson (van Es & Sherin, 2008), it is impossible to be aware of and respond to everything that is happening within one lesson (Carpenter et al., 1989; Jacobs et al., 2010; van Es & Sherin, 2008). It has been found that teachers can change what they notice, through various interventions, and move from a focus on teachers' activities to students' thinking and to develop from responding to particular moments with evaluative comments to ones interpreting what has occurred in the classroom (Sherin & Han, 2004b; van Es & Sherin, 2008). Such a change in what teachers see during their practice is counted as teacher learning (Corcoran, 2011b; Jacobs et al., 2010). Within lesson study, teachers are provided with

the opportunity to begin to interpret how students are thinking about mathematics (Meyer & Wilkerson, 2011). Teachers also improve their observational skills in, for example, realising that they may tend to miss students' non-verbal communication, compared with what fellow teachers notice (Lewis, 2002).

Teachers often identify students' incorrect answers and interpret how a student arrived at an answer (Ball et al., 2008; M. K. Stein et al., 2008) but may not always have the opportunity to reflect on these responses and thus may not always build such observations on student learning within their teaching. Observation of research lessons is at the core of lesson study (Corcoran, 2009; Murata, 2011) and observing students within research lessons allows teachers a unique perspective in witnessing students' interactions within a lesson, but also allows teachers to reflect as a group (and as an individual) on the teaching and learning of mathematics. Within lesson study, teachers have the opportunity to reflect on how students came to a particular answer and can decide how to respond to the student perhaps in that lesson or incorporate an additional activity within a subsequent lesson to address this particular strategy or way of thinking (Meyer & Wilkerson, 2011).

In lesson study teachers take notes within the research lesson and reflect on what they observed in the post-lesson discussion. The intended focus of these meetings is for teachers to discuss what they have noticed about student learning in the post-lesson discussion (Lewis & Hurd, 2011; Tepylo & Moss, 2011).

2.8.6.4 Reflecting on the Research Lesson

Mason (2002) notes that the purpose of reflection can influence that which is noticed and so both practices are closely linked. He suggests that in noticing students' thinking it is not enough that teachers pay attention to students' responses and interactions within a research lesson, they should also examine and reflect on what these responses might convey, and this stance echoes those suggested within the lesson study literature (C. Fernandez et al., 2003; Lewis et al., 2009; Lewis et al., 2006). Reflection is a key learning activity for teachers where they can reconstruct, re-enact and recapture the events, emotions and accomplishments of the lesson (L. Shulman, 1987; L. S. Shulman & Shulman, 2004). While teaching certainly involves making judgements about what did or did not go well in a lesson, it is critical for teachers to notice what is significant in a classroom interaction, interpret that event and then use that interpretation to inform their

future pedagogical decisions (Ball et al., 2008; van Es & Sherin, 2008). Reflection on the research lesson is thus an important part of the lesson study cycle and the literature suggests that it should take place as soon as possible after the research lesson (Lewis et al., 2006; Sloane, 2005).

Frameworks developed in order to utilise observation and reflection (such as that developed by Santagana et al. (2007)) cannot necessarily apply to lesson study since teachers are already familiar with the learning goals and have been involved in constructing the lesson to accomplish a learning objective. In lesson study teachers are encouraged to reflect on the research lessons in terms of the pedagogical approaches employed, the content knowledge included and the engagement of students by sharing their observations, comments, and suggestions. Teachers' attention is focused on students' engagement in the lesson and how students' thinking was explicit within the lesson. Episodes of what teachers noticed about student thinking within a lesson can be professionally significant and provide further opportunity for discussion and learning (Mason, 2002) and this should provide a basis for teachers' conversations in their post-research lesson discussions. In focusing on the students, teachers are encouraged to think about:

1) How students understood the problem, and
2) How to facilitate students' understanding.

Teacher reflection can lead to a change in their teaching and learning approaches (Lynch, 2011; Meirink, Meijer, Verloop, & Bergen, 2009). By participating in group reflections, teachers can gain insights into the experiences of colleagues and build on their own knowledge of teaching and learning approaches, strategies or content knowledge (Breen et al., 2014). In this regard, lesson study provides teachers with a structure through which they can experiment with and reflect on new teaching and learning approaches or reflect on student learning within the classroom.

2.8.7 Critical Lenses within Lesson Study

Teachers view teaching and learning practices from different perspectives and these perspectives can be changed and enhanced through participation in professional development and through lesson study (Corcoran, 2011b; Jacobs et al., 2010; Murata, 2011; van Es & Sherin, 2008). Fernandez et al. (2003) introduced the idea of developing

critical teacher lenses within lesson study when a group of American teachers were guided in their participation of lesson study by Japanese teachers. Within their study, Fernandez et al. (2003) note that these participating teachers were encouraged to see themselves as researchers conducting an empirical investigation around the teaching and learning of mathematics and came to view their practice with a *researcher lens* in investigating what strategies and approaches best suited developing students' mathematical thinking. In that research teachers were also encouraged by their Japanese counterparts to think about students' entire learning experience and the sequencing of content before planning a lesson (C. Fernandez et al., 2003). Teachers were encouraged to view what students had been taught and how it had been taught prior to a research lesson in order to best build students' thinking and by doing this were developing a *curriculum lens* on their practice (C. Fernandez et al., 2003). Building on this research, Kamina and Tinto (2011) in their research on fifth grade teachers' engagement with lesson study found that teachers were beginning to develop a curriculum lens in viewing students' learning across grade levels.

The Japanese teachers participating in Fernandez et al.'s (2003) research also conveyed the importance of viewing the entire learning experience from a student's perspective – from the overall learning goal all the way through the lesson study process (C. Fernandez et al., 2003). Developing a *student lens* was an important element of teachers' learning and teachers were encouraged to view all aspects of the research lesson from a student's perspective (Corcoran, 2011b).

2.8.7.1 Critical Student Lens

Professional development interventions highlight the importance of teachers' knowledge of how students construct content (Carpenter, Fennema, Peterson, & Carey, 1988; Carpenter et al., 1989; Fennema et al., 1996) and literature on noticing students' mathematical thinking were incorporated into Ball et al.'s (2008) framework of PCK discussed in Section 2.7.5. Japanese teachers note having "the eyes to see children (*kodomo wo miru me*)" (Lewis, 2002, p. 21) as one of the major benefits of lesson study. During research lessons, teachers scan the classroom for evidence of student learning, motivation, and behaviour, particularly noticing if students "eyes were shining" (Lewis, 2000; Lewis, 2002) when they engaged in an activity or if students were making exclamations of understanding under their breath (Lewis, 2000, 2002). As teachers carefully observe students' learning and engagement during a research lesson, they have

the chance to think more deeply about students than is usually possible in the normal workings of daily classroom life. Teachers are provided with the opportunity to see the lesson through a student's eyes and this focus on students' thinking can be a source of learning for teachers (Corcoran, 2011b; Jacobs et al., 2010; van Es & Sherin, 2008). Teachers develop or enhance their critical lens through which to see mathematical thinking, teaching and learning within the classroom.

A focus on student thinking during lesson planning positively effects classroom learning and engagement of students and eventually leads to an improvement in the understanding of students (Cerbin & Kopp, 2006; Fernandez, 2002; Lewis et al., 2009). In lesson study, teachers examine student learning by anticipating their responses and by collecting data on students' engagement in activities during research lessons (Fernandez & Chokshi, 2002; Maria Lorelei Fernandez, 2010; Lewis & Hurd, 2011; Lewis et al., 2009; Meyer & Wilkerson, 2011). Research on lesson study has shown that a focus on student thinking during lesson study positively effects classroom learning and the engagement of students, and eventually leads to an improvement in the understanding of students (Cerbin & Kopp, 2006; Fernandez, 2002; Lewis et al., 2009). This is interesting to note since separate literature on professional development has shown that teachers who learn how to learn from their own students' perspectives can continue learning after the formal professional development support ends by maintaining this student lens on their practice (Franke et al., 2001).

For teachers undertaking lesson study for the first time, it may not be a natural perspective for teachers to view their practice through the eyes of a student (C. Fernandez et al., 2003), nor is it necessarily a natural stance for pre-service or newly qualified teachers to view their practices from a students' perspective (Jacobs et al., 2010) but with structured professional development support, this lens may develop over time (Sherin & Han, 2004b; van Es & Sherin, 2008).

In this research it will be investigated how a focus on student thinking, a critical student lens emerges through teachers' participation in lesson study and how this might enhance teachers' practices.

2.8.8 Facilitator in Lesson Study

In lesson study the facilitator plays an important role in guiding the community through the cycle, maintaining a focus on the over-arching goal and learning objectives of the research lesson, and providing opportunities for all participants to engage in the cycle. A facilitator may be an internal member of the group or may sometimes be invited from outside the school to participate because of their strong content, pedagogical, or curricular knowledge (Lewis, 2002; Lewis & Hurd, 2011). The role of the facilitator should be rotated within the group within each lesson study cycle or meeting since such shared leadership ensures that all members of the community can participate equally in discussions and observations (Louis & Marks, 1998).

In terms of educational reform, the role of external expertise is crucial in providing important information to the teachers and in assisting their development within a reform (Cordingley, 2005). An external facilitator should: co-ordinate an agreement on the purpose of the common work of the community, allow the ownership of the community's agenda and work within the meetings to develop, and bring 'formal' theory into the work of the community (Erickson, Minnes Brandes, Mitchell, & Mitchell, 2005). The external facilitator may also be instrumental in bringing new curricular knowledge and research literature to the community that may otherwise be too difficult or time-consuming for teachers to access (Corcoran, 2007; Erickson et al., 2005; Fernandez, 2002; Tepylo & Moss, 2011). In this research I acted as a facilitator of lesson study, providing access to research literature, providing curricular knowledge, and introducing teachers to the mechanisms within the lesson study cycle (see Section 3.11.2).

Teachers that are new to intensive collaboration have few tools to use and little competencies for structuring the work of the community or for overcoming difficulties. The external facilitator has an important role in handling any conflicts (between teachers and with the reform) by bringing them to the surface and resolving them (Cordingley, 2005; Thousand & Villa, 1992). Because the facilitator is outside the school culture, they can mediate conflicts differently to those bound by behavioural patterns (P. Grossman et al., 2001) and can ensure that certain people do not dominate the conversation (Arbaugh, 2003).

2.8.9 Lesson Study in Ireland

Lesson study is growing in popularity all over the world from Singapore to the USA (Doig et al., 2011). Introduced formally in Ireland by Conway and Sloane in their 2005 report to the NCCA (Conway & Sloane, 2005) and as a paper in the first national conference on mathematics education (MEI 1) (Sloane, 2005), lesson study has since been utilised in primary school settings in developing pre-service teachers' teaching and learning approaches (Corcoran, 2007; Aisling M Leavy, 2010; Aisling M. Leavy, Hourigan, & McMahon, 2010; O'Shea & Leavy, 2013) and content knowledge (Hannigan, Gill, & Leavy, 2013).

Lesson study is not yet common in schools nor has it yet been a common focus of research in post-primary schools. Many differences exist between primary and post-primary systems and lesson study practices may therefore be different in both settings. Post-primary teachers do not teach only one class but may have as many as nine different groups of classes within one day. In Ireland teachers do not just teach one year group but may teach all of the six year groups within one day. Also, unlike primary teachers, post-primary teachers in Ireland are not yet required or incentivised to participate in professional development and although subject specific departments are encouraged to operate, there is often no formal structure within which common subject teachers meet or plan collaboratively. This creates a striking difference between the lesson study literature based in predominantly primary schools in the USA and post-primary schools in Ireland.

2.8.10 Alternative Models to Lesson Study

As well as lesson study, other similar models of collaborative professional development exist but none seem to incorporate the same aspects of community, micro-curriculum planning, student noticing, reflection of teaching, collaboration and dialogue. Three similar practices will be expanded upon here.

2.8.10.1 Teacher Design Teams

A teacher design team (TDT) is a group of at least two teachers, from the same or related subjects, working together on a regular basis with the goal to (re)design and enact (a part of) their common curriculum (Handelzalts, 2009). The TDT model provides teachers with a creative space in which to reflect on the teaching of their subject through collaboration (Mooney Simmie, 2007) but, in contrast to the community of teachers engaged in lesson study, the goals of professional development are seen as secondary to

the main design goal of a TDT. A TDT can be seen as an ad hoc functional unit and as soon as their task is completed, they are not considered as a TDT any longer. While the work of the teacher design team shares many comparisons with the practices of the community engaged in lesson study, there are integral differences in that the community in this research should be a sustainable object of professional development within a subject department.

2.8.10.2 Learning Study

In a learning study, teachers jointly work in an iterative process to gain knowledge about how their teaching practice in the classroom affects student learning. In this approach, teachers formulate a research question based on their students' misconceptions or reflections. Similar to lesson study, a learning study is a collaborative and systematic process of inquiry in classroom practice, however the focus is more on variations in student understanding (Davies & Dunnill, 2008). Learning study combines a model of collaborative 'plan-teach-review' teacher development with a focus on the structure of learning outcomes (Davies & Dunnill, 2008). Learning study emphasises the importance of recognising the gap between students' initial understanding and the ideas that teachers hope students will develop during a lesson.

A more superficial aspect of learning study may be implemented by teachers in lesson study within their reflection of and engagement in lesson study. Lesson study may also provide a broader introduction to this form of collaborative planning for teachers that may, in time, develop into a learning study depending on a community's own goal and objectives. Lewis et al. (2006) note that lesson study should be regarded as 'local theory' where what works for those particular teachers in that particular school with these particular students should be adhered to and developed.

2.8.10.3 Lesson Explaining

In the Chinese practice of 'lesson explaining' a teacher explains the teaching process and related issues about the lesson to colleagues and mathematics experts who then comment and discuss what was explained. This process has gradually become an extra kind of evaluative resource for teaching, emphasizing teacher reflective practice. The essential feature of lesson explaining is then the articulation of the teacher's epistemological view of the content (Peng, 2007). In general, the mathematics content, the intended teaching method and the learning method, as well as the whole teaching procedure, form the main

components of a session of lesson explaining (Peng, 2007). The teaching goals include the expected teaching results, analysed from the point of view of knowledge and skill, process and method, emotion and attitudes as required by the mathematics curriculum standards. If the goals are definite and the teaching design is reasonable, it suggests that the explaining teacher understands the content well. However, this is a very individual (and could be seen as perhaps a risky) process for one teacher. This method contrasts greatly with the community approach of lesson study where shared and collaborative practices and goals encourage a development of content and pedagogical knowledge in an open and engaging environment.

2.8.11 Conclusions from Lesson Study

Lesson study is a model of professional development that is based on the premise that teachers collaborate in planning, conducting, observing and reflecting on research lessons. Within lesson study teachers are provided with opportunities to enhance their knowledge (Fernandez, 2005; C. Fernandez et al., 2003; Lewis et al., 2009), both subject matter (Tepylo & Moss, 2011) and pedagogical (Meyer & Wilkerson, 2011; Tepylo & Moss, 2011), particularly when teachers participate in iterative cycles of lesson study.

One of the most important aspects of lesson study is that teachers enhance their ability to see lessons from different perspectives and see their teaching and learning practices through different lenses (C. Fernandez et al., 2003). It is particularly relevant in terms of teachers' PCK that teachers begin to view lessons from the perspective of a student, begin anticipating how students might respond within lessons (Meyer & Wilkerson, 2011), and begin to focus more on students' strategies within lessons (Tepylo & Moss, 2011).

While the process of lesson study is well explained in the literature (V. Shulman & Armitage, 2005), the underlying moments of transformation where teachers' knowledge and perspectives develop and change have not been addressed in detail. The literature to date has also not detailed how teachers learn or enhance their knowledge through lesson study and any references to teacher knowledge have not been addressed according to any frameworks of teacher knowledge. In addition, although it has been intimated that teachers change their perspective on their teaching through lesson study, this has not been addressed in the processes of lesson study nor has this changed perspective been mapped to the literature on teacher knowledge.

2.9 Conclusions from Literature Review

With the introduction of a new mathematics curriculum, Project Maths not only changes what post-primary students are learning in mathematics but also embeds the teaching and learning practices within a constructivist approach (Lynch, 2011; Oldham & Close, 2009). At a time of curriculum change it is important that teachers be supported through continuous professional development, particularly since successful curriculum reform relies on changing teachers' knowledge and beliefs. It is particularly important within a time of such reform that the model of professional development be relevant to teachers' own knowledge and practices and relevant to their school structures (Putnam & Borko, 2000; Remillard & Bryans, 2004). Professional development encompasses the activities that develop a teacher's skills, knowledge and expertise and simultaneously affects teachers' professional characteristics as well as their classroom learning and student experience. Effective professional development is continuous and includes practice and reflection. Establishing communities of teachers has become an important initiative in professional development since communities provide collegiate support while also allowing a school structure to develop where teachers can share ideas on professional and pedagogical practices. Within these communities, teachers of all experiences and background will benefit, in different ways, from the collaborative approach.

Lesson study provides a vehicle for such a community of teachers to collaborate in planning, conducting, observing and reflecting on their instructional practices. Particularly in a time of curriculum reform it is imperative that professional development be situated within the classroom (Putnam & Borko, 2000) where teachers can observe a curriculum and its associated materials being enacted (Hanley & Torrance, 2011; Remillard & Bryans, 2004). Lesson study provides teachers with a structure within which they can reflect on and analyse their own practices in relation to this reform. Teachers' knowledge should benefit through the exchange and developing of pedagogical approaches and ideas and through observing and reflecting on students' thinking within research lessons (Lewis et al., 2009; Meyer & Wilkerson, 2011; Tepylo & Moss, 2011).

Within lesson study teachers are provided with opportunities to see the curriculum being enacted (an important aspect of curriculum reform) but also have opportunities to develop and plan lessons that are relevant to their own students and school. Within lesson study teachers are provided with opportunities to enhance their knowledge through: a

detailed planning of lessons where students' thinking is anticipated and prior knowledge is detailed, where the content of lessons is sequenced, where teachers focus on attending to students' thinking and strategies within research lessons, and where teachers reflect on students learning and review how teaching and learning can be further enhanced. Irish teachers are not yet familiar with participating in in-school models as forms of professional development but these approaches to CPD are being recognised and furthered in policy as especially important in times of impending educational reforms.

With the introduction of Project Maths, lesson study may provide teachers with opportunity to engage with the curriculum in a way which enhances their knowledge and develops their perspectives of their teaching in a way which benefits their teaching and learning practices. Although the lesson study literature is detailed in its analysis of the mechanisms within the model, the literature is lacking in detailing the exact features of how teacher knowledge is enhanced through their participation in lesson study. This research sets out to fill this gap within the literature and to map teacher learning from lesson study to a framework of teacher knowledge.

Chapter Three

3 Methodology

3.1 Introduction to Methodology

The aim of this chapter is to provide a methodological overview for the introduction of lesson study as a model of professional development based on teacher community within two post-primary school Mathematics departments. The chapter begins with a research summary and outline of the research questions. The approach of the research then is investigated and a methodological framework for the school based study is outlined. Following this, a brief overview of the theoretical framework identifying the qualitative research methods employed is explored. Data generation and collection techniques are outlined and the chapter discusses some ethical considerations related to the study. Finally, the analysis of the data will be discussed.

Yin states that "the desired result is for the investigator to create a rich dialogue with the evidence" (Yin, 2014, p. 73). It is intended that the methodology of this research will provide opportunity for an illustrative relationship between the data generation, data analysis and interpretation of the research.

3.2 Research Summary

As part of the Project Maths curriculum, teachers are encouraged to develop their classrooms as a learning environment which "engages students, facilitates discussions, supports risk taking and encourages inquiry" (National Council for Curriculum and Assessment, 2012a, p. 8). These explicit skills identify new practices and engage new pedagogical methodologies for the majority of post-primary mathematics teachers in Ireland (Oldham, 2001). Curriculum reform is best supported through professional development (Fullan & Hargreaves, 1992; Hopkins & Reynolds, 2001) and in identifying and enacting these new curriculum approaches, teachers have been offered a number of professional development in-service training days funded by the Department of Education and provided external to the school (Section 2.2.1). This research investigates whether an in-school model of professional development, that of lesson study, may provide richer opportunities for teacher learning which enhances their mathematical knowledge for teaching.

This research is an empirical inquiry investigating the evolution, development and outcomes of a professional development intervention introduced within the backdrop of a major curriculum reform. The purpose of the research is threefold:

1. To investigate the impact of participating in lesson study on teacher knowledge.
2. To assess the impact of lesson study as an instrument of professional and pedagogical development and curriculum implementation in a specifically Irish post-primary context.
3. To identify the evolving features of a teacher community (although this will be addressed in future research).

The research was conducted within two Mathematics Department with participating teachers engaged in lesson study as members of two distinct teacher communities. The research was approached as a multi-site case study with qualitative data generated through mixed methods. Conversations between participants in the teacher community will be the foundational source of qualitative data in discovering moments of transformation in the learning of participating teachers. The data for this study is the set of 36 audio recorded lesson study meetings in both research sites and the corresponding 33 interviews conducted with participating teachers.

In the event of a positive assessment of this model as a form of professional development enhancing teacher knowledge, it is hoped this research will conclude on the necessary structural framework and seminal ingredients around which a sustainable teacher community can develop within an Irish post-primary school subject department.

3.3 Research Questions

Within this project the research questions have been continuously modified and changed throughout the course of this study. In particular, I have modified the main question of this thesis relative to what has emerged from the data but also by taking note of comments and questions I have received during various conference presentations and seminars.

For the purposes of this project one question formed the basis of this research. Two sub-questions within this main question formed the basis of the analysis. Another question, outlined below, will later be explored external to this thesis.

1) How is teacher knowledge, particularly pedagogical content knowledge, enhanced through lesson study?
    a. How are knowledge of content and students (KCS) and knowledge of content and teaching (KCT) enhanced within lesson study?
    b. Are other forms of teacher knowledge, external to subject matter knowledge, employed within lesson study?

2) How does teacher community develop (P. Grossman et al., 2001) through members' participation in lesson study? (not addressed within this research)

3.3.1 Investigating the Research Question

Q1. How is teacher knowledge, particularly pedagogical content knowledge, enhanced through iterative cycles of lesson study?

A principal objective of this research is to identify how PCK, particularly features of KCS and KCT, are utilised and enhanced within lesson study. In attempting to answer this question specific practices within lesson study which invoke and develop these features of teacher knowledge will be explored.

The reasons for choosing the practice of lesson study as the focus of this thesis are threefold:

1. It is an established international practice (Yoshida, 2012),
2. It provides scaffolding for the structured collaboration of teachers that is teacher-led rather than externally-generated (Perry & Lewis, 2008),
3. It changes teacher knowledge and beliefs (Lewis et al., 2009; Murata et al., 2012; Tepylo & Moss, 2011; Zawojewski et al., 2008).

However, while research has stated that teacher knowledge is enhanced through participation in lesson study (Lewis et al., 2009; Meyer & Wilkerson, 2011; Murata et al., 2012; Zawojewski et al., 2008) and that KCS and KCT can be developed through lesson study (Tepylo & Moss, 2011), the research literature to date has not detailed the specific features within KCS and KCT that are developed. Nor does it explain how teacher knowledge is developed while mapping teacher learning to a specific framework of teacher knowledge. This research aims to identify the specific features of KCS and KCT which were utilised and developed through participating in lesson study and maps

these to an existing framework of mathematical knowledge for teaching (Ball et al. 2008).

While the impact to teachers' subject matter knowledge is not investigated in this research (see Section 3.10), within the research question there is a possibility that 'teacher knowledge' enhanced by lesson study may not be encompassed by SMK and PCK. Through the data generated in this research, an additional element of teacher knowledge utilised and enhanced through lesson study is explored.

Q2. **How does teacher community develop through members' participation in lesson study?**

While it is beyond the remit of this thesis to investigate this question, it is of interest to me to track how teacher community may develop through participating in lesson study. Based on a model of evolution of community suggested by Grossman et al. (2001) (see Section 2.6.3), it will be interesting to note if and how teacher communities may evolve from beginning to mature phases through engagement with iterative cycles of lesson study. I hope to continue to research this question following an investigation of the previous question.

3.4 The Tradition of the Research

3.4.1 Hermeneutic and Phenomenological

This research was broadly based in the hermeneutic tradition. The data was constructed and generated with the background of this tradition, with the assumption that the data comprised of teachers' interpretations of their own work (Merriam, 2009; Patton, 2002). The analysis was then also hermeneutic in constructing meaning from and interpreting what the teachers were saying, doing and meaning through their actions and interactions in lesson study.

The research was also partly phenomenological in that the data generated was allowed simply to emerge. The model of lesson study introduced to both schools had the capacity to change according to each school's environment and the data was not approached with a rigid structural process. The research was not testing a hypothesis, but instead relied on the participants' view of the situation and their interpretations of how their own knowledge has been enhanced (Blaikie, 2007), as well as my interpretation of how their

knowledge was enhanced through their discussions with their colleagues. The research was a focus on an experience, interpreting the essence or moments of transformation in teachers' participation in a professional development intervention (Merriam, 2009).

The research was based in a social constructivist epistemology where participants developed subjective meanings of their experiences of and in the research (Creswell, 2007). These meanings were varied and multiple, since not all teachers were affected in the same way through their participation (Jacobs et al., 2010; Zawojewski et al., 2008), and led me to look for a complexity of views to interpret and develop meanings of the process and participants as a whole.

3.4.2 Hermeneutical-Phenomenological Researcher

In light of the descriptive, evaluative and interpretive perspectives of this study, I took a social constructivist approach as a hermeneutical, phenomenological researcher (Creswell, 2007; Merriam, 2009). I aimed to interpret the generated data through a deep understanding with the subject of my inquiry: teacher learning as participants in lesson study in an Irish post-primary context situated in a background of curriculum change (Miles & Huberman, 1994).

Taking this hermeneutical, phenomenological approach I was wary about condensing the vast amount of generated data and of formulating codes which may not have fully encapsulated the essence of what occurred. I was fully aware that my interaction with the data and participants in this research impacted on me as a researcher and as a person and my interpretation of this data unquestionably applied my own sense of meaning to the dialogue generated between the data and I. As a researcher, my own understanding of teacher learning and of lesson study deepened as the research progressed (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003). The measures which I employed in attempting to reduce bias in my analysis is further described in Section 3.11.

3.5 The Research Approach

3.5.1 Outlining the Research Approach

The procedures of qualitative research are characterised as inductive, emerging, and shaped by the researcher's experience in collecting and analysing the data (Creswell, 2007; Newby, 2010). A qualitative case study approach was employed in this research based in two separate sites due to the necessarily continuous and qualitative nature of the

data and due to the separate stages of curriculum reform which both schools were engaged in. As a participant-observer I participated as a facilitator of lesson study and participant of lesson study in both schools while also employing the research instruments (Section 3.11.2). While the research was designed as a multi-site case study employing mixed methods, it can also be viewed as multi-tiered since the teachers examined and reflected on classroom situation, and I as researcher and facilitator investigated teachers' practices and reflections in deriving principles for professional development that may be transferrable to other contexts (Zawojewski et al., 2008).

The main qualitative data was generated through recordings of teachers' lesson study meetings and through interviews with teachers and analysis of this data was supplemented by other lesson study artefacts (Section 3.10). Audio recording, field notes, and artefact collection were also useful as data tools in an approach like lesson study where the mechanisms within lesson study enhancing teacher knowledge is not yet understood and where individual teachers are expected to differ in their outcomes of participation (Jacobs et al., 2010; Lewis et al., 2009; Zawojewski et al., 2008).

3.5.2 Qualitative Case Study Approach

3.5.2.1 Qualitative Inquiry

Since the foundational focus of this research is teacher learning through conversations and interactions based within lesson study, the data generated and analysed had to be qualitative in nature. Within teacher conversations, the research aimed to highlight and identify episodes of learning where teachers engaged in dialogue which called on their knowledge of teaching and developed their knowledge of teaching. This research built on previous investigations of teacher learning through teacher conversations (Dudley, 2013; P. Grossman et al., 2001; Murata et al., 2012; Sherin & Han, 2004b; Tepylo & Moss, 2011). Within the interactions episodes of surprise, reflection, and evidence of changed practices were of particular interest. It is therefore qualitative inquiry which informed the data generation for the purposes of this research.

Qualitative inquiry, which focuses on meaning in context (Creswell, 2007), requires a data collection instrument that is sensitive to underlying meaning when gathering and interpreting data (Merriam, 2009). The instruments for this research in informing this qualitative inquiry were designed to isolate and identify these learning moments for

teachers and are discussed in further detail in Section 3.10. The holistic and intensive qualitative inquiry was approached as a multiple case study implementing these research instruments and triangulating generated qualitative data. The context of the research was important as was my approach in engaging with participants and in interpreting meaning from the generated data (Bogdan & Biklen, 2007; Marshall & Rossman, 2011; Merriam, 2009).

3.5.2.2 Why Case Study?

Case study approach is best situated to this research since it is an in-depth description and analysis of a bounded system of mathematics teachers engaging in school-based lesson study and an empirical inquiry that investigated a contemporary phenomenon within its real-life context (Merriam, 2009; Yin, 2014). The research looked for meaning within and understanding of lesson study as a model of professional development, where the researcher was the primary instrument of data collection and analysis, and where the end product aimed to be richly descriptive (Creswell, 2007; Merriam, 2009).

According to Yin (2014), employing a case study is the preferred method when:

a) "how" or "why" questions are being posed.
b) The investigator has little control over events.
c) The focus is on a contemporary phenomenon within a real-life context.

With regards to this research, each point was satisfied in identifying the key moments of transformation of participants engaging in lesson study. The research asked "how" teachers developed their knowledge through their participation in lesson study. Besides providing a framework for and facilitating the introduction of lesson study within the two schools, as participant-observer I had little control over how teachers engaged in the model since teachers themselves dictated what topics they discussed, which students they focused on, the curriculum materials employed and developed, and the number of lesson study cycles they participated in. The focus of research was on contemporary and contextualised phenomena attempting to in-depthly describe and analyse the bounded system of teacher learning within a community engaged in lesson study in the real-life context of two school Mathematics Departments. According to the methodology employed, there were also multiple sources of evidence, where data would converge and triangulate (Bogdan & Biklen, 2007; Seale, 1999). Each of these points led me to inarguably decide on employing case study as the approach for this inquiry.

This qualitative case study approach identified me as engaging with and interpreting data collected through a variety of instruments. Within this qualitative case study I, as participant observer, observed what was occurring, but also attempted to interpret the change over time of particular forms of interaction through qualitative observation instruments.

3.5.2.3 Multi-site Case Study

Multi-site case studies, also known as collective case studies, cross-case, multi-case, or comparative case studies, involve collecting and analysing data from several cases (Merriam, 2009; Schofield, 2000). Multi-site studies are likely to produce results that are a good basis for generalisation to many other situations depending on the number of sites studied and the heterogeneity of the sites chosen for study (Schofield, 2000). While a single-case study of one school would provide rich qualitative data in examining teacher learning due to their participation in lesson study, the study of a single case cannot provide valid evidence that is generalisable to all relevant possibilities (Donmoyer, 2000; Gomm, Hammersley, & Foster, 2000; Lincoln & Guba, 2000). It was therefore decided that more than one school should be involved in this research in attempting to address the research questions in a way that was applicable to the majority of post-primary Mathematics Departments in Ireland. Due to temporal and personnel resources, it was possible to include only two schools in the research and sampling of schools and boundaries of the research will be discussed in detail in Section 3.8. Furthermore, with the phased introduction of the Project Maths curriculum, I thought that generating data on two separately phased school sites could provide contrasting data for cross-case analysis (Merriam, 2009). I also hoped the contrasting and comparable elements of both sites would strengthen the external validity of the research.

This multi-site case study was carried out in two post-primary Mathematics Departments (Doone and Crannóg – see Section 3.7.1 and 3.7.2) that were reasonably heterogeneous in their population of mathematics teachers but also comparative in their previous experiences working as a teacher community as phase 1 and phase 2 schools engaged with the Project Maths curriculum reform. It was envisaged that conducting this multi-site case study would:

- provide a literal replication of the framework for teacher community engaged in lesson study (Yin, 2014)

- provide representative, heterogeneous cases for Irish post-primary schools (Schofield, 2000)
- provide comparative cases for Irish post-primary schools in that the sample contained both: state funded and private funded schools, mixed gender and single sex schools, a Project Maths phase 1 and phase 2 school, and teacher populations which varied in their academic mathematics qualifications
- strengthen the external validity of the theoretical framework and data analysis (Hammersley, Gomm, & Foster, 2000; Merriam, 2009; Schofield, 2000)
- allow for rich data generation from one site should one community disband during the research (Giles & Hargreaves, 2006)

Bogdan & Biklen (2007) recommend doing fieldwork one site at a time rather than simultaneously collecting data from several sites to initially define the parameters of research from one site and to avoid confusion. However, due to the temporal constraint of this research this was not be possible and the multi-site case study was conducted in both schools in parallel.

3.5.2.4 Unit of Analysis

The unit of analysis in this research is mathematical knowledge for teaching, specifically based on Ball et al.'s (2008) model. This was analysed within teachers' interactions in their teacher community (Meyer & Wilkerson, 2011). It is the changes to teacher knowledge which remained a focus of the data generation and analysis for the purposes of this research.

3.5.3 Reliability of Findings

The premise of this research was to inform practice. For this to be true, it was important to operate under the constraint that an effective intervention should be able to migrate from the classroom in the research to the typical classrooms and from the lesson study research meetings to teachers' practices outside of the research (A. L. Brown, 1992). Approaching this research on two separate sites aimed to increase the generalisability and reliability of the research. In strengthening the external validity of the research (Donmoyer, 2000; Schofield, 2000), it was also important to consider the heterogeneity and contrasts of the two sites, how they were typical in relevant respects, how practices of the Mathematics Departments varied, and how the participating teachers differed in their experiences within the research. Contrasts and similarities of the two teacher

communities and qualifications and teaching experiences of the participating teachers will be further explored in Section 3.7.3.1 of this chapter.

3.5.4 Researcher Approach

In qualitative studies, the researcher is often the instrument of research (Marshall & Rossman, 2011). In order to provide a "thick description" (Creswell, 2007) in investigating how teachers' knowledge was developed through lesson study, it was necessary to act as participant-observer in this research. In this situation, I was immersed in the activities of the group and was known to the participating teachers as a researcher, former mathematics teacher, Project Maths facilitator, and pedagogy lecturer (Newby, 2010).

3.5.4.1 Tensions between Participant and Observer

I entered each teaching community as an early-career mathematics teacher from a phase 1 school, Project Maths facilitator, and mathematics pedagogy lecturer. I entered the research at a point of purposeful evolution of both communities and had hoped to become a member of both communities in order to fully immerse myself in the data and in order to benefit in the analysis of data from being familiar with all of the nuances, occurrences, relationships, and practices within both communities. Tensions due to my dual role of participant and observer were likely since I was an external other entering the environment and also openly researching the mechanisms of the community as interviewer and observer.

In order to become a "cooperative fieldworker" (Bogdan & Biklen, 2007) I aimed to engage with the schools as an additional (if temporary) member of staff. Once accepted as a trustworthy (if perhaps more peripheral) member of the community (Lave & Wenger, 1991), I hoped that any tensions existing while acting as observer in recording observation or field notes would dissipate. It was also necessary to resolve the researcher stance of expert and learner (Blaikie, 2007) since I was acting both as an 'expert' in facilitating the lesson study meetings being familiar with the literature, but also as learner in engaging with lesson study for the first time and acting as an educational researcher for the first time. As facilitator and researcher it was necessary to sometimes hold-back and allow teachers' creativity to emerge (Mooney Simmie, 2007) but to also ensure that no one voice (including my own) was dominating the conversation within the lesson study cycles (Lewis & Hurd, 2011).

Through my visits to both schools throughout the academic year, my regular presence became accepted by and familiar to staff. This provided a unique opportunity to experience the ethos and environment of both schools and an 'insider' perspective while at the same time, being neither staff nor student, being an 'outsider'. This balance of both participant and observer provided advantage in interpreting the themes which emerged from the data but also highlighted bias which may have emerged from a consequential lack of neutrality.

3.5.4.2 Continuum of Participant Observer

The role of participant observer can be described as existing on a continuum (Bogdan & Biklen, 2007; Glesne, 2011). Similar to Grossman et al.'s (2001) three year study of teacher community, I saw myself as a cross between "project organiser" and "project teacher" occupying an ambiguous line as researcher and participant.

In this research, I envisaged that at different points during each lesson study cycle, my role would migrate from one as more of a participant to one as more of an observer, as modelled in the following figure:

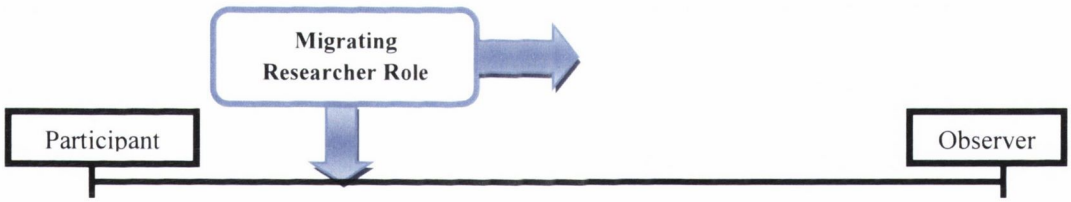


Figure 8 Continuum of the Participant Observer

During the first lesson study cycle I realised it was difficult to simultaneously record field notes as an observer within the meetings and participate as a member of the teacher community. I decided that my observations would instead be recorded as part of a researcher log (see Section 3.10.5.2).

At the start of this study, my participatory role was necessary as a facilitator of the research meetings and, as hoped, I became more of an equitable participant in the meetings in both schools (although at different stages) as teachers became more familiar with the lesson study process. This migration of roles from participant to observer was important in providing teachers with opportunity to be sustainable in this model once the

research was completed (Bogdan & Biklen, 2007). This also provided me with the opportunity to move to more of an observer role as the teacher community engaged in lesson study without the necessary direction of the 'outsider' facilitator and in the final lesson study cycles in both schools I purposefully contributed less to the lesson study conversations.

3.5.5 Triangulation of Data

Seale (1999) advises using triangulation when a subject is complex and is not easily understood by employing a single research methodology. Due to the limitations associated with observation data, a range of complementary instruments including teacher conversations and teacher interviews were employed in this multiple case-study of a task-based teacher community. The instruments employed in generating data to address the research questions included the following:

1. Recordings of teacher conversations held during lesson study meetings
2. Interviews with participating teachers consisting of three separate interviews held at different stages (beginning, middle and end) of the research.

In this research various sources of evidence led to a fuller understanding of teacher learning when data from teacher meetings and teacher interviews were compared and cross-checked during different stages of the analysis (Bogdan & Biklen, 2007; Merriam, 2009). Documents from meetings of the teacher community including constructed lesson plans, teacher notes, and researcher field notes were also used within this research in defining teachers' practices within lesson study but did not form part of the analysis.

3.6 The Research Context

With the key methodological frames in place, the participant and researcher roles detailed, and the understanding that data generation was to be undertaken through teachers' participation in lesson study as a model of professional development, the next step was to find schools to participate in the research.

3.6.1 Contacting the Schools

According to Lyons et al. (2003) it was necessary to choose schools and classes where teachers and students were interested and willing to be part of the research process. As a multiple case study, it was necessary to find two schools to participate in the research project for the academic year 2012/2013.

Teacher communities are especially difficult to establish and maintain in post-primary school where collaborative relationships are particularly hard to achieve in the face of an historical legacy of top-down administration and fragmented departmentalized subject-based communities (Giles & Hargreaves, 2006; Hargreaves, 2001; M. W. McLaughlin & Talbert, 2006; Morrissey & D'Ette, 2004). The success of setting up a teacher community within a school is dependent on the school management to support the organisational structures within the school to allow a community to develop, interact and support individual members (Fullan, 2003). It was crucial to the success and sustainability of teachers' communities of practice within schools, that management agreed with and then supported teachers in actively participating in this research through organisational structures within the school. To this end, I compiled a list of schools in the Dublin and Wicklow locality that could be suitable to participate in the research project. These schools were chosen on the basis that they represented a broad demographic of student population, teacher population, Mathematics Department history, and that they were also engaged in the roll-out of the Project Maths curriculum at different phases (Section 2.2.1). As discussed in Section 2.6.6 it was important that management be engaged in supporting teacher community and in April 2012 I sent a letter to the principals of these schools summarising the research and inviting schools to participate (Appendix A1). During that time, I was also facilitating a four week Project Maths Modular Strand 3 course to 20 teachers at the Blackrock Education Centre and requested and received permission from the PMDT National Coordinator to also present the research idea to the teachers on this course. Teachers interested in participating in the research following the presentation were given a letter to share with their colleagues (Appendix A3).

From these two separate approaches, three schools expressed interest in participating in the research. I visited these schools in May 2012 and gave a presentation to the mathematics teachers on what the research would entail (Appendix A2). Following the presentations in the three schools, I asked that teachers deliberate on the idea for a number of days before I contacted the head of department or group leader the following week. Two of the schools, Crannóg and Doone, expressed their interest in participating in the research during the academic year 2012/2013.

In September 2012, I wrote to the principals of both schools thanking them for their participation in the project and informing them of the details of the NCCA funding (Section 3.12) (Appendix A4). I met with the teachers in September 2012 to review

details of the research and to obtain consent from teachers for their participation in the research (Section 3.9).

Both schools were assumed to differ in their initial experiences of working as teacher communities due to their curricular experiences (phase 1 and phase 2 of the curriculum reform) as Mathematics Departments. Giles & Hargreaves' (2006) prompt that schools which may adopt practices in developing new learning communities are often seen by their peers as not "real schools" because of the additional resources and attention favoured on that school. It was therefore favourable to introduce a comparative angle to this study and it was hoped that this systematic selection of cases for study as a complementary approach would reference any problem of empirical generalization in this research (Gomm et al., 2000; Schofield, 2000).

3.7 The Participating Schools and Teachers

All school and teacher names used in this thesis are pseudonyms. I have kept my own name within the data as researcher and as participant in the lesson study cycles in both schools.

3.7.1 Crannóg

Crannóg is a fee-paying school with a mixed gender cohort of approximately 800 students. Crannóg had been part of the Project Maths pilot or phase 1 scheme (Jeffes et al., 2013; National Council for Curriculum and Assessment, 2012c) (Section 2.2.1) and seven of the 10 mathematics teachers at the school chose to participate in the research. The mathematics teachers taught only Mathematics with two of the participating teachers also teaching Applied Mathematics, a subject whose curriculum was related to the prior mathematics curriculum and constitutes a form of mathematical physics taught only to senior cycle students. The Mathematics Department was well structured with a head of department, Walter, who filed all of the student mathematics assessment records, student assessments, and details of year plans and other relevant Mathematics Departmental documents. Mathematics classrooms had a designated area within the school building where the majority of mathematics lessons were held and where teachers' resources were stored. Within the school there was also a designated meeting room and working room for teachers.

All of the teachers at Crannóg had participated in the professional development workshops that had been offered to phase 1 schools as part of the curriculum roll out and were also participating in a compulsory hybrid form of lesson-study in the academic year 2012-2013 which differed in its structure and execution to that involved in this research (Project Maths Development Team, 2013).

3.7.2 Doone

Doone is a non-fee-paying boy's school of approximately 550 students and was engaged in the national (phase 2) roll-out of the new curriculum (Section 2.2.1). Five of the nine mathematics teachers chose to participate in the research, one of whom was a retired teacher, Nora, volunteering at the school. Excepting Nora, all of the participating teachers taught more than one subject including Business, Science, Religion, Accounting, and Applied Mathematics. All teachers except Nora had participated in professional development workshops offered to phase 2 schools as part of the reform. It was not common for the teachers to meet as a Mathematics Department and during the year teachers noted that they usually met once a year as a group of mathematics teachers to discuss end-of-year assessments or assigning classes. At the beginning of the research there was no head of Mathematics Department and teachers communicated only informally with their mathematics colleagues.

Contrasting the organisation structures of Doone with Crannóg, while there was a smaller designated work room for teachers there was no room in which teachers could meet and work together and mathematics classes were spread all over the school.

3.7.3 Teachers Participating in the Research

All of the teachers participating in this research taught mathematics and volunteered to take part in the research for the duration of the academic year 2012-2013 (Section 3.9).

Similar to Fernandez's (2002) study, both groups of teachers were provided on an ongoing basis with information on lesson study (Appendix C1) and research literature (which will be referenced when relevant within the data analysis). The teachers in both groups were allowed to engage freely in lesson study, to discuss the lessons they were planning in any way they saw fit, to choose what elements of the curriculum to focus on, to decide on the class groups of students that would participate in the research lessons, and were allowed to opt out of the research at any stage (Section 3.9).

In both schools the teachers participating in the research did not comprise of the whole Mathematics Department. Four of the teachers in Doone (who taught mostly senior higher level students) did not wish to participate in the research. In Crannóg, three of the mathematics teachers in the department did not wish to participate. These teachers did not attend any of the information meetings but I did begin to meet some of these teachers in informal conversations in the staff-rooms during the research. Teachers participating in this research, according to their initial interviews, were motivated to engage in the study in order to have an opportunity to meet and work with their colleagues in a new and structured way and to develop resources that would be of immediate use within the classroom (Handelzalts, 2009; Mooney Simmie, 2007). In Doone teachers were further motivated in order to access more support in their engagement with and enactment of the Project Maths curriculum.

Not all of the teachers who attended the information presentation in May 2012 participated in the research. In Doone a number of staff had been reassigned from mathematics to science classes and were no longer in a position to participate in the research. Lisa instead invited Owen and Nora to participate in the research and a summary profile of all of the teachers will be provided below (Table 1). In Crannóg, Gerald and Judy were interested in the research but due to time constraints with other roles in the school they informed the group that they would be unable to attend all of the meetings. They were however interested in joining in the lesson study cycles when it was possible.

3.7.3.1 Profile of Participating Teachers

The teacher groups participating in both schools were not homogenous but included teachers of varying experiences and qualifications in both schools as detailed in Table 1.

Table 1 Participant Teacher Qualifications and Experience

School	Teacher	Years of Experience	Graduate Qualification	Years at that School
Crannóg	Walter	12	B.Eng. Civil Engineering & Ph.D. Transport	12
	Fiona	31	B.Sc. Mathematics	31
	Dave	5	B.Ed. Mathematics & Physical Education & M.Ed.	5
	Stephen	9	B.Bus. Business & Economics	9
	Eileen	3	B.Bus. Management Science & Information Systems	3
	Gerald	27	B.A. Mathematics & Geography	27
	Judy	20	BSc. Mathematics	11
Doone	Lisa	7	B.Bus. Business & Finance	7
	Kate	3	B.Eng. Mechanical Engineering	3
	Michael	4	B. Commerce & Diploma in Marketing	4
	Owen	1	B.A. History & Masters in Theology & Philosophy	1
	Nora	35	B.A. Latin, French & Mathematics	4

In Crannóg, Walter was the head of the Mathematics Department, had 12 years teaching experience, and was interested in introducing a way for teachers to work together in a structured manner. Fiona had 31 years of teaching experience, was interested in experimenting with classroom practices and, at the start of this research, felt confident in teaching the revised curriculum. Gerald had 27 years teaching experience and although he now felt confident in his teaching, did not feel that he had really embraced the teaching and learning approaches encouraged by the new curriculum. Ronan and Dave both had approximately nine years of teaching experience and Eileen was a newly

qualified teacher with two years teaching experience, one of those as a qualified teacher. This group of teachers incorporated a vertical team with teachers teaching all class groups of students from 1st to 6th year and at all levels of assessment from foundation to higher level.

The teachers at Doone were not used to referring to themselves as a Mathematics Department and did not have the structure of collaboration evident in Crannóg. Junior cycle and senior cycle teachers were viewed as separate entities and the teachers participating in the research taught Mathematics only in junior cycle with the exception of Kate who taught senior level Mathematics and Applied Mathematics. These teachers saw themselves as the more junior teachers and saw themselves as having less content knowledge and experience than the teachers not participating in the research. There were no mechanisms for staff to communicate with one another for things such as resources or in-service training days; although some of the teachers participating in the research did informally share resources with one another. Both Lisa and Michael had worked in different industries before becoming mathematics teachers. Neither Kate nor Owen were recognised as qualified mathematics teachers by the Teaching Council (i.e. were 'out-of-field' teachers (Ríordáin & Hannigan, 2011)) and, of the two, Owen had been accepted to participate in a post-graduate mathematics course in order to qualify as a mathematics teacher. Nora, the fifth member of the group, had 35 years of teaching experience and acted as a voluntary learning support teacher in the school.

During the research, teachers at Doone attended in-service professional development workshops given by the PMDT throughout the year and their experiences often permeated the conversations within the lesson study meetings (as will be referred to within the data analysis).

3.7.3.2 Participating Teachers within Teacher Communities

Each of the participating teachers brought with them their former experiences in being a teacher of Mathematics, their personal anecdotes, and their personal relationships with their colleagues into the meetings. Having a mix of teachers of varying experiences ensured that teachers would engage with and be impacted on in different ways through their participation in the professional development intervention (Jacobs et al., 2010). Having a mix of teachers of different teaching experiences also provided opportunity for different ideas, approaches and opinions to be discussed during lesson study.

While Thousand and Villa (1992) recommend only four to six members be participant in a learning community, the usual school mechanisms of administration and extracurricular activities meant that not all teachers could be present at every stage of lesson study and hence having more than six teachers wishing to participate in the research in Crannóg did not pose a problem.

3.8 Research Boundaries

The objects under investigation were the two teacher communities engaged in lesson study, analysed as two separate case studies. The research was constrained by natural boundaries occurring due to the study being school-based. These boundaries fall under the headings of temporal, cultural, demographic, geographical and curricular.

3.8.1 Temporal Boundary

This research study was bounded in terms of one academic school year, September 2012 until May 2013. It was important to begin the professional development intervention at the beginning of the school year when both teachers and students were familiarising themselves with new class groupings and were positive about engaging in new practices. Data generation was completed in April 2012 in order to minimise disruption to summer assessments and state examinations.

The research was also temporally bound to the daily school timetable. Teachers decided when meetings would be held and in Doone it was necessary to hold a number of these meetings outside of the school time-table due to the clashing nature of teachers' timetables. Research lessons were necessarily held during the school day.

Due to time constraints in this research, there was no opportunity to introduce and familiarise participating teachers to the practices and discussions within lesson study during the summer, as recommended by Lewis & Perry (2008) in the form of an initial teacher workshop. Instead a briefing document was provided to the teachers as an introductory and reference document for the duration of the research (Appendix C1).

3.8.2 Geographic Boundary

The location of both schools was geographically bound since it was important that I be able to travel regularly to both schools. The lesson study research meetings and research lessons were bounded by the classrooms and facilities of the school. While Crannóg had a specific area where teachers could collaborate with one another, meetings in Doone

moved from classroom to classroom according to where teachers could find an available space.

3.8.3 Demographic Boundary

The research was bounded by the demographics of the participating teachers. The 12 teachers in both lesson study sites were all members of the Mathematics Departments but differed in terms of their age, years of experience, mathematical backgrounds, role in the department etc. as was detailed in Section 3.7.3.1 of this chapter.

3.8.4 Cultural Boundary

The two schools which participated in this study had differing student populations and their own distinct ethos. One school was a community school while the other was a privately funded school and the cultural differences in both schools were likely factors in the resources teachers had available to them during the research.

The research was also bound by teachers' own practices. This research asked Irish teachers, for whom working in independent isolation is most often the norm (OECD, 2009), to work collaboratively and to open up their classrooms for observation and candid discussion with peers. This cultural boundary seemed to dissipate as teachers' familiarity with lesson study developed but it did initially impact on how teachers engaged with lesson study.

3.8.5 Curricular Boundary

During the research teachers chose specific topics of curriculum which would be the focus of each lesson study cycle. According to the year group and due to the phased roll-out of Project Maths, teachers restricted their decisions based on elements of the Project Maths curriculum that had been introduced. For this reason, for example, no topics from Strand 5 were chosen in Doone. Neither of the schools chose to focus on Strands 1 or 2 since these had been extensively covered within external professional development workshops teachers had participated in (see Table 2).

School	Crannóg			Doone		
Lesson Study Cycle	**Number of Meetings**	**Lesson**	**Project Maths Strand**	**Number of Meetings**	**Lesson**	**Project Maths Strand**
1	6	Introducing quadratic patterns	4	6	Introducing x-squared	4
2	5	Factorising quadratic expressions	4	5	Fractions: sense of measure	3
3	1	Factorising quadratic expressions (2)	4	4	Exploring quadratic expressions	4
4	4	Introducing differentiation in Calculus	5	5	Introducing Pythagoras' Theorem	4

Table 2 Curricular Strands of Project Maths Focused on within the Research

3.9 Ethics of the Research

The ethics procedure laid out in the School of Education (Trinity College Dublin) Handbook was followed and the nature of the project, the information given to the participants and the consent asked of them was cleared by the School's Research Ethics Committee.

The nature of the research was transparent at all times to the participants and they were made aware that the overall enquiry related to their participation as mathematics teachers in a community of practice, how they worked together as professional colleagues engaged in dialogue about planning mathematics lessons and how they may have been transformed through these experiences.

3.9.1 Informed Consent

Informed consent entails informing the research participants about the overall purpose of the investigation and the main features of the design, as well as any of the possible risks

and benefits from participation in the research project (Kvale & Brinkmann, 2009). As well as an initial lesson study presentation (Section 3.6.1) and the lesson study briefing document (see Appendix C1), participants were given an information form outlining the aim of the research, expected participation in the research, data storage and confidentiality (Appendix A5). Participants were asked to voluntarily participate in the research and were also informed of their right to withdraw from the study at any time (Kvale & Brinkmann, 2009). Participants were each given a consent form (Appendix A6) and these were collected prior to their first interview.

Each participant was given my professional email address and the initial group leader (Lisa in Doone) or head of Mathematics Department (Walter in Crannóg) was also supplied with my personal mobile phone number. As the research progressed, other teachers involved in the research contacted me by phone or text to confer about meeting or interview times.

3.9.2 Confidentiality

Participants were assured of the confidentiality of their identity in the analysis and write-up of this research. All of the names of participating teachers or other members of staff referred to in the data are pseudonyms and any student names referred to in the research have also been changed to protect their anonymity.

Teachers are referred to here by name (as by Murata et al. (2012)) instead of number (for example, Fernandez et al. (2005)) in recognition that these teachers had both professional and personal relationships with their colleagues and students and were entering the research as persons with social and cultural histories.

3.9.3 Data Storage

Audio data and the associated transcriptions were stored on my personal laptop computer and on an external memory drive and memory cloud. Electronic data on the laptop, drive, and cloud could only be accessed using a personal password and printed copies of the transcripts were stored in a locked cabinet.

3.10 Discussion of Research Instruments & Data Generation

Qualitative inquiry, which focuses on meaning in context, requires data generation instruments which are sensitive to underlying meaning when gathering and interpreting data (Merriam, 2009). Within this research, lesson study was employed as a model of

professional development to engage teachers in dialogue around their practices where the subtleties and nuances within teaching could come to the surface through teachers sharing their ideas, experiences, comments and questions. Accessing this dialogue made visible the knowledge they utilised in teaching and the knowledge they enhanced through their participation in lesson study (Dudley, 2013; C. Fernandez et al., 2003; Sims & Walsh, 2009). For this reason, the conversations between teachers during their meetings around lesson study were the main source of data in this research. Audio recordings of lesson study meetings, along with interviews with participants were included as instruments of data generation in this research. In addition to these, researcher field notes also contributed to the research data as well as documents and artefacts from lesson study, such as teacher notes from lesson study cycles and samples of student work from research lessons.

Although this research attempted to identify changes to teacher knowledge, teachers were not summatively assessed on changes to their subject matter knowledge or their pedagogical content knowledge. The reasons for this were threefold:

1. The intervention was designed to be situated within the school environment where teachers could be supported and not evaluated in their engagement in in-school continuous professional development.
2. Research on the implementation of Project Maths suggested that in the majority of schools the approaches encouraged by the curriculum were not being introduced into the classroom. It is likely that teachers already felt a lack of confidence in their knowledge due to the curriculum reform (Charalambos & Philippou, 2010; M. Fetters et al., 2002) and testing teachers on their knowledge would likely further negatively impact on teachers' confidence and engagement in the research.
3. If testing of teachers' knowledge were incorporated within the research it would require a detailed and appropriate assessment, which has yet to be researched for teachers at post-primary level (Howell, 2014).

Following previous research literature on investigating changes to teacher knowledge due to lesson study I decided that changes to teachers' PCK would be informally assessed through their conversations around lesson study (Dudley, 2013; C. Fernandez et al., 2003; Lewis et al., 2009; Meyer & Wilkerson, 2011; Tepylo & Moss, 2011).

3.10.1 Lesson Study as a Structural Framework

The data was directed by a framework of lesson study and teachers were made aware of this framework at the beginning of and throughout the data generation. It is important to highlight that within this research teachers were not submitting reports or evaluative outcomes to me as a researcher nor to any external persons who would evaluate their experiences and development. Instead teachers were becoming familiar with lesson study as a model of professional development and were provided with the freedom throughout the research to modify or adapt the model in order to provide an optimal structure for practice within their school.

Research suggests that the effectiveness of an innovation may be increased through cycles of refinement and testing (Bannan-Ritland, 2003; Lewis et al., 2006; Tepylo & Moss, 2011), therefore in this research it was suggested to the participants that they would participate in more than one cycle of lesson study throughout the research. Teachers themselves would decide how many cycles they would participate in and had the autonomy to choose the curriculum topics and student groups on which they would focus.

"Outsiders" to a group affect the "insiders" and vice versa (Miles & Huberman, 1994) and in introducing lesson study I was bringing about a set of activities that would not have otherwise occurred within the two teacher communities. It has not been ignored that what occurred during this research was characterised by teachers' relationships with one another and involved people who knew each other both professionally and personally and worked alongside one another every day. As an outsider, I hoped to work with the communities in both schools in a way which would not seem like a 'top down' approach to professional development (Fullan, 2003; Voogt et al., 2011), but rather where the power of their participation lay in the grassroots approach of this intervention. Through my participation over a full academic year I had hoped that I would become something of a peer or insider with the participating teachers but this was not assumed as an outcome of my role in the research.

3.10.2 Framing the Analysis

The data for this research was generated in a manner that was relevant to answering the research questions and was derived from the research literature. While there was a directed textual approach which governed much of the data analysis technique, there

were also elements of intuitive researcher interpretation of the data which were not recorded within the text.

3.10.3 Teacher Learning in Lesson Study: Audio Recordings of Teacher Meetings

This research focused on moments of transformation that were discernible through changes in teachers' verbalised understanding, a verbalised realisation, or changes made to a lesson plan (Lewis et al., 2009). Talk is central to teacher learning (Dudley, 2013) and teachers' conversations around lesson study were audio recorded in order to provide insight into how and what teachers were learning through their participation. By following teacher's learning in their conversations during iterative lesson study cycles, this research aimed to understand at a detailed level how teacher learning (specifically of PCK) could result from teachers' participation in lesson study (Murata et al., 2012).

Akin to Lewis et al.'s (2009) research on the impact on teachers through their participation in lesson study, the choice of focusing on an "auditable trail" (p. 289) of data generation was based on the consideration that I would act as a participant-observer in the research. In her investigation of teacher community, Little (2003) recorded teacher talk in analysing how teachers represented their classroom practices in their collegiate conversations. In Grossman et al.'s (2001) investigation of the evolution of teacher communities, they also analysed teachers' learning through conversations with one another. Therefore in identifying the features and nuances within lesson study which may lead to teachers enhancing their knowledge, this research mirrors the methodology of previous research in recording and analysing teachers' conversations within lesson study as the main source of generated data (Corcoran, 2011b; C. Fernandez et al., 2003; Murata et al., 2012; Tepylo & Moss, 2011).

3.10.3.1 Audio versus Video

Zawojewski et al. (2008) state that videotaping, field notes, and artefact collection are more useful for a design study approach like lesson study which is not yet well understood and where individual teachers are expected to differ in what is learned. While video recordings may have provided more information on how teachers were participating in lesson study (Dudley, 2013) and may have recorded more of the nuances of conversation and the physical interactions between participants (Perry & Lewis, 2008), audio recording was chosen over video recording. The use of audio recordings of

teacher meetings required minimal intrusion into the teacher community partaking in lesson study (Lewis et al., 2009) and allowed me, as researcher, the freedom to participate in the community without focusing on solely taking observation notes. This led to less of a risk of reshaping the process or unwittingly affecting teachers' participation in recording the teacher meetings in audio format.

3.10.3.2 Teacher Meetings within the Research

Teacher meetings around lesson study were held in teachers' naturalistic setting. Teachers worked in rooms they were familiar with and conducted research lessons in their own schools and classrooms in contrast to many professional development interventions which occur outside of the school.

In Doone teachers completed four cycles of lesson study. In Crannóg teachers completed three cycles of lesson study including a re-teaching of their second research lesson since they were not satisfied with the outcome of that lesson.

Below is a table of when teacher meetings were held during the research and the duration of these meetings. Teachers decided when meetings would be held and how long they would make themselves available for. While there may have been a lessening of the duration of meetings in cycles 3 and 4 in Doone, teachers were happy to participate in meetings over two class periods (average 80 minutes).

Table 3 Teacher Meetings: Dates & Times

School	Crannóg		Doone	
Lesson Study Cycle	**Date**	**Duration (mins)**	**Date**	**Duration (mins)**
1	10/09/2012 20/9/2012 27/9/2012 8/10/2012 18/10/2012 7/11/2012	0:36 1:19 1:44 1:08 1:23 1:03 (p-l)	11/09/2012 21/9/2012 26/9/2012 5/10/2012 17/10/2012 19/10/2012	0:44 0:41 1:49 0:35 1:55 1:19 (p-l)
2	29/11/2012 10/1/2013 17/1/2013 24/1/2013	0:57 1:27 1:01 0:47 (p-l)	14/11/2012 26/11/2012 30/11/2012 3/12/2012 14/12/2012	1:13 1:27 1:16 1:04 0:31 (p-l)
3	8/2/2013	1:01 (p-l)	21/1/2013 25/1/2013 28/1/2013 18/2/2013	1:13 1:16 0:30 0:23 (p-l)
4	21/2/2013 28/2/2013 7/3/2013 15/3/2013 18/4/2013	1:18 1:16 1:57 0:29 (p-l) 0:43	25/2/2012 27/2/2013 28/2/2013 13/3/2013 9/4/2013	1:07 0:45 0:38 1:29 0:13 (p-l)

(p-l) = post lesson discussion

Within this research post-lesson discussions were held directly after a research lesson and provided teachers with opportunities to reflect on what they had noticed about students' thinking in the lesson. Teachers began feeding back incidents which they found significant within the lesson. These accounts were exchanged with their colleagues, allowing for a deeper introspection on the incident but also encouraging interpretation of the incident, especially if other colleagues engaged in discussing the particular student

interaction. As can be seen from the table, the duration of post-lesson discussions lessened as the cycles continued. This was not necessarily to an advantage where Doone's final post-lesson discussion (where I was not present) lasted only 13 minutes.

In total, 18 hours and 9 minutes of teacher meetings in Crannóg were recorded while in Doone 20 hours and 8 minutes of teacher meetings were recorded, totalling 38 hours and 17 minutes of teacher meetings around lesson study.

All of the teacher meetings were audio recorded using two devices, one as a primary recording device and the other as a substitute or back-up. The audio data was downloaded to my computer at the earliest convenience and the data was stored in two separate files and on an external drive. The audio data was later transcribed and stored as a physical document for the purposes of analysis. This comprised of one book of transcripts of teacher meetings for Crannóg of 647 pages and two books of transcripts of teacher meetings for Doone totalling 966 pages. An interview book was also printed for each school.

3.10.3.3 Approaching the Analysis of Teacher Learning

The analysis presented in this research rests on investigating and examining the conversations of two groups of teachers during their participation in lesson study cycles and identifying the moments within the research which show teachers enacting and/or enhancing their mathematical knowledge for teaching (Fernandez, 2005). The analysis is based on the theoretical position that learning can be mediated through groups of individuals participating in rich conversations (Section 3.2.2) (Cobb & Bowers, 1999; Cobb et al., 1996; Greeno, 1998). Moreover, the key rationale for teacher community is that it provides an ongoing venue for teacher learning (P. Grossman et al., 2001; M. W. McLaughlin & Talbert, 2001) and the venue, in this case, is situated within teachers' conversational interactions within lesson study.

Initially, I began transcribing the data generated within teacher meetings since I felt it to be a key phase of the data analysis in familiarising myself with the content (Braun & Clarke, 2006). However, due to the multiple voices in the research meetings it proved to be a cumbersome task and instead the data was transcribed by an external party with the identity of the schools and teachers protected.

In answering the research question "*How is teacher knowledge, particularly pedagogical content knowledge, enhanced through iterative cycles of lesson study?*" the analysis of the data was both emic and etic in the approach as will be further detailed in Section 3.13.

3.10.4 Teacher Interviews

Since this research was practice based and situated within the realities of the schools, teacher interviews were identified to provide the least intrusive form of triangulating data generated in teacher meetings. Interviews were conducted at the beginning, middle, and end of the research in order to identify detail within the data that were not obvious or may not have emerged from teachers' conversations with one another during lesson study meetings.

As already referenced in the literature review (Section 2.5), professional development which impacts teachers' beliefs of their own philosophies and teaching approaches are most effective (M. Fetters et al., 2002). Interviews provided opportunities for participating teachers to individually identify features of lesson study which they deemed important or necessary as part of an in-school model of professional development, but also provided teachers with the opportunity to self-report any knowledge they believed they had enhanced and any of their professional practices that had changed as a result of their participation in lesson study (Wallace & Priestley, 2011). Interviewing participants allowed teachers to directly input their own opinions and beliefs of the research process, providing them with a certain ownership of the project, but also provided participants with a further opportunity to play a part in developing the relationship between the data and the researcher. In creating a rich dialogue with the evidence (Yin, 2014), interviews were an important and necessary instrument in strengthening the data generation and analysis in this research.

In a qualitative interview, where both facts and meaning are significant, it is important that a relationship be established between the two interview players where possible (Seidman, 1991) and in this research, as participant-observer it was important that familiar and equitable relationships were established with the teachers. As interviewer, I researched and attempted to equip myself with appropriate questioning skills, for example asking questions in a familiar language (Merriam, 2009; Patton, 2002) and made myself and the participants aware of the ethical implications of the interview (Kvale &

Brinkmann, 2009; Merriam, 2009). The interviews were semi-structured conducted according to interview guides (Appendix D), allowing me more freedom to follow ideas and themes suggested by the teachers.

3.10.4.1 Piloting the Interview

As suggested by Seidman (1991), I piloted the interview prior to commencing the research project. This pilot interview, held in May 2012 prior to the research commencing, proved to be useful in assessing the length of time for each interview and for the re-wording of two questions to a language that was more familiar to teachers (Patton, 2002).

3.10.4.2 Interviewing the Teachers

The initial interview was based on themes emergent in the literature on lesson study, teacher community and teachers' experience with professional development as part of Project Maths. As suggested by Merriam (2009), the first interview began with relatively neutral, descriptive information to lay the foundations for more questions accessing teachers' opinions and values. The purpose of this interview was to elicit teachers' experiences and opinions of professional development, to identify the current practices of the Mathematics Department, to ascertain what practices teachers employed in their classroom, to identify their philosophy of mathematics teaching, and to elicit any questions or anxieties they had about participating in the research. The interview questions were categorised in terms of: personal history, continuous professional development and participation in the research (Appendix D1).

The second interview was held at the beginning of the second term of the academic year and was constructed in order to identify how teachers were engaging with lesson study and with the research. Teachers were asked if there were any practices which were frustrating them since I was cognisant that there were perhaps sensitive issues that could not be addressed publically within the meetings. Teachers were also asked if their participation in lesson study was impacting on their teaching and learning approaches outside of the classroom. In Crannóg an additional question was included since the PMDT had introduced a hybrid form of lesson study as part of the school's association with the phase 1 curriculum reform and I was aware that this may have impacted on teachers' participation in this research (Appendix D2).

The final interview was a summary of teachers' engagement in lesson study incorporating elements of teacher community that may be relevant in future research but also incorporating what teachers believed about any changes to their knowledge or how they viewed their own teaching. Teachers were asked if they were focusing more on their students' mathematical thinking due to their participation in lesson study. Teachers were also asked if they would recommend this form of professional development to other teachers and were asked if they would continue to participate in lesson study without being part of the research (Appendix D3).

All of the interviews were held at a time that was convenient to the participants and were held in the school building. Interviews were audio recorded using two devices, one as a primary recording device and the other as a substitute or back-up. The audio data was downloaded to my computer at the earliest convenience and the data was stored in two separate files and on an external drive. The audio data was later transcribed.

The scheduling and times of the interviews are detailed in Table 4. Total interview times amounted to 13 hours and 42 minutes. Transcribed, they produced 203 pages (Crannóg) and 252 pages (Doone) of interview booklets to be analysed.

Table 4 Teacher Interviews: Dates and Duration

School	Teacher	Interview 1 (mins)		Interview 2 (mins)		Interview 3 (mins)	
Crannóg	Walter	31/9/2012	0:16	17/1/2013	0:18	18/4/2013	0:24
	Fiona	11/9/2012	0:18	7/2/2013	0:34	18/4/2013	0:28
	Dave	13/9/2012	0:20	17/1/2013	0:19	18/4/2013	0:31
	Stephen	13/9/2012	0:19	24/1/2013	0:28	23/4/2013	0:32
	Eileen	13/9/2012	0:14	17/1/2013	0:15	23/4/2013	0:21
	Gerald	7/11/2012	0:28			18/4/2013	0:43
	Judy	13/9/2012	0:13				
Doone	Lisa	14/9/2012	0:14	10/2/2013	0:26	25/4/2013	0:36
	Kate	14/9/2012	0:19	28/1/2013	0:31	20/5/2013	0:30
	Michael	14/9/2012	0:24	10/2/2013	0:30	25/4/2013	0:24
	Owen	27/9/2012	0:28	25/1/2013	0:30	25/4/2013	0:29
	Nora	14/11/2012	0:41	10/2/2013	0:24	25/4/2013	0:28

As can be seen from Table 4, Gerald was interviewed only twice since he was not available to participate fully in lesson study cycle 2 and was not available to be interviewed at that mid-point stage. Judy was interviewed once since her schedule allowed her to participate only in the first lesson study cycle.

3.10.5 Other Data Generated within Lesson Study

As well as generating qualitative data from teachers' lesson study meetings and teachers' interviews, a number of other forms of generating data were utilised within this research such as field notes, audio researcher log, teacher notes and samples of students' work. Researcher generated data, such as the field notes and researcher log, was not included within the analysis since these were included in the research as reflective instruments of a participant observer and were likely to include bias. Some documents compiled by the teacher community were collected as artefacts of lesson study (Lewis et al., 2009) such as: diagrams of a lesson unit, teacher notes from research meetings, lesson plans (including questions and activities for students), teacher observation notes, and samples of student work. Not all documents were made available to me since, on occasion, teachers were not willing to share all of their notes within meetings or from observations from lessons. Student work was also not always available to be collected following research lessons, depending on how teachers wished to assess students' participation in the lesson. Since cameras could be used in an uncomplicated and unobtrusive manner to take inventories of objects in setting (Bogdan & Biklen, 2007), visual recordings were also taken of some of these documents where it was not possible to retain the physical data.

Samples of student work were not included within the analysis since the lesson study research lessons were not a focus of the research and only work referenced within post-lesson discussions were of interest in referencing teachers' conversations. While other lesson study artefacts and researcher generated data were not included within the analysis, they benefitted in the structuring of the analysis and in strengthening the relationship between the data and I and in enriching the context of the analysis.

3.10.5.1 Field Notes

Field notes provided a record of what occurred in the lesson study meetings as a participant in lesson study. Initially I had planned that the field notes would be illustrative of comments and occurrences during meetings and lessons (Merriam, 2009)

but this proved to be impossible as it was also incumbent on me, as the facilitator of lesson study and as a participating teacher, to note details of the planning meetings in order to reference this information for subsequent meetings. In recording the conversations as a lesson study facilitator, it was not possible to also record field notes as a research observer and therefore these notes were not designed to generate data but were contextual and situational aspects that assisted in the analysis of what occurred during lesson study meetings. They were recorded in a specific researcher notebook with the date, place and classification of the meeting, teachers in attendance (Bogdan & Biklen, 2007; Glesne, 2011; Merriam, 2009) and were recorded in a format that allowed me to find desired information easily (Appendix E4).

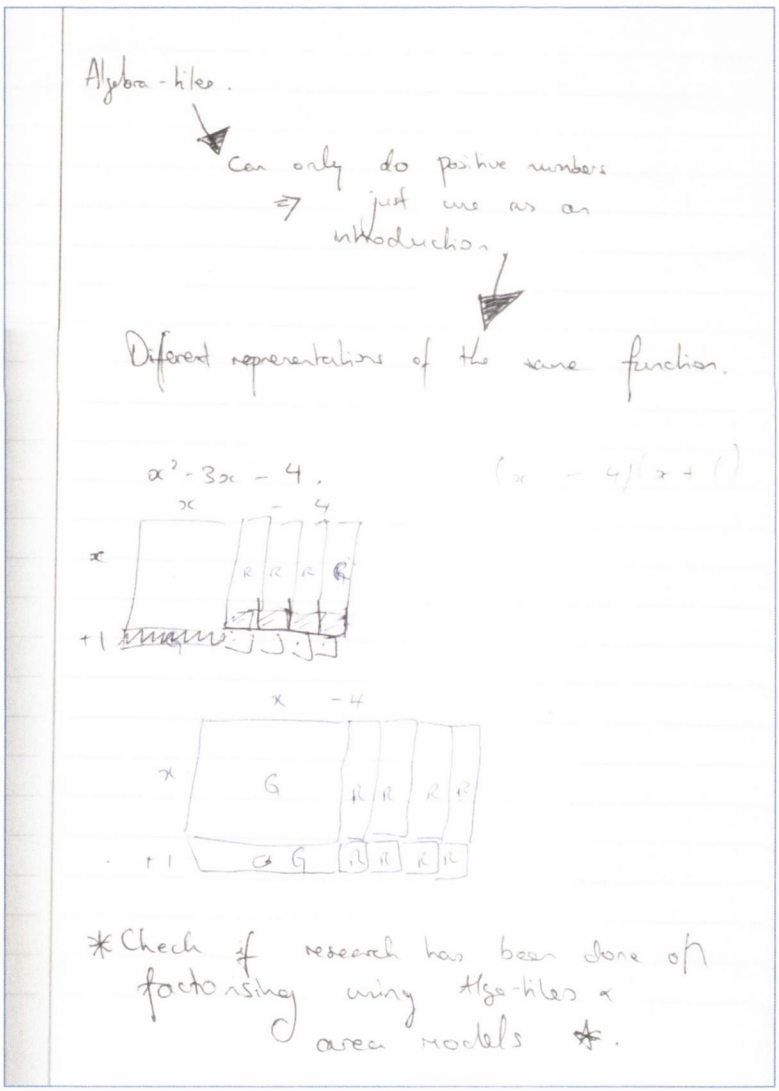


Figure 9 Sample of Field Notes from meeting: Doone cycle 2. Appendix E4

With reference to the research lessons, it had been my initial methodological decision that I would act as an observer of the process into how teachers were observing their students and to record what I saw of this stage of the lesson study cycle (Bogdan & Biklen, 2007). This did not occur however since as a member of the teacher community, I was assigned to observe groups of students as other teachers were doing within the lesson. This also had benefits in terms of my engagement in the research since it inferred that I was in fact a member of these teacher communities

but also that I could contribute to and understand the post-lesson conversations which focused on specific observations of student engagement within the research lessons.

While Miles and Huberman (1994) recommended that the field notes be transcribed, I did not find this was necessary since these hand-written notes contained little detail of what actually happened within the meetings and instead acted as a reminder what occurred. The field notes instead proved to be useful in the analysis as a tool of corroboration in noting the purposes of meetings and in recording the content that was discussed within the meetings.

3.10.5.2 Audio Researcher Log

Personal reflections are integral to the merging analysis of a cultural group, because they provide the researcher with new vantage points and with opportunities to make the strange familiar and vice versa (Glesne, 2011). While I attempted to integrate my own reactions and perceptions into the core of my field note writing (Corwin & Clemens, 2012), at a very early stage of data generation this proved to be difficult in operating as both facilitator of the meetings and researcher and I required another instrument in order to incorporate my own interpretations of the data.

Researcher diary comments can include feelings, reactions, initial interpretations, speculations and working hypothesis and this setting for autobiographical notes create a reflective record of the researcher's behaviour and emotions as a participant observer in the research (Glesne, 2011; Merriam, 2009). For these reasons and as the sole researcher in this project, I decided to utilise a researcher log during this research since it was important that I had a way of recording my personal observations of and reactions to meetings held as part of the teacher community participating in lesson study external to the field notes. This researcher log was an audio recording of my interpretation of what occurred within the meetings and provided me with an opportunity to critically assess what was occurring within the research. The log was recorded immediately after each session in the field and contained an assortment of opinions, views, remembered conversations etc. (Gary Thomas, 2011). The log allowed me to analytically and systematically interrogate my thinking, actions and assumptions and my interactions with the participating teachers (Mooney Simmie, 2007) in a confidential environment and thus allowed me to engage in some preliminary data analysis (Merriam, 2009). The log also allowed me to reflect on the workings of each community and to decide how I could best

facilitate subsequent lesson study meetings in, for example, ensuring each person had opportunity to voice their opinions or to find appropriate research literature for teachers to use in their next meeting (Appendix E3).

While data from the log did not directly contribute to the analysis within this research it proved to be very useful in the analysis of the data generated within teacher meetings in providing a rich summary of what occurred during each meeting and data from the researcher log was included in the final transcript of the data generated during the teacher meetings (Section 3.13.2.4).

Date of Event: 30/11/2012

Length of audio filY: 1:16

3: Lots of breaks trying to get everyone together for the meeting. Once the meeting begins, Kate Lisa and Owen get straight into organising the introduction to the lesson. Lisa is typing the lesson plan and is quieter than usual as a result. Kate, Owen and Myself work out the ins and outs of the lesson. Owen has decided on differentiated groups and some discussion over his first trying of group work. Kate and Lisa contributing and sharing tips with Owen on classroom management. The bones of the lesson discussed and structured. Team splits into two groups to plan the lesson work.

Log: Best session so far with Doone. Helped that there was a clear lesson objective. Lisa doing plan, Kate taking Notes. Michael in half-way and helped that everyone had to explain. Not model the question so the students have to problem solve. Break through in the second cycle of this lesson. Everyone in good form. No loud shouting. Might be that Nora doesn't work well with everyone. Don't think she's enjoying it – hope she doesn't feel like she has to do it. Shows progress with the community.

Figure 10 Sample of Researcher Log included in transcripts, Appendix E3

Teachers were not requested to keep a reflective journal in this research since I considered that it would add to their workload in participating in the project. Instead the teacher interviews (Section 3.10.4) provided teachers with opportunity to reflect on their participation in lesson study.

3.10.5.3 Teacher Notes

As part of the lesson study cycle teachers were encouraged to record notes as part of the planning meetings, observation of research lessons, and post-lesson discussions. As part of planning research lessons teachers often worked on lesson

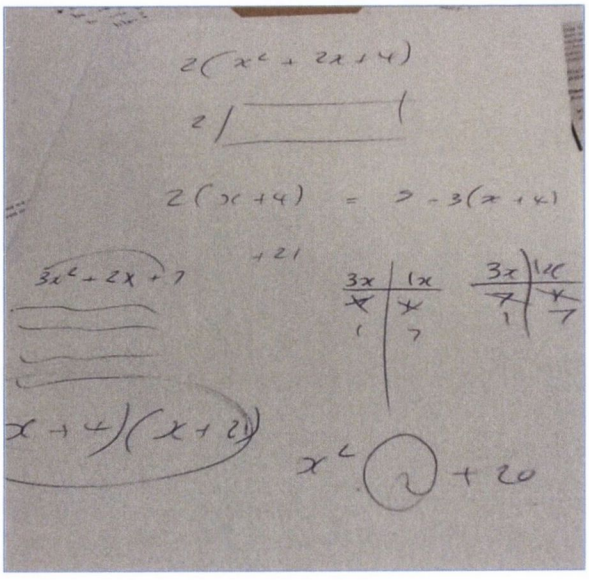


Figure 11 Sample of Fiona's planning notes, Doone cycle 2

plans electronically with a replica template from the teacher booklet (Appendix F1 4). Teachers often also recorded their own written notes during meetings where they practiced mathematics questions, mapped series of lessons, or jotted down ideas.

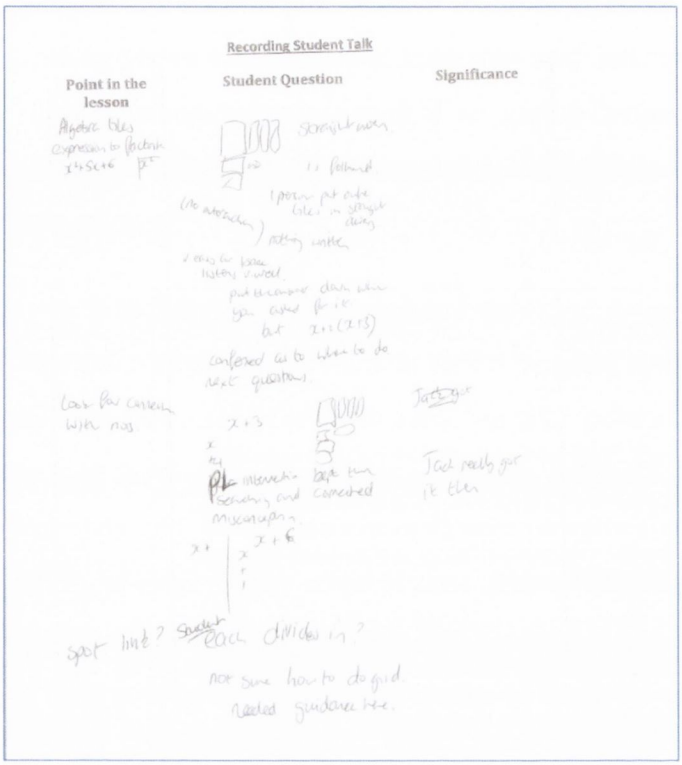


Figure 12 Sample of Teacher Observation Notes, Crannóg cycle 2 Appendix F3 1

During the research lessons teachers recorded observation notes according to what they had decided to focus on during the planning of the lesson, again utilising a template provided to them within the teacher booklet (Appendix F3 1).

It was not always possible to record all of these teacher notes since (as already mentioned) teachers were sometimes not willing to share or copy what they had written during meetings. Available teacher notes were recorded and formed part of the data generated in this research. These notes were not included as part of the data analysis as explored at the beginning of this section.

3.10.5.4 Student Work

Since research lessons were not a focus of this research, it was not relevant to include students' work from these lessons as part of the data. However in their post-lesson discussions, teachers referenced and focused on students' work generated within the research lessons. Therefore student work was included as illustrative within the data analysis but did not form part of the data analysis itself.

3.11 Researcher Reflexivity

As the researcher, I became a common feature in both schools, particularly in the staff rooms. I spoke to other teachers, spent tea breaks discussing various items in the news or educational matters and became familiar with the corridors and classrooms through the various research meetings and lessons. While this was not an ethnographic study, there

was an element of ethnography in the research approach by becoming familiar with the day-to-day activities within both schools. I dressed in attire that was appropriate for the school environment, that is, clothes a teacher would wear rather than student or business clothes (Corwin & Clemens, 2012), and was present as a former teacher and researcher within the schools. I have attempted to remain aware of any personal bias that may have developed during my interactions with the teacher communities during the research (Newby, 2010) and of the emotional attachments to the communities where more social relationships developed between myself and the participants through the regular engagement with the teachers in their staff rooms and around the school either side of teacher research meetings (Fernandez, 2002).

As a facilitator of lesson study within the research (present for each of the lesson study meetings and interviews) and as a former phase 1 teacher, I am aware that my contributions to the teacher meetings may have impacted on how teachers engaged with the model. I am also aware that the social constructs of the language which I utilised throughout the research, my gender etc. may have had an impacted on how teachers perceived me and my role throughout the data generation.

3.11.1 My Role as Broker in the Research

Fundamental to generating valid data is the need for trust to exist between the researcher and the teachers engaging with the research (Newby, 2010). The bond of trust is frequently initiated through discussion about the project and the process of data gathering (Newby, 2010) but takes time to develop and hence research involving participant-observer researchers should be conducted over a long period of time. In this study I entered as a former teacher from a phase 1 school and facilitator of Project Maths, and engaged with teachers over the duration of one full academic year. While it is not necessarily implied that trust between the researcher and participants was fully developed within that period of time, a significant peer relationship developed. It may also have been important that as researcher, I engaged with teachers as a former teacher and not as one with an authoritative role or no classroom experience. Throughout the research I articulated my gratitude to the teachers for participating in the year-long investigation and also thanked the principal of both schools for allowing my access to the schools during the year. At each meeting I provided refreshments for the teachers as a token of gratitude for teachers giving of their time.

3.11.2 Reflexivity as a Facilitator of Lesson Study

Within the constraints of the school-day, I assured teachers that they had freedom to choose which topics to teach, which lessons to focus on, who would conduct the lessons and what students would be involved. Teachers were also constantly reminded that they did not have to attend all meetings nor all of the research lessons.

As suggested by Corcoran (2011), my authority as primary researcher and facilitator of lesson study was offset by my dependence on teachers' co-operation within the research for the duration of the academic year. In my role as facilitator I attempted not to direct meetings and handed over facilitations of meetings as soon as it was possible.

In lesson study the facilitator should be invested in the lesson planning process and input ideas, have ideas discarded, laugh along with the rest of the participants (Corcoran, 2011a; Lewis & Hurd, 2011). To this end I participated in the research as a committed participant in lesson study and although I myself was only familiar with this form of professional development through my review of the literature, my participation was authentic as a qualified mathematics teacher interested in developing new ideas for teaching topics and in engaging students in the research lessons.

3.11.3 Reflexivity in Analysis

Audio recording of the teacher conversations allowed me to access and understand at a detailed level the development of teacher knowledge as they interacted with their peers within the structure of lesson study. I deemed it important that I was present at these meetings in order to record the nuances in conversations and to note points of reference, not captured by the audio recordings, within the data[1]. It was also important that I had participated in the lesson study cycles in noting the changes to the teaching environments as the research continued and to note the changes to personal interactions between teachers which also developed during the research.

Writing in the first-person in the analysis allows for an active voice in the research which communicates the inquirer's self-aware role in the inquiry (Patton, 2002) and using the subjective did not mean that the research was not objective in its analysis. While I participated in this research, I was also aware of my background as a mathematics

[1] Unfortunately, due to time-table constraints, I was unable to attend the final post-lesson discussion in Doone but teachers audio recorded the meeting themselves and Michael electronically sent me the file that same day.

teacher, Project Maths facilitator, and pedagogy lecturer, and my experiences in my former school's Mathematics Department as part of a Project Maths phase 1 school. My interpretation of the happenings in and analysis of the research was as an educational researcher but also acknowledged my teaching experiences. Use of the researcher field notes and researcher log (Section 3.10.5.1 and 3.10.5.2) assisted in reflection of reflexivity, since these forms of data captured my initial reactions to and thoughts on various interactions during lesson study, and included questions of my influence and interpretation of events (Marshall & Rossman, 2011; Seale, 1999).

I am also aware that within teacher meetings or interviews, teachers may have expressed ideas in order to please me, the researcher and facilitator (Newby, 2010), adding further value to the triangulation of data within this research. The duration of the research for one full academic year hopefully negated this impact as teachers became more familiar with lesson study, with me as researcher, and with the purposes of the research.

Analysis of the data did not begin until after data generation was completed in order to allow me, as sole researcher, time to critically review the conversations, interactions and activities within the teacher community. The first phase of analysis (Section 3.13.2.2) began in September 2013 when the majority of the data set was transcribed and this analysis continued for 7 months. Data analysis also included a self-critical account of how the research was done, an auditing exercise in reflexivity (Seale, 1999). My own value-laden perspectives of teaching and learning have also been analysed on the embarkation of this research.

3.12 Funding for the Research

On finalising the research idea, it was obvious that participating teachers would invest a great deal of additional work and time in participating in the research project. Previous research projects on introducing lesson study to schools for the purposes of professional development of in-service teachers provided external support to participants (Fernandez, 2005; Lewis et al., 2009; Murata et al., 2012) and I considered it important that teachers were supported in their participation in this school-based research. I thus attempted to secure funding from an external source to provide the schools with some substitution or supervision costs necessary for teachers to fully engage with lesson study.

In January 2012 I met with the Director of Curriculum & Assessment, Education Officer for Project Maths and Deputy CEO of the NCCA with a summary of the proposed research (Appendix B1) and they expressed interest in the project. In March 2012 I contacted them again to present a more concrete research idea and to request funding for the participating schools (Appendix B2). The NCCA representatives responded positively with an allocation of funding for each of the participating schools amounting to €2000 per school over the academic year, provided they maintained involvement for the full research term. They also requested a mid-term report and presentation on the progress of the research which was provided in January 2013 and again at the end of the data generation in May 2013 (a further final report will be provided to the NCCA during the academic year 2014/2015).

In September 2012, each of the two participating schools received a cheque for half of the allocated funding to be used as supervision/substitution cover and/or to acquire additional resources for the Mathematics Department. Again in January, each of the two participating schools received the final cheque to support their continued participation in the research.

In both schools teachers gave of their own time in holding meetings scheduled outside of their timetables, however in Doone the funding provided by the NCCA was not sufficient in providing for all of the substitution required for teachers to participate in the research. Nonetheless the school management maintained their support by providing sufficient resources for teachers to continue their participation in the final cycle. In Crannóg there was a surplus of funding at the end of the research which teachers were encouraged to avail of in purchasing additional resources for the Mathematics Department.

3.13 Analysis of Data

There are many variables of interest in this study such as the dialogic interactions between teachers, the development of the teacher communities due to their participation in lesson study, and the role of the facilitator in introducing lesson study in these contexts. However, in order to address the research question detailed in Section 3.3.1, the data was analysed addressing an interpretative framework of teacher knowledge highlighted by Ball et al. (2008).

Instead of assessing teachers' pedagogical knowledge (as Jacobs et al. (2010) did with 131 pre-service and in-practice teachers), this research attempted to identify how teachers used their PCK in their planning and post-lesson discussions in lesson study and examined how this was enhanced through their iterative experiences. In an attempt to provide a rich and detailed account of the data, the data was analysed thematically as a chronological evolution of teacher learning through iterative cycles of lesson study. Utilising both emic and etic approaches in the various phases of analysis, initial emergent themes were strongly linked to the data themselves and were not constrained by the theoretical framework of the research questions. While the analysis is data-driven, the themes and codes identified were also situated within the literature referenced within Chapter 2 and further detailed in the following section. Patterns were analysed and identified within the data (Clarke & Braun, 2013) according to themes identified within the research literature on lesson study and teacher learning.

This analysis was undertaken within a sociocultural epistemology on how participants were affected by their engagement with the intervention through their pedagogical conversations. In addressing the development of teacher community, a thematic decomposition analysis may be performed on this data in future to identify patterns within the dialogue between participants.

As previously noted, the datum was not coded until all of it had been generated. Following the generation of the data all of the audio files were transcribed and analysis of the data was then carried out in four phases. While the teacher interviews also formed part of the data corpus, this data served as a form of triangulation of findings from teacher conversations and thus the analysis of these interviews did not follow the same structure. The analysis of teacher interviews will therefore be detailed separately to the analysis of the conversations around lesson study meetings.

3.13.1 Reliability of Analysis

As the sole researcher, it is likely there was a lack of reliability in checking the coding of the data (Weston et al., 2001). As suggested by Lincoln & Guba (2000), a robust literature review served to mitigate the potential to overlook key codes and themes within the data and within the analysis of this research. I hoped that through my review of the relevant literature and my familiarity with the data, the research participants, and the research setting would provide a consistency to the analysis (Miles & Huberman, 1994).

Furthermore, the physical hand-coding of the data provided further structure in coding the data in an attempt to increase the reliability of analysis.

3.13.2 Analysis of Teacher Meetings

3.13.2.1 Overview of Analysis

Analysis of the qualitative data was pursued through both thematic and directed content analysis as will be described in the following sections. Coding systems evolved within the initial emic phases of analysis and were incorporated and developed utilising a framework from the literature in the final phases of analysis (Weston et al., 2001). Analysis involved reading all of the transcript texts, identifying if the text qualified as a legitimate code, and deciding if the text was relevant to the codes within each evolving framework of analysis. Gibbs (2007, p. 47) provided a useful list of items that could be included within the coding and during the analysis I included marginal remarks (Miles & Huberman, 1994) throughout the texts which added meaning and clarity to the coding (Appendix E1).

In this analysis, as the sole researcher, the coding system began with a very large range of features encompassing almost all of the themes and patterns within the data (Appendix G3). As an initial researcher, I had naively attempted to incorporate all of the occurrences within the data which were relevant to the literature but not necessarily relevant to the research question. The first two phases of analysis were a thematic analysis of the data utilising an emic approach identifying common features within teachers' planning and reflection conversations around research lessons incorporating teacher learning, teacher community, and the lesson study cycle. In contrast to the previous two phases of analysis, this third phase incorporated an etic approach based on a framework constructed from Ball et al. (2008)'s and Hill et al. (2008)'s research, but including relevant codes from the previous two phases. Phase four of the analysis built on the previous phase and incorporated both an etic and emic approach adapting the previous framework and incorporating other elements derived from the data which had not been identified in the previous framework.

While I made an effort to access elite informants by including multiple individual teacher interviews (Miles & Huberman, 1994), it was difficult to isolate individual teacher learning through coding conversation chunks in teacher meetings since not all

participants were as vocal in their participation in the research meetings. However, I believe the analysis of the research does not favour any participant over another and provides a balanced analysis of how teachers' knowledge was impacted through their collaborative participation in lesson study. Further analysis contrasting newly-qualified to experienced teachers' individual participation and self-reporting may provide a future avenue of research.

3.13.2.2 Units of Analysis

Research on discourse analysis points to the importance of considering the broader context of the conversation in which individual utterances are made (van Es & Sherin, 2008), while research on quantifying analysis of verbal data (Chi, 1997) highlights the value of conceptualizing segments at different grain sizes. In deciding how to analyse the data of transcripts of teacher conversations it was important to represent the collective dialogue within lesson study and I wished to further develop the analysis of teacher learning through peer communication.

Sentence chunks, as utilised by Murata et al. (2012), while isolating what individual teachers learned through participating in lesson study would not have captured how teachers were enhancing their knowledge through collaboration with their colleagues. Analysis of teacher conversations line by line did not work for Schoenfeld et al.'s (2011) analysis of teachers' decision making during class and he argued that a more macro approach, parsing data into episodes, was more beneficial in analysis of people's problem solving strategies. This parsing approach was followed by Cjakler et al. (2013) in their analysis of pre-service teachers' learning in lesson study and I decided that this unit of analysis would be established as any episode of conversation where:

A) the conversation was relevant to the lesson study cycle (planning, conducting, observing, or reflecting on a research lesson) and/or

B) the conversation was relevant to constructing content of a lesson from the perspective of the student or from a teacher pedagogical perspective.

Based on these criteria, not all of the teacher dialogue within the generated data was coded.

For example in anticipating students' mathematical responses, teachers in Crannóg articulated the strategies they expected students to utilise when prompted with a particular problem. This conversation chunk constituted one unit of teachers building their KCS through their collective conversation.

Fiona	*How are we going to get them to start thinking about the patterns involved? Do you think they are automatically going to go and look at first difference straight off?*
Stephen	*Some will and some won't. That kind of depends on the kind of class you are pitching it at, but some will you would think. You don't know for sure but you would think after being exposed to it last year that they would start looking at the differences... Because some would look, if it is 1, 4, 9, some will look at it as being squared. Some will look at it as being –*
Fiona	*Add 3 and 5*
Stephen	*And 7, and that is – if you give them a pattern, if you give them a list of numbers of 1, 9 that is what they do. They will actually see how much it is going up whereas some don't relate it to being anything squared.*
Fiona	*So they need to be doing. If we are going to do that all in the one class they need to be doing an activity...*
Eileen	*If you do that in the previous class and just do the linear pattern with the bases like 1, 3, 5 and then you are coming back and tabulating then the total number of boxes and then leading on the x^2, just play with the one pattern in different ways.*
Fiona	*Well we are looking at x^2 first but then if we could have another one that is going to lead to maybe x^2 and a constant.*

This conversation included 26 lines of transcribed text which excluded an element of the conversation which meandered away from anticipating students' strategies, but returned to the original context after 30 lines of unrelated conversation. This element of the planning conversation resulted in the teachers deciding on and designing a contextualised problem where students could identify a pattern that was not linear with additional follow on activities for students to attempt during the research lesson.

Conversations occurred during the lesson study meetings which were not coded since they were neither related to planning or reflection on the construction of the research lesson, nor to students' mathematical thinking around the lesson:

Lisa *I have had that aspiring Higher class, Michael has the Ordinary level class. I've just only 16 in it and already looking at it I would say 2/3s of those will want to do the Ordinary level and just Kate then has found a few in her Higher level class that she believes should be doing more. So we reckon that it should be two smaller classes of Ordinary level and two bigger classes of Higher level.*

Nora *I don't especially know that Junior Cert year group, how the ability is spread. But generally it's not a good idea for a very average student to aim to get a pass in the Junior Cert Higher level. They're not reasonably good let's say at getting, let's say, an A at the Ordinary level, then you could focus on one or two extra things that are going to be very relevant down the road.*

Kate *That kid in my class, Damien, he needs to do Ordinary level.*

While this conversation was relevant to these teachers' in discussing their own students and class groups, it was not directly relevant to the lesson study cycle nor to constructing the content of a lesson from a student or teacher perspective. The percentage of transcript coded in relation to the entire transcript is detailed in 3.13.2.6.

3.13.2.2.1 Coding Decisions

The two previous contrasting conversation chunks served as exemplars of coding decisions made throughout the analysis of the qualitative data in this research in incorporating elements of teachers' conversations within the data.

In providing an example of the coding decisions made in later phases of the analysis, a sample code of 'anticipating students' mathematical responses' (specifically mathematical strategies) is illustrated here. In this second cycle of lesson study Fiona explicitly detailed how a student might answer a particular question providing two different (correct) strategies which students could construct within the lesson. As a result of this conversation teachers agreed to provide a different question for students where only one correct strategy could be given by students as they anticipated this would lessen any mis-interpretation by students in their introduction to this topic.

Fiona *For example, I will give them something like that* $2x^2 + 4$

Eileen *Plus 4x*

Fiona *Plus 4x, and I will say "Ok now put that into a rectangle". So they might put it like that [utilising algebra tiles at desk], where they are. So you have 2x and x+2. Or they could put it like x times 2x+4.*

Walter *Which gives it to you then.*

Fiona *Now you see with these ones, if I gave them something like x(x+3) I gave them that there now – there is only one way they can get it like that. With the 2x(x+2) there is going to be – they can do it that way or they can do it the other way. So they can see that there is two different ways. So I am going to do algebra tiles. I want to start off the factorising by giving them an expression they have to put into algebra tiles in the common factor.*

The decision to code the segment in this way was due to the first two features detailed earlier in this section:

1. The content of the conversation relating to the lesson study cycle
2. The conversation was relevant to constructing content of a lesson from the perspective of the student or from a teacher pedagogical perspective.

And two additional features associated with this particular code:

3. The use of "them" and "they" in referring to students
4. Interpreting the activity from a students' perspective and anticipating how the student might respond.

In the final two phases of analysis conversation chunks continued to be chosen according to these initial two features with varying additional features according to the individual code.

3.13.2.3 Phase 1

In beginning the first phase of data analysis and familiarising myself with the data (Braun & Clarke, 2006), I listened to the original recordings of the teacher meetings and teacher interviews in order to identify participants and attempted to note significant moments of teacher learning within the lesson study cycles. These initial 'significant moments' were chosen where teachers explicitly noted "*I always wondered about that*" or "*I never knew that before*" during their conversations with their colleagues or self-reported in their interviews of learning which had impacted on their content knowledge or pedagogical practices.

As the second stage of this first phase of data analysis I again listened to the original recordings of the teacher interviews and research meetings in conjunction with reading the transcripts to check for accuracy and to make notes on emerging features of teacher learning (Braun & Clarke, 2006). While the interviews did allow for certain themes to appear, such as changes to classroom practices or an increased focus on student thinking, these themes were evident of the semi-structured questions asked during the interview. A broad thematic analysis of the teacher meetings at this phase led to features and patterns emerging from the data which included multiple codes (Appendix G3a) around the themes of:

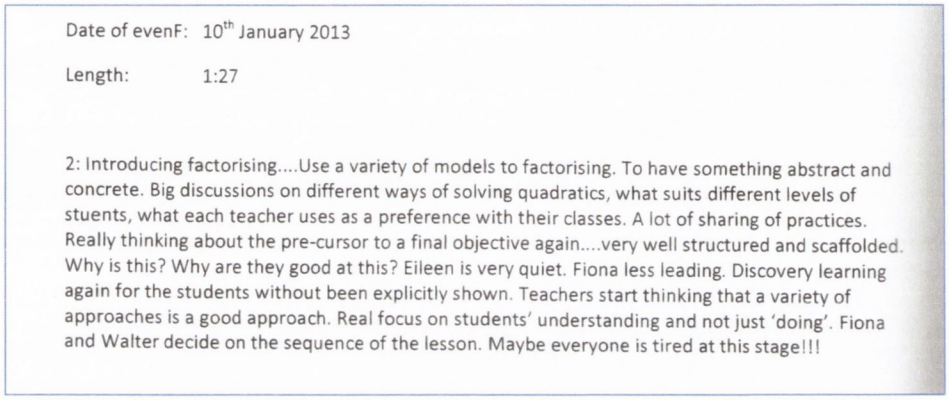
- Curriculum reform
- Teacher learning
- A focus on students

These codes were solely prompted by the data and were not related to the literature around teacher learning or lesson study during this first phase of analysis.

3.13.2.4 Phase 2

In beginning the second phase of analysis, I again listened to each of the research meetings and began making more detailed notes of key incidents of teacher learning as found in phase 1. In conjunction with this phase, I listened to the researcher log in parallel to the research meetings in order to ensure that the essence of the meeting was captured (Miles & Huberman, 1994) and that no important incidents were neglected. The researcher log, which acted as a contact summary form (Miles & Huberman, 1994), assisted in summarising each of my meetings with both schools and highlighted my

immediate researcher interpretations of significant occurrences during the meetings or lessons. The researcher log also provided a summary of the interactions and overall atmosphere of the meetings (for example when Katie and Nora seemed at odds in planning research lesson 2 or when teachers at Crannóg were too busy correcting in-house examinations to proceed with lesson study). From listening to the researcher log I made shallow descriptions of what occurred in each meeting and included these in the physical print of transcripts which was the basis of phase 3 of the analysis (Appendix G1).



Date of evenF: 10th January 2013

Length: 1:27

2: Introducing factorising....Use a variety of models to factorising. To have something abstract and concrete. Big discussions on different ways of solving quadratics, what suits different levels of stuents, what each teacher uses as a preference with their classes. A lot of sharing of practices. Really thinking about the pre-cursor to a final objective again....very well structured and scaffolded. Why is this? Why are they good at this? Eileen is very quiet. Fiona less leading. Discovery learning again for the students without been explicitly shown. Teachers start thinking that a variety of approaches is a good approach. Real focus on students' understanding and not just 'doing'. Fiona and Walter decide on the sequence of the lesson. Maybe everyone is tired at this stage!!!

Figure 13 Sample of Shallow Description: Crannóg cycle 2, Appendix G1

As an aside, it was at this phase of the analysis I realised that the reading of the data provided additional perspectives to only listening to the data alone. For example, from listening to the first research meeting of cycle 2 in Doone I had initially thought it to be a very tense and unproductive meeting, however in reading the script it was in fact rich in moments of teachers defining and unpicking students' understanding of multiplying fractions.

In this phase while listening to each of the teacher meetings I logged changes to each lesson study group's practices from one cycle to another such as: teachers deciding to build a sequence of lessons in preparation of a research lesson, or teachers identifying how to construct contextualised questions for their students relevant to a lesson objective. Similar to phase 1 of analysis these coding chunks were chosen on the relevance of conversation content within the lesson study cycle (planning, observing, reflecting) and/or relevance to constructing content from the perspective of the student or from a teacher pedagogical perspective. These notes were recorded in a 'marginal remarks' document which became a helpful tool in the subsequent phase of data analysis since it

clearly identified changes in teachers' planning and reflection conversations (Appendix G2).

Building on the themes of teacher learning and curriculum reform from phase 1, additional codes were identified in this second phase of analysis such as (Appendix G3b contains a full list of codes):

- Planning for increased communication in the classroom
- Creating mathematical questions
- Noting student misconceptions
- Reflecting on student talk.

Similar to the first phase of analysis, these themes were solely prompted by the data and, while many were expected from the preceding review of the literature, these codes were not linked in an etic approach to the literature during this phase.

Referencing the research question I attempted to link these codes around teacher learning to the Project Maths curriculum in order to identify the pedagogical content knowledge and practices that were being affected through teachers' participation in lesson study (Appendix G3b).

3.13.2.5 Phase 3

In this phase of analysis physical transcripts of all of the teacher meetings and teacher interviews were printed, including associated shallow descriptions of the meetings and notes from the researcher log. These printed transcripts were physically coded over the next two phases of analysis.

Because of the very broad and varied codes identified in phase 2 and in order to identify how teachers were learning in terms of their PCK, I decided to group these codes into 3 themes (Appendix G5) (Braun & Clarke, 2006) directly related to the literature on teacher learning in lesson study where Fernandez et al. (2003) had identified critical lenses through which teaches began examining their practice through participating in lesson study: For this analysis these were labelled as:

1. Teacher perspective
2. Student perspective
3. Curriculum lens

In examining how teachers' pedagogical content knowledge was enhanced, I compared these codes derived from the data with those associated with KCS and KCT from the Ball et al.'s (2008) and Hill et al.'s (2008) studies. While the majority of codes identified within the literature had emerged from the data:

- Interesting and motivating students (Ball et al., 2008)
- Anticipating students' responses (Ball et al., 2008)
- Interpreting student talk and action/productions (Ball et al., 2008; Hill et al., 2008)
- Sequencing content for instruction / developmental sequences (Ball et al., 2008; Hill et al., 2008)

A number of codes present in the literature were absent from those emergent from the data in the previous two phases of analysis and were incorporated at this time:

- Common student errors (Hill et al., 2008)
- Evaluating examples (Ball et al., 2008)

This framework thus incorporated both an inductive and deductive approach to the data. Hand coding of the data text was then undertaken utilising IF codes (Dudley, 2013) associated with these descriptors (Appendix G4).

During this third phase of analysis given the new theme structures of 'student perspective' and 'teacher perspective', changes to teachers' pedagogical content knowledge became apparent within their conversations around lesson study. It became obvious from the coding of the data through these critical lenses that as teachers participated in more cycles of lesson study, they began to focus more on the content and construct of lessons from a students' perspective and began to notice, interpret, and reflect on the research lessons from a student perspective (Corwin & Clemens, 2012). Teachers began to more regularly anticipate students' mathematical responses, began to incorporate students' prior knowledge within their planning, and began reflecting on students' mathematical strategies in post-lesson discussions. This explicit focus on students' perspective, associated with Fernandez' et al. (2003) reference to 'critical student lens', became an important focus of analysis within the subsequent analysis and

corroborated Murata et al.'s (2012) research where conversations moved from a focus on the teacher towards an in-depth focus on student thinking.

As an additional point, the notes created during phase 2 (Appendix G2) were useful as an additional review in ensuring no significant moments within the data were missed and this set of notes acted almost like an assistant researcher in providing a cross-check of themes of analysis.

3.13.2.6 Phase 4

The analysis in phase 4 involved a restructuring of the coding framework applied in phase 3 to more directly align with the KCS and KCT framework explored by Ball et al. (2008) and Hill et al., (2008) (See Appendix G6 for rough notes on the development of this framework). While the majority of codes had already been derived from the data in an emic approach, this final phase of analysis was more etic in its approach with the framework derived from the literature but encompassing codes derived from the data which had not been incorporated within the literature.

It was expected that some codes would not have been included in the literature since the original framework was related to teachers' PCK utilised in-class while this research references teachers' PCK in the planning and reflection stages around the conduction of a lesson. Specific codes which were not included within the literature but are referenced in this proposed framework include:

- Anticipating students' mathematical strategies
- Attending to student talk
- Sequencing learning trajectories over series of lessons
- Developing contextualised activities
- Identifying engagement/motivation for particular students
- Identifying how to incorporate more verbal communication of mathematical thinking for particular students.

These newly identified features of PCK were found to be related to separate literature on teacher learning and thus these varying strands of literature around teacher knowledge and teacher knowledge were incorporated in this final framework. Literature emphasising teachers' focus on student mathematical thinking was relevant to those codes which were prompted by the data (Carpenter et al., 1989; Jacobs et al., 2010; van Es & Sherin, 2008):

- Interpreting students' mathematical responses
- Reflecting on student talk.

Further literature related to the design of contextualised activities and to teachers' analysis of mathematical activities (prompted by the data) was also incorporated within the coding frame (Galbraith, 2013; Stein et al., 2008).

Unusually the feature of 'highlighting common conceptions' which was prominently outlined by Ball et al. (2008) and Hill et al. (2008) did not emerge as a strong or common feature of the data and is thus absorbed in teachers' anticipation of students' thinking in teachers' planning of a research lesson.

It was also during this phase of analysis that the notion of TKS also emerged since it did not conform to definitions of PCK present in the literature. Teacher's continued references to their own students, noting students by name and incorporating their knowledge of those students' typical classroom responses, was an important feature of teachers' conversations around lesson study and led to the inclusion of this form of teacher knowledge in the final coding scheme.

The final coding framework is outlined below and can be seen in Appendix G7. This framework is further explored further in Chapter 4 since it is in itself a contribution to knowledge in identifying the elements of KCS and KCT that are part of teachers' planning and reflection pedagogical conversations, and not focused on the PCK incorporated and utilised within teachers' conducting of mathematics lessons as originally identified by Ball et al. (2008) and Hill et al. (2008).

Table 5 Features of KCS and KCT as analysis codes

PCK	Code	Sub-code
KCS	Identifying and incorporating students' prior knowledge	
	Anticipating students' mathematical responses	*Anticipating students' mathematical thinking and strategies*
		Highlighting common conceptions
	Noticing students' mathematical thinking	*Noticing students' mathematical strategies*
		Attending to student talk
KCT	Sequencing learning trajectories	*Sequencing a series of lessons*
		Sequencing student learning within research lessons
	Developing contextualised content	
	Evaluating mathematical activities	
TKS	Student engagement as a function of learners' sociocultural history	
	Encouraging student communication in class groups.	

Within the analysis, particularly in the fourth and final stage, I tried to maintain an awareness of inputting my own theories and judgements onto the data. Reading Mason's 'Discipline of Noticing' (2002), I became cognisant that I may have only noticed what I wished to see within the data and was perhaps neglecting other features within the data. For this purpose, I began to again listen to the data in the fourth phase of analysis in order to allow myself an opportunity to notice items that I would not necessarily see in reading. Listening to the data also offered further opportunity to focus on how the groups of teachers in each school developed in their conversations around PCK and TKS through their participation in iterative cycles of lesson study.

3.13.2.6.1 Frequencies of Codes

As a final stage of the analysis, each of the codes was tallied for all of the teacher conversations in each cycle (see Appendix G8) and the percentage of transcript coded in relation to the entire transcript was calculated.

Table 6 Percentage of Transcript Coded

	Doone (%)	Crannóg (%)
Cycle 1	17.96	37.27
Cycle 2	36.98	39.3
Cycle 3	45.31	---
Cycle 4	51.42	42.98
Total Percentage	36.56	

As a percentage of the total qualitative data coded for this thesis 36.56% of the final transcript was coded in relationship to the entire transcript for the final coding.

It is notable here that for the earlier cycles the percentage of transcript coded in Crannóg was greater than that in Doone. This is due to the fact that in Doone more of their conversations did not focus on the lesson study cycle but rather on general teaching issues unrelated to lesson study and such talk could not be coded. It is likely due to the fact that teachers in Doone were unused to collaborating as a teacher community and lesson study provided these teachers with an unusual opportunity for them to share conversations around their professional teaching worlds as well as planning and reflecting in their lesson study cycles. This is in contrast to the teacher community in Crannóg which was already established as an evolving community where teachers regularly met as a Mathematics Department and where more of the lesson study meetings were relevant to the planning and reflection of the research lesson.

3.13.3 Referencing Teacher Conversations

Within the analysis chapters, teacher conversation excerpts will be referenced according to the content of those conversations but without including intonation, volume, or characterisation of manners of that speech (P. Grossman et al., 2001). A notation has been ascribed to the discussions in the textual transcript within the analysis drawing on, but not adhering, to the work of Grossman et al. (2001).

Not all elements of the same conversation were relevant to that excerpt and where elements of the conversation have been omitted a "..." will denote the cutting of interim parts of that conversation. Oftentimes, participants overlapped in their conversations and within the transcript where a speaker was mid-sentence and where another participant interrupted that flow of conversation a "-" will mark that point. Where it is relevant to denote that a speaker is referring to a physical item, a note will be included within that quoted excerpt and occasionally italics will denote where a word was emphasised.

To summarise:

1. ... : elements of the conversation deemed irrelevant to the analysis is omitted from the excerpt
2. - : participant interrupted when speaking by another candidate
3. [ ]: will describe a physical item the speaker is referring to
4. *Italics*: words given emphasis when spoken.

3.13.3.1 Analysing Teacher Interviews

The analysis of the interview data preceded that of the teacher meetings and was continued in parallel to analysis of the teacher meeting transcripts. Teacher interviews were analysed iteratively following a phenomenographic convention (Fai Pang, 2006). In beginning the first phase of data analysis, I listened to the interviews of each of the participants in real-time in order to identify important themes highlighted by the participants and to log changes in their pedagogical practices that teachers noted themselves. In the second phase of analysis the transcripts were checked for accuracy (Braun & Clarke, 2006) and read holistically. In this phase, themes highlighted by participants were noted (Wallace & Priestley, 2011) and formed the basis of codes that were initially used in coding teacher meetings in Phase 1. Significant quotes were marked within these themes to form "a pool of meaning" to note any similarities and differences between teachers (Fai Pang, 2006) (Appendix E1). The third phase of analysis of the teacher interviews was done in conjunction with the third phase of analysis of the teacher meetings where I attempted to triangulate the data in ascertaining the development of teachers' knowledge through their participation in lesson study. Finally, in the fourth phase of analysis of teacher interviews, the interviews were analysed according to the framework which had emerged from the analysis of the teacher meetings (Section 3.13.2.5).

The analysis of the teacher interviews in parallel with the analysis of teacher meetings provided a holistic picture of the data but also allowed a form of triangulation of the data to occur, where any inferences made on the data from teacher meetings could be corroborated from data in the teacher interviews.

3.14 Conclusion of Methodology

This chapter has detailed how this research was conducted, how data was generated, and has outlined how that data was analysed. The research investigated how participating in lesson study impacted teachers' knowledge and was conducted as a multi-site case study in two post-primary schools that provided contrasting developments as well as having some comparable, heterogeneous properties. The data for these case studies was generated by the participating teachers through recordings of their lesson study meetings and through semi-structured interviews with each of the participants. Additional data was also important in verifying the structures and processes of introducing lesson study into both schools and that data included teachers' lesson plans, observation notes, and samples of student work. In order to incorporate researcher reflexivity, field notes and a researcher log were also employed during the research but were only used in the interpretation (and not the analysis) of changes to teacher knowledge during the research.

The research was broadly based in the hermeneutical tradition and the data generated by teachers was interpreted by the researcher. There was also a phenomenological element to the research since the model was not rigid and changed according to the participants during the research in terms of, for example, the number of meetings held in preparation of a research lesson. I acted as a participant observer in this research as a member of each of the communities, as facilitator of lesson study, and as a mathematics teacher. The role of participant observer was changeable throughout the research where I became more of an observer as teachers became more familiar with lesson study and as they adapted the model to their particular school environments and organisational structures. Researcher reflexivity played an important part in the reliability of the analysis and a number of measures were identified within the analysis of this research to minimise bias.

The following Chapter 4 will outline the final framework of analysis of teacher knowledge which emerged during this research as a hypothesis for testing. As well as incorporating more detail to how Ball et al.'s (2008) model of mathematical knowledge

for teaching, specifically PCK, was utilised and changed through teachers' participation in lesson study, another dimension of teacher knowledge, that of teacher knowledge of student (TKS), will be put forward as adding a depth to teachers' PCK that has not yet been identified or highlighted within the research literature. The following two Chapters 5 and 6 will detail how, within this research, teachers' KCS, KCT and TKS were incorporated and developed through teachers' participation in iterative cycles of lesson study.

Chapter Four

4 A Framework of Analysis of Teacher Learning in Lesson Study: A hypothesis for testing

4.1 Introduction to Chapter

This chapter outlines a framework as a hypothesis for testing teacher learning in lesson study of which findings will be explored in Chapters 5 and 6. The chapter proceeds by first engaging with Ball et al.'s (2008) model with particular emphasis on the dimensions of KCS and KCT. Through focusing on these forms of PCK, the chapter takes a close view of the nature and structure of the 'critical student lens' and explicitly embeds this teacher perspective within a framework of KCS and KCT. Finally, an additional element of MKT, that of teacher knowledge of student (TKS), is proposed as an important part of teachers' knowledge in the teaching and learning of mathematics. Analysis Chapters 5 and 6 then use this hypothesised framework to detail how teachers' knowledge was enhanced through specific characteristics of this project.

The first section of this chapter is devoted to the final stage of the analysis of the data in respect of the research question:

> ***How is teacher knowledge, particularly pedagogical content knowledge, enhanced through iterative cycles of lesson study?***

A hypothesis for testing teacher learning in lesson study, developed around Ball et al.'s (2008) model of PCK is detailed and extracts from the data are included as exemplars of the analysis. The chapter constitutes an outcome claimed for this research in identifying features of PCK included within teachers' conversations around lesson study but is also linked with a second outcome of making a contribution to the understanding of practice.

The initial contribution to theory concerns a refinement and enhancement of a well-known model in the literature of mathematical knowledge for teaching (MKT) as presented by Ball et al (2008) and discussed in Chapter 2. It will be argued in this chapter that the results of this research point to the possibility of enriching and developing this model by providing empirical evidence of 'knowledge of content and students' (KCS) in teachers' lesson study practices and a greater understanding of the dimensions of KCS

and KCT incorporated within lesson study. This enriched understanding is achieved by including teachers' focus on student mathematical thinking, that of employing a 'critical student lens' as proposed by Fernandez et al. (2003) in their analysis of teachers' initial engagement within lesson study, as part of their planning, observation, and reflection on research lessons. The idea of employing a critical student lens has been referenced by other research as an outcome of lesson study (Fernandez & Robinson, 2006; Murata et al., 2012; Murata & Takahashi, 2002; Sims & Walsh, 2009), but has not been explicitly linked as relative to enhancing teachers' KCS and KCT. In this research, however, it was a dominant theme within both generated data sets as an element of how teachers employed and enhanced their knowledge through lesson study.

Again referencing the research question, in this chapter an additional element of MKT is proposed. Teacher knowledge of student (TKS), which is closely related to PCK and is close both to professional practice and professional learning, is suggested as an additional form of knowledge which is utilised and enhanced within teachers' participation in lesson study. This form of knowledge is in addition to SMK and PCK within a framework of mathematical knowledge for teaching (Ball et al., 2008) and came from emergent themes within teachers' discussions around planning and reflection of research lessons in lesson study.

4.2 Developing Teacher Knowledge: A Framework

4.2.1 Mathematical Knowledge for Teaching

The framework proposed and employed within this research builds on Ball et al.'s (2008) representation of mathematical knowledge for teaching as has been detailed within the literature review (Section 2.7) with definitions of these sets of knowledge taken from Ball et al.'s (2008) and Hill et al.'s (2008) representation of this model.

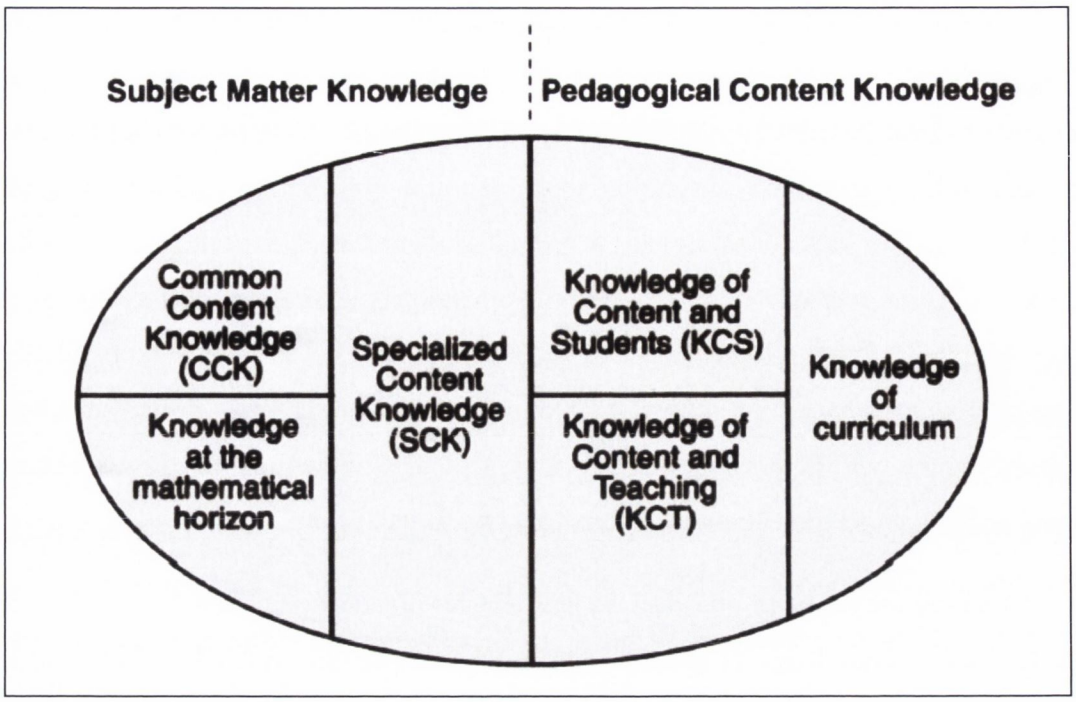


Figure 14 Mathematical Knowledge for Teaching (Ball et al., 2008)

This research aims to extend the framework of knowledge originally provided by Ball et al. (2008) (Figure 14) by understanding how teachers' knowledge can be developed through lesson study. Within this research there is not scope to fully explore each of the individual elements of MKT and therefore only two elements of teachers' PCK (those most prominent within the data) will be focused on: those of KCS and KCT. A further element not contained in Ball et al.'s (2008) model but mentioned in previous theoretical constructions of teacher knowledge and lesson study (P. L. Grossman, 1995; Murata et al., 2012; L. S. Shulman, 1986) will also be detailed as an additional part of MKT as teachers' knowledge of their own student (TKS).

It is important to re-state that within this proposed model it is not suggested that elements of teacher knowledge are separate and distinct from one another, but rather overlap and complement one another during teacher learning, i.e. when teachers' KCT is being developed, this may also be impacting on teachers' SCK and KCS (Baumert et al., 2010; Rowland et al., 2005; Rowland 2014).

4.2.2 Critical Student Lens as part of Teacher Knowledge

The framework proposed in this chapter demonstrates the need to extend Fernandez et al.'s (2003) frame of revealing critical lenses within lesson study. As described in Section 2.8.7, Fernandez et al. (2003) refer to 'adopting the student lens' (pp. 179) as

examining all aspects of a lesson through the eyes of a student. While the emergence of the employment of a critical student lens was expected during teachers' participation in lesson study (C. Fernandez et al., 2003), it was not originally an explicit focus of this research. However, concentrating on elements of mathematics from the perspective of a student emerged as a key feature within the data in the development of teacher knowledge. As has been explored within Chapter 2, professional development interventions can provide teachers with opportunity to increase their attendance to, interpretation of, and developing responses to children's mathematical thinking (Jacobs et al., 2010). In their participation in lesson study teachers move from a focus on teachers' actions to a focus on students' thinking (Carpenter et al., 1989; van Es & Sherin, 2008); reflect on their practice; and move from evaluative comments to interpretive comments on students' mathematical thinking (Murata et al., 2012; Sherin & Han, 2004b; van Es & Sherin, 2008). Teaching which builds on students' ways of thinking can lead to rich learning environments (Corcoran, 2011b; Fennema et al., 1996; Gearhart & Saxe, 2004), particularly in incorporating constructivist stances where student autonomy and communication is valued (Cobb et al., 1991) as is embedded in the Project Maths curriculum. In defining this critical perspective, it does not infer that teachers in this research did not already focus on their students and focus on how to develop their students' mathematical thinking, but rather that this perspective was developed more through teachers' participation in iterative cycles of lesson study.

In the following section, a conceptual analysis of employing a critical student lens will be undertaken in order to include this perspective within an analytical framework for the data. The employment of a critical student lens will be analysed in terms of teacher PCK, as relevant to the research question, with examples drawn from the data to argue for the legitimacy of employing this framework. In this chapter, the process of developing or enhancing teacher knowledge through the employment of a critical student lens in teaching will be analysed in relation to investigating features of KCS, KCT, and TKS utilised and enhanced through teachers' participation in lesson study over the course of one academic year.

4.3 Knowledge of Content and Students

Focusing on students' mathematical thinking provides teachers with an opportunity to enhance their knowledge of how content can be best developed in order to build students'

understanding (Carpenter et al., 1989; Jacobs et al., 2010; van Es & Sherin, 2008). Employing a critical student lens throughout a lesson study cycle led to an enhancement and deepening of teachers' KCS and KCT.

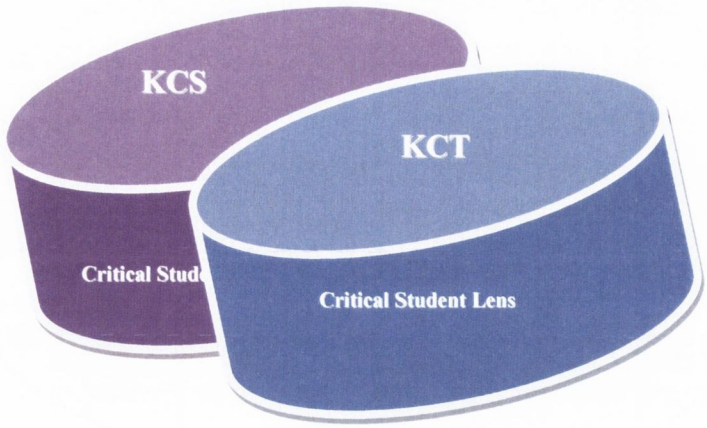
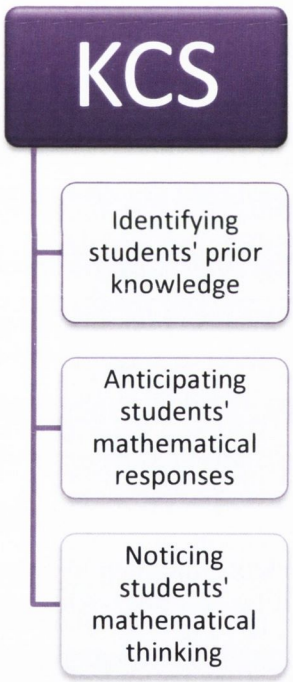


Figure 15 Critical Student Lens adding depth to KCS & KCT

The following section identifies features of KCS utilised and enhanced within teachers' conversations around planning, observing, and reflecting on research lessons.



From an analysis of the data generated within this research three features of KCS utilised and enhanced within lesson study are proposed (the first two features are associated with planning conversations and the latter with reflection conversations but none are exclusive to either phase of the lesson study cycle):

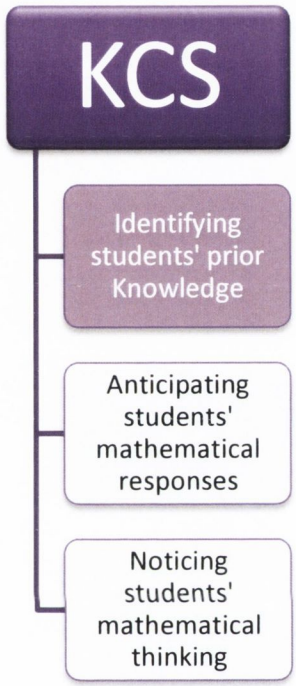
1) Identifying students' prior knowledge
2) Anticipating students' mathematical responses
3) Noticing students' mathematical thinking.

These features are supported by separate literature highlighting students' mathematical thinking (Carpenter et al., 1989; Jacobs et al., 2010; Sherin & Han, 2004b; van Es & Sherin, 2008) and teacher knowledge (Ball et al., 2008; H. C. Hill et al., 2008; Krauss et al., 2008):While this may not be an exhaustive list of all elements contained in focusing on students' mathematical thinking, it reflects what emerged from the qualitative data

generated from these teachers' discussions in their participation in this research and is particularly relevant in examining changes to teacher knowledge when participating in lesson study.

4.3.1 Identifying Students' Prior Knowledge

In this section, teachers' identification of students' prior knowledge will be explored as an element of teacher KCS utilised and enhanced through teachers' participation in lesson study. This is the first of three features detailed within KCS in this research.



This first feature of KCS was based on literature around mathematics teacher knowledge (Ball & Hyman, 2003; Baumert et al., 2010; Krauss et al., 2008) and which aligned with data. What students learn in the classroom depends to a large extent on what they already know (Ball & Hyman, 2003; Holten & Thomas, 2001) and in mathematics students can be challenged by drawing on their prior knowledge and building on the assumptions they have already made (Baumert et al., 2010). Teachers identify, highlight, and detail what students already know in building students' mathematical thinking (Carpenter et al., 1988; L. Shulman, 1987; L. S. Shulman, 1986) and articulating students' prior knowledge constitutes part of teachers' KCS (H. C. Hill et al., 2008). Teachers regularly have to make decisions on their practice (Schoenfeld, 2011) and identifying students' prior knowledge is an integral part of a teacher's decision in how to begin and develop a lesson (Krauss et al., 2008).

Failure to identify students' prior knowledge can create confusion within a lesson as in Fernandez' (2002) study where teachers at the New Jersey site could not agree on what students already knew which led to a confusion of learning objectives within the research lesson. In Kamina and Tinto's (2011) study, through participating in lesson study, fifth grade teachers realised that they needed to know about their students' prior knowledge and how they could build on that knowledge.

Within the lesson study booklet supplied at the beginning of this research, teachers were asked 'what do students currently understand about this topic?' (Appendix C1). The template reflected the detailing of prior knowledge suggested by Lewis & Hurd (2011) in their guide to lesson study but also reflected the detailing of prior knowledge in the teaching and learning plans supplied by the PMDT (Project Maths Development Team, 2011). This provided a point from which teachers could begin planning their lesson and prompted teachers to suggest what students had already met during their experiences of mathematics. Teachers were initially unfamiliar with identifying students' prior knowledge and, particularly in the case of Doone, became better equipped in identifying and incorporating this element of students' learning as their participation in lesson study continued.

4.3.1.1 Identifying Students' Prior Knowledge: Lesson Study Data

In Doone, teachers were unsure of students' mathematical knowledge when they arrived to begin their post-primary education. In these teachers' opinions, students' mathematical experiences in primary school were often dependent on their teacher and the group of teachers could not decide on what they expected students to know prior to the first research lesson nor what they should revise prior to that lesson.

Lisa	*So prior knowledge. They have to know what a square number is.*
Kate	*Yes, they would have done a bit of area.*
Aoibhinn	*So that's not going to be part of the lesson, they'll know some area?*
Lisa	*Or revision.*
Aoibhinn	*Or we can revise it.*
Kate	*Yeah.*
Lisa	*We will, really, because I think -*
Kate	*They'll know that from primary school.*
Lisa	*Yeah.*
Kate	*4 by 4 is 16 and, like, the square...*

Owen *I'd assume they know it.*

Lisa *You'd assume they'd know it, so that if you introduce the concept of X squared they would know it's X by X?*

Owen *Yeah.*

Aoibhinn *I thought that was the point of the lesson, was it?*

Kate *Yeah.*

Lisa *Yeah.*

Michael *I definitely wouldn't assume they know it.*

Kate *I wouldn't assume they'd know it but, like, I'd assume... say the lesson before we do this lesson you maybe have done a bit of, like, shapes, squares, rectangles, you've done things like perimeter.*

Teachers were unsure of what students may have or may have not known prior to this lesson, but also did not agree on where the lesson should have begun in accessing and building on students' mathematical understanding. Teachers later discussed what students' prior knowledge might have been and stated that they also wanted students to have met x as a variable before the research lesson, however Kate negated this suggestion stating that some students would already have met the concept of x as a variable. Despite discussing the elements of the lesson which students needed to know, teachers did not plan for the fact that some students may not have been familiar with the concepts of perimeter, area, or x as a variable. This led to teachers' reflection of that lesson as a failure since they realised that some students had already sufficient prior knowledge of these topics.

In this case, teachers had not correctly identified students' prior knowledge and on reflection of the research lesson, many students had already met and were familiar with the concept of squaring numbers. The conducting teacher, Lisa, judged that this lesson had not been very useful for students since their prior knowledge had been incorrectly anticipated and built on by the teachers. While teachers had begun to employ a critical student lens in attempting to identify what knowledge was required of students in order to build on their mathematical understanding of x^2, they had not included activities to

assess students' prior knowledge and therefore did not successfully build on students' understanding within the lesson

In Crannóg, prior to introducing quadratic patterns to second year students, teachers detailed exactly what students had met in first year related to this topic:

Fiona	*Just, before we even do this, what have they done in algebra up until this? Or does it matter?*
Walter	*I would have it here [referencing year plan records]. They've done standard linear equations, they've done expressions.*
Fiona	*So we would have done kind of, you know, work from a practical context and brought that into linear. We would have done simplifying expressions, so they have done a little bit already.*
Walter	*Hmm mm.*
Fiona	*Because if the variables aren't, things are just going to come out of the blue to them...*
Walter	*They've done substitution, they've done -*
Fiona	*They've come across the variables in linear patterns already*
Walter	*They've done some multi-graphing. That sort of stuff.*
Fiona	*Yeah, they would have done substitution.*
Walter	*And they've done tables and graphs of linear patterns.*
Fiona	*Yeah. So they've come across the idea of tabulating and trying to organise their information but in a much more structured way...*
Walter	*And solving linear equations.*
Eileen	*And did we do basic statistics with them? Did they, they would have come across tables and things like that.*
Walter	*They'd have done tallying last year.*

In this excerpt, Walter reviewed their year plan from last year but Fiona and Eileen also added detail to the specific mathematical skills students had met, such as graphing and

tallying, outside of that particular curriculum strand. Fiona also pointed out that students should have been familiar with the idea of variables, intimating that were they not, it would have to be covered prior to introducing this research lesson on quadratic patterns. These three teachers were detailed in reflecting on the mathematical knowledge that would be required of students in their investigation of quadratic patterns. Highlighting this detail of mathematical content which students should have already met allowed teachers to accurately anticipate students' conceptions and responses in this lesson (Section 4.3.2).

In Crannóg, teachers had detailed records on what student year groups were taught what specific topic and, since all teachers were familiar with these year plans and they were readily accessible, it was reasonably straight forward in identifying what mathematics students had already met for this topic. In their reflection of this research lesson, teachers were very positive about and pleased with how students had been able to successfully begin their exploration into quadratic patterns at a point that teachers had suggested. The ability to identify students' prior knowledge allowed teachers to better predict how students' mathematical understanding could be built upon within the lesson and teachers employed a critical student lens in detailing the knowledge students required before engaging in the lesson (Ball et al., 2008; Holten & Thomas, 2001). It also allowed teachers to plan a sequence of lessons which revised these topics for students in advance of the research lesson (Section 4.4.1).

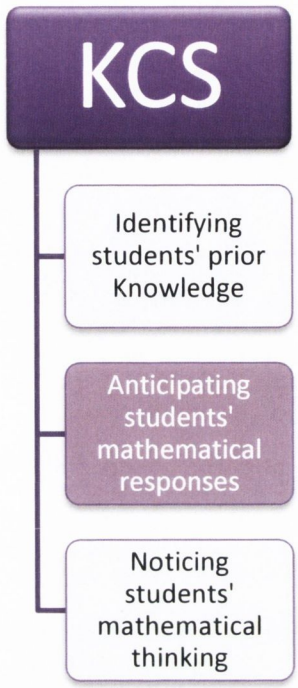
Composing a lesson from the view point of a student in identifying their prior knowledge forms an important part of teachers' KCS in deciding on where to start and how to build on students' mathematical thinking (Ball et al., 2008; H. C. Hill et al., 2008). These contrasting examples highlight the importance of teachers having knowledge of, or being able to properly judge, students' prior knowledge and being able to incorporate this into constructing a lesson focused on developing students' mathematical thinking.

The data suggests that in this project, as teachers participated in continuing cycles of lesson study, there was a development of teacher knowledge in more accurately identifying students' prior knowledge: that which was already met by the students and that which was required in order to build on their mathematical thinking during a research lesson. The data suggests that as the lesson study cycles progressed, teachers became more cognisant of the need to incorporate required prior knowledge into the

lessons prior to the research lesson (developing their KCS) as will be investigated in further detail in the following chapter.

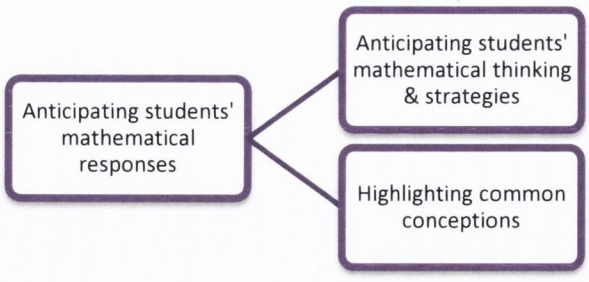
4.3.2 Anticipating Students' Mathematical Responses

This section accounts for the second of three features detailed as part of KCS within this framework. This feature will be separated into two sub-sections as detailed below.



This second feature was identified by Ball et al. (2008) as part of KCS and in this research is developed as a specific feature of teacher learning in lesson study. Through their classroom experiences, teachers build on their knowledge of how students are likely to understand or have difficulties with the topic at hand (Schoenfeld, 2011). An experienced teacher will know what issues are likely to be problematic and what conceptions are common to students (Ball et al., 2008; Krauss et al., 2008; L. S. Shulman, 1986). Teachers with developed PCK can draw on examples which can clarify a point for students or can unravel an activity to units which are understandable for the student. In this research it was found that teachers who did not articulate or identify students' mathematical responses initially in lesson study were incorporating this element of KCS as part of their planning and reflection conversations as lesson study continued.

Anticipating students' responses includes having a sense for what errors might commonly arise with what content, identifying different strategies or routes students might take in answering a question, and identifying and providing explanation for common student conceptions[2] (Ball et al., 2008; Fernandez & Chokshi, 2004; H. C. Hill et al., 2008; Schoenfeld, 1994; M. K.



[2] In this frame the term 'errors' as referenced by Hill et al. (2008) in their detailing of KCS is omitted since students can often be correct in their observation but incorrect in their understanding of a term.

Stein et al., 2008). These anticipations stem from teachers' PCK, built from their own teaching experiences (Ball et al., 2008; L. S. Shulman, 1986) or from a teacher's ability to view an item from a particular student perspective (Carpenter et al., 1989; C. Fernandez et al., 2003; Jacobs et al., 2010; Schoenfeld, 2011).

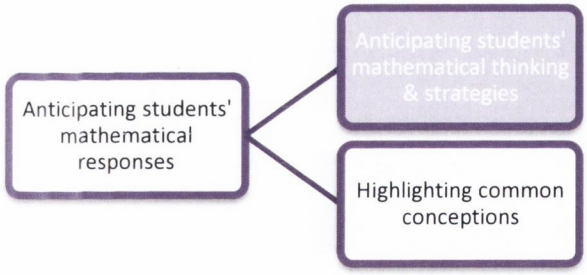
Within this framework of KCS, anticipation of students' mathematical responses is separated into two sections according to the data generated in this research:

1) Anticipating students' mathematical thinking & strategies
2) Highlighting common conceptions

In the lesson template (Appendix C1) supplied to teachers, explicit reference was made to anticipating students' responses and highlighting common conceptions (again reflecting teaching and learning plans supplied by the PMDT and by Lewis and Hurd (2011)), and teachers were encouraged to reference these points within their planning. While teachers did anticipate other student responses, such as their non-mathematical questions and interactions as also highlighted by Meyer & Wilkerson (2011), in this framework specific attention is paid to how teachers anticipated their students would mathematically respond to activities within the class.

4.3.2.1 Anticipating Students' Mathematical Thinking and Strategies

As referenced in the literature review (Section 2.8.6.2), anticipating students' thinking prior to a lesson can equip teachers in deciding how to respond to students and act as a facilitator in their learning. Neglecting to do so can lead to a teacher struggling with students' unanticipated confusion as in Nelson's lesson on exponents in Schoenfeld's (2011) study of teachers' decision making, where the teacher had neither anticipated nor planned for students' responses to a question and eventually resorted to having to give students a rule without building their mathematical understanding of that rule.



4.3.2.1.1 Anticipating students' mathematical thinking & strategies: lesson study data

An excerpt from the data shows that anticipating students' mathematical strategies is something which Fiona regularly did by standing in her students' shoes to see what they might do. In this conversation from cycle 2 in Crannóg, teachers were deciding on what quadratic expressions to give students who would then use algebra tiles to try and factorise.

Fiona	*Like for example, I will give them something like that,* $2x^2 + 4$
Eileen	*Plus 4x*
Fiona	*Plus 4x, and I will say "Ok now put that into a rectangle"....so they might put it like that, where they are. So you have 2x and x+2. Or they could put it like, x times x+4; x times 2x+4*
Walter	*Which gives it to you then...*
Fiona	*Now you see with these ones, if I gave them something like x(x+3), I gave them that there now, there is only one way they can get it like that. With the 2x(x+2) there is going to be, they can do it that way or they can do it the other way. So they can see that there is two different ways. So I am going to do algebra tiles. I want to start off the factorizing by giving them an expression they have to put into algebra tiles, in the common factor.*

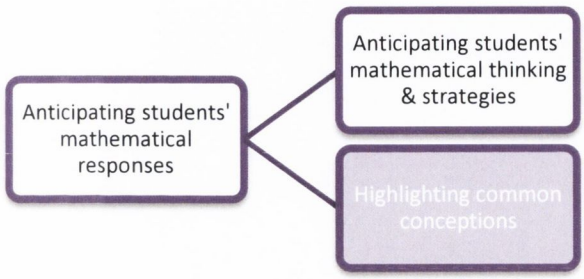
Fiona highlighted to her colleagues the two options and routes that students could have in trying to factorise the expression $2x^2 + 4x$. Fiona used this example to identify to her colleagues that this expression may not be the best one to begin with in introducing algebra tiles as a tool for factorisation, since students had a choice of how to factorise. Instead, she suggested using an expression like $x^2 + 3x$ where students had only one route by which to factorise.

In this research, participating in detailed discussion of content and particular questions when planning a research lesson allowed teachers to unravel their own mathematical thinking and assisted them in focusing on and anticipating students' mathematical thinking (Mason, 2002, 2009; M. K. Stein et al., 2008). Teachers discussed and interpreted their own and each other's responses to mathematics questions, which then allowed them to expand on what they anticipated of students. Through listening to more

experienced teachers articulate and anticipate what a student might do with a question, other teachers gained insight from their colleagues and improved their own pedagogical knowledge.

4.3.2.2 Highlighting Common Conceptions

Highlighting common conceptions is an important element of PCK and KCS since it assists teachers in identifying which questions may provide the best examples for students or can influence teachers in choosing particular strategies for teaching a particular topic. Hill et al. (2008) refer to a lack of familiarity with the conceptual errors made by students as a "thinness of teacher knowledge" (p. 376). Interestingly, this feature of KCS while explicitly identified with the literature of Ball et al. (2008) was not often evident within the data and was notably only expressed by more experienced teachers.



4.3.2.2.1 Highlighting Common Conceptions: Lesson Study Data

As a veteran teacher, Nora was familiar with common conceptions that students constructed when learning a topic for the first time. In designing their first research lesson on squaring terms, the teachers at Doone talked about the examples they would introduce the lesson with. In an exposition within the research lesson, Lisa was going to start with '4' as an example to find the square root and was describing this presentation to her colleagues. Nora reminded Lisa and Kate that this would not be a good example since students could easily confuse finding the square root with dividing by 2.

Lisa *This is what, when I was doing this the other day, I was thinking where are we going with this when we're finished? We are really talking about the square root of 4 to get back to our x, but do we need to tell them that?...*

Nora *Can I just suggest that rather than 4 there.*

Lisa *Yeah.*

Nora *I prefer to go for 9 because an awful lot of people half things and double things.*

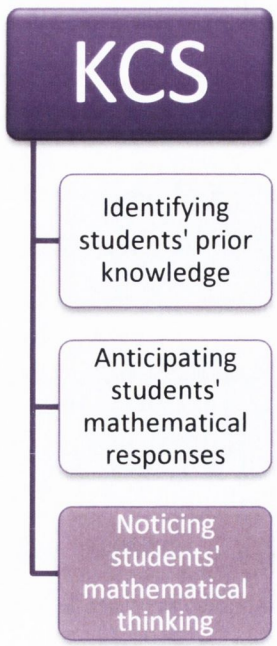
Lisa *That's right, we said that before. We said that before.*

Kate *Yeah, a lot of people just think 4 is 2 by 2, or the 2 squares. So avoid 4s.*

Lisa *We left the 4 out, we discussed that the last day for that same problem. So I think that's a good point – it's 9.*

Interestingly, in a previous planning meeting Michael had identified this common conception to Kate, but could not clearly articulate why this could be a problem for students in mathematically reasoning what was occurring within the situation. Kate had learned from that previous conversation that '4' might not be a suitable number to use as an example for students and agreed with Nora. In this final planning meeting it was Nora, with over 30 years of experience, who clearly identified that students commonly made that connection between squaring and doubling and suggested that beginning with a different number could provide a clearer example to students of what 'squaring' mathematically constituted. Following this conversation, Lisa changed the number to be square-rooted to '9', suggesting that her KCS had benefitted from this conversation and also that she was now beginning to think about this question from a students' perspective, utilising her critical student lens.

4.3.3 Noticing Students' Mathematical Thinking



This is the third of three features identified in the data within the framework of teacher KCS utilised and enhanced through teachers' participation in lesson study. This feature will be further separated into two sub-sections which are detailed below.

While Ball et al. (2008) and Hill et al. (2008) acknowledge that their frame of mathematical knowledge of teaching is influenced by literature

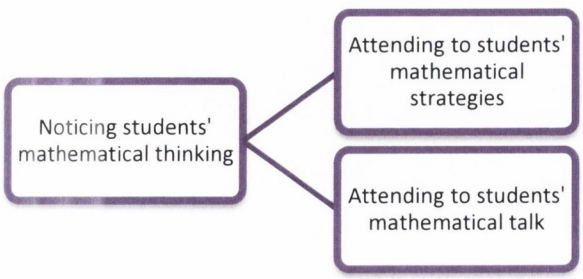
on student thinking, noticing of student thinking is not a prominent feature within their detailing of KCS and is here presented as a contribution to the framework proposed in empirically verifying teacher development of PCK in lesson study. 'Noticing' incorporates teachers interpreting how students think about mathematics within a lesson and teachers reflect on the learning that occurred within that 'noticed' event (Breen et al., 2014; Jacobs et al., 2010; Mason, 2002). Within the proposed frame in this research, noticing students' mathematical thinking brings together separate literatures of lesson study and professional development. Noticing students' mathematical thinking incorporates attending to and interpreting students' thinking: (Jacobs et al., 2010; Mason, 1998, 2002, 2009). Within this proposed framework, attending to students' mathematical thinking is included as a feature of KCS where teachers incorporate what they have previously noticed about student thinking on a topic within their teaching, but also learn from further noticing, interpreting and reflecting on how students thought about the mathematical topic (see Section 2.8.6.3.1).

Within this research, participating in lesson study encouraged teachers to identify students' responses to particular questions and activities within research lessons. In their observations teachers identified and reflected on students' responses and incorporated students' mathematical thinking in arriving at a particular (incorrect) answer and could thus incorporate this observation as a tool in their future teaching

Within this framework of KCS noticing students' mathematical thinking is separated into two sections:

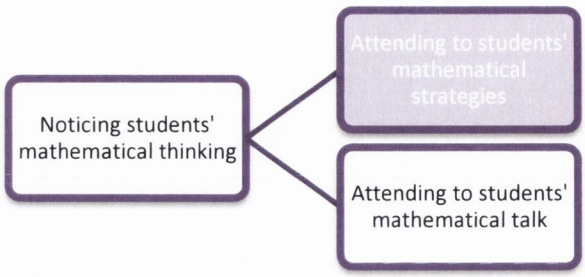
1) Attending to students' mathematical strategies
2) Attending to students' mathematical talk

These sub-sections are modified from Jacobs et al.'s (2010) framework of professional noticing of children's mathematical thinking but do not include that framework's feature of deciding how to respond during a lesson since that was only relevant to the conducting teacher in each research lesson and this data was not incorporated as part of this research. Since Jacobs et al.'s (2010) research was focused on



early childhood (K-4) their reference to 'children' is modified here to 'student' since this particular research is based in post-primary education.

4.3.3.1 Attending to Students' Mathematical Strategies



When students try to construct their own mathematical understanding in their own terminology and in their own time, there are important learning moments for teachers to attend to and gain insight into their students' mathematical thinking (Carpenter et al., 1989; Mason, 2009). Within lesson study, teachers are offered a unique opportunity to not only observe students' actions within another teachers' lesson, but observe in a lesson which they have participated in constructing. In this research, since all teachers observing in the room were involved in the planning of the lesson they were familiar with the lesson objective and were familiar with the ideas of how students might respond and engage with certain activities, teachers were able to focus their attention more on students' mathematical thinking.

4.3.3.1.1 Attending to Students' Mathematical Strategies: Lesson Study Data

In Crannóg's first research lesson, students were asked to find how many blocks would be required in the bottom layer of a pyramid with 49 layers and were given Lego with which to model the question if they wished (see Figure 16). Teachers, observing a research lesson for the first time, noticed that students were readily identifying the constructed quadratic pattern, even though they had not seen such a pattern before. Teachers had planned this research lesson as a "*discovery lesson*" where very little information was given to students in the teachers' presentation at the start of the

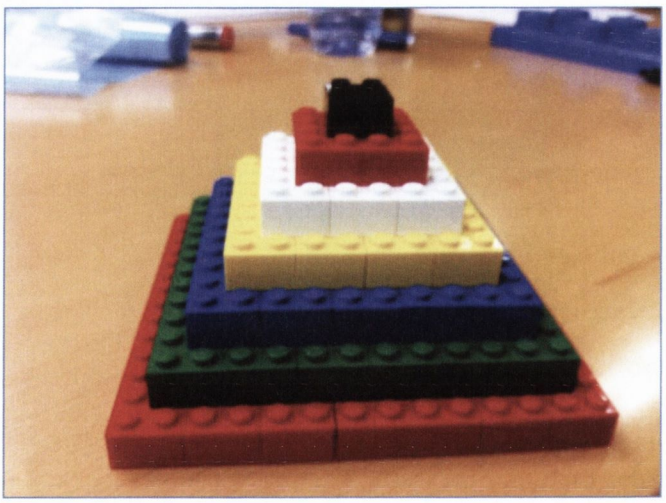


Figure 16 Modelling a square pyramid: Crannóg cycle 1

lesson and teachers were very pleased to notice that students were correctly identifying features of the pattern for themselves without the traditional teacher guidance. They reflected on students' strategies in their post-lesson discussion:

Judy *Like straight away when they went to that second difference and say it wasn't zero, they were all going "It is not linear."..They were delighted with themselves that they actually saw that.*

Judy noticed how a group of four students came to the realisation that the pattern was not linear, and noted that the students' strategy was to immediately look for a pattern in the first difference. Observing the research lesson which she had collaborated in planning led her to attend to and interpret how students were engaging with the mathematics in the research lesson.

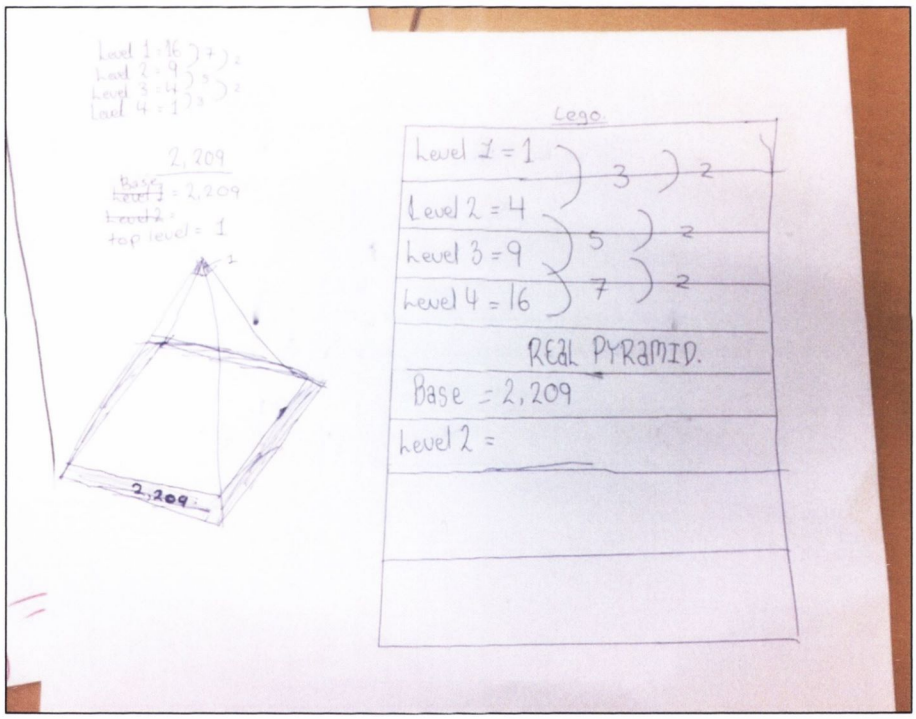


Figure 17 Student work: Crannóg cycle 1, Appendix F4 4

Within the meeting each teacher shared their observations of students' strategies in finding, for the first time, a pattern that had a common second difference. Dave, the conducting teacher, shared his observation of how one particular group of students arrived at a conclusion that the pattern was not linear.

Dave *Do you remember that girl? She drew this [referencing student work from Figure 18]. They shot ahead...and she started saying "It is not a line", they had all this stuff done within three minutes, it was*

great...and she asked me and I said "It is not a line, it is not linear" and she said "if you do a graph of it will it look like this?."

Dave repeated back what students had told him during the lesson and reflected that these students' strategy was to graph their data which eventually led them to defining the pattern as having a second common difference and identifying the relationship: $y = x^2$. Attending to students' thinking expanded on his knowledge of how students might engage in this topic, building his KCS in future teaching of this topic.

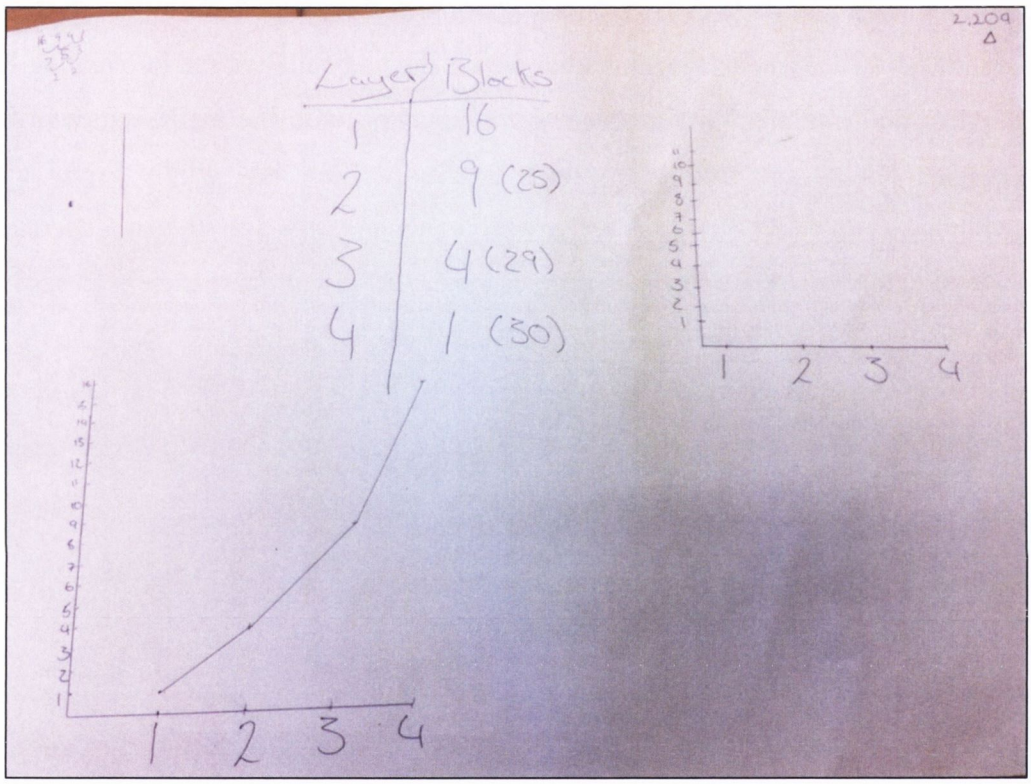


Figure 18 Students graphing information: Crannóg cycle 1, Appendix F4 4

In attending to students' strategies, teachers identified an element of the expression $y = x^2$ that may have been confusing to students:

Walter *They might be confused though since the second difference was say '2' in front of the x^2, they might be thinking that 2 of the power of 2...*

Stephen *because one of my group came up with the formula as $2x^2$ and I think it came from that 2 being the second difference, and then they actually, and then the thought process then was that "oh actually it is x^2 when they actually looked and they tried $2x^2$, which is interesting to see.*

Teachers had not previously anticipated this point (although as facilitator I had pointed it out during planning) and this thinking from a student perspective only came about during the observation where teachers had attended to, interpreted, and reflected on what the students they had observed had done within the lesson. They agreed that at the end of the lesson the conducting teacher should spend time addressing this point in future marking a development of teacher knowledge in teaching this subject.

In attending to students' strategies, teachers also became aware of students tackling a question using a strategy unanticipated by teachers where they began examining the pattern of bricks from the bottom as opposed to the top of the pattern. Fiona reflected how the group of students she was observing began trying to work out the problem but then reverted to the method teachers had anticipated. Stephen had noticed the same strategy with his group and examined this group's work while sharing his reflections with his colleagues.

Dave — *I think two or three of the groups got the thing [backwards], the levels 1,2,3,4, your group did and yours as well, yeah*

Eileen — *Mine did, yeah.*

Walter — *I don't think we spotted it as a potential.*

Dave — *No we didn't.*

Fiona — *Interestingly, mine started off, actually I meant to say that, mine started off trying to build it from the bottom up. So they counted all the blocks first to try and work out what size would the bottom base need to be if they were going to make a pyramid out of this number of blocks. And so they were trying to figure out how many layers they were going to have if they were going to use this number of blocks.*

Walter — *Fascinating.*

Fiona — *And then one of them said "Well why don't we start from the top down?" and "Oh yeah, that would be much better" so they were actually looking at it a bit like they were looking at it from the bottom up rather than the top down.*

Stephen *That was exactly the same with mine. It was from the top down as well. I think that is actually my groups here. So if you turn it over I don't think there is anything on the back.*

Reflecting on student work was an important element of this conversation around noticing students' strategies since teachers could further interpret and analyse how students had gone about examining the pattern. Teachers noted how students utilised their prior knowledge of linear patterns to tabulate and often graph the values that they found. Teachers also incorporated student talk in highlighting students' strategies in attempting to answer the question and this will be further explored in the next section of this chapter.

It is obvious from the data that an important element in noticing students' thinking was for the teacher to situate themselves in a particular activity in order to be able to see it from a student's perspective. In this regard, lesson study again offered rich opportunities for teachers to work together on the mathematics they would be engaging with in class (Ball et al., 2008; Borko, Jacobs, Eiteljorg, & Pittman, 2008; Mason, 2002, 2009; M. K. Stein et al., 2008) since teachers, in the planning of this lesson, looked at all of the various different ways students might have interpreted a pattern in the Lego. Teachers had discussed the possibility of students counting the dots on the Lego bricks as opposed to the blocks themselves or if they investigated a two dimensional version of the model. Working on activities and creating questions during the planning phase for this particular group of students sensitised teachers to noticing students' mathematical thinking and offered more depth in their employ of a critical student lens and will be discussed as a feature of KCT in Section 4.4.3.

In intentionally observing how students understood the content and sharing it with their colleagues, teachers built on their own understanding of how students thought about the question. Teachers built on their KCS and incorporated this knowledge into their own teaching of this lesson, which Fiona, Stephen and Judy taught the following day. In participating in lesson study, teachers began to attend more to students' mathematical responses in planning and to build on their interpretation of students' thinking in their reflections as lesson study continued.

4.3.3.2 Attending to Students' Mathematical Talk

Gaining access to students' dialogue or unofficial talk in the classroom is one aspect of gaining insight into how students are thinking about the mathematics at hand (Houssart, 2009). Usually as teachers we can only access students' written tasks, which do not always allow us to assess their understanding or analyse how they arrived at that particular point. In gaining access to student talk, teachers can: make assessments about what students know and understand, identify errors, difficulties and misconceptions, and decide on the next steps which could be followed within that lesson taking particular students or groups into account (Briggs, 2009; Houssart, 2009).

While Jacobs et al. (2010) did not reference student talk within their research, Ball et al. (2008) note that teachers are expected to hear and interpret students' emergent and incomplete thinking as expressed in the ways that students use language. In attending to student talk, teachers become aware of how students are thinking about and structuring their mathematical tasks. In this research, teachers were aware that the Project Maths curriculum (National Council for Curriculum and Assessment, 2012a, 2012b) encouraged student communication within the classroom and in both schools teachers made conscious decisions to incorporate more student group work within the research lessons as their participation in lesson study progressed, thus providing teachers with more access to student talk.

Since teachers' post-lesson conversation was constructed as a reflection of the lesson, teachers shared particular moments they had noticed by 'highlighting' or making a 'call out' (Jacobs et al., 2010) during the meeting. These 'call outs' are noted within the data analysis according to what teachers attended to and if or how they interpreted these students' strategies.

4.3.3.3 Attending to Students' Mathematical Talk: Lesson Study Data

In Crannóg's final research lesson teachers observed separate groups of students working together to try and find an athlete's exact speed at a particular moment. This lesson was an introduction to calculus, leading students to a concept of instantaneous rates of change and a concept of limits. In their post-meeting discussion, many of the teachers reflected on what they had heard students saying to one another and these excerpts of student talk were discussed in assessing whether students understood the concept of calculating the rate of change (adapted from a PMDT resource) at an exact point. Below is an example

of the question posed to students and an excerpt of the conversations reported on by Fiona and Eileen to their colleagues.

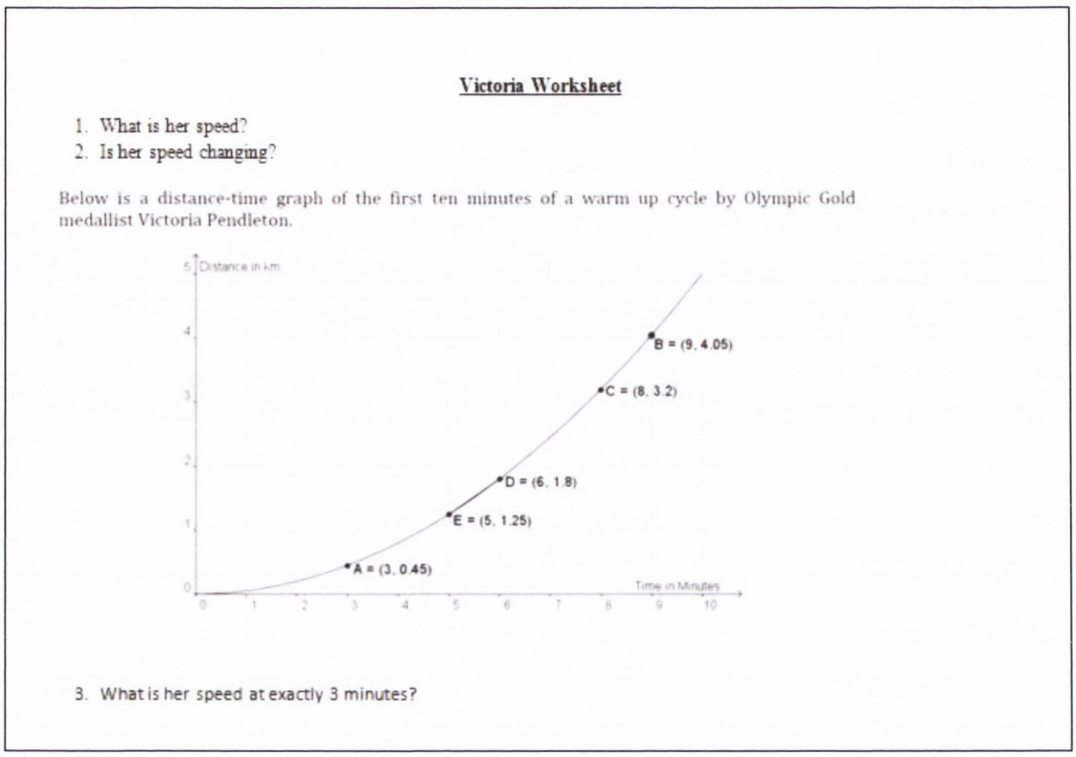


Figure 19 Student Worksheet: Crannóg cycle 3, Appendix F2 7

Students had not tackled the question as teachers had expected in finding the slope between A and B and then finding the slope between A and C etc. (Figure 19), but instead went directly to calculate the slope either side of point A.

Fiona *They didn't quite look at it the way we, we were looking at it at them trying to go from A to E and then A to D, they didn't -*

Eileen *No they straight away wanted to go either side of it.*

Fiona *They wanted to go either side of it pretty quickly.*

Stephen *Yeah.*

Aoibhinn *I think they were frustrated as well that couldn't get it exactly.*

Fiona *Mmmm*

Aoibhinn *But that was good.*

Gerald *Siobhan here was saying you know "can you not just get the distance at D?"...*

Fiona *But Simon is, that's what Alan wanted to do "can you not just use the slope formula at D" and Simon was saying "sure you can't it's only one point, you can't use the slope if you only have one point."*

Eileen *Yeah well remember Stephen at the very beginning "can we not just go from A to A do you know, not move and get the slope from A to A."*

Teachers reflected that students were showing frustration at not being able to calculate the speed exactly at the point A and paraphrased the conversations they had heard during the research lesson. Teachers were pleased with this confusion noted in students' conversations since it led students to think about limits. Fiona also reflected that one group of students also spoke about trying to use GeoGebra to zoom in on the graph and try to find points closer and closer to A in order to identify the most accurate speed possible. From this student talk, Stephen suggested that in their re-defining of this lesson, they incorporate a GeoGebra file in order to demonstrate this to students and also consolidate their understanding of limits.

Having access to student talk within the research lessons allowed teachers to view how students embarked on answering the question, adding to their KCS in how they might anticipate in future how students engage with that question. This observation role allowed teachers to employ and reflect on their perspective of student mathematical thinking. Accessing student talk and then reporting back on students' conversations during post-lesson discussions demonstrated that teachers wanted to understand how students were engaging with a particular topic and how they were reflecting on and interpreting students' strategies through repeating and interpreting their conversations.

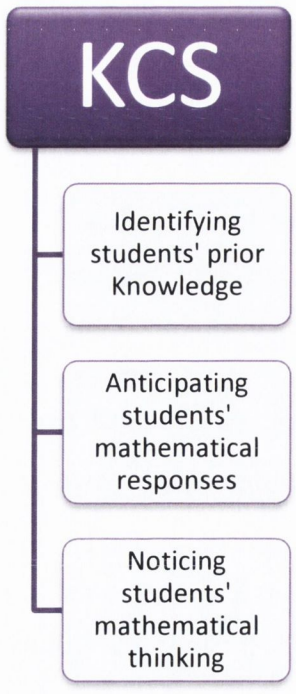
4.3.4 A Critical Student Lens & KCS: Conclusion

KCS is an important element of teachers' PCK and is defined by teachers' understanding of how students learn particular content and encompasses teachers' knowledge of common student errors and conceptions or misconceptions about mathematical topics (Ball et al., 2008; H. C. Hill et al., 2008). In this research, teachers were provided with opportunity to build on their KCS in collaboratively planning, conducting, observing, and reflecting on lessons.

Teachers were grateful of the opportunity to observe individual students' thought processes during research lessons. Gerald reflected on this point in his final interview and was positive about having had the opportunity to follow students' thinking during a mathematics lesson.

> *Gerald* *It is great to be able to sit there and watch that group working, and they were engaged and you could follow the train of thought and you can give them time because you are not under pressure to run off to another group but while if you are a single teacher, you can't follow the train of thought around all the groups, and if you are delving in and giving assistance it is quite a hard thing to do, you know, to do that effectively.*

Through the planning, observation, and reflection on how students were thinking about mathematics during class teachers began to focus more on how students attempted a question, what items were causing them difficulty, and how they were communicating their thinking with the teacher and their fellow students. This learning led teachers to make changes in their participation of lesson study in: identifying students' prior knowledge, anticipating students' mathematical responses, and noticing students' mathematical thinking during lessons. These changes to teachers' KCS also led to further developments in their own teaching and learning practices such as allowing students more time to answer a question or work on a question on their own as reported by Walter and Eileen in their final interviews, or explicitly incorporating more whole-class discussions as reported by Owen in his final interview.



The analysis of the data suggests a framework of KCS which is additionally supported by research literature external to lesson study. This framework of KCS is identified in three features, two of which are further detailed into two sub-sections:

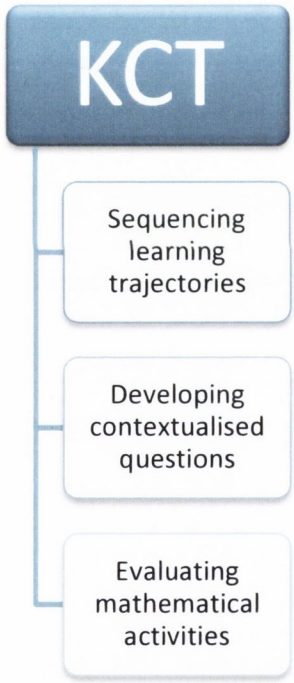
- Identifying students' prior knowledge

- Anticipating students' mathematical responses
    - Anticipating students' mathematical thinking & strategies
    - Highlighting common conceptions
- Noticing students' mathematical thinking
    - Attending to student strategies
    - Attending to student talk

Within the boundaries of analysing changes to teacher knowledge through their participation in lesson study, a framework of KCS is here proposed. Furthermore, analysis in Chapters 5 and 6 suggests that by focusing on students' mathematical thinking and employing a critical student lens in their practice, KCS is strengthened and developed.

4.4 Knowledge of Content and Teaching

Knowledge of content and teaching (KCT) combines knowing about mathematics and knowing how to build a learner's understanding of that mathematical idea (Ball et al., 2008). Designing mathematical questions, sequencing content and acting as a facilitator of learning are all elements of KCT - content knowledge which is specifically relevant to mathematics teachers (Ball et al., 2008). KCT is included within this model of teacher knowledge which has the possibility of being enhanced through teachers' participation in iterative cycles of lesson study through the employ of a critical student lens in their practice.



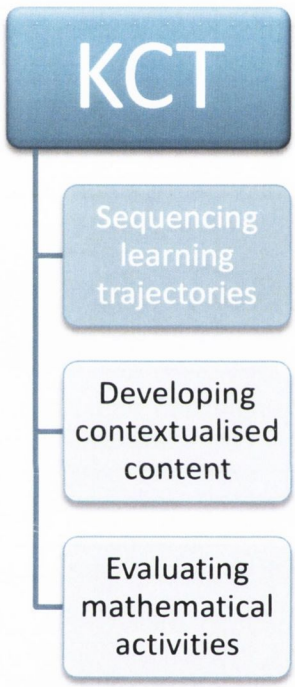
From an analysis of the data three features of KCT utilised and enhanced within lesson study are proposed:

1) Sequencing learning trajectories
2) Developing contextualised questions
3) Evaluating mathematical tasks

These features are emergent from the data but are also supported by research literature. While these features may not be exhaustive in building a whole picture of KCT, they are relevant to this data generated in this research through teachers' participation in iterative cycles of lesson study.

4.4.1 Sequencing Learning Trajectories

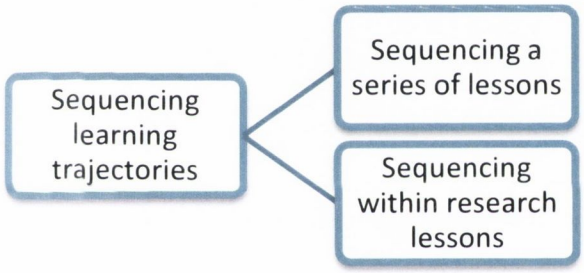
This is the first of three features of KCT detailed within this suggested framework. This feature of sequencing learning trajectories, which emerged from the data generated within this research, will be further broken into two sub-sections within this frame.



Sequencing material for instruction requires teachers to make connections between the mathematics being taught and how the student should be introduced to that topic. The teacher has to be aware of the relative demands that different topics and tasks make of students (Rowland, 2007) and that extending and building students' mathematical understanding of a topic requires teachers to fundamentally understand that topic. Sequencing content within a lesson is contained within a Knowledge Quartet frame of 'connections' (Rowland et al., 2005), but is here included within a modified framework of Ball et al.'s (2008) PCK. As referenced in Section 2.8.6.2, sequencing of students' learning incorporates anticipating what students already know, what common difficulties students make with a particular topic and pre-empting students' responses to particular questions or activities (Ball et al., 2008; Hiebert et al., 2005; H. C. Hill et al., 2008; Krauss et al., 2008). The sequencing of mathematical content in designing students' learning trajectory combines KCS, KCT and knowledge of curriculum, since teachers must build into the sequence: what they believe a particular group of students have already met about a topic, what content knowledge students need in order to build on their mathematical understanding of the topic to be taught, and also include what students need to know about the content according to the assigned curriculum.

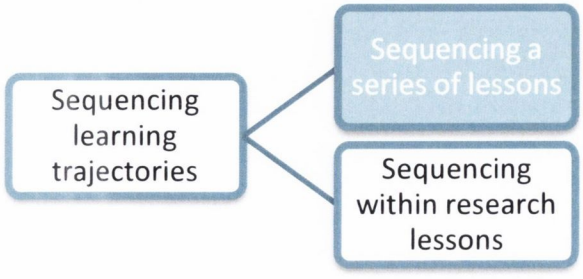
In introducing lesson study to their American colleagues, Japanese teachers conveyed the importance of connecting research lessons to students' prior and future lessons (C. Fernandez et al., 2003). They also suggested that lessons should be concentrated on core topics which would have antecedents and developments across grades thus facilitating more curriculum based discussions (C. Fernandez et al., 2003). Krauss et al. (2008) and Ball et al. (2008), in their construction of a definition of PCK, also highlight the importance of choreographing sequences of activities which build on learning opportunities for students within lessons. This knowledge of how to sequence learning trajectories for students is a key dimension of PCK and is incorporated within this frame as a feature of KCT.

In this research these series of lessons, which included the research lesson began to incorporate students' prior knowledge and provided opportunity for teachers to also plan students' learning following a research lesson as teachers' participation in lesson study continued. Teachers also began to plan in greater detail the sequence of activities which would be contained within the research lessons. Thus in this section the theme of sequencing learning trajectories will be analysed under the following two sub-sections:



1. Sequencing learning trajectories over a series of lessons
2. Sequencing student learning within a research lesson

As part of this sequencing, teachers in this research anticipated different mathematical strategies and decided what order of questions and activities would best guide students towards a learning objective incorporating other elements of their KCS and KCT.



4.4.1.1 Sequencing Series of Lessons

Sequencing a learning trajectory for students develops teachers' KCT in creating new ideas on how to build on content for the benefit of student understanding

(Simon & Tzur, 2004; Tepylo & Moss, 2011). Developing content while focusing on how students' mathematical thinking can be developed over a number of lessons enhances teachers' KCT in understanding how to build students' mathematical thinking of a particular topic. Planning such a sequence of lessons within which a research lesson lies is not previously referenced as an outcome for teachers initially engaging with lesson study, although it was highlighted as a positive outcome of Japanese teachers experienced in lesson study by Fernandez et al. (2003).

4.4.1.2 Sequencing Series of Lessons: Lesson Study Data

In Doone's fourth research lesson, teachers planned to gradually build students' mathematical understanding of Pythagoras' theorem. Previously, teachers had taught this topic from the textbook with an introduction of the equation $a^2 + b^2 = c^2$, where students found various values for the hypotenuse given the adjacent and opposite sides and later found one side given the hypotenuse and one other side. In their first planning meeting, teachers discussed the issues around teaching the topic in this way.

Lisa	*The... thing about learning for the students is they can learn the theorem, but then it is confusion when the diagrams are labelled in any given way...*
Owen	*They can learn it, but -*
Lisa	*We think it's saying, $a^2 + b^2 = c^2$, but it's meaningless to them when you give them a thing and and a is the hypotenuse and then you go, $a^2 + b^2 = c^2$.*
Owen	*Yeah.*
Lisa	*They don't actually understand.*
Owen	*So that's rote learning.*
Kate	*Yeah, concept rather than formula.*
Michael	*That's what I was saying about [that student in 6th year].*
Lisa	*Concept rather than formula.*
Michael	*He said it to me and "I remember", he just rattled it off. I asked him, "what does it mean?." It didn't mean anything.*

Lisa *Yeah, exactly.*

Lisa articulated the fact that she had never really thought about this equation from a students' perspective before. She had never thought how changing labels of the sides of the triangles could confuse a student since she had only really taught the topic based solely on the formula:

Lisa *We know it. We know that this is the formula, but we don't look at it from the kids -*

Owen *The concept around the formula rather than the formula.*

Lisa *And it's only that you, talking about that today, if they label the hypotenuse a, I hadn't actually realised that that is what's causing the problem. They don't, I keep -*

Kate *I always just label it X or just some random letter....*

Lisa *We, I think we, always teach by formula.*

Lisa realised that in sequencing the lesson from a student's perspective and focusing on slowly building students' understanding of what the hypotenuse represented would be far more beneficial in building students' mathematical thinking and understanding. This was a very important moment within the data, demonstrating a learning moment in Lisa's experience where through building on her KCT through sequencing the lesson she also enhanced her knowledge by (for the first time in this research) seeing this topic from the point of view of one of the students in the classroom. She changed her approach to the topic and wanted to teach the theorem from context to abstract instead of the other way around. She no longer wanted to focus on the formula, but instead wanted students to have a deeper understanding of what the formula represented.

The teachers within the group began to build a new developmental sequence for Michael's second year class, who were designated as being low ability[3]. The teachers planned a sequence of lessons leading to students solving word problems in order to find the hypotenuse. Employing a critical student lens and incorporating their experiences of previously teaching this topic to students, teachers began the lesson with a 'hook' to

[3] Within this research teachers often referred to students as being of "weak" or "strong" ability according to their prior interactions with a particular class.

engage students with a story about how Pythagoras first calculated the theorem travelling between two girlfriends houses (Appendix F2 4). Teachers then looked towards modifying curriculum materials available on the Project Maths website (Project Maths Development Team, 2011) and gave students an exercise to identify the hypotenuse side in a number of triangles using highlighter pens. Teachers hoped that this exercise would familiarise students with the fact that the hypotenuse did not always represent the 'missing' side (as identified in the previous conversation excerpt) and the conducting teacher would also draw from the student that the hypotenuse is always the longest side.

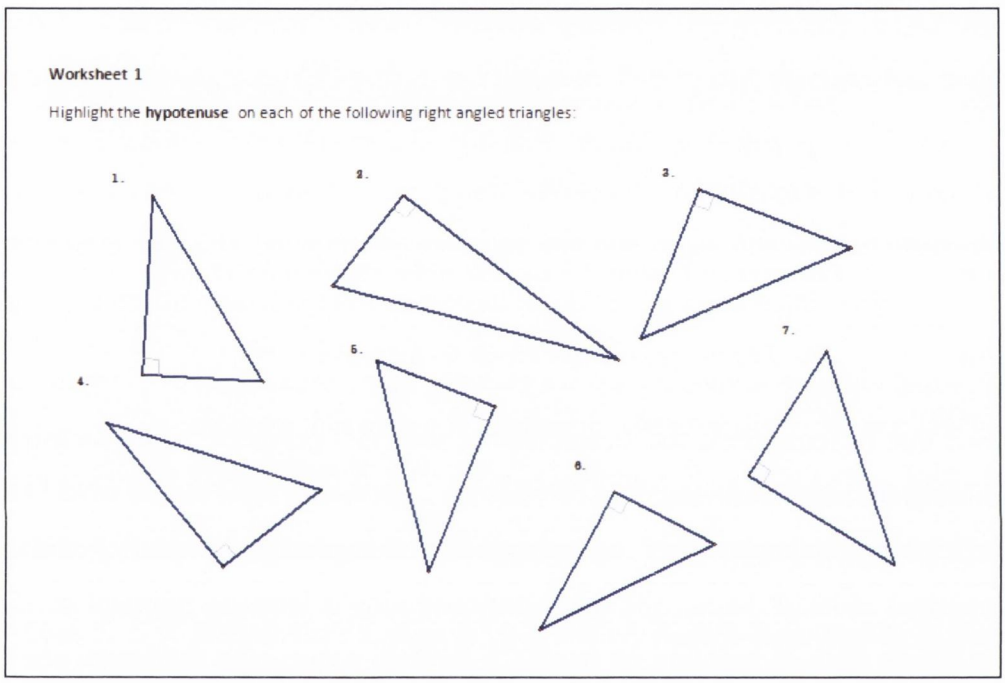


Figure 20 Worksheet 1 calculating the hypotenuse: Doone cycle 4 (PMDT 2011)

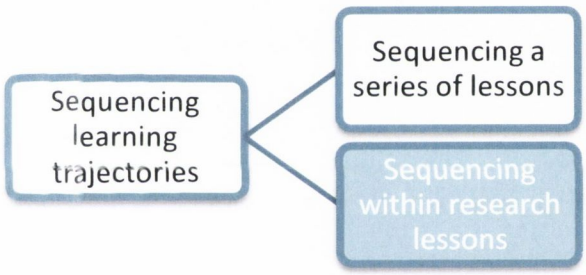
Students were then to be given the same exercise sheet but this time with numbers where they could calculate the hypotenuse given the two other sides. Within this learning trajectory, students worked together in pairs while the teacher asked questions to the groups. In the third lesson of this sequence (the research lesson), students worked on contextualised problems, designed by the teachers, utilising their new knowledge to calculate the hypotenuse. Teachers also decided to avoid calculating the adjacent or opposite sides (given the other two lengths) until students had encountered solving algebraic equations.

This sequence of lessons included teachers identifying what knowledge was required of students in order to understand the utilisation of Pythagoras' Theorem. Teachers

highlighted what students already knew and detailed what they should teach to them within the lesson sequence. Teachers anticipated student responses and also noted items to attend to following their designing and working out of questions. This sequence of building on students' mathematical understanding was a very different approach to how these teachers would have previously taught this lesson, where the equation would have been utilised in its entirety according to a common textbook layout, which did not emphasise students' understanding of the hypotenuse. This sequenced learning trajectory was based on teachers re-thinking and revising how they taught this topic incorporating how a student would best understand utilising this equation. These teachers' approach to developing a learning trajectory was greatly changed through their experience of participating in lesson study.

4.4.1.3 Sequencing Content within a Lesson

It is an important element of teaching that questions be chosen that are appropriate to highlight the intended learning outcome within a question (Ball et al., 2008). Questions should have a match between the task and the student (Holten & Thomas, 2001). If the question does not provide any difficulty for the student, they have no need to engage with it. On the other hand if the question is too difficult, students are unlikely to be able to engage with it in any meaningful way. Sequencing content within a lesson was identified by Hill et al. (2008) as a component of teacher knowledge emergent from their generated data on KCS but is included here since sequencing this content is, within this research, planned as an activity of teaching to a whole class group instead of focusing on how individual students construct meaning for themselves.



4.4.1.4 Sequencing Content within a Lesson: Lesson Study Data

In Doone's final research lesson, students solved problems (as defined by the teachers) using Pythagoras' Theorem. Teachers created a number of contextualised questions for this lesson and in the final planning meeting, debated over the sequencing of these questions within the lesson in order to best build on students' mathematical thinking and simultaneously scaffold their learning. They decided that the lesson would begin with a sample question from the conducting teacher where they would extract the important

information from the question as a class discussion. Teachers wanted to begin this lesson trajectory by highlighting to students the necessary skills required in identifying information from a problem solving or "wordy" question.

The lesson began with an example of a question teachers call 'Maid in the Mist' as and students were asked to identify the important pieces of information using a modified problem-solving worksheet (Appendix F2 2). In discussing why to begin this lesson with this example, teachers employed a critical student lens in identifying what should be useful to students in building their mathematical understanding.

Lisa	*Put Maid of the Mist up and get them to say what's important*
Michael	*I like the way you did [in the previous research lesson]. So, the first one, when we did, with the TY's, we had them on the board and we extracted the information*
Kate	*Yeah...we showed -*
Michael	*How to extract information.*
Kate	*How to, yeah, underlining and we said, "oh this is how you can do that."*
Lisa	*Yes, ok. So, we're not going to solve, you're just going to click and highlight.*
Michael	*Yeah.*
Lisa	*The kids are going to write down on that worksheet that Owen has: "What's important to help them solve?", you're not telling them, they're just [going through it], and we're watching do they write down*
Michael	*What's the most -*
Lisa	*"the measurements, a right angle" and click - it's Pythagoras.*
Michael	*Yeah.*
Lisa	*That's what we want to see, that connection being made...*

Teachers wanted students to be able to find and define the hypotenuse and right angle in this sample question and wanted students to understand why Pythagoras' theorem might be useful in solving this question. They debated the sequence in which problems should be given to students. They decided to begin with a question on an airplane because they perceived it to be more straightforward for students since it had an obvious diagram and students were looking for the distances, which closely mirrored the 'Maid in the Mist' example the students had already seen (Figure 21).

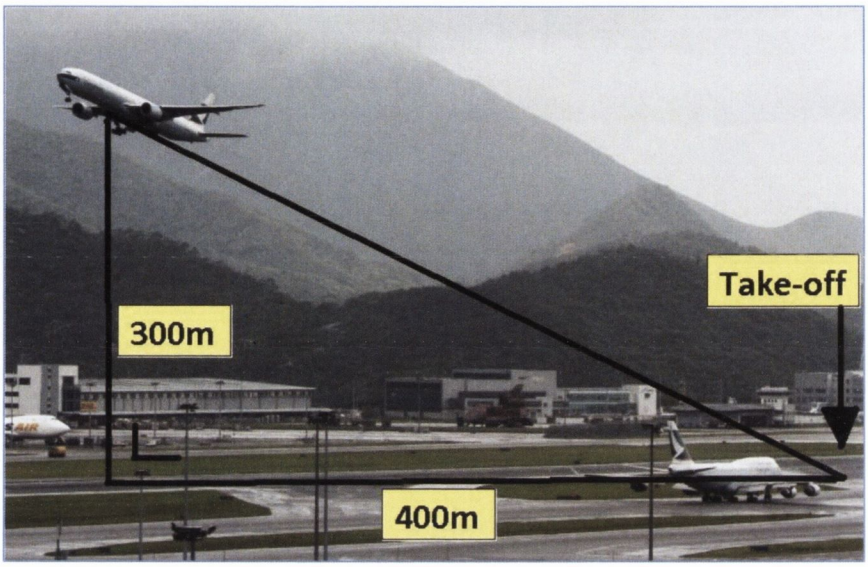


Figure 21 Sequencing questions for students: Doone cycle 4, Appendix F2 5

Teachers then sequenced gradually more difficult questions with the scaffolding questions, such as "Can you identify the right angle?" removed for problems later in the question sheet.

Michael *Isn't there a nice point where we stop the scaffolding around question 4 or so?*

Aoibhinn *Yeah.*

Michael *And we drop all the cues, stop there and then give them the last 2 or 3 without all the blurb.*

Kate *Yeah.*

In that extract teachers designed the students' problems to give fewer hints to students as their mathematical understanding of when and how to use Pythagoras' Theorem improved through their successful completion of previous questions.

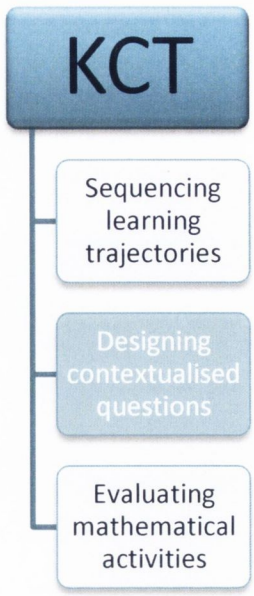
Teachers discussed in what order students should meet the problems and left a particular question on athletics until the end, since it had the possibility of being extended and provided opportunity for differentiation for students who may have answered the problems before their classmates.

Owen	*Should this one maybe come before the previous one?*
Kate	*Yeah, I think maybe if you put this one last.*
Michael	*Yeah...*
Kate	*Yeah, maybe like four and a six and a five, or something like that.*
Lisa	*Yeah, keep five last 'cause you can do more with it.*
Owen	*Yeah.*
Michael	*Ok. So you want to put this as...four. So that's four, ok. That's one, that's two, that's three, that's five now...and the other one is six.*
Owen	*So, will we add in then that last part?*
Michael	*And how far does -*
Kate	*Oh yeah, you could also say, "how far does he run in one lap?"*
Lisa	*One lap.*
Kate	*Now the answer is 182 because 13, 84, 85...*
Nora	*That's quite good I think, making it more in context for students.*

In this extract, teachers debated the sequencing of the problems within the research lesson to ensure that students experienced success with the initial questions and gradually gained confidence in building on their mathematical understanding of the topic as they investigated the questions in the worksheet (Appendix F2 5). In sequencing these questions, teachers employed a critical student lens in identifying what question was best for a student to build on their mathematical thinking and what questions best extended this knowledge as students continued the activity. Teachers built on their KCT in debating the contexts and reasons for making such decisions with their colleagues.

4.4.2 Developing Contextualised Questions

This is the second of three features detailed within KCT within the framework of analysis developed in this research and suggested by the data.



The problems that teachers pose to students, as well as the precise way these problems are presented, affect how students will approach and think about the content of a lesson (C. Fernandez et al., 2003). When students do not see the relevance of mathematics to their real life they are less likely to continue to engage with the subject (M. Brown et al., 2008). Polya (1973) suggests that the teacher should challenge the curiosity of their students by setting problems proportionate to their knowledge but that are also stimulating. It is thus important that teachers develop questions that are both relevant to the mathematical content in question and to the student while simultaneously building on students' thinking (Galbraith, 2013).

As this research continued, teachers often preferred to develop their own questions instead of sourcing them from other curriculum materials or textbooks since they could then be constructed as relevant for their particular students in terms of their prior mathematics understanding and in terms of the research lesson objective. Developing mathematics questions was a new activity for many of the teachers and in doing this they not only focused on the particular mathematics skills they wanted their students to engage in, but also developed their own knowledge. This is not to say that teachers did not have sufficient content knowledge to teach a particular mathematics subject, but developing mathematics questions relevant and specific to a particular research lesson focused teachers on the exact mathematical skills in question. In addition, providing and designing contextualised questions for students did not mean teachers were providing 'cover stories' for students (Schoenfeld, 2011) in providing a context to a question for the sake of it, but rather were providing purposeful and meaningful activities in designing activities specific to their students (Galbraith, 2013).

Developing curriculum materials and questions can be a difficult task and often one too daunting for a teacher to do by themselves, particularly those who are relatively new to

teaching (L. S. Shulman & Shulman, 2004). In designing particular questions on multiplication of fractions in Doone's 2nd lesson study cycle, Owen exclaimed "thank God I'm not doing this by myself" since the construction of that amount of questions was daunting to him as a newly qualified teacher who was not confident in his mathematical ability. Developing questions collaboratively allowed teachers to build on their KCT, relevant to their own students, while also building their confidence in their ability to design questions which drew on their knowledge of content, knowledge of their students, and were also relevant to the new curriculum.

4.4.2.1 Developing Contextualised Questions: Lesson Study Data

At the beginning of the year, teachers in Doone wanted to engage their students with questions that gave "meaning and purpose to their mathematics" and they were conscious of providing such tasks throughout the year. In that initial conversation Nora noted that it was of no use to students to provide irrelevant applications since it had negative implications on students' experiences of mathematics.

Nora *I think the most important thing right throughout is meaning. I definitely think things don't have to be made in every case 100% applicable to everyday life for them to have meaning, and for it to be satisfying to kids. I think all our learning, almost everything, is based on something happening, having a meaning that the youngsters are comfortable with.*

In their third research lesson, teachers in Doone decided to develop a question focused around an Irish international rugby player. Teachers had been discussing constructing questions with a quadratic function and were toying with ideas around diving boards and hot-air balloons. Since Doone is an all-boys school where rugby is the main sport and since the research lesson would be held during a popular rugby championship, Michael put forward the idea of using a rugby move (a Garryowen) where the ball takes a negative quadratic trajectory. Teachers were enthusiastic about designing this question.

Michael *Could we do a football?*

Kate *Football? Yeah. A rugby ball!*

Michael *A rugby, an up and under in rugby!*

Aoibhinn *Like a what?*

Lisa *An up and under.*

Kate *A Garryowen.*

Michael *An up and under is you kick, Garryowen, yeah, you kick it up, it goes up high, they chase it, and try to catch it....*

Kate *Aw, let's do a Garryowen. Love it. Will he get there on time to catch it, if this is the path the ball takes?*

Michael *Or something like that, yeah.*

Owen *Brilliant.*

Kate *Let's do it Michael! Yes! And they'll love that.*

Teachers purposefully designed a question that they felt students would engage with because it had context and meaning for them, while also prompting students to develop in their understanding of quadratic trajectories, and teachers were palpably enthusiastic about designing this question themselves.

Kate and Michael investigated the question in detail to ensure that each of the variables was realistic and that the answer would also be believable to students. They wrote out the activity and played around with the numbers to ensure that they made sense, Michael decided to increase the distance that the player had to run.

Kate *We may, we might want to double-check the height is high enough.*

Lisa *Right.*

Michael *Yeah.*

Kate *How high would you get a Garryowen?*

Michael *40 metres, 50 metres.*

Kate *Ok.*

Kate *Maximum, yeah?*

Michael *Yeah. He could kick it further. Let's take it further.*

Teachers finally arrived at the question demonstrated below (Figure 22 & Appendix F2 3). In designing this question, teachers enhanced their KCT in constructing the question in a manner that would build on students' mathematical thinking and in investigating the numbers in the question that were both useful and realistic to students.

During the recent Ireland v England match in the Aviva Stadium as part of the Six Nations competition, Jonathan Sexton kicked a Garryowen before he went off with an injury. A Garryowen, also known as an "Up and Under", allows the attacking team to disrupt the defensive line, take the defense's pressure off themselves and put offensive pressure on their opponents. However, the kicking team risks losing possession of the ball, after which the opposing team may counter attack.



George Hook and Brent Pope, as well as being famous for their rugby commentary are also keen mathematicians who have calculated that the height of the ball above ground during the Garryowen can be described by

$$H = 25t - 4.9t^2$$

where H gives the height of the rugby ball above the ground at any time, t, in seconds.

a) How long is the ball in the air for?

b) What is the maximum height reached by the ball?

c) If Jonathan runs at a speed of 7m/s and is not tackled or hindered in any way. The ball will land 26m away. Will he catch the ball?

Figure 22 Constructed contextualised question: Doone cycle 3, Appendix F2 3

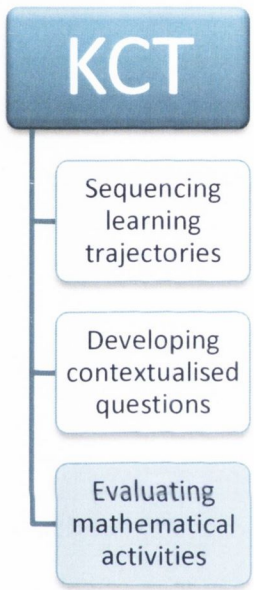
In this case teachers utilised a critical student lens in designing a question that was of particular interest to this specific group of students, included the mathematics relevant to the research lesson objective, and built on students' understanding of quadratic functions.

While developing contextualised questions is contained within this frame as an element of KCT, it is also relevant to TKS (explored later in this chapter, Section 4.5), since teachers had a personal knowledge of their students and anticipated what would engage and interest them in developing a question that held meaning and motivated those students.

4.4.3 Evaluating Mathematical Activities

This is the third and final feature of KCT detailed within this suggested framework of analysis.

Evaluating mathematical tasks before asking students to tackle those tasks in class benefits teachers in anticipating students' strategies, but also in identifying points which could be unclear for students (Stein et al., 2008). Doing questions builds teachers' KCT since attempting mathematics questions with other teachers makes visible other strategies or aspects of a questions that may be unfamiliar (Ball & Hyman, 2008; Borko et al. 2008; Mason 2009). When teachers collaborate on mathematics questions, be they for their students or for their own purposes, teachers can listen to how others construct mathematical understandings and therefore decide on ways to build students mathematical thinking through choosing particular questions or highlighting particular aspects of questions (Ball & Hyman, 2003; Borko et al., 2008). Mason (2009) agrees that in practising doing mathematics questions, teachers strengthen and deepen their understanding of the ideas and teachers become sensitised to their own students' struggles in doing mathematics. While doing mathematics questions obviously entails teachers using and developing their SCK, in seeing other strategies of how to answer a question teachers are also developing their KCT.



4.4.3.1 Doing Mathematics Questions: Lesson Study Data

In Crannóg's second research lesson, teachers discussed how to utilise algebra tiles and array models in factorising quadratic expressions, which led to a sharing of strategies in how to factorise quadratic expressions. Fiona had already introduced the 'squiggles' method to Dave and Stephen a number of years previously, but neither myself, Eileen nor Walter knew what it meant. Walter had heard of this strategy and noted that when he mentioned 'squiggles', some of his students were familiar with the strategy but he was not sure what it was. Fiona described the method to the group:

Fiona *Because if I was factorizing that I wouldn't go straight into brackets, I would put another two or three lines in between...*

Walter *You do that even with the 'a' as 1?*

Fiona *Any old thing.*

Dave *Everything squiggles.*

Fiona *Yeah. Go for squiggles every time...*

Aoibhinn *What do you mean squiggles?*

Fiona *Where you have got something like, say* $x^2 + 4x + 3$*, there are two numbers multiplied together to give 3, add to give 4. So you get three times 1, so you split that up – [Writing notes - see Figure 23]*

Dave *To get what is common, which is the easiest type of factorizing for them.*

Fiona *Then you squiggle it, that is the squiggles.*

Dave *And they love that, they never forget that word.*

Aoibhinn *Ok, I have never seen that.*

Fiona *So I would always do quadratics like that and they would know how to do it every time.*

Stephen *I never forget the word but I forget how to do it!*

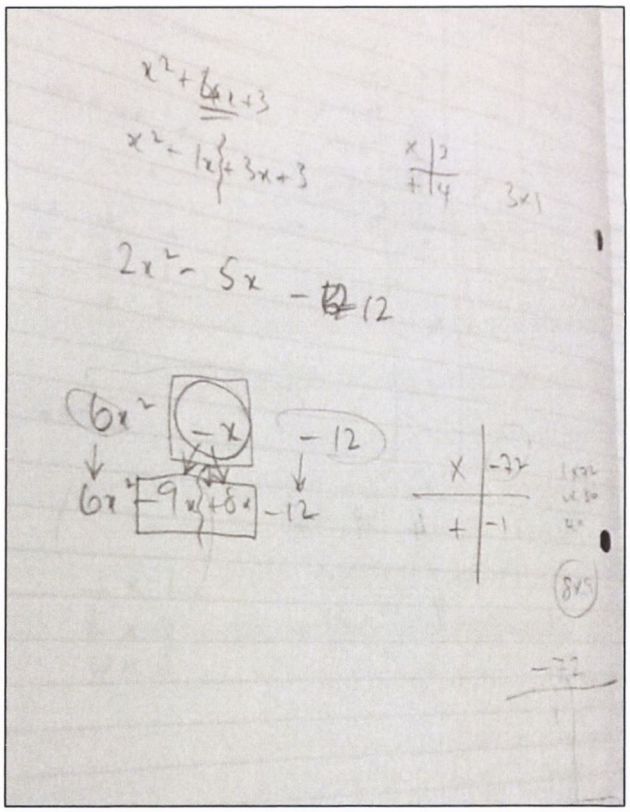


Figure 23 Fiona's notes explaining 'squiggles', Crannóg cycle 2 Appendix F1 5

Walter *I say it to fifth years, I say "So we squiggle it" and half the class go sort of "Oh yeah"*

The conversation continued with Dave and Stephen sharing how they usually introduced and taught this strategy and Stephen further noted how his senior students enjoyed the systematic approach that was required.

These conversations about solving quadratics continued outside of the research meetings and Dave in his mid-point interview a week later noted that he had learned more about factorising and teaching of factorising in that past week than he had in his previous teaching experience.

> *Dave* *I have looked at more ways of solving quadratics in the last week than I have looked at in the last ten years. I have tried more ways of solving quadratics in the last week than I have looked at in the last ten years. I had two methods, the method I used at school for ten years and the method I met with Fiona...and I knew there were others out there...Now I am just reading different types with algebra tiles and with area models and with squiggles and without squiggles.*

Through participating in lesson study and as part of the planning process exploring different mathematical strategies of factorising, Dave (as one example) enhanced his KCT in realising more ways in which he could approach teaching this topic.

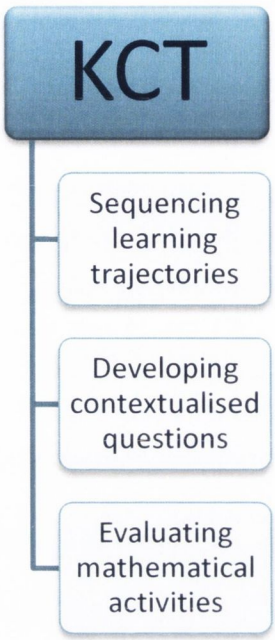
Through engaging in doing mathematics within the lesson study planning meetings, teachers were developing their KCT which, in turn, impacted on their KCS since teachers had more strategies from which to choose in deciding how to best build on and anticipate students' mathematical thinking. Teachers commented that different strategies would be useful for different groups of students depending on their prior mathematical understanding and depending on the students' preferences in representing quadratic expressions i.e. in utilising a visual method such as array models or in preferring the step-wise instructions of factorising using brackets or squiggles.

In doing mathematics questions within the planning stages of lesson study, teachers developed their specialised content knowledge in finding out new ways and strategies of representing mathematics ideas to their students. Evaluating activities themselves also assisted teachers in developing more mathematics questions because it enabled teachers to think about activities from a students' perspective and thus construct relevant questions for this particular lesson and this particular group of students. Teachers learnt

about how their colleagues interpreted the same mathematics concepts and thus developed a better sense of how other students would perhaps answer the same or similar types of questions.

4.4.4 Conclusion of KCT

In this section a framework of detailing KCT utilised and enhanced through teachers' participation in lesson study was developed.



In planning detailed research lessons teachers saw the merits of sequencing mathematical content in a particular way which would motivate students to continue in an activity but also would build on students' mathematical thinking. Designing contextualised questions for students was an enjoyable activity for teachers within lesson study which focused them on thinking about mathematical contexts from their particular students' perspectives but also allowed teachers to build on their own mathematical thinking in identifying important variables and skills during the construction and investigation of these questions.

4.5 Teacher Knowledge of Student

This section of the chapter refers to a broader element of the research question which is referenced by the second sub-question:

> ***How is teacher knowledge, particularly pedagogical content knowledge, enhanced through iterative cycles of lesson study?***
>
> *ii) Are other forms of teacher knowledge, external to subject matter knowledge, employed within lesson study?*

From the analysis of the data generated within this research, it is proposed that there is an additional form of teacher knowledge – that of teacher knowledge of student – which can be added to a framework of mathematical knowledge for teaching and which is utilised in and enhanced by teachers' participation in lesson study. While this research did not assess what knowledge teachers had of their students, it emerged from the data that the

knowledge teachers had about the students in their class was an important factor in planning, conducting, observing and reflecting on the research lessons. The proposal of this additional element of mathematical knowledge for teaching is a major contribution of this research.

As referenced in Section 2.7, Shulman (1987) and Grossman (1995) identified knowledge of learners and their characteristics as one of the categories of knowledge held by teachers. Shulman also identified "knowledge of educational contexts, ranging from the workings of the group or classroom...to the characters of communities of cultures" (p. 8) encompassing the facilitation of the classroom and the social and cultural backgrounds of students. These categories of knowledge of learners and learning are not included within Ball et al.'s (2008) framework of content knowledge for teaching.

Taking Ball et al.'s (2008) original model for mathematical knowledge for teaching, the framework in this research suggests that as well as subject matter knowledge (SMK) and pedagogical content knowledge (PCK) there is another valuable knowledge that teachers hold and employ in their teaching, that of teacher knowledge of student (TKS). This notion has emerged from the qualitative data generated in this research where, over the duration of one academic year and throughout teachers interactions with cycles of lesson study, teachers constantly referred to individual students within their classrooms, discussed the student interactions within the classroom, talked about the physical classroom environment, and very often referred to the engagement of their students within the lesson. The data would therefore suggest that the framework of mathematical knowledge of teaching suggested by Ball et al. (2008) should be expanded to incorporate an additional form of knowledge where teachers' knowledge of their own particular students is referenced and is a contribution to knowledge of this research.

While Ball et al.'s (2008) model is based on the content related categories of Shulman's categories of teacher knowledge, the content being taught should not be separated from who it is being taught to, hence within this modified model, TKS is included within the frame.

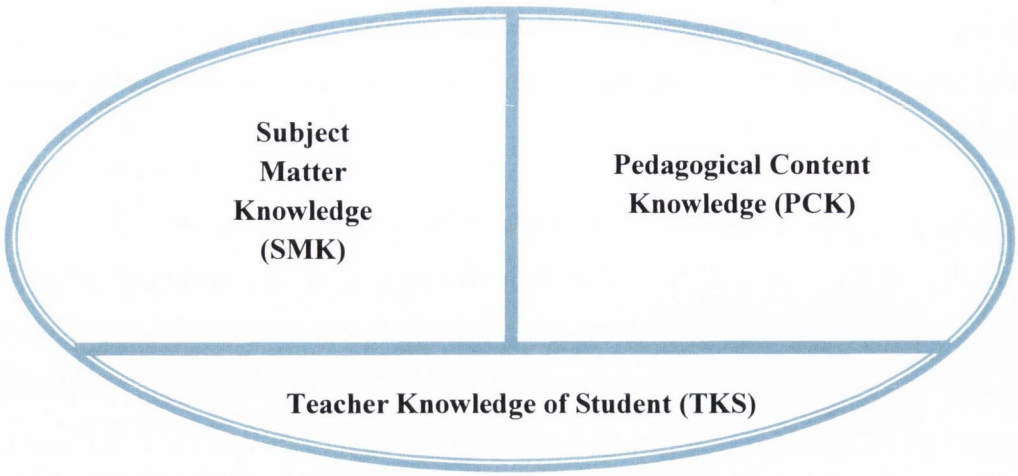


Figure 24 Modified model of Mathematical Knowledge for Teaching

While Ball et al. (2008) also note that within KCS it is important that teachers anticipate what students will find interesting or motivating, in analysing this data of these teachers' conversations as part of lesson study, it became apparent that teachers' knowledge of their own students, their social and cultural backgrounds, was very important in designing mathematics lessons that were relevant to these students, that engaged and motivated the students and that also built on their mathematical understanding. The data thus suggest that teacher knowledge of student overlaps with PCK and bolsters teachers' KCS and KCT.

While the idea of employing a critical student lens or focusing on mathematical thinking from the students' perspective is one which is present in the literature (C. Fernandez et al., 2003; Lewis et al., 2006), few scholars have suggested that mathematics teachers' knowledge of their own students is something which is integral to their practices. This finding is a significant contribution to the research literature in highlighting this knowledge as an integral part of teaching.

4.5.1 Teacher Knowledge of Students and Critical Student Lens

Teachers are far better able to interpret their own students' understanding than, for example, teaching a class of new students and attempting to interpret their understanding (van Es & Sherin, 2008). Teachers can make more informed decisions on their practice during lessons where they are familiar with their students (Schoenfeld, 2011). Teacher knowledge and specific pedagogical content knowledge about particular students is required in interpreting and developing students' understanding. It is an important

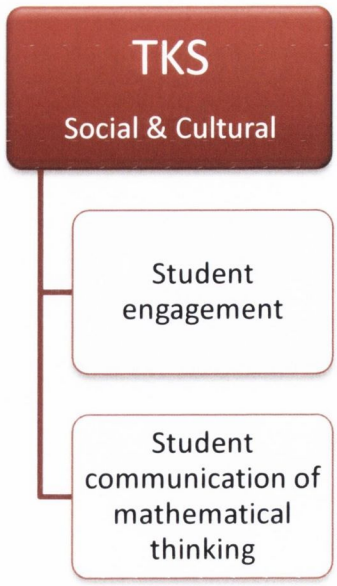
element of lesson study that teachers have an opportunity to reflect and focus on *their own* students' mathematical thinking as opposed to, for example, interpreting other students' thinking through video studies (Sherin & Drake, 2009; Sherin & Han, 2004b; van Es & Sherin, 2008) or other professional development interventions (Jacobs et al., 2010).

School based lesson study provides teachers with a unique opportunity to not only observe students' thinking in a lesson that teachers have actively participated in planning and are familiar with the construct and objectives of, but teachers can also observe students that are familiar to them and whose social and cultural contexts are familiar. Within lesson study teachers are provided with unique opportunities to anticipate and reflect on their own students learning. In focusing on their own students' learning, teachers incorporate and enhance their TKS through lesson study.

As an aside, lesson study within Irish post-primary schools may be unique in the fact that teachers often teach more than one subject and therefore meet students, who are participating in research lessons, in environments other than as their mathematics teacher. Teachers thus have relationships with students that are not based solely on mathematics, but also on other subjects or extra-curricular activities.

4.5.2 Framework of Teacher Knowledge of Students

Within this new framework, TKS will be detailed in terms of social and cultural knowledge of students. In this regard it is suggested that teachers can further develop their mathematical teaching and learning practices due to knowledge of the students whom they are teaching. Within this frame, teachers' knowledge of students will be analysed under the following two features:



1) Student engagement
2) Encouraging student communication of mathematical thinking within the classroom

Within this research the data suggests that teachers' employ of a critical student lens also added depth to TKS, as it did for PCK, since by viewing teaching and learning practices from the

perspective of a student, teachers could better anticipate how to encourage student engagement and student communication. It is not suggested that this framework of TKS is exhaustive, but within the remit of this research the detailing of TKS is bounded by teachers' planning and reflection conversations as part of lesson study.

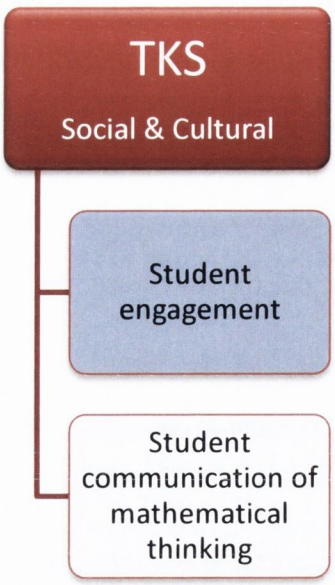
4.5.2.1 Social and Cultural Knowledge of Students

During teachers' planning and reflection of research lessons, it was particularly noticeable that teachers had a familiarity with and knowledge of the students that were being taught and that this impacted on how they approached the lesson, the strategies they employed within the lesson, and how they interacted with individual and groups of students within the lesson.

Within this research, teachers were provided with unique windows of opportunity (Meyer & Wilkerson, 2011) to be present in another classroom other than their own, a practice that is not common within Irish post-primary schools, particularly in terms of professional development (Corcoran, 2011b; Gilleece et al., 2009). Through observations of teaching students in groups, teachers built on their own knowledge of how to facilitate student group work, but also enhanced their social knowledge of students in how they interacted and performed with their classmates.

4.5.2.2 Student Engagement

This is the first of two features which are included in TKS in this proposed framework.



Affective issues have been of considerable interest in mathematics education research (Leder & Forgasz, 2006) and, as already explored in the literature review (Section 2.2.5.2), students should feel that what they are learning is relevant and of practical use (Galbraith, 2013; Noyes, 2007). Within the new Project Maths curriculum, one objective states that learners should develop "a positive disposition towards mathematics" (National Council for Curriculum and Assessment,

2012a, p. 6) and it is highlighted that "through the use of meaningful contexts, opportunities are presented for learners to achieve success" (National Council for Curriculum and Assessment, 2012a, p. 12).

Within this research, teachers' reflections continuously referred to a critical issue for them which was the 'interestingness' (Ainley, 2001) of the learning situation for students and how they could maximise student engagement as potential for learning, referred to as *hatsumon* in Japanese lesson study (Doig et al., 2011). Teachers were conscious of keeping students engaged in the mathematics lessons and of providing students with motivation to interact with both the planned mathematics activities and with their peers in communicating their understanding of a particular topic.

At the beginning of this research teachers at Crannóg wanted to encourage students to be autonomous learners. Teachers spoke of students' fear of making mistakes which discouraged them from engaging with mathematics in lessons (Figure 25).

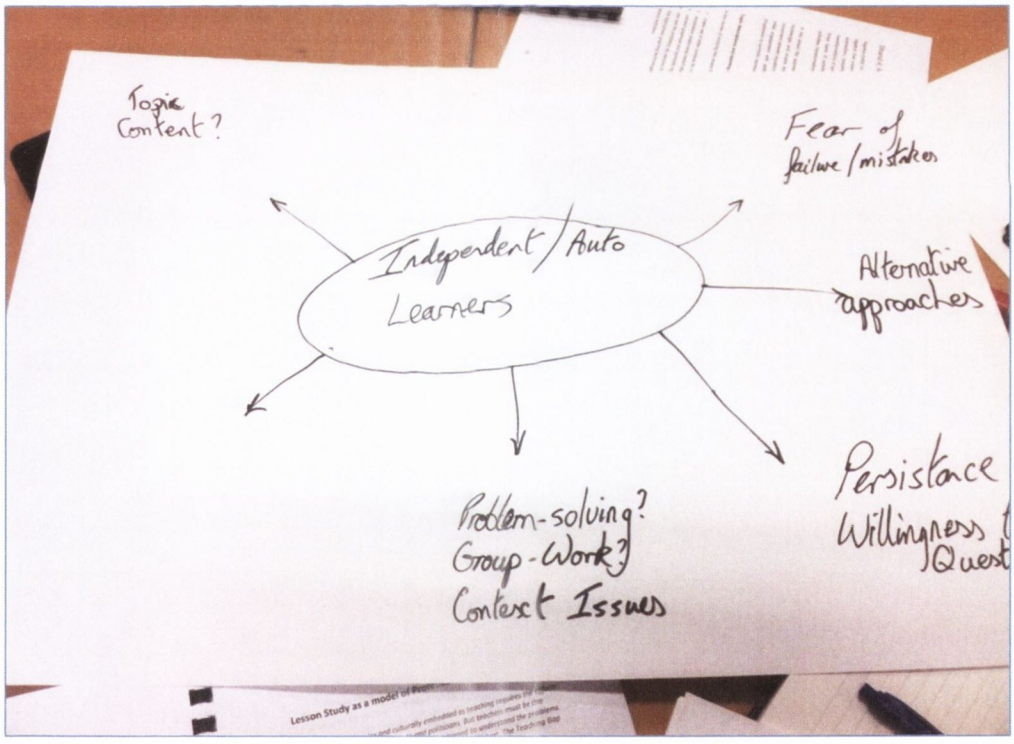


Figure 25 Teachers' overall goal of lesson study: Crannóg cycle 1, Appendix F1 2

Following a brain-storming discussion on the issues they felt most pertinent to their teaching, teachers agreed on the following goal of their participation in lesson study for the academic year in their first meeting:

Creating a culture which fosters independent thinking and fearlessness which leads to autonomous learners.

In this research teachers at Crannóg worked to provide a learning environment which would allow students to exercise and develop these qualities where students would be encouraged to work with one another, with guidance from the teacher, in attempting mathematics questions and building confidence in their own ability (Solomon et al., 1993).

Similarly in Doone, teachers wanted to provide students with opportunities to engage with mathematics that was purposeful and relevant to students as their overall goal of lesson study and agreed on the following over-arching goal in their first meeting:

For our students to find purpose and meaning in their mathematics that leads to their enjoyment and confidence in the subject.

Teachers wanted students to see the importance of the subject but also to enjoy the mathematics they were engaging with in school.

In both schools involved in this research, engaging students in their learning of mathematics was a fundamental goal of their participation in lesson study and this was maintained as a focus (although sometimes implicit) throughout the research.

4.5.2.3 Student Engagement: Lesson Study Data

From the very first research lesson in Crannóg, teachers were enthused when they saw that students were actively engaged in investigating the pattern of blocks required to build a new pyramid for the pharaoh as previously described in Section 2.2.4.8. Directly following the lesson (before even all of the students had left the room) teachers were excitedly reflecting about the success of the lesson in terms of student engagement and in terms of students discovering the pattern.

Walter *I think it was more exciting for them than I thought it would be!*

Eileen *Yeah!*

Fiona *Well I think the way you started it off was brilliant. I think you got them interested by talking about the competitive Pharaohs and as they came up with all the answers you wanted, which was great.*

Dave *They came up with the questions themselves!*

Walter *Their imagination of it, they just sort of plugged into that more.*

Fiona *No, it was great...It worked. Those guys over there I was watching, they really helped each other. They hugely helped each other, they were listening to each other...they had great discussions*

Later, in the post-lesson discussions, Gerald reflected that students were engrossed in the activity and also reflected on the excitement that was in the room because students had to attempt to figure something out by themselves.

Gerald *They were so engrossed in what they were doing, you know. So all that dynamic is great, to watch how, you know, they are hitting off each other and when they have their own ideas and it is good. There is an excitement about it, you know. There was a degree of excitement in there about just getting stuff worked out.*

Gerald in that extract used the language of "excitement" and "engagement" to describe the students' behaviour. Judy also reflected that within the lesson she sensed students felt "satisfaction" in completing the activity by themselves in the group, without having had tips or hints from the teacher or others outside their group.

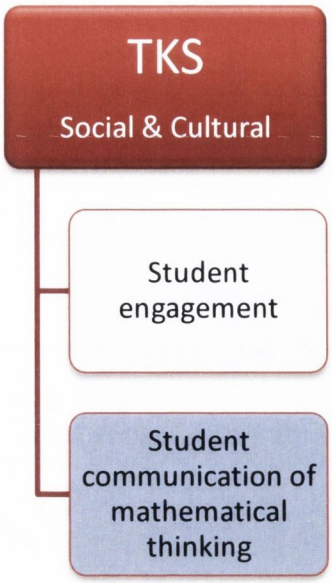
These instances of student behaviour, reflected on by the teachers, note the positive impact of the lesson on students and aside from the pedagogical and subject matter knowledge of the research lesson, it was important to the teachers that the students enjoyed the lesson and were motivated to pursue, as a group, the answer to the question 'How many more blocks do I need to build the next pyramid?'.

Planning, observing, and reflecting in lesson study extended teachers knowledge of engaging their own students in their mathematical learning.

4.5.2.4 Student Communication of Mathematical Thinking

This is the second of two features of TKS which are included within this suggested framework.

Within the Project Maths curriculum it is explicitly stated that "learners are encouraged to discuss approaches and solutions to problems and are expected to consider and listen to other viewpoints" (National Council for Curriculum and Assessment, 2012, p. 11). A classroom where student communication is encouraged can be beneficial to building students' confidence in their mathematical ability and building their relational understanding (Boaler, 1998; Goos, 2004) but may be a very different environment to the traditional Irish post-primary classroom (Oldham, 2001).



It is suggested within the literature, that as teachers participate in lesson study, they begin to attend more to encouraging student communication and to how student groups should be constructed (Bruce & Ladky, 2011; Kamina & Tinto, 2011). A number of teachers within this research were wary of introducing such an environment into their classrooms where they were not confident in facilitating and managing students' discussions. At the beginning of the research, Owen stated that he did not enjoy or see the benefit of a "*noisy classroom*" however, in employing a critical student lens in his practice and in viewing communication from a students' perspective, he began to see the benefits to students' learning and engagement through encouraging student talk. In Crannóg, Judy reflected from her observation of the first research lesson that despite having 20 minutes to work together as a group, students' conversation did not deviate from the mathematical task at hand as she would have expected, providing her with new insight into how her students would engage with a similar mathematical task. Through participating in lesson study, teachers' TSK was enhanced, encouraging students' communication of their mathematical thinking, which in turn positively impacted on their PCK.

4.5.2.5 Encouraging Student Communication: Lesson Study Data

Most of the teachers in this research were unused to teaching students in groups and despite the fact that communication within the classroom is highlighted within the Project Maths curriculum and that teachers are encouraged to incorporate student group

work into their mathematics teaching, many of the teachers in Crannóg (a phase 1 school) were not yet comfortable or confident in teaching students in groups.

In Crannóg's first research lesson teachers wanted to conduct a lesson that was suitable for students to work in groups and where students were encouraged to work without the step-wise guidance of the teacher. Dave was the conducting teacher and had never taught students in groups. Prior to the lesson, he re-arranged the classroom so that students could sit in groups of four and work together for the duration of the class. Since Dave had never taught his students in groups before, going around to each of the groups during the lesson and asking each of them questions was a new experience for him and one that frustrated him in how time-consuming it was. But he also realised that he had usually based his lesson on the abilities of the two strongest students in the class, Jack and Jane. Working in groups forced him to pace the lesson according to the majority of the students and he noted that through employing a critical student lens, he should think of mathematics in terms of more than just Jack and Jane. He realised that his usual pace, while it might suit a minority of students, in fact may not be prescient for three quarters of the class and Walter echoed this sentiment.

Dave — *But because they were in groups and because I wasn't standing up at the front asking everybody in the room "Does anyone know what the formula is?" Jack didn't say anything but you knew he had it. Jane didn't say anything but their group had it and they went off getting more anxious about the colours of it... which is interesting because group work stuff therefore, you have to be aware, takes longer. But the quality may be off what goes on at the time....*

Walter — *But all the learning was so much better in all the other groups. Otherwise, I have been tempted if I had been teaching it in the first way, ok we have 'y', we have got 'x' squared.*

Dave — *And then move on. You have got further, maybe you have lost three quarters of the class.*

Through participating in the research lesson, both the conducting and observing teachers enhanced their TKS through seeing the benefit of incorporating student communication from the perspective of students within the class.

Dave considered that in allowing students to work at their own pace, their mathematical understanding was more visible to him as a teacher and also students constructed their own understanding without the usual time pressure dictated by others as he reflected in the post-lesson discussion.

> *Dave* *And you see because I have never worked like that with groups, I have done that same class in virtually the same way, with just them all sitting in front of me and Jack and Jane would have after five or ten minutes would have put their hands up and gone "it's x^2" and I was basing my predictions for how quickly the class would go...on that.*

Dave was also surprised at how students, usually considered to have a 'weaker' ability in mathematics, performed surprisingly well in the group activity in that lesson.

> *Dave* *The middle group had the thing built straight away before anyone else had even worked out that it was colours, and there was two girls at the table that would be among the weakest in the class....she came so confident and came out with some good stuff.*

From that same research lesson in Crannóg, other teachers observed the benefit to students of participating in group work. Instead of showing the students the answer as they would have traditionally taught it, the students discovered it for themselves. Walter, Judy and Stephen reflected on how they would have traditionally taught this lesson and, having observed students working it out for themselves, reflected on the benefit of such an approach for students where they discovered and learned together. Stephen also noted that the teachers' role in this form of lesson was to construct the lesson and activities in such a way that students met a certain level of success within the question, which they did for this lesson keeping in mind what level of students they were planning for.

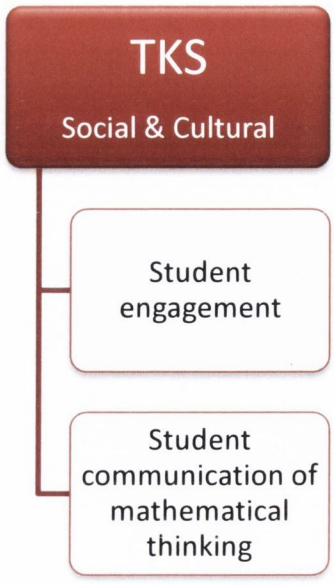
> *Stephen* *They were all engaged and I think part of that as well for me was just how structured we had been and obviously we have put a lot of*

time into it, but that the lesson was so structured, they did, there was always a progression for them, and there was that end point which was beautiful where sometimes maybe when we give group work, we give things that maybe are over their heads and people get stuck. It actually built up nicely for that level, for that class. But sometimes, and obviously as we are teaching all week, where we go we don't maybe prepare as much. And when you give a group something to do and it goes straight over their heads and they aren't actually even able to discuss, and so it was actually beautifully in terms of the class and the natural progression I thought was brilliant.

This was a new aspect of their teacher knowledge in considering how students working in groups and conversing about mathematics with one another enhanced their experiences of mathematics (Boaler, 1999, Goos, 2004). Participating in this first cycle of lesson study provided teachers with a new knowledge of how their students learned through communicating with their peers. This experience provided new perspectives to teachers in thinking about how different students build their mathematical understanding.

4.5.3 Conclusion of TKS

Teachers' knowledge of their own students has not thus far been explicitly included in models of teacher knowledge (Rowland, 2014). Within this research, from analysis of teachers' conversations throughout iterative cycles of lesson study, the data suggests that teachers' knowledge of and relationships with their own students impacted how they approached a lesson, how they taught the class, and how they attended to how specific students engaged within that lesson.



In this research, teachers' knowledge of students was referred to in encouraging students to engage more in their mathematics and in encouraging more communication of mathematics within the classroom. The frame of analysis of TKS in this research is thus based on those two features.

In participating in lesson study, teachers in this research built on their knowledge of students through various aspects of their teaching of mathematics but also in their personal knowledge of students sitting in their classrooms. Teachers found out that student engagement could be enhanced through keeping students active and autonomous in their learning, and in employing a critical student lens saw that engagement and enthusiasm within a lesson did not mean that students were "*off task*" but rather that it could enhance their experience of mathematics.

In teaching students in groups, these participants both enhanced their knowledge of how to conduct a lesson where students engaged in group work through conduction and observation of research lessons, but also came to examine how different students participated in different groups and during their reflection conversations, teachers discussed the construction of these groups and became aware that the members of a student group would impact on how students participated and engaged with the mathematical content and their peers. Through participating in lesson study and through more often employing a critical student lens in constructing and reflecting on lessons, teachers built on their existing knowledge of the students they were teaching.

Throughout the data, teachers' TKS was intertwined with their KCS and KCT, suggesting that TKS is an important element of teachers' knowledge but that TKS also overlaps and perhaps enhances teachers' pedagogy. Further analysis of teachers' utilisation of TKS will be discussed in Chapter 6.

4.5.4 Lessons from the Data for the Original Model of MKT

Within this chapter two contributions were suggested by the data to frame the investigation of changes to teacher knowledge due to their participation in lesson study cycles over the course of one academic year. These contributions concern a refinement and enhancement of a well-known model in the literature of mathematical knowledge for teaching (MKT) as presented by Ball et al (2008).

The first concerns a greater understanding of the dimensions of KCS and KCT due to teachers employing a critical student lens (Fernandez et al., 2003) on their practice due to their participation in lesson study. These subsets of PCK have been explored in the literature (Hill et al., 2008) and separate literature has investigated teachers' noticing of student mathematical thinking (Carpenter et al., 1989; Fennema et al., 1996; Jacobs et al.,

2010; van Es & Sherin, 2008). Within this research, these two sets of literature are combined and amalgamated in considering a focus on students' perspectives (including their mathematical thinking) as enhancing and deepening teachers' KCS and KCT through iterative cycles of lesson study.

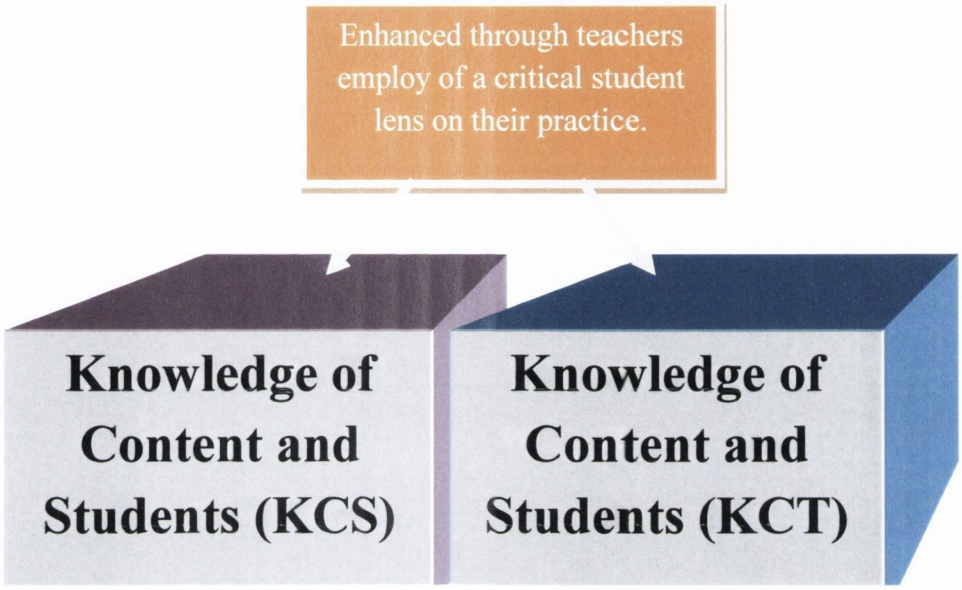


Figure 26 Enhancing KCS and KCT through employing a critical student lens

KCS brings together teachers' knowledge of content and how students might best engage in that content (Ball et al., 2008). Within this research, it emerged that in employing a critical student lens to their practice, teachers enhanced their KCS in the dimensions of: identifying students' prior knowledge, anticipating students' responses, and attending to students' mathematical thinking (Figure 27).

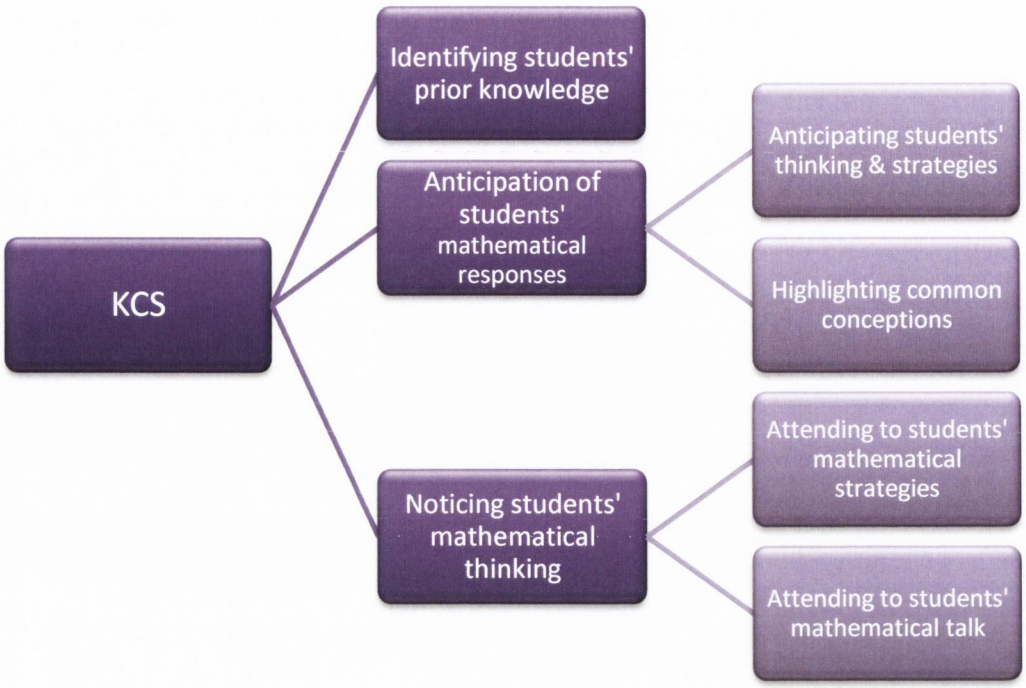


Figure 27 Features of KCS identified within the data

While there may be more dimensions to KCS, these three dimensions form the basis of the analysis of data generated through teachers in Crannóg and Doone participating in iterative cycles of lesson study.

KCT brings together teachers' knowledge of content and teaching (Ball et al., 2008) and within this research it is suggested that through employing a critical student lens on their practice, teachers enhanced and strengthened their knowledge of teaching particular mathematics topics. The dimensions of KCT which developed as teachers in Crannóg and Doone participated in iterative cycles of lesson study were proposed: sequencing of student learning trajectories, designing contextualised questions, and evaluating mathematical activities (Figure 28).

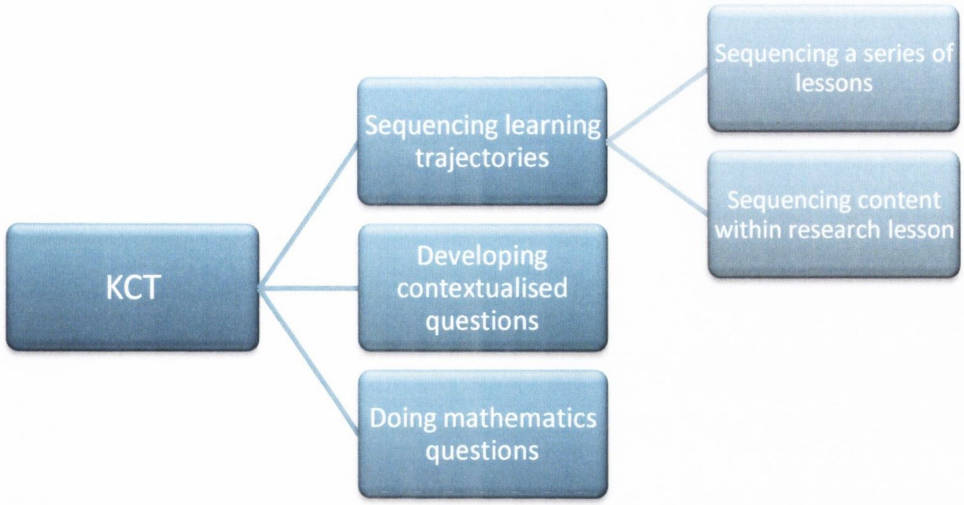


Figure 28 Features of KCT identified within the data

While there may be more dimensions to KCT, these three features form the basis of the analysis of data generated through teachers in these two schools participating in iterative cycles of lesson study.

The second contribution concerns the addition of another element of mathematical knowledge for teaching to be included with SMK and PCK, that of teacher knowledge of students, TKS (Figure 29). This additional element has emerged from the data within this research and is supported by Shulman's (1986) and Grossman's (1995) original hypothesis for teacher knowledge which both separately include a knowledge of the learner. It is suggested within this research that the teaching and learning of mathematics is further enhanced through teachers' knowledge of their own students in: constructing engaging mathematics activities and lessons that are relevant for these particular students, and providing students with opportunities that specifically engage them in communicating with their peers (Figure 30).

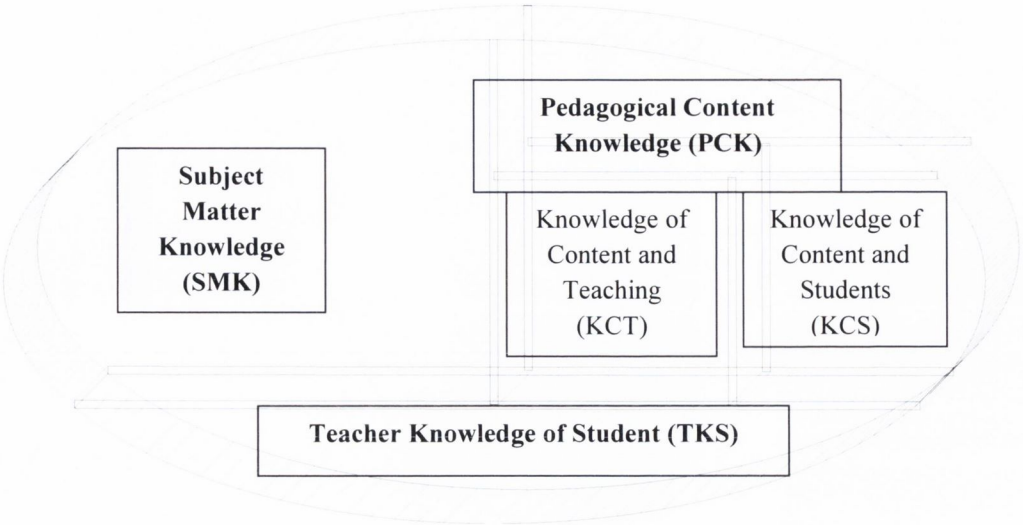


Figure 29 New model of MKT, adapted from Ball et al. (2008)

This contribution concerns a refinement and enhancement of a well-known model in the literature of mathematical knowledge for teaching (MKT) as presented by Ball et al. (2008).

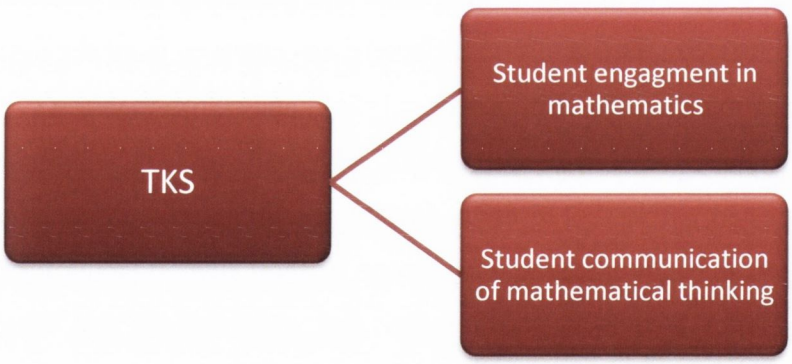


Figure 30 Features of TKS

These models shown in the figures above are based on the literature but the detail has emerged from the data generated in this research. I do not suggest that these models nor this framework are complete nor encompass all teacher knowledge, however, I do believe that within this research these frameworks access how teachers' pedagogical content knowledge was developed through their participation in iterative cycles of lesson study over the course of one academic year.

From the data generated in this research it is suggested that as teachers continued to participate in iterative cycles of lesson study, their knowledge was enhanced through the increase of KCS, KCT, and TKS in the previous lesson study cycle as represented by the figure below.

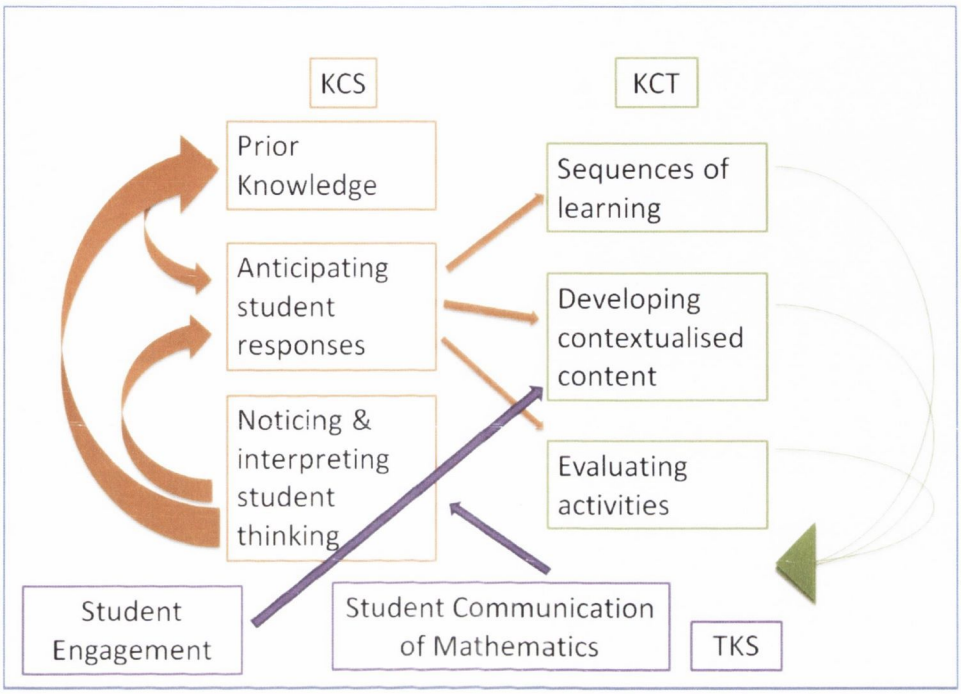


Figure 31 Suggested model of teacher learning in iterative cycles of lesson study

For example, as teachers began to notice more of students' strategies within a research lesson this influenced them in more frequently anticipating students' responses in the following research lesson. This in turn led to teachers developing more detailed sequences of learning etc. as suggested in the table above.

4.6 Conclusion of Chapter

Within this chapter, an analytical framework was constructed, detailing the elements of teacher knowledge impacted upon by teachers employing a critical student lens (Fernandez et al., 2003) to their teaching throughout their participation in lesson study cycles.

In this research, evidence is provided on a professional development programme (a model of lesson study) which can have a positive impact on teacher learning. A fine-grained detailing of elements of PCK, KCS and KCT, is proposed as knowledge which is utilised and enhanced by teachers in their participation in iterative cycles of lesson study.

Furthermore in this chapter a modified analytic framework based on Ball et al.'s (2008) model of MKT is proposed.

Employing a critical student lens builds on teachers' pedagogical content knowledge and refers to all aspects of teaching mathematics combining both KCT and KCS (Ball et al., 2008; Krauss et al., 2008). As part of a teacher's KCS, students' prior knowledge is identified and students' responses are anticipated in planning a lesson. In reflecting on a lesson, students' thinking and interactions are interpreted and reviewed and can be built upon in future teaching and learning opportunities. As part of a teachers' KCT, teachers construct sequences of learning where content is chosen according to developing students understanding. Teachers develop questions and activities that are contextualised and relevant to their students and attempt and evaluate mathematics questions in their planning which allows them to further tweak or modify questions and anticipate how students might engage with such mathematical activities. From the data, teachers incorporated their knowledge of students within their planning and reflection of students' mathematical thinking. It was particularly relevant for teachers to include how to engage students in their learning and to build on incorporating more communication of mathematical thinking within lessons.

As stated in the introduction, employing this analytical framework to the data generated in this research does not imply that teachers in this research were not focusing on their students or on their students' mathematical thinking. To the contrary many teachers participating in this research employed a well-developed critical student lens to all aspects of their practice. However, teachers who were not obviously doing so began to more frequently articulate aspects of their practices from students' perspectives and self-reported that they thought more about their mathematics teaching from how a student might construct their understanding. This, I believe, would not have occurred had teachers not participated in lesson study within their post-primary Mathematics Departments.

In the following chapters, the data will demonstrate how teachers deployed their knowledge within lesson study cycles, but also, more importantly for the purposes of this research, will demonstrate how teachers' knowledge was developed through engaging in conversations based within the model of lesson study. In the following chapter a chronological analysis of how teachers employed and developed their KCS through

participating in iterative cycles of lesson study will be investigated. Analysis of KCS is more extensive than those of KCT and TKS as is proportionately visible within the data generated as part of this research. In Chapter 6, an analysis of how teachers employed and developed their KCT through participating in iterative cycles of lesson study will also be investigated and that chapter will further include an investigation of how teachers' TKS was included within their practices.

Chapter Five

5 Analysing the Development of KCS through Iterative Cycles of Lesson Study

I think getting together like this formally is vital. We talk all the time about how we're teaching and what we're teaching now.

Dave – Final meeting in Crannóg

5.1 Introduction to Analysis Chapters 5 & 6

The purpose of the following two chapters is to employ the analytic framework, as described in the Chapter 4, to investigate the changes to teacher knowledge due to their participation in multiple lesson study cycles during one academic year on the basis of the data sets generated in this research. In this and the following chapter, the analytic frame will follow and explore in more detail changes to teachers' mathematical knowledge for teaching (MKT), specifically pedagogical content knowledge (KCS and KCT), and teacher knowledge of students (TKS) evidenced in the data. While the previous Chapter 4 explored a static framework, the following two Chapters 5 and 6 will employ this same framework to a process of changing teacher knowledge during the course of the research as a function of the change in teacher talk during their participation in iterative cycles of lesson study.

The analysis of the following two chapters is organised into two broad sections. The first is an analysis of changes to teachers' PCK, in particular their knowledge of content and students (KCS) and knowledge of content and teaching (KCT). The second smaller section is an analysis of TKS.

In this chapter, how teachers incorporated and enhanced their KCS through their participation in lesson study will be explored and this forms the main portion of analysis within this research. In Chapter 6, how teachers incorporated and enhanced their KCT through their participation in lesson study will be explored in a similar fashion to the analysis of KCS. Chapter 6 will also include an analysis of how TKS was employed and enhanced through teachers' participation in this research. These changes to knowledge

are based on teachers employing and developing a critical student lens on their practice (as explored in the previous chapter) during their participation in lesson study.

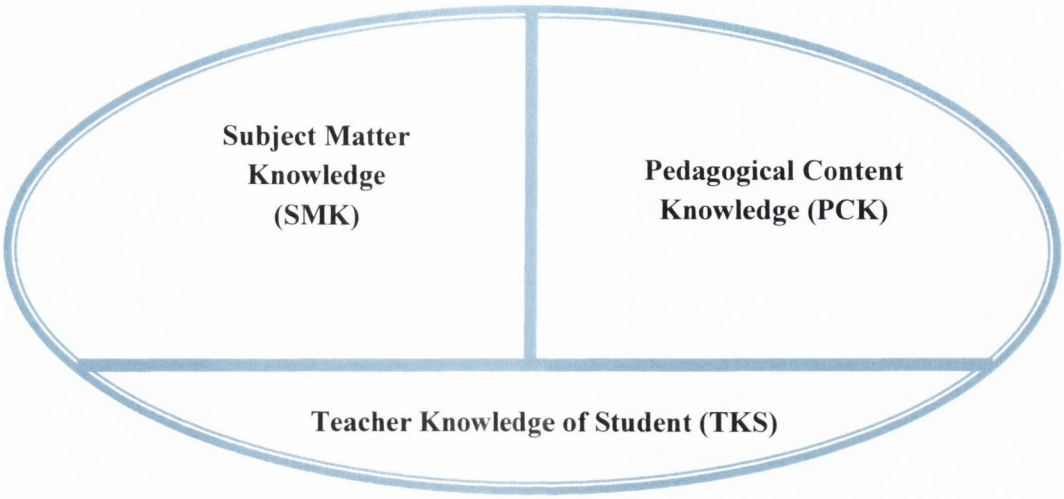


Figure 32 Model of Mathematical Knowledge of Teaching, Ball et al. (2008)

5.2 Chapter Structure

Within this chapter, changes to teachers' KCS will be investigated and the analytic framework will be applied to both Doone and Crannóg schools separately. Within this framework the analysis proceeds by way of a chronological analysis of teachers' conversations and interviews during the research and over iterative cycles of lesson study.

Within this analysis, examining what influenced teachers to make changes to their practices, through their developing knowledge, was critical in attempting to fully understand how we can utilise and improve this model in mathematics teachers' learning.

5.3 PCK: Knowledge of Content and Students

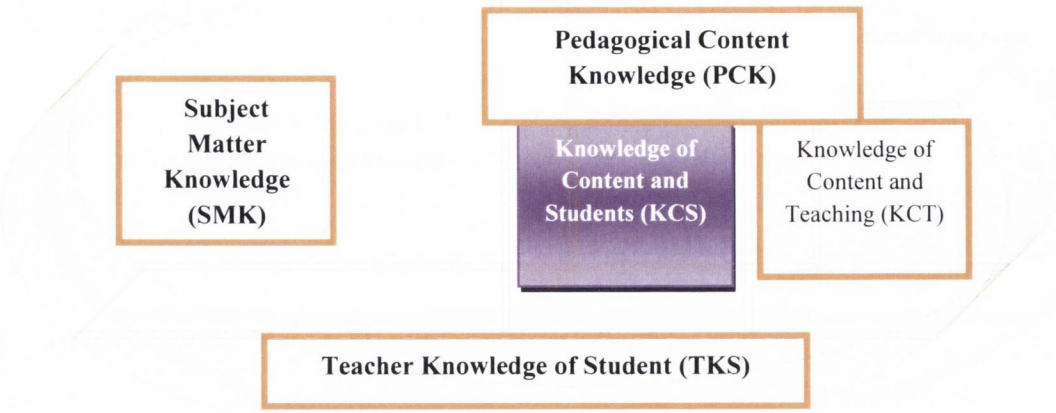
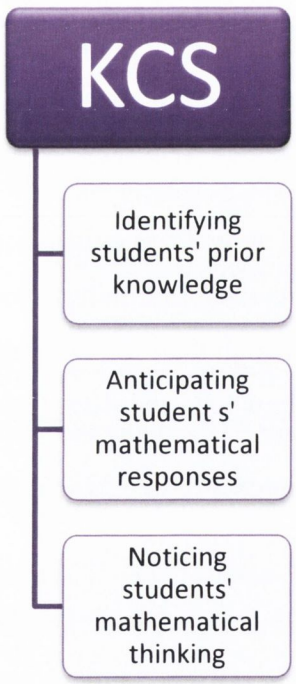


Figure 33 An Analytic focus: KCS

5.3.1 Applying the Analytic Framework: KCS

In the following sections, the development of teachers' KCS will be analysed under the following headings and according to the framework provided in Chapter 4:



- Prior Knowledge
- Anticipating students' mathematical responses
    - Anticipating students' mathematical thinking & strategies
    - Highlighting common conceptions
- Noticing students' mathematical thinking
    - Attending to student strategies
    - Attending to student talk

The following table outlines the frequencies of each knowledge code related to KCT occurring in different lesson study meetings throughout the data generation in this research.

Table 7 Frequency of KCS codes

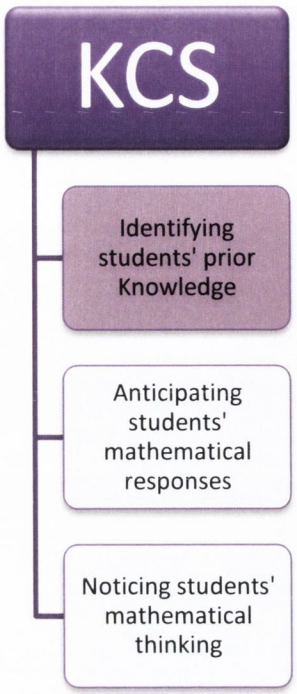
		Doone					Crannóg		
Cycles		1	2	3	4		1	2	3
KCS									
Prior knowledge	Identify	16	21	13	15		10	15	17
	Incorporate	4	6	6	15		10	7	14
Anticipate student math responses	Strategies	5	5	4	6		24	24	23
	Mathematical thinking	17	38	32	40		38	42	36
Highlighting common conceptions		4	6	1	7		1	27	1
Noticing students' thinking	Interpreting strategies	10	11	12	11		16	8	16
	Attending to student talk	4	8	6	7		10	7	12

As is discernible from the table, as the cycles continued both communities began to incorporate more of students' prior knowledge within their planning of research lessons and began to more frequently anticipate students' mathematical responses within research lessons. As teachers began to notice students' mathematical thinking within research lessons, this prompted teachers to anticipate more of students' thinking in subsequent research lessons. A narrative illustration of each of the codes will now be explored in the following sections.

5.3.2 Identifying Students' Prior knowledge

In this following section, as the first of three features of KCS the development of teacher knowledge in detailing and incorporating students' prior knowledge will be analysed from the data on teachers' participation in iterative lesson study cycles.

5.3.2.1 Doone



At the beginning of this research teachers in Doone identified what prior knowledge would be required of students for a lesson introducing square terms to students but did not include this information as part of the research lesson. In their second planning meeting Kate made reference to the fact that an understanding of area and perimeter would be required of students but this suggestion was not incorporated in the lesson plan. Michael also noted that indices would also be important to discuss within the lesson but this suggestion was not followed up. Teachers did not emphasise the merit of students revising and understanding these topics prior to the research lesson, believing that students may have met these topics in primary school. In planning the lesson, Nora also suggested that the idea of 'x' as a representative value or variable would be difficult for students to understand and put forward the idea of exploring this concept before the research lesson, but this was not included in the teaching of the research lesson. Teachers did not identify what these first year students already knew prior to the research and as a result, none of the pre-requisite topics were revised in advance of the research lesson.

It arose in the post-lesson discussion that during the research lesson, some students did mistake finding the area of a square with the perimeter and that students were unsure of why they were multiplying a length by itself in finding the area of the square. Nora noted that one student she observed had begun to confuse perimeter and area, but managed to correct himself before too long.

Nora *Lisa, I'm sorry, I have to tell you another very interesting observation. The little fella beside me, he was squaring them up to 5 and that was grand but when he got to 6, he started to multiply*

Lisa *By 2?*

Nora *By 4*

Aoibhinn *Why?*

Kate *By 4?*

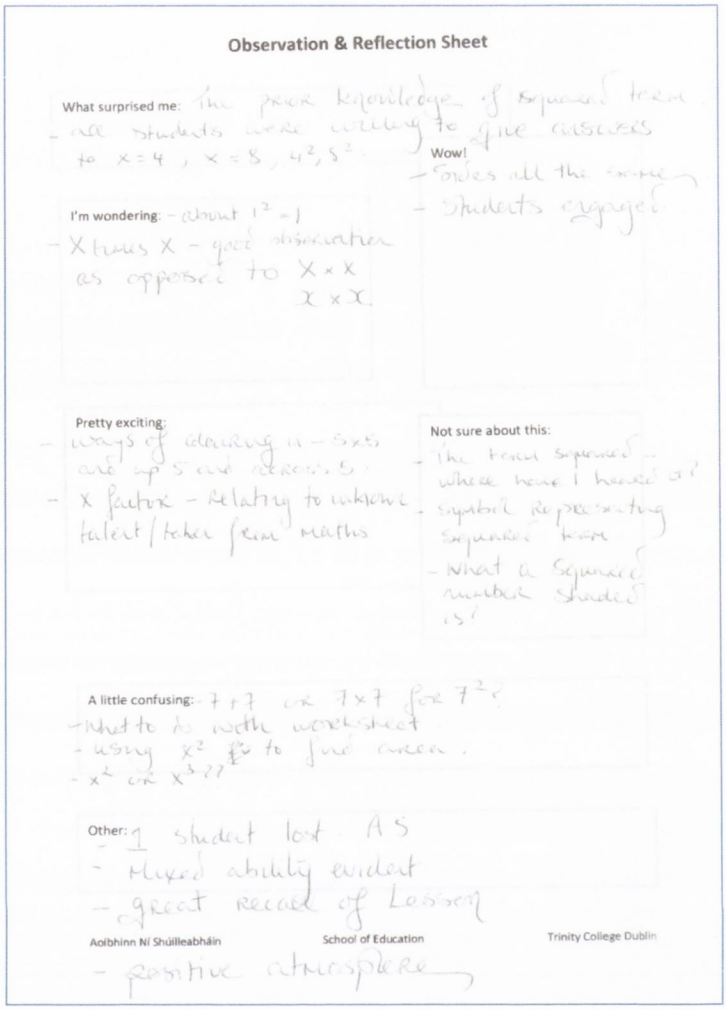
Nora *For perimeter! I was actually going to say that to you before class. I'd be surprised if somewhere along the line you're not going to get someone to get the perimeter from the area. It was funny though, when he started, it was very interesting... At a certain point he seemed to know...he seemed to realise he'd gone off the rails and he was getting them wrong and he went back to them up top when he was halfway through filling in the squares and he started correcting them.*

Nora had been absent in the two planning meetings prior to this lesson and had not had a chance to contribute her perspective that these students might confuse area and perimeter since they were 1st year students. We see that Nora was apologetic in reflecting on this incident to Lisa, who had been increasingly nervous about conducting this research lesson. None of the teachers, including myself, had realised that within the planning of the lesson we had neglected to factor in students' prior knowledge of area and perimeter in building their mathematical understanding of x^2.

While student confusion can often be positive in building students' mathematical thinking (Anghileri, 2006; Schoenfeld, 2011), not all students were sure enough in their understanding to re-route their thinking during the lesson. Neglecting to revise the concept of area also led to some students having difficulty in shading in the correct area of, for example, 4 x 4 on their graphing sheet – an activity designed to provide students with a visual representation of the squared term (Figure 36), as reflected on by Nora in the post-lesson discussion. Her observation of students' difficulty with this activity led her to again suggest that the concept of 'x' should have been introduced to students prior to this lesson and, she notes, had she been present at all of the planning meetings, she would have suggested this sooner.

Teachers began to see within the first post-lesson discussion that it might have been a good idea to revise students' prior knowledge before teaching the research lesson. Owen shared his reflection that he was surprised at how many students already knew what 'squared terms' were and yet he was surprised that so many of them were confused within the lesson noting "another one lost student" in his observation sheet (Figure 34).

This variance in students' understanding may have been avoided had teachers spent time assessing, reviewing and building on this particular class of students' prior knowledge. Teachers reflected on what they had observed and even though teachers had spent 5 meetings planning this lesson (within which Lisa had taught two mock lessons to her colleagues), they concluded that the lesson had not been a success.


Observation & Reflection Sheet

What surprised me:

Wow!

I'm wondering:

Pretty exciting:

Not sure about this:

A little confusing:

Other:

Aoibhinn Ní Shúilleabháin School of Education Trinity College Dublin

Figure 34 Observation notes: Doone cycle 1, Appendix F3 1

In their second research lesson, it was easier for teachers to detail what students had already done within a topic since Owen had been following the Common First Year Course suggested within the curriculum document (Project Maths Development Team, 2012). Teachers detailed what knowledge would be required of students in general in order to understand the value of the result of multiplying two fractions and then employed a critical student lens in ascertaining what this particular class of students already knew about fractions. Viewing prior mathematical knowledge from a students' perspective was an important difference to the previous lesson where teachers had effectively detailed the prior knowledge required of students but did not ascertain what that class group of students already knew nor incorporated it within the research lesson.

In planning research lesson 2, Nora highlighted her opinion that students should have met multiplication and division of whole numbers by fractions as part of their prior knowledge leading to this lesson dividing one fraction by another. The conducting teacher, Owen, shared with his colleagues the strategy he had employed in teaching this

element of the topic to these students in detailing that instead of multiplying a whole number by a fraction, he always converted the whole number to a fraction and then multiplied them. This information changed his colleagues' opinion of how they should conduct the lesson and Nora responded in suggesting they build the research lesson around this strategy:

> *Nora* *I think, from what you were saying...that you were using the technique of multiplying a fraction by a fraction, multiply the two top and multiply the two bottom. So, it wouldn't make any sense to start with multiplying a fraction by a whole number as a step towards multiplying a fraction by a fraction.*

In noting what strategy students were already familiar with, teachers altered their approach of incorporating the multiplication of fractions by whole numbers to one which built on these particular students' prior mathematical experiences.

The data shows that Nora, a veteran teacher, regularly employed a critical student lens in her practice and in this cycle she led the conversation in detailing and incorporating students' prior knowledge. In their post-lesson reflections, teachers acknowledged that they had identified reasonably correctly what students already knew and were able to build on that knowledge within the lesson. This marked a change in how they had approached identifying and incorporating students' prior knowledge when compared to the previous research lesson.

During the planning of research lesson 3, the conducting teacher Kate, noted that students were familiar with tabulating values and graphing quadratic expressions from their Junior Cycle syllabus. The teachers discussed this group of students' prior knowledge and incorporated it within their planning conversations. Teachers also detailed the skills which would be required of students in order to attempt to solve the previously unseen problems within the lesson. The articulation of these skills was guided by the 'problem solving worksheet' (Appendix F2 2) which Owen shared from the professional development course he was attending (Section 3.7.2):

> *Owen* *They will have done something in the algebra in 3rd year, wouldn't they?*
>
> *Kate* *Which?*

Owen	*They analyse information presented verbally and translate it to mathematical form. You know, the part c [referencing typical exam question construct].*
Kate	*Yeah, like, they'd be able to find the maximum.*
Owen	*They would have done, so they'd know some –*
Lisa	*Know the range of values. X is greater or equal to.*
Owen	*Yeah.*
Kate	*They'd know that, they'd be able to do that. But I think, yeah, it's those new questions like the massive paragraph, the scary looking formula, a blank page for a graph followed by 20 questions or, you know. They're so intimidating looking and they will never have seen anything like that before. They'll be able to get the function. I'd say the formula they'd be, like, but they'll type it into the calculator and they'll figure that out themselves. Drawing the graph will be fine.*
Lisa	*It's interpreting it.*
Kate	*it's the interpreting that we want to focus on isn't it?*
Aoibhinn	*Okay, yeah. And interpreting what kind of graph: quadratic or cubic?*
Kate	*I think just quadratic. Well, we could do a linear and then a quadratic and maybe a cubic as well...*
Lisa	*And the students working on it should be able to find first, second. Linear, quadratics.*

Following their detailing of students' prior knowledge and prior skills, teachers' positively responded in their reflection of the lesson. The more positive post-lesson discussion of this research lesson was likely due to the fact that they had spent much of the planning meeting anticipating students' responses and building on what knowledge they expected students' to have.

By research lesson four, teachers built on their lesson study experiences throughout the academic year and identified exactly what was required of students to know in order to understand and implement Pythagoras' theorem. In particular, the data from the lesson

planning meetings show that teachers attempted to make note of what this particular class of students had already experienced in terms of prior mathematical knowledge – employing a critical student lens in their planning of this research lesson.

Kate	*....when you're doing the example on the board you'll have to go through "the square root of." Would you have done it with them before?*
Lisa	*Yeah.*
Michael	*Square roots? No I wouldn't have done that.*

In identifying the prior knowledge required of students in this lesson, teachers decided, of their own accord, to not just plan one research lesson but rather to plan a sequence of lessons. Within this series of lessons, two lessons led to the main research lesson where teachers could then be certain of what students already knew (Appendix F1 1). Within these lessons teachers wanted to ensure that the class of students had a relational understanding (Skemp, 1976) of terms such as 'hypotenuse', 'right angled triangle' and 'theorem' before they met and began to apply Pythagoras' theorem.

Within this final lesson study cycle the majority of planning meetings focused on the mathematical content of the lesson and identified in detail the prior knowledge that would be required of students. Kate articulated the need for students to be familiar with surds and decimals in calculating the hypotenuse. In designing the questions for that lesson, Michael noted his expectation for students to be able to identify directions as being 90 degrees apart. Kate, Lisa and Michael discussed students having an understanding of squaring numbers and of area, but negotiated that this would not be included within this particular sequence of lessons. Nora, who was only present for the final planning meeting, flagged these topics as necessary prior knowledge but as a community of teachers they decided on this particular trajectory of learning that would not reference area and were confident in building on students' mathematical thinking in applying Pythagoras' theorem.

An important moment in highlighting students' prior knowledge was teachers' realisation that in finding the lengths of sides other than the hypotenuse, students would have to be able to manipulate equations. Michael and Kate discussed this matter and made a landmark decision to not teach all of this topic at once, but return to it next year. I

describe this decision as 'landmark' since teachers traditionally would have taught this entire topic as a whole, taking the textbook lead without initially spending time on developing students' knowledge and building students' understanding of what the hypotenuse represented. Instead, in this research teachers began to plan a vertical trajectory of learning for students, where they could gradually build on students' mathematical understanding at a future stage instead of trying to teach the entire topic at once at this point.

Kate	*Because the book, even here, there's three questions for hypotenuse.*
Michael	*Yeah.*
Kate	*And then it's straight in hypotenuse, other sides. They really mix it up quite quickly and this is Ordinary Level...*
Michael	*You're right and just as we say it again, going to that is to be done later on.*

Teachers realised that the textbook was not suited to their particular students since the content questions moved too quickly to a new section without building on students' mathematical thinking. Teachers agreed that in reflecting and building on this group of students' prior knowledge, it would be best to structure a cyclic curriculum to return to this topic at another stage and build further on students' understanding and following Kate's observation, teachers constructed their own questions to use within the lesson.

In their post-lesson discussion teachers reflected that all students were able to participate in the activities during the lesson, likely due in part to the fact that they had built on students' prior knowledge within the research lesson. The majority of this post-lesson discussion was devoted to discussing students' strategies and interpretations of questions, a major shift in the conversation from previous lesson study cycles, and was likely possible since teachers had purposefully constructed the lesson in building on this particular group of students' prior knowledge.

Through their participation in lesson study, teachers in Doone began to build students' prior knowledge into their teaching (Ball et al., 2008) and in doing so, incorporated the employ of a critical student lens in their practice in thinking about what students already knew, what they understood, and how this could be built upon (Carpenter et al., 1989; C.

Fernandez et al., 2003). As outlined in the analysis above, there was a progression of teachers detailing students' prior knowledge and incorporating and building on their knowledge before planning a research lesson. In cycle 1 teachers spent a cursory few minutes debating but not identifying students' prior knowledge and did not include a revision or building on prior knowledge within the research lesson. In contrast by cycle 4, teachers detailed the knowledge required of students and built a sequence of lessons around identifying and consolidating students' prior knowledge for this final research lesson. In incorporating prior knowledge into a research lesson, teachers came to plan a sequence of lessons, something which they greatly valued in their post-lesson discussion and a feature of KCT which will be later explored in the following chapter (see Section 6.3.1). In highlighting what prior knowledge was required of students in general and then identifying what students within the research lesson already knew, teachers enhanced their KCS in identifying how to develop students' mathematical thinking of different topics.

5.3.2.2 Crannóg

In Crannóg, detailing their students' prior knowledge was a very different activity for teachers compared to that in Doone. Teachers in this school taught the same topics to all year groups according to one time-line (which was flexible according to class groupings and examination levels). The head of department, Walter, also kept detailed records on what was being taught in each topic to each year group and had these details available electronically at all of the meetings.

In their planning meetings, from the very first research lesson, teachers explicitly detailed, discussed, and dissected the knowledge which would be required of students in order to build their understanding in a particular topic. Teachers were also able to call upon their records in order to detail what students in 2nd year had already experienced as prior knowledge of investigating quadratic patterns as initially included in Section 4.3.1.1.

Fiona — *Just, before we even do this, what have they done in algebra up until this? Or does it matter?*

Walter — *I would have it here [referencing year plan records]. They've done standard linear equations, they've done expressions.*

Fiona *So we would have done kind of, you know, work from a practical context and brought that into linear. We would have done simplifying expressions, so they have done a little bit already.*

Walter *Hmm mm.*

Fiona *Because if the variables aren't, things are just going to come out of the blue to them...*

Walter *They've done substitution, they've done -*

Fiona *They've come across the variables in linear patterns already*

Walter *They've done some multi-graphing. That sort of stuff.*

Fiona *Yeah, they would have done substitution.*

Walter *And they've done tables and graphs of linear patterns.*

Fiona *Yeah. So they've come across the idea of tabulating and trying to organise their information but in a much more structured way...*

Walter *And solving linear equations.*

Eileen *And did we do basic statistics with them? Did they, they would have come across tables and things like that.*

Walter *They'd have done tallying last year.*

For research lesson 1 teachers decided to revisit linear patterns for one lesson prior to the research lesson in order to remind students of the skills of tabulating, attempting to identify a pattern, and graphing which will be required of them in the research lesson. From the very beginning these teachers saw merit in planning a sequence of lessons, building on students' prior knowledge in order to develop students' mathematical thinking (see Section 6.3.1).

In their second research lesson, teachers decided to teach factorising of quadratic expressions using algebra tiles and array models, strategies that none of the teachers had previously taught. Because teachers were unfamiliar with these strategies, they identified the prior knowledge required of students by doing out a number of questions themselves within planning meetings using both strategies, a planning activity which led to teachers

developing their KCT as well as their KCS (see Section6.3.3). They decided to introduce area models to students using numbers initially to build on a concept of multiplication of factors and area.

Walter	*But in algetiles, if they are trying to learn how algetiles work, that confuses them.*
Fiona	*Yeah, it does, so does that make sense to kind of do an introduction to the grid...they have probably done some kind of numbered grid multiplication in primary school.*
Stephen	*They do it in first year actually. We do it with them, it comes up.*
Fiona	*So we could just give them a number, a simple one, one like that and then go onto n+2 by n+3 or something...*
Walter	*And then do multiplication with algebra tiles?*
Fiona	*Yeah, just to show that these are the factors.*

In that excerpt, teachers decided that in order to introduce the concept of using algebra tiles, students needed to first be familiar with an area model using numbers, then leading to an array model with algebraic expressions (Appendix F1 4). Teachers also determined that students needed to be familiar with the concept of factors and Fiona decided that she would introduce students to factorising by using an array model strategy, very different to her traditional strategy of teaching this topic by factorising using 'squiggles' (Section 4.4.3.1 or Appendix F1 5) or brackets. By detailing all of the prior knowledge and skills which students required for the lesson, teachers arrived at a plan which was ambitious for that particular class of students (which led to them re-teaching the lesson with pared back content), but did achieve in the objective of providing students with an alternative, more visual way of factorising quadratic expressions.

In the third research lesson, the data revealed that teachers incorporated the prior knowledge required of students in introducing calculus into a very detailed sequence of lessons. Teachers built on what they expected these students to already know in terms of recording, tallying, tabulating and graphing data into a 'Tractor Class' where students recorded and presented their own data and then analysed it in terms of finding instantaneous speed at a particular point in time. Teachers employed a critical student

lens in detailing these transition year students' prior knowledge. Similar to Doone where Nora made a point of identifying students' prior knowledge, it was the most experienced teacher Gerald who wanted to clarify what this particular group of students knew before planning a lesson which would build on their prior knowledge.

Gerald	*At this stage have they, have they got a notion that speed is represented by the slope of that curve?*
Fiona	*Yeah, they've done that with driving tractors.*
Stephen	*Driving tractors on the hockey pitches and stuff.*
Gerald	*Excellent.*

Gerald had been absent from the previous planning meetings and wanted to clarify with his colleagues that this group of students' prior knowledge had been ascertained and built upon for the research lesson. Teachers then built on students' prior knowledge in order to lead them towards identifying an instantaneous speed for an object that was, in fact, accelerating. Building on students' prior knowledge provided teachers with opportunity to plan for a number of lessons which would gradually build on students' mathematical thinking. Teachers also enhanced their KCS in incorporating students' prior knowledge into an entirely new approach to teaching this topic.

Because teachers in Crannóg were, from the beginning, accomplished in detailing students' prior knowledge as a community of teachers it allowed the community to plan for inquiry-based, student-centred lessons and allowed the teachers to attend to students' mathematical strategies during research lessons as will be discussed further in this chapter (Section 5.5.4.1).

5.3.2.3 Conclusion of Incorporating Students' Prior Knowledge

In summary, the analysis of the data in the light of this theme seems to demonstrate that as teachers participated in iterative lesson study cycles throughout the year, the importance of incorporating what students already knew within the research lesson became more apparent to teachers. In Doone, while teachers began by discussing (but not incorporating) the general prior knowledge required for a particular lesson, as the lesson study cycles continued they began including what students actually knew within their lesson plans. The articulation and inclusion of students' prior knowledge in planning

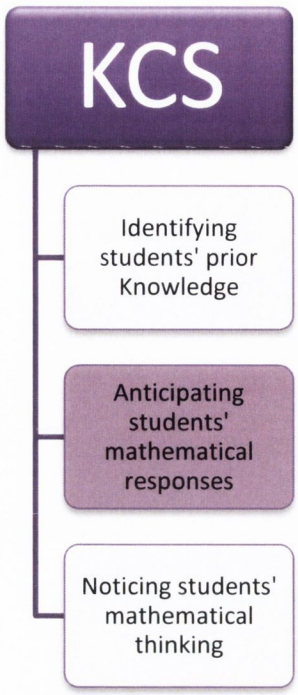
meetings led to teachers focusing on students' mathematical thinking and teachers began to explicitly build on how students understood a particular topic as they began to contain prior knowledge within their discussions. In Crannóg, teachers were well equipped in building students' prior knowledge within research lessons since all students had experienced the same mathematics topics within each year group and it was straight forward for teachers to track what topics had been taught according to the department's year plans. Teachers rarely required these year plans however in discussing what topics students had already met and what would be further required of them since the majority of teachers had a familiarity of year plans and content from due to their regular Mathematics Department meetings.

Not all teachers enhanced their KCS in incorporating students' prior knowledge since it was a regular feature of Nora, Fiona, and Gerard's practices. But other teachers such as Eileen, Lisa, and Owen, who had not highlighted students' prior knowledge in early lesson study discussions, later participated in identifying, highlighting, and incorporating what students already knew.

Within the template provided to teachers at the beginning of this research, teachers were unfortunately not asked to outline what was required of students to know in order to build their mathematical understanding of a topic which was the focus of a research lesson (Appendix C1). This is something which could be remedied within future templates provided to teachers where students' prior knowledge should be outlined and explicitly incorporated within the lesson plan. Nonetheless, teachers came to see the relevance of including students' prior knowledge within the lesson plans as in Doone's final research lesson and Crannóg's research lessons where building on students' prior knowledge and experience was incorporated within a planned sequence of lessons.

5.3.3 Anticipating Students' Mathematical Responses

The second theme of the analysis in this chapter is that of anticipating student responses, as detailed as a feature of KCS in Chapter 4.



As part of the research design, the lesson study templates (Appendix C1) were provided to teachers which explicitly highlighted anticipating students' responses within the lesson plan (see Figure 35).

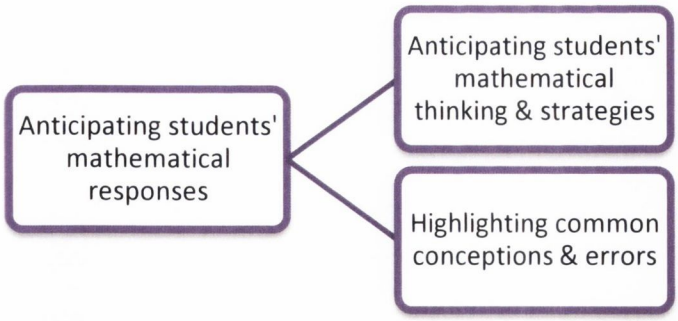
In both schools at different stages during the research, individual teachers stated that it was difficult to anticipate what students were going to suggest. Teachers in Doone improved in their ability and frequency of anticipating what students were going to do within the research lessons as their participation in lesson study progressed. Throughout all of their lesson study cycles, teachers in Crannóg were attentive to anticipating how students would react with and engage with mathematical activities and although Fiona had described this planning activity as difficult, she was the one who most frequently anticipated how students would interact with their mathematics throughout all of the lesson study cycles.

Approximate Time Guide	Student Learning Activities	Anticipated Student Responses	Anticipated Teacher Responses	Points to notice & evaluate

Figure 35 Initial Lesson Design in Teacher Information Booklet, Appendix C1

The analysis of the data reveals that it is possible to distinguish two characteristics of anticipating student responses that have not yet been treated in any depth with reference to the literature on lesson study. These are discussed under the following headings as outlined in Chapter 4:

- Anticipating students' mathematical thinking and strategies (C. Fernandez et al., 2003; Schoenfeld, 2011)
- Highlighting common conceptions (Ball et al., 2008; H. C. Hill et al., 2008)



Within this section the data on teachers' anticipation of students' mathematical thinking and mathematical strategies will be analysed over the course of their participation in the research as a subtheme of anticipating students' mathematical responses. In Crannóg teachers regularly incorporated anticipation of students' mathematical thinking and mathematical strategies within their planning and thus this sub-section will be deconstructed into these two separate elements of mathematical thinking and mathematical strategies for this community of teachers. In Doone, as there were no initial significant moments of anticipating students' mathematical thinking or mathematical strategies, these will be investigated as one theme.

5.3.3.1 Doone

In Doone's initial research lesson teachers did not spend time discussing how their students would think about mathematics or the strategies they would employ within the research lesson. Within the conversations teachers referred to what students' commonly found difficult about squaring numbers but did not balance these conversations with how they expected students to think about or respond to the specific mathematical activities within the research lesson. Occasionally teachers did begin to think how students might answer particular questions, but were vague in their articulation of how they anticipated building on students' mathematical thinking.

Michael *You're going to be saying to them..."Why is the room square?" And they're going to say, "well, because the length and the width are the same thing."*

Lisa *Exactly.*

Michael *You're going to say that to them and they're going to say "Well, why is it square, why?" We know it's squares. We know they're the same size. So why do you say four? That where's they're going to guess.*

Kate *Yeah -*

Lisa *Okay, so what does it mean to square a number? It's where the length and the width are the same measurement.*

In this excerpt Michael attempted to anticipate how students might think about the features of the square room and also attempted to highlight why this was important but his articulation was unclear. His anticipation of teachers' questions and students' responses were intermingled. His colleagues attempted to join him in this conversation and Lisa tried to summarise his point but in planning this element of the lesson there was no definitive content discussed and no clear plan as to how it could build students' thinking about the model of a square and the concept of x^2. At this early stage of participating in lesson study, teachers were not yet familiar in conversing with one another about the teaching and learning of mathematics and were not yet familiar with articulating theirs or their students' mathematical thinking.

In research lesson 1, teachers anticipated elements of the lesson which would confuse students such as incorporating units within their answers, and made reference to what language students might use in identifying sides of squares, but did not anticipate how students would respond to the mathematical content within the lesson. Much of the discussions in the planning of research lesson 1 revolved around how the white board would look, what would be included in the PowerPoint presentation, what language the conducting teacher should utilise, and what activities should be included to keep students 'busy'. Even when Lisa taught two mock lessons where the teachers had acted as students, the focus remained on the mechanics of the lesson. During the first mock lesson Owen drew teachers' attention to how students might go about filling in their square grid and suggested that reflecting on this strategy with students, as a class discussion, could be a useful exercise but this was the only occasion in the four planning meetings where teachers detailed a student's possible mathematical strategy.

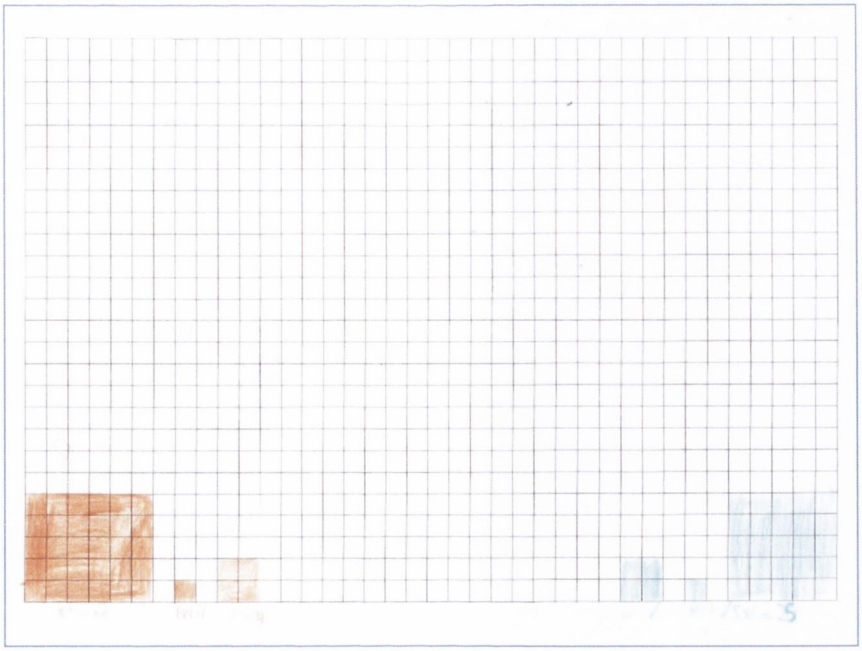


Figure 36 Sample of student work: Doone cycle 1, Appendix F4 1

The teachers in Doone took a very traditional approach with this lesson in 'giving' students the idea of x^2, and taking this approach negated the need to anticipate students' thinking or strategies since the lesson would ultimately be led and focused on the teacher. Anticipating students' mathematical thinking was not required within the traditional, didactic style of teaching (Oldham, 2001) where students worked individually on questions, where there was no focus on communication of mathematical thinking, and where the focus of the lesson was on the conducting teacher.

Teachers' way of thinking about planning a research lesson changed in the second cycle. It was particularly noticeable from the transcript of teachers' conversations in the second planning meeting that teachers began focusing on how students within the lesson might think mathematically about multiplying fractions together. From reading an article from 'Mathematics Teaching in the Middle Schools' (Johanning, 2011), which I had brought into the planning meeting as the facilitator of lesson study (Lewis & Hurd, 2011; Tepylo & Moss, 2011), Kate began thinking about the different strategies of solving a question on fractions within the article. This was the first time Kate tried to solve something from a students' perspective within lesson study.

Kate *I was just thinking of different ways of doing it, this, the way the kids did it.*

Nora also enjoyed reading literature on multiplying fractions since it detailed how the topic could be taught but also outlined students' responses to questions. Reading the research literature also led Owen to think about how his own class would react to a question he had read about. Teachers took 10 minutes within the planning meeting to read various articles and then reflected on what they had read with one another. Michael had been particularly interested in one article which had corroborated his anticipation of student thinking in the previous planning meeting. They each shared observations from what they had read in the research literature and Michael read an excerpt to his colleagues.

Michael *I really kind of decided to look at what the issue was, em, "many students and adults think that these situations do not involve multiplication but rather division or subtraction because the resulting quantity is smaller than the quantity named in the problem. They come to believe the problem that asks them to find a fraction of a quantity are solved by division. For students to understand that multiplication can solve problems with fraction multipliers requires them to extend and reconceptualise their understanding of multiplication to include multiplication by fraction multipliers." So I suppose yeah, that really kind of is, when I just read that I said God yeah, that really is what it's all about.*

Kate *It's, yeah.*

Lisa *Yeah. It's good. Yeah, yeah.*

Michael *It's a total mind-shift.*

Nora *It is yeah....All my experience would say that, I would subscribe to that.*

Lisa *So that could be the learning outcome of the lesson.*

Michael *You've already touched on this here and we discussed this last week [reading from notes] 'Multiplication by a number less than one, there is also a product that is less than the original number'*

Accessing the research literature seemed to give these teachers a new impetus to plan a mathematics lesson, and in planning this lesson teachers were far more student-focused with conversations revolving around "them" (the students). Michael's realisation led to an activity being included within the lesson which teachers hoped would build on students' understanding of multiplication of fractions as not always leading to a larger number. Teachers anticipated what students would find difficult and constructed an activity around that basis.

Lisa *So, to start off, we multiply something by one – nothing happens. But what happens if we multiply by a fraction greater than one? And what happens if we multiply by a fraction less than one.*

Michael *That's really what we're trying to convey...*

Lisa *Instead of just giving them something as complicated as that table, to use this as a kind of an introduction to estimation "What happens if I multiply a number greater than one?" So just get them to say "will the number get bigger or smaller?" So get them to estimate... so then they're forming an impression.*

Lisa focused on the student and while the main planning conversation still revolved around constructing activities for the lesson, she began to think about students' mathematical thinking in getting students to "*form an impression*" of the different outcomes which were possible in the multiplication activities. Teachers were beginning to pre-empt students' strategies in how they might read and interpret particular questions and were more student-focused in designing and choosing activities to be included within the lesson.

In their third planning meeting, teachers began to construct an activity within the lesson by pre-empting how students would think about a question. From an exercise in multiplying a number by a fraction less than 1, teachers extended the activity to first look at multiplying whole numbers and then lead to multiplying by fractions in order to build on students' mathematical thinking.

Kate	*We do 3 or 4 of these where the number is getting smaller because it's multiplied by a fraction....And then we do a few where it's multiplied by whole numbers and it's getting bigger.*
Aoibhinn	*I would just say if we're going to do a ½ by a ½, would it be nice to start with a whole number first?*
Kate	*Start with a whole number, yeah.*
Owen	*Yeah, because it's getting smaller, you can show that there.*
Kate	*Like a ½ of a 100.*
Owen	*Yeah so we could do half of 100 is 50, half of 50 is 25, half of 25 is 12.5. And then you could say to them at that point "What have we noticed?"*
Kate	*Well, we could have a little box you know, like answer is and then something than the original. And they have to fill in bigger or smaller. You could even have greater than or less than, but they can fill in the blank then for each of the sums. ...And then we do the other way around, multiply by a number bigger than one and the answer gets bigger. And we kind of build to this so they're coming up with this themselves.*
Owen	*Good, that sounds great. That's not complicated.*

While teachers were not necessarily anticipating students' strategies, they were beginning to build the lesson from the perspective of the student, constructing activities by thinking about the mathematics from a students' point of view.

Following the success of students working on a question in groups in research lesson 2, teachers wanted to build more student group-work and inquiry into the third research lesson as will be explored further in Section 6.5.2. Again it was Nora who initially focused on how students might think mathematically about a question (Figure 37). Nora incorporated students' prior knowledge of graphing linear expressions into her planning and guided the community to structure the lesson in a way which would lead students towards identifying the associated graph as quadratic, anticipating how students would interpret this new form of question.

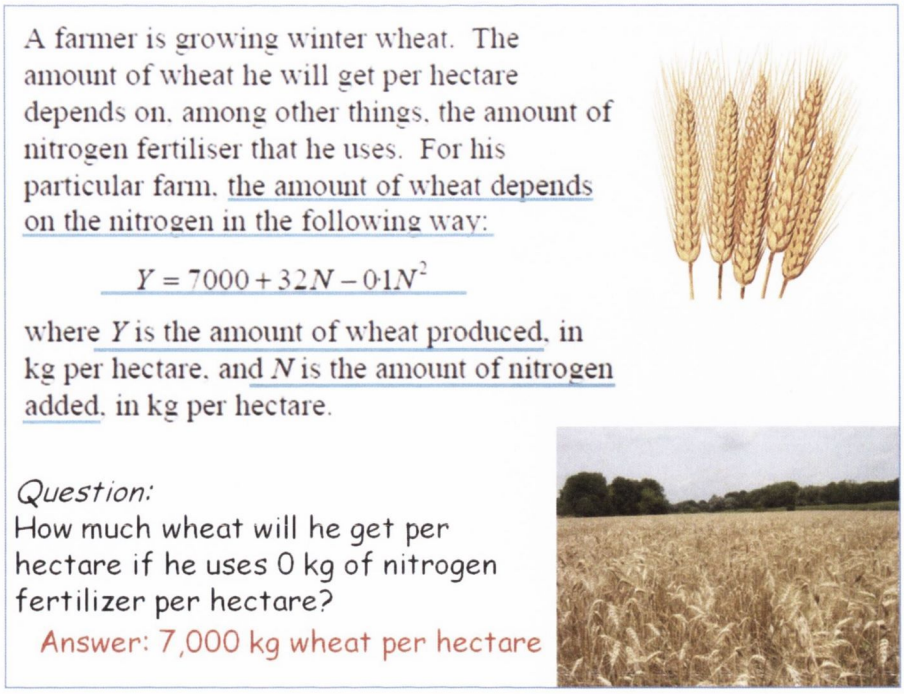


Figure 37 Question presented to students to begin research lesson 3

Nora *[Referencing the question] I would ask 'how much wheat would he have if he uses 200kg of nitrogen? Will he get a better yield?... Will you get a poor yield if you, if he uses less?' Just even to get...them to see a pattern... I think you must maybe somehow highlight the fact that it's going to be a curved thing. It's not going to be a direct relationship like the linear one you've just done.*

Nora wanted to engage the students in a discussion around the question in order to develop students' understanding of the question and to note the differences between the linear graph students were familiar with and the new quadratic graph, building on their prior knowledge. Nora anticipated students' thinking in attempting to build a picture for them of a graph that was curved instead of the linear graphs they had already met. Following Nora's advice from the first planning meeting in this cycle, Kate modified the original wheat question which Nora had brought to the lesson to guide students into choosing variables within the questions. Kate also included scaffolding questions (Anghileri, 2006) such as 'How much wheat will he get if he uses zero nitrogen' since she anticipated that students would not identify a pattern within the values without some guidance.

In another question to be investigated by the students in their groups, it was also Nora who anticipated a difficulty for students in deciphering between the variables of

temperature and time within a question and teachers decide to change the representative variables for the purposes of that question (Figure 38).

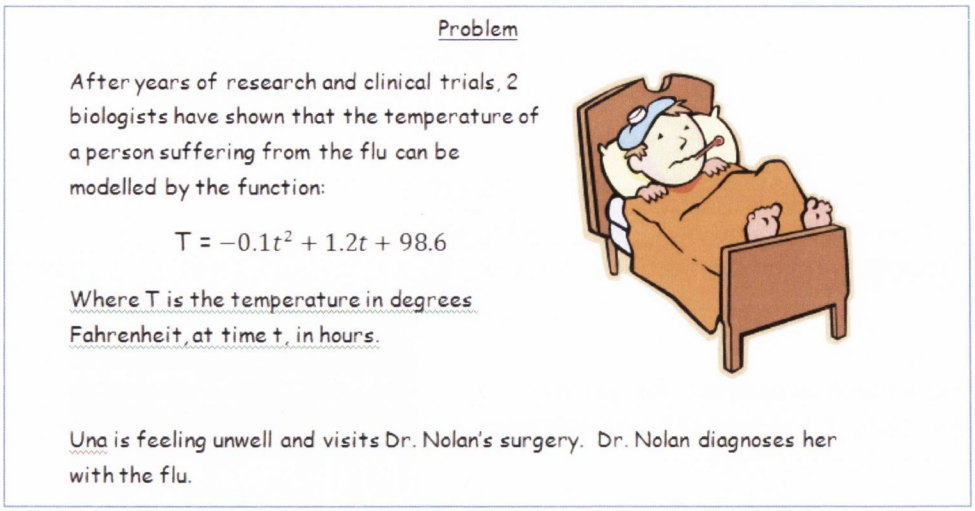


Figure 38 Temperature question: Doone cycle 3, Appendix F2 3

Nora *Would they have a problem with that 't' 'T'?*

Kate *'t' 'T'*

Nora *'t' and 'T'. I hate that*

Kate *Can we, I hate the same letters, me too, yeah. Temperature and time is it?*

Teachers decided on using different variables in this first example and also changed the variables in some of the other questions in order to avoid any unnecessary mathematical confusion.

In planning this lesson, other teachers apart from Nora began to anticipate students' mathematical strategies and thinking. Kate realised that students would choose a strategy of graphing (since that would have been in their previous activity) in order to answer the rugby question (Appendix F2 3) and decided to include a differentiated final task where students were encouraged to utilise algebra at the end of the question. As Lisa was filling out the lesson plan template, she asked her colleagues to answer the questions on the problem solving resource sheet (Appendix F2 2) which they did employing a critical student lens instead of listing what they wanted students to do as they had in planning previous lessons.

Lisa	*Now then, there's a whole lot of anticipated student responses and anticipated teacher response [referencing lesson plan template, Appendix C1]. So what are you actually being asked to do or find here? Student responses?*
Kate	*They'll probably just read out the question that's on the board. They'll read it out word-for-word. "How much wheat will we get if he uses 100kg of nitrogen?."..*
Aoibhinn	*But some would, do you think someone will go "just 100."*
Kate	*Ask to find a Y, given that N is 100. Probably.*
Nora	*I would ask how much would he have if he uses 200? Will he get a better yield if he uses 200 thing per thing? Which you could do.... Will you get a poor yield if he uses less? Just even to get 50 and 200 maybe?*
Kate	*Yeah, we could even then –*
Nora	*Them to see a pattern you know....*
Kate	*One of the questions we could pop up is... "What is the maximum possible amount you can ever get? The maximum yield?*
Nora	*They probably will estimate "What did I do? It was 160 so it probably would be – ."..*
Kate	*Maybe I'm being highly optimistic but I think someone's going to say when they see the function they'll say "Oh, that's a frown, that's curved, it's a parabola."*

While Lisa did not feed these anticipated responses into the lesson plan, Nora and Kate discussed how they expected students to think about and respond to the question. Articulating anticipated student responses assisted teachers in further developing the questions to ask within the activity which would build on students' understanding and focus them on what the graph might look like.

Unfortunately, the teachers only gave themselves two meetings in which to plan this lesson, which they later regretted. Also, although teachers had set out to build in more student inquiry within the research lesson, some teachers (such as Lisa and Nora)

anticipated that students would find it too daunting to work on an unseen problem without any guidance. Teachers moved from their original inquiry-based lesson and decided to talk through one example, highlighting important terms and beginning to tabulate results, reverting to a more traditional exposition than they had planned. If teachers had spent more time planning this lesson there may have been opportunity to discuss a less traditional approach and teachers would have had opportunity to anticipate how to develop students' mathematical thinking in a way that did not require teacher show and tell.

In research lesson 4 there was a dramatic change in how teachers anticipated students' mathematical thinking. From the transcripts of teachers' planning conversations, it is evident that this lesson was concentrated on the student, on building students' mathematical knowledge, identifying students' prior knowledge, and on anticipating how students might think about, engage with and do the mathematics questions contained within the sequence of lessons which teachers planned around Pythagoras' theorem. During the planning of this research lesson teachers attended to the more macro mathematics issues, for example all of the teachers discussed how questions would be ordered for students to build a better understanding. Teachers also attended to the more micro mathematics issues, for example Kate suggested that they avoid using the numbers 3, 4, 5 in an introductory activity since students might think they are just consecutive numbers. Through their experiences of planning, observing and reflecting on the previous research lessons, teachers had built on their KCS and had begun attending to how students thought about the mathematics instead of how they taught the mathematics.

It was within the planning of this lesson that there was a clear moment of realisation for Lisa that she had never thought about Pythagoras' theorem from the point of view of a student before. She discussed how her students constantly made mistakes in identifying the hypotenuse when they were given questions that were slightly different to those they had seen in exposition, particularly if the sides were labelled differently. It is during this planning conversation that she realised she had not previously thought about how a student might mathematically interpret these questions.

Lisa *The other thing about learning for the students is they can learn the theorem but then it is confusion when the diagrams are labelled in any given way...*

Owen	*They can learn it, but –*
Lisa	*We think it's saying* $a^2 + b^2 = c^2$ *but it's meaningless to them when you give them a thing and a is the hypotenuse and then you go,* $a^2 + b^2 = c^2$.
Owen	*Yeah.*
Lisa	*They don't actually understand.*
Owen	*So that's rote learning.*
Kate	*Yeah concept rather than formula.*
Michael	*That's what I was saying about your man in* 6^{th} *year, he said it to me and he just rattled it off. I asked him "what does it mean?" – it didn't mean anything.*
Kate	*Yeah, exactly. We want them to know the English sentence in their own words.*
Lisa	*....they know the formula but they can't conceptualise it. We know that this is the formula but we don't look at it from the kids -*
Owen	*The concept around the formula rather than the formula.*
Lisa	*And it's only that you, talking about that today, if they label the hypotenuse A, I hadn't actually realised that that is what's causing the problem. They don't, I keep... We, I think we always teach by formula.*

From listening to her colleagues discuss this lesson, from focusing on building students' understanding of this topic and deconstructing all of the content to be contained in the lesson, Lisa realised she had not before employed a critical student lens in teaching this topic. She had not previously realised that when triangles were labelled with sides 'a', 'b', and 'c' that differed from 'c' being the hypotenuse, that students could not decipher that difference in the diagrams. For the first time she understood that her students mislabelled sides because they only knew the formula but not what each of the letters represented in terms of the right angled triangle.

Lisa's realisation of this equation from a student perspective was a very important learning moment in her participation in lessons study. She had not before vocalised a mathematics idea from the point of view of a student within lesson study and it was in planning this final research lesson that she suddenly had a moment of realisation about her teaching. It is possible that she had never thought about this mathematical content from the perspective of a student in her class and, although she was a very enthusiastic and organised teacher, this focus on student mathematical thinking constituted a learning moment for Lisa and an enhancement of her KCS.

Throughout the planning of this lesson Michael articulated exactly how he felt the content should be broken down in order to build students' mathematical thinking. For example, he clarified the layout of the equations in order to maintain coherence with what his students have already seen. Both he and Kate discussed the merits of keeping the unknown value on the left or right hand side of the equation. Lisa also advocated the use of angleg resources in class for students as a visualisation tool for students. It is interesting that in introducing the angleg resources Lisa, employing her new-found student lens, argued with Kate over the merit of the resource for students that would find the concept of measure more challenging.

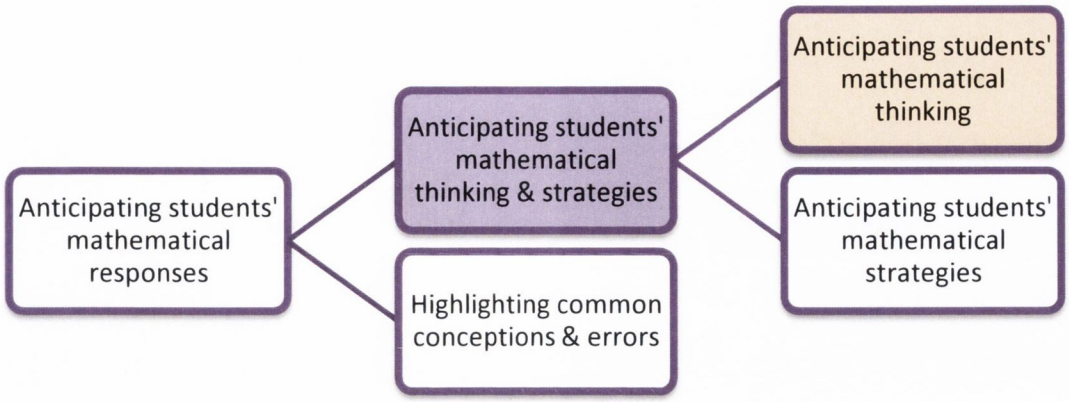
In this lesson, teachers sequenced the activities within the lesson in order to build on students' understanding and modified curriculum resources available to them in order to make them suit this particular group of students. Teachers were not only very student-focused in their planning but in this final research cycle also focused on how students would interpret the mathematics and build on their understanding.

Due to teachers' participation in lesson study cycles, within the same community of colleagues and over the duration of one academic year, teachers became more aware of students' mathematical thinking and began to better anticipate and incorporate students' mathematical understanding and strategies within their research lessons.

5.3.3.2 Crannóg

The subtheme of anticipating students' mathematical thinking and strategies (under the theme of anticipating student responses) is now investigated for Crannóg. Within the data there were far more examples of teachers anticipating students' mathematical thinking and students' mathematical strategies throughout their participation in lesson study and hence this sub-theme is broken into two separate categories:

1. Anticipating students' mathematical thinking
2. Anticipating students' mathematical strategies



This fine-graining of the features of KCS employed within teachers' planning and reflection conversations is an original contribution of this research.

A. Anticipating Students' Mathematical Thinking

In Crannóg, teachers began their participation in lesson study by planning research lessons through anticipating students' mathematical thinking. In research lesson 1, teachers interrogated how students might think about their first meeting of a quadratic pattern and discussed all of the various different options they had in investigating patterns of Lego bricks built as square pyramids. Teachers wanted to choose the best presentation of the content that would give students a solid understanding of why a pattern was quadratic and not linear. The focus of planning conversations was on how students would think about and interpret the topic and teachers regularly referenced "*them*" (the students), "*you*" (as the student), or "*we*" (the students and teacher) in anticipating how they might interpret the mathematics. In the second planning meeting Walter wondered aloud how they might encourage students to investigate the quadratic pattern instead of a linear one.

Walter — *There must be some way. I mean, it's still the same question of suddenly "why is there an x-squared?" I'm just trying to get them to arrive at that...*

Dave — *Isn't it because the variable has to multiply itself, which you haven't met at linear. There's a constant multiplying the variable, for the first time we're meeting a variable multiplying itself.*

Walter *Okay, so you can model that. You can see that in the numbers that the number multiplies itself can you? Can you break that down? Can you, within your table and patterns of numbers, can you see that one number is multiplied by itself?*

Both Walter and Dave discussed the mathematics from the perspective of a student and were trying to anticipate whether the student would be able to identify that the pattern now contained a second common difference instead of only one common difference. Teachers employed a critical student lens in attempting to anticipate how students in 2nd year would react to this new topic.

This approach to planning continued in the next two research lessons where teachers were at pains to anticipate how students might think about algebra tiles and array models, since few of them had previously taught using these resources. Some teachers did wonder if using tiles would in fact confuse students even more since they would also have to learn the rules associated with the tiles and approached this lesson as research into investigating these strategies. Stephen and Walter taught sketch lessons (lessons on the same topic feeding into the planning of the research lesson), using both tiles and arrays in order to input to the planning meetings how students were engaging with these new approaches. These sketch lessons proved beneficial in assisting teachers to anticipate students' thinking within the research lesson. Later during the planning of research lesson 2, Stephen suggested that it might be beneficial for students to revise area models prior to using tiles in order to build on their mathematical thinking for this topic. As Stephen was explaining his idea of developing students' understanding, he and his colleagues anticipated how students might interpret factorising quadratic expressions using algebra tiles employing a critical student lens in, at times, seeing themselves as the student.

Stephen *I was thinking it would be quite nice if you have that $x+1$ and $x+3$ that we had earlier. I was wondering how much, so if you fill that in, how much of a benefit; and I think it might be a big benefit if you just start adding, so if we have that one, if you add a 1 onto one of the other sides*

Walter *So, just make it bigger then?*

Stephen *so we have that done and then add a 1 and see what happens*

Walter *In the middle bit, or -*

Stephen *Just onto one of the sides.*

Dave *So you are moving to x+4.*

Stephen *So moving to x+4 ...so actually I think it might be of help seeing how it happens as you add from that very base where you start off with x+1...But also the factorizing way as well. Looking at the numbers at the end, how they, the relationship, I am just wondering if they can see that relationship, so you have your 4 and your 2 and looking at that relationship with the numbers at the end and how effective that is. So if you actually looked at it and actually built up a pattern of, if everybody had that in front of them -*

Aoibhinn *How do you mean? Sorry Stephen.*

Dave *Are you looking at this 8 here and saying it is 4 by 2*

Stephen *Yeah.*

Dave *And a minute ago it was 3 by 1 and it gave us our 3, and now 4 by 2 and it is giving us 8; our quadratic has changed and they can see that it is all to do, as this number changes, directly affected by this and this...That is your pattern then...*

Walter *And finish out a table almost, a whole table of loads of multiplications...*

Dave *So this is $x^2 + 6x + 8$, em, what are its factors? Well I just need to come up here and put those there and those there, and I read those off and they are the factors. So I can factorize.*

Here Dave interpreted Stephen's idea of how to develop students' understanding and as a group of teachers they anticipated their students' mathematical thinking and developed a plan for the lesson, enhancing their knowledge of teaching this particular topic using the strategy of array models and algebra tiles. Through planning this lesson these teachers regularly employed a critical student lens in unpicking the mathematical activities in this lesson.

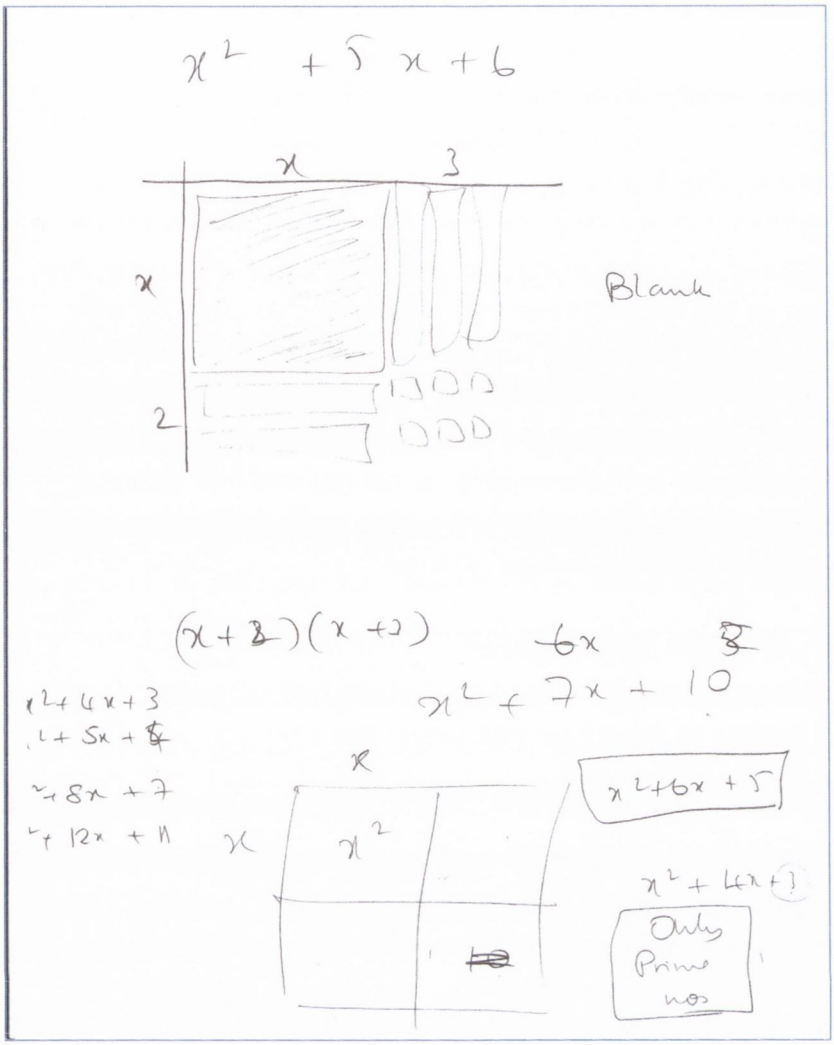


Figure 39 Planning notes: Crannóg cycle 2, Appendix F1 4

Fiona, employing a critical student lens, later asked why students would place the x^2 value in the top left corner. In doing the activity herself (explored further in Section 6.3.3) as demonstrated by Figure 39 and in thinking of the activity from their point of view, she wondered why a student would be pre-empted to always place that expression in that part of the diagram. Walter responded, feeding from his reflections on a sketch lesson he had taught, that this had to be emphasised to students.

Fiona *Why are they going to put that in the top corner?*

Walter *Well when I was doing the multiplication with them I always talked about the x² goes to the top corner...*

Fiona *Maybe I will emphasise that with them on Monday.*



Figure 40 Teachers doing activity using algebra tiles: Crannóg cycle 2

From this interaction, Fiona decided that this element of the lesson would have to be included within an introductory lesson for the students and that the rules associated with the tiles would have to be explicitly explored with the students, enhancing her KCS for this particular strategy and building into the series of lessons planned towards the research lesson, impacting also on her KCT (see Section 6.3.1).

At the beginning of the year in Crannóg, Eileen barely contributed to the planning of the first research lesson but in their last cycle, she regularly contributed to anticipating students' responses and focusing on students' mathematical thinking. In planning the final research lesson she discussed the resources they were developing and suggested modifications to the graph they were supplying to better support students' learning. Eileen suggested these modifications to activities by anticipating students' thinking and wanted to focus students' attention on graphing the data without having to also attend to scaling the graph, since she found it a common error that students forgot to scale their graphs correctly.

Eileen *It might be better to have it scaled for them so that...*

Stephen *I would say...*

Dave *It's difficult for them.*

Eileen *Well you will just get a load of...*

Stephen *You will get...*

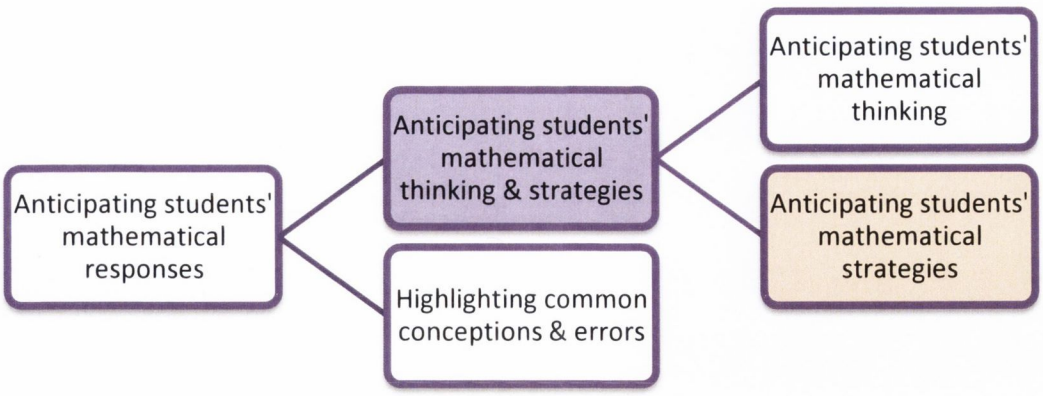
Eileen *You will get ones like tiny like here, they won't use the whole [axis] .*

Dave *Unbelievable. Scale of axis, they do that.*

Here Eileen and Dave anticipated students' thinking in identifying that scaling a graph may prove to be difficult for some of the students. Eileen highlighted this common student difficulty with this from a sample of student work she had brought with her to the planning meeting from a class she had taught earlier. She wanted to avoid this happening in this research lesson and incorporated the scaled graph within the lesson for students to record their results following the 'Tractor Class'. Later in planning the research lesson, Fiona noted that it would be a very difficult idea for students to understand that the concept of limits and teachers modified a curriculum resource in guiding students towards this idea but avoided telling students the reason for this in order to encourage their inquiry of the topic.

B.Anticipating Students' Mathematical Strategies

In this section, the second of the two categories under 'anticipating students' mathematical responses', that of anticipating students' mathematical strategies, is analysed for Crannóg.



As well as anticipating students' mathematical thinking during planning, teachers in Crannóg also spent time anticipating what different strategies students might employ within lessons. For example in research lesson 1 teachers tried to decipher all of the different ways students might respond to finding a pattern in the pyramid blocks and during this conversation decided to use Lego to model the question, as opposed to matchsticks, since there was less options for confusion in setting up the pattern. (The idea of using matchsticks for this particular quadratic pattern had originally come from research literature I had brought to the meeting as lesson study facilitator (Peterson, 2006)).Teachers played with the Lego bricks themselves in order to anticipate how students might engage with the various patterns and modelled each of the extension questions within the planning meeting in order to identify how students might respond to each of the activities. This practice continued right throughout the lesson study cycles in this school where teachers paid attention to trying to identify how students might go about a particular activity by doing out the mathematical activities themselves and identifying markers where students could possibly choose a different strategy (a topic further developed as a feature of KCT in Section 6.3.3).

Initially in cycle 1, Fiona rejected Walter's idea of using $y = x^2$ as the base equation because she anticipated that students would find it too easy to derive that pattern, but teachers decided that it was more beneficial the class to begin with a simpler expression which could develop into a class discussion instead of a more difficult one that not everyone would be successful in deciphering. More difficult questions were then included in as an extension or homework activity for that lesson (Appendix F2 6). Fiona's anticipation of students' responses to the question initiated a discussion where

teachers anticipated how the majority of students in the class would succeed with the activity. In responding to Fiona, Stephen anticipated that since students had met linear expressions the previous year (as recorded in their detailed year plan) they would try to find the first difference. Here the teachers imagined what students might do with such a pattern.

> *Stephen* *Because some would look... if it is 1, 4, 9 some will look at it as being squared; some will look at it as being...*
>
> *Fiona* *Add 3 and 5...*
>
> *Stephen* *And 7, and that is... if you give them a pattern, if you give them a list of numbers of 1,9, that is what they do, they will actually see how much it is going up, whereas some don't relate it to being anything squared...*

Stephen employed a critical student lens in planning and saw the mathematics from the perspective of a student who would not have previously seen this type of pattern. Stephen drew on his experience as a mathematics teacher in anticipating what a student would do with this mathematical activity (Jacobs et al., 2010) and was later surprised in his reflection of teacher work in how students were in fact successful in interpreting this pattern. It was important that as part of his reflection that Stephen had anticipated students' strategies and it enhanced his KCS in observing how a student might interpret that particular pattern.

Fiona was an important figure in the planning of this lesson in anticipating that students would easily decipher a squared pattern if they were asked to calculate the number of blocks in the 100^{th} layer. She anticipated that students would by-pass finding a 1st and 2nd difference in the pattern in order to calculate this and to overcome this astute employ of a critical student lens, teachers decided to use the more unusual number of the 47^{th} layer of the pyramid (2209 blocks).

Teachers attempted to anticipate the different strategies which students could engage with: they might count the blocks (as intended), they may count the number of dots on each block (which would also give a quadratic expression), or they might take the total number of bricks as the pattern (which would lead to a cubic equation). Teachers also detailed the strategy they wanted students to employ in finding the quadratic pattern in

this lesson, anticipating the various strategies students might use in identifying this pattern.

Dave *It would be lovely if they do this, if they subtract one from the other, and one from the other and one from the other, and see did it, it's about 3, 5, 7, 9. They'll see those. Why would anybody go and look at the gap between them? It's there, isn't it? It's staring at them. It's two. They're all odd numbers.*

Eileen *But they'll see then that the next level will add nine, and then they'll add eleven, and then they'll add thirteen*

Dave *Yeah.*

Eileen *You'll know that they know that if you ask them.*

Stephen *Yeah.*

Eileen *Yet they won't physically know, two, two, two. But they will know what the next thing would be.*

Dave *Ok, that's ok, so they got that. They now know that, that that thing is going up.*

Stephen *If they got a first difference because of 'by 2' that's what you really, that's what they know.*

Eileen *They've got it.*

Dave anticipated and articulated how he wanted the students to go about finding the quadratic pattern but wondered why students might look at the second common difference, what might entice them to investigate that pattern. Eileen suggested that it would be the recurring pattern of addition which students would notice and Stephen anticipated that students would realise that the first difference is increasing 'by 2' every time. Teachers anticipated that this would be students' strategy in finding a second difference in the pattern, and was in fact the way that the majority of students went about their investigation within the research lesson. Teachers' successful anticipation of students' strategies was a major part of their planning and successful implementation of meeting the learning objectives of that lesson.

In research lesson 2, teachers anticipated students' strategies by playing with the algebra tiles themselves within the planning meetings (Figure 41). In doing this teachers employed a critical lens to their practice by seeing the activity from a students' perspective, when many of the teachers had never used tiles before. Stephen jokingly anticipated what students might incorrectly do with the tiles, placing all of the representative small yellow tiles to cover the entire area. He anticipated that students would equate the tiles with the area of the surface they covered.

> *Stephen* *If I was a student I would be saying "Why don't we just put in all the small ones and then we know what it is equal to? Numerically?"*

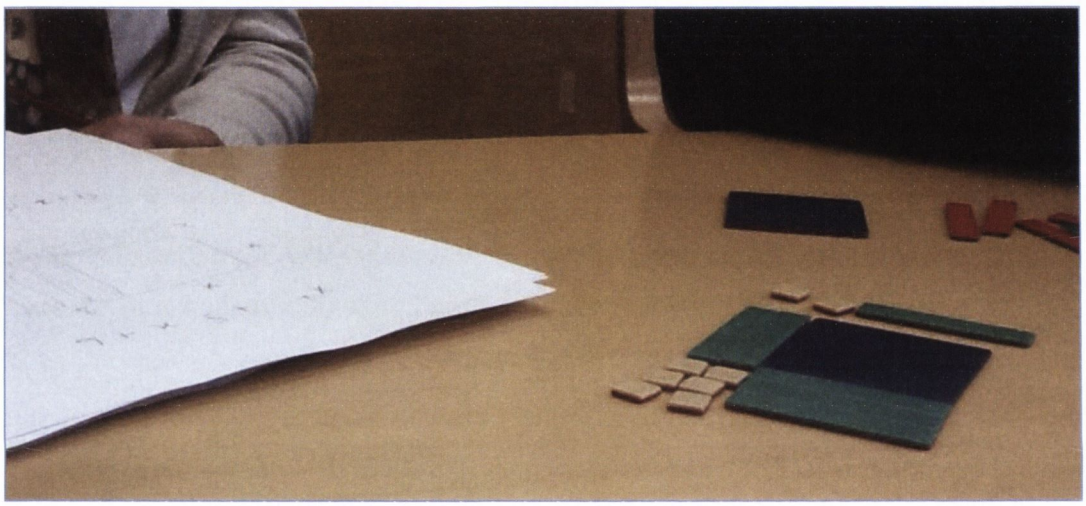


Figure 41 Doing activity using algebra tiles: Crannóg cycle 2

Stephen put himself in the position of a student confused by what the tiles represented or where they should be placed and articulated this in a student voice. In fact, teachers used a student voice regularly in the planning and this point will be explored further in Section 6.4.3. In anticipating students' strategies within the planning of this lesson, teachers also decided that they should begin with only prime numbers as the 'key number' in order to reduce the number of routes available to students in factorising the expression.

By lesson cycle 3 teachers were comfortable and adept at anticipating how students would mathematically respond with particular strategies to questions. In designing the sequence of content within the lesson teachers began asking themselves during the planning "*Realistically, what are they going to come up with there though?*" (Eileen, planning meeting 2, cycle 3). Teachers anticipated that students would attempt to average the speeds when asked to find the speed at a particular point and although this would be

fine for a linear graph, it would cause students a necessary cognitive dissonance in trying to find the instantaneous speed for a quadratic graph where the speed was constantly changing.

Stephen *And you can even get to that point of asking what is their speed, just asking the general question what is their speed and there should be an interaction in terms of, somebody says their average speed, and say so is that their speed then whatever and actually get them thinking.*

Fiona *Mmm, it says then to help answer the question do the following, using your ruler draw on the sequence A B, A C, A D and A E.*

Walter *Oh yeah okay so, A to B.*

Fiona *And they're getting closer and closer, so that's exactly what you do if you're doing first principles.*

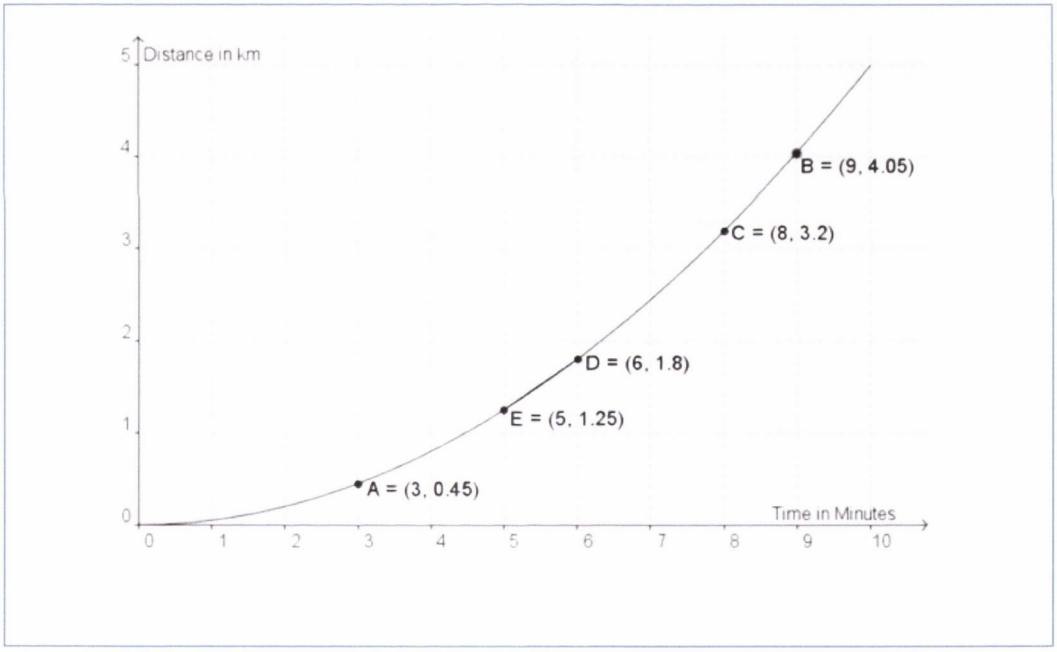
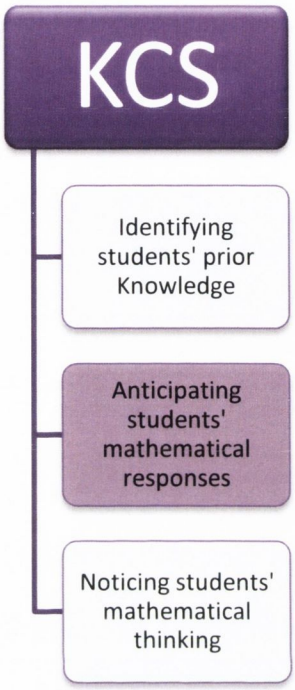


Figure 42 Student activity: Crannóg cycle 3, Appendix F2 7

Teachers anticipated that students would use the strategy spoken about above to guide themselves towards finding the speed at A i.e. by finding the slopes between points that grew closer and closer to A. While this was not exactly how students went about trying to identify the instantaneous speed at A during the lesson, anticipating students' strategies assisted teachers in sequencing the lesson in order to build on students'

mathematical thinking and benefited teachers in attending to and reflecting on students strategies after the research lesson.

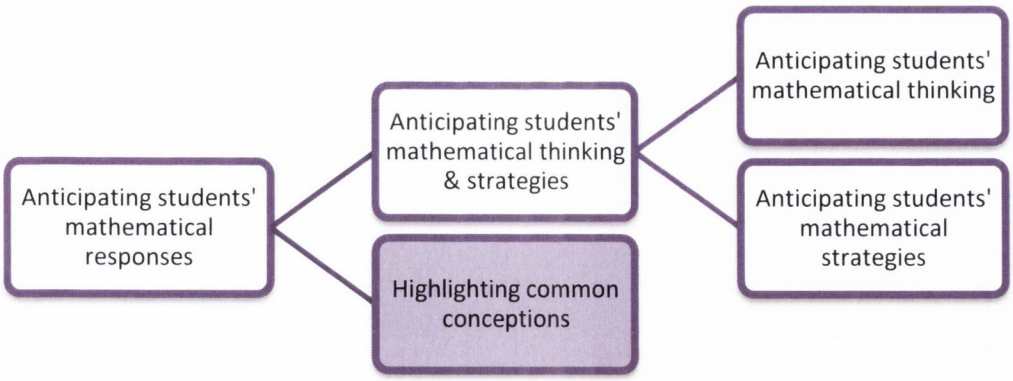
Teachers were so detailed in their anticipation of students' mathematical strategies that they even predicted how students would calculate the slope when they had graphed their data as Fiona did in their penultimate planning meeting saying that "*they will go for rise over run.*"



While these teachers were proficient in anticipating students' mathematical thinking and strategies over the course of the lesson study cycles throughout the year they were not, however, prepared for how their senior students would not engage well in participating in peer groups during the lesson. Nor were they prepared for senior students' discomfort in being observed by a number of teachers since the junior students had never seemed to particularly mind the fact that there were far more teachers in the classroom than was usual. These examples of teacher knowledge of student will be explored in further detail later in the following analysis Chapter 6, Section 6.5.2.

5.3.3.3 Highlighting Common Conceptions

Continuing the analysis of KCS according to the framework outlined in Chapter 4, this section analyses how teachers improved in highlighting common student conceptions as part of their lesson study planning.



Overall in the data there were not many examples of teachers highlighting common conceptions, even though this is highlighted within the literature as a main feature of KCS (Ball et al., 2008; H. C. Hill et al., 2008). There were a number of moments when mainly experienced teachers, such as Fiona and Nora, highlighted mistakes that students commonly made when doing a particular question and according to the literature it would be expected from the data that it was the more experienced teachers who would exercise this facet of knowledge in their practice (Jacobs et al., 2010).

5.3.3.3.1 Doone

Nora, a teacher who taught for over 30 years and was experienced as a learning resource teacher, highlighted common student conceptions throughout the lesson study cycles. For example, in cycle 1 Nora suggested avoiding the number 4 in finding the square root as students commonly confused it with multiplying by two as previously highlighted in Section 4.3.2.2.

Lisa	*This is what, when I was doing this the other day, I was thinking where are we going with this when we're finished? We are really talking about the square root of 4 to get back to our x, but do we need to tell them that?...*
Nora	*Can I just suggest that rather than 4 there*
Lisa	*Yeah*
Nora	*I prefer to go for 9 because an awful lot of people half things and double things.*
Lisa	*That's right, we said that before. We said that before.*

Kate	*Yeah, a lot of people just think 4 is 2 by 2, or the 2 squares. So avoid 4s.*
Lisa	*We left the 4 out, we discussed that the last day for that same problem. So I think that's a good point – it's 9.*

Similarly, Michael (who also had experience in teaching small groups of students who were perceived as having a lower ability in mathematics) regularly put forward suggestions to avoid common conceptions that students made. In research cycle 1, in a meeting prior to the example referenced above, Michael suggested to Kate that she should avoid using '2' as an example for squaring since students often mixed up the operation of squaring with multiplying that operation by 2. In this excerpt Kate talked about presenting the idea of squaring to students.

Kate	*If I had the example of the 2 by 2, I can have underneath the thing, I can have 2 by 2 and then I can have that disappear and replace it with 2 squared...*
Michael	*Should we skip the 2 by 2, sorry.*
Aoibhinn	*Go to 3 by 3.*
Michael	*Yeah.*
Aoibhinn	*Yeah, probably.*
Lisa	*That's what I was saying earlier on.*
Kate	*So I'll do 3 by 3.*
Michael	*Do it by 3 squared, because 2 by 2 is 2 and 2 squared is – sorry, 2 by 2 is 4 and 2 squared is 4 and they're going to go "well then, it's 3 by 2"*

While initially in the research Michael was not very clear in articulating what students might not understand within the planning meetings, as the lesson study cycles continued he improved in detailing and articulating what students commonly found difficult.

In the first two lesson study cycles it seems to have been easier for this group of teachers to reflect on what students commonly misunderstood than to anticipate how to build that understanding. Lisa in research lesson 1 put forward the suggestion of focusing on what

students commonly misunderstood by x^2 being the same as $2x$. She wanted incorporate it as part of her exposition within that lesson.

Lisa *The other thing that I thought of, when I was doing this, is: what is 3^2 not? You know, so when they'll be colouring in their squares, they're doing it and when I'm doing it on the board, what I'm going to say is that "well 3^2 is a row of three times three and 5^2 is a row of five times five." What most students think when they see 3^2? They think of six. So let's do what is 3 2s. 3^2 is 9 but 3 by 2 is 6.*

Kate *Because that's the most common and irritating mistake that is made.*

Lisa's observation resulted in a discussion about an activity where students would model the multiplication on worksheets in order to identify the difference between 3 by 2 and 3 by itself. Unfortunately, this activity was not incorporated within the research lesson.

In the second lesson study cycle, Nora often shared with the group the difficulties she saw students have in multiplying fractions. She saw language as being a very big issue since many of them would not have come across the word "of" as meaning "multiply." She noted that for students "*it means part of, it means divide*" and it led to difficulties in multiplying fractions since it contradicted their usual experience of multiplication causing a value to increase. Kate incorporated Nora's suggestion within the plan and maintained a focus on language throughout the planning of this research lesson.

Neither Owen nor Lisa contributed greatly in conversations detailing students' conceptions. This may be understandable in Owen's case since he had very little teaching experience of mathematics (Jacobs et al., 2010). Lisa, with 6 years of teaching experience, would have been expected to be more forthcoming in her ideas of common student conceptions but perhaps this was due to her not employing a critical student lens and not thinking about the teaching and learning of mathematics from a student's perspective. As discussed in Section 5.5.4.1, there was a change in Lisa's perspective towards the end of the year and this may have accounted for her lack of prior input in these particular conversations.

5.3.3.3.2 Crannóg

As previously detailed, teachers in Crannóg regularly anticipated students mathematical thinking and strategies during their planning of research lessons. Within these

conversations, since teachers were teaching topics in a very new way to how they would have traditionally taught them, there was not much opportunity for teachers to articulate difficulties students commonly had with a topic. Throughout the planning of the lessons however, teachers often articulated the point that students usually learned how to do something without fully understanding why they were doing it. In planning these new approaches teachers were attempting to build students' relational understanding of their mathematics (Skemp, 1976). In fact, teachers in Crannóg encouraged students being confused, particularly in research lessons 1 and 3. Teachers had wanted students to be confused in research lesson 1 in order to motivate them to investigate the pattern. In the post-lesson discussion on the third research lesson, teachers reflected that students left confused about the concept of how to find the instantaneous rate of change at a particular point and suggested that this was a positive outcome, since participating in the lesson activities had changed their way of thinking about average and instantaneous rates of change and would perhaps motivate the students to research more of the topic independently.

In Crannóg's second lesson study cycle teachers spoke about errors which students commonly made in factorising quadratic expressions. Fiona discussed the fact that students often did not understand the meaning of the factors they found.

Fiona *And the other thing that they don't really seem to get into their head is the whole idea of why it has to equal to zero and the whole thing about that...And like even for something like the quadratic $2x^2 - x - 3$. Quite a number of them get the -6 and -1 and if they do it by that method, but they change it then and do $x^2 - x - 6$.*

Eileen *Yeah.*

Stephen *Yeah.*

Aoibhinn *Do they?*

Fiona *Some of them do that, they use the -6 to get their pair of factors and then they just put 6 down and change it.*

Stephen *I was getting my fourth years to actually sketch it and putting the y intercept as -6 then as well, which they were doing, which was interesting. It is a complete misunderstanding.*

Fiona *Just not thinking, not just doing, obviously doing things mechanically as opposed to actually thinking about what they are doing.*

Fiona and Stephen suggested that students were often factorising without an understanding of why there were doing so and this led teachers to designing a lesson which would, they hoped, give students more of an understanding of factorising quadratic expressions. It is interesting here to note that teachers did in fact blame students for "*not thinking*" without questioning the teaching approach to this topic but, on the other hand, within this research cycle teachers did begin to investigate new approaches to teaching this topic as a way to build on students' understanding of factorising.

In the third cycle, Walter noted that his students were having difficulty with the concept of rate of change and differentiation. Both Eileen and Fiona agreed with his observation since students commonly tried to calculate speed from an equation based on a constant speed instead of differentiating when acceleration was occurring.

Walter *My sixth year's ordinary class are struggling with that idea and rate of change. They have totally messed up that whole section of the [exam] paper.*

Eileen *I have kids who didn't differentiate anything on paper one.*

Fiona *Well it is hard for them because before it used to say "Differentiate."*

Walter *Well we just told them "If in doubt on questions 6, 7 or 8; differentiate."*

Fiona *But you can't just put it down for every question on the paper: differentiate.*

Eileen *No, no ,no, but some of these are words like max and min and I told them so many times that max, you know, that if they want to 'max', differentiate it.*

Fiona *Maximum translated into max means differentiate. It actually means to, use your maths translator, that is what the word says, don't worry about the maximum.*

Eileen *So I have loads who did 'Dad's Silly Triangle', and when they were asked to find speed, they divided the distance by time. The distance was a quadratic so it was rubbish.*

It was in highlighting this common conception within this discussion that teachers focused on building students' understanding of identifying rates of change within the third research lesson.

Highlighting common conceptions in both schools often led to ideas to develop activities which would improve students' understanding of a topic. This form of KCS however was not often noticeable within the data and when it was, these occasions were more often by experienced teachers. This may suggest an important point that in order to enhance teachers' KCS a range of teachers of varying experiences should participate in the community in order to have a sharing of this knowledge.

5.3.3.4 Supporting Teachers in Anticipating Students' Mathematical Responses

It emerged from the data that teachers were supported in their anticipation of students' mathematical responses by utilising research literature during their planning. In Doone's second research lesson teachers read research literature which impacted on how they constructed the lesson in terms of designing activities that would build on students' understanding, but also impacted on how teachers anticipated what students would find difficult about multiplying fractions. In the first and second cycles of lesson study in Crannóg teachers accessed literature that impacted on how they planned the lesson but also highlighted how students might go about answering particular questions or engaging in particular activities.

5.3.3.5 Utilising Research Literature

In Doone's second research lesson, as facilitator I brought in literature for teachers to read, some of which included children's responses around a topic (Coughlin, 2011; Cramer, Monson, Whitney, Leavitt, & Wyberg, 2010; Foster, 2008; Gregg & Gregg, 2007; Johanning, 2011; Li, 2008; Petit, Laird, & Marsden, 2010; Son, 2012; Taber, 2007). As referenced in Section 5.5.4.1, Kate, Nora and Michael enjoyed referencing this literature, quoting some extracts from what they read within the meeting, since it corroborated what they had already identified as difficult concepts for students in

multiplying fractions and since it detailed what students often found difficult about a topic.

Nora — *You know, conceptually it's good. And it also tells you about children's responses around it [referencing the literature she was reading (Taber, 2007)]...*

Michael — *I really kind of decided to look at what the issue was, em, "many students and adults think that these situations do not involve multiplication but rather division or subtraction because the resulting quantity is smaller than the quantity named in the problem. They come to believe the problem that asks them to find a fraction of a quantity are solved by division...." So I suppose yeah, that really kind of is, when I just read that I said God yeah, that really is what it's all about...it's a total mind-shift.*

Teachers incorporated many of the ideas which they read in their research literature into their lesson.

In Crannóg's first research lesson, I brought research literature into the lesson which led to teachers creating the story and idea around investigating quadratic patterns, but which also gave teachers an idea of how students might respond to a particular task (Bryan, 2010; Mittag & Taylor, 2009; Peterson, 2006). In describing two of the articles, Fiona developed an idea where students could explore quadratic patterns in the research lesson. Again in their second lesson I sourced and contributed research articles which teachers referenced in planning their second research lesson in terms of anticipating students' responses and also in sequencing the activities (Core Plus Mathematics, 2013; Leitze & Kitt, 2000; Phelps & Edwards, 2010; Vinogradova, 2007).

As a researcher introducing lesson study to both schools I did not continue the practice of bringing in research literature since I hoped teachers would source resources for themselves as part of continuing lesson study as a sustainable practice within schools. I regret that decision since teachers did benefit from the literature in terms of gleaning ideas for their teaching, but also in terms of anticipating students' responses. In future lesson studies, I would encourage both the teachers and facilitators to access research literature and incorporate it within their planning. This may also suggest that further research is necessary in this regard since in Ireland, in general, teachers do not access research literature and the majority of teachers do not have access to such literature.

From the specific occasions within this research where teachers accessed literature it had a positive influence on their planning of research lesson (Tepylo & Moss, 2011) and their employ of a critical student lens in their teaching and learning practices.

5.3.3.6 Conclusion of Anticipating Students' Mathematical Responses

In a traditional approach to teaching mathematics, such as that in research lesson 1 in Doone, it was not necessary to anticipate students' mathematical thinking since the teacher dictated the pace of the lesson and showed students the mathematical content within the lesson. In planning lessons where students collaborated and communicated with their colleagues and worked without teacher interruption, it was important that teachers anticipated how students would respond to the mathematics within the lesson.

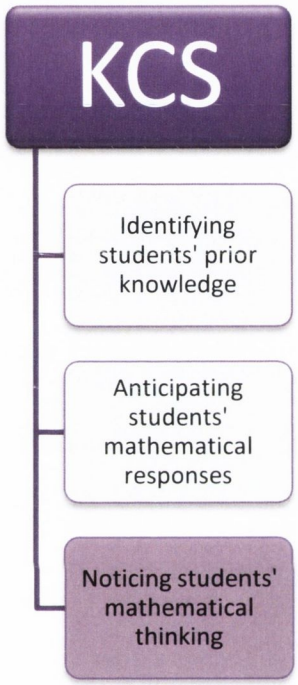
In Doone, while teachers did not initially spend time in anticipating students' mathematical responses, this occurred more and more often during the lesson study cycles and culminated in teachers anticipating how students would respond to each individual activity within the sequence of lessons which they planned, including those within the research lesson. Teachers enhanced their KCS in anticipating students' mathematical thinking as the lesson study cycles continued because they learned from their colleagues who were employing a critical student lens in their practice. As the cycles continued, the lessons in Doone became more and more student-centred with teachers, such as Nora and Michael identifying areas where students commonly erred in their thinking, and built these into the lesson plans. In their third and fourth research lessons, teachers began to speak about students "*discovery*" elements within the lesson. The planning of research lesson 4 included teachers anticipating how students' mathematical thinking could be progressed and also in anticipating how students would answer the questions that were developed specifically for them in that particular lesson.

The consistent anticipation of students' mathematical responses within Crannóg, where teachers such as Fiona and Stephen regularly employed a critical student lens in identifying how students might respond to particular activities within planning, allowed teachers to detail the mathematical content which would be included within research lessons and build their KCS in identifying how students' mathematical thinking and understanding could be developed during particular episodes within lessons. In witnessing the anticipation of student mathematical thinking in early research lessons, other teachers such as Eileen and Walter began to also anticipate how students would

think about mathematics activities. This anticipation of students' mathematical responses allowed teachers to develop student-centred lessons where students' own inquiry within a lesson would guide them in their learning towards a particular objective.

Although teachers sometimes found it difficult, anticipating students' mathematical responses was an important part of planning research lessons within lesson study and this form of KCS was enhanced through teachers' participation in lesson study.

5.3.4 Noticing Students' Mathematical Thinking

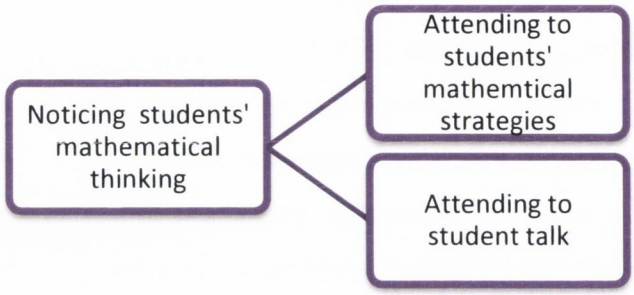


One of the most unique and important aspects of lesson study is the live research lesson (Meyer & Wilkerson, 2011; Murata, 2011). In this project, the data reveals that observing live research lessons which teachers had participated in planning, afforded teachers the opportunity to notice and attend to how students were thinking about mathematical content and how they were responding to mathematical questions from their teacher and from their peers.

In this research, not all teachers within both communities could attend and reflect on each of the research lessons but over the course of the academic year, each of the teachers had at least three occasions to participate in an observation and post-lesson discussion of a research lesson (excepting Judy as described in Section 3.10.4). As the lesson study cycles continued, teachers' noticing and interpreting of students' mathematical strategies within post-lesson discussions increased. In Doone, teachers moved from initially attempting to record various elements of a lesson, to each teacher focusing on one group or pair of students. Similarly in Crannóg, teachers initially began with a model where all but one teacher focused on one group of students within a research lesson. As teachers' participation in lesson study continued, they began to purposefully notice how students were employing strategies, what questions they were asking within lessons, and how they were communicating within the lesson.

The noticing, interpreting and reflecting of these student activities within the lesson enhanced teachers' KCS which then led to a development of how teachers contributed to the planning of subsequent research lessons. This professional noticing of students' mathematical thinking saw a shift in how teachers began considering their lessons (both research lessons and other lessons outside of lesson study) but also in how teachers began attending to how their students would engage with the mathematics contained within a lesson.

This theme of noticing students' mathematical thinking refers to teachers observing, interpreting, and reflecting on the mathematics students engage with during research lessons (Breen et al., 2014; Mason, 2002). In the categories of noticing students' mathematical thinking, 'attending to' incorporates teachers' observation, interpretation, and reflection of students' mathematical activities during the research lessons. This section will investigate teachers' noticing of students' mathematical thinking under the following headings:

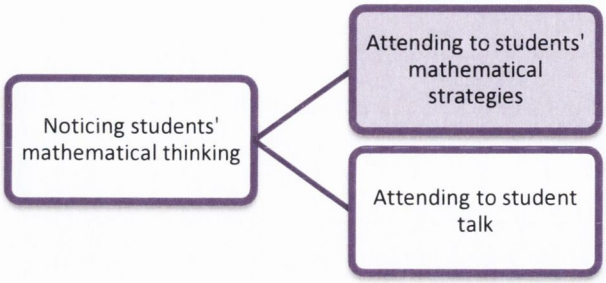


- Attending students' mathematical strategies
- Attending student talk

Each of these headings will be analysed in the chronological progression of each individual school as in previous sections.

5.3.4.1 Attending to Students' Strategies

In this section, teachers' observation, interpretation, and reflection of students' mathematical strategies during research lessons will be investigated according to the data generated in teachers' lesson study conversations. This is the first of two categories under the feature of 'noticing of students' mathematical thinking' and the data will be chronologically analysed for each separate school.



5.5.5.1.1 Doone

Initially in Doone, teachers did not plan for how students might mathematically engage with activities which were to be included within a research lesson. As detailed in the previous section (Section 5.5.4.1), teachers did not incorporate anticipation of students' mathematical thinking into the first lesson. However, in reflecting on that first research lesson, teachers began to attend to how students had worked out particular activities and were interested in seeing how students, who had been using an incorrect strategy of multiplying numbers by two instead of squaring, were able to independently correct their work. Teachers also paid attention to and interpreted how students incorrectly carried out activities.

In attending to students' strategies teachers began to interpret how students were thinking, employing a critical student lens in observing students within the research lesson. In this post-lesson conversation, Nora and Owen interpreted why one particular student had incorrectly shaded in the whole sheet instead of a number squared (Appendix F4 1).

Owen	*One of mine just did that with the hundred in a box. He was just shading the whole box.*
Lisa	*Ok.*
Owen	*So, he didn't know what to shade.*
Nora	*Yeah 'cause they looked like little squares.*
Owen	*Yeah.*
Nora	*He was just shading.*
Owen	*He was shading that square.*
Nora	*What he thought all the squares were.*
Lisa	*Oh, really?...*
Nora	*Even though in actual fact the weren't exactly weren't quite, he was, he was -*

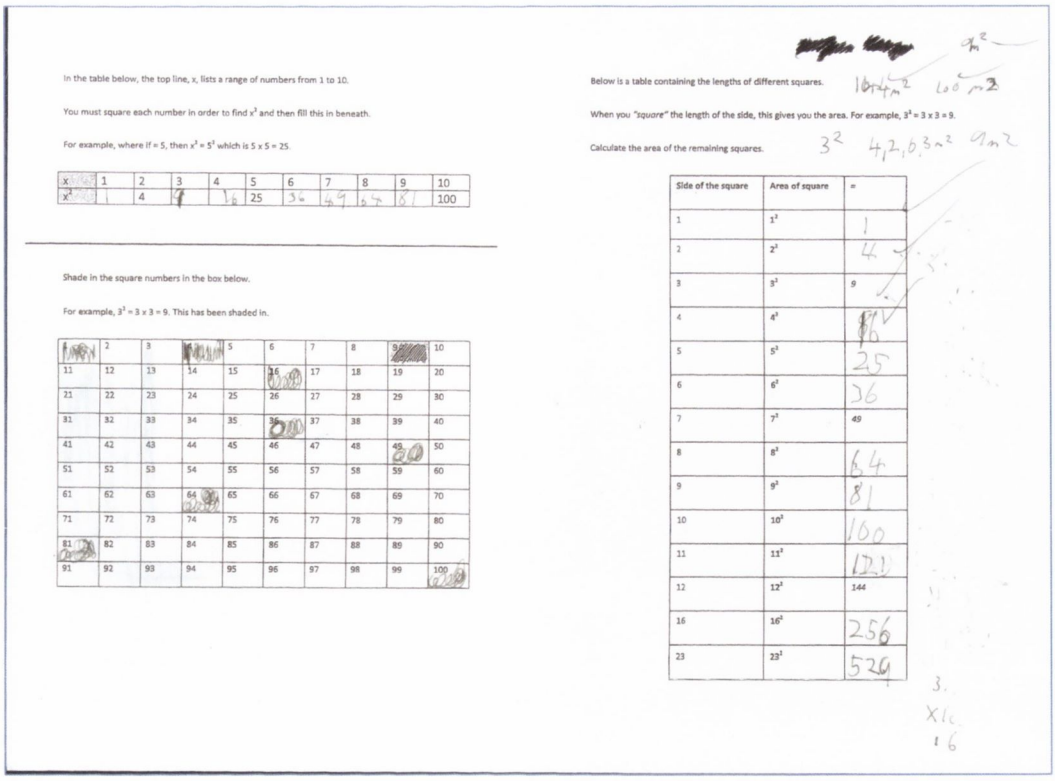
Owen *Yeah, he was shading 'cause it was a square.*

Nora *No, it wasn't actually quite a square, was it? You know, to an untrained eye, they all looked like squares.They were all little squares...*

Kate *What do you mean he did?*

Owen *Do you know in this sheet, he shaded the whole thing. In his eyes, it was a square, yeah.*

Both Owen and Nora noticed and interpreted an incident where instead of shading in a model of a number squared, a student shaded in the entire page since, in his eyes, the whole page represented the squared number. Owen reported this to the group in order to highlight his point that the idea and model of squaring a number could have been emphasised more before this student activity. Owen later noted that students would have benefitted from more of an emphasis on the visual representation of 'squaring' before shading in their own squares. Owen explained to Kate his interpretation of what the student had understood about the activity. This sharing of the employ of a critical student lens may have been a learning moment for Kate in interpreting mathematics from a students' perspective as she began to articulate a critical student lens more and more throughout the lesson study cycles.



In the table below, the top line, x, lists a range of numbers from 1 to 10.

You must square each number in order to find x^2 and then fill this in beneath.

For example, where if = 5, then $x^2 = 5^2$ which is 5 x 5 = 25.

x	1	2	3	4	5	6	7	8	9	10
x^2	1	4	9	16	25	36	49	64	81	100

Shade in the square numbers in the box below.

For example, $3^2 = 3 \times 3 = 9$. This has been shaded in.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Below is a table containing the lengths of different squares.

When you *"squore"* the length of the side, this gives you the area. For example, $3^2 = 3 \times 3 = 9$.

Calculate the area of the remaining squares.

Side of the square	Area of square	=
1	1^2	1
2	2^2	4
3	3^2	9
4	4^2	16
5	5^2	25
6	6^2	36
7	7^2	49
8	8^2	64
9	9^2	81
10	10^2	100
11	11^2	121
12	12^2	144
16	16^2	256
23	23^2	529

Figure 43 Student work, research lesson 1 Doone, Appendix F4 1

Later in that discussion, Kate shared what she noticed about that student correcting his own mistake. The student, in filling out the excerpt from the worksheet in Figure 43 saw that when he got to the completed number five, his work had been incorrect to this point. Kate noted that "*he obviously realised then he'd gone wrong*" and interpreted that through the construction of the worksheet the student had the opportunity to correct his own work. Up to this point in the meeting Kate's observations had been focused on students' engagement and vocal reactions within the class. However, from noting the number of points reflected on by Nora and Owen, she began to start reflecting on and interpreting the student work she had noticed during the lesson.

Michael was unable to be present in this post-lesson discussion and of this group and Owen and Nora most often articulated the noticing and interpreting of students' strategies within that reflection meeting. Identifying and interpreting students' strategies was a new activity to all of these teachers who had not previously observed students outside of their own classroom, but those who regularly employed critical student lenses (like Nora) positively influenced those they were working with.

By research lesson 2, Lisa joined her colleagues in beginning to articulate her noticing and interpreting of what students were doing with the mathematics within the lesson. In

this research lesson, students had been asked to go to the board and place on a number line (from 0 to 1) where they thought a fraction might lie. For the first time in this research, Lisa interpreted students' responses to a mathematical activity contained within a research lesson and put forward the idea that students only thought of the distance between 0 and 1 as divided into two parts but not anything else. Lisa shared a number of reflections of what students had been thinking about and doing within the research lesson.

Lisa	*They put all the fractions in relation to a 1/2 so when they were asked to do 2/3, they didn't take the number line and say, "well, there's a 1/3, a third and so there's 2/3." They just said, "it's greater than a 1/2."*
Owen	*Right.*
Lisa	*So, I didn't think they got the, well, maybe they were just a short cut but they weren't doing 1/5s, they weren't doing 1/10s, they weren't doing 1/11s, they were just doing it in relation to a 1/2.*
Kate	*Mmm. That's ok though.*
Lisa	*You think that's all right?*
Kate	*Yeah.*
Lisa	*I just didn't think they got the sense that the number line was to be divided up into so many more parts.*

While Kate did not see a problem with this form of interpretation, what is of interest in this excerpt that Lisa, for the first time, began to interpret mathematics from a student's perspective. Lisa had been the conducting teacher in the previous research lesson and was more pre-occupied with her role as the teacher than on how students were engaging with and thinking about the mathematics within that first research lesson and post-lesson discussion. As an observing teacher her perspective of learning had changed and we see a difference in her perspective of teaching and learning mathematics in suggesting this point be highlighted to students differently.

By research lesson 3 all of the teachers had begun, to varying degrees, thinking about how students would engage with mathematical activities during the planning of the

lesson. As noted in the previous section on anticipating students' responses, for the first time in their planning in this research teachers began thinking about the mathematical strategies that students might employ in tackling a question. Teachers also brought their observations of how students worked together in research lesson 2 into the planning of this lesson where students were each given a copy of the question but only one worksheet to answer as a team. Teachers also planned their facilitation of students presenting their work based on their reflections of research lesson 2 (explored in further detail later in Section 6.5.2). Within the research lesson, teachers focused their reflections on how students engaged with these mathematics questions and noticed and interpreted the strategies employed by students within the research lesson. Unfortunately Owen could not attend the post-lesson discussion but within that meeting all of the other teachers explicitly detailed how students were attempting to answer questions within the lesson.

Kate, as the conducting teacher, shared her interpretations of how the class as a whole generally attempted tackling the questions while Michael interpreted how one particular student in the group he was observing had attempted to answer the question.

Michael *It's just the same with Shane in the group that I was watching he was the guy I was saying, he said " nah, I, there's just not enough information there for me to do that question", but he was calculating how high the ball would go and all that stuff and then the guys who were straight into the maths were like, "oh yeah, that makes sense." So everybody was offered -*

Nora *Everyone made a contribution, yeah, from their own strengths, yeah.*

Kate *I was really impressed when I gave them this out and majority of groups, bar two I think, didn't even draw a graph. They went straight into, they obviously got the table of values and went, "ok, well this is complete symmetrical, so we have the answer." Another one solved this and then solved this [referring to questions in Appendix F2 3]*

Teachers began attending to students' strategies more by using students' work as reference points within their post-lesson discussion, something that had not previously occurred in these meetings. Teachers used samples of student work from within the lesson to discuss how it was that students had attempted to answer the mathematics questions. This detailing of student mathematical thinking focused teachers more on the

potential of a lesson to impact and build on students' understanding of a mathematical topic. Teachers were enthused by observing how students had been working on the questions in this lesson, enhancing their KCS in highlighting how contextualised questions motivated students to work on the problem and teachers then viewed and interpreted students' strategies (explored further in Section 6.3.2).

By research lesson 4, while the post lesson discussion was much shorter (13 minutes as opposed to 23 and 31 minutes in the previous two cycles) than the other meetings, all of the conversations revolved around the student: how students were thinking, about the mathematics within the lesson, and what could be altered in the lesson to build on students' understanding. Unfortunately, teachers did not share their observation notes of that lesson, but from excerpts of the post-lesson discussion it is clear that they focused on how students had gone about answering the questions the teachers had constructed for that particular lesson.

> *Lisa* *In the golf one, they didn't read the context because the numbers were in the picture of the map. They went straight to working it out without bothering reading the context. So that made me think. But then when we went to the boat question, there was no numbers and they had to read the context and they had no problem reading the context. And it was good watching them saying which one was the tall side.*

Lisa made reference to the fact that students were not provided with a necessity to read the context of a particular question about a boat but also detailed how students went about answering that question. Each of the teachers, all of whom had been present for the lesson and reflection, focused on how their students had attempted to answer questions within the lesson. In this final research lesson Owen was also beginning to reflect and interpret the strategies students had employed within the lesson.

> *Owen* *Another thing they didn't pick up on was* $h^2 = 25$. *They just put in the square root before writing down* $h = \sqrt{25}$.
>
> *Lisa* *They weren't seeing the connection between them*

While Owen had focused on how students might interpret mathematics in the planning of prior lessons, he had not adopted this perspective in the reflection of these lessons. By research lesson 4, Owen had begun to not only focus on how the lesson was implemented

but also on how students had engaged mathematically within the lesson. His comment led the conversation to include an additional sequence within students' prior knowledge of manipulating equations in any re-teaching of that lesson. Owen's observation led to an enhancement of the community's KCS in highlighting how to incorporate another element within the lesson in order to develop students' mathematical understanding.

Kate reflected on a teaching point that may have been better utilised in order to encourage students to estimate what length the hypotenuse of a right angled triangle should be before they went to calculate it. Kate suggested that instead of the conducting teacher asking them to guess a random number, which she thought would be difficult for students, that instead Michael could give them a range of values to choose from in order to provide a better estimate. Michael had in fact given students the option of suggesting whether the hypotenuse would be higher or lower than 170 (the sum of 120 and 50) since he saw this as something students commonly mistakenly did. This was important since neither Kate nor Lisa had previously realised a relationship between the lengths and from their observation and post-lesson discussion knew that they could highlight the fact to students that the value of the hypotenuse will always be less than the sum of the other two sides.

Kate *When you were saying at the start "Estimate for me what size is it going to be?" and someone said "70" because they were just minused, I think maybe to even add in for them "Do you reckon the answer will be higher or lower than 120?"*

Lisa *Yes*

Kate *So they only have to give you a number because that might be a bit kind of..."Oh, I don't know and I'm going to be way off the answer." So just higher or lower than 120, and then say higher but lower than – I actually liked the way you did lower than 170*

Lisa *Yeah. I liked that 170. I have never seen, I never thought of it like that*

Kate *I never thought like that either...*

Lisa *I thought that was a very nice way to do it, that the answer is going to be less than the two of them added together*

Michael *Oh yeah, because physically it has to be. We discussed that yesterday [in class]*

In this excerpt Kate interpreted what the students did (subtracting one length from the other) and suggested that Michael modify this question in order to provide students with an opportunity to provide a more reasonable estimate of the hypotenuse. Lisa and Kate had not previously realised that there was a relationship between the length of the hypotenuse and the sum of the two sides and as part of this conversation, developing her knowledge of this topic but also her KCS in realising how students' understanding of the concept could be further developed.

By the final research lesson, all teachers participating in this research in this school had begun to attend, interpret and reflect on how students were going about answering mathematical questions within post-lesson discussions suggesting that teachers were focusing more on students' mathematical strategies within research lessons.

5.5.5.1.2 Crannóg

Due to the nature of the first research lesson in Crannóg, which was planned as a lesson where students would 'discover' the pattern, and due to the anticipation of students' mathematical strategies in planning the lesson as described in the previous section, teachers' post-lesson discussions from the outset incorporated noticing, interpreting and reflecting on students' strategies. In the first post-lesson discussion teachers had been very interested in how engaged students were in the lesson but also explored how students had gone about investigating the quadratic pattern at the focus of the lesson. Judy, Fiona, Gerald, Stephen and Dave all reflected on and discussed the various strategies that students had employed during particular points in the lesson. Walter may have been absent from this line of conversation since it had originally been his duty to record teachers' questions but in fact he had been noting and observing the various dynamics of group-work around the classroom. Within the discussion Stephen reflected on how the students he had been observing went about identifying the pattern of the pyramid.

Stephen *There one kid in the group...he actually did come up with a curve. He looked at it through a curve, which is interesting to see that is how he looked at it, and he actually drew the graph to see what was happening. The other, they kept coming up with, two of them came up with that diagram, so they had the pyramid with 1, 4, 8 and 16 and then they have differences on the side... Then Daniel came up with: 'I think it is 2216' and they also said 'it is Daniel, yeah" without asking why or where this magical number had come from... And I think he just added on 7 for some reason. I think because the last block had last on 7. So from 9 to 16 and then on.*

Stephen had noticed how this particular group of students were attempting to identify the pattern within the blocks and find out how many blocks would be in the last layer. This group of students were led by one student, Daniel, whose answer had been incorrect and Stephen interpreted how Daniel might have come up with his pattern and final value. This detailed report of how this particular group of students attempted to identify the pattern of the blocks was repeated by the other teachers mentioned above and dissected when students began with a strategy that had not been anticipated by teachers in the planning of the lesson.

In the post-lesson discussion Dave had interpreted only one of the students' strategies which he had encountered within the lesson, he did however report on incidents of student talk which he had heard during his facilitation of group work and these will be discussed later in Section 6.5.2. This was a contrast to Lisa in Doone who as the conducting teacher in their first research lesson, did not at all attend to or reflect on how students had been thinking about the mathematic activities within the lesson.

Eileen, a newly qualified teacher, contributed very little to the conversation in this first post-lesson discussion. This was expected from the literature since as an initial teacher it was likely her professional noticing had not yet been attuned to focusing on how students were thinking about this activity (Jacobs et al., 2010; Mason, 2002). On the other hand, teachers such as Gerald, Judy and Fiona (all experienced teachers) had enjoyed the observation process and all shared strategies that their observed groups of students had employed during the lesson.

Noticing of students' mathematical thinking began to be incorporated within the planning of subsequent research meetings. While Walter had not contributed much in the way of noticing and interpreting students' thinking in the first post-lesson discussion, he began to play a role in feeding his noticing of students' mathematical thinking into the next lesson study cycle. Before the group had planned the second research lesson, both Walter and Stephen began using the array model (or area model) method of factorising expressions in their own teaching as sketch lessons. They reflected on how students had reacted to this strategy in subsequent planning meetings, which added to the cycle of teachers' reflection of student work being incorporated into the anticipation of student thinking within another lesson. In this manner, Walter and Stephen both enhanced their KCS of this particular topic while sharing this knowledge with the rest of the lesson study community.

Walter *Using the area model, they almost seemed to intuitively get that idea and it just seems to sort of arrive at it more easily. It doesn't seem as abstract.*

Fiona *But you didn't multiply letters together using the area model did you?*

Walter *Yeah, I did because I just drew a square of x by x and said 'Ok, what is the area of that square?' and they knew and said it was* x^2*...*

Dave *So what kind of stuff are you doing with the area model? What sort of questions are you doing?*

Walter *Well today we did all the squares actually, which is just interesting. We did* $(x + 1)^2$ *and* $(x + 2)^2$ *and stuff like that and just looked for patterns.... immediately everybody, the weakest person just writes down* x^2 *straight away there, every time no problem.... A couple of them were really trying to think and they were adding things together and subtracting...*

Walter reported back to the group how his students had engaged with the concept of using the array models for factorising quadratic expressions. He noted the strategies that

students initially engaged with (noting x^2 in the top left box first) and also reflected on how students further investigated the factorised model. Stephen later added to this conversation in reflecting that his students responded to this strategy since they found it easier to think about and visualise: "*some of them just find it easier to see.*" Here, the teachers in the group recognised that noticing and reflecting on how students were engaging with particular strategies could be incorporated into subsequent research lessons and other lessons in their own teaching.

In the post-lesson discussion of research lesson 2, all of the teachers who observed the lesson reflected on the strategies students had used to attempt to model a quadratic expression using the algebra tiles or using the array model. Eileen, who had been quiet in the previous lesson study cycles, contributed in this discussion in reflecting on and interpreting the strategies of the students she had been observing. Stephen also reflected on how students were using the tiles by "*walking across them*" with their fingers, counting as they went and he was surprised by all of the ways students thought of using algebra tiles when they were unsure of the rules. Eileen joined Stephen and for the first time reflected on a strategy she had noticed a student using within the lesson.

Stephen *A lot of them were walking across them...and I actually found that they had to have a rule for it because some of the things they did with them are amazing.*

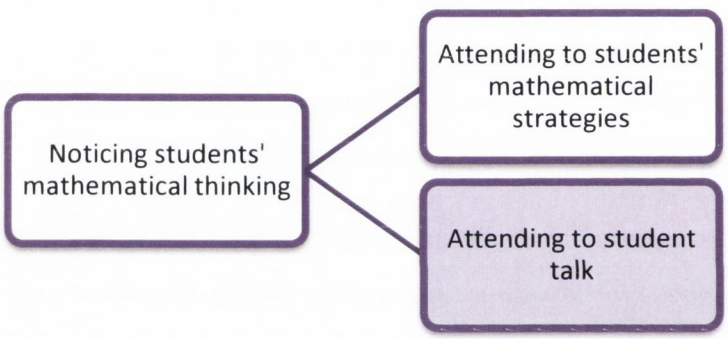
Eileen *And then you would use that as well as across the top, because as Andrew was saying as he was reading across the top, he wasn't seeing* $x + 6$*, he was seeing* $x^2 + 6$ *and as he read the side he read* $x^2 + 1$.

Eileen interpreted in detail how one of the students was attempting to factorise the expression $x^2 + 7x + 6$ but was misreading the tiles. This was an important change in Eileen's contributions since her observation within the research lesson was now attuned to attending to and interpreting how the students engaged mathematically with a question. In this lesson and the subsequent re-taught lesson, both the conducting teachers (Fiona and Walter) also noticed, interpreted, reflected on and shared how students within the lesson had gone about attempting to answer questions within the lessons.

By the final research lesson, all of the participating teachers in Crannóg reflected on the strategies that students had employed within the lesson. Walter referenced one student gradually finding more accurate values of the slope at a point and more of his colleagues referenced student talk in describing the strategies they undertook during the research lesson (described in the following section) discussing how students had gone about doing the activity of finding the instantaneous speed at a particular point and reflected on how engaging students with a strategy of finding the speed closer and closer to a particular point brought to life a concept of limits for the students.

5.3.4.2 Attending to Student Talk

In this section, teachers' observation, interpretation, and reflection on student talk during research lessons will be investigated according to the data generated in teachers' lesson study conversations. This is the second of two categories under the theme of 'noticing students' mathematical thinking' and the data will be chronologically analysed for each separate school as in previous sections.



Teachers can assess students' mathematical thinking by listening to a group's conversation as they work (M. K. Stein et al., 2008) and within this project the live research lessons provided an extraordinarily rich resource for teachers to access students' talk and thus take note of what students were thinking about and communicating to their classmates.

The data suggest that attending to student talk provided teachers with invaluable access to how students were responding to and thinking about activities included within the research lessons. In attending to students' strategies, listening to student talk provided teachers with a clear route into interpreting how students were thinking within a lesson, adding to their KCS. However, attending to students' talk was not necessarily useful unless teachers used this access to interpret what students were saying in order to allow them to hypothesise about how students were thinking.

It must be noted that attending to student talk, as referred to in this section, is different from teachers utilising a student voice within planning meetings which will be discussed

later in Section 6.4.3. Under this feature, teachers had access to how students communicated their mathematics with one another and recorded or paraphrased students' talk in order to highlight or reflect on an incident within the lesson with their colleagues.

5.5.5.2.1 Doone

In Nora's very first contribution to the first post-lesson discussion she incorporated a students' conversation with himself, identifying his own mistake. Throughout the research Nora, a teacher experienced in teaching students in a one-to-one or small group basis, often focused on how students were communicating their mathematics in order to identify how they were thinking about a topic.

> *Nora* *He was doing a lot of multiplying by 2, but when we were doing the exercise – this one here – he went "oh no, no. I'm multiplying by 2. I multiplied by 2." So he was stopping himself doing this.*

Nora's regular incorporation of student talk in her post-lesson discussions is interesting since it is a strategy she herself used in highlighting to her colleagues how students within the class were thinking. It may have also influenced her colleagues in them recording and sharing how students they had been observing were thinking about their mathematics, thus enhancing teachers' KCS. Within that same first research lesson Owen identified one student's comment during the lesson which he thought had demonstrated the student's understanding that all sides of the square were the same, but this comment was in answer to the teachers' direct question and did not demonstrate that this was new information to the student. Owen was not yet focused on identifying student learning or in interpreting students' mathematical thinking that suggested a developing of their understanding.

Within this lesson Kate's role was to record students' questions within the lesson and this made her focus on student commentary within the lesson. She reflected on one particular student's question which she found very insightful in interpreting how that student was thinking about the operation of squaring (Figure 44). She reflected on this incident with her colleagues as she thought it this student's question could have played a bigger role within the lesson in talking about what the index number represented.

Kate *One I think, which would have been great was, "Is it always going to be 2 up in the top corner?."...It would have been lovely to say "Well, have you had 3 x 3 x 3?...7 x 7? And have a guess what number will be there." That would have been a little lovely addition.*

From recording this students' question, Kate suggested that the conducting teacher Lisa could have explored this students' thinking more by asking further questions including more indices. This access to student talk provided Kate with more knowledge of how this content could be developed for students, enhancing her KCS for when she would perhaps teach this particular lesson again.

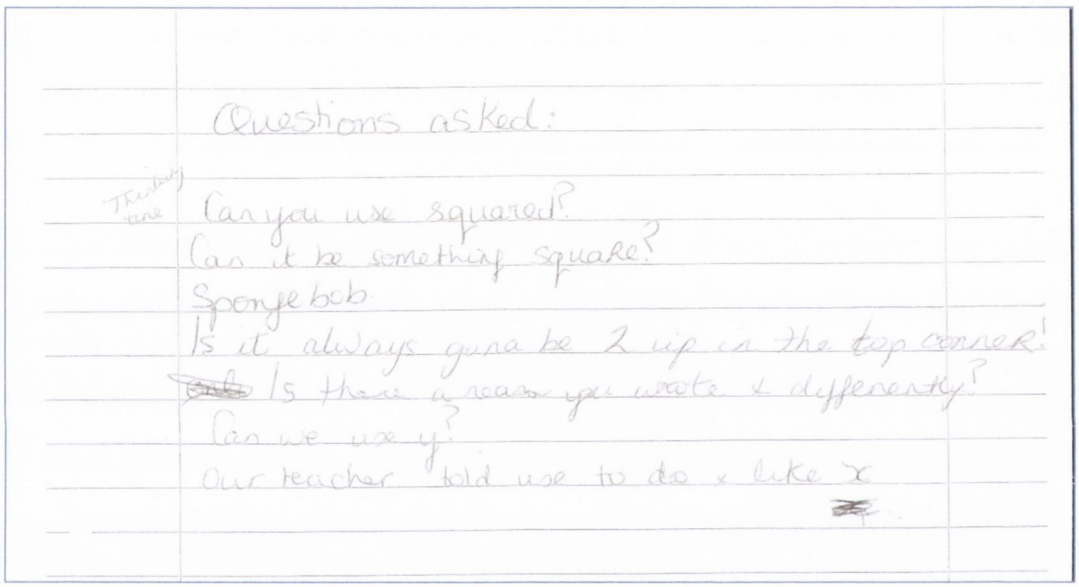


Figure 44 Kate's observation notes: Doone cycle 1, Appendix F3 1

While as a community they later decided that it was more beneficial to follow one group of students' throughout a research lesson, the activity of noting students' questions within the lesson had been important in building Kate's KCS and focusing her on reflecting on how students were thinking about the mathematics within the lesson.

Apart from Nora and Kate, teachers did not incorporate student talk into their reflective conversations and, as previously outlined, did not reflect on students' mathematical strategies within the lesson. This could be due to the fact that teachers were not used to focusing on how students were thinking and also due to the fact that within that lesson, students had no particular activities which encouraged conversation with their school-mates or as a whole-class discussion. However, teachers were most likely not focusing

on students' talk since they were not focusing on students' strategies because they had not anticipated student thinking in the planning of this lesson. Anticipating student thinking, a key feature of KCS already explored in this chapter, was an important factor in teachers' motivation and impetus to notice students' mathematical thinking within research lessons.

In research lesson 2 students worked together in groups with two groups presenting their finished work to the class. There was therefore far more student talk which teachers had opportunities to notice and reflect on. Lisa reflected a number of isolated student comments which demonstrated both what she had enjoyed about observing the lesson but also in identifying moments of learning for the students.

> *Lisa* *It was lovely to hear Adam say when he was working on the board "now I get it!" I just loved it, you know? And when you were asking about the ½ multiplied by 8 and 8 multiplied by a 1/2 he went: "It's the same."*

Lisa's attention to student talk is more demonstrative of the fact that this student learned something in the lesson instead of utilising student talk in order to decipher how this student was thinking about a particular question. But, it is of import that Lisa was attended to how students were talking and thinking about the mathematics. Later, after her colleagues had reflected on students' strategies, Lisa did take note of how students had thought about a particular activity and reflected on their strategy in terms of student talk. In this case she was not repeating what she had heard students say but was using student talk to exemplify how she interpreted how students were thinking.

> *Lisa* *They put all the fractions in relation to a half so when they were asked to do 2/3, they didn't take the number line and say "Well, there's 1/3, 1/3 and so there's 2/3." They just said "It's greater than ½."*

Utilising a student voice allowed Lisa to focus her thinking about the mathematics from a students' perspective, an idea which will be referenced in the next chapter (Section 6.4.3).

In this lesson the teachers seemed more entertained by the fact that they had access to students' talk rather than using it as an insight into how students were thinking. Kate reflected on what the presenting group had said when they began their presentation to the whole class.

> *Kate* *They were brilliant, they were fantastic, and when they said "We went to the end of the story to get the answer", that was perfect!*

In this excerpt Kate commented on the fact that she was impressed by the students but did not delve into how students might have gone about the question by "*starting at the end.*" While Lisa extols the fact that they got to listen to students "*arguing, discussing it among themselves*", teachers did not bring these student conversations into the post-lesson discussions.

By the third research lesson there was a change in how more of the teachers attended to students' talk. In this post-lesson discussion Kate, Michael and Nora brought in conversations that students had about the mathematics questions they were working on within the lesson. Michael reflected on how his group of students spoke about the mathematics to one another and compared it to how they were actually able to do the mathematics question.

> *Michael* *It's just the same with Stan in the group that I was watching. He was the guy I was saying – he said "Nah, I, there's just not enough information there for me to do that question" but he was calculating how high the ball would go and all that stuff and then the guys who were straight into the maths were like "Oh yeah, that makes sense."*

Michael used this example to reflect that students had little confidence in their own ability for the first question they worked on, but that this improved as students were able to see how other students had answered the question during the presentation phase of the lesson. Later Michael also attended to student talk in reflecting on how students were engaging with the mathematical question:

> *Michael* *The kids were having a great chat about well, if he was tackled or you know if he was charged down and all that sort of stuff, and they were talking about all these while they were doing the problem. And*

here they were talking about the temperature rising and then they were saying, well, that they were joking like "Ahh! She is going to explode"

Attending to students' talk not only gave these teachers access to how students were thinking about the mathematics within the question but also how they were engaging with the mathematics question and with its content, investigated further in Section 6.5.2.

This trend of attending to student talk and incorporating student commentary within the post-lesson discussions continued in the next cycle. While in the first research lesson Nora had been the only one to reflect on students' talk in identifying their own mistakes, in the final post-lesson discussion more teachers followed this example. Lisa reflected on students noting and correcting their own mistakes within the lesson. She also utilised student talk in highlighting how students' went about answering a question:

Lisa *On the mobile phone question they were saying "Where is the right angle?" and they were talking about the hypotenuse.*

It is interesting here to note that Lisa attended to students' strategies and talk in this lesson study cycle, since it was in planning this lesson that Lisa realised she had not previously been thinking about the mathematics from a students' perspective (Section 5.5.4.1). Within this post-lesson discussion Lisa was attuned to reflecting on how the particular students she had been observing had thought about the mathematics within the lesson.

Owen did not utilise student talk in accessing student thinking in the post-lesson discussion but had begun to focus on the strategies students were using within the lesson and had begun to reflect on students' mathematical thinking within the lesson and sharing his observations with the rest of the community. Because this post-lesson discussion occurred after students' lunch break, a number of class periods following the research lesson, Owen had had unusual access to students' talk outside of the classroom setting. He reported back to his colleagues the student conversation about Pythagoras which he had witnessed out in the school yard.

Owen *I was on yard duty there and Alan and the little boy beside Alan, they were showing Dan how to do it both of their ways. I went over*

and I was like "What are you doing?" because I had seen them draw the right angled triangle against the wall outside and they were showing Dan

Kate *That is amazing!*

Owen *And they were showing him the two ways that they could do it. There you go.*

Lisa *Wow!*

Owen *We just achieved something, well Michael [the conducting teacher] achieved something there.*

The community of teachers were enthused by this incident where students taught Pythagoras' theorem to their friends after the research lesson. Because teachers' had anticipated student activity within this lesson, accessing and attending to student talk became important in highlighting to the teachers how students were thinking about the mathematics questions contained within the lesson.

By the end of the research, all of the teachers were beginning to attend to student talk as a way of accessing how students were thinking about the mathematics contained within the research lesson.

5.5.5.2.2 Crannóg

In their first post-lesson discussion teachers from the beginning began reporting back on what students had said within the lesson and utilising this as a way of explaining to their colleagues how particular students had interpreted and gone about identifying the quadratic pattern within the lesson. Teachers were delighted when verbal responses they had anticipated within the planning actually occurred during the research lessons. For example, in planning lesson 1 Dave wondered "*Would anyone say 'when you plot them, you don't get a line?*"" and later in the post-lesson discussion Dave reflected on the fact that a number of students did in fact plot their pattern and communicated their findings saying that "*it's not linear*" (Figure 45). This corroborated the fact that revising students' prior knowledge allowed teachers to more accurately anticipate students' work.

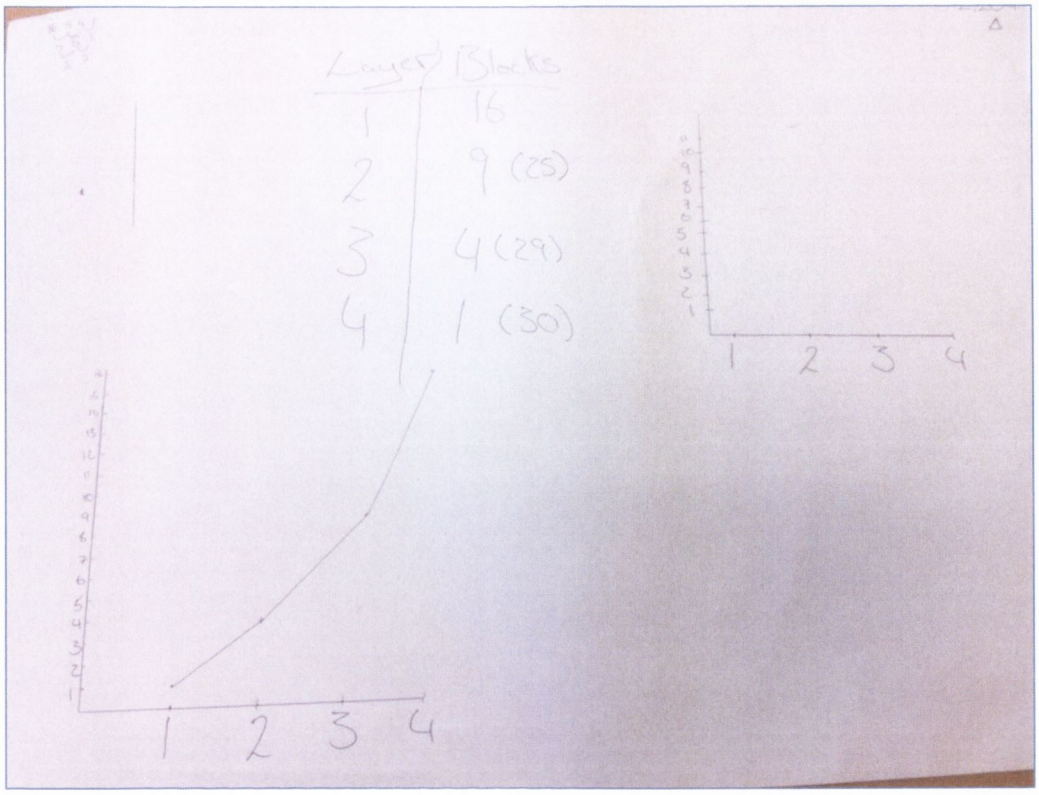


Figure 45 Sample of student work: Crannóg cycle 1, Appendix F4 4

Gerald was also delighted to reflect on the moment where students realised there was a pattern to the blocks.

> *Gerald* *It was only when they came to the point where they realised there was only going to be one at a top and four in the second and nine in the third that they started realising "Oh, there is something going on here – there is a pattern!"*

Detailing what students had been saying to one another and how they had said it became important demonstrations within post-lesson discussions of students building their understanding within research lessons.

Teachers also noted episodes of student talk where students had engaged with an idea which was not correct as recorded in Stephen attending to students' strategies earlier in Section 5.5.5.1.2. Fiona also noted where students had said that "*The difference doesn't really matter – that's just a coincidence*" and had not in fact realised that the second difference would be common for all of these strategies. From this conversation Fiona concluded that it would take a number of lessons with students investigating such patterns for them to realise that it was not in fact a coincidence but would always be the

case, an important point since accessing students' talk and interpreting students' strategies positively fed into teachers' practices.

Interestingly in this post-lesson discussion, Gerald also reflected on the inadequacy of the recording instrument for the lesson, which I had designed to record students' questions based on the research literature (C. Fernandez et al., 2003; Lewis & Hurd, 2011). Gerald suggested in the post-lesson discussion that in the future research lessons, the observers should record all student interactions or moments of interest that they as observers found interesting and not just student questions (Appendix F3 2). The instrument was then modified slightly to record all student comments (Appendix F3 3).

In the second research lesson, teachers noted that students had not really spoken to each other nor was there much impetus given to them to converse with one another. Walter, Stephen and Dave all suggested questions that the teacher might instead ask in order to encourage students to communicate their thinking and these were incorporated into the re-teaching of that lesson. In the re-teaching of this lesson teachers again had little to reflect on in terms of student talk, however, and this may have been due to the fact that there were only very small numbers of students (8-10) in both classes for research lesson 2. Also, both conducting teachers reflected within the respective post-lesson discussions that these classes contained students with very many learning difficulties and the majority of students preferred learning on their own as opposed to having to collaborate with one of their school mates.

In the final research lesson, the teachers all commented on the fact that these senior students did not seem to enjoy working in groups and that there was much lesson communication between students than they had anticipated. In Dave's observation sheet he noted "*silence*." Nonetheless, students had been encouraged to work and communicate with one another in their groups and teachers did attend to some of the student talk which had been observed within the lesson and reflected on the insight it gave them on how students were thinking about the instantaneous rates of change which they were attempting to calculate. Fiona and I reflected on how students were thinking about limits through their conversations with one another.

> *Aoibhinn* *That girl, the blond girl, was saying "you're never going to exactly get there" which I thought was quite good that they were introduced to that concept.*

Eileen *Yeah*

Fiona *Yeah, because you never do. The formula that we use is based on a limit which is getting very very close, but not quite getting there... They were talking about zooming in. They were looking at the two graphs and said "Okay, this one zoomed in is the bigger graph" and Alan said "If you just got on the computer and kept zooming in and zooming in, and then you could actually get it to."*

Here, two students had two different ideas: that you could never actually arrive at the limit or that you could if you could zoom in close enough. In attending to students' conversation in this way, Stephen had the idea of incorporating GeoGebra within the lesson and showing students what would happen if you continued zooming in on a particular point. Here, attending to students' talk gave teachers the opportunity to reflect on how students were interpreting the mathematics and to suggest how they could build students' thinking at that particular point in the lesson. This activity was incorporated by Eileen, the conducting teacher, in her next day's teaching following the research lesson.

5.3.4.3 Conclusion of Attending to Student Talk

While attending to student talk was not possible if students were not encouraged to converse with one another, accessing students' conversations with one another and with the teacher did allow teachers to make assessment of how students were thinking and how that thinking could be developed. Noticing how students' thought about the mathematics through accessing how they spoke about the maths with one another, provided teachers with opportunity to modify or develop activities within the research lesson.

5.3.4.4 Conclusion of Noticing of Students' Mathematical Thinking

When teachers had anticipated how they expected students to act and react during the lessons, they attended to students' mathematical thinking during their observation, and reflected both on students' thinking and on their predictions as teachers within the planning. Attending to students' strategies in the form of noticing, interpreting and reflecting on how students answered particular questions, was an important part of the post research lesson discussions in building teachers' knowledge of how students understanding of a certain topic could be enhanced. In Crannóg's first research lesson

and Doone's fourth research lesson (where teachers had planned a sequence of lessons in order to build on students' prior knowledge – Section 5.5.3), teachers could more accurately and succinctly anticipate students' responses within the research lesson and thus attended to and reflected on students' mathematical strategies within these lesson study cycles. In their reflections teachers focused on and dissected how students went about the activities, what particular questions students asked of the content, of the teacher and of each other, and reflected on how students communicated their mathematical thinking during the lesson. These features incorporated within the lesson study model impacted positively on teachers' KCS when they began viewing the teaching and learning of mathematics through the eyes of students.

Where lessons had been planned without specific reference to students' prior knowledge or without spending time anticipating how students would respond to questions and activities within the lesson, teachers' reflections consisted of discussing the implementation of the lesson - such as in Doone's first research lesson where the post-lesson discussion revolved mostly around the conducting teacher. In this post-lesson discussion teachers did discuss some changes that could have been made to the lesson and its sequencing, but teachers in the whole did not focus on students' mathematical thinking. This contrasted greatly to the penultimate and final research lessons in Doone where teachers' post-lesson discussion revolved around how students engaged with the mathematics due to the fact that more of the planning conversations had anticipated how students would interact with the mathematics within the lesson. In Doone as teachers continued their engagement with lesson study, they incorporated students' prior knowledge, anticipated students' responses and became more aware of noticing and interpreting students' mathematical thinking and built this into their subsequent research lessons. As this community of teachers participated more in lesson study, their KCS was enhanced and as part of this knowledge, teachers began thinking more about mathematics from the perspectives of students they were teaching.

In Crannóg the majority of teachers had begun their participation in post-lesson discussions by attending to students' strategies and by the end of the academic year all of the teachers were attending to how students were going about answering mathematics questions. In the final research lesson Eileen, who had initially been very quiet in planning and reflection meetings, participated with the rest of the community in anticipating how students might answer questions and in what sequence activities should

be planned in order to best build on their understanding of rates of change. Within the post-lesson discussion of that final research lesson, she also contributed in reflecting on strategies which students had employed within the lesson. Walter thought that the observation was the most important part of the lesson study cycle since allowed teachers a window into watching how students actually thought about and engaged with mathematics within the classroom. Dave felt that the post-lesson discussions were the most important part of the cycle since it allowed teachers to 'fine tune' the lesson in analysing its success afterwards. In Dave's final interview he reported that due to his participation in the lesson study cycles, he had changed how he thought about his students and changed what he thought what mathematics they were capable of.

There was a difference between the two schools in how they observed their students from the outset. Teachers in Crannóg had decided that they wanted each teacher to follow one group of students as observers within the research lesson. Even I, as researcher, had been included by the teachers as an observer in the first and final research lessons. This impacted on how reflective conversations revolved around students' engagement, both mathematical and personal, within the research lessons and teachers continued with this strategy of observation throughout the academic year. In Doone, teachers decided to also follow this model in research lessons 2, 3, and 4 and it was during these cycles that teachers' conversations (both in planning and reflection) became more focused on the mathematical thinking of their students. It may be that this model of observation, where teachers follow a specific student group for the duration of the research lesson, focuses teachers on identifying the strategies, questions and conversations which give teachers an insight into how students are thinking about and engaging with mathematical tasks. This may be a point to investigate further in future research.

Attending to students' strategies and questions within a research lesson emerged from teachers anticipating students' responses, but it also built on teachers' ability to anticipate students' mathematical thinking and the strategies they would utilise in later lessons, contributing to the cyclical element of lesson study and to the importance of participating in iterative cycles of school-based lesson study. Doone's second post-lesson discussion involved far more concentration by teachers on students' mathematical thinking which, in turn, led to teachers thinking more about students' mathematical thinking than had occurred in the first post-lesson discussion and this trend continued in the following two lessons. The professional noticing of mathematical thinking, which in itself developed as

the cycles continued, also led to teachers having a better understanding of how to anticipate how students would react and respond to mathematical questions and activities included in subsequent research lessons. From this analysis it is obvious that observing research lessons within lesson study enhanced teachers KCS through accessing, attending, interpreting and reflecting on how students thought about particular mathematics topics which could be incorporated into their future and further teaching of these topics.

5.3.5 A Focus on Students' Mathematical Thinking Enhancing Teachers' KCS

Through participating in iterative cycles of lesson study, the data suggest that teachers enhanced their KCS in terms of: incorporating students' prior knowledge into their teaching, anticipating students' mathematical thinking in planning lessons, attending to how students approached and communicated mathematics within the classroom and reflecting on how this might impact on future teaching of this topic. As teachers participated in lesson study with more experienced members of their Mathematics Department, they became more attuned to how students thought about mathematics and incorporated this employ of a critical student lens into the elements of KCS listed above. Anticipating students' mathematical thinking within planning meetings led teachers to professionally notice how students were thinking within the research lessons. If they had anticipated or imagined what those strategies might have been in the planning of the lesson, teachers focused on students' mathematical strategies and utilised student talk to identify these strategies. Teachers' KCS was enhanced through the process of engaging in conversations around what students might do within a lesson and was further enhanced by finding out how students went about solving those same mathematics questions within the research lesson. Participating in the stages of planning, observation and reflection was particularly important in providing teachers with the opportunity to posit, interpret and reflect on how the particular students in this class in their school engaged with the mathematics. It was also important that teachers participated in iterative cycles where their observations and reflections could feed into future lessons.

The data shows that, similar to investigations of video studies as professional learning (Sherin & Han, 2004a), teacher conversations in lesson study began to move from an initial focus on the teacher and teacher activities, to one where the primary focus was on

the student and student learning. As their participation in lesson study continued, teachers began to articulate their ideas on mathematical strategies, classroom practices and teaching approaches in terms of student learning and were less restricted by the curriculum or by traditional approaches. This was particularly evident in Doone where their initial research lesson very much focused on the activities of the conducting teacher with only 21 minutes of the planning meetings focused on building students' mathematical thinking (according to the features of KCS and KCT utilised in this framework) compared to 70 minutes of the planning of their final research lesson constructed with focus on the student and on building students' understanding and mathematical thinking. Interestingly, planning in the first cycle took a total of 4 hours by which time teachers were "*sick to the teeth*" (Lisa during lesson study cycle 1) of the lesson while planning for the final research lesson took almost the same amount of time but much more of which was focused on the student.

A shift came in teachers' in Doone thinking in the second planning meeting of lesson study cycle 2. In this lesson, teachers other than Nora began really thinking about the mathematical content from students' perspectives as opposed to just focusing on content or focusing on the activity of the teacher. In this lesson teachers such as Kate and Lisa highlighted common student errors, anticipated students' mathematical thinking and the group began to sequence the lesson in a way which would build on students' understanding. Within this meeting, teachers read literature focusing on students' perspectives of learning which may have been an important influence in directing how teachers began to think about building this research lesson around the group of 1st year students (Section 5.5.4.4). In cycles 3 and 4, teachers held the student at the centre of the planning and built research lessons around how they could best build on students' mathematical thinking. In cycle 2 Lisa incorporated student thinking into planning after hearing commentary from Nora in a previous planning meeting. In cycle 4 Lisa identified for the first time Pythagoras' theorem from a students' perspective. Michael became more detailed and clear in his anticipation of students' thinking and in cycle 4 incorporated his anticipation of how students thought about mathematics into the planning of the lesson series. Owen also began to view mathematics not from his own perspective as teacher, but from a students' perspective. This impacted on his classroom practice in becoming more attuned to how students communicated mathematics with one another and how this benefitted teacher learning as will be explored in Section 6.5.2

Nora did not feel her attention to student thinking had changed throughout the research and it would be fair to say that her KCS may not have improved due to her participation in lesson study, but Nora did feel she had gained in sharing different ideas of sequencing lessons and creating contextualising questions as will be further discussed in Chapter 6. While Michael and Kate reported in their final interview that they had always thought about mathematics from their students' perspectives, both teachers also felt they had gained insight into students thinking from observing students within research lessons. Michael noted that participating in lesson study reinforced the importance for him in employing a critical student lens in all of his teaching, not just of mathematics.

Michael *It just reinforces to me that it's really important to always think 'How do kids see it?'. You know we're doing final accounts there on the board and for me in school, you know, you get lost in there – so I'm always conscious of the fact that this is how I felt at that time. How do they feel? Where are they at? It's really important...It does remind me that it is important but I always do think 'how do they look at it?'*

In Crannóg, the community of teachers began their engagement in lesson study with more experienced teachers leading but not dominating conversations with an embedded focus on the student. As their participation in lesson study continued, more of the individual participants within the community began incorporating a critical student lens to their planning, conduction of and reflection of research lessons.

All of the teachers felt that they were thinking more about how their students were thinking about mathematics, as self-reported in their final interviews. Walter felt that it was attending to student talk in observing research lessons that had brought about this enhancement of employing a critical student lens in his practice. Dave and Gerald also both agreed that they were thinking more about mathematics from their students' perspectives and that planning with their colleagues, hearing different ideas, benefitted them adopting that perspective in their classrooms. Eileen noted that she now employed a critical student lens within her own teaching outside of the lesson study cycles:

Eileen *Yeah, I probably would ask myself a bit more how would they react to this or what questions will they have – pre-empt their questions or*

pre-empt their confusion. Yeah, I would think about that a little bit more.

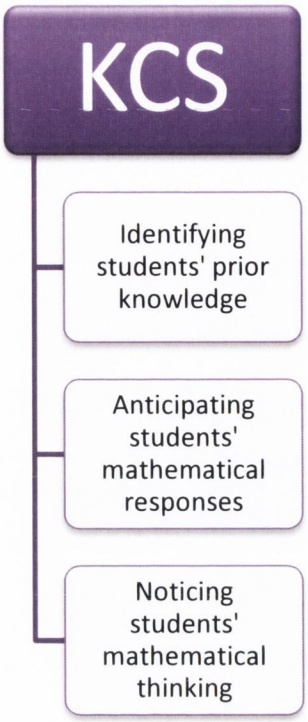
Surprisingly, both Stephen and Fiona, who throughout all of the planning and reflections focused on how students were thinking and doing mathematics, both felt that they were thinking more about how students were thinking due to their participation in lesson study. In her final interview Fiona, who led the charge on inquiry learning and in dissecting content in order to plan it in a way which would build on students' mathematical thinking, says that observing students gave her an insight into the strategies students used in their mathematics.

Stephen also suggested in the final two teacher meetings in Crannóg that he has begun to think more from a students' perspective. A point that the data would suggest is not entirely true since Stephen's interjections from the beginning were always focused on the student, but perhaps he had not realised this until participating in multiple planning and reflection discussions with his colleagues.

Within the boundaries of this research, teachers who were not employing a critical student lens in their practice prior to participating in lesson study, were more aware of students' mathematical thinking by the end of the research and were incorporating the employ of this lens in their teaching as evidenced in the data by teachers' own self-reporting in their final interviews. Experienced teachers who had already been incorporating students' mathematical thinking in their practice felt they had benefitted from participation in lesson study through collaborating with their colleagues in devising lessons that were creative and engaging for students.

It is important to note that the learning in both communities were likely positively impacted by the fact that experienced teachers such as Nora and Fiona who employed a critical student lens on their practices were active participants in the community. In both schools teachers such as Michael and Stephen also positively influenced conversations in focusing on how students would think about and respond to particular topics. This would lead me to conclude that for an in-school model of lesson study, it would be important that newly qualified or pre-service teachers would participate in lesson study communities that included a broad range of teaching experiences in order to provide teachers with opportunity to enhance their KCS through discussions with other teachers.

5.4 Conclusion of Chapter



In this chapter, the data findings on enhancement of teachers' KCS through their participation in iterative cycles of lesson study were discussed. Three features emerged from the data and are corroborated by the literature on professional development, mathematical knowledge for teaching, and professional noticing of students' mathematical thinking. Through these features of: prior knowledge, anticipating students' responses, and attending to students' thinking established in the previous chapter it can be concluded from this research that teachers' KCS increased as they participated in the planning, conducting, observation and reflection of research lessons in a number of distinct ways not previously examined or detailed within the literature.

In an environment where teachers were not familiar with working with one another, with sharing ideas, with articulating mathematical content etc. it was ambitious to expect teachers to discuss items other than how best to construct content, the mechanics of a lesson and their individual roles within the lesson study. This was the case in Doone where mathematics teachers neither met as a department nor shared resources prior to their participation in lesson study. As teachers become more and more familiar with one another and with each other's contributions to each lesson study cycle, teachers became more focused on developing content within research lessons and on developing contextualised and relevant resources within those lessons. As their participation in lesson study cycles continued, teachers began reviewing their practice more critically as well as viewing the mathematics they were teaching more from a students' perspective.

More experienced teachers, such as Nora, Fiona and Gerald, did not seem to gain as much in terms of KCS when compared to their less experienced colleagues, but their colleagues greatly benefited from these teachers sharing their insights. Newly qualified teachers such as Kate and Eileen greatly benefited from participating in lesson study and we saw a shift in how their focus on teaching was on the student at the end of the

research. Lisa had an important learning moment in planning of cycle 4 where she realised she had not been thinking about mathematics from a students' perspective and self-reported that she was doing this more-so in her practice. Owen had incorporated more of students' prior knowledge into his teaching. Dave, Stephen and Walter all self-reported that they had begun thinking more about their teaching of mathematics from their students' perspectives at the end of the year.

Participating in lesson study over the course of one academic year over a number of lesson study cycles had a positive effect on teachers' PCK in building on their KCS which impacted on their teaching outside of lesson study.

Chapter Six

6 Analysing the Employment and Development of KCT and TKS through Iterative Cycles of Lesson Study

6.1 Introduction to Chapter

This chapter is a continuation of the previous data analysis Chapter 5 which employed the theoretical framework described in Chapter 4 of teacher knowledge to the data generated in this research. The purpose of this chapter is to employ the analytic framework, as described in Chapter 4, to investigate the changes to teacher KCT due to their participation in iterative lesson study cycles on the basis of the data sets generated in this research. Evidence of teacher knowledge of students (TKS) will also be explored according to the data sets generated in this research. Similar to the previous analysis chapter, changes to teacher knowledge are also based on the development and employ of a critical student lens on teachers' practices. As a conclusion to the analysis, changes to teacher knowledge and the cross-over in these forms of knowledge (KCS, KCT, and TKS) as evidenced in the data will also be discussed.

6.2 Structure of Chapter

This chapter is separated into two sections. In the first section, changes to teachers' KCT will be investigated and the analytic framework will be applied to both Doone and Crannóg schools separately. Within this framework, the chapter proceeds by way of a chronological analysis of teachers' conversations and interviews during the research and over iterative cycles of lesson study as was done in the previous Chapter 5.

In the second section, how teachers utilised and enhanced their knowledge of students will be investigated and the analytic framework will be similarly applied to both Doone and Crannóg schools separately.

Within this analysis, examining what influenced teachers to make changes to their practice, through their developing knowledge of content and teaching and of their students, was critical in attempting to fully understand how we can utilise and improve this model of lesson study in mathematics teachers' learning.

6.3 Knowledge of Content and Teaching: a Focus on Deconstructing Mathematics for Students

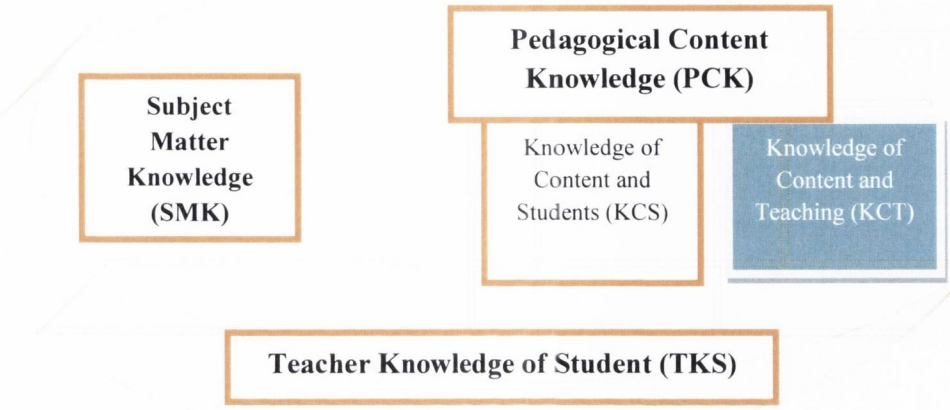
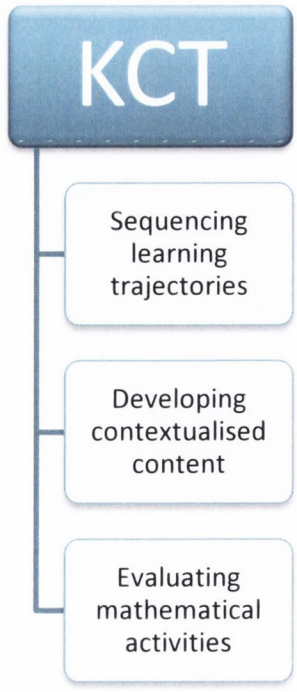


Figure 46 An Analytic Focus on KCT

In this section, evidence for the enhancement of teachers' KCT will be explored. Within this element of PCK, I claim that teacher knowledge was enriched through their employ of a critical student lens in the planning, conduction, observation, and reflection of research lessons.

This analysis will follow a framework, explored in Chapter 4, which emerged from the coding of the generated research data under the following three features:

1. Sequencing learning trajectories
2. Developing contextualised content
3. Evaluating mathematical activities



As in the previous chapter, the data will be analysed chronologically for both case sites.

The following table outlines the frequencies

of each knowledge code related to KCT occurring in different lesson study meetings throughout the data generation in this research.

Table 8 Frequency of KCT codes

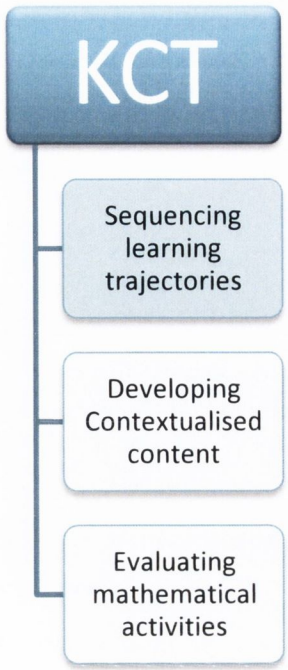
		Doone					Crannóg		
Cycles		1	2	3	4		1	2	3
KCT									
Sequencing learning trajectories	Sequencing content within research lesson	3	16	11	16		15	21	35
	Sequencing content over series of lessons	1	6	10	22		8	5	20
Developing contextualised content		3	13	20	41		23	11	36
Evaluating mathematical activities		1	18	15	28		21	29	29

As is discernible from the table, as the cycles continued both communities began to incorporate planning of a series of lessons and began to more frequently evaluate the mathematical activities within the research lesson from the perspective of the student. Contrasting the first and final cycles in both schools, teachers also began to develop their own contextualised content within research lessons. A narrative illustration of each of the codes will now be explored in the following sections.

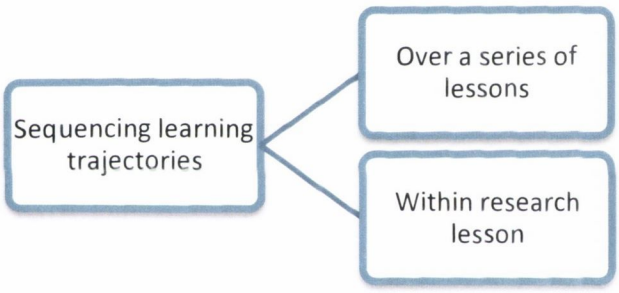
6.3.1 Sequencing Learning Trajectories

The importance of sequencing learning trajectories emerged as a feature of the data as teachers began developing series of lessons within the lesson study cycles. These series of lessons, which included the research lesson, built on students' prior knowledge and provided opportunity for teachers to also plan students' learning following a research lesson. As teachers' participation in lesson study continued, they also began to plan in greater detail the sequence of activities which would be contained within the research lessons. Thus in this section the theme of sequencing learning trajectories will be analysed under the following two categories:

1. Sequencing learning trajectories over a series of lessons
2. Sequencing student learning within a research lesson



As an aside, unlike teachers in Fernandez et al.'s (2003) study, teachers were not always concerned with planning a lesson which could be used by all members of the community. Teachers in this research, however, were willing to spend time planning a lesson that they might not have opportunity to teach themselves since they felt they could adapt it for other year groups if and as the need arose.



While this mapping of a series of lesson within which the research lesson resides has been detailed in Lewis & Hurd as mapping out a unit plan (2011, p. 49) within the research literature this has not been highlighted as an important aspect of lesson study. In both schools, teachers arrived at the decision themselves to design a number of lessons, sequencing students' learning trajectories through this planning activity. This feature of KCT of designing a series of lessons within lesson study is a contribution to the literature in expanding a model of lesson study which may be relevant for post-primary teaching.

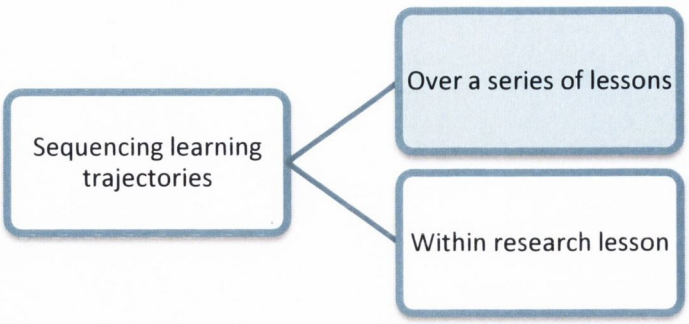
Teachers also regularly referenced the Project Maths syllabus in planning these series of learning in order to identify which learning outcomes were being included within the learning trajectory.

In deciding how to sequence content within lessons teachers often shared their experiences of how they taught particular topics or utilised particular strategies. It was during these conversations that experienced teachers shared their approaches with one another but also with more newly qualified teachers who often had innovative ideas on

approaching a topic. In this regard, everyone participating in these conversations added to their KCT in hearing new ideas about teaching particular topics.

6.3.1.1 Sequencing Learning Trajectories over a Series of Lessons

This is the first of two categories within the feature of sequencing students' learning trajectories as part of teachers employing and enhancing their KCT. The data will be analysed chronologically on both case study sites and changes to how teachers began approaching sequencing content within lesson study cycles will be detailed.



6.3.1.1.1 Doone

As outlined in Chapter 5, research lesson 2 was structured a continuation of lessons introducing fractions to 1st year students. Following the first lesson study cycle where teachers felt the isolated research lesson did not succeed in building students' understanding of x^2 and stood out like a "*sore thumb*", I suggested that the teachers track where the research lesson might be situated in terms of a larger series. The lesson template (Appendix C1) suggested situating the research lesson within a unit of lessons, but the community of teachers still balked at the idea of planning a sequence of lessons to lead up to their lesson on multiplying fractions. However, they did incorporate the teaching and learning plan 'Introduction to Fractions' and 'Multiplication of Fractions' (part of the curriculum materials developed by the PMDT (2011)) as the lead in to this research lesson. As part of this planning process teachers discussed continuing the series of lesson with a deeper focus on division of fractions. While this lesson could have been planned with more attention on content, the conversation suggested that teachers were beginning to think of students' learning trajectories for more than one 40 minute lesson.

Research lesson 3 was planned in a slightly different context to the other lessons since Kate was teaching these Transition Year students as part of an 8 week problem solving module unlike other classes taught from September to May. This made it more straightforward for the teachers to plan this research lesson as a component of the module and Kate suggested that this research lesson lead from students investigating linear

equations to investigating quadratic equations under the title of 'Problem Solving'. Thus this lesson was situated within a sequence of lessons and while this happened almost by chance, teachers saw the benefit of this process and repeated it in the following cycle. Teachers originally planned the research lesson to include a progression to students investigating cubic equations. In the second planning meeting Nora, who had been absent in the first planning meeting, suggested that this might be too much development of content for one lesson and thus the cubic equations were moved to a future lesson within the series. Thus, almost by chance, teachers had outlined a series of lessons, detailing students' learning trajectory, of investigating linear, quadratic and then cubic expressions.

By research lesson 4, teachers were intent on purposefully designing a series of lessons where students' understanding of applying Pythagoras' theorem would be developed. Lisa initially suggested that a one-off lesson without a lead-in of content would not be suitable for this particular group of students and Kate followed on this idea in suggesting that they begin planning a sequence of lessons within this cycle.

Kate *Do we want to just, like, because if we're just going to observe a single class? Because to kind of introduce it, teach them how to use it and them teach them where to use it would be three classes.*

Michael *Oh yeah.*

Kate *You know, it wouldn't just be a single. So what single would we focus on?...We can't focus, we can't see all three.*

Michael *No.*

Lisa *It's the application of Pythagoras, would then be demonstrating to the observers that they understand what this guy [the equation] is all about.*

This is the first time that this group of teachers agreed and decided on planning a sequence of lessons which would build students' understanding of a topic. This was an important learning moment for this group of teachers in realising that a one-off, isolated lesson did not necessarily benefit students as much as purposefully situating a research lesson within a sequence of lessons. Teachers wanted to track these students' learning through a trajectory, building on their prior knowledge and building on their

mathematical thinking. This series of lessons was developed with a particular class in mind and teachers focused on how these students would respond to particular activities as a group of "*fairly weak*" (Michael, planning meeting 3) 2nd year students.

In this final cycle, teachers discussed which part of the sequence would be most useful and beneficial for them to observe and they agree on choosing the 'problem solving' lesson as the research lesson since Lisa suggested that observing students applying their knowledge to contextualised problems would be the most interesting for the teachers. Teachers here displayed a researcher lens (C. Fernandez et al., 2003) in beginning to utilise lesson study as a way of investigating student learning and investigating particular approaches to teaching a topic.

For the first two lessons within this sequence, teachers built on curriculum materials which were available to them on-line (Project Maths Development Team, 2011) and modified activities to make them more relevant, both in content and context for this particular class of students. The community incorporated a story as a hook to the lesson which they anticipated would engage students and provide a memorable context for Pythagoras' equation (see Section 6.5.1.1). Kate had originally told this story during a planning meeting in cycle 1 at the beginning of the research. She told it again as part of this lesson study cycle since it was an item within her own practice which she was very proud of and her colleagues were happy to incorporate it within this series. For the research lesson itself, teachers developed a number of questions for students to work on. As will be detailed in Section 6.3.2, teachers designed each of these questions themselves during the planning of this lesson and also sequenced the questions in terms of providing scaffolding questions to guide the student in tackling the problem, but also in providing extension activities as a mechanism of differentiation within the lesson.

What was most important about teachers designing this series of lessons was that for the first time, teachers analysed the questions within the textbook for this topic. Employing a critical student lens for the students within this particular class, Kate was critical of the way the textbook moved too soon from focusing on finding the hypotenuse to finding the other two sides. As discussed in Section 4.4.1.2 of the previous chapter, Michael made a landmark decision that instead of teaching the entire topic in one block, as was his traditional approach, he would only teach as far as manipulating an algebraic equation to find the hypotenuse and would return to the next section in the following year.

Michael *I really like the idea of doing it with just the hypotenuse, I have to say I really like that.*

Lisa *Yeah, yeah.*

Michael *I really like the idea of coming back to it maybe at the end of second year or the start of third year, or at some stage there and bring in the other option.*

Kate *Yeah.*

Michael *Because that is where they get lost, when you do it early on.*

Lisa *That is...when you try and teach it too early.*

Kate *And there is no point in racing, rushing it, because this is probably the most important theorem in the whole of maths.*

This is the first time that these teachers realised they could choose not to teach this entire topic at one time. They employed a student lens on their teaching and in detailing the mathematical content, isolated a difficult item for students to understand. They made an n important decision to purposefully plan a vertical trajectory of learning (Suh, 2014) where they would return to this topic the following year with these students.

In their final planning meeting for this cycle, teachers discussed why the series was so important.

Kate *So it's us seeing do they know what they know. So is our - has our plan for the plan for the previous two lessons worked?*

Lisa *Yes.*

Kate *Is that maybe a good way of teaching?*

Michael *Well that's if you were to, so we're essentially testing a scheme here.*

Kate *If we came in and say "Right they couldn't do it...what did you do in the last two classes?"*

Owen *But it's dependent on the first 2 then, for it to work.*

Due to the careful building of content within this sequence of lessons and within the research lesson itself, teachers in their post-lesson discussion reflected on the successes of students in being able to answer the questions. Teachers were also afforded the opportunity to reflect on how students were engaging with and thinking about the questions because of the teachers' familiarity with what students' had experienced prior to this research lesson and because of teachers' familiarity with the singular and focused objective of this research lesson. Because of their detailed planning of content in this cycle, teachers were now afforded the opportunity to reflect on how to better develop students mathematical understanding, a point that was absent in prior cycles.

Planning a series of lessons within their final lesson study cycle afforded teachers the opportunity of employing a critical student lens to the mathematical content and teachers thus detailed how student mathematical thinking could be developed over a number of lessons. In detailing the content, teachers were provided with opportunity to reflect on this mathematical content and to suggest how this could be improved further within the research lesson – an occasion that was absent from prior lesson study cycles in this community.

6.3.1.1.2 Crannóg

In Crannóg's last meeting, where teachers discussed their experiences of lesson study, Eileen suggested that it was important for them to have planned and followed series of lessons throughout the year and her colleagues agreed.

Eileen *And for each of them we always managed to come up with a series of lessons, we never just planned one lesson, we always had at least a week or six classes.*

Fiona *So even though you're looking at a lesson as the focus of it, you have all the –*

Eileen *But for each of the three sessions or whatever we did this year, there was always an intro in to it and the afterwards was planned as well.*

Dave *We had a topic planned as well as a lesson planned.*

Fiona *And focused on one getting very specific detail for.*

Eileen *You do get more out of it than just one lesson plan, you were getting a section.*

The teachers in this school felt that they had gained more and learned more from planning units of learning instead of single research lessons. By following the suggested planning steps in the teacher booklet (Appendix C1), teachers began the process of lesson study by planning series of lesson within which one research lesson would be chosen. They continued this activity for the duration of the academic year and found it very beneficial to their practice to design learning trajectories which ran for a number of classes (sometimes over a number of weeks) instead of for one 40 minute lesson.

Teachers began their participation in this research by identifying the area of quadratic patterns as one which they wanted to focus on since, as teachers, they did not yet feel confident in teaching Strand 4 of Project Maths (National Council for Curriculum and Assessment, 2012a, 2012b). It was Stephen who initially suggested that they plan a series of lessons instead of just focusing on one isolated research lesson. This idea stemmed from the fact that there were very many ideas about teaching quadratics and, in deciding what elements would be best to focus on, Stephen suggested that the content chosen would very much depend on which class of students were being taught. While teachers had been introduced to the idea of planning a unit in their booklet, the idea of planning a series of lessons made sense when Stephen thought about the mathematics from the perspectives of his own students. He suggested that his students, who were not considered one of the 'top' classes, would better understand the mathematics if they could spend more time on certain topics, but still following the same sequence of learning.

Stephen *I'm just thinking in terms of, who we're going to [teach to], just how much you would get done in one lesson. It may be an issue in terms of, if we're concentrating on one lesson. Or else we can, we'll all to an extent, be planning a series of lessons. And here we're really looking at one lesson.*

Walter *Is it a good idea to plan the series of lesson in broad strokes? And do that, put together our whole sort of thinking on it.*

Stephen *That's what I was thinking.*

Walter *And then at the next stage say "Okay, which lesson are we actually going to do as an observation"*

Aoibhinn *And focus on part of it. Yeah.*

Stephen *Yeah. Exactly. That's what I was thinking. The whole process, as a group for once, with a particular emphasis on the one lesson.*

As a facilitator, I had not given too much thought to the idea of planning a series of lessons, not wanting to restrict the community or demand more from them, but the teachers seemed very enthused about this idea. The idea of planning a series of lessons suited this group of teachers, who taught streamed classes, because they could follow a learning trajectory with any group of students but at a different pace. This idea stemmed from Stephen's employ of a critical student lens in thinking about a particular lesson from the perspective of students in his own 2nd year class.

Following on from this conversation, teachers began outlining the content of quadratic patterns and discussed which area of the curriculum they would most like to focus on and develop as a research lesson. They graphed their conversation in order to best highlight where they collectively saw the most difficult area to teach and the most difficult topic for students to understand (Figure 48).

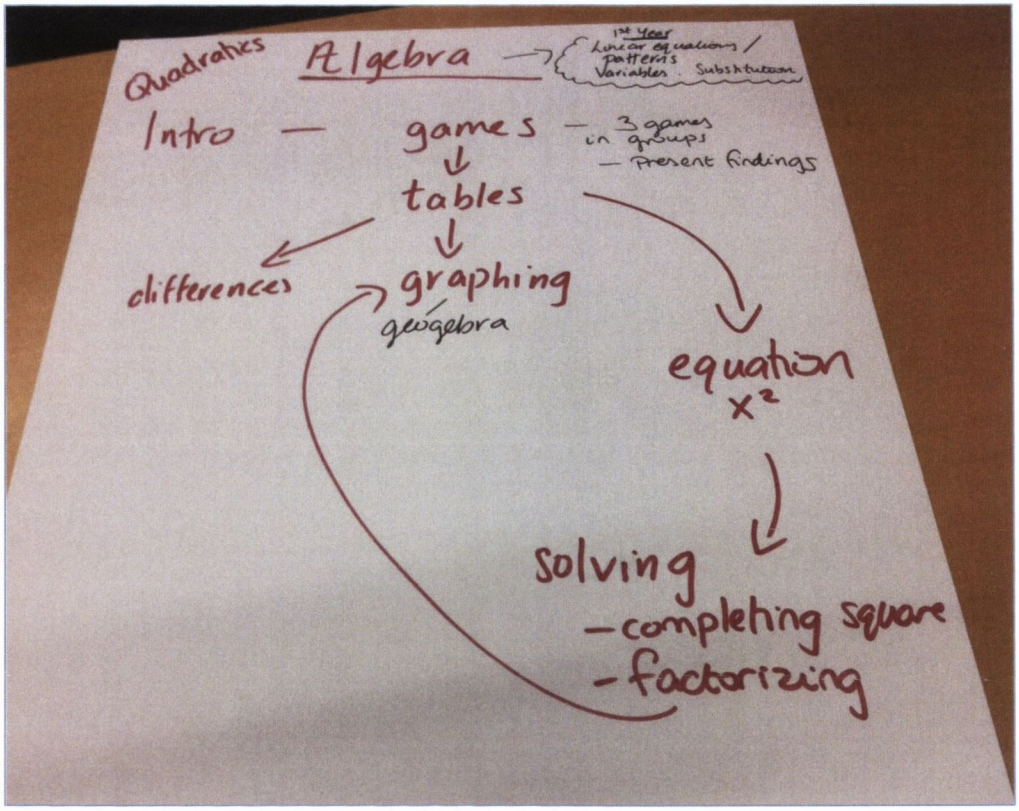


Figure 47 Crannóg Cycle 1: Planning a sequence of lessons, Appendix F1 3

In the first cycle, teachers focused on students identifying quadratic patterns, related to tables and graphing, which would be preceded by linear patterns and would be followed by leading students to factorising quadratic expressions.

In cycle 2 the factorisation of quadratics using algebra tiles and array models was originally planned for a small class of students who their teacher Fiona had described as having "weak" mathematical skills. Teachers wanted to focus on this topic since they had not previously used algebra tiles or array models in their teaching and had noted that these strategies were being promoted by the PMDT team at workshops and in curriculum materials. Teachers also noted that their senior students often had difficulty in understanding the concept of factorising and wanted to see, employing a researcher lens (Fernandez et al., 2003), if using models for factorising benefitted students in consolidating their understanding. They discussed how to build students' understanding in lessons prior to and following the research lesson.

Fiona *You see I suppose what we have to do in that pre-runner class, we have to go back with them over the concept of factors – what are factors? And then they need to look, because we kind of gloss over that a bit when we go into factorising usually.*

Stephen *That we don't make up two numbers in algebra.*

Fiona *Yeah, so go back into factors...*

Gerald *I think the discussion of factors should start with prime numbers myself because they have only got two factors to talk about then.*

While a series of lessons was outlined, unfortunately it transpired within the post-lesson discussion that Fiona had not necessarily followed this series and that the sequence of activities within the lesson did not actually build on students' prior knowledge and experiences. The lesson was re-taught along the same series of content, but with less new concepts being introduced to students within the research lesson and instead delayed until the following lesson. The idea of following a series of lessons allowed that transition of content to occur since students would eventually experience the same ideas, but at a pace which better suited their mathematical understanding.

While the research lessons within this cycle did not seem to be very successful in achieving the learning objectives for these particular groups of students, other teachers in the community taught the topic based on the planned series of lessons and reflected on enjoying a measure of success with the new strategies with their own classes. The teachers in this group were using lesson study as a vehicle of experimentation in finding out how these new strategies worked for students and throughout the planning process, teachers had been doing sketch lessons to trial some of the approaches which they then fed into the planning of the research lesson. In reflecting and revising on the lesson, other teachers such as Stephen and Dave noted that they would teach the series of lessons at a slightly different pace to better suit their particular classes of 2nd year students.

The teachers concluded in their post-lesson discussions that following this learning trajectory in factorising quadratic expressions, more students understood the concept of factorising (as opposed to learning it by rote). This sequencing of teaching area models, algebra tiles, array models and eventually factorising by 'squiggles' and brackets was one which teachers concluded would follow again.

Planning a sequence of learning over a series of lessons did not only benefit students' learning but also enhanced teachers' KCT. In the post-lesson discussion of Crannóg's final lesson, it seemed that through planning the sequence of lessons and observing students completing the activities also improved teachers' knowledge of the content and

how it could best be taught, Dave reflected within his final interview that he had never taught Calculus properly before and from participating in the lesson study process, designing a learning trajectory that was building students' mathematical thinking, that he himself had a better understanding of the topic. Participating in the collaborative planning of this introduction to differentiation, where teachers took an extraordinarily detailed, active and measured approach to the topic, built on and enhanced Dave's knowledge of how to teach this particular content.

Dave *I have met loads of other ways of perceiving or thinking about say quadratics or calculus or anything else. I have turned how I would introduce topics completely on their head instead of starting with just a formula or something, you are opening up a problem and you are letting them go out and do things... When I was doing my Leaving Cert of whatever followed procedural methods, I just learned how to do the sum and practiced them over and over. To fully understand the concept of rates of change and how it applies to calculus or things like that, I never really fully understood. I learned as much as those 4th years did from some of those classes.*

While other teachers were not as explicit about what content they had learned during their interviews, it is likely that others similarly benefited from this process of collaborative planning.

In their final cycle, teachers modified and built on a series of lessons which had been developed as part of the curriculum materials (Project Maths Development Team, 2011). Teachers attempted to anticipate how students would attempt to find an instantaneous rate of change if they had not already done technical graphics. It was at this point that teachers decided that this would be the focus of the research lesson, constructing a way for students to investigate how to find the instantaneous rate of change at a particular point. Fiona felt it important that they again plan a series of lessons and not just an isolated research lesson.

Fiona *We need to have a kind of a number of classes when we are going to teach it over. We are not just going to interrupt a class and say*

"That was it" and gone. We need to have a little module that we can continue on with from it.

As a group, the teachers wanted to begin the series with students generating and recording their own data which they could then analyse in terms of investigating the rate of change of a linear expression. Building on this series, teachers then incorporated curriculum materials from which students could investigate rates of change but modified these materials in order to encourage a more inquiry-based approach to the lesson (Project Maths Development Team, 2011). Teachers continued their discussion of the phases of the classes within their planning meeting.

Constantly in the sequencing of learning activities and in the planning and detailing of learning trajectories, the students and their mathematical understanding were at the centre of the planning process. Because teachers had planned and detailed how students' mathematical thinking would be developed over a series of lessons, teachers were able to reflect on students' thinking and the mathematical content in their post-lesson discussions.

At the end of the year many teachers at Crannóg, when asked if they thought they had increased their mathematical knowledge due to participating in lesson study, acknowledged that what they had learned was how to approach teaching a topic in different ways. Referencing the planning of sequencing a learning trajectory for particular topics, Stephen noted that he sometimes taught mathematics as he had been taught since it was all he was familiar with. Participating in lesson study made him more aware of how a topic could be approached and developed. Dave and Eileen echoed his opinion in their final interviews with Dave, Stephen and Gerald explicitly noted that they had learned so much about teaching calculus from their final research lesson.

What is particularly interesting that the two most experienced teachers, Fiona and Gerald, felt that their knowledge had been increased by planning new learning trajectories in lesson study. Gerald responded to the question:

Aoibhinn *Do you feel that participating in this [research] has increased your maths knowledge?*

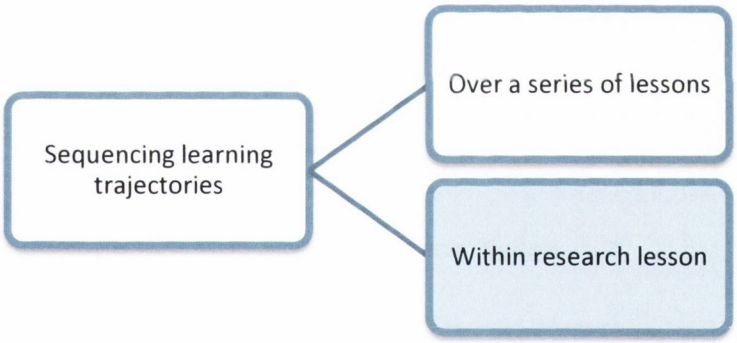
Gerald *...Has it increased my knowledge? Hard to say. In terms of mathematical content of that I suppose not. But in terms of just looking at things and bringing things together in different ways, yes.*

Gerald noted that he was particularly interested to see how the group were able to design a sequence of lessons on calculus, something he had been independently trying to do for some time. Teachers' KCT was enhanced through participating in the collaborative sequencing of building students' understanding.

6.3.1.2 Conclusion of Sequencing Learning Trajectories over a Series of Lessons

Within this research, when teachers planned a series of lessons and not just an isolated lesson, the research lesson itself became more detailed and teachers felt that they were more equipped in including the lesson within a series of lessons where the students' learning trajectory for a particular topic could be mapped out.

6.3.1.3 Sequencing Learning Trajectories within the Research Lesson



This is the second of two categories under the theme of sequencing students' learning trajectories which will be analysed in terms of teachers employing and enhancing their KCT throughout their participation in lesson study.

In this section, the data on sequences of learning planned within research lessons will be analysed according to teachers employing a critical student lens and incorporating other elements of teachers' PCK such as prior knowledge and anticipating students' mathematical thinking, into their planning (as detailed in Chapter 5 as features of KCS).

Within this research, since teachers were focusing on developing research lessons which would build students' mathematical thinking of a particular topic, teachers were provided with the opportunity to use and enhance their KCS and KCT in developing a sequence of learning for students within each research lesson. Through articulating what students

already knew, anticipating how they might respond and through different iterations of attending to and reflecting on student thinking, teachers began to sequence learning trajectories which were specific to the class of students within the research lesson.

This sequencing of content within research lessons also benefitted from teachers developing learning trajectories over series of lessons as detailed in the previous section.

6.3.1.3.1 Doone

The first research lesson in Doone was an unruly mess of content when compared with their final research lesson. The first lesson attempted to incorporate a wide range of content (both new and as revision) including area, indices, an introduction of x as an unknown variable, perfect squares, and finding the square root of a number. In teaching mock versions of this research lesson, Lisa attempted to identify how students would learn about x^2 within this lesson, but the large amount of content concentrated within this one lesson did not allow for a clear learning trajectory to exist.

The sequencing of content within the research lesson began to improve in the second cycle. As mentioned previously in planning this lesson Michael reflected on the first research lesson as a "*sore thumb*" which stood out on its own without being incorporated within a sequenced learning trajectory and, as Nora had suggested in the first cycle, without definitively outlining the prior knowledge of that particular group of students brought to the lesson. For research lesson 2, teachers decided to teach a lesson on the multiplication of fractions which would follow on from Owen teaching 'An introduction to Fractions' and 'Multiplication of Fractions' teaching and learning plans from the curriculum materials (Project Maths Development Team, 2011). Because teachers could outline from these plans the mathematics that these students had already experienced and could thus outline their prior knowledge in arriving at this research lesson, teachers had the opportunity to focus on building the mathematical content within the research lesson. In building on these curriculum materials teachers were also familiarising themselves with the syllabus and curriculum content as has been recommended by the curriculum reform literature (Charalambos & Philippou, 2010; Hanley, 2010).

In the first planning meeting Nora shared her opinions of what prior knowledge would be required of the students in order to focus on multiplying two fractions and detailed the content of this to her colleagues.

Nora	*I think multiplying and dividing by a whole number comes first in somebody's thinking, you know...The meaning of multiplying something by 3 is take 3 of them and add them. That already has a meaning for a student... Then I think you are ready to talk about what it means to multiply by a fraction*
Owen	*Oh, I love that. Yeah, I think that's great.*
Nora	*So you multiply it*
Kate	*It's the same as multiplying by a number*
Lisa	*Yeah*
Kate	*Half of something is the same as something divided by 2?*
Nora	*Yeah*
Lisa	*Because it links in with. I think that's very good.*

Nora's experience of working with students in learning support capacity impacted on the sequence of the lesson plan since her colleagues Lisa, Owen and Kate were impressed by her ability to break down the content in terms of students' thinking and these teachers began to see the mathematics from the perspective of the students. Nora's suggestion was incorporated into the lesson plan where students first revised multiplication of whole numbers before moving to fractions. It did, however, take a considerable amount of teachers to agree on what point in students' prior knowledge the lesson was beginning but teachers were at least clear in their positioning by the end of that second planning meeting.

There were a total of four planning meetings within this second cycle and it is interesting to note that in the planning phase of this lesson study cycle, Lisa repeated in meeting 3 exactly what Nora had said in meeting 2. Lisa repeated that in their sequencing of this lesson, they should not introduce the concept of a "fraction by a fraction" too early since then students would then focus only on the mechanics of the expression without understanding what was happening to the final value. Nora had made this point that students often did not understand what they were being asked to do in this topic and hence learned the instrumental skill by rote. As has been suggested by literature on video studies, this was a building of Lisa's knowledge in enhancing her own KCT from

participating in discussions with others who were more experienced and who employed a critical student lens more often in their teaching (Jacobs et al., 2010).

The second lesson avoided a traditional opening exposition by the teacher (as had occurred in the previous research lesson) and instead students were immediately involved in the lesson by having to place fractions on the number-line on the whiteboard in order to revise a sense of measure of fractions. The lesson continued by highlighting to students, through exposition and activity, that multiplication did not always lead to a bigger answer - a concept that teachers had anticipated students would not be familiar with and revised again at the end of the lesson. The lesson progressed to students working in groups of five on a problem involving the multiplication of fractions. Within this lesson teachers focused more on students' engagement in the lesson than they had in the first cycle and also the strategies which they might use in some of the activities, but not on how students' understanding of multiplying fractions would develop since there was no clear focus on how mathematical content should be sequenced within the lesson.

In the sequencing of content within the third research lesson teachers focused on sequencing activities and content within the lesson. The sequence of learning within this lesson built on students' prior experiences of tabulating, graphing and investigating linear expressions and on students' prior knowledge of graphing quadratic expressions from junior cycle. The sequence of learning activities remained very traditional with an exposition from the teacher followed by students attempting two questions. Following from the success of research lesson 2, student communication was incorporated within the lesson with students working together in groups of four and presenting their work to the class.

The sequencing of a learning trajectory within research lesson 4 was very different to all of the other lessons that had been planned by this group of teachers. As explored in Section 6.3.1.1.1, teachers outlined and planned students' experiences prior to the research lesson giving them a better knowledge of students' mathematical understanding of the topic at the point of the research lesson. Teachers spent four planning meetings outlining the detail of content to be included within each of the lessons but also anticipating students' responses to certain activities. Within the research lesson, students' learning trajectory was sequenced in a way which would revise their prior knowledge by

incorporating it within new problems which students had to solve. Questions designed by the teachers (see Section 6.3.2) were sequenced according to their level of difficulty.

Lisa *Do you think you might use maybe the Niagara Falls one as the first one.*

Michael *I was thinking that actually.*

Lisa *And then move this down to two and three.*

Michael *Yeah, I thought we should do that on all the first one...*

Nora *There's too much in the boat, so the first one.*

Michael *Yeah.*

Teachers included scaffolding questions within the activities where students were guided in identifying the right angle and hypotenuse and were then asked to calculate the length of the hypotenuse using Pythagoras' theorem. The number of scaffolding questions was reduced as students continued solving problems.

Michael *Isn't there a nice point where we stop the scaffolding around question 4 or so, and we drop all the cues. Stop there and give them the last 2 or 2 without all the blurb...*

Nora *And they will be on their own then.*

Michael *Do the last few without all the prompting.*

Teachers thought carefully about which questions to omit within the sequence and were enthusiastic about leading students to a point where they could answer a question independently.

The development of teachers' ability to sequence the learning trajectory within a research lesson and to include the research lesson within a series of lessons, allowed teachers to think about how a student's mathematical understanding of a topic might be built and afforded teachers the opportunity to think about the mathematical content from a students' perspective. Teachers discussed detailed learning trajectories when they adopted a critical student lens in deciding how to build students' mathematical understanding about a topic. Through their participation in lesson study, teachers in

Doone developed their KCT through identifying particular mathematical activities that would build students' understanding of a topic and sequencing them in way which built on students' mathematical thinking.

6.3.1.3.2 Crannóg

In the first research lesson, this community of teachers spent three planning meetings detailing the content to be included within the research lesson and then the sequence of activities which they anticipated students would follow within the lesson. In order to engage the students, a story was devised with which to begin the lesson and the lesson was planned as a discovery activity (similar to those used in Phoenix Park school, (Boaler, 1999)) for students working together in groups of four. Because teachers spent the majority of their planning time anticipating how students might think about a particular activity, this led teachers into sequencing their questions and activities in the lessons based on building students' mathematical thinking. Because teachers anticipated the difficulty for students to initially decipher a quadratic pattern, teachers decide on using a 'simpler' equation such like $x^2 + 1$, and although Fiona thought that a pattern of x^2 may be too easy for students, as a community teachers decided to begin with this question from which all students could then build on their understanding with gradually more difficult patterns in their extension sheet (Appendix F2 6).

In the second lesson study cycle, teachers detailed what they would like students to have experienced in lessons prior to this research lesson and discussed how the research lesson, focusing on a concrete model of factorising quadratics, could then lead to the more traditional abstract way of factorising. In planning this lesson, teachers detailed and discussed the activities and strategies which students would engage with within the lesson. It was agreed that it would be better for the students to begin working with algebra tiles since it gave them a physical representation of factorising an expression before moving to the more abstract strategy of using array models. Employing a critical student lens to their planning, Fiona and her colleagues discussed this sequencing of the learning trajectory in terms of their own particular students' learning.

Fiona *I was just going to do the whole idea of factorising. Give them all different expressions...*

Stephen *That is a lot, like with my group that is a lot even just to get through. I think you will struggle to get through that with -*

Fiona *So do you think I would be better to start them off with just common factors and give them an expression and just get them to do that maybe?*

Stephen *Yeah.*

Aoibhinn *With an area model?*

Fiona *No, with algebra tiles and then do that and then lead that onto, just go into an area model and that. And I can go from there to quadratics because that kind of leads nearly better into quadratics.*

In this conversation Stephen incorporated his observations of students when he taught a sketch lesson to his own class. Teachers sequenced these activities according to what they anticipated from their own students' perspective. Teachers discussed each of the activities in great detail in order to also sequence them in terms of gradually increasing levels of difficulty within the lesson.

Within the research lesson students were to investigate factorising using algebra tiles and teachers discussed the particular questions which should be attempted by students in their initial activities. While teachers had originally chosen random expressions to factorise and had focused on the expression of $x^2 + 5x + 6$ for almost an entire meeting, Fiona then suggested using an expression with a prime number as key number to factorise in order to make it more straightforward for the students.

Fiona *Am I not better to start them off this first one with a prime number, I am -*

Walter *So give them no options.*

Fiona *5, 6x, I have done that.*

Eileen *That is what we have just done.*

Fiona *No I haven't.*

Eileen *3 and 4.*

Fiona *$x^2 + 6x + 5$.*

Walter *No it was the other way around -*

Eileen *Oh yeah.*

Fiona *Give them a prime number. Because we were trying to work out Dave earlier on -*

Dave *Go 3 and 4 just to get away from these numbers in case there is confusion, 4 in the middle and 3 at the end.*

Teachers sequenced the questions within the lesson in order for students to build on their understanding of factorising with gradually more numbers to choose from. Within the research lesson Fiona deviated from this order of questions, which may have added to students' confusion, but the sequence had originally attempted to align to students' understanding of the topic.

This first teaching of research lesson 2 then moved to using array models to factorise different quadratic expressions. A number of teachers including Dave and Stephen reflected that students had been very confused by this and in the second teaching of this lesson the sequence of learning activities within the research lesson was shortened to only focus on students' understanding the mechanisms within and the mathematics of manipulating algebra tiles to model quadratic expressions.

In the third lesson study cycle teachers again planned a series of lessons which led up to the research lesson. The research lesson itself focused on students investigating finding an instantaneous rate of change within a quadratic function. Teachers used and modified curriculum materials in planning this research lesson by choosing two specific activities from within 'An Introduction to Calculus' teaching and learning plan and from within teacher workshop materials (Project Maths Development Team, 2011). Students' learning trajectory over a series of lessons was sequenced first by revising their investigations of a linear function and then moving this investigation to a quadratic function. The sequence of learning within the research lesson was based on the curriculum resource guiding students to calculate the speed at one point by finding the slope in interval points that were closer and closer to the point of interest (Appendix F2 7). Again, this sequenced activity was planned by thinking about the activity from a students' perspective as articulated by Stephen.

Stephen *So how could we lead them to start thinking, we need some way, they won't come up with anything that way at all, so how do we get them to actually start thinking about -*

Walter *Sections?*

Stephen *Sections.*

Fiona *Just throw it at them and somebody will come up with it, somebody will come up with something to do with it. I think.*

Walter *Are they more likely just to start drawing tangents because they have the idea of slopes and some of them.*

Fiona *Well so be it, but if they do that, if they draw approximate tangents they can structure tangents, curve at that point, perfect. That's a way of doing it.*

Stephen *Could we split, to get them thinking, could we think of the average speed for sections first? As in, so what's her average speed for the first three minutes, what's her average speed, would that be, I'm just trying to, thinking in terms of my class, if I give them that it's just going to be.*

Eileen *Silence.*

Stephen *Silence and it's going to be average speed for the whole thing. Where if we were to split it up in to her average speed over the first three minutes, or between each of those points, sections.*

Dave *And make those intervals smaller?*

Stephen *And so they should be able to see that her average speed is getting faster obviously between those.*

Eileen *Like between A and B.*

Stephen *And then what's her speed at this particular point. How could we potentially do that? I know it's a bit of a stretch anyway but just for my class certainly there needs to be something in between what's her speed and then.*

Stephen, again employing his critical student lens, identified that students would require some guidance at this point within the lesson and built a number of questions around the activity in order to guide students along this learning trajectory. Through anticipation of students' responses, teachers developed a detailed sequence of activities for students to engage with which brought them towards a concept of limits.

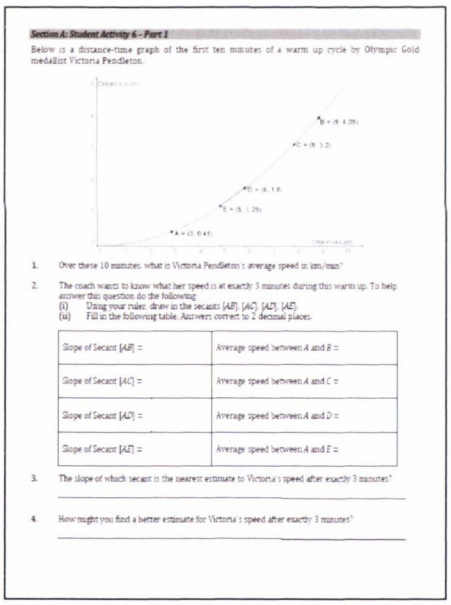
Through detailing students' prior knowledge, anticipating students' thinking and responses and through discussing the mathematical content of the lessons, teachers in Crannóg constructed research lessons that focused on building students' mathematical thinking.

6.3.1.4 Sequencing Learning Trajectories: Utilising and Modifying Curriculum Materials

Many of the teaching and learning plans supplied as curriculum materials to teachers on the Project Maths Development Team website (Project Maths Development Team, 2011) provided mathematical guidance and learning opportunities for teachers. In detailing the mathematical content included within a lesson, in suggesting teacher questions, and in anticipating students' responses, the plans provided an informative guide for teachers. However, these materials were commented on by teachers within this research as being too detailed and time-consuming to read. Prior to the research, teachers such as Lisa and Fiona, had referenced a number of the teaching and learning plans within their teaching of the new curriculum and Kate reflected that she only ever used the worksheets contained in these documents without looking at the detail of the lesson plan as she found them overly detailed. Participating in lesson study and planning research lessons, teachers began to unpick some of the resources supplied to them, utilising certain elements or modifying certain activities to make them relevant to their research lesson and to their students (Hanley, 2010).

In Doone's final research lesson, teachers designed a sequence of lessons to build students' understanding of using Pythagoras' theorem incorporating a number of resources, designed by other teachers, available on-line (Project Maths Development Team, 2011). Teachers in Doone took these resources and modified them in order to best suit their group of 2nd year students. Due to the detailed sequencing of the content of each of the lessons leading up to the final research lesson, teachers modified the resource in a way that matched their plan of building on students' mathematical thinking.

In Crannóg's final research lesson, teachers discussed the ways that students could be introduced to concepts in calculus such as calculating instantaneous rates of change and the concept of a limit. There was much discussion on how this could be best achieved and Stephen put forward the idea of utilising a Project Maths resource that had a video and graph of Usain Bolt that he had seen in a PMDT professional development workshop. Teachers discussed using this presentation and Fiona suggested that the resource could be improved. Teachers choose elements from the Project Maths teaching and learning plan 'Introduction to Calculus' and from the teacher resource booklet from Workshop 8 (which as facilitator I introduced to the teachers since they were no longer attending workshops as a phase 1 school). Incorporating their own ideas with activities from the curriculum materials, teachers modified the lesson and students' worksheets in order to suit this particular group of transition year students and in order to develop the concept of rates of change first in linear functions and then developing on to quadratic functions (Hanley, 2010).



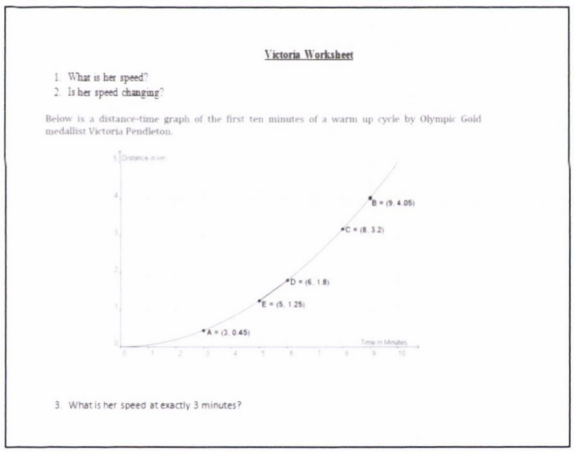


Figure 48 Original & modified curriculum resource: Crannóg cycle 3, Appendix F2 7

In modifying the resource, teachers employed a critical student lens in re-designing the worksheet to remove what they felt to be questions that were too leading for their students. Stephen wanted students to inquire into the process of finding the instantaneous speed rather than be directly shown what to do as suggested in the curriculum resource. Here Fiona described what the question in the curriculum resource asked of students and Stephen suggested some modifications since Walter already noted that students would make a connection between average speed and slope in terms of a linear function.

Stephen also anticipated that students would make a connection between slope and average speed despite the fact that they were now investigating a quadratic function and he wanted this to emerge as a point of discussion for students in order to develop their thinking and understanding of instantaneous speed in this new case.

Fiona *So you get your slopes, average speed between A and B, average speed between A and C, average speed between A and E. So you can see which is nearest...*

Stephen *I think that's a very interesting question to ask them and to see what they come up with in terms of "what is their speed at that point" and actually see if they can, rather than leading them to draw this, draw that.*

Walter *Yes.*

Fiona *Yeah I think maybe just say to them, leave it open and see what they come up with.*

Stephen *Exactly.*

Fiona *Rather than structure it.*

Stephen *Yeah completely and so they will already have done linear so they'll be thinking in that term.*

Fiona *And ask them, maybe ask them those two questions.*

Stephen *Yeah.*

Fiona *Say, "What is, over the ten minutes, what is her average speed?" And then, and hopefully they'll be able to figure that out.*

Stephen *Yeah, but also not even ask the second question just ask them..."what is their speed." Because I guarantee you, half of them will say that's their speed.*

Walter *The average speed?*

Stephen *The average speed is their speed, and so there should be then that talk of does their speed not change, so they should be able to get to that point.*

Fiona *Yeah yeah.*

Stephen *Through discussion as opposed to actually just saying what is their speed after that point.*

Teachers utilised a critical student lens in identifying what students would already know in this lesson, in anticipating how they would answer questions, and in modifying the resource to allow students to investigate the topic themselves as opposed to being directly guided as the curriculum resource originally suggested.

In modifying this particular curriculum resource teachers extended their knowledge of the resources available to them since none of them had previously seen or used this particular worksheet nor had these teachers the opportunity to attend this particular professional development workshop, further developing their KCT (Ball et al., 2008; Hanley, 2010).

6.3.1.5 Conclusion of Sequencing Learning Trajectories

Due to the number of teachers of varying experiences in lesson study in these two schools, there were always a number of ways in which learning trajectories could be and were constructed. The sequences arrived at for the final research lessons in this research were the culmination of numerous planning meetings where teachers discussed the content they wished to include within the lesson but also discussed the construction of that content in relation to their particular groups of students.

It was an unexpected finding that planning series of lessons, within which students' learning trajectories could be detailed, would emerge as a benefit to both teachers and to students. Teachers benefitted by detailing the sequences of questions and activities, thus enhancing teachers KCS. Students benefitted from experiencing mathematics in lessons which could be tailored to their particular prior knowledge and abilities, but which could also be situated within a planned series of lessons.

As teachers in Doone continued their participation in lesson study they became more aware of and focused on sequencing a particular learning trajectory that was suited to the particular class of students. While research lesson 1 did not have a clear direction in what students were learning, research lesson 4, in comparison, had a very definite trajectory through which students could build their mathematical understanding over a number of lessons.

In Crannóg, teachers seemed exceptionally clear on what they wanted students to learn and how they wanted them to go about learning it. The sequences of learning were designed clearly and with purpose in the first and third research lessons. The second research lesson seemed to have had less clarity for the teachers and perhaps this was due to the fact that teachers were incorporating a new strategy using algebra tiles and array models within their teaching which they had not previously attempted. Nonetheless, teachers felt that they gained from investigating this approach in the second research lesson as they reported in their mid-way interviews. It was quite surprising that these teachers identified at a very early stage of their preference to plan a series of lessons instead of an isolated lesson. It is also interesting that they continued with this trend throughout the research and reflected on the benefit of this practice which they hoped to continue with in the following year. In his final interview Walter suggested that developing these series had been a very interesting and useful learning process of sharing ideas and took the planning of research lesson 2 as an example.

> *Walter* *Say just factorising a quadratic, I think we could have said that in a meeting a couple of years ago and we would all have had our own idea of what a series of classes looked like on that, but by having had the discussions we now know what that means for us all...we have more understanding.*

In Doone, while it took some time for this group of teachers to independently decide to pursue a process of planning which would include a series of lessons, teachers found it far more beneficial to discuss and detail students' learning over a longer time frame than a single class. In their final planning meeting Kate realised that they had not planned a unit before and Nora reflected that this practice was both positive for the teacher and the student.

> *Kate* *This is the first time we've actually kind of planned a little scheme.*
>
> *Nora* *For the whole thing, yeah. Take you, follow it, follow it through, yeah. Because I think you know exactly where, where you stand or where they should stand.*

It is a significant finding of this research that within these schools research lessons were of far more benefit to teachers in explicitly planning a learning trajectory for students as

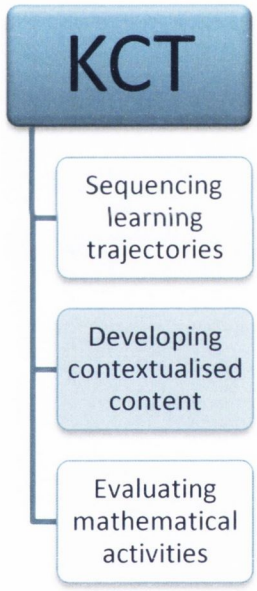
part of a series of lessons as opposed to, as the literature regularly suggests, one isolated lesson (C. Fernandez et al., 2003; Lewis et al., 2006; Murata et al., 2012).

Sequencing lessons with a purposeful trajectory brought teachers' KCT into the planning of the research lessons but in a symbiotic way, both KCT and KCS developed through the continuous planning of research lessons over a number of lesson study cycles as exemplified by the improved clarity of learning trajectory in Doone's fourth and final research lesson.

6.3.2 Developing Contextualised Content & Questions

In this section, the progression of developing contextualised content and questions within lesson study cycles will be analysed in both schools as part of developing teachers' KCT. This is the second of three features of KCT as described in Chapter 4.

6.3.2.1 Doone



Originally in Doone's first research lesson, while they wanted to incorporate a real-life application of the mathematical content, a farmer finding the area of his fields may not have been the most relevant context for a first year group of urban, male, teenaged students. In the second lesson study cycle the teachers' began thinking about mathematics questions in more than content terms but also involving relevant context for students (Galbraith, 2013). In the third planning meeting of cycle 2, Kate put forward the idea of giving students "*a scenario, like problem solving questions*" and originally suggested having questions based around Christmas, since the lesson would be held close to the school holidays. Instead, in the fourth meeting, teachers broke into groups of two groups of two and designed and devised mathematical questions that were both relevant to the mathematical content of the lesson and to the students in the class. Each group of students were to be given a different question which they could work on and teachers enjoyed designing these questions and sharing them with their colleagues. Lisa read the questions to the group.

Lisa	*Michael had a birthday party yesterday, a quarter of the birthday cake was eaten.*
Kate	*Yeah, because don't forget they're going to have to –*
Lisa	*This morning Michael ate a third of the remaining ¼ of the cake. What fraction of the whole cake did Michael eat this morning?*
Kate	*Yeah, 'cause you're saying it might take them two minutes to solve it.*
Owen	*Yeah.*
Kate	*But by the time they draw it up, write it out, do the maths, explain it, yeah.*
Lisa	*Yeah. I think they're [the questions] are brilliant.*

These questions included one scenario where a class group had to attend the school nurse. This question was immediately relevant to students since they were each receiving mandatory inoculation at the time of the research lesson (Figure 50).

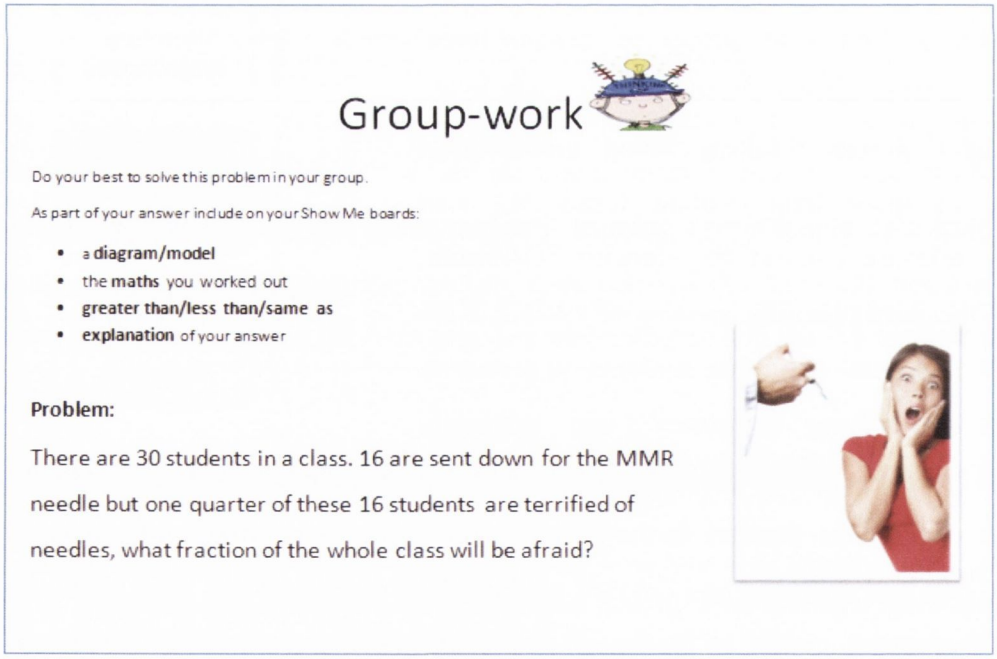


Figure 49 Contextualised questions: Doone cycle 2, Appendix F2 1

Questions also included a question on a recent sports game which students in this class had likely watched. In contextualising these questions, teachers were employing a critical

student lens in identifying what question scenarios might engage this particular class of student. This was a turning point of learning for this group of teachers since in each of the subsequent research lessons, teachers were focused on creating contextualised questions that had meaning for the students (as teachers had expressed as their over-arching goal for the year).

In developing these questions, teachers enhanced their KCT in the construct of questions. Lisa originally began writing a question for the class and Kate agreed that it was a very good example.

Kate — *That is a good one... and they are calculating the fraction and then getting the fraction of a whole number as well so it is linking all three.*

Lisa — *Yea, so 2/3 haven't had the injection and 1/2 of those are to go down to the library on Monday - how many students go down to the library on Monday?*

Kate — *Yeah. So they have to get 2/3 of a 1/2 and then multiply by the number of students in the class....*

Lisa — *Yeah, now is that - 30 students, they will get a 1/3 which is 10 and then 1/2 which is 5...*

However, later Kate realised that her example only required students to multiply a whole number by a fraction. She pointed this out to Lisa and, from their conversation, Lisa realised that her mistake in designing this question lay in including the number of 30 students within the question. This was a learning moment for Lisa and her colleagues as they talked through the question.

Kate — *You see my problem with that is they will say "Well there is 30 in the class, 2/3 is 20. 1/2 of that is 10" we don't want them to do it like that... We want them to do the fraction by fraction...*

Lisa — *So we need to give them the answer then. If we say that there was 'x' number of students in the class. 2/3 of them weren't injected, 1/2 of them went down, so that was 6 students went down. So how many students were there at the beginning? You have to work back to whole is it?*

Kate — *But then they can still work back...*

Lisa *Ok well, just what fraction of the class goes down?*

Kate *But they still might do easy whole numbers...*

Lisa *But no, we don't tell them how many is in the class, a maths class where 2/3 have not had the injection,*

Kate *Yeah, that's it.*

Lisa *A 1/3 go down on Monday, what fraction of the whole class goes down on Monday?*

Kate *Yea...That is it... without any numbers because they will just...*

Michael *Yeah, that'll work....*

Kate *Just look for the, just give them three fractions to multiply together, you know, give them something that 1/2 the class go down and 1/4 of them are sent back, how many were sent back?*

Unfortunately, two meetings later Lisa created two new questions which again led to students multiplying a whole number by a fraction. However Kate, in noticing Lisa's mistake, pointed out how the question could be improved which also enhanced other teachers' KCT since they were now also participants in this discussion.

Lisa *The Ireland Rugby squad, 1/5 of these have eye problems. Of this 1/5, ½ wore contact lenses. What fraction of the players wore contact lenses...*

Kate *But that's going to be, they're going to get, so they're getting thirty players.*

Lisa *30 multiplied by 1/5 is 6.*

Kate *Well, they're going to divide by 5. Just so you are aware... Unless we said, a squad, instead of saying 1/5 of the players.*

Michael *Mmm.*

Kate *Don't give them a number of players, 'cause they'll divide by 5 and get 6, and they'll divide and get 6.*

Lisa	*I think that's what we're doing wrong. No, I think that might be why that, that problem with the class, we just want a fraction.*
Kate	*Yeah... when they say 30, you're going to, they're going to divide by 5 and get 6, divide by 2 and get 3 and then say 3/30, 1/10.*
Michael	*Yeah.*
Lisa	*And that's not what we want.*
Kate	*You want them to say, the squad...And 1/5 of them...*
Lisa	*Yeah.*
Kate	*And then 1/2 of them.*
Michael	*Yeah.*
Kate	*So you want them to do fractions by fractions.*
Owen	*That's why the cake was..*
Lisa	*Worked, because it was a whole, yeah.*
Michael	*Grand.*
Lisa	*So there was only one of them...*
Owen	*'Cause we're not talking about a total number*
Lisa	*Yeah.*
Owen	*So it's one whole squad.*

Through Kate anticipating what students would do with the question ("*they'll divide by five*"), she again identified that for the purposes of this lesson they should just talk about a rugby squad instead of giving the total number of players. Lisa then remembered that they should not be giving the total numbers in any of the questions. Within this extract, Owen also realised that this why the question on the cake (Appendix F2 1) worked, because there was only one cake from which to find the fractions and again students were not dealing with whole numbers from which to calculate fractions. Through their

collaborative construction of this mathematics question, teachers understood how to construct the questions in order to build their students mathematical thinking, but also consolidated their own KCT in accurately representing mathematical ideas (H. C. Hill et al., 2008).

Teachers had enjoyed the creative process of designing questions in research cycle 2 and this was continued in the following lesson study cycle which included more advanced mathematical content for senior cycle students. In planning these questions teachers focused on the mathematical content and context of the questions and ensured that the values contained within the questions would be realistic. Lisa initially suggested a question based on a diving board and Kate began thinking about questions involving a hot-air balloon since both had a visual connection to a quadratic trajectory. Michael then suggested creating a question on rugby as discussed in Section 4.4.2.1. Kate and Michael were both excited and enthusiastic about designing this question and as a group of teachers this activity seemed to enhance their confidence in their own knowledge and ability in creating mathematically sound and interesting questions.

Michael *Could we do a football?*

Kate *Football? Yeah. A rugby ball?*

Michael *A rugby, an up and under in rugby.*

Aoibhinn *Like a what?*

Lisa *An up and under.*

Michael *An up and under is...*

Kate *A Garryowen.*

Michael *you kick, Garryowen, yeah, you kick it up, it goes up high, they chase it, and try to catch it.*

Kate *Aw, let's do a Garryowen. Love it. Will he get there on time to catch it, if this is the path the ball takes?*

Michael *Or something like that, yeah.*

Owen *Brilliant.*

Kate *Let's do it Michael! Yes! Ok. I love it. Let's do it. And they'll love that.*

Lisa *So we're not doing a hot air balloon anymore, it's a Garryowen. (laughing)*

Kate, Lisa and Michael ensured that the figures included within the rugby question would be realistic for students. Teachers modified their equation in order to make it more believable for the students.

Kate *We may, we might want to double-check the height is high enough.*

Lisa *Right.*

Michael *Yeah.*

Kate *How high would you get a Garryowen?*
Michael 40 metres, 50 metres.

Kate *Maximum yeah?*

Michael *He could kick it further. Let's take it further.*

Teachers altered their equation in order to make the distances and timings believable for the students (Appendix F2 3). As another question to be included in that lesson, some of the teachers had been wary about including a medical question based on temperature (Appendix F2 3). However, they decided to include it within the lesson as something which students may not be familiar with but would allow students to see the utility of their mathematics in a new context (Galbraith, 2013).

In their planning of these contextualised questions teachers anticipated students' non-mathematical responses to the questions. Meyer & Wilkerson (2011) included these responses with teachers anticipating their mathematical responses. Kate anticipated that if they referred to a specific game, students would know whether Johnny Sexton had made a certain kick during the match or not.

Kate *They'll probably start saying "He didn't do a Garryowen in that match miss.*

Michael *Yeah yeah.*

Kate *When were we last in the Six Nations?*

Owen *During the England game, yeah, they'd have played England by then.*

Kate *we'll say a previous Six Nations, in case they start going "He didn't do Garryowen."*

Michael *He definitely will!*

While these were not mathematical responses, they added to the employ of a critical student lens where the teachers saw mathematics through the eyes of their students. Teachers engaged in creating that question of believable context, anticipated how students would respond to the question, and modified the question according to their employ of a critical student lens.

In their post-lesson discussion, teachers reflected on how engaged the students had been in the questions – discussing the mathematics in context and not only seeing the question as a mathematical abstraction. Some of the students found the context of the question funny since it referred to a well-known sports pundit as a mathematician:

Kate *It is so relevant and they were even laughing! I could hear them saying "George Hook's not a mathematician!"*

Michael *Oh yeah, yeah, yeah.*

Nora *George Hook doesn't know...*

Michael *The kids are having a great chat about well, if he was tackled or, you know if he was charged down and all that sort of stuff. And they were talking about all these sorts of, whilst they were doing the problem.....But that shows that they understand what they are being asked!*

Teachers enjoyed accessing students' talk in attempting to solve these problems (see Section 5.5.5.2) since they could see that students were engaged with the problems because of the context provided within them.

Contextualising these questions enhanced teachers' KCT in realising how students can be further engaged in their mathematical learning (*hatsumon*, (Doig et al., 2011)) but also

benefitted teachers' TKS in understanding what particular topics these students found interesting and could be further incorporated into future lessons (see Section 6.5.1). Teachers had employed a critical student lens in devising these questions for the research lesson and it had benefitted the lesson in terms of student engagement and in terms of developing mathematical content for the research lesson.

In their final research lesson teachers devoted an entire planning meeting to designing questions that would be of relevance and interest to this group of 2nd year students, but that also contained applications of Pythagoras' theorem. Teachers originally began with traditional questions involving gable walls of houses or cliff heights, but then contextualised these questions for this particular group of students to contain a question involving a mobile phone screen and a question involving a prominent Irish golfer, topics that were of more interest to this class group (Appendix F2 5).

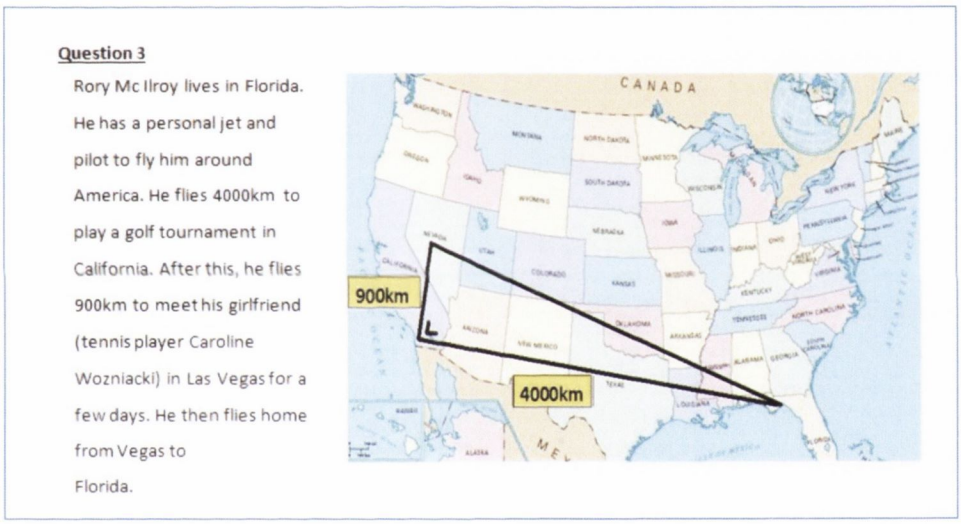
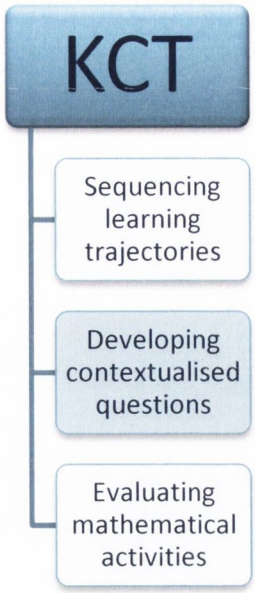


Figure 50 Contextualised question: Doone cycle 3, Appendix F2 5

In planning this lesson and providing contextualised scenarios such as the story of Pythagoras and his two girlfriends (Section 6.5.1.1), teachers again anticipated students' responses and engagement. Although these were not mathematical responses, teachers were employing a critical student lens in attempting to view the lesson from a student's perspective within this class and developing their KCT in devising new ways to introduce and teach this topic.

Designing contextualised questions throughout the lesson study cycles engaged teachers further in planning lessons within the lesson study cycle, but also assisted teachers in employing a critical student lens since they viewed questions from students' point of

view in firstly anticipating how interesting the student would find the question and secondly anticipating the realism and mathematical difficulty of that question (Ainley, 2001; Blackhouse et al., 1992; M. Brown et al., 2008; Galbraith, 2013; Noyes, 2007). Creating specifically contextualised questions enhanced teachers' KCT in understanding how particular mathematical content could be taught in a socially and culturally relevant way to the students in this school.



6.3.2.2 Crannóg

Designing contextualised questions will now be analysed according to the data generated at Crannóg in terms of teachers employing and enhancing their KCT.

In their very first research lesson, teachers in Crannóg wanted to create a situation which would engage students in a way which would lead them to inquire more about the question. Since this was to be a student-led discovery lesson, it was imperative that the question have an interesting context for students and while the scenario was not something that was relatable to these second year students, teachers felt that they would engage with the story of the question. It was Stephen who devised the context of this story, partly based on the research literature from the professional magazine Mathematics Teacher (Peterson, 2006) which I had brought to the meeting in anticipation of teachers' planning conversations, and pitched this idea to his colleagues.

Stephen *What I was pitching was doing it actually building up... and have a number per in a pyramid, much the same but building a pyramid... was what I was pitching...*

Fiona *So you start with the bottom, you start with the four...*

Stephen *So start ... so just there [referencing the literature]... so my thing was Egyptians making pyramids... how many do they need per layer... and actually start with... so... and have one for the top layer, you have one, you have four underneath, you have nine underneath that...*

Aoibhinn *What does it look like?*

Stephen *So... a three dimensional thing...I was thinking... as an issue...*

Dave *Is that still a quadratic?*

Stephen *Yeah, 1, 4, 9...*

Aoibhinn *Could it be an extension though? Because I think that might...*

Stephen *I was just thinking of it in terms of a general introduction... introducing it with a general aim which is pyramids and Egyptians and something like that, and they have square... each layer is a square, but it is the same numbers that would go, rather than just kind of building these... and then it is all in one in front of them...*

Eileen *And then they have the bricks at the bottom.*

Fiona *They would see the x^2 nicely there...*

Stephen *That is what I was thinking...*

The story was to engage students in investigating the quadratic pattern of the blocks contained in each layer of the pyramid and teachers spent much time anticipating students' mathematical responses but also their responses to the question. As soon as they had constructed the detail of the story, teachers started anticipating what students would ask:

Stephen *Straight away "Sir, what's a pharaoh?"*

Eileen *Yeah...*

Stephen *"What's a pharaoh? What's a pyramid?"*

Teachers also anticipated that students would wonder about the shape of the base of the pyramid. In anticipating these responses and in designing this question, teachers were employing a critical student lens in viewing the gender-neutral question and content from the perspective of this particular group of 2nd year students.

Similar to the second post-lesson discussion in Doone, teachers in Crannóg shared their surprise and enthusiasm after the research lesson in seeing students so engaged in a mathematics question and lesson. This was a moment of learning in teachers' KCT in

realising that providing an interesting context for students could engage them in the mathematical content of the lesson.

> *Walter* *I think it was more exciting for them than I thought it would be...Their imagination of it, they just sort of plugged into that more!*

In research lesson 2, however, no context was provided for the questions in investigating new strategies to factorise quadratic expressions. This may have been due to the fact that the content of this lesson was particularly in abstract and it would have been difficult to provide context for all algebraic expressions contained within the lesson. Also, since the focus of the lesson was in familiarising these students with a strategy of factorising using algebra tiles and array models, embedding this within a context may have been confusing for these students. In saying that however, teachers did discuss the lessons which would lead up to the research lesson and Stephen had suggested included area models (a numerical-only form of array model) within this sequence (Appendix F1 4). This may have provided a physical context for students to visualise why algebra tiles and array models may have been useful in algebra but was not incorporated within the series of lessons in the first teaching of this research lesson.

The final lesson study cycle in Crannóg was very focused on providing students with meaningful context for the need for differentiation in calculus. In the lead up to this research lesson, teachers devised the 'Tractor Class' which would involve students in generating their own data on the hockey pitch with the school care-taker driving a lawnmower tractor at a constant speed. This lesson was designed to engage students but to also illuminate the difficulties that come with generating data. The following lesson in this series repeated this objective in documenting how volume increased with height for various containers of liquid. Finally for the research lesson, students were engaged in content based on the Olympics which had occurred in London the previous summer. Students were given data based on Usain Bolt and Victoria Pendleton's races and graphed and analysed the data (Appendix F2 7). In providing students with real-life contexts, teachers wanted to engage students with mathematical content that they had traditionally taught as abstract ideas.

> *Dave* *We could tie this in with your applied maths... You need never do calculus with your 4th year. If you are just introducing a topic or*

something you would never go off down the road of produce rule and quotient rules or anything else.

Walter *No.*

Dave *Whereas you could tie it back into your practical applications: distance, speed, time and acceleration.*

Walter *Absolutely...*

Fiona *And you could also have them graphing those, the expressions that you have done in applied maths and you could lead them into instantaneous change by getting them onto graphs and actually getting them up and putting in the tangents and moving the tangents. You could do that kind of first principles idea using distance, velocity, time graphs.*

In planning this lesson, Dave commented that he found the textbooks unhelpful in introducing calculus and as a community of teachers they immediately began suggesting different ideas of how they could introduce an idea of rate of change.

Walter originally suggested attempting to model the trajectory of a rugby ball or hockey ball that one of the students would kick. Eventually teachers decided on the sequence of lessons which began with students investigating linear data and moving to quadratic data as described in the previous section. This series of lessons was focused on providing students with a context in which they could realise a necessity for calculus and was powerful in its ideas since students began by investigating data they had generated themselves.

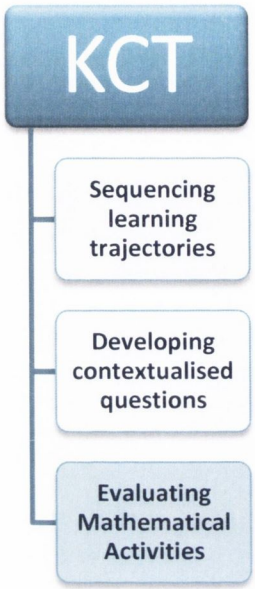
6.3.2.3 Conclusion of Developing Contextualised Questions

In devising questions that were relevant to their students and which provided students with a real-life content for their mathematics, teachers had enhanced their KCT of teaching particular topics to the students within their school. The content they had planned was specific to certain classes and year groups and also enhanced teachers' TKS (Section 6.5) in finding out what particularly interested students and how this could be incorporated into future teaching. As the lesson study cycles continued in Doone, teachers increasingly used tasks that elicited students' thinking and supported students' exploration as had been suggested in Perry and Lewis (2008). Teachers in Crannóg had begun their engagement in lesson study with a focus on the stimulus activity and its

presentation to students and, with a change in research lesson 2 likely due to their unfamiliarity with the curriculum materials, continued with this process in cycle 3.

6.3.3 Evaluating Mathematical Activities

This is the third of the three proposed features of how teachers employed and enhanced their KCT through teachers' participation in iterative lesson study cycles over the course of one academic year. These features contribute to the literature on teacher knowledge and lesson study in identifying how teachers employed and enhanced their KCT through planning and reflecting on research lessons.



In planning research lessons and in designing questions, teachers increased their KCT by identifying what exactly they wanted students to encounter in term of mathematical content. Evaluating mathematics questions within planning assisted teachers finding out how this could be achieved.

6.3.3.1 Doone

As the lesson study cycles continued, teachers began designing more questions (as detailed in the previous section) but also began attempting the mathematics questions which students were to undertake within the research lessons. In doing so, teachers were able to dissect their own mathematical thinking which allowed teachers to employ a critical student lens and highlight how students might answer these particular questions. In the second lesson study cycle in Doone teachers dissected how they themselves thought through multiplying fractions and this led to teachers to incorporate this point within the research lesson.

Lisa *Posing "is a 1/2 of 5/4 the same as 5/4 of a ½." Now technically it is, it's 5/8ths, but it's not really. That's not what we want to know. Is it the same answer? Is the result of multiplying a ½ by 5/4 bigger or smaller?*

Owen *Than 1.*

Kate *Than 5/4ths...*

Lisa *Yeah, than 5 1/4s... Is a 1/2 of 5 1/4s bigger or smaller than 5 1/4s?*

Aoibhinn *So you're taking, this number [on the right] is what you're checking to see if it's bigger or less than 1?*

Kate *Yes.*

Aoibhinn *Okay, because I was thinking of it the other way.*

Michael *So was I, I think....yeah....*

Kate *So this is the number we're checking is greater than 1, aren't we?*

Aoibhinn *Well, that's what I was asking.*

Owen *Which number?*

Kate *Which way would most people go?*

Michael *In my head I had it that way.*

Kate *Would you see it as 3 by a ½ or would you see ½ by 3?*

This discussion of the question may have seemed a little meandering but actually clarified to all of the teachers in which order they were viewing the multiplication of two fractions. This was important since the objective of the associated student activity was to develop a sense of measure of what happened when multiplying a number by a fraction. Doing the question allowed teachers to clarify the mathematics as they wanted students within the lesson to view the lesson. Doing mathematics questions prior to the research lesson also allowed teachers to anticipate any difficulties students might have in attempting to do the question themselves (Mason, 2009; M. K. Stein et al., 2008), as detailed in Section 5.5.4.1.

In designing and doing a particular question within that second research lesson, Lisa clarified her own knowledge of the content and how to construct the question in order to meet a particular objective for students. This was an important moment in building Lisa's KCT of this particular topic and was facilitated through the collaboration of teachers in designing the research lessons and associated curriculum materials. As described in Section 6.3.2.1, following various conversations around the same issues in designing

questions for this particular lesson, Lisa realised how to construct the questions so that the students would multiply fractions instead of whole numbers.

Lisa	*The Ireland rugby squad.... 1/5th of these have eye problems. Of this 1/5th, ½ wore contact lenses. What fraction of the players wore contact lenses?*
Kate	*But that's going to be, they're going to get, so they're getting thirty players.*
Lisa	*30 multiplied by 1/5th is 6 –*
Kate	*Well, they're going to divide by 5. Just so you are aware.*
Michael	*Yeah.*
Kate	*Yeah, so they are yeah. Unless we said 'a squad' instead of saying 1/5th of the players. Don't give them a number of players because they'll divide by 5 and get 6.*
Lisa	*I think that's what we're doing wrong. No, I think that might be why that, that problem with the class [previous question]. We just want a fraction...*
Owen	*So it's one whole squad.*
Lisa	*So it's 'a class', 'a squad', 'a pizza'. The answer. Brilliant.*

By designing and doing out all of the questions that teachers had created for the lesson, teachers realised that how they were phrasing the questions by including the total numbers in the problems was not meeting the objective of multiplying fractions. This was an important learning moment in enhancing teachers' KCT, in particular Lisa's as she concluded her finding in that planning conversation.

Unfortunately, in the third lesson study cycle teachers only devoted two meetings to planning this lesson and did not attempt to do the questions themselves prior to teaching the lesson. As the teachers had designed one of the questions (the rugby question described in Sections 6.3.2.1 and 4.4.2.1, Appendix F2 3) within the planning, they were assured of the content within this question but did not investigate the other question further. While in their post-lesson discussion of this lesson teachers did not record any

student difficulties with the content, teachers would most likely have been able to anticipate students' responses within the lesson instead of, for example, Kate being surprised by some students' strategies to identify a pattern by table alone, without having to draw a graph (Section 5.5.5.2.1).

In the final cycle, all of the teachers involved in the planning of the lesson designed questions and then spent a portion of the subsequent planning meeting doing out the questions prior to the research lesson. This proved to be a very beneficial activity "*doing it like students*" (Owen in the final planning meeting) since teachers made small adjustments to the questions in terms of their language, but also focused on the numbers within the questions. In doing the questions teachers employed a critical student lens and adjusted the questions in order to better suit this particular group of students.

Kate	*Okay, so question 1 [Figure 52]. Maths – no problem. We had the, the sentence "the aeroplane has travelled 400 metres in straight line distance on the ground." We just should replace that with "horizontally"?*
Michael	*Oh yeah.*
Kate	*And has travelled 300 metres vertically.*
Nora	*I hadn't done that.*
Lisa	*But you see this is a problem. Because you're saying that the plane has gone along the runway and then taken off, but actually it took off from a point. So, you know, if you say it travels horizontally, they'll think it's like a plane on a runway and then taking off.*
Owen	*Okay.*
Lisa	*So we have "At a certain point in the sky, it's 300 metres above ground from a point 4000 metres from the take-off point"*
Owen	*That sounds right*

By doing out the questions that they, as a community, had created, teachers were afforded the opportunity to adjust the wording of the question in order to better clarify the activity for students.

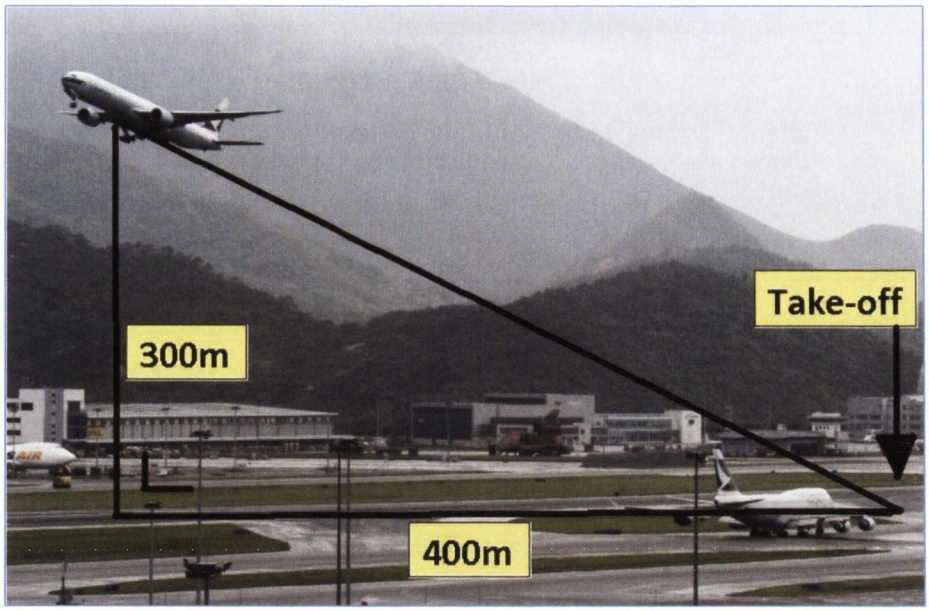


Figure 51 Student activity: Doone cycle 4, Appendix F2 5

This conversation then led to teachers sequencing the questions, in order to build on students' mathematical thinking and successes in answering the questions, including differentiated tasks for students who may complete the tasks prior to their colleagues.

6.3.3.2 Crannóg

It was a strong feature within the data from Crannóg that teachers spent time during their planning doing out the mathematics questions prior to students attempting the questions. In the first lesson study cycle teachers modelled the questions using the Lego that students would use in class. Through doing these activities, teachers were able to clarify exactly what pattern they wanted students to look for. Walter joined his colleagues in playing with the Lego and through this activity clarified the pattern of bricks he was investigating.

Walter *We are going 1, 2, 3. Sorry, I was looking at the base still thinking that we are going up 1, 3, 5, but we are not, so we are overlapping the bricks, okay.*

Fiona *We could get the – the total number of bricks is a cubic.*

Aoibhinn *That is a nice extension.*

Eileen *Yeah.*

Fiona *Just even to get the third difference – there's no need to do it but there's no harm in doing it.*

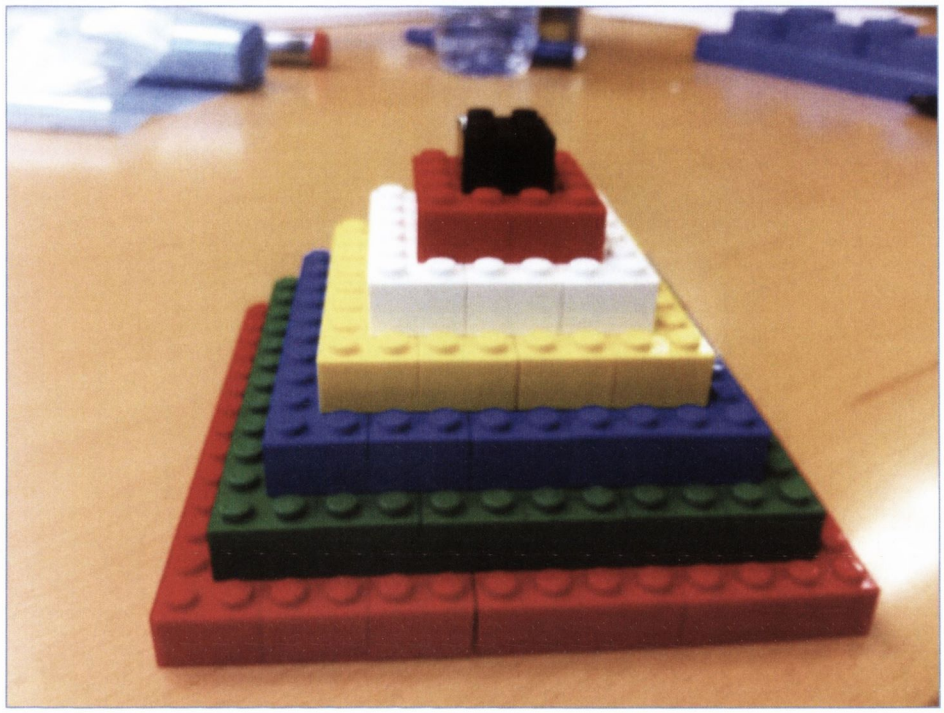


Figure 52 Teachers doing the activity: Crannóg cycle 1

In doing the questions, teachers made sense of what was being asked but, as seen in the excerpt above, also got ideas to extend questions and provide additional differentiated content if it were required (Appendix F2 6).

By doing and talking through the questions, teachers also clarified what would be asked of students and modified it according to how they anticipated students would respond. For example, over the course of the third planning meeting the issue of the '100^{th} layer' continuously returned as being too easy for students (Section 5.4.4.2 B). Teachers eventually agreed to change this in order for students to have to calculate the number of bricks in the 49^{th} layer, a number which they would not know the square of and would have to be able to identify the quadratic pattern in order to calculate.

In the second lesson study cycle it became particularly relevant for teachers to do out the mathematics questions since teachers were investigating new strategies of factorising which they were not familiar with themselves. Within these planning meetings, the majority of time (70 minutes of a total of 310 minutes of planning) was spent doing out questions. Some teachers, such as Walter, were initially unsure of how to use algebra

tiles and he spent time within the planning meetings asking questions and clarifying the strategy to himself. Teachers taught each other the new strategy by physically manipulating the tiles themselves during the planning.

Walter *Do we want to link the factorising thing into the model, the multiplication model?*

Aoibhinn *The square?*

Walter *Yeah, I have never taught it that way.*

Fiona *I haven't either...*

Walter *You try putting it into the box and then start. You put in your x^2 and your 6 and then you look for factors of 6 that add upto 5. I mean you are doing the same thing.*

Aoibhinn *Yeah, it is just a different –*

Dave *Model.*

Aoibhinn *Way of visualising it.*

Dave *I would be interested in looking at that stuff, I would be interested. Are those arrays you called them?*

Eileen *The array model.*

Dave *And algetiles – is it the same thing?*

Teachers were not sure of the differences between the array models and the algebra tiles and found out the nuances in both strategies through repeatedly doing out quadratic factorisation questions during the planning.

Walter taught his class how to use algebra tiles for factorising as a sketch lesson to one of his classes and fed back his observations to the lesson study group. Stephen used the array model, developing from an area model, to his own class in another sketch lesson and again fed back his observations to the lesson study group. These two teachers did this of their own accord and it added to the planning process and to the learning processes of the teachers. As none of the teachers had taught these strategies before, doing the

activities themselves and then teaching portions of the research lesson enhanced their KCT of this topic.

Stephen *I didn't do tiles, I just did area models....But actually I think it worked really well with that actually having the set method for the difference of two squares, for everything. And actually it cleared things up for them, where in previous years if you do it by a difference of two squares it is like something else to remember. ...*

Walter *I think though the algebra tiles definitely for the weaker students, gives them a structure to lead into the grid. I think with my guys –*

Fiona *Yes, I do think so.*

Walter *At that level again I definitely would do it that way.*

Eileen *Whereas I think with my class, even when you would multiply those out in a grid that they didn't really need the tiles to know that the* x^2 *always went there and the number always went there.*

Because teachers kept factorising questions during the four planning meetings in this cycle, they enhanced their knowledge of content since they were becoming aware of and familiar with new strategies of factorising. In his second interview, when asked if he had engaged more in the new curriculum, Dave responded saying that he had learned more about quadratic patterns during this lesson study cycle than he had in his whole career up to now.

Dave *I certainly had to engage more with quadratics or x-squares or whatever it might be than I ever thought about them before. I have looked at more ways of solving quadratics in the last week than I have looked at in the last ten years. I have tried more ways of solving them in the last couple of weeks than I have over the last ten years. I had two methods, the method I used at school for ten years and the method I met with Fiona... Now I am just reading different types with algetiles and with area models and with squiggles and without squiggles.*

Similar to Remillard's (2000) research, it was not the curriculum materials themselves from which teachers learned, but rather the fact that they had to rethink aspects of their teaching and interpret new ways of thinking in exploring these two new strategies of solving quadratic expressions.

Teachers did not do out the questions in research cycle 3 to the same level of detail as they had in the previous two cycles. Instead, teachers talked through the questions that would be asked of students based on the linear and quadratic expressions. Talking through the questions allowed teachers to see the questions from students' point of view and generated anticipated responses from students, but teachers neglected to note any difficulties students may have encountered during the lesson. From the process of talking through questions, teachers did adapt questions in order to better scaffold students in building their mathematical understanding of rates of change without telling the students directly. In planning the progression of the series of lessons teachers discussed these various anticipated responses of students as a community and adapted the lesson materials from these conversations.

Fiona *... then we would go onto something maybe where it is not [linear], you are going to have different speeds and you are going to have a non-constant.*

Eileen *Like a rugby ball or -*

Fiona *Do we want to come straight in there? Or do we want to have a class introducing that concept and then go into it? What do we actually want to do in this class? Which part of it do we want to do in this class?*

Walter *So we just want to, I mean, it is just we want to end up with a non-linear graph.*

Fiona *An instantaneous rate and being able to find out an instantaneous rate of change.*

Walter *If you graphed it and were looking at it, would a lot of them intuitively come up with the instant rate of change rule?*

Fiona *How do you get it?*

Walter *Well the guys who do TG [Technical Graphics] and stuff would just say "Well let's just construct a tangent of that" but how will they do that?*

Fiona *If you haven't got it graphed -*

Walter *I don't actually know.*

Fiona *Yeah, it is hard to know what is going to happen,*

Walter *Hmmmm.*

Fiona *So do we want to, this class to be, where we have got - are we going to do a practical on this? it depends on what we are going to do, whether we going to do a practical on the quadratic as well?*

Fiona *If we gave them a graph with no dots I think they would be very, it would be kind of stretched to do it.*

Eileen *Yeah, don't give them the fraction with the structure of that table but if you needed to guide them say "what about the speed between A and B"*

Fiona *Yeah.*

Eileen *And then make it smaller and smaller. And then if they go to put in another F point or G point or H point, closer and closer to A, then great.*

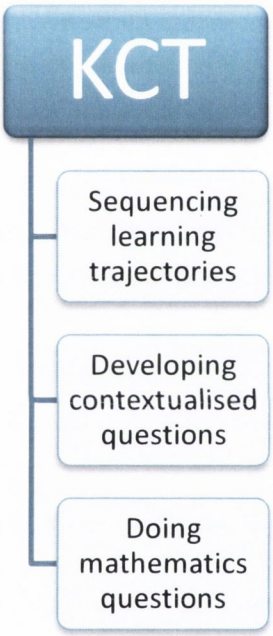
It may be unfortunately that teachers did not actually do out the mathematics questions that were to be contained within the lessons. Doing these questions as the students would most likely have focused teachers on what to attend to within the observation of the research lesson and would have focused teachers' attention on attending to students' strategies but nonetheless, talking through the questions themselves did provide teachers with sufficient knowledge of and around the questions prior to students attempting them within the research lesson.

6.3.3.3 Conclusion of Doing Mathematics Questions

As is suggested by Stein et al. (2008) and Mason (2009), doing questions prior to asking students to do those questions is extremely beneficial to teachers in anticipating how students might begin to answer that question and in pre-empting any mistakes students

might make. In doing questions teachers were able to view the exercises from students' perspectives, both employing and developing their critical student lens. Doing mathematics questions during the planning of a research lesson allowed teachers to construct or modify existing questions that were available to them as part of the curriculum reform resources. After following the activity, teachers were able to modify questions to specifically focus on developing relevant skills for students. Doing mathematics questions developed teachers' KCT in experiencing how best to build students understanding in terms of activities and in choosing questions that were relevant to their particular students' learning.

6.3.4 Conclusion of KCT



In this research it became apparent that the teachers participating in this research enhanced their KCT through their participating in lesson study.

The analysis reveals that the standard understanding of KCT can be further differentiated into these three features of sequencing learning trajectories, developing contextualised content, and doing mathematics questions in planning.

Through sequencing series of lessons and sequencing activities within research lessons teachers understood more of how to build students' mathematical understanding through developing a learning trajectory which preceded and followed the research lesson that was the focus of teachers' planning, observation and reflection. Through discussing various ideas with their colleagues, teachers enhanced their knowledge of ways in which particular content could be taught and developed their KCT.

In Doone, Michael felt that he had increased his mathematical knowledge through teachers doing questions but doing them using different strategies or looking at the question in a different way as expressed in his final interview when asked if he felt that participating in lesson study had increased his mathematical knowledge.

Michael *It's just by listening to different people looking at the same thing you're going to get different perspectives on it. And you'll always get some sort of take out to move away from it to say 'Oh God, I never looked at it like that'*

In her final interview Eileen in Crannóg also agreed that her knowledge had been enhanced through discussion of ideas on teaching different topics with her colleagues.

Eileen *I definitely learned from watching the others teach and I have definitely come up with different ideas and strategies in my teaching...I think it's good to hear how other people approach things, even if we didn't go with someone's idea, just to hear different people's approaches, different methods that they would use. Yeah, definitely, heard loads of ideas and do you know, thought about them more and tried to bring them into my own teaching.*

Eileen had not previously known about or taught factorising using an array model but was the first teacher to design and use a template to introduce this strategy, which she shared with the teachers in the group. She also felt that in dissecting how to introduce calculus as a topic to her transition year class, she had thought about content in a way which she would not previously have done. It is important to note that for Eileen, a newly qualified teacher, she realised that she had benefitted from the conversations in planning sequences of learning for students that were creative and new to how she would have traditionally taught a topic.

Through developing contextualised questions teachers had benefitted their KCT in terms of constructing questions that were relevant to their students' mathematical ability, motivated those students to engage in the question, but also developed students' mathematical skills relevant to a particular topic. In designing the rugby ball question both Kate and Michael enhanced their KCT in designing scaffolding questions around the exercise but also in determining realistic variables within the question which were relevant to their students. In Crannóg, designing the Tractor Class and subsequent volume and Olympics questions provided a new context for students to engage with calculus.

Through doing mathematics questions teachers enhanced their KCT. In his mid-point interview Dave suggested that he had learned more about the teaching of quadratics in the previous 10 days from doing questions using different strategies – conversations and exercises that had been prompted by planning conversations in Crannóg. In doing mathematics questions within the meetings teachers in Doone teachers had developed their KCT by realising the relevant and pertinent details within questions, such as when Lisa was designing and checking the fraction question in cycle 2 or when teachers at Doone all practised the questions within research lesson prior and modified some of the language to be more accessible to students. In developing the Lego exercise, teachers in Crannóg developed their KCT in terms of constructing a discovery learning activity where students could derive a number of patterns, not all of which were quadratic, from investigating a particular structure which they themselves had attempted in their planning.

Through engaging in lesson study, teachers developed their KCT by engaging in planning meetings where they could be creative with ideas of content and questions for their students. In observing and reflecting on how students then engaged with the content and questions, teachers were able to make valuable judgements on what activities, sequenced in particular ways, were of benefit to students and thus were provided with opportunity to build on their KCT in their everyday classroom practices.

6.4 Enhancing PCK and Employing a Critical Student Lens

As has been detailed throughout this analysis of teachers' enhancement of their PCK, the employment of a critical student lens on their practices was an inherent point of teachers developing their knowledge. There have been multiple samples from the lesson study data of teachers employing a critical student lens in their planning and reflection of research lessons for example, Lisa's first realisation of how students viewed the equation associated with Pythagoras' theorem in the planning of their last research lesson or Stephen's suggestion to modify the calculus lesson to include zooming in on the graph of the equation in his reflection of their final research lesson.

At the end of the research, many of the teachers noted that they had become more aware of students' thinking due to their participation in lesson study and that this was now impacting how they approached teaching and learning in their everyday practices.

6.4.1 Doone

In Doone, while there was evidence within the teacher meetings that teachers were employing more of a critical student lens in their practice, not all of the teachers thought that they had begun thinking about mathematics from a students' perspective. As expected neither Nora nor Michael, who regularly viewed mathematics from the perspectives of their students and particularly from the viewpoint of a student who may find mathematics difficult, felt that they had not gained in their practice in terms of employing a critical student lens. Michael suggested that participating in lesson study had reinforced to him the importance of wondering how students saw mathematics but did not enhance how he employed this perspective in his practice. Kate, who although a conscientious teacher had not employed a critical student lens in her initial participation in lesson study, did not feel that she had gained this perspective from lesson study although data generated in teacher meetings would suggest this was not the case. Instead she particularly focused on how engaged students were in the lessons and focused on this as a positive outcome of her participation in lesson study.

On the other hand both Owen and Lisa reported that they had very much benefitted from lesson study in beginning to think about mathematics from a students' perspective. In Lisa's mid-interview she reflected that she had begun to think more about her teaching from the perspective of student in her class and this employ of a critical student lens began to grow as was evident in the final lesson study cycle where Lisa realised the mathematics from a students' perspective and also made Kate aware of how a student might find a particular activity difficult. When asked if she was thinking more about how students were thinking Lisa reflected in her mid-point interview about how she was beginning to think from the perspective of students in her classroom:

> *Lisa* *You are more aware of how it sounds to the kids...Because I know in my head what I'm trying to say and I know that maybe sometimes when I'm saying it it's right, but maybe I'm speaking too fast....I am more conscious of how a teacher sounds at the top of the room to kids who are not engaged...So I'm more conscious now that if I try to pack too much in to a class.*

Lisa suggested that thinking about mathematics from the perspective of a student the most important aspect of her learning from participating in the research. In reply to the

same question Owen noted that participating in lesson study had brought about a huge change in his practice since previously he had been "*totally oblivious*" of how students were thinking about mathematics.

6.4.2 Crannóg

All of the teachers in Crannóg reported that they were thinking more about how students were thinking in their teaching. This was surprising since it was not expected that Stephen or Fiona, who had constantly employed a critical student lens throughout the year, would feel that lesson study had benefited them in this way. However, both explicitly pointed this out they felt they had benefitted from this practice and enhancement of their knowledge of teaching in their final interviews.

Walter, as head of department, held a meeting to discuss the reasons for possibly continuing with lesson study in the following year and asked teachers if they felt they had benefitted from participating in the research. He and Stephen agreed that one of the most significant benefits was thinking more from students' perspectives.

Walter — *Do you think your teaching has benefited from doing this, and in what ways? Any thoughts on that?*

Stephen — *I think definitely so in probably two ways: one being in terms of your preparation for your class...thinking in different ways about it rather than being too regimented in your own way...And the other one obviously is a big benefit being in with other people and seeing how, I kind of got it from how kids react and what kids think and kind of think in terms of my lesson preparation how kids think and actually just being able to sit beside them and see what they do, I found that very beneficial....*

Walter — *So we've been opened up to different ways of doing things through our discussions together but also then we're thinking more about how the students are reacting to it.*

Eileen, a newly qualified teacher in Crannóg, felt that she had very much benefitted from the collective employ of a critical student lens during lesson study and that she now approached all of her teaching in a way where she tried to see the students' perspective,

anticipate their responses, and facilitate their learning accordingly as she reported in her final interview.

> *Eileen* — *Yeah, I probably would ask myself a bit more like how would they react to this or what questions will they have. Pre-empt their questions or pre-empty their confusion, yeah I would think about that a little bit more.*

Participating in lesson study, planning and reflecting on meetings had benefitted Eileen's KCS in terms of anticipating students' responses but had also benefitted her KCT in terms of making her aware of the various ways of introducing and sequencing topics in mathematics.

Dave responded to the same question that he was thinking more about how his students were thinking but also that "*what I thought they were thinking was completely wrong.*" Through participating in lesson study he felt that he had previously underestimated students in his practices.

> *Dave* — *In the same way, in the same reasons, I think it benefitted my maths knowledge, it benefitted, it changed, had an effect on my teaching styles, my perception of what a 2nd year or a 5th year or a 4th year was capable of. My perception of how people lean, like presenting a problem at the start rather than a method of a procedure you know is completely flipping my standard pedagogical mind.*

All of the teachers in this school felt that they were focusing more on how their students were thinking in their regular practices as a consequence of their participation in iterative cycles of lesson study throughout the year.

6.4.3 Student Perspective and Student Voice

As a brief aside, in Fernandez et al.'s (2003) paper, researchers briefly referenced a teacher using a 'childlike voice' (p. 180) in anticipating how a student might respond to a particular question. From the data in this research, it emerged as a strong feature that teachers very often used a student voice during their planning when articulating to their colleagues how they expected students to react or to ask a question. In their reflections teachers also used a student voice in reflecting on hearing student talk within the research

lesson. This may be one way for teachers to access their critical student lens in attempting to situate themselves as a student and anticipate what that student might say.

While Kate was the most regular use of a student voice in her conversations during the year, her colleagues also began (likely unknown to themselves) using the same tactic as their participation in lesson study continued. In research lesson 4, Lisa, Michael and Kate all engaged as students in a conversation where they saw themselves in the class asking questions and experiencing success in doing a mathematics question.

Kate *Why are we saying this, all of this?*

Lisa *Can I see a right angle in my problem?*

Michael *I can work something out here.*

Lisa *If I can happy – days!*

This may not be an element of teachers knowledge of content and students, but it may be worthy of further research in exploring teachers' imitation or emulation of students with reference to an increase in their anticipation of student mathematical thinking or mathematical strategies.

6.4.4 Conclusion of PCK

Research literature has suggested that teachers' knowledge is enhanced through participation in lesson study (Kamina & Tinto, 2011; Lewis et al., 2009; Meyer & Wilkerson, 2011) and although some of this knowledge has been linked to increasing KCS and KCT (Tepylo & Moss, 2011) it has not been specifically detailed *how* teachers' knowledge was enhanced through their participation nor has it been explicitly mapped to a frame of teacher knowledge.

In this research there was a deliberate focus on how teachers developed their PCK through their participation in the lesson study cycle. Using a frame of KCS and KCT developed from the data, how teachers used and developed their KCT and KCS through their participation in lesson study has been detailed in this analysis.

In terms of KCS (Ball et al., 2008; H. C. Hill et al., 2008), through identifying students' prior knowledge, anticipating students' responses, and noticing and interpreting students' thinking, teachers enhanced their knowledge of student mathematical thinking and of

how to build on students' understanding of particular topics. In Doone, changes to teachers' KCS was not immediately apparent until their second and third cycles when their observations of students began to impact on their planning and this process continued throughout the academic year. In Crannóg, teachers began in their first lesson study cycle by detailing students' prior knowledge and anticipating their responses within the research lesson. This benefitted teachers in their noticing and interpretation of students' interactions within the research lessons and teachers subsequently continued to improve in drawing from and building on their KCS through the following lesson study cycles.

In terms of teachers' KCT, teachers enhanced their knowledge of instruction (Ball et al., 2008) through sequencing particular content for instruction and particularly in sequencing students' learning over a number of lessons, through designing contextualised questions for students, and through doing mathematics questions during planning meetings. In Crannóg, teachers had begun their participation in lesson study by designing a learning trajectory over a series of lessons for their students in meeting quadratic patterns and continued this practice for all of their lesson study cycles. In Doone, teachers did not detail a sequence of learning for their students until their final cycle but, in doing so, employed a critical student lens in defining a sequence of activities which were suited to their students' abilities and built on their students' mathematical thinking. Teachers also enhanced their KCT through designing contextualised questions and doing questions prior to conducting a research lesson. These elements of planning research lessons benefitted teachers' in terms of thinking about instructional activities from a student perspective and in providing teachers with opportunity to realise new ways of teaching particular content.

Through participating in lesson study, teachers enhanced their PCK by engaging in planning, conducting, observing, and reflecting of research lessons but also from thinking about their practices from the perspective of students. In employing a critical student lens on their practice, teachers began detailing the content of lessons with the student and student learning at the centre of the process. Teachers, in the main, acknowledged changes to their practices in viewing their teaching from perspectives of students and also noted that they were employing a student lens in their everyday practices. Teachers also acknowledged that they had enhanced their knowledge of teaching through

developing new ideas on how to teach certain topics (KCT) and in anticipating and reflecting on students' interactions with mathematics within the classroom (KCS).

While it has been shown that teachers enhance their PCK through engaging with lesson study (Kamina & Tinto, 2011; Lewis et al., 2006; Meyer & Wilkerson, 2011; Murata et al., 2012), this research has contributed to knowledge by identifying and detailing specific features of KCT and KCS employed and enhanced by teachers participating in iterative cycles of lesson study within their school.

6.5 Teacher Knowledge of Student

Referencing the research question a new dimension of teacher knowledge not relevant to PCK or SMK, as defined by Ball et al. (2008), emerged from the data generated in this research.

In this research, the data suggested that teachers' knowledge of their students was an important part of teachers' planning and reflection of research lessons and enhanced teachers' ability to employ a critical student lens within their practice. Throughout this research, teachers' knowledge of students added to the planning, conduction and reflection of research lessons as has been identified in previous sections in this chapter. It was important to teachers *who* it was they were teaching. The sequences of learning within series of lessons, scaffolding content, and the context of questions depended on the students would be present within the classroom.

TKS has been intertwined throughout the previous two sections of KCS and KCT as, within this proposed framework, there exists an intersection between teachers' PCK and the TKS which impacts on how they teach particular topics to particular students.

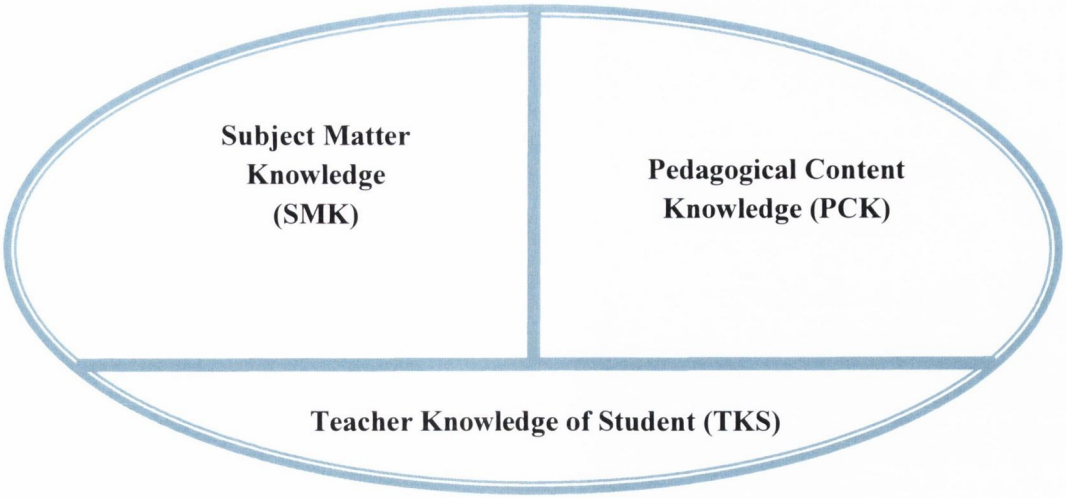
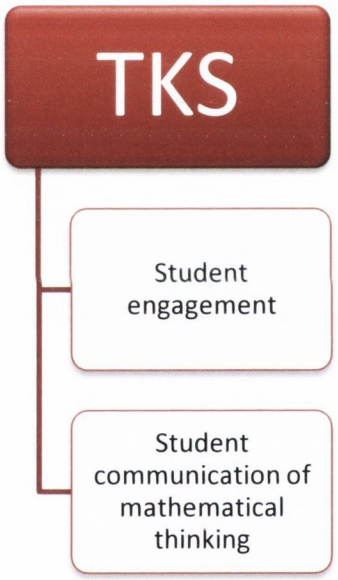


Figure 53 Model of Mathematical Knowledge for Teaching

Teachers' knowledge of students, existing prior to their participation in lesson study, allowed teachers to develop the content of lessons that would be interesting to students. Reflecting on these lessons also enhanced teachers' knowledge of students in various different ways from deciphering how different students reacted to working in groups to understanding how certain students could be more engaged within a lesson through incorporating relevant contexts for the mathematics being learned.



The following table outlines the frequencies of each knowledge code related to TKS occurring in different lesson study meetings throughout the data generation in this research.

Table 9 Frequency of TKS codes

		Doone					Crannóg		
Cycles		1	2	3	4		1	2	3
Student Engagement		12	24	21	26		7	15	16
Communication of Math Thinking		1	16	25	26		8	12	22

As is discernible from the table, as the cycles continued both communities began to incorporate more communication of students' mathematical thinking into their research lessons. This was done through an increase of planned student peer work and whole class discussions during research lessons. As the cycles continued, both teacher communities also began to more frequently focus on student engagement in their planning and reflection conversations around research lessons. A narrative illustration of each of the codes will now be explored in the following sections.

In this next section TKS will be briefly investigated under the following themes as emerged from the generated research data:

1) Student Engagement
2) Communication of students' mathematical thinking

6.5.1 Student Engagement

As detailed in Section 2.2.5.2, it is important that students are engaged with and motivated in the mathematics they experience in their classroom (Brown et al., 2008; Galbraith, 2013; Mason, 1988). In both schools, student engagement emerged as a very strong theme in teachers' planning and reflection of research lessons.

It was important to teachers to provide relevant context for mathematics questions in order for their own students to engage with them. This was also prevalent in teachers' KCT (Section 6.3.2) and is hence an area of overlap between PCK and TKS since in developing content and in sequencing lessons it was important to teachers that their students be engaged in the mathematical content of the lessons. Referencing student engagement, Fiona and Dave in Crannóg began asking their students of their opinion on different mathematics strategies, an unexpected finding in teachers' attention to students' engagement. Teachers' knowledge of individual students will also be briefly referenced in this section.

Furthermore, through teachers' specific and relevant knowledge of their own students, content was differentiated in a small number of lessons in order to engage students who may not otherwise have been as involved in the lesson at the same pace as their classmates. Differentiation is highlighted as a very important element of the new curriculum where the needs of all students in the classroom should be met through varying mathematical strategies, questions, assessment, and approaches (National Council for Curriculum and Assessment, 2012). While differentiation was not explicitly emphasised within the lesson plan template nor in the lesson study booklet, some teachers of their own accord made regular reference to differentiating work within the classroom in order to allow all students to engage with their mathematics and experience measures of challenge and success within the classroom.

6.5.1.1 Doone

From the very first meeting this group of teachers was focused on engaging their students in mathematics. Teachers spoke of wanting their students to enjoy mathematics more and wanted to see the applications of mathematics in real life rather than focus on exam results. Lisa spoke of "*capturing their imagination*" in the first few lessons of a topic in order to engage students more in their own learning.

This was a very important issue for these teachers and it formed the basis of their over-arching goal for the year:

> *For our students to find purpose and meaning in their mathematics that leads to their enjoyment and confidence in the subject.*

Throughout the lesson study cycles teachers occasionally introduced novel ideas into their lessons that were relevant and context based for particular class groups of students. In the first lesson study cycle Kate suggested referencing the variable x as having "the X-factor" i.e. something special about it that allowed it to represent all types of numbers that connected with a popular reality TV show.

At the beginning of this research however, the first research lesson was taught to first year students of whom teachers did not have any knowledge of and whom their conducting teacher, Lisa, was not yet familiar with. Since teachers' knowledge of these students was not yet established, teachers failed to correctly identify these students' prior knowledge and also failed to anticipate how students would best engage with this lesson.

In their second research lesson, teachers began differentiating student tasks in order to cater for different students within the lesson. Lisa referred to the brighter students known to her in the class as the "*yawners*" who could be bored within the lesson since it was a revision of multiplying fractions. She and Kate decided that these students should be given more challenging questions within the lesson in order to engage them in their learning.

Lisa	*So if you have the yawners with what we have been doing, I would put those together with really challenging tasks so that they don't feel so smug that they have got it.*
Kate	*Yea give them almost... one that we would have trouble with just to shut them up so they are not going "This is easy"... because that annoys me sometimes, this look of "Aah, it is so easy"*
Owen	*Yeah.*
Kate	*So we will find the most hard problem we can find!*

Owen later decided that one particular group of students (Aidan, Shane and Cian) would not be succeed in the planned activity of multiplying fractions by fractions and would instead be given questions involving multiplication of a whole number with a fraction. This was in fact a correct decision, made through his knowledge of this class group of students, since in the post-lesson discussion it was noted that these particular students were not completely successful in this endeavour.

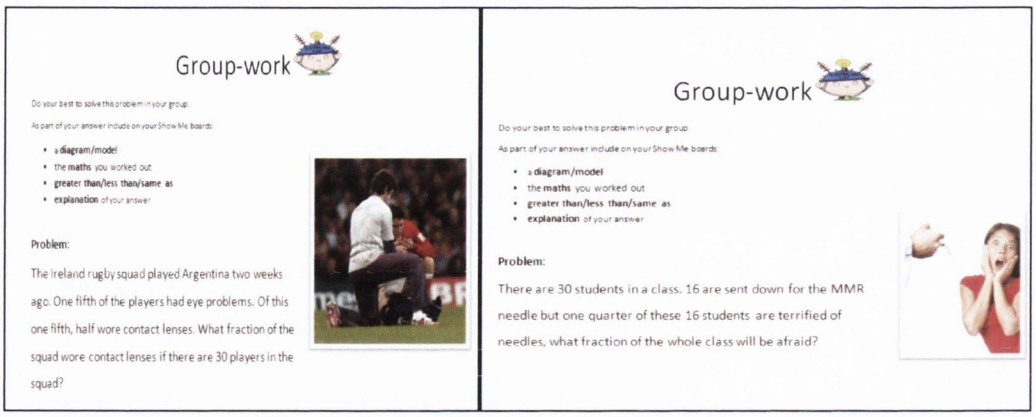


Figure 54 Differentiated tasks: Doone cycle 2, Appendix F2 1

Owen, in planning this lesson, had employed a critical student lens in viewing the lesson activities from these particular students' perspective and from his prior knowledge and relationships with these students he had wanted them to experience a measure of success with a task.

Owen	*Well, I gave them[Aidan, Shane, and Cian's group] the easiest problem because one of the problems actually had a number to start off with, 30, so they had a total...so I kept that one until the end and went to them last with that one.*
Lisa	*Right.*
Owen	*because I knew that they wouldn't be able for any of the other ones...It needed to be easier, yeah.*
Lisa	*Yeah. But at least they got something in.*

In engaging her students by research lesson 3 however, Kate was familiar with these students and wanted to introduce an element of competition into her lesson where groups would compete for the best presentation of their work. Owen came up with the idea of hosting a 'Come do Maths with me' (based on another popular reality television show) competition within the lesson that students could have fun with using monopoly money and a tray. Although teachers did not use this idea within the research lesson it is an idea that they felt could work in the future in engaging students in maths lessons.

Throughout their planning, as detailed in Section 6.3.2, teachers attempted to incorporate contextualised content and questions within the research lessons. Stories were developed around mathematical content such as that involving Pythagoras and his affair with two women who led him to finding a shorter route (the hypotenuse) between their two houses. Kate convinced her colleagues that the context of this story made it very memorable for students and further provided an interesting context for students when teachers included well known actresses as the characters within the story.

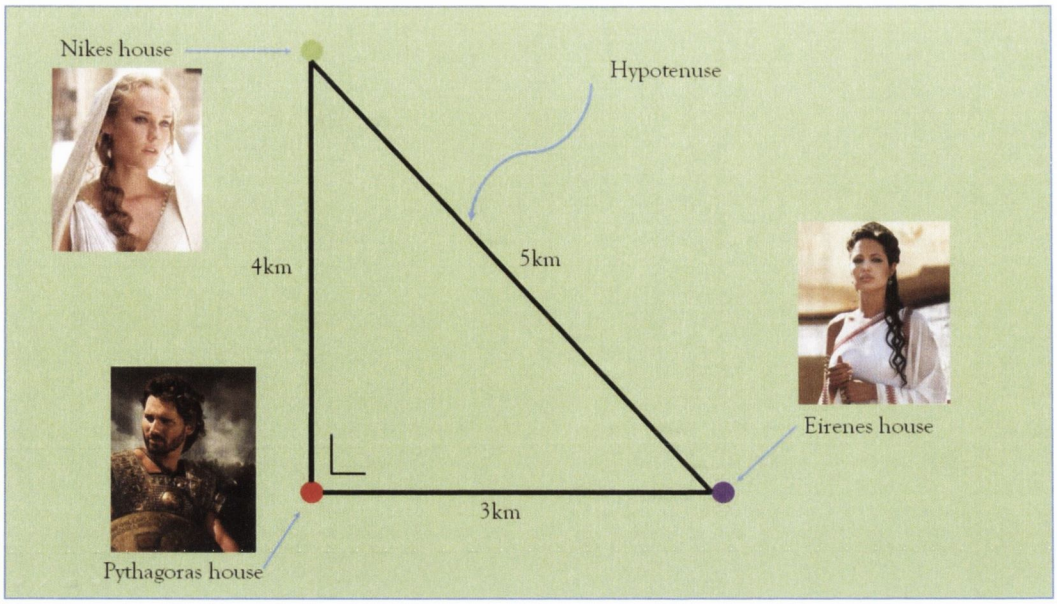


Figure 55 Contextualised presentation for students: Doone cycle 4, Appendix F2 4

In the final research lesson teachers were also concerned about considering these students' motivation and confidence in attempting and being successful in solving questions. As a group the teachers wanted to scaffold and sequence questions where students, working together, would enjoy solving the problems.

Lisa *I think in an ordinary [level] class we'd get a huge amount of satisfaction.*

Kate *Yeah.*

Owen *Being able to get it.*

Kate *And once they get it, they get it and they really do get it, I think. Well, with the words. And they'll be delighted with themselves.*

Teachers incorporated thinking about the affective issues of mathematics including terms like "*satisfaction*" and "*delighted*" in anticipating how they would hope students to engage with the activities.

Throughout their participation in lesson study cycles, teachers maintained a focus on attempting to engage students in the mathematics by situating questions within real-life contexts as they did in cycles 2, 3, and 4 (Section 6.3.3). It was obvious from students' drawings in cycle 2 that students had been engaged in working in their groups, but perhaps the mathematical content was not sufficiently challenging or detailed enough for

all students since there had been ample time for students to create their own versions of drawings.

Kate *My group weren't really working. They were all just drawing on the Show Me boards.*

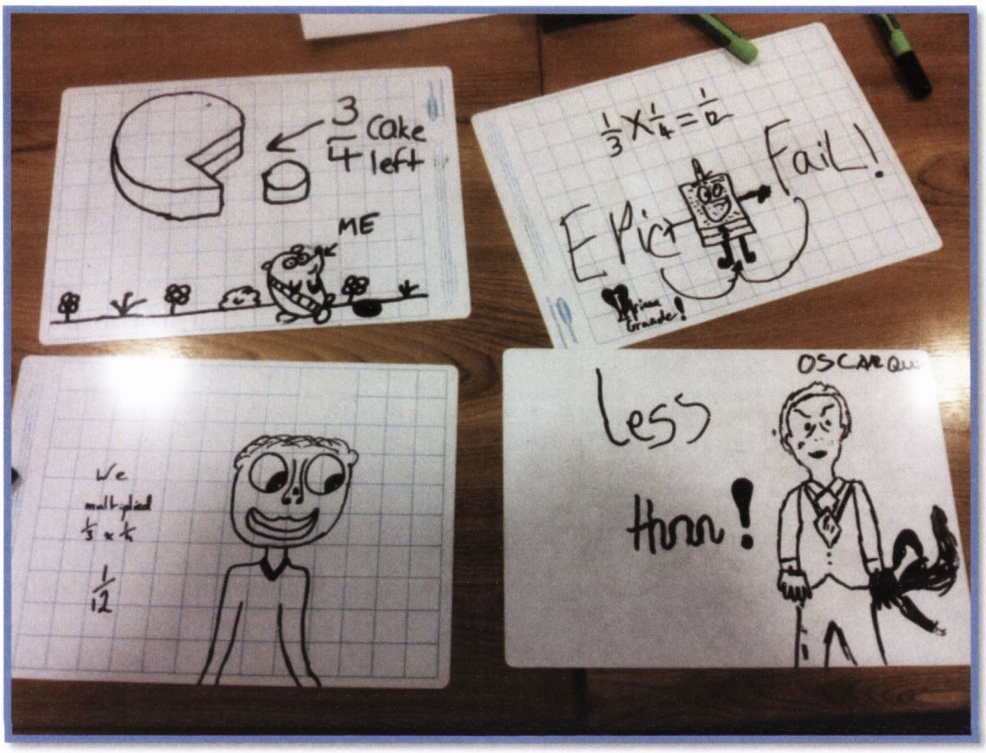


Figure 56 Samples of student work: Doone cycle 2, Appendix F4 2

Teachers developed questions that were particularly relevant to class groups of boys attending a school where rugby was a popular sport. In cycles 2 and 3 the teachers created questions based on the Irish rugby team. In cycle 3, teachers decided to include a question which would be slightly unfamiliar to students but would provide them with an insight into how mathematics could be utilised in the real world. Such authentic content is important in motivating students to engage in mathematical problems (Galbraith, 2013).

Throughout teachers' post-lesson discussions they referred to how well particular students were engaged with the lesson. Following research lesson 3, teachers commented on how engaged students had been in the rugby question but also in the medical question. Students had been enthused by the context in which the mathematics was situated which prompted them to further investigate the question. For example, in attempting the problem involving temperature, students enjoyed hypothesising about the context of the

question (Appendix F2 3). Michael, Kate and Nora reflected on how student groups they had observed (referencing their observation notes) had engaged with the questions in their post-lesson discussion.

Michael *Here they were talking about the temperature rising and then they were saying, well, they were joking like "ah, she is going to explode."...*

Kate *Yeah because one person was saying "Is this Celsius?" and I was like "99. Something degree Celsius, then she would be bubbling and boiling" and they were like "oh yeah! Fahrenheit"... Even one group were saying "ah, this couldn't be true."... He was like "the temperature would clearly rapidly increase and then slowly come down."*

Nora *Sometimes they can make kind of fairly valid observations.*

Michael *But, as you said there about the, at the start of the year when we said about aims and that sort of stuff...being able to visualise them and being able to just conceptualise what it is that we'd ask, that's ticked two of the boxes of over-arching aims that we had kind of without actually attempting to do them.*

Michael concluded that in including these forms of questions that had relevance and context for this particular group of 4th year students, they were providing meaning and purpose for the students to want to attempt the questions.

In their final research lesson, engaging students in the application of Pythagoras' theorem was at the core of the research lesson. In creating questions, teachers altered the contexts so that they were relevant to their particular students. For example, a question traditionally about television screen sizes was changed to focus on an iPad screen and question about a sports pitch was included since it was a familiar context for these students from a sports-oriented school (Appendix F2 5).

Teachers in Doone began their participation in lesson study wanting to engage students in their mathematics. Throughout the year teachers became more and more skilled in creating questions that were relevant and contextualised for the particular students in their school, enhancing their KCT their but also enhancing their TKS in becoming familiar with the type of questions engaged students. Kate reflected on how the research

lessons benefitted these students' engagement in the research lessons in her final interview:

> *Kate* — *Looking at the last two, the functions and the Pythagoras, there was great engagement, there was. You kept them on task, there was no looking out the window, they were all focused and they were on task for the lesson...the lessons were interesting and we had made them as fun as maths can be with interesting problems.*

Teachers had wanted to provide students with a purpose and a meaning for their mathematics as their over-arching goal for their teaching throughout the year, and while the overall lesson study aim had not been overtly referenced following the first research lesson, teachers felt they had achieved this aim through the various cycles of lesson study they had participated in.

6.5.1.2 Crannóg

At the beginning of the year in Crannóg teachers spoke about their students being afraid of mathematics and having a palpable fear of being wrong.

> *Stephen* — *What I really want to get better at...[for them] to think independently, so that they're not too worried about the process...*
>
> *Fiona* — *Just trying to get them to be prepared to have a go. Not to be afraid...just to take the fear element out of it, where they think "I can't do this", just to have a go and see what happens...*
>
> *Stephen* — *We're trying to get out of the culture of right or wrong.*
>
> *Dave* — *Right and wrong. "Ah, I can't do it."*
>
> *Fiona* — *That's the fear thing...if they don't think they're going to be right then they're just not going to do it.*

Teachers wanted to develop students' confidence in their mathematics so that they would become less afraid of getting an incorrect answer and instead enjoy the process of finding out how to answer a question. Throughout the year, as has been detailed in previous sections within this chapter, teachers developed research lessons where students' own curiosity in the mathematics was nurtured. In research lesson 3 in particular, students were not trying to construct a correct answer but instead teachers were nudging the students towards identifying a need for the mathematical construct of differentiation.

During this research teachers focused very little on students achieving the 'correct' answer and instead attended to looking at how students went about doing the mathematics.

Following their first research lesson teachers were incredibly enthusiastic about seeing students fully engaged in a lesson and began excitedly reflecting on the lesson as students were leaving the classroom. Teachers had been surprised at how engrossed students had been in that class. Gerald reflected on how enthusiastic students had been about trying to solve the Pharaoh puzzle in research lesson 1 (as initially introduced in Section 4.5.3.3).

Walter *It was more exciting for them [students within the lesson] than I had thought it would be*

Eileen *Yeah.*

Fiona *Well I think the way you started it off was brilliant. I think you got them interested by talking about the competitive Pharaohs and as they came up with all the answers you wanted, which was great.*

Dave *They came up with the questions themselves!*

Walter *Their imagination of it, they just sort of plugged into that more.*

Gerald *There is an excitement about it, you know. There was a degree of excitement in there about just getting stuff worked out, you know.*

Later, both Walter and Eileen reflected in their second interviews about the fact that students had continued working on the problem for over 20 minutes. Walter would not have previously thought was possible and this caused him to reflect on his teaching of Junior Cycle when asked if he thought that observing research lessons had impacted on his own practices.

Walter *Since then, if I have set a task I have probably let the task run for longer than I would have previously. I think I'd always have been very much: let it run, get it back in, let it out, get it back in, let it out. And doing that sort of model as opposed to actually letting it run a bit further.*

Eileen also reflected on the fact that following the observation of students within this lesson she was allowing students more time to work on questions by themselves before

interrupting them and giving them the answer. The observation experiences increased teachers' TKS in observing that allowing students more time to work on a topic once they were engaged in the question.

In reflecting on the second research lesson, while teachers noted that students had been engaged in manipulating the physical model of the algebra tiles, Fiona was not convinced that her students had learned anything factorising quadratic expressions from this model. Teachers debated the usefulness of the strategy where students were engaged but may not have gained anything from this strategy and they decided to re-design and re-teach the lesson. It was important to these teachers to detail a valid trajectory of learning, enhanced by student engagement, where students' mathematical thinking was enhanced.

In the re-teaching of that lesson Walter reflected that the students in his class seemed to enjoy the manipulation of the algebra tiles as a strategy of factorising more than they would have enjoyed a traditional abstract factorisation of a quadratic expression.

Fiona *How do you think they found the algebra tiles? You hadn't used them before?*

Walter *We have used the algebra tiles before, they - I mean, they seemed to enjoy –*

Fiona *The physical –*

Walter *The physical putting them together. That was, and so they definitely, they got through a series of those questions and got them factorised with them and I think they enjoyed that more than they would have...Mathew and John just didn't say boo through the whole class.*

It was important to Walter that his students had engaged in the lesson and he felt that through utilising these mathematical resources, his students enjoyed using the resources and had realised a strategy of factorising and had thus increased their mathematical understanding of that topic. In that post-lesson discussion Fiona concluded that she did not find the algebra tiles had added to developing her students' understanding of factorising and debated using that strategy again in the future. Walter argued the case with her that before dismissing the algebra tiles, she should consider the fact that the

strategy engaged students more and thus had the potential to also benefit students' understanding.

Fiona *So I just, like I feel if I think of classes that I had before that were weak, em, I think they would have had, having spent the length of time I have spent on it -*

Walter *They would be in the same place.*

Fiona *They would have a, they would have made a better go at those than the one they did today, I think.*

Walter *But have they enjoyed the classes more?*

Fiona *Possibly, but...*

Fiona *But as I just said to -*

Walter *Did they feel more engaged?*

Fiona *No, he is right.*

Walter *They will stay more engaged if they had, if they have fun activities along the way.*

Through participating in this lesson study cycle, Walter has gained new knowledge of how engaging his students in their learning had the potential to enhance their understanding of factorising. Fiona seemed to be disheartened at how her students had engaged in the research lesson and questioned the utility of factorisation itself as a topic in mathematics but, as previously outlined in Section 6.3.1.1.2, Fiona had not followed the series of lessons which had been constructed to lead into the research lesson and this most likely had an effect on how her students understood the process of factorising using algebra tiles.

In the final research lesson, this community of teachers designed a whole series of lessons which would enthuse senior cycle students to engage in generating and analysing their own data. The 'Tractor Class' was one which teachers themselves were very proud of and involved a whole year group of students recording the data on a tractor travelling the length of the hockey pitch, tabulating and graphing it and finally analysing it. Teachers concluded that this would be a memorable class for their students and would

improve students' opinions of mathematics as a subject but would also provide students with a meaningful context for a need for calculus. The series of lessons continued with videos of volume of liquid increasing over time and of Usain Bolt running in the Olympics where students had to pre-empt what the graph would look like before investigating the data. Incorporating a video into the class would, teachers hope, engage the students.

In devising ways to engage the students in lessons, teachers such as Stephen took this as an opportunity to see how his students would react to such activities, employing a researcher lens to his practice.

> *Fiona* *It's just a video starting the class, to catch, try to catch their attention from the video of that and get them to okay and "what do you think that graph looks like as a distance time graph?"*
>
> *Stephen* *Yeah and it will be interesting to see what the different people draw.*

While engaging students had not been an explicit focus of this community of teachers, they had been interested in and enthused by seeing their students engaged in research lesson 1. Throughout the lesson study cycles, teachers did focus on getting students engaged in their mathematical tasks from building a pyramid to manipulating tiles to recording the speed of a tractor on the school's sports pitch. Teachers realised the benefit of engaging students in the mathematics through contexts that were interesting, relevant and sometimes unusual to students.

6.5.1.3 Gleaning Students' Opinions

It was interesting to see that in Crannóg, teachers had begun to ask students for their opinions of what they were learning. During lesson study cycle 2, Dave taught two different strategies of factorisation to his students: one which he had traditionally taught and a new strategy using array models. At the end of the unit, Dave asked his students which method they preferred. He expected them to choose the array models but in fact the majority of students chose the opposite. Dave was very surprised by their feedback and also realised that within his own teaching, he should not guess what students preferred but instead ask them for their opinions. He reflected on this in their final meeting as a lesson study community and in his final interview.

Dave *I asked my class yesterday, having done both methods and moving on to factorising with coefficient for x^2, you know what I mean, harder ones, which method they preferred or enjoyed or understood more, whatever you want to call it, which method they were more comfortable with and they were virtually to a person said squiggles. Which I was surprised by that.*

Similarly following this topic, Fiona asked her students if they preferred algebra tiles or array models. She was not surprised that they chose algebra tiles but in finding out their opinion, she decided that she would continue teaching class groups that were similar to this cohort of students using algebra tiles.

This seems to have led Fiona to incorporating her students' opinions into her teaching since she had recently, following Dave's example, asked her students which operation they preferred.

Fiona *I think the observing of small groups meant that you got a handle on how they were thinking. Like, even Dave's comments earlier that he said about his group with the quadratics, where he said he did it two different ways assuming that they'd find one way much easier and actually they preferred the other way. In that you can make assumptions often about them, thinking that they prefer one way when they don't.*

Having access to students' mathematical thinking gave Fiona the impetus to ask opinions of her students about the mathematics they are learning. She trialled this exercise with her 5th year students and assumed that students would choose the opposite of what they did. It seems that focusing on students' mathematical thinking also drove some teachers to focus on students' personal opinions about the mathematics they were engaging with.

Asking students for their preference and opinion in how they learned mathematics added to teachers' knowledge of their students which, teachers reported, impacted on their thinking and teaching practices. In Dave's final interview he reflected that he had underestimated students' abilities until he had observed them in research lessons and then asked them their opinions on their learning.

Dave *My perception of what a 2nd year, of a 5th year, or a 4th year was capable of, my perception of how people learn, like presenting a problem at the start rather than a method of a procedure, you know, is completely flipping my standard pedagogical mind.*

While none of the teachers in Doone directly asked students for their opinions on how they learned mathematics, Lisa had been taken aback when her students were so excited to have a *'fun maths class'* in research lesson 1. Lisa concluded from their reaction that they did not think any of their other lessons were fun.

Lisa *The shocking revelation to me was, when I said them for maths we were having a fun maths class and it went around the room "A fun maths class." So we don't normally have a fun maths class!*

Shortly after this reflection Lisa changed the layout of her classroom in order to facilitate students working together in groups. This change of her classroom environment (see Section 6.5.2.4) may have been due, in part, to this new insight into students' opinions of their mathematics classes in general. While this may not always be a realistic for teachers to interview students on their perspectives during the already busy school day, it is interesting that teachers had begun to incorporate students' own opinions and reflections within the research.

6.5.1.4 Knowledge of Individual Students

In planning, observing and reflecting on research lessons teachers were provided with opportunities to focus on particular students. In this research teachers often referred to students by name, and even when they were not teaching this student themselves, were interested in discussing that student in terms of their usual classroom interactions, their mathematical ability, and their usual approach to participating in learning activities. It is perhaps particular to this set within two post-primary schools that many of the participating teachers taught more than one subject and it occurred that teachers interacted with particular students or class groups in lessons other than mathematics. Other teachers, such as Dave and Stephen, were engaged in many sports activities and interacted with students outside of their own taught class groups.

In Crannóg, these conversations may have been particularly relevant where students were streamed after first year. In the third planning meeting in cycle 1, teachers discussed one particular student and his progress.

Fiona *James Donovan got 90% in his test, his volume test. Was he in your class last year? I thought he was.*

Dave *James Donovan, no.*

Walter *James, was the one who came up [from a lower streamed class]*

Fiona *Yeah, he came up from the little weak group. He got 90% in his test.*

Walter *Brilliant.*

Stephen *I think maybe he got a little shock.*

Fiona *I thought he was asleep, I thought he was asleep in the back of the room for the last two weeks. He was well chuffed.*

TKS may have been enhanced through discussions held by teachers in the planning and post-lesson meetings since different teachers had taught students at various stages in their schooling and lesson study provided teachers with opportunities to see students in different settings other than their own classroom.

In Doone's first post-lesson discussion, teachers identified a student who may have needed additional learning support. Because teachers had knowledge of the student from different subjects, they were able to confer and discuss his engagement in other classes. Lisa, Kate and Nora discussed a student who may have required individual learning support. Following this conferral, Kate followed up the matter to investigate providing learning support for the student.

Lisa *There's the guy who was colouring in the whole square at the start, look-it, in blue, Sam. Yeah, he'd be, he'd be weak enough as well.*

Aoibhinn *Yeah, 'cause he didn't do four by four 'cause I was just -*

Lisa *Oh, you saw him doing that?*

Aoibhinn *Well I went over and I said, "well, let's draw it out", and he didn't even do that then and I came back and he had it filled in, but he*

hadn't it drawn and then he was stuck on six by six and I said "well we can do the same thing here."

Lisa *Ok.*

Aoibhinn *I'd say he'd need a lot of individual attention.*

Kate *I have him for Science, I totally agree.*

This student may have been identified as requiring additional learning support following such shared teacher observations. In reflecting on the first research lesson, Nora (in her role as a learning support teacher) and Lisa discussed how a particular student, who had come to their attention prior to the research lesson, had engaged in the class.

Lisa *I was particularly happy with the little fellow there, Niall.*

Nora *Yeah, yeah. He's the boy that we were talking about yesterday. He was doing a lot of multiplying by two, but when we were doing the exercise, this one here, he went "oh no, no. I'm multiplying by two. I multiplied by two." So he was stopping himself doing this.*

Lisa *He was getting a benefit from the class.*

While I think it important that lesson study not be seen as a vehicle to analyse students' ability, it is noteworthy that observing students in different lessons provided teachers with new insight, perspective and knowledge of the students they were teaching. Lesson study provided teachers with unique chances to observe specific students in a new environment and develop new knowledge about that student. Observation of research lessons facilitated teachers in making professional judgements about a student due to having two teachers collaborating and reflecting on a particular student. Teachers were also able to use the opportunity to participate in research lessons to discuss particular students.

Throughout this research teachers regularly identified and commented on individual students in their post-lesson discussions. While reflecting on these individual students may have simply provided examples for teachers to demonstrate students' responses and activities within research lessons, there were occasions where focusing on individual students allowed teachers to build on their knowledge of this particular student and to

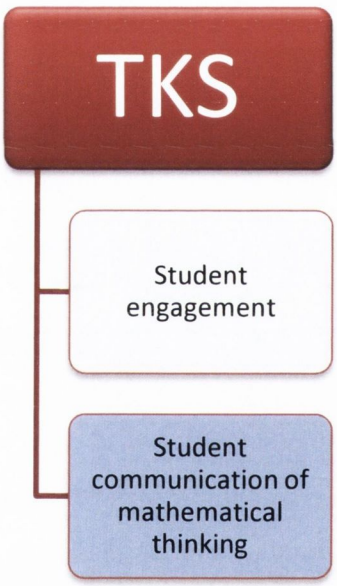
incorporate this knowledge in their further teaching interactions with that student. This identification of students within lesson study may be worthy of further research.

6.5.1.5 Conclusion of Student Engagement

A focus on individual students, who were familiar to other teachers in other subjects, also seemed to generate productive teacher collaboration. This contextualisation of teacher learning (Dudley, 2012) provided an opportunity for teachers to share observations on specific students and to deepen their teacher knowledge.

6.5.2 Student Communication of Mathematical Thinking

Within this research, teachers were provided with unique windows of opportunity to be present in another classroom other than their own, a practice that is not common within Irish post-primary schools, particularly in terms of professional development (Gilleece et al., 2009). Through observations of teaching students in groups, teachers built on their own knowledge of how to facilitate student group work, but also enhanced their social knowledge of how students interacted and performed with their classmates, building their knowledge of how to involve their particular students in communicating their mathematical thinking.



In the following section students' communication of their mathematical thinking will be analysed in terms of students working together in groups. An additional element of communicating mathematical thinking, that of changing classroom environment, emerged from the data and will be briefly noted in this section.

In a change from the traditional post-primary mathematics classroom (Lyons et al., 2003; Oldham, 2001), within this research students' communication of mathematics became more and more the norm within research lessons. In Doone, by lesson study cycle 2 teachers planned questions to ask students as a whole-class activity. In Crannóg, research lesson 1 involved students working together and communicating with one another for the majority of the lesson. In this section the development of teachers' knowledge in

encouraging and facilitating students' communication of their mathematics will be detailed. Access to student talk further informed teachers' incorporation of student communication and student group work within other lessons (Section 5.5.5.2).

6.5.2.1 Students Working Together in Groups

It emerged in this research that as teachers began constructing students' interactions with one another, they also became aware of and interested in how their students could best learn from these interactions. As suggested by Kamina and Tinto (2011) and Bruce and Ladky (2011), teachers also began to attend to how student groups should be constructed as their participation in lesson study develops.

6.5.2.2 Doone

By lesson study cycle 3, teachers in Doone had begun to naturally encourage their students to communicate their mathematics to one another. This was a development in teachers' approaches and they had gained knowledge from the previous two cycles of how students could be best motivated to communicate their mathematical thinking with one another and were also convinced of the merit of this approach.

Originally, teachers had focused on the actions of the conducting teacher and the teachers' orchestration in the first research lesson. They had not employed a student lens in planning the lesson. In that research lesson Lisa and her colleagues had included student pair work within the lesson plan where students would attempt a task together. Unfortunately, the activity of calculating the square of numbers from 1 to 10 did not require students to communicate or work with one another and within the research lesson students did not cooperate since each had their own worksheet (Figure 36) and Michael reflected on this in the post-lesson discussion. The inclusion of 'pair work' within the lesson and seemed to have been included as an aside rather than as a vehicle through which students could develop their mathematical thinking. Within the mock lesson Lisa did attempt to encourage students to communicate their mathematical thinking in explaining how they went about modelling squared numbers on graph paper, but this sharing of strategies was not included within the actual research lesson. There was also no reflection of how students worked together or spoke about their mathematics during the post-lesson discussion.

In research lesson 2, Owen, a teacher who had originally stated that he did not enjoy teaching students in groups, took the opportunity of the research lesson to attempt a new

approach to his teaching. Owen had originally thought about communication of mathematics from his own perspective as an authoritative figure within the classroom and not as a benefit of students' learning. This perceived risk to his teaching (Dudley, 2013) in attempting a very new and different approach, was facilitated through his participation in this lesson study community.

In teaching the second research lesson Owen realised that it was difficult to relinquish the traditional role as didactician in the classroom to one where students were doing most of the work. In their third planning meeting, following their discussion of including group work and differentiating questions in the previous meeting, Owen summarised his intention for the lesson. Owen, focusing on one particular student Jose in the class, articulated what he saw as a real difficulty in getting students to work in groups and his colleagues offered him constructive advice in encouraging students to work together.

Owen *I've decided myself that I'm going to put them in the groups to challenge their ability. To have more difficult questions for the two top groups because I think they will all be kept going, whereas I know there is a couple of them in the class that will just do everything. As well, I'm doing the group work at the moment, you have one or two of them...they're just, they don't include anyone. That's what annoys me about group work. They just have it done and they don't even talk about it.*

Lisa *Well they have to be taught group work.*

Owen *Well I'm trying to, everyone has a...a role to play but it's still, I have one or two of them – do you know Jose and he's just like, it's done.*

Kate *But you could, you should actually probably have a quick word with him and saw "you're obviously flying along, why not give the help to someone who needs it?" You know, that will really reinforce the material. I'd love to see you doing that... I can imagine he'd be like "arrrgh, he's asked me for help!"*

Owen *Yeah, that's what it was, that's what I'm working on myself at the moment is the group work.*

Kate gave Owen constructive and useful advice as to how he could improve students' engagement in his class and Lisa advised that students need to be taught group work (Johnson & Johnson, 1999). Conversations similar to this would not have occurred in this school had teachers not had the opportunity to sit together and discuss their teaching approaches through planning a lesson. Owen enhanced his TKS in realising how to approach introducing group work into his mathematics class but also in receiving advice on how to attend to particular students. In the post-lesson discussion Owen admitted that "*it's hard to let go*" of, as he saw it, control of the classroom but in later research lessons was happy to share advice with other teachers on facilitating group work and encouraging them to incorporate more student communication in research lessons 3 and 4. This was a major finding of this research that through this enhanced knowledge, Owen had dramatically changed his practice as he reported in his final interview.

Owen *Because at the start I would have never done group work and now I'm trying and trying, like even today I was like 'deep breaths Owen, deep breaths'. You know we can do this, we can do this! You know that they are learning and I just have to accept that.... I just think I'm more open-minded to the fact that they will learn. And the fact that I've seen it in practice and been able to be an observer, so standing in behind someone's class and say "this is fantastic" that helped big time. Standing in Lisa's room was great. Just to even see them even when they were off-task, to see what they were talking about and how can you get them back on task in a group work situation.*

Owen felt he had broadened his TKS through attempting to incorporate group work in his own class but also through participating in lesson study as an observer in other research lessons. Through participating in lesson study Owen changed his practice as a teacher and his knowledge of incorporating more student communication of mathematics.

Lisa reflected that students within the lesson had learned far more working in their groups than they would have from a traditional exposition class. This was a new aspect

of her teacher knowledge in considering how teaching in groups and how students conversing with one another may in fact have enhanced their experiences of mathematics (Boaler, 1999; Goos, 2004; Johnson & Johnson, 1999).

Participating in this second cycle of lesson study provided teachers with a new knowledge of how students, particular students they were familiar with, worked in groups with their peers. This experience provided new perspectives to teachers in thinking about how different students build their mathematical understanding.

In that second research lesson, teachers also decided to have students present their work to the class. This was a new practice for all of the teachers and they were very positive about how it worked. It was obvious that students had enjoyed presenting their work, that a number of students in the lesson had gained from watching their peers present, and also that students were communicating their mathematical thinking. Teachers repeated this activity in research lesson 3 where students presented their work after having worked on it in groups together. Teachers had decided in their reflection of research lesson 2 that having 5 students work together was too many and hence decided on groups of 4 students collaborating in research lesson 3, as is supported by the literature (Johnson & Johnson, 1999). In facilitating student presentation, Kate built on their experiences from research lesson 2 and discussed how to choose what groups of students to choose to present. From her knowledge of her own students, she categorised her class into separate groups and strategized about which students she would call on to present during the research lesson.

Kate *I think I'll pick a group who have a lovely, nice, clear graph that's done well and who, maybe I was just listening and they were kind of talking about it in a very logical way or something...I think I'll just pick them.*

Michael *Pick them on the day*

Kate *based on what's happening, yeah. But I won't pick the strongest group...because then everyone's like "Uch, she's only picking the guys"*

Michael *Yeah, yeah. They'd definitely be, yeah.*

Kate *So I think I'd pick from a medium group.*

Kate viewed her facilitation of the student groups from the point of view of the students within the class and anticipated how to facilitate the student groups prior to students presenting their work, an approach recommended by Stein et al (2008) in encouraging classroom discussion in mathematics. Kate planned to choose students to present within this research lesson according to the work they completed as a group and wanted to avoid choosing the 'good' students in order not to dis-incentivise other students in her class. This was another enhancement of TKS with teachers employing a student lens on their incorporation of group work within the lesson and in facilitating students' learning.

Incorporating students' presentation of their mathematical work became the norm within Doone's research lesson and was an important element of encouraging students' communication of their mathematical thinking as part of their learning (Forman, 1996; Forman, McCormick, & Donato, 1997).

It was hugely important that in the final research lesson that Michael, who was originally not an advocate of group work, changed his opinion of having students working together and incorporated it within his lesson. He and his colleagues had concluded from observations of other research lessons that it was of benefit to students' learning that they work together in communicating and thinking through their mathematics. In planning the lesson Michael originally negated the option of students working together but then, within the same conversation with persuasion from his colleagues and in utilising his critical student lens in imagining his own students during the research lesson he changed his mind.

Owen *Is it going to be in a group work setting?*

Michael *No, no. They're in, they're in the back, the 'U'. There's no point in changing just because we're. Well, I don't know. I don't mind. I mean I did go to groups so I could do it again. Sorry, sorry, it can be groups, of course it can be in groups.*

Lisa *I think it would be useful, even pairs, right?*

Kate *Even pairs.*

Lisa *I would. I would do it in pairs just to keep them interested and we saw the transition years who looking at it, if they were on their own, would be still looking at it.*

Michael *Yeah, no, I would prefer it.*

Kate *I'd say pairs.*

Owen *Pairs*

Lisa *Pairs.*

Michael *Yeah.*

From his observations of students working in groups in research lessons 2 and 3, his opinion of the benefit of student group work had changed and he was now more willing to incorporate group work into his own teaching practices.

Despite the fact that student group work had not been well-incorporated into the research lesson she had taught, Lisa had a very positive opinion of group work which strengthened throughout her participation in lesson study. In planning research lesson 2, Owen noted that he was consciously working on incorporating more group work in his teaching and was encouraged by Kate and Lisa in this endeavour. Lisa shared her reflections on how her students were now assisting each other in their learning since she had changed her desks for research lesson 1 (Section 6.5.2.4).

Lisa *Oh I love it – I would never go back.*

Throughout the reflections on research lessons 2, 3, and 4, teachers discussed the dynamics within the student groups and on how certain individuals had engaged with the cooperative learning. Employing a critical student lens, Nora reflected that group work was not necessarily always beneficial to students but she had noticed positive benefits during research lessons 2, 3, and 4. Reflections on students' communication of their mathematics benefitted TKS in providing teachers with additional knowledge and options when assigning students to groups in their other mathematics lessons.

It is interesting to note that in the first lesson study cycle in Doone, Lisa was not familiar with the students in this lesson as they had only been in the school for a few weeks. Lisa did not have a knowledge of these students: their prior knowledge, what they found interesting, how they interacted with one another. This lack of knowledge due to the timing of the lesson with this particular 1st year group of teachers may have been a factor in the failure of this lesson to cater to any clear learning objectives or to develop

students' mathematical thinking in a coherent way. However, during this lesson Kate did enhance her KCT in terms of facilitating students' presentation of their work (Ball et al., 2008; M. K. Stein et al., 2008).

6.5.2.3 Crannóg

While teachers in Crannóg had been teaching the Project Maths curriculum for over three years, they had not yet all put into practice the idea of students communicating their mathematics by working together in groups. In the first lesson study cycle teachers explicitly set out to design a lesson that would include students working in groups together. Experienced teachers wanted to improve how they employed group work in their classroom and although teachers did not incorporate this into their over-all aim for the year, it had been an important part of the discussion of teacher aims.

Fiona	*The [2nd year] class and myself both have to develop in improving group work techniques.*
Walter	*What questions to ask, what tasks to set them.*
Fiona	*Yeah.*
Walter	*How to sort of facilitate that best.*
Fiona	*Just to develop group teaching with group work -*
Stephen	*Yeah, I think we're all trying to get that.*
Dave	*It's a different classroom culture.*

These teachers had sensed that they did not know how to teach students in groups and wanted to change the culture of students working together in their classrooms.

Fiona and Stephen had suggested that students should present their work not to talk about getting the correct answer but to talk about how they went about working through the mathematics. While this did not occur in the final planned first research lesson, it was important that teachers had begun to realise that students should become more familiar with communicating their mathematical thinking during lessons. In reflecting on this lesson Fiona pointed out to her colleagues that giving students time to work with one another in groups was helping them in developing their mathematical thinking.

Fiona *This is what they need to do, to speak it and say it.*

Throughout the course of the year, teachers acknowledged a change in their classroom practice in engaging students more in mathematics conversations. Teachers encouraged students to talk about maths, particularly as a new activity with the senior cycle students who were encouraged to discuss problems while solving them in groups.

As discussed in Section 4.5.3.5, Dave conducted the first research lesson and had never taught in groups before. Prior to the lesson, he re-arranged the classroom so that students could sit in groups of four and work together for the duration of the class. Since Dave had never taught his students in groups before, going around to each of the groups during the lesson and asking each of them questions was a new experience for him and one that frustrated him in how time-consuming it was. He realised, however, that he usually based his lesson on the perspectives of the two strongest students in the class, Jack and Jane. Working in groups forced him to pace the lesson according to the majority of the students and he noted that in previously only focusing on Jack and Jane, he had not been thinking about the pace of the lesson from the perspective of any other student. Teaching students in group brought Dave to a realisation of the importance of employing a critical student lens in thinking about more than the top two within the class. Dave was also surprised that students he usually considered "*weak*" had fared much better in the mathematics activity than had been expected.

Dave *The middle group had the thing built straight away before anyone else had even worked out that it was colours, and there was two girls at the table that would be among the weakest in the class....she came so confident and came out with some good stuff*

Dave enhanced his TKS in understanding that collaboratively working on a mathematics problem benefitted students' learning and also ensured that the top students, Jack and Jane, were not the only ones dictating the pace of his teaching. Dave also considered that in allowing students to work at their own pace in their groups, students' mathematical understanding was more visible to him as a teacher.

From that same research lesson in Crannóg, other teachers observed the benefit to students of participating in group work. Instead of showing the students the answer as they would have traditionally taught it, the students were encouraged to explore solutions

independent of the teacher. Walter, Judy and Stephen reflected on how they would have traditionally taught this lesson and considered how beneficial an approach group work was for the particular groups of students they had observed during the lesson.

Walter *But all the learning was so much better in all the other groups. Otherwise, I have been tempted if I had been teaching it in the first way "ok we have y, we have got x squared"*

Dave *And then move on...*

Judy *But it was, it was interesting, in terms of, I know that I would rush them along. I do group work once in a while and I would always be "Oh my God, the level of noise in this room, come on!" and I would rush them along - let's get them to the merry end. So it was actually interesting to just sit back, observe and watch them get there themselves. Ok, it took them that bit longer, but they learnt it for themselves. There wasn't the intervention where I would be coming over going "Is it this, is it that, is it the other thing?" trying to push them along and get them there quicker.*

Fiona *And the noise was genuine working noise. You know they are chatting away, they were all stuck in.*

From observing the lesson teachers saw group work from the students' perspective which made them think about their usual practices in rushing students and telling them answers before students had time to process the question for themselves. This activity and reflection enhanced teachers' TKS of incorporating student group work in their own teaching.

Teachers also noted how specific students interacted and engaged with one another in communicating their mathematical thinking as noted by Stephen in the post-lesson discussion.

Stephen *That is actually funny because it was Conn was the one who actually came up with most of it but David was trying to take control but didn't really know what he was doing, but he was in*

control. That was very interesting. Even at one point Conn, I think he said "I think it is 48 by 48, I am probably wrong though" to which David ignored and said "Oh my previous..." (Laughing). It was actually interesting, the dynamic but the lack of confidence he had where David didn't really know what he was doing was funny, you know, he was backing himself completely. So it is interesting...

Gerald *What was interesting was they spent quite a while, before you were saying about, you know, you thought you should have got more done. They needed time at the start to sort of work around what they were actually doing and they were talking about counting the numbers around the edges and they were doing all sorts of things.*

A number of teachers re-taught this first research lesson in their own classes with differing measures of success. Eileen reported back that in teaching this lesson, she did not re-arrange students into groups but instead students worked as normal in their individual desks. The lesson did not work since one student immediately spotted the answer and announced it to the class. It would suggest that teachers were beginning to take note that teaching certain topics in a particular way benefitted by students working together in groups.

Similar to Doone, teachers wondered about how students should be assigned to groups. Walter asked if students should be put into groups for a whole term, if the group should be mixed ability, the size of the group etc. Teachers came to decisions among themselves about how they thought groups of students should be best structured and that students should also be provided with specific training in such approaches as cooperative learning (Johnson & Johnson, 1999) or in creating classroom communities (Goos, 2004; Watkins, 2005).

In research lesson 2, it had been planned that students would work with one another in attempting to factorise a quadratic expression using algebra tiles but this did not occur in the lesson. The lesson was taught to a small group of approximately 10 students and Fiona reported to her colleagues that some of these students had mild learning difficulties and preferred working on their own. In re-teaching the second research lesson, Walter

planned to continue with the series of lessons re-enforcing students' learning by getting them to "*explain it to each other.*"

Eileen *And a couple of them had made the connection that the 2 by 4 makes 8, and 3 by 4 makes 12, and a few of them had seen that...*

Walter *And I got Debbie to explain that to the rest of the class as well!*

This exemplified a change in Walter's teaching throughout his participation in lesson study from one where he was wary of how to build students' communicating their mathematical thinking within his lessons to one where he actively encouraged and promoted students talking through their mathematics with one another. Walter acknowledged this change in his own teaching in his final interview when asked if he is thinking about how students are thinking:

Walter *Well I suppose it is all the different approaches with Project Maths, the fact that you are not just, it isn't so much teaching from the front and students working away, the fact that I have them in groups, I am going around engaging them, asking them to talk about stuff and there is more, they are talking about the maths more.*

In holding research lessons for differing year groups of students, teachers realised that there were differences in how junior students reacted to group work as opposed to senior students. Following their last research lesson, teachers noted that their senior students were more shy and hesitant in working together but were also more wary of putting forward a suggestion in case it was wrong.

Stephen *Yeah there is a strange dynamic where it is very different for the second years. When we had it with the second years and even with you there, they were more willing to actually just to say something, to have a go...*

Dave *I had an interesting class [observation], Irene...*she sat there for ages and eventually she sat forward and said "I'm not comfortable with this".

Stephen had noted the cultural difference between senior and junior cycle students in that senior students, used to a traditional approach to teaching, just wanted to be shown mathematical strategies to learn it by rote while junior students were more open to approaching mathematics as a social activity. Teachers collectively resolved to work on the issue of senior students working together in groups.

Stephen also noted that there was one student, Rebecca, who was trying to hide in her group since she was not sure of what she was doing and hence had nothing to offer her group. Although this issue was not resolved, Stephen had noticed and reflected on this event which may have influenced his own facilitation of senior student groups in his class. Other teachers had had similar observations from previous research lessons noting the group dynamics and individual student's interactions within different groups. Teachers had enhanced their TKS by establishing that students enjoyed working together on a problem and recognised the various dynamics of these situations.

Due to teachers' participation in lesson study where, as a community, they collectively focused on incorporating more student group work and communication within their classroom, teachers felt that they were encouraging more student communication of their mathematical thinking in all of their teaching practices. Eileen reflected on this in her final interview and commented on her re-assurance that since they were approaching this as a community, it gave her the impetus to continue with the approach.

> *Eileen* *I would say I am a bit more willing to put them in pairs, in groups now just to see how it goes, just let them at it and even, I suppose, it was factorising this week - it's not saying much and seeing what they come up with themselves. I would say maybe this time last year I would have been like "do this."*

Participating in lesson study also increased Gerald's TKS in providing him with opportunities to observe other teachers employing students working in groups and communicating with one another in research lessons. In observing these teaching approaches, Gerald felt more confident in introducing these approaches in his own teaching.

> *Gerald* *from sitting in a classroom observing how groups work it is easier to know what to expect, you know, when you go to do it yourself, but*

you observe how students learn...I suppose in terms of planning future lessons, it does help to maybe get sort of the timing right, the timing I mean of, if it's a lesson discovery, when to give out information, maybe also how to maybe get the kids interested... So I think in terms of timing of the lesson and in terms of expectations of feedback and that sort of thing...what I saw in those lessons is where the teacher couldn't see all groups at once, therefore there's a necessity to move around groups fairly frequently and that's a hard skill, do you know. Not to get bogged down in one group.

It is interesting to note that despite attending professional development courses supplied by the curriculum development team for three years and despite the support provided to the school by the PMDT, Gerald had not felt confident in changing his teaching practices until he participated in the planning of and observation of incorporating more student communication within his classroom. Enhancement of teacher knowledge in this manner demonstrates the power of participation in iterative cycles of lesson study, as part of a community of colleagues situated within the same school.

In addition to these changes to classroom practice, both Walter and Eileen self-reported in their final interviews that they had begun to extend their 'wait time' (Brown et al., 2008) in class for students to give a verbal answer to a question. Through enhancing their TKS, teachers were aware that students required time in developing their communication skills and were encouraging more talk within their classrooms. This is a considerable and important change in teachers' practice due to their participation in iterative cycles of lesson study.

6.5.2.4 Classroom Environment

How teachers organise and manage their classrooms impacts on students' academic growth and social development (Solomon et al., 1993) and situative theorists have supported this view that the physical and social contexts of where learning take place are an integral part of that learning (M. Brown et al., 2008; Lave & Wenger, 1991; Putnam & Borko, 2000). As referenced in Section 2.3.2, teachers' own classrooms are powerful contexts for their learning (Ball & Cohen, 1996; Putnam & Borko, 2000) and in this research classrooms benefitted from teachers' participating in lesson study cycles.

In teaching students in groups in research lesson 1, Lisa realised that her classroom of "*slanty desks*" was not conducive to students collaborating with one another. Lisa swopped desks with Owen to desks where students were able to work together. She considered this to be of great benefit to her own teaching but also to students' learning. Owen, who at this stage in the research was still adverse to teaching students in groups, asked her about the change of classroom layout during their planning of research lesson 2, over one month after Lisa has made the changed.

Owen	*Are you happy with yours?*
Lisa	*Oh I love them. Even Ralph [another teacher] came into my room the other day, he thought it was a great layout.*
Kate	*Oh, you still have that layout?*
Lisa	*Oh I have, yeah. They were lined in fours and sixes, it worked brilliantly.*
Aoibhinn	*And are the students are enjoying it?*
Lisa	*Third years are just unbelievable...Because they can all see each other's work, [they say to each other] "you're not doing that right." I don't have to go as far as the group.*
Owen	*That's great.*
Lisa	*Oh I love it....I would never go back.*

Lisa reflected that due to the research lessons they planned as a community of teachers, she had adapted her classroom environment to better suit students as learners in groups. Her knowledge of the classroom environment was altered due to thinking about learning from a different student perspective. Employing a critical student lens spurred Lisa on in thinking about teaching and learning approaches which, in turn, led to a change in her classroom environment.

This employ of a critical student lens on her classroom environment led to Lisa changing the layout of the classroom to better facilitate her students' learning. Lisa in her final interview also commented on other teachers in Doone who changed their classroom

environment to be more conducive to students communicating their mathematics with one another.

Lisa *I think we've all looked at the way our classrooms are set up to facilitate cooperative learning. All of our classrooms were the traditional single look-at-the-board and everybody has used cooperative learning practices as a result of the lessons that we designed. Very little of our work was done for the kid to do on their own. So from that aspect it would have changed in the classroom.*

Throughout the research lessons in both schools, the environment of the classroom impacted how the lesson was carried out. During the planning of research lesson 2 in Doone, teachers realised that the room the class was held in would greatly affect the lesson. Some rooms only had old-fashioned desks (similar to those Lisa had had in her classroom) and were not conducive to students working together. Teachers planned the lesson for a particular day when Owen taught in a different room that had flat desks which could be moved around for group work. Also in Crannóg, teachers commented on the classroom environment for their first research lesson

Dave *It was a good classroom to do it.*

Fiona *It is...it is so bright, like my room is so dark and dull and dreary.*

The classroom was spacious and bright and was conducive to moving the desks into groups as opposed to single desks. Dave noted that had he had his own classroom, he would have kept this layout of desks for the rest of the academic year.

Michael had also changed the layout of his desks during the year and while originally he had arranged his desks in a horse-shoe shape for the benefit of his own moving around a classroom, he had thought about changing this layout following his observations of students working in small groups. He reflected on his point in his second interview.

Michael *I don't know if it's because of this, maybe it's because of this I don't know, but I have been doing more group things in the last few months than I had been doing. Which is good. I have thought about going to the group desk situation, but I don't know if I will.*

Aoibhinn *Oh okay. But you're thinking about it.*

Michael *Well I'm definitely going to change the layout of my room after the midterm.*

The teaching environment impacted on the approach that teachers introduced within their mathematics lessons. While the majority of teachers involved in this research had their own base room where they could control the layout, a number of the junior teachers in both schools moved around classrooms and thus found it difficult to plan approaches that would often benefit students due to teaching in different classrooms throughout the week.

6.5.3 Conclusion of TKS in Lesson Study

Through teachers' participating in lesson study, they enhanced their knowledge of students: how they worked together, how they engaged with learning, and how this could be improved upon and encouraged. In participating in lesson study, teachers built on their knowledge of students through various aspects of their teaching of mathematics but also in their personal knowledge of students sitting in their classrooms. Teachers found out that student engagement could be enhanced through keeping students' active and autonomous in their learning and, in employing a critical student lens, saw that engagement and enthusiasm within a lesson did not mean that students were 'off task' but rather that it could enhance their experience of mathematics. Teachers utilised their knowledge of students in differentiating certain tasks for particular students but also built on this knowledge through the shared observation and reflections of other teachers who interpreted these students' actions in a research lesson.

In teaching students in groups, teachers enhanced both their knowledge of how to conduct a lesson where students engaged in group work through conduction and observation of research lessons, but also came to examine how different students participated in groups. During their reflection conversations, teachers came to realise the benefit of students communicating their mathematical thinking with one another. Teachers also discussed the construction of these groups and became aware that the members of a student group could impact on how students participated and engaged with the mathematical content and their peers. Through participating in lesson study and through more often employing a critical student lens in constructing and reflecting on lessons, teachers built on their existing knowledge of the students they were teaching.

In Doone there was a marked increase in the communication of students' mathematical thinking in progressive research lessons and there was an obvious increase in the amount

of work that students did with their peers during lessons. In Crannóg, teachers encouraged more whole class discussions in research lessons and while there was not an explicit focus on designing how students should work together, teachers felt that they had learned a lot about how to introduce more student group work in their own classes from their involvement in the planning and observation of research lessons.

Throughout the research in both schools it was important to teachers that students were engaged in the mathematics activities and questions that were included within the research lessons. In both schools teachers learned the value of student engagement in a lesson and teachers learned how to encourage student engagement through developing contextualised and relevant material to their particular students. This was a valuable element of TKS enhanced through teachers' participation in lesson study.

6.5.4 Summary of Findings & Conclusion: Enhancing Teacher Knowledge through Iterative Cycles of Lesson Study

Teachers must know about the subject they teach and must know about the various ways in which they can teach it, but teachers also engage their knowledge of their students in deploying their knowledge of how to teach this subject. While many have argued that there is a need to map teachers' PCK (Ball et al., 2008; Baumert et al., 2010; Rowland et al., 2005) and others have argued that lesson study enhances teachers' knowledge (Lewis et al., 2006; Meyer & Wilkerson, 2011; Murata et al., 2012; Tepylo & Moss, 2011), this research provides detail into how specific features of teachers' knowledge, KCS and KCT, is enhanced through their engagement with in-school lesson study in which teachers of varying experiences participated.

Through participating in iterative cycles of lesson study with their colleagues, the data generated in this research highlighted the features of KCS employed by teachers in their planning and reflection of research lessons and those features of knowledge were improved as teachers continued their participation in lesson study. These three features of KCS (identifying students' prior knowledge, anticipating students' mathematical responses, and noticing students' mathematical thinking) emerged from the data and were supported by the literature. The analysis within this research also revealed that understanding of KCT can be further differentiated since the features of sequencing learning trajectories, developing contextualised content, and doing mathematics questions, emerged from the data generated in this research. As teachers in both Doone

and Crannóg continued their participation in lesson study, these elements of KCS and KCT were developed in different ways for individual teachers depending on their prior teaching experience and their perspectives on their teaching (Jacobs et al., 2010). Teachers improved in their planning of research lessons as their participation in lesson study continued as these highlighted features of KCS and KCT developed and were incorporated within the research lessons. This detailing of these elements of PCK is a contribution to literature on lesson study but also a contribution to literature on teacher knowledge, particularly within this post-primary setting.

As originally highlighted by Fernandez et al. (2003), incorporating a critical student lens on one's practice is an important element in lesson study. This is further corroborated within this research but in addition, the analysis of this data suggests that as teachers participate in iterative cycles of lesson study, they further develop and incorporate the employ of a critical student lens within their practice. As highlighted in Chapters 4 and 5, teachers collectively began to incorporate more of a focus on students' mathematical thinking (Jacobs et al., 2010; van Es & Sherin, 2008) and students' engagement and communication as their participation in lesson study continued. The majority of teachers also self-reported that due to their participation in lesson study, they were focusing more on their own students' thinking within their teaching. It is an important contribution of this research to incorporate the employ of a student lens as an element of teacher pedagogical knowledge and knowledge of students.

Finally, the analysis of the data generated in this research suggests that teacher knowledge of students (TKS) should be acknowledged and incorporated within a model of mathematical knowledge for teaching. Within the data, teachers constantly referred to their individual students and incorporated their social and cultural knowledge of their students within their teaching. Within the confines of this thesis, two features of TKS are suggested within this research as student engagement and communication of mathematical thinking. As teachers continued to participate in lesson study they incorporated more ways of engaging students in their own mathematical learning and the focus on students communicating their mathematical thinking grew within planned research lessons. While I do not suggest that these features are exhaustive, I do think it important that teachers' social and cultural knowledge of the students within their classrooms is acknowledged and included within a framework of teacher knowledge and this modification of Ball et al.'s (2008) framework is a contribution to knowledge.

Chapter Seven

7 Findings, Contributions, Recommendations & Conclusion

7.1 Introduction

Within this chapter, findings from the literature, data, and analysis in light of the research questions will be reviewed. Recommendations to practice and policy in light of this research will also be outlined. The contributions of this research to the literature on lesson study and teacher knowledge will also be discussed.

7.2 Chapter Content

This chapter will extend the commentary and conclusions offered in Chapters 4, 5, and 6 by first presenting the collective findings in parallel with the contributions of the research under the headings of the research questions. The findings of this research will be mapped to the relevant literature as a means of validating and articulating how this research contributes to our understanding of teacher learning as a community within lesson study and of the mathematical knowledge for teaching that is employed and developed through participating in iterative cycles of lesson study.

7.3 Restatement of the Research Questions

While the research questions were addressed and answered during the discussion of data within the data analysis chapters, this chapter will consolidate and extend the commentary and conclusions of these chapters around these questions.

The findings, contributions, and recommendations of this research are based on one main research question, containing two sub-questions, as introduced in Chapter 1.

1) How is teacher knowledge, particularly pedagogical content knowledge, enhanced through iterative cycles of lesson study?
    a. How are KCS and KCT utilised and enhanced within lesson study?
    b. Are other forms of teacher knowledge, external to subject matter knowledge, utilised within lesson study?

Prior to discussing the specific findings and contributions, it is necessary to restate the value of these research questions within this particular educational setting.

This research focused on teacher learning in community and was grounded in teachers' view of themselves as active participants in their ongoing professional development. The overarching goal of this intervention was to help teachers develop their mathematical knowledge for teaching through their participation in iterative cycles of lesson study with their colleagues. This professional development objective emerged from literature on curriculum reform (Cohen & Barnes, 1993; Hopkins & Reynolds, 2001; Putnam & Borko, 2000; Remillard, 2000; Remillard & Bryans, 2004) which is relevant to the introduction of Project Maths currently underway in the Irish post-primary system (Cosgrove et al., 2012; Jeffes et al., 2012). As part of this research, teachers were provided with opportunities to develop new perspectives on their teaching (Corcoran, 2011b) in employing a critical student lens on their practice (Fernandez et al., 2003; Lewis et al., 2009).

In choosing a model of professional development, lesson study met the features of teacher community (Grossman et al., 2001; Stoll et al., 2006) and provided teachers with a structure to investigate their own practices, based within the classroom (Christou et al., 2004; Dudley, 2013; Putnam & Borko, 2000). In order to promote teacher learning and to build on teachers' already developed knowledge and experiences, this project provided teachers with structured opportunities to build and reflect on their own teaching and learning (Sherin & Han, 2004b), espeicially in referencing the pedagogical approaches encouraged by the Project Maths curriculum (Lynch, 2011; Oldham, 2005). Through planning research lessons as a community, teachers shared their experiences of teaching particular topics, collaborated in creating and modifying mathematical activities, and designed sequences of mathematical learning for particular classes of students in their own schools. Teachers also became aware of the importance of identifying and incorporating students' prior knowledge within mathematics lessons and of anticipating students' mathematical responses prior to teaching. Through observing research lessons, teachers became more aware of how particular students engaged with mathematical activities and of how students extended their mathematical thinking. Through reflecting on research lessons as a community, teachers became aware of different perspectives of mathematical thinking, realised different mathematical strategies, began noticing students' mathematical thinking during lessons, and developed their knowledge of their own students.

This research took place in two post-primary schools and involved mathematics teachers (of varying teaching experiences) within two Mathematics Departments. 12 teachers participated in the research which took place over the course of one academic year. Qualitative data was generated through recording all teacher meetings around lesson study and through interviewing teachers at various stages throughout the research. A qualitative approach was selected in order to authentically assess how teachers would engage in lesson study as a model of professional development external to any research and to minimise any extraneous pressures on their engagement in such professional development (Dudley, 2013; Murata et al., 2012). Teachers' learning was framed by a model of mathematical knowledge for teaching as suggested by Ball et al. (2008).

7.4 Limitations of the Research

In presenting the findings, contributions, recommendations, and conclusions, it is necessary to recognise the limitations of this research.

One limitation of this project was the sample population of the study. Firstly, only two schools were involved in this research and while phase 1 and 2 of the Project Maths curriculum reform were represented, these two Mathematics Departments cannot be stated as representative of all Mathematics Departments that may exist around the country. In addition, these schools were both situated within the Dublin area and represented schools of large populations (greater than 500 students). Further research may be necessary to investigate how smaller or more rural schools with smaller populations of mathematics teachers might be impacted through participating in iterative cycles of lesson study.

This research began at different stages of developing teacher community (Grossman et al., 2001) for both Mathematics Departments. In Crannóg, teachers were familiar with collaborating with one another on teaching and learning at the commencement of the research. As a school involved in phase 1 of the curricular reform, the Mathematics Department had established practices of professional collaboration and were likely at an evolving/mature stage of teacher community (Section 2.6.3). This may have contributed to how teachers began incorporating more features of KCS and KCT within their initial lesson study practices when compared to the initial lesson study cycle in Doone. In Doone, teachers were unfamiliar with working with one another on a formal basis and there were no established structures of practices of the Mathematics Department when

the research began. These differing stages of development of teacher community likely impacted on how teachers engaged with lesson study and are thus acknowledged here as a limitation of the research.

Another limitation of the research is that teachers were not summatively assessed on how their knowledge had changed due to their participation in this research. One reason for this lack of summative data has already been stated within the methodology (Section 3.10) since it may have led to the participating teachers feeling increased pressure as part of their participation. Furthermore, instruments assessing post-primary mathematics' teachers' subject matter knowledge or pedagogical content knowledge have not yet been designed (Howell, 2014). The evidence for teacher learning was therefore based within the qualitative data of audio recorded teacher meetings, not tracking individual teacher learning but rather analysing their collective learning as a community of teachers engaging in lesson study (Dudley, 2013; C. Fernandez et al., 2003; P. Grossman et al., 2001; Murata et al., 2012) and including their self-reporting of their learning in teacher interviews (Wallace & Priestley, 2011).

7.5 Findings and Contributions

While findings and commentary were included within the data analysis Chapters 4, 5, and 6, this section will bring together the findings and contributions of the three data analysis chapters and will situate the findings within the literature.

There are three main contributions within this research:

1. Features of KCS and KCT utilised and enhanced through teachers' participation in lesson study are identified and mapped to a framework of mathematical knowledge for teaching.
2. Teachers' knowledge of their students (TKS) is identified as an additional feature of mathematical knowledge for teaching.
3. Participation in lesson study encouraged teachers to employ a critical student lens on their practice in viewing their teaching and learning practices from the perspective of a student (Fernandez et al., 2003), which further developed teachers' knowledge. This occurred despite teachers having prior experience or knowledge of lesson study and occurred without a 'knowledgeable other' being included with the lesson study model.

7.5.1 Lesson Study and Pedagogical Content Knowledge

How is teacher knowledge, particularly pedagogical content knowledge, enhanced through iterative cycles of lesson study?

The research question of this study was not interested in *if* teachers learn through participation in lesson study since this was already addressed by previous lesson study research projects (Corcoran, 2008; Dudley, 2013; C. Fernandez et al., 2003; Lewis et al., 2009; Murata et al., 2012). Literature has also shown that teachers' pedagogical content knowledge is enhanced through participation in lesson study (Meyer & Wilkerson, 2011; Tepylo & Moss, 2011). This research thus focused on *how* teachers' PCK was enhanced in detailing how features of KCS and KCT, proposed in this research, were employed and developed during teachers' participation in iterative cycles of lesson study as a community. It is a major contribution to the literature on lesson study in detailing and fine-graining these feature components of KCS and KCT that were utilised and enhanced through teachers' participation in lesson study. Furthermore, the mapping of this knowledge to Ball et al.'s (2008) framework of mathematical knowledge in teachers' planning and reflection conversations is an additional contribution of this research.

In this research, there were discernible differences between schools in how teachers employed their PCK as part of their lesson study planning and reflection conversations and how the features of PCK developed in both communities. In Crannóg, there was an evolving/mature teacher professional community (Grossman et al., 2001) where teachers were familiar with collaborating with one another around teaching. In this community, teachers incorporated far more features of KCS, such as identifying and incorporating students' prior knowledge and anticipating students' mathematical responses, in their initial lesson study cycle and immediately began sequencing students' learning trajectories (KCT) during their engagement with lesson study. In contrast, teachers in Doone were collaborating as a beginning teacher professional community (Grossman et al., 2001) and the first two lesson study cycles incorporated teachers learning about negotiating and communicating within that community. Interestingly, the incorporation of features of PCK from their initial to their final lesson study cycle was more apparent in Doone since they were including features such as identifying students' prior knowledge for the first time rather than building on how this feature was incorporated within student learning as occurred in Crannóg. Relating to the second research question

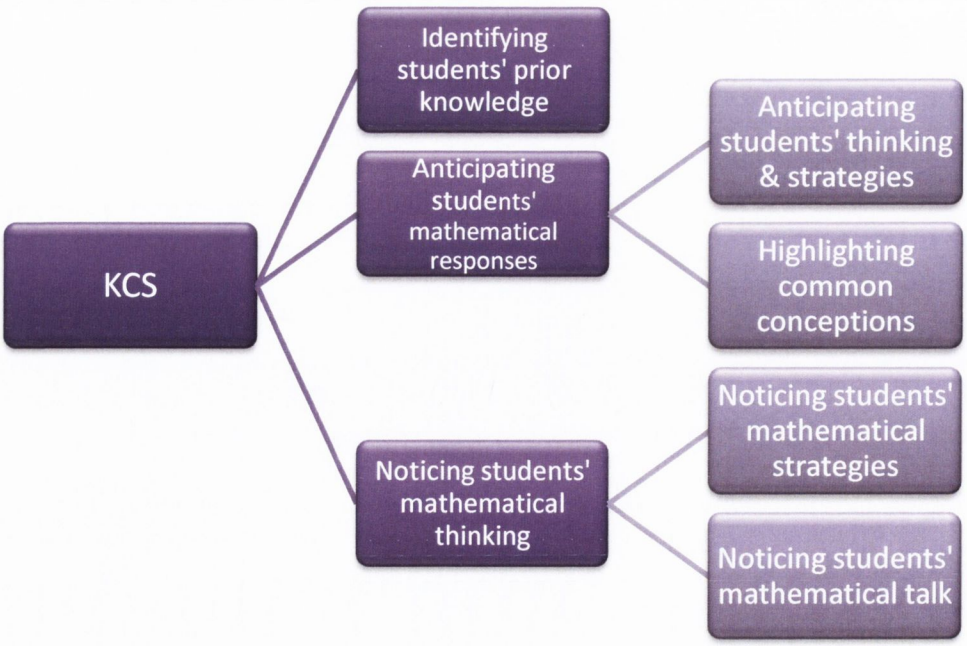
around evolution of community (Section 3.3), a shallow analysis of the data generated in this research would suggest that as the teachers in both schools participated in iterative cycles of lesson study, the communities within which they were working developed. Further analysis will be required to investigate this point and will likely make further contributions to the literature on lesson study.

Within the data generated in this research it is clear that teachers employed and enhanced specific features of their KCS and KCT as a community due to their participation in lesson study. While Hill et al. (2008) provided detail on what teachers' KCS encompassed, and while Tepylo and Moss (2011) referenced this framework in their research, no further mapping of this knowledge framework to lesson study was provided in the literature. In this research detail of KCS and KCT that teachers used and enhanced through their participation in iterative cycles of lesson study was developed.

7.5.1.1 Detailing the Features of KCS

The features of KCS which emerged from the data are summarised as:

- Identifying students' prior knowledge,
- Anticipating students' mathematical responses,
- Noticing students' mathematical thinking.



As teachers continued to participate in lesson study, these features became more prominent in planning and reflection conversations and in teacher interviews. As

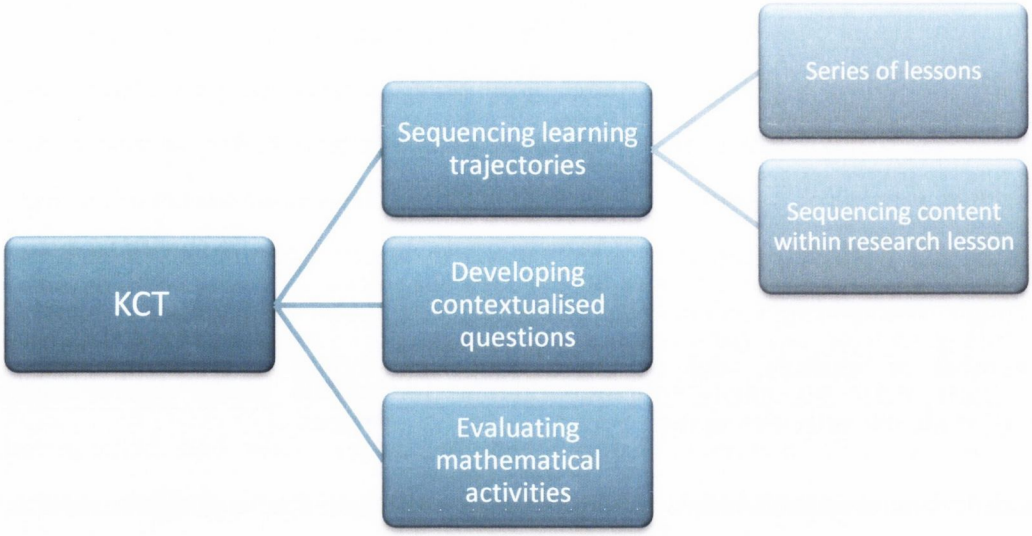
teachers' participation in lesson study continued, they began to identify and incorporate more of students' prior knowledge within their planning. Anticipating students' responses during planning occurred more and more often as teachers' participation in lesson study continued and teachers were more aware of questions or difficulties students may have within a research lesson. Anticipating students' responses during planning research lessons also positively impacted on how teachers sequenced and planned lessons (KCT). Through anticipating students' mathematical responses in planning research lessons a number of teachers, such as Owen and Dave, began to incorporate this anticipation of students' activities in their general teaching practices.

Noticing students' mathematical thinking within research lessons had a positive impact on teachers' KCS since teachers observed how students engaged in particular mathematical activities, interpreted how students thought about the mathematics, and reflected on how students' mathematical thinking could be improved upon in future lessons. This noticing of students' mathematical thinking encouraged teachers to view the teaching and learning of mathematics from their own students' perspectives and impacted on how they approached teaching particular topics. In their final interviews the majority of teachers in both schools reported on how they had learned from viewing students' engage in mathematics within the research lessons and in their interviews reported that this perspective had impacted on their teaching practices outside of lesson study. While teachers' beliefs were not analysed within this research, these findings would suggest that their beliefs were impacted as a result of the communities' participation in lesson study (Lewis et al., 2009; Tepylo and Moss, 2011).

7.5.1.2 Detailing the Features of KCT

The features of KCT which emerged from the data are summarised as:

- Sequencing learning trajectories,
- Designing contextualised questions,
- Evaluating mathematical activities



An important finding of this research was that in building students' understanding of a topic, teachers found it far more beneficial to plan a series of lessons in detail within which the research lesson was situated. In this regard, teachers built on their own ideas of how a topic could be approached but were also forced to be detailed in their planning of content and in viewing the content from the perspective of their students. Teachers were also creative in their sequencing ideas due to the collaboration of colleagues of different experiences and qualifications. Anticipating students' mathematical responses (part of teachers' KCS) also impacted how students' activities were designed and in what order activities should be presented.

This element of KCT has been encouraged as an activity by Lewis & Hurd (2011) and by the Japanese teachers in Fernandez et al.'s (2003) study but has not been otherwise noted within the literature. It is thus a contribution to knowledge of lesson study and our understanding of teacher knowledge that sequencing students' learning trajectories over a series of lessons is an important element of teacher learning within lesson study. As this finding has not been referenced previously within the literature it may only be relevant to post-primary teaching of mathematics. It may also be a finding particular to Irish schools since all of the teachers participating in lesson study in this research taught various different year groups at different levels of assessment grading which may have contributed to how teachers imagined that lessons could be sequenced for students and may require further investigation.

Designing questions that were both relevant to the content and relevant to the students' learning needs benefitted teachers' knowledge of content and knowledge of how to best develop that content from the perspective of a student. Engaging in constructing these contextualised mathematics scenarios built on teachers' knowledge of approaching new topics and in engaging in the curriculum content. This knowledge also overlapped with TKS in recognising and acknowledging that teachers' familiarity with their students' social and cultural backgrounds and their students' interests were beneficial to them in constructing questions and activities that would be interesting and engaging for their students.

In evaluating mathematical activities teachers enhanced their KCT, as was suggested by literature on teaching mathematics (Mason, 1988, 1998; M. K. Stein et al., 2008). Through doing questions during the planning of research lessons, teachers built on their knowledge of how to develop students' mathematical understanding of a topic but also, overlapping with KCS and TKS, developed an understanding of how a student might engage with that question. Evaluating mathematical activities during planning allowed teachers to anticipate what strategies students might use in attempting to answer a question.

Through participating in iterative cycles of lesson study teachers began incorporating these features of KCT more and more in their planning of research lessons. Both teacher communities developed series of lessons incorporating a research lesson and became more aware of sequencing activities within a research lesson. Teachers began to build confidence and knowledge in designing contextualised questions. Doing mathematics questions prior to the research lesson began to be incorporated as an important part of the lesson study cycle. While this research does not claim that all participating teachers learnt equal amounts in the same features of PCK (Jacobs et al., 2010), the data does suggest that all teachers, regardless of their years of teaching experience, benefitted in terms of their KCS and KCT from participating in these school-based iterative cycles of lesson study.

7.5.2 Adding to the Framework of Teacher Knowledge: Empirical evidence of linking content to particular students

7.5.2.1 Teacher Knowledge of Students

It is a major contribution to theory of this research that in addition to SMK and PCK, a new dimension of mathematical knowledge for teaching has been identified from the data as relevant to these Irish teachers' participation in lesson study. As discussed in Chapters 4 and 6 this dimension of knowledge, that of teachers' knowledge of students (TKS), was hypothesised as one which teachers employed throughout their planning and reflection discussions within lesson study and which also impacted on how teachers designed, conducted, and reflected on research lessons.

TKS, as hypothesised within the proposed framework (Chapter 4) and demonstrated within the data (Chapter 6), impacted on how teachers taught mathematics to their own students and had an impact on the approaches and strategies teachers employed in attempting to develop their students' relational understanding of the mathematics at hand. This form of knowledge relates to what Shulman (1986) and Grossman (1995) would have referred to as knowledge of student. This research thus contributes to the literature in extending Ball et al.'s (2008) framework of teacher knowledge by suggesting that there exists a third dimension of mathematical knowledge for teaching that can be added SMK and PCK (Figure 58):

1. Subject Matter Knowledge
2. Pedagogical Content Knowledge
3. Teacher Knowledge of Students

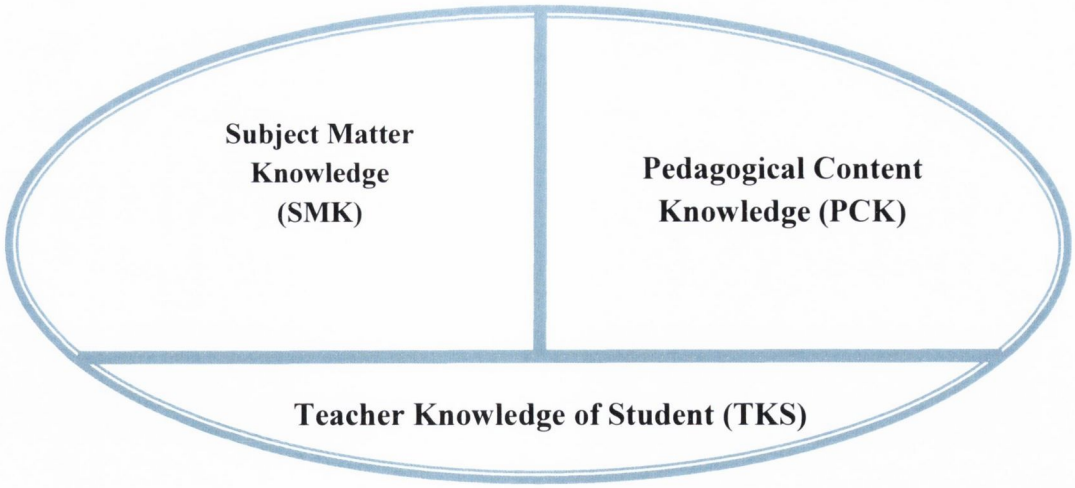
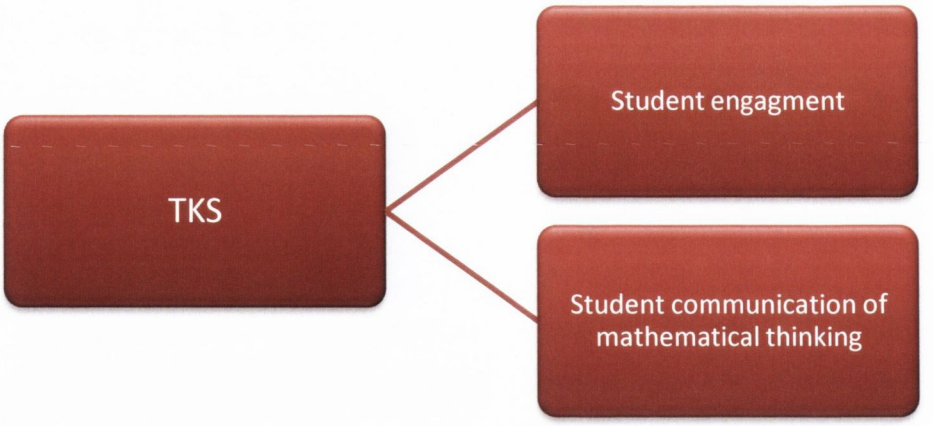


Figure 57 Contribution to Theory: an adapted model of MKT

As is suggested by Ball et al. (2008) and Rowland (2005), this model of teacher knowledge is not disjoint but rather each dimension of this knowledge overlaps and intersects with the others and no one section of this knowledge is, in isolation, sufficient in teaching mathematics in a way that is meaningful for students.

In this research, teachers' knowledge of students was broken into two subsections: student engagement and encouraging students to communicate their mathematical thinking. Both of these elements of the knowledge are explicitly highlighted within the Project Maths curriculum (National Council for Curriculum and Assessment, 2012a, 2012b) and are also identified within the literature that students should be provided with relevant context to engage them in their learning (Blackhouse et al., 1992; M. Brown et al., 2008; Galbraith, 2013) and should communicate their thinking in order to develop their understanding (Boaler, 1998, 1999; Goos, 2004).

As teachers' participation in lesson study continued and by incorporating their knowledge of their own students, teachers were motivated to engage their learners in particular content and benefitted from developing relevant, contextualised stories, activities, and questions that would encourage students to become more involved in the mathematics within the lesson. This was especially relevant in the final research lessons in both schools where in Doone teachers included a context building story around Pythagoras' theorem which they anticipated students would enjoy and in Crannóg where students began initially generating their own data and then working on real-life data based on the Olympic games in order to develop an understanding of rates of change and limits.



Through participating in lesson study, teachers enhanced their TKS in incorporating more communication of students' mathematical thinking within the classroom. Experienced teachers such as Gerald, who were initially reticent about employing group work in the classroom, benefitted from observing how group work could be executed within the classroom and learned the strategies that could be introduced into his own classroom. Walter and Eileen both explicitly encouraged more communication within their classrooms and self-reported that they were extending the time they waited for a student to verbally answer questions. In Doone, Lisa, Michael and Owen had changed their classroom environments to encourage students to participate more in group-work and to encourage more communication of mathematics between students. During this research Owen had dramatically changed his outlook on student group-work from one that was difficult and unnecessary to one that he thought was advantageous to students' learning which he made specific effort to include within his teaching practices. Similar changes to other teachers' perspectives on teaching students in groups were detailed in Chapter 6.

I do not wish to suggest that the detail and framework of TKS suggested within this research is exhaustive nor extensive, and it is certain that further research on how TKS is employed and enhanced through teachers' practices is necessary. However, it is a contribution to the literature in suggesting that this third dimension of teacher knowledge exists in the totality of knowledge necessary for the teaching of mathematics and employed within these teachers' participation in lesson study.

7.5.2.2 Critical Student Lens

The data generated in this research supports the research literature in suggesting that through participating in lesson study, teachers were encouraged to employ a critical student lens on their teaching which impacted on their learning (Corcoran, 2011; Fernandez et al., 2003). Within this data, not all teachers initially routinely noticed students' mathematical thinking nor did they view teaching and learning from the perspective of a student. These were skills that were enhanced as teachers participated in iterative and continuous cycles of lesson study as a community over the course of one academic year.

Within the data it is evident that teachers began to incorporate a critical student lens in all of the features of KCS and KCT outlined within this thesis as their participation in lesson study continued. While in research lessons 1 and 2 teachers in Doone were developing research lessons from the viewpoint of the teacher, in planning and reflecting on their final two lessons teachers utilised a critical student lens in developing mathematical content. This was represented by a change in percentage of codes of critical student lens of the total qualitative data from 0.46% to 1.2% in Doone. In Crannóg, where experienced teachers began planning their first research lesson utilising a critical student lens, the less experienced teachers had begun to articulate a critical student lens in their planning and reflection conversations in cycles 2 and 3 and the occurrence of the critical student lens throughout the lesson study cycles was maintained as a percentage of the total qualitative data at an average of 1.7%.

Employing a critical student lens encouraged teachers to more frequently anticipate students' mathematical responses and notice students' mathematical thinking within live research lessons (Carpenter et al., 1989; Jacobs et al., 2010) and allowed teachers to attend to, interpret, and reflect on how students within the class responded to and thought through mathematics ideas. Teachers included these observations and unique perspectives of student learning into planning future research lessons, but also incorporated it in their own individual teaching practices as exemplified by Eileen's and Lisa's admissions to thinking more about how their students were thinking within her own teaching.

A further theoretical contribution of this research has been to align literature on professional noticing of students' mathematical thinking (Carpenter et al., 1989; Jacobs

et al., 2010; van Es & Sherin, 2008) with that of developing new perspectives on teaching such as employing a critical student lens in lesson study (Corcoran, 2011b; Fernandez et al., 2003) and to include this within an adapted framework of mathematical knowledge of teaching (Ball et al., 2008). As explored in Chapters 4, 5, and 6, the data generated in this research suggests that when teachers viewed teaching and learning from the perspectives of students in the classroom, it deepened and strengthened their PCK and TKS.

While employing a critical student lens may not have impacted on teachers' SMK, in this research developing and utilising a critical student lens benefitted teachers in enhancing their awareness of their practices and in building their PCK and TKS. In contributing to the research literature I suggest that incorporating students' perspectives of mathematical thinking is an important element of mathematical knowledge for teaching, supported by lesson study, which strengthens how teachers can approach and introduce mathematics to their students.

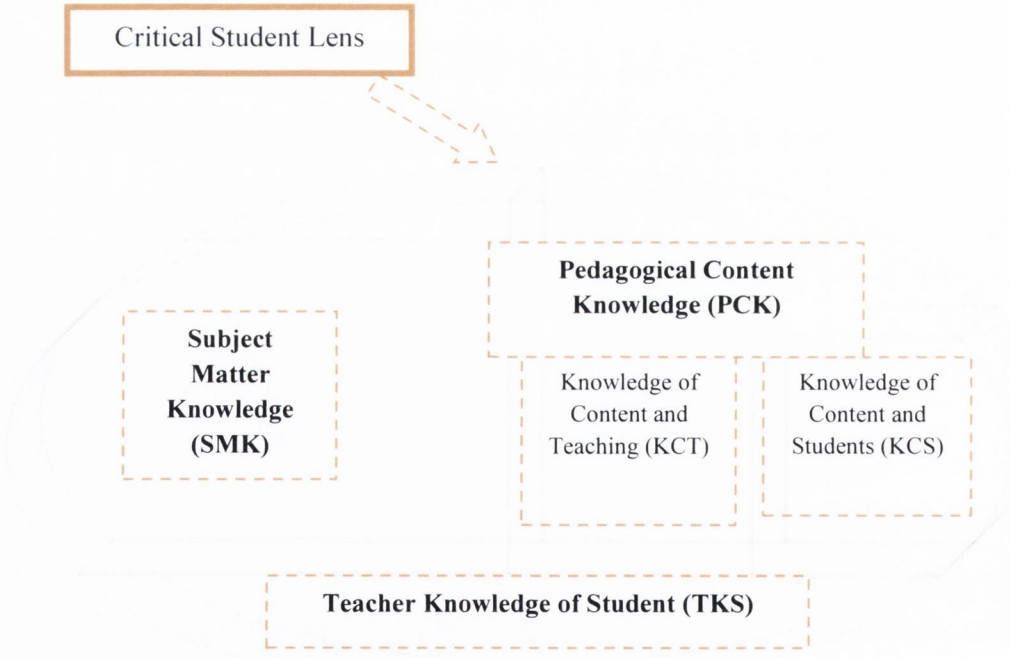


Figure 58 Framework of MKT: critical student lens providing depth to knowledge

This finding is an important contribution to theory and an expansion of knowledge of how teachers employ their own knowledge in their practices. Further research on how a critical student lens can be developed in pre-service mathematics teachers or newly

qualified mathematics teachers outside of participating in lesson study may also contribute further to research literature on teacher learning.

7.5.2.3 Who Benefitted from Participating in Lesson Study?

This research analysed teacher knowledge and learning as a community and through participating in lesson study, different teachers gained knowledge in different ways. Did every teacher benefit from their participation? Nora, the most experienced teacher (35 years) involved in the research, expressed in her final interview that did not feel she had gained in terms of content knowledge but had developed a familiarity with the new curriculum and had developed new ideas on teaching topics. Fiona, the most experienced teacher participating in Crannóg (31 years), also reported in her final interview that while she did not feel she had gained in terms of content knowledge, she had begun to think more about mathematics from students' perspectives and had enjoyed coming up with new ideas to teaching topics. In terms of the newly qualified teachers in both schools, Eileen and Owen both reported that they were thinking more about mathematics from their students' perspective and both had made changes to their classroom practices engaging students in more communication of their mathematical thinking and in structuring activities that gradually build on students' understanding. While Stephen, Walter and Lisa felt they had gained from observing students within research lessons and anticipating more how students would engage in activities in their own class, Dave, Gerald, and Michael felt they had learned about teaching approaches in terms of teaching class groups of students. Kate reported that she had gained a familiarity and confidence in teaching the new curriculum but it is apparent from the data generated within the teacher meetings that Kate had also developed her KCT in sequencing lessons in a way that required less exposition and more input from her students. The most obvious change in teachers' confidence was in incorporating more communication of mathematics in their classrooms, particularly in Doone were all teachers changed the layout of their classrooms to encourage students to communicate their understanding during lessons.

From the data, each of the teachers participating in this research benefitted in increasing their pedagogical content knowledge. Further research is required in specifically analysing the learning of one or more teachers as a function of their participation in the lesson study community, however, teachers themselves acknowledged that they had learned from their experiences of lesson study, albeit in different ways.

7.5.2.4 What were the Mechanisms within Lesson Study through which Teachers' Knowledge was enhanced?

As was explored within the analysis of the data within the framework of the lesson study cycle, teachers' knowledge was enhanced in three separate ways:

1. The conversations with their colleagues around the planning and reflection of research lessons as a community,
2. The observation of students within research lessons,
3. Teachers' internal, personal reflections on their own teaching as a result of these conversations and observations.

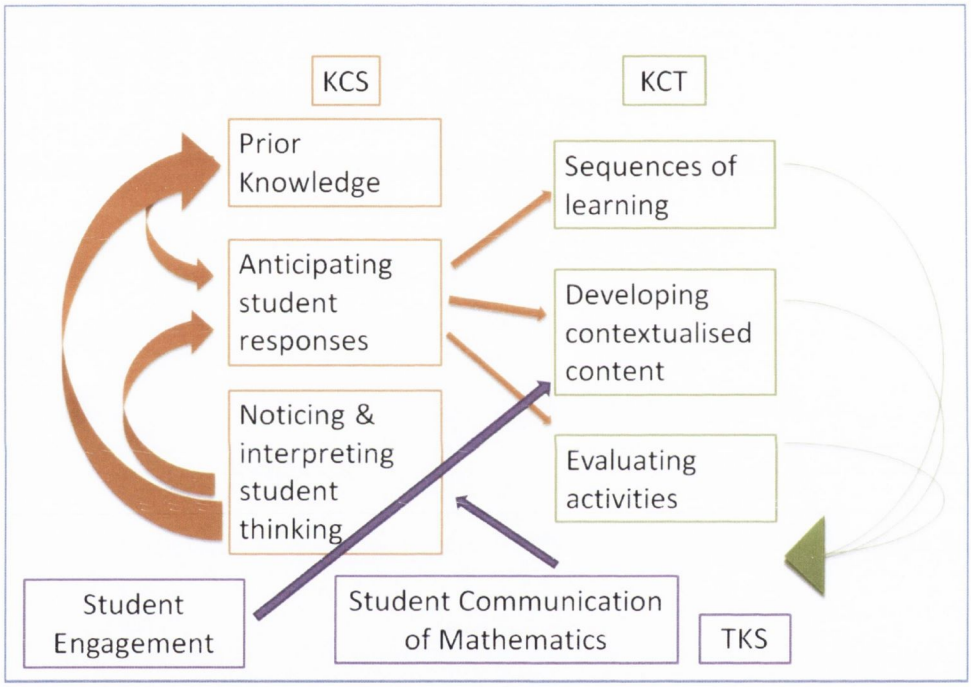


Figure 59 Suggested model of teacher learning in iterative cycles of lesson study

Through these mechanisms, this research brought together three separate literatures of teacher community, professional noticing of students' mathematical thinking, and lesson study. Through participating in more than one cycle of lesson study with their particular group of colleagues, teachers had the opportunity to build on their knowledge which impacted on their participation in subsequent lesson study cycles. As demonstrated in the chronological analysis of the qualitative data in Chapters 5 and 6, certain teachers such as Lisa, Eileen, Walter and Owen changed in their interactions, observations, and ideas as the lesson study cycles continued.

In teachers' conversations with their colleagues, discussions were often peppered with teachers' experiences of teaching similar topics or similar class groupings. In sharing their experiences and ideas, teachers enhanced each other's awareness of the different strategies and practices each had in teaching mathematics. In observing students' engagement within a research lesson teachers gained unique access into watching students' attempt to answer mathematics questions where the observing teachers were aware of the lesson objective and had anticipated how those students would act, a unique element of the lesson study model of professional development (Murata, 2011). In collectively reflecting on their observations of the research lessons teachers shared what they had noticed or what had seemed important with their colleagues, implying a level of personal reflection. Through their individual interviews teachers also suggested a further level of personal reflection where they had brought practices of lesson study into their teaching, such as Eileen's anticipation of students' activities in her own classroom and Owen incorporating more whole-class discussions and student group work in his classes. From their observations, Walter, Eileen and Stephen reported that they had consciously extended the time they waited for students to verbally respond in their own classrooms – a point Kate also noted and admired of Michael when he conducted research lesson 4 in Doone.

To borrow a phrase used by Meyer & Wilkerson (2011), lesson study provided teachers with 'windows of opportunity' to collectively plan the content of lessons, to observe students engaging in mathematics in a unique and informed way, and to reflect on their own teaching and learning practices in a way that impacted on their teaching.

7.6 Impact on Curriculum Reform

The premise of this research was to support teachers in implementing a new curriculum where teacher learning would support the enactment of Project Maths within the Irish post-primary classroom. While curriculum reform was not an explicit focus of the research, it was important to reflect on some curricular issues in this concluding chapter. This research did not look at the large scale dimensions of Project Maths, but rather took a look at the micro level of curriculum implementation where teachers discussed their practices and investigated the learning approaches encouraged within the Project Maths syllabus.

While teachers' engagement with the new Project Maths curriculum was not an explicit focus of this research, the data would suggest that participation in lesson study enhanced teachers' confidence in engaging in the new curriculum and in employing the teaching and learning approaches encouraged by the curriculum. As teachers participated in lesson study, they referenced the syllabus in deciding what area of the curriculum to focus on and to clarify what was included within the syllabus in terms of students' learning outcomes. All but one participating teacher agreed in their final interview that they had engaged more with the Project Maths curriculum due to their participation in lesson study. The only person to disagree was Fiona who felt that her knowledge of the curriculum was already strong when she began participating in the research and hence she did not feel there had been any additional change.

The majority of teachers self-reported that they had more confidence in teaching the new curriculum and all teachers participating in the research felt that they had more awareness of and had engaged more in the teaching and learning approaches encouraged by Project Maths such as: encouraging more communication within the classroom, cultivating problem solving approaches, and employing more student group work within their teaching. These points will require more investigation but it may be worth noting for policy makers in introducing curriculum reform that providing teachers with collaborative, school based, models of professional development, such as lesson study, strengthens their engagement with curriculum and increase their confidence in teaching that curriculum, one of the most important elements of ensuring successful curriculum reform (M. Fetters et al., 2002).

As stated in Section 2.2, within the Project Maths curriculum it is an explicit aim that teachers identify links within the curriculum as opposed to teaching separate, localised and isolated topics. In this research teachers identified curriculum links during lesson study cycles in topics that they would traditionally have taught in isolation. It is the case that teachers were aware that this was an explicit objective of the curriculum, but had they not participated in this research it is unlikely that they would have discussed or constructed these links within the curriculum in sequencing learning trajectories as part of research lessons.

While there was not an explicit focus in this research on teachers' implementation of teaching approaches encouraged by the Project Maths curriculum, a cursory analysis

would suggest that teachers focused more on developing students' relational understanding, encouraged more problem solving approaches within the classroom (Prusak et al., 2013; Schoenfeld, 1992, 1994) and developed students' skill in communicating their mathematical thinking (National Council for Curriculum and Assessment, 2012a, 2012b). When this research commenced teachers in Crannóg were already utilising textbooks only as a question resource, while teachers in Doone were depending on textbooks as a curriculum source as is common in post-primary teaching (Cosgrove et al., 2012). At the end of the research Lisa and Michael commented that, as a Mathematics Department, teachers in Doone were planning not to use a textbook as a basis of their teaching in the following academic year. Further analysis will be required in definitively suggesting this point but it is likely that participating in lesson study encouraged teachers to be less dependent on textbooks as curriculum resources and increased their confidence in creating their own teaching and learning resources.

7.7 Recommendations for Policy and Practice

The analysis and discussion of the data in Chapters 5 and 6 show that collaboration as a teacher community in lesson study is of benefit to mathematics teachers in a post-primary setting. As an outcome of this research, a number of recommendations for educational policy and practice are suggested under the following headings:

1. Professional Development
2. School-based Mathematics Departments
3. Duration of Mathematics Lessons
4. Curriculum Reform and Curriculum Materials
5. Research Literature in Lesson Study

7.7.1 Professional Development

Professional development is not compulsory nor is it incentivised for post-primary teachers in Ireland (Gilleece et al., 2009; OECD, 2009). In this research teachers were willing to give of their free time and resources to participate in research in which they hoped they would learn. Teachers participating in this research often expressed the wish for new forms of professional development to be provided, both external and school-based, but noted that such participation should also be recognised. It is a recommendation of this research that teachers be incentivised and encouraged to

participate in a wide range of professional development opportunities, including lesson study, which should be made available to them, as a compulsory component of a teachers' career.

School based professional development is not the norm in Ireland (Gilleece et al., 2009) but in-school professional development that is based in teacher community is beneficial to teachers in terms of their learning and in terms of changing teachers' practices. Within this research teachers unequivocally agreed that participating in lesson study had increased their awareness of teaching approaches, had changed how they engaged with their colleagues in talking about the content of their teaching, and would recommend this form of professional development to their colleagues. Many of the teachers suggested that this form of professional development should be incorporated within any contractual hours[4] they were obliged to participate in. Based on this research I recommend that teachers be provided with opportunities of in-school learning, based within teacher communities, which provide structure in reflecting on and observing the teaching and learning of mathematics.

Within this research, many of the teachers stated that they resented participating in professional development workshops where they had to miss teaching their students for a full school day, as was suggested by the literature in Section 2.5 (Guskey, 2002; Sugrue, 2006). Within this research, teachers gave much of their own free time in order to participate in lesson study planning and reflection meetings, and observing or conducting research lessons. Other teachers within the school were supportive of their colleagues participating in this research in supervising lessons as a favour to their colleagues. From these occurrences I conclude that there is a willingness for teachers to participate in both mathematics education research and professional development. I believe however, that this professional development should take forms other than in-service training (Gilleece et al., 2009) and that forms of in-school professional development should also be made available to teachers. In stating this, it is also important that there remain an element of choice and professional autonomy for teachers in participating in school-based professional development, something which the teachers in this research themselves stated. Furthermore any assessment or rating of teachers in their participation in such

[4] Teachers specifically referenced the 33 'Croke Park' hours they were obliged to participate in during the academic year 2012/2013 and hoped lesson study could be included within these hours in future.

professional development would have to be carefully orchestrated and designed in conjunction with teachers (Hopkins & Reynolds, 2001; Putnam & Borko, 2000).

Following the cessation of data generation in this research, teachers in Crannóg began to participate in lesson study again during the academic year 2013/2014 but this was not the case in Doone. These may have been due to the various differing school structures mentioned in Section 3.7, however, teachers in Doone had noted that there was no culture of planning within their school and while they were happy to give of their free time for this research, some suggested that this should be acknowledged or incentivised to ensure that all teachers had the option to participate and learn, not just the same enthusiastic teachers who drove all initiatives within the school. Based on the limited data within this research, I would recommend that teacher communities and teacher collaboration be explicitly encouraged by school management and school structures within the Irish post-primary school system (Fullan, 2003).

7.7.2 School Based Mathematics Department

As mentioned in Chapter 3, in the event of a positive assessment of this model as a form of professional development enhancing teacher knowledge, the research would conclude on the necessary structural framework and seminal ingredients around which a sustainable teacher community can develop within an Irish post-primary school subject department. This research has contributed to literature in demonstrating that school-based lesson study can be successfully introduced to post-primary schools in Ireland. Mathematics Departments offer a ready-made structure within which teacher community can be developed as exemplified by the strong community in Crannóg. While the teachers participating in the research benefitted in Doone, there was a disjoint between teachers who taught junior and senior cycles. I would therefore recommend that within each school, a Mathematics Department be formally structured including all mathematics teachers within the school. It is not necessary that all teachers participate in all events and activities of the department, but that people are aware of the existence of the department and of their membership in a designated community of teachers of common practices and resources (Lave & Wenger, 1991; Wenger, 1998; Wenger et al., 2002).

Contrasting the two schools participating in this research, it was beneficial having a definitive role of 'Head of Department' as exemplified by Walter who organised all of the teacher meetings, cover when necessary, and facilitated meetings outside of the

lesson study meetings. Walter also maintained detailed records of all of the activities, assessments, and resources of the Mathematics Department. In Doone, teachers collaborated in the organisational aspects of arranging teacher meetings throughout the research but this may have been more efficient if one person had the role of 'Head of Department'. Furthermore, when this research began the resources and records of the Mathematics Department in Doone were not stored in one particular location. It is hence a recommendation that a Head of Department role be established in schools as a formally recognised and incentivised position with additional administrative and CPD responsibilities.

While teachers are restricted by resources within their schools, it was beneficial to the introduction and development of the model of lesson study that teachers had a designated meeting room and an area where all of the mathematics resources were stored. It was also beneficial in Crannóg that the majority of mathematics lessons were held in one area where teachers could easily find one another and discuss issues. Based on the generation of data in this research I would recommend that mathematics lessons be held in one area if possible, where mathematics resources are stored and available to all teachers, as is suggested by the subject inspection reports by the DES (Department of Education and Skills, 2009, 2014c).

It was of benefit to the teachers within the Mathematics Department in Crannóg that they had two consecutive periods within the week where all mathematics teachers were not teaching. This time was used by teachers for Mathematics Department meetings and, during this research, lesson study meetings. In contrast, teachers in Doone found it difficult to find a common time within the school day where they could meet and collaborate. From the data generated in this research I would recommend that, where possible, teachers be scheduled with at least one common period a week which can be utilised by the Mathematics Department as a time for collaboration or professional development.

The more experienced teachers, Fiona and Nora, could almost be seen as the captains of the ship within these two lesson study communities. Both of these teachers, with their developed employ of a critical student lens, impacted on their colleagues in how they also began to view their teaching from students' perspectives. They were important players within the community in that they were able to (unconsciously) guide their

colleagues in developing their knowledge. Nonetheless, both of these teachers also felt they had gained from participating in lesson study in becoming more aware of approaching teaching through student group work and in encouraging more communication within the classroom. For this reason I would recommend that for any in-school model of lesson study, it is important that all members of the Mathematics Department would be encouraged to participate with their colleagues in the endeavour since there is something different to be gained by all teachers within the community. Furthermore, many retired mathematics teachers could be included within lesson study cycles as invited external experts in order to share their knowledge and experiences with their colleagues.

7.7.3 Duration of Mathematics Lessons

In both schools teachers participating in this research lamented the fact that classes were so short (typically 35 to 40 minutes). As part of this research the most successful lessons in teachers' opinions were those which occurred during double-class periods, such as the third and fourth series of lessons in Doone or the final series of lessons in Crannóg. As part of this research, and reflecting the duration of lessons in other countries (Stigler & Hiebert, 1999), I recommend that mathematics lessons be lengthened to one hour in order for teachers to more successfully incorporate teaching and learning approaches which encourage students' communication of their mathematical thinking and problem solving, such as those encouraged by the Project Maths curriculum.

7.7.4 Curriculum Reform and Curriculum Materials

Most reforms that rely on teachers' voluntary contributions do not last (Hargreaves, 2003). While the teachers participating in this research gave freely of their time, they were acknowledged for their contributions by the funding providing to the schools by the NCCA. In introducing curriculum reform, I recommend that policy makers incorporate school-based professional development, situated in teacher community, in order to strengthen teachers' engagement in the reform and that teachers should be acknowledged and recognised for this participation.

Ball et al. (1996) advocate the development and use of curriculum materials which support teachers in focusing on and interpreting students' responses and to anticipate what learners may think when meeting particular mathematical activities. However, while the Project Maths teaching and learning plans available to teachers include

references to 'Student Activities: Possible and Expected Responses' teachers participating in this research did not always see the merit of these statements. During teacher meetings in both schools teachers also referred to the Teaching & Learning plans supplied by the PMDT (2011) as being too long and too detailed to be workable within the classroom. It is therefore important that as part of curriculum reform, teachers are explicitly encouraged to modify the materials supplied to them or construct their own lesson plans around any material supplied to them (Hanley, 2010). This is corroborated by the findings in this research which suggest that it is important to incorporate teachers' knowledge of their own students in the developing and contextualising of mathematics questions and of lesson plans.

7.7.5 Research Literature in Lesson Study

As referenced in Section 5.5.4.4, it was beneficial to teachers in developing ideas for their lessons and in doing mathematics questions out that teachers had access to and referenced research literature as part of their planning process. Reading research literature supported teachers in anticipating what difficulties or conceptions students might have of a particular topic (Tepylo & Moss, 2011). As part of this research, it is thus a recommendation that reading research literature become a part of the lesson study planning process but also that post-primary teachers are encouraged to regularly access and reference research literature in their practices. In continuing this point it may be beneficial to post-primary schools in Ireland to establish collaborative links with universities in order to access such research literature.

This research should perhaps have included more time in training teachers to observe for, interpret and reflect on these specific activities in order to prepare teachers to professionally notice students' mathematical thinking, to attend to their mathematical communication and strategies. Teachers may have benefitted from incorporating a video-study analysis of teaching in order to identify the features they saw as important within mathematics lessons prior to participating in any lesson study cycles (Sherin & Han, 2004b; van Es & Sherin, 2008). Future investigation of lesson study as professional development could incorporate these exercises where teachers discuss particular vignettes as in Borko et al. (2008) or Sherin & Han (2004), in order to earlier encourage teachers to notice students' mathematical thinking within live research lessons.

7.8 Conclusion of Findings, Contributions, & Recommendations

Lesson study encompasses many of the most important facets of teacher professional development: collaboration, noticing of students' thinking, and reflection. Through introducing lesson study into two different schools at a time of major curriculum reform, this research has established how teachers develop their pedagogical content knowledge through participating in iterative cycles of lesson study as part of a teacher community in an Irish post-primary context.

This research has contributed to theory in identifying and detailing how teachers' PCK (KCS and KCT) is developed through participating in iterative cycles of lesson study within a teacher community and by mapping this knowledge to a framework of mathematical knowledge for teaching as suggested by Ball et al. (2008). The research has also contributed to theory in identifying another dimension of teacher, that of teacher knowledge of student, which has been neglected in recent literature on mathematical knowledge for teaching. Furthermore, this research contributes to the literature on lesson study in a particularly Irish context where no previous research has been situated within post-primary schools involving mathematics teachers.

I entered this research as a teacher of mathematics and as someone who wanted to support mathematics teachers in the backdrop of a major curriculum reform. In embarking on this research I have learned much as an educational researcher but also as a mathematics teacher. As a facilitator of lesson study within this research, I gained much from participating in conversations with other teachers and in sourcing literature that teachers could reference within their meetings. As a mathematics teacher and pedagogy lecturer, it was incredibly motivating and interesting to see how teachers arrived at and built on new ideas for their teaching and how teachers' separate experiences could inform one another's practices. As a researcher I have learned much about working with teachers in order to generate qualitative data and about analysing such data.

Lesson study, while seemingly simple to describe, holds much power in how it can engage teachers in investigating their own practices and in providing an environment within which mathematics teachers can be creative in their teaching. It also provides teachers with unique opportunities to access how students respond to and act within mathematics lessons.

This is a model of professional development which I believe holds great potential as a vehicle for continuous professional development for teachers and as an environment within which teachers can enjoy, learn, and reflect on the importance of the noble profession of teaching.

Chapter Eight

8 Bibliography

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