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Enhancing the Teaching and Learning of Mathematics

in the
Senior Cycle

Two Volumes

Volume II

David Flannery

A Thesis Submitted for the degree of
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Part II

Chapter 6

Topics

A major proposal of this work, which has already been elaborated upon in detail in the Introduction, relates to the notion of a Topic and its envisaged central role as part of the teaching of a new syllabus. In that section it was maintained, in relation to the instructional sequence:

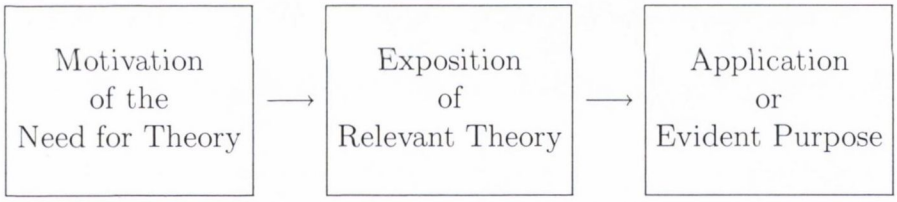

Figure 6.1: Instructional sequence guiding the design of a Topic.

that the predominant practice at the moment is an almost total (technical) concentration on the middle element, with little or no attention being given to the other two vital components.[1]

By contrast, it was explained that "A Topic is seen as the expansion of an existing syllabus entry, or a newly proposed one, to such an extent that the

[1]See page 31.

reason for teaching it is indisputably evident."[2] According to the Canadian mathematician Ed Barbeau:[3]

> Rather than fragment mathematics, it may be more productive to take an integrated approach, in which students are encouraged to dwell on a mathematical topic long enough to sense how it is put together and what its proper context is.

It was also indicated in the Introduction that "the meaning, elaboration and practical illustration of this notion constitutes an essential element of this thesis, and dictates much of the mathematical content and discussion appearing in its core section."[4] Furthermore, it was promised that the nature of a Topic would be made clear by comparing "the current treatment of one segment of the present syllabus ... with a suggested more substantial treatment which devotes more time to initial motivation, requires no more technical knowledge than the present syllabus and which justifies itself by both solving the practical problems raised initially and the showing of fresh applications."[5]

The term 'exposition of relevant theory', as it features in the second stage of the above instructional sequence guiding the design of a Topic, simply means the description/development of the mathematical theory needed to solve the general class of problems presented and (in PISA03 terminology) 'horizontally mathematised' in the motivational stage. This three-stage instructional sequence puts the all-important 'why' of mathematics before

[2]See pages 31-32.
[3]Ed Barbeau, *Polynomials*, (USA: Springer-Verlag, 1989), vii.
[4]See page 32.
[5]See page 32.

its 'how', whereas, according to Conway and Sloane, the traditional model (with its origins in the New Math approach) is one where "the sequence of instruction often proceeds from a generalisation to specific examples, and to applications in context."[6]

In this latter approach a 'ready-made' body of mathematical theory is first presented as the means of solving a category of problems of both a technical and applied nature. The task for the teacher is then to guide and assist the student in assimilating this body of received knowledge through the repeated solving of carefully chosen examples most of which, in the context of current Irish provision, tend to be of the technical kind resulting in a very impoverished treatment of the more authentic real-world applications. When this practice becomes the accepted norm there is a danger, as has already been indicated, that the 'how' of mathematical teaching dominates, almost to the point of obscurity, the equally, if not more, important 'why' of mathematics instruction.

For example, in relation to the topic of Quadratics, the four current texts (examined previously) adopt this traditional approach. They begin *ex abrupto* by simply telling the reader (teacher and/or student) what a quadratic equation is, and then they reveal how such equations are solved using the celebrated quadratic formula. This is then illustrated by a number of purely technical examples which are followed by a raft of exercises of the

[6]Paul F. Conway, & Finbarr C. Sloane, *International Trends in Post-Primary Education*, (Dublin: NCCA, 2005), 142.

same nature. If there is a problem of an applied type it is almost always connected with area (quadrature).

On the other hand, as will be demonstrated presently in relation to quadratics, the instructional sequence displayed above equates to a policy whereby quadratic polynomials (equations) are first revealed in context. It is only when the student is acquainted with this family of problems (having the same underlying mathematical structure) and the need of a general theory for their solution, that he or she is introduced to the relevant powerful theory. In terms of the PISA03 'mathematisation cycle' this motivational stage involves a crucial part of the problem-solving process – the formulation or horizontal mathematising stage whereby the real-world problem gets translated horizontally into the mathematical problem. From this perspective the first $\longrightarrow$ in the sequence may be seen as the horizontal mathematisation aspect whereas the second $\longrightarrow$ incorporates the element of 'vertical mathematisation', which is mathematics acting on itself through the application of general theory to specific intra-mathematical problems. In this viewpoint, the horizontal element of backward interpretation of the mathematical result in the real-world situation also falls under the umbrella of this second arrow.

With regard to the terms 'problem space' and 'solution space' as used by some writers, one might consider the term 'problem space' as broadly corresponding to the motivational stage with the term 'solution space' referring to the second phase in the instructional sequence – the setting out of the relevant theory by which problems residing in the problem space are eventually

resolved.

It is not the intention of this writer to elevate the instructonal sequence underpinning a Topic to that of a new pedagogical model of teaching, nor is there any need to, since as will be pointed out momentarily, it is but one individual's visualisation of a concept which resides in a number of existing models of effective teaching. In fact, the connection that can be made between this instruction sequence and the PISA03 mathematisation cycle has already been indicated.

The Conway and Sloane quote cited a little earlier continues: "Mathematics in Context reverses this sequence; mathematics originates from real problems."[7] Since the rationale for this reversal is an essential feature of the instructional sequence underpinning the notion of a Topic, the pedagogical stance adopted in this work could be viewed has having an affinity with that of Mathematics in Context (MiC). In that:[8]

> ... *Mathematics in Context* is really a combination of three things: the NCTM *Curriculum and Evaluation Standards*, the research base of a problem-oriented approach to the teaching of mathematics, and the Dutch realistic mathematics education approach.

the philosophy underlying this work can be squarely seen has having close ties with both RME and the recently espoused principles and standards of the NCTM. Needless to say, there are other modern models of effective teaching which differ from those just referred to. Some of these are described at length

[7]Ibid., 142.
[8]Ibid., 140.

in Conway and Sloane.[9]

The remainder of this chapter is dedicated to describing the rationale and general nature of each of the three pieces written to illustrate the notion of a Topic. These extensive expositions are to be found in the core section of this treatise which resides on an accompanying CD. The three subject areas from the current sylabus chosen for this purpose are

1. Quadratics

2. Difference Equations

3. The Newton-Raphson Algorithm

These headings are the names given to each of the Topics respectively.

It should be clear from the amount of material devoted in the core to each of these Topics that, in each case, the relevant discourse is much more extensive than the corresponding one to be found in current textbooks. One reason for this is that it was felt necessary to elaborate in as explicit a manner as possible on much of the proposed content and related background information so as to ensure that the teacher could easily assimilate any theory and ideas that might be new to him and thereby achieve a level that would guarantee him that higher viewpoint so neccessary to any form of teaching. As Freudenthal says in this regard:[10]

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[9]Ibid., 139-170.

[10]Hans Freudenthal, *Mathematics as an Educational Task*, 661.

> It is an axiom of teachers' training that the teacher should know more than merely what he teaches. This "more" does not only aim at the subject matter. The teacher has to know the things he knows in a form different from that in which he is teaching them. He shall not only stand above the subject matter which he teaches but also above its logical form. To reach his goal he shall be able to fathom the logical depth of subject matter.

Another reason for the expansive treatment is that it was considered important to describe a considerable number of diverse examples in an effort to make each Topic a rich syllabus entry, which would allow the selection of a sample of applications for a given cycle rather than being constrained to use all of the examples from a more limited collection. While the overall thrust of the discussion is aimed at the teacher, it is hoped that pupils would also be able to read most parts of these expositions on their own and certainly after being taken through it once by the teacher.

As will be evident from these core treatments, a feature of the presentations is that they integrate, at various appropriate junctures, with subject matter from other areas of the mathematical programme. This is in keeping with views expressed by Freudenthal, who wrote:[11]

> I would recommend integration or coordination of mathematics not *with* other disciplines but integration *around* mathematics, that is with mathematics as a nuclear discipline that attracts subject matter of other disciplines to have them worked on by the student as fields of mathematical organization. This does not make other disciplines dispensable nor are they devalued in any way, but at least the student would learn what he can do with mathematics.

[11]Hans Freudenthal, *Mathematics as an Educational Task*, 139.

6.1 On Quadratics

The current treatment of quadratics by the textbooks and examinations has been described in some detail.[12] In brief, it was argued that the topic receives no concrete motivation (in the texts) by way of indicating where quadratic equations arise, that the celebrated 'quadratic formula' is simply revealed and that there is an almost total absence of genuine applications. Furthermore, it was pointed out that no references are made to historical sources.

It has already been claimed here that it is a major omission in the current provision not to derive the quadratic formula from first principles using 'the method of completing the square', particularly when the algebra involved is straightforward but so empowering. Judging from an IMTA Newsletter of November 2001, it would appear that some secondary school teachers may not be familiar with this very effective technique. In that issue, one such teacher presents a single page article on how to find the critical point of a quadratic polynomial using this method which he remarks he learned from one of his Chinese students.[13] Motivating and deriving the 'completed square' form of an arbitrary quadratic polynomial is central to the discussion of Quadratics to be found in the core section of this work. The overall nature of this treatment is now elaborated upon under various headings.

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[12] See pages 295-298 and Appendix E.

[13] Patrick Quill, "The Turning Point of a Quadratic Function", *IMTA Newsletter* 100 (November, 2001)

Sequence of Presentation: The discussion is divided into three major stages and embraces a number of themes relating to the use of history, formulation, terminology and notation, transformations, technique and applications (which include the use of physics). It is proposed here to first describe each of the three stages and afterwards to discuss the various themes.

`Stage I:` This first stage "How Quadratics Arise" is a motivational one which presents the teacher/student with a number of different problems which, when analysed, amount to asking mathematical questions about quadratic polynomials. This policy of presenting examples before the statement and proofs of theorems is in line with that advocated by Freudenthal who criticised the opposite approach adopted by many textbook writers. He believed strongly that it is through examples that one first gleans what is central and what is peripheral.

The first example discussed does not give rise to a 'quadratic problem' but appears for the purposes of providing a contrast to the second problem, deliberately chosen to be the 'inverse' of the first and is a genuine 'quadratic' problem. This is done so the student can appreciate the significant technical differences that exist between the relatively straightforward task of multiplication and the considerably more involved operation of extracting a square root. The practical difficulties involved in extracting a square-root are not dwelt on by the texts reviewed, presumably because of the existence of calculators. However, the core treatment discusses this matter in some detail because it affords a number of opportunities which are elaborated upon in

that section. Briefly, it allows the teacher to talk a little about the practical and the abstract; the former in relation to aspects of 'computation' and the latter by way of revealing (historically) how the concept of number had to be extended to accommodate irrational numbers. Later it is emphasised that not alone does one encounter the realm of irrational numbers when attempting to extract the square root of a natural number as simple as 2 but that one is also led to discover the much vaster domain of complex numbers when faced with the similar problem of extracting the square-root of the negative number -1. It is important that students be given such overviews from time to time as it provides them with a broader perspective of the general mathematical landscape.

These two examples are followed by three problems whose ultimate solutions require no more than this ability to extract square-roots. Thus they reinforce for the student the central importance of this operation. However, to arrive at the final equation requires that the student be able to formulate the underlying context in mathematical terms – a skill often referred to as mathematisation. As already stated, it is intended to discuss the theme of equation formulation (as it relates to this Topic) in more detail under a separate heading later. The first two of these three problems are 'practical' relating as they do to the construction of a box of a certain volume and how one might measure the speed of a river. The third, while dressed in the guise of a puzzle whose context is known to have appealed to many a student and teacher, requires clear physical thinking for its solution and is perhaps more

demanding than its precursors.

At this juncture a number of different directions the teacher might take are indicated before presenting an algorithm for extracting square roots believed to be known to the Ancient Babylonians but almost certainly known to the Greeks of the first century A.D. The reasons for this aside are explained in the relevant section and are reproduced here for the sake of completeness. Doing this

1. addresses the mathematics curriculum injunction to teach something of the history of mathematics

2. serves as a special case of the Newton-Raphson algorithm which follows later in the course

3. shows how, with no more than elementary reasoning, one can fashion a very useful computational tool

4. explains the meaning of *algorithm* as a *recursive* procedure

5. affords pleasure to those students who revel in calculations, of whom there are many.[14]

After a brief reference to the hierarchy of computational abilities encompassed in the operations

$$+ \ ; \ - \ ; \ \times \ ; \ \div \ ; \ \surd$$

[14]Elizabeth Oldham, private communication.

and which are the only five operations that can be executed geometrically using no more than a compass and straight edge, a problem relating to *optimisation* is introduced. It should be stressed that since the HLC treatment of the quadratic does not touch on the method of completing the square, problems of this type which are amenable to solution from a knowledge of quadratics alone are not set on the examinations. This problem concerning maximising a `Garden Plot` subject to certain constraints is related in the form of a story designed to engage the student's interest. It is developed in detail both experimentally and analytically, so as to reflect what often is done by practicing mathematicians when confronted with similar problems. It will be noted that the discussion of this example is considerably longer than that afforded to any similar one treated by the textbooks and is used as a vehicle by which to introduce in a natural fashion some terminology and notation – another theme discussed at greater length under a separate heading. This problem is also used to show how quadratic *equations* also arise.

The subsection which follows shows how one and the same mathematical problem is often encountered in different guises, some of which appear more 'relevant' than others. Some problems of this latter type drawn from Babylonian sources are given and prove most useful for pedagocical reasons to be explained in due course under the theme heading of history.

The HLC often sets questions on quadratic equations involving parameters but given without context. At this juncture the Topic treats an example

which shows how an equation of this type arises in an applied context. It also demonstrates a very important technique by which such an equation is rendered into an equivalent but parameter-free quadratic. This is something that is done very often when solving 'real-world' problems and at least one other example of this is to be found in this work in relation to a cubic equation connected to a calibration problem.

Having by this stage of the development been exposed to a number of different situations requiring the optimisation of a quadratic or the finding of its roots, the reader's attention is now turned to clasifying the equations found to date. This requires introducing further terminology such as *polynomial, roots, degree* to name but a few. Considerable pains are taken to explain these terms, which are too often glossed over or mentioned without explanation in some texts. This done, the notion that certain quadratics can be factorised easily is demonstrated and exploited in a particular case to verify that a certain quadratic has exactly two roots. An exploration of whether or not factorisation holds out the prospect of finding a method to solve all quadratic equations is then undertaken and ends with the discovery that such is not the case. This is revealed by examining a classical problem related to the golden section which provides an example of an irreducible quadratic (over the rationals).

The treatment goes on to explain under the heading 'Using an Equation' how certain equations can be made to reveal characteristics of their solutions even when these solutions are not explicitly known. The pedagocical point

here being that even if one does not know how to solve an equation, it can still be a valuable possession in its own right.

The final segment of the first stage allows the student to design quadratics of his own. This experimental exercise can give the students a greater appreciation of the nature of a quadratic polynomial and can prove useful when finding the expression for a curve known to be generated by such a polynomial. A problem of this type arises in stage III.

By setting before the student a considerable number of examples where quadratics arise in practice, one hopes to impress upon him the need for developing a relevant theory which can solve these problems as well others of a similar type. This is the business of the second major division in the treatment of this Topic.

`Stage II`: The second portion of the Topic entitled 'Theory of the Quadratic' is concerned with deriving the key identity

$$p(x) = ax^2 + bx + c \equiv a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} \quad ; \quad (a \neq 0) \tag{6.1}$$

and then revealing the pivotal theoretical role played by this relationship.

The discussion begins by describing how the Arab mathematician Mohammed ben Musa al-Khowârizmî solved the particular quadratic equation $x^2 + 10x = 39$ geometrically circa 830A.D. Since the process of solution actually involves forming a geometrical square to arrive at its final result, the student is afforded a tangible *visualisation* of 'the method of completing the

square' which is central to the algebraic solution of the general quadratic equation. If, in the derivation of the general formula, this step is carried out purely algebraically then it can appear unmotivated and consequently mysterious. However, by revealing the inspiration of this simple but ingenious step, this historical problem helps demystify the process, thereby making it easier to understand and appreciate.

With the phrase 'the method of completing the square' thus introduced, the exposition focuses on the 'normal' form of the expression $p(x) = ax^2+bx+c$ solely as a polynomial function and slowly unfolds the algebraic derivation of the identity (6.1), taking care to explain the vital role of the completing the square step. When this is done, it is advised that "the teacher should hail this result as an important advance, whose full significance is about to be revealed in a few short strokes".[15] This process is begun under the heading "The Anatomy of an Expression" where it is clearly revealed how the 'completed-square form' of $p(x)$ given by

$$p(x) = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

is much more informative than its normal form "despite having the appearance of being more cumbersome and artificial".[16] This is done by showing how to ascertain and classify the critical point of $p(x)$ with no more than simple logic, and as such it is an excellent exemplar of the mathematical method.

[15] *On Quadratics*, 84.
[16] *On Quadratics*, 86.

At this juncture there is an ideal opportunity to cross-reference the results just obtained with their alternative derivation using calculus. This is covered briefly in the aside 'Confirming with Calculus' which concludes with the remark: "All-in-all, a wonderful illustration for students of different mathematical methods reaching the same important conclusions."[17]

Before revealing a second major consequence of the identity (6.1) which is that of the 'quadratic formula', the discourse revisits and solves the optimisation problem connected with the `Garden Plot` problem. Doing this gives a concrete application of the theoretical results just derived and confirms for students what they strongly suspected to be the case from their own numerical investigations. Then the famous quadratic formula of 1544 is obtained and highlighted to stress its significance. Again it is urged that teachers give the result the due emphasis it deserves as young minds may not, of their own accord, appreciate its powerful generality. The formula is then set to work on solving all of the quadratic equations encountered in the introductory section. Naturally the treatment is accompanied by comments when they are deemed necessary and by the explanation of further points of detail along with any added insights. For example, once the golden section problem is solved, the opportunity is availed of to talk a little about the golden ratio ϕ and the golden rectangle which appears both in art and architrcture.

After the numerous concrete illustrations of the quadratic formula, the discussion returns to its final but significant technical application of the iden-

[17] *On Quadratics*, 89.

tity (6.1) by way of using it to deduce "The Shape of a Quadratic". The nature of this long section will be elaborated upon in greater detail under the theme of Transformations. However, as explained in the introduction to that piece, "it appears to this writer that the approach adopted by the textbooks is to *assume that it is obvious* that the graph of $p(x)$ has the U-shape (or $\cap$-shape) indicated in the drawings that accompany their examples. Besides being logically unsatisfactory, it is poor pedagogy to make this assumption about the nature of the graph, particularly without explicitly admitting that it is something which needs to be established. That it can be demonstrated is the task at hand."[18] The importance of this section is that it provides the student with a concrete image of the graph of the general quadratic $p(x)$ which proves invaluable when the nature of the roots of $p(x)$ are being discussed. It also gives a clear visualisation of the minimum (maximum) of $p(x)$ at the apex with coordinates given by $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$.

The subsection which follows connects this shape with that of a parabola. This is done so that one can justifiably refer to the shape of a quadratic as parabolic and vice versa and the importance of this becomes apparent in stage III in relation to the shape of an arch and more significantly to the path of a projectile. This writer would have liked to expand upon the special characteristics of the shape of a parabola which underpin the design of satellite dishes because of its obvious application but considerations of time and space did not allow. However, there are Internet sources which do

[18] *On Quadratics*, 107.

just this and which the teacher could call on should he wish to expand on this theme. It would also make for a worthy school project to reveal and explain these properties since the mathematical prerequisites are all accessible at senior cycle level.

In the next subsection on "The Nature of the Roots of a Quadratic", the advantages of having first given the student an understanding of the graph of a quadratic becomes evident. With the aid of a number of these graphs the three different cases that can arise in relation to the roots of a quadratic are made clear visually. The treatment does however re-derive these results purely algebraically for the sake of completeness.

The final subsection of part two presents the general quadratic in factorised form, that is, as

$$\boxed{p(x) = ax^2 + bx + c = a(x - \alpha)(x - \beta)}$$

where

$$\alpha,\, \beta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

It is pointed out that this result shows that there is not just *one* quadratic satisfied by the roots α and β but a whole family characterised by the parameter a, an important observation that is not made plain in a number of the texts. One particular example appearing in the final part of this dicourse on quadratics gives a tangible application of this result to a problem connected with a parabolic arch.

By way of concluding this section on the theory of the quadratic, it may be stated that the clear advantage of the organic (from the ground-up) treatment is that the student sees the formula for obtaining the roots of any quadratic polynomial *derived* using no more than elementary algebra. Furthermore, he gets to see why the graph of such a polynomial is a parabola, something which makes it all the easier to see that a quadratic can have at most two real roots. In addition he is able to decide whether the polynomial has a local maximum or minimum and can give this value. There is none of the mystery that an approach which simply hands students results out of the blue can engender, a practice that has to be unsatisfactory for those able students who like to see as much as possible explained. For others, there is the danger that they will come to think of this as an acceptable scientific approach. By contrast the approach adopted here shows a substantial piece of mathematics being developed in a consistent fashion, achieving significant insights in a few short steps due to one simple ingenious insight – the method of completing the square. This is much closer to the spirit of mathematical thought and it seems a shame not to adhere to it in cases such as this where the derivation of the result is accessible and highly educational. According to the great Russian mathematician Andreĭ Nikolaevich Kolmogorov:[19]

[19]American Mathematical Society, *Kolmogorov in Perspective*,(USA: AMS, *History of Mathematics Series* 20 (2000)

> I regard formal rigour to be *obligatory*, and I think that in the final analysis, after a lot of work (which is usually *helpful* for a final understanding), rigour can be always combined (in the exposition of *important* results, that is, in essence, *simple* results) with complete simplicity and naturalness. The only way to realise these ideals is strictly to require logical clarity even where it is burdensome for the present.

By way of a section summary it may be said that as soon as the hugely informative identity given by (6.1) is either established (as recommended here) or simply placed before the student, one can derive results which otherwise would require the use of calculus. As already indicated this approach enables the teacher to

- demonstrate the existence of a unique local minimum (maximum) point
- derive the quadratic formula
- explain the shape of the graph of the general quadratic.

which, it is fair to say, is a very good return for the labour involved in assimilating the method of completing the square.

Stage III: The third stage of the discourse is entitled "Applied Quadratics" and, as its title suggests, treats a number of applied problems which are a little more substantial than those presented in stage one. However, the extra sense of realism which these possess make their purpose self-evident and so should appeal to the student. The treatment begins with four problems relating to optimisation which could be regarded as examples of 'how to

solve max-min problems *without* the use of calculus'. The first, dealing with a cinema maximising its revenue, is expounded at length in order to guide the students to the correct formulation of the relevant quadratic. The second is essentially concerned with determining a rectangle of maximum area and while presented in an historical context is subsequently given various guises, some of which may strike the student as more relevant than others. The third problem is related to the second and receives consideration because it introduces the theme of recognising that a certain quartic equation may in fact be regarded as a quadratic. The fourth example shows the theory of the quadratic deriving a statistical result relating to the mean and variance of a sample and is designed to give students a greater appreciation for the connection betwen these two measures. While it is recognised that "this example is unquestionably at a level which is higher than that which students are exposed to currently, it represents a genuine application of the theory of the quadratic to a significant aspect of statistics."[20]

Next a problem regarding a network of computers is chosen because it leads to a quadratic inequality as opposed to a quadratic equation. Its solution is approached initially through an examination of small cases which are illustrated with appropriate diagrams. It requires an elementary knowledge of combinatorics or the formula for the sum of first n natural numbers in order to formulate the relevant inequality whose solution is then obtained with some care. This example affords the teacher a context to show a use for

[20] *On Quadratics*, 161.

the 'floor notation'. This specific example is then generalised and its elegant solution presented as an example of a mathematical vignette, which is decribed by Mankiewicz as a sutra — " a short poetic aphorism which strives to give the essence of an argument in the most condensed and memorable form".[21] The result, which is expressed concisely with the aid of the floor function, is connected to two sequences of natural numbers which the teacher and student may find interesting for no other reason than the possible novelty of what they portray.

The next subsection shows how the evaluation of certain values of the trigonometric functions can be accomplished with the aid of appropriately formulated quadratics. The treatment shows how early mathematicians blended algebra and trigonometry rather skillfully to find these values which were of such practical concern to surveyors and navigators. This section contains a vignette linked to a short aside on the golden ratio and the pentagon which may hold some appeal for the mathematically stronger student and might also be made the basis of a student project.

Under the heading 'Quadratics and Calculus' a number of problems requiring a knowledge of the quadratic and the basic notions of the calculus are presented. The first relating to maximising a volume is discussed in some detail and is afterwards harnessed to provide a realistic question involving

[21]Richard Mankiewicz, *the story of mathematics*, (London: Cassell & Co, 2000), 40. A number of mathematical vignettes have been written for this work to convey the ingenuity present in many arguments relating to or at the level of school mathematics. These are described in due course in a relevant section and are accompanied by suitable examination questions.

a cubic equation whose solution requires the use of the Newton-Raphson algorithm. As such it provides subject matter for another of the Topics considered in this treatise and will be taken up again. The second offering concerns a parabolic arch and has already received mention earlier in this work as furnishing a 'more vibrant and relevant alternative' to the type of question being examined currently. It is a problem in two halves; the first reveals to the student the necessity of being able to describe the shape mathematically by a quadratic polynomial while the second shows the integral calculus finding an area. The valuable lessons this problem can be made to teach are discussed at some length and involve, among other things the notion of a solution being dimensionally correct as well as the concept of linearity. This arch problem is then linked to the calculus exercise of obtaining the area subtended by the function $\sin x$ over the interval $[0, \pi]$ which is designed to have students make an observation. The subsequent exploration of this finding reveals how well a certain quadratic polynomial can be made to approximate the sine function over the given interval under consideration. Among other things, this exercise demonstrates to the student how important mathematical interconnections can be discovered by playful mathematical experimentaion. These deliberations are followed by two further problems relating to parabolic arches and tangents to these curves, so that a little differential calculus is called on. Afterwards by way of giving these problems a contexts it is pointed out that they arise in connection with the drawing of parabolic curves using a certain typesetting package.

Continuing with the theme of the parabola the following subsection is a substantial one treating projectile motion in the absence of friction and is presented at this juncture simply because it can be cross-referenced with calculus. However, since the entire discussion can be carried out *without* any use of calculus it could appear at any point in a section dealing with applications of the theory of the quadratic. As stated in the introduction to this piece:[22]

> Everybody has seen a football kicked into the air or a golf ball driven along a fairway and observed the path taken by the projectile in question without, perhaps, reflecting on or knowing much more about its geometrical shape. Many students are interested to learn that the shape of this trajectory is parabolic and thus is a particular example of the graphs of certain quadratic polynomials already discussed in the classroom. Making such a connection reveals to students a tangible link between the abstract and the concrete and as such is an invaluable lesson.

Besides being a grade one application of the theory of the quadratic, it is also a rich illustrator of the application of many other quite simple mathematical facts covered in the senior cycle programme. As the treatment shows, a discussion of this simplest form of projectile motion allows the teacher to illustrate the use of:

- `History` – to act as a possible motivator
- `Coordinate Geometry` – to embed a problem within a framework
- `Vectors` – to resolve a vector into orthogonal components

[22] *On Quadratics*, 200.

- `Trigonometry` – to reveal just how useful it can be to have a basic knowledge of the trigonometric functions

- `Parametrisation` – to introduce parametric equations in a natural way

- `Calculus` – to either derive results *ab initio* or confirm those obtained by other methods

- `Mathematics in action` – providing, with the help of a basic fact from physics, a deeper understanding of a natural phenomenon.

The final example treated would have to be considered a 'modern' one in the sense that the application to be described was first conceived of as recently as 1977. It deals with the use of quadratic polynomials to an easily understood and visibly practical problem in cryptography which, in the experience of this writer, students find very appealing. The mathematics underpinning the application involves no more than the simple but fundamental fact that any three points in the plane with distinct abscissae determine a unique quadratic polynomial. After discussing how this observation can be made evident, the student is introduced to the problem to be solved which is that of 'How to Share a Secret'. The nature of what is involved along with the central notion of a 'threshold' is explained, after which the student is invited to come up with mechanical schemes for solving the problem at hand. These solutions require logical thinking as opposed to specific mathematical knowledge and as explorations in the classroom they can provide a welcome change

to the normal fare. Once the students through their own investigations come to understand what is involved, the discussion can then be broadened by the teacher to show how simple but clever combinatorial reasoning can be used to gain greater insight into the limitations of the mechanical schemes being proposed.

Because several pages are devoted to explaining the ideas involved the teacher/student does not see how quadratic polynomials enter the fray until the combinatorial reasoning used convinces one and all that the proposed mechanical schemes are totally impractical. As pointed out at this stage of the development "the student gets to see a general argument showing that a proposed solution is *computationally infeasible* because of its inherent *complexity* – two notions which pervade the practical implementation of algorithms". There is an accompanying PowerPoint presentation which parallels the written description but which is bound to be more appealing for visual and dynamic reasons. It might perhaps be used by the teacher in the first class or two by way of an engaging introduction or used in the third class or so after the students have had a chance to make their own independent discoveries.

With the stage thus set, the students are shown the elegant fashion by which quadratic polynomials can be made to implement a $(3, n)$ scheme – a protocol by which a secret number is shared among any number (n) of persons but in such a way that it takes the cooperation of 3 or more before the secret can be recovered. The manner in which the number is shared is

described in detail with the aid of many diagrams while the procedure by which this same secret is recovered reveals to the student one reason why it is crucial to know how to solve a 3×3 system of simultaneous linear equations. With the theory explained and illustrated with simple examples, some real-world applications of such a scheme are described, one being the digital signing of cheques. In this way the student gets to see an effective use of quite elementary mathematics.

The last page of the rather extensive discourse contains a list of other applications which fall under the umbrella of Quadratics and which over time might be described and made part of the content of this Topic.

This section describing the nature of the proposed Topic on Quadratics is brought to a close by giving short descriptions of a number of the themes permeating the exposition.

Use of History: Historical sources provide many problems whose solutions require the solving of a quadratic equation. The earliest such problems date from Babylonian times and are invariably linked to squares or rectangles. Even if they appear contrived they are not without their uses. For one thing they provide reasonably simple exercises in formulation for the student while linking with the mathematical past where insights into the ancient mind can be gleaned. A number of these are presented in stage I, with one in particular being harnessed to introduce the concept of factorisation and to suggest that a quadratic equation has exactly two roots.

Another historical example from the middle ages due to Al Khowârizimî proves of particular value in stage II where, as has already being mentioned, it is used to introduce the ingenious idea of completing the square. This accessible example blending algebra and geometry is studied because it acts as a precursor to the general algebraic method which leads to the quadratic formula that solves all quadratic equations. As such it is judged to be crucial to the discussion, while also affording an historical cameo that reveals a little of the legacy of mathematics. The historical progression is continued when Michael Stifel is credited in 1544 with fusing all that was known about quadratics into the one single quadratic formula that is now so commonplace.

Stage III also contains its historical references, in particular to the trajectory of a projectile where the observations and speculation of such luminaries from the mathematical past as Niccolo Tartaglia, Galileo Galilei, Bonaventura Cavalieri and Evangelista Torricelli are described in some detail and act as a springboard for the subsequent discussion. By way of giving a glimpse of more recent history reference is also made to the living mathematician Adi Shamir in connection with cryptography, something which may be regarded as noteworthy by some students.

Formulation: Since the first part of the Topic is concerned with revealing how quadratic expressions and equations arise from the consideration of different problems there is a greater emphasis placed on formulation than is the case with the current HLC programme where students are not normally

required to derive such equations but simply solve them. The final section on applications offers more sophisticated examples which it is believed are described in sufficient detail to allow teachers to impart this particular skill without undue difficulty.

Terminology and Notation: Considerable care is taken, whenever it is judged informative, to explain terms which in the current HLC provision are often used without any mention of their etymological origins. Almost from the outset stress is laid upon functional notation, considering it better pedagogy to place the emphasis on polynomials rather than equations, since, after all, the latter normally arise in connection with special questions relating to the underlying polynomial. While this approach is not predicated on the assumption that students will already have been introduced to functions and their graphs, (something it would appear is not being done at present judging from the order in which material is presented in the textbooks), it does provide an 'as early as possible motivated introduction' to the function notation. This affords the teacher greater flexibility in handling subsequent material in a concise manner faithful to context. As Siu writes in his article, "Felix Klein strongly advocated an emphasis on the function concept in teaching as a unifying idea permeating all mathematics."[23]

The parabolic arch and the projectile problems discussed in part III are

[23]Man-Keung Siu, "Concept of a Function – Its History and Teaching" ,*Learn From The Masters*, edited by F. Swetz et al. (USA: Mathematical Association of America, 1995), 105.

used to illustrate how effective function notation can be.

Transformations: The gradual process by which the graph of the general quadratic is arrived at helps introduce students to the general topic of the transformations of functions in a particularly straightforward case. The function notation in particular is seen to be most convenient when the completed-square form of the the general quadratic polynomial is used to show that the graph of $ax^2 + bx + c$ is simply a translation of that of ax^2. By way of exhibiting the shape of this latter curve, the treatment has the student sketch the graph of the fundamental curve of $y = x^2$, followed by a number of the form ax^2 for different values of a, carefully chosen to reveal the essential features of what holds in general. This done, the general shape of the quadratic need no longer be taken as an act of faith, as is the case at present, but becomes clear to the student and provides him with a strong visual guide by which to appreciate the different cases that can arise in relation to the nature of the roots of a quadratic.

Currently there is little or no treatment of the transformation of functions on the HLC, although it is clear from Internet searches that many of the American High Schools devote much time to this very dynamic subject matter. The treatment here shows students how to effect reflections, translations and dilations. This is as it should be, because exploring these transformations reinforces the concept (of the graph) of a function in a very visual manner which, of course, can be made that much more attractive by

being executed almost instantly on a computer using a mathematical package. Thus an important aspect of this Topic is the manner in which it helps students arrive at an understanding of the transformation of functions in an accessible manner.

6.2 Difference Equations

Another recommendation of this work is that the current syllabus segment relating to difference equations be expanded into a substantial and engaging Topic. As a step in this direction, first-order recurrence relations drawn from recreational and applied contexts should be introduced with the equations being derived from actual contexts that reveal their importance and usefulness. In this regard, it is suggested that, in particular, the recurrence relation governing the amortisation of loans be treated in depth. As already claimed, "this is perhaps one of the most useful applications of mathematics lying outside of the scientific situation that bears on a student's (future) personal life and which can be undertaken at the level of senior-cycle." [24]

It is also advised that the treatment of some second-order difference equations be retained as part of any future syllabus. However, by contrast to the current practice, it is urged that the examples studied be presented in context so that their purpose is clear. In particular, it is recommended that the difference equation connected with the Fibonacci sequence receive some attention because of its historical roots and because it arises in the study of

[24] See page 256.

a wide variety of problems. It is also suggested that the Topic be examined by one full question devoted to it alone.

It is hoped that the core treatment of the Topic on difference equations is envigorating by having much to offer by way of intrinsic interest. Its nature and rationale is discussed at length after a short review of the current programme's provision.

Current Treatment: The following is the fourth and final entry appearing under the subheading `Discrete Mathematics and Statistics` on the HLC Mathematics Syllabus:[25]

> Difference equations: If a and b are the roots of the quadratic equation $px^2 + qx + r = 0$, and $s_n = la^n + mb^n$ for all n, then $ps_{n+2} + qs_{n+1} + rs_n = 0$ for all n.
> Examples of difference equations $ps_{n+2} + qs_{n+1} + rs_n = 0$ $(n > 0)$, (with s_0, s_1 given) to be solved, p, q, and r being specific numbers and the quadratic equation $px^2+qx+r = 0$ having distinct roots.

This entry seems to amount to no more than asking for the solution of second order homogeneous difference equations with constant coefficients and specified initial conditions subject to the restriction that the characteristic-equation have distinct roots. It is not explicitly stated that n is assumed to be a non-negative integer nor is it clear whether or not difference equations whose characteristic equations have complex roots are to be excluded. It may be that the intention was to have a characteristic-equation with positive discriminant rather than one with distinct roots.

[25]Department of Education, *The Leaving Certificate Mathematics Syllabus (Higher Level and Ordinary Level)* (Dublin: Government Publications Sales Office, n.d.), 11.

In this syllabus abstract there is no explanation as to why the equations to be solved are called Difference Equations. Likewise the texts, while defining a difference equation in the manner indicated by this syllabus entry, fail to mention why it is so called. Furthermore, the extract makes no attempt to justify itself by setting the topic of difference equations in a context nor does it give guidance to young and possibly inexperienced teachers on how to interpret it, except by reference to sample or past examination questions. As one secondary school teacher wrote in an IMTA article of 1995: "At first glance one might wonder what this is all about." In the same article the author O' Loughlin writes:

> What intrigued me about the method of solution was:
> how might it have been developed?
> why use a quadratic equation as a starting point?
> why use a linear combination of powers of the roots?

The major portion of the remainder of the article is devoted to answering these questions and concludes with a discussion of "an example of a difference equation taken from the computer science field".[26]

6.2.1 First Order Recurrence Relations

This work advocates that first-order recurrence relations be part of a future syllabus segment devoted to the Topic of Difference Equations, since many 'first-order' problems are ideal for illustrating the method of *recursion.* This powerful technique where the solution to a problem depends on finding the

[26]M. O'Loughlin, "Difference Equations", *IMTA Newsletter* 86 (1995): 9-16.

solutions to smaller instances of the same problem is not explicitly catered for in the current syllabus.

The first part of this account describes a number of problems which illustrate the effective use of 'recurrence relations' or 'first-order difference equations with initial condition' as they are also styled. Some of these problems are treated extensively either in the core section or by means of PowerPoint presentations to help the teacher understand the intent and actual implementation of the suggested new syllabus entries.

`The Tower of Hanoi Puzzle:` The first problem considered is presented by means of a PowerPoint slide show to be found on the CD accompanying this work. It treats the `Tower of Hanoi` puzzle, a game invented in 1883 by the French mathematician Edouard Lucas. This engaging recreational problem is chosen because it is an ideal vehicle for introducing many of the key notions and concepts used in discussing recurrence relations.

The presentation first describes the objective of the game which involves a 'toy' consisting of three pegs set in a base with a tower of eight disks, stacked in decreasing size on the first peg. It then recounts how Lucas furnished his puzzle with a romantic legend about a much larger Tower of Brahma, which supposedly has 64 disks of pure gold resting on three diamond needles. The method of solving the problem is then shown over a series of slides. While doing so, track is kept of the minimum number of moves required because this acts as a prelude to the mathematical problem about to be introduced

and which is the real purpose of considering this children's puzzle. The problem for the senior-cycle student is

> Determine how many moves are necessary and sufficient to reconstruct a tower of height n according to the stipulated conditions.

It is understood that n denotes a non-negative integer.

The would-be solver is advised at this stage to attack this problem by first examining small cases with the exhortation that "smart mathematicians are not ashamed to think small, because general patterns are easier to perceive when the extreme cases are well understood (even when they are trivial)".[27] When this is done the student is instructed to employ the tactic of 'name and conquer', which is that stage in solving a general problem where one introduces appropriate notation. Specifically, he is told to let $t(n)$ denote the minimum number of moves needed to reconstruct a tower of height n according to the stipulated rules.[28] With this notation in place the student displays the results of his analysis of small cases in the form

$$t(1) = 1 \quad ; \quad t(2) = 3 \quad ; \quad t(3) = 7 \quad ; \quad t(4) = 15$$

He might prepend to this list the entry $t(0) = 0$, after noting, perhaps on a prompting from the teacher, that it requires zero moves to reconstruct a

[27] R. Graham, D. Knuth D. & O. Patashnik, *Concrete Mathematics* (USA: Addison - Wesley, 1989), 5.

[28] It should be noted that throughout the discussion of this Topic the function notation is used rather than the subscript notation employed on the HLC.

tower having no disks. In this way the student obtains the sequence whose leading terms are

$$0,\ 1,\ 3,\ 7,\ 15,\ \ldots$$

Now it is time to use this knowledge of the 'little' to go 'large', but to do this one needs to be fortunate to have made a key observation from the study of the small cases.

From an examination of the numerical data just presented one insight, which many students have is that the current value is obtained by doubling the previous value and adding 1. It is here that the teacher may have to show the students how to formulate this insight mathematically though some students may well do this for themselves. Specifically, if $t(n)$ denotes the current value, so that $t(n-1)$ is the previous one, then the 'conjectured' rule, believed to be valid for $n \geq 1$, can be written as

$$t(n) = 2t(n-1) + 1$$

However, it is preferable that the reason why this relation holds in general be established by making a key observation regarding the process itself. A considerable number of slides are devoted to having the student see that the problem relating to n disks can be related to the smaller problem concerned with $n-1$ disks. This is where the concept of recursion is revealed and it is this understanding of the underlying mechanism of the game that allows mere conjecture to be turned into fact. It is crucial at this stage that students fully appreciate how justifying the relation on physical ground turns conjecture

into proof.

Whatever route is taken, and normally both are, the student has been shown a *recursion relation* – so named because of its connection with the process of recursion. The relation is more commonly referred to as a *recurrence relation.* In fact, as the treatment explains, strictly speaking it is the relation

$$t(0) = 0$$

describing the 'initial' condition, combined with the general relation

$$t(n) = 2t(n-1) + 1$$

valid for $n > 0$, often written together as

$$t(n) = 2t(n-1) + 1 \quad ; \quad t(0) = 0$$

that constitutes the recurrence relation.

The manner in which the recurrence relation can be used to generate successive values of $t(n)$ is reinforced by calculating further terms of what is often called the recurrence sequence.

However, under the heading "But It Takes Too Long", it is pointed out that, starting from the initial value $t(0) = 0$, it takes 64 iterations of the general recurrence equation to reach $n = 64$; a prohibitive amount of calculation to be carried out by hand or with the aid of a calculator. While mentioning that a computer program could be used, it is noted that changing the value of n from 64 to $1,000,000$ results in the amount of calculation

becoming prohibitive even for a machine. The purpose of this exploration is that for n 'large', it simply takes too long to generate values recursively to obtain a desired $t(n)$. In this way, it is hoped the student will realise that it would be much better to have a closed form for $t(n)$, that is, a formula which allows $t(n)$ to be calculated for each value of n, without reference to any other values. Such a formula, if it exists, would be a solution to the recurrence relation.

The student is told that one way of solving a recurrence relation is to *guess* a solution and then try to prove that it is correct! It is important that young people be told that much of mathematics is discovered in this way, intuitively as it were, and that this is a perfectly acceptable way of doing business provided one can subsequently copper-fasten one's guesses with irreproachable logic.

In this case, many students guess that $t(n) = 2^n - 1$ in general because this formula fits the data accumulated for the first few values of n starting with the value 0. The teacher can then explain that all that is needed to prove this new conjecture is to verify that $t(n)$ satisfies the recurrence relation since, as the students now know, this has been firmly established. This is an elementary but valuable exercise. Leaving to one side questions of uniqueness, the general problem can then be considered as solved.

After pointing out that finding the solution in this manner requires an inductive leap, that is, a lucky guess about the possible general answer, one must consider how to proceed when such flashes of inspiration are not

forthcoming. This leads to the final portion of the discussion of this problem which shows how the recurrence relation itself can be used to provide the general formula for $t(n)$ in a systematic fashion.

It is first explained that the manner in which the recurrence is used to obtain successive values of $t(n)$ can be described as 'forward recursion' since it moves forward through the sequence of values that $t(n)$ attains as n itself moves forward through the natural numbers. Then it is shown in great detail how the value of $t(8)$ can be calculated using 'downward recursion' or 'backward recursion'. A number of important points are revealed in this discussion including the notion of a 'recursive call', common in programming parlance. Then in 'Unfolding the General Recurrence' it is shown how one obtains, with the aid of this technique and the general formula for the sum of a *geometric series*, the previously conjectured formula for $t(n)$. The student is reminded that it is vital that it too be checked in the manner indicated.

After remarking that the `Tower of Hanoi` recurrence relation is typical of many that arise in applications of all kinds, the three stages involved in solving a problem of this type are reviewed as follows:

1. The examination of small cases which give insight into the problem and help with the second stage.

2. To find and justify a recurrence relation connected to the quantity of interest.

3. To find a closed form satisfying the recurrence relation.

As can be seen, this problem alone has a number of valuable features warranting its study and which might be synopsised as follows:

1. Exploration and the Scientific Method: the exploration of small cases teaches something of the scientific method where one gathers data by careful enumeration with the intention of examining it afterwards in the search for patterns which might prevail in general.

2. Conjecture and Proof: upon analysis the accumulated data may lead to the formulation of a recurrence relation which, if it is to be subsequently used with any degree of confidence, needs to be proved.

3. The Value of a Recurrence Relation: once justified, the recurrence relation is a valuable possession in its own right since it allows one to generate answers by the process of recursion, which is quite feasible for moderate values and often suffices for many practical problems.

4. The Limitations of a Recurrence Relation: however, if one desires a method for specifying an arbitrary value without recourse to previous values, then the recurrence relation alone is not enough to furnish such a general solution to the underlying problem.

5. The Art of Guessing: from an examination of even a small amount of numerical data one can often 'induct' a general formula which can be immediately checked on the recurrence relation. This method of arriving at an answer is perfectly acceptable and can prove hugely

time-saving. However it needs to be stressed for the student that such 'inductive leaps' only prove their worth after they have been verified.

6. Solving a Recurrence: the method of downward recursion often allows one to find the general solution of a recurrence relation in a systematic fashion.

7. Need for Technical Mathematical Knowledge: to arrive at the general solution of a recurrence relation often requires knowing basic facts concerning certain sums, in particular those connected with arithmetic series or, as in this case, geometric series. Thus the solving of recurrence relations shows the student technical mathematics being called upon.

`The Planar Regions Problem`: The second problem treated in detail, also by means of a PowerPoint presentation, is a geometric one entitled the `Planar Regions` problem. The task is to

> Determine the number of regions into which the plane is divided by n lines if
>
> 1. no two lines are parallel
> 2. no three lines intersect at a common point.

Like the problem just discussed, this is another example of an *enumeration* task where one must enumerate or count a complete set of possibilities. The method of recursion is very often used in such 'counting' problems because the strategy of reducing a larger problem to a number of smaller but similar

problems often proves very successful, as is illustrated for a second time with this example. Since the treatment of this problem teaches most of the same lessons as the `Tower of Hanoi` puzzle, the commentary here is confined to briefly describing those aspects of this example in which it differs from the first offering.

After passing through the examination of small cases, students soon conjecture on the basis of the data gathered that the relevant recurrence is

$$r(n) = r(n-1) + n \quad ; \quad r(0) = 1$$

where $r(n)$ stands for the number of regions created by n lines subject to the stipulated restrictions.

By contrast with the first problem where some students actually hit upon the physical explanation for the relevant recurrence almost immediately, it is unlikely that this will happen with the same alacrity in this case. However, as part of the search to establish the validity of this recurrence relation the teacher should ask the students to interpret the recurrence in the context of the underlying geometrical context. Some students note that general equation can be cast in the form

$$r(n) - r(n-1) = n$$

and interpret the 'difference', $r(n) - r(n-1)$, in this equation, as the number of new regions created by the introduction of the nth line. In so doing they provide the ideal opportunity for the teacher to mention that a recurrence relation of this type is also called 'a difference equation with initial condition'.

The observation also points the way to proving the conjecture, and is a task on which the exposition spends some time by way of explaining this difference geometrically and making it clear that once this is accomplished, the recurrence relation can be regarded as fully justified.

The other facet of this problem not found in the previous one occurs at the stage where the recurrence relation is used to provide its own solution by downward recursion. In this instance, the student needs to know the formula for the sum of an arithmetic series as opposed to a geometric series.

Other Recurrences: It is recommended that a number of the recurrences listed below be part of the discussion of first-order recurrence relations at senior cycle level:

1. Permutations:

$$p(n) = np(n-1) \quad ; \quad p(1) = 1$$

2. Compound Interest:

$$a(n) = (1+r)a(n-1) \quad ; \quad a(0) = A$$

3. Mortgage Repayments:

$$a(n) = (1+r)a(n-1) - p \quad ; \quad a(0) = A$$

4. Tower of Hanoi Puzzle:

$$a(n) = 2a(n-1) + 1 \quad ; \quad a(1) = 1$$

5. The Regions Problem:

$$r(n) = r(n-1) + n \quad ; \quad r(0) = 1$$

6. Determinant Evaluation:

$$M(n) = n[M(n-1) + 1] \quad ; \quad M(2) = 2$$

7. Even Number of Zeros:

$$a(n) = a(n-1) + 3^{n-1} \quad ; \quad a(1) = 2$$

It should be remarked that the topic of 'integration by parts' can also be harnessed to present a number of first-order recursions in connection with the evaluation of certain families of anti-derivatives.

The first recurrence listed is connected with the problem of counting the number of different ways n distinct objects can be permuted among themselves. It requires no more for its formulation than a clear understanding of the *multiplication principle* which underpins all counting procedures. Furthermore, it is easily solved by the method of backward recursion. As a consequence it might be used as an introduction to the theme of recurrence.

The second recurrence concerns the total amount of capital available after a principal of A units has been invested for n years where interest is compounded annually at the decimal rate r. It is a practical problem from the 'personal and financial' domain and as such is an invaluable example

bearing on the lives of all students irrespective of their future career paths. Its solution is easily found by downward recursion and verified, if required, by the method of mathematical induction. It is also a vital stepping stone for the even more important recurrence appearing next on the list.

The third recurrence relating to mortgage repayments is treated in some detail in the core section, while the rationale for its inclusion will be elaborated upon.

The fourth and fifth recurrences have already been described and are each the subject of a presentation.

In order to derive the fifth recurrence students would need to know about determinant evaluation for arbitrary $n \times n$ determinants. This is not covered on the current HLC. However, in a future framework where this might be the case, the recurrence allows the teacher to generate successive values recursively and show how quickly $M(n)$ grows. The teacher is then in a position to mention that mathematical algorithms based on determinant evaluation, such as Cramer's rule, are totally impractical for even moderate values of n. Having done this, he might instruct his pupils on how to obtain the formula

$$M(n) = n! \left[1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n-2)!} + \frac{1}{(n-1)!} \right]$$

recursively. They could then check that this formula for $M(n)$ reproduces the same values as those obtained recursively using the recurrence relation. It is also an excellent exercise for students to verify that the formula does indeed satisfy the general recurrence relation.

The final recurrence concerns codewords of 'length' n formed using characters from the *ternary* alphabet $\{0, 1, 2\}$. Here $a(n)$ denotes the number of these codewords in which the character 0 appears an *even* number of times. It is a challenging but not overly difficult exercise, particularly if guided, to derive the given recurrence. However, it is an easy exercise to show that

$$a(n) = \frac{3^n + 1}{2}$$

and easier still to check that this formula satisfies the given recurrence. The importance of the example is the powerful manner by which it illustrates the recursive technique solving a counting problem. It is the second of the first-order recurrences treated in some detail in the core section.

The first example treated in the core is perhaps the primary show piece of a recurrence relation being used to solve a practical problem which impinges on almost everybody's life at some stage. The exposition is written in such a way that renders it independent of the two presentations already described, but on those points which have been amply elaborated upon in these presentations it is less expansive. At the small cost of being repetitious, if not irksome, this gives the material greater flexibility in that it can be regarded as a self-contained unit. However, as mentioned in the introduction to the piece, it is still advised that one or both of these two problems form part of the students' preparation for the study of this particularly important example.

`The Mortgage Problem:` One method by which mortgage repayments can be calculated is through the formulation and solution of a first order recurrence. Since most young people will face the problem of raising a loan at some time or another, be it to purchase a car or buy a house/apartment, it is truly of genuine benefit that part of their mathematical education at school be made to inform them on how the repayments on such loans are calculated. Once understood, the solution obtained provides the student with the ability to work out such repayments and frees him from having to trust to someone versed in financial matters. As such it is a worthwhile objective and should be part of any mathematical syllabus which aims to provide a general education as part of a preparation for adult life.

This is perhaps one of the most useful applications of mathematics lying outside of the scientific situation that bears on a student's future personal life and which can be undertaken at the level of senior-cycle. In the language of PISA03, this application is as an extra-mathemtical one belonging to the 'personal situation'. Besides being of value to all members of the classroom, it is also likely to appeal to a broader band of students.

The core discussion begins by explaining why the word mortgage is used in connection with house loans and presents a scenario of a typical loan a first-time buyer might want to raise. After this, the problem is stated and treated in abstract terms so as to achieve the greatest generality. The algebra used subsequently is quite elementary and within the compass of present HLC students. The method of solution via recursion draws on nothing more

technical than the formula for the sum of a *finite geometric series*, something which is part of the current HLC syllabus. Thus, outside of the possibility of taking some time to become accustomed to the mode of reasoning, the diligent student need fear no other difficulties.

Considerable care is taken in introducing appropriate notation and revealing its efficacy, first through specific examples before being generalised. The purpose is to have the student see the function notation facilitating reasoning and circumventing excessive algebraic manipulation which is a crucial lesson.

Once the recurrence relation has been derived, it is emphasised how it gives a concise description of the simple but fundamental mechanism by which the amount owed at the end of one year is connected to that of the previous year and how it incorporates both the effects of interest accruing on the outstanding balance and the reduction due to repayment. The importance of taking time to discuss equations in their own right is a point which is made a number of times throughout this work.

By contrast with the previous problems discussed the treatment here proceeds directly to solving the recurrence rather than, as heretofore, using the recurrence itself to generate values. The reason this is done is explained. Then it is shown how the general solution can be guessed at and the student is reminded once more that guessing general answers from particular cases is a much valued skill in any form of investigative work, as it often proves to be the fastest route to an overall solution. Then a direct derivation using the

method of downward recursion is demonstrated and it is here that knowledge of a finite geometric series is needed. Thus again the student is shown the importance of knowing technical results.

Empowered by the possession of a general formula, one reaches the exciting stage where the question of how much a loan costs can be answered. This is done in stages over the next few paragraphs and is where the student gets 'a real education' in layman's terms. It is shown in an illustrative example that a moderate loan of €160,000 over twenty-years at an interest rate of $r = 0 \cdot 05$ costs in excess of €250,000 – an amount which often has students reeling in shock upon realising that such a loan appears to cost nearly a hundred thousand euro. For some this is a moment of deep realisation concerning the cost of borrowing money.

Under the heading 'Mathematical Theory and Computer Implementation', the student is told that mortgage repayment tables are best compiled with the use of a package such as Excel. In fact, it is recommended that students be taught how to use such a package to numerically investigate a wide range of possibilities, which would be too time consuming to do manually. In addition to gaining an appreciation for the power of technology, this would enable them to investigate the repayments on different sized loans, at different rates of interest, and over varying loan-durations. However, the overall purpose of these computer-aided explorations should be that the student comes to understand that it is the mathematical theory which enables one to carry out all these informative calculations, ably assisted, of course,

by the computer.

The discussion concludes by displaying a table giving both the outstanding balance and reduction of principal at the end of each year during the term of the loan given in the illustrative example. This table often holds a number of other surprises for the student, one being that after ten years of repayments amounting to $€128,388 \cdot 10$, a borrower still owes close on a hundred thousand euro.

It is believed that the core description of the mortgage problem is a substantial one which illustrates the power and relevance of mathematical thinking. But what recommends it above all else is that it an example of school mathematics being of educative value to all members of the classroom.

`Even Number of Zeros`: The other first-order recurrence presented in the core arises out of a counting problem of the type that is often of concern to computer scientists. While the exposition of this problem teaches most of the same lessons taught by the three problems discussed to date, this problem has an 'open-ended' feature which is the sole focus of this short section devoted to its rationale and nature.

After the preliminary stages of investigation, students invariably come up with a recurrence relation which accounts for the data gathered empirically but which they fail to justify from the context of the problem. On the other hand, the teacher is able to show a different observation leading to a recurrence which, after some thought which he may need to supply, can be firmly established from the context of the problem in a straightforward manner.

When this latter recurrence is solved and its general solution checked, it is then revealed that this general solution also solves the original recurrence.

Thus students have to part company with the problem unable to justify the recurrence they have observed, and yet knowing it to be right. It is this 'unfinished business' aspect of this problem which gives students a genuine glimpse of the nature of the mathematical endeavour; often while solving one problem, another crops up for which a solution is not readily forthcoming.

Nomenclature: It should be pointed out that the exposition to date has consistently used the term recurrence relation rather than 'difference equation with initial condition', because the way in which such relations are typically derived connects with recursive thinking. However, as mentioned, the dicussion relating to the `Planar Regions` problem does suggest why the relevant recurrence can quite naturally be regarded as a difference equation with initial condition. It is only in the next section dealing with second-order recurrences that time is taken to expound on the terms first- and second-order as well as explain why a second-order recurrence relation is also referred to as a second-order difference equation with initial conditions. This is done by means of an historical aside on the 'difference operator' introduced by mathematician George Boole in the mid-nineteenth century.

6.2.2 Second Order Difference Equations

The practice of asking for no more than the solution of a second-order difference equation (with specified initial conditions) presented *without* any reference to a context has been criticised in this work as being no more than an academic exercise of doubtful value and a mis-use of a syllabus segment which could be greatly invigorated by a more imaginative interpretation.

The manner by which a difference equation is arrived at is more often than not a greater practical skill than the routine by which the solution of the difference equation is obtained. While it undeniably essential to be able to solve such equations, it is of little educational import if that is all the student is able to do.

It will be noted that many of the educational points made heretofore in connection with first-order difference equations are often repeated in the core examples dealing with second-order difference equations. This makes them somewhat self-contained and would prove a virtue in the event that first-order recurrences were not sanctioned for a future airing.

`Tiling a` $1 \times n$ `Hallway:` This first example concerns another counting problem and focuses on revealing how recursive thinking leads to a second-order recurrence relation or a second-order difference equation with specified initial conditions. It emphasises that the possession of such a recurrence is of great value in itself while the matter of finding a closed-form solution, though dealt with, is regarded as one of secondary importance.

The problem is presented in the form of a puzzle and the recursive man-

ner by which one obtains a valuable difference equation is developed. This accomplished, it is shown how other problems with different contexts are in fact connected to this puzzle problem; in that each is underpinned by the same difference equation governing the puzzle, albeit with different initial conditions. One such problem provides an applied context for the foregoing discussion and, as such, may prove more engaging for students. It is concerned with the transmission of codewords.

Besides revealing the isomorphism that can exist between apparently unrelated problems, this example is chosen because it also links with the celebrated Fibonacci sequence. A short paragraph on Leonardo of Pisa tells a little about the early history of the sequence and this renowned medieval mathematician's important contributions to mathematics.

Considerable space is then given over to establishing the truth of the conjectured recurrence by means of an argument which ably illustrates the method of recursion. Students are usually very impressed by the reasoning and gain an added appreciation of the effectiveness of recursive thinking. Because this demonstration is substantive it makes for a good examination question requiring a certain skill to describe properly. Hence asking for its description provides an excellent means of testing the Communications Standard.

It is at this juncture that issues relating to nomenclature are dealt with by way of an historical aside which may be of interest to teachers and students alike for the introduction it provides to certain operators much used in

mathematics.

This detour is followed by a discussion revealing the need for a general method for solving second-order recurrences where it is reiterated that the method of recursively generating specific values is a task that may be hugely labour intensive even with the assistance of a computer. Futhermore, it is added that the method of downward recursion, which so often leads to a closed-form expression for a first-order recurrence, is of no avail at the level of second-order (or indeed higher-order) difference equations. In this way the student is made to realise to an even greater extent the need for a theory by which it is possible to solve equations of this 'higher' type.

One systematic procedure for solving equations of this order is the classical method (due to the Swiss mathematician Euler) which is simply presented as a given in the DES syllabus segment. Whereas one might like to see some form of motivation for the method, if it be the case that time is at a premium, then it is better to opt for the source of equations being revealed rather their method of solution. The core treatment takes the classical method of solution as a given since this is the one aspect of recurrence relations that does receive adequate attention on the HLC.

However, when the classical method of solution is applied to the recurrence of this example it leads immediately to a quadratic polynomial which is irreducible over the rationals. Since this quadratic does not factorise over the rationals, any recurrence with this characteristic equation would *not* appear as question on the current HLC examination where the practice is to

invariably choose test questions containing recurrences whose characteristic equations split over the rationals. This despite the almost ubiquitous presence of this recurrence in all manner of different contexts and despite the fact that one can use the quadratic formula to determine the two distinct real roots of its quadratic to obtain its solution basis.

This is done in the core and yields the *de Moivre-Binet* formula for the general term of the Fibonacci sequence as a function of the golden ratio ϕ and its irrational conjugate. It is remarked that students find it quite remarkable that this formula, specified as it is in terms of irrational quantities, yields an integer value for every natural number. However, it is also stated that this celebrated formula is not ideally suited for computation and is often rendered in a form more amenable to this purpose.

Tiling a $2 \times n$ Hallway: The second example of a second-order recurrence relation presented differs from the one just treated in that its associated characteristic equation is easily factorised over the rationals. As such it would accord with the intentions of the current syllabus and might be made the basis of a future examination question on the understanding that the specific example would be explicitly mentioned in the syllabus segment.

It is treated at some length but teaches essentially the same lessons as the problem just discussed. Initially it affords the student plenty of room for careful exploration leading to the recognition of a certain pattern which results in a conjectured recurrence. This is followed by using the recurrence

and then the search for its proof. In addition, the problem has the virtue of being easily solvable by the classical method to yield an amenable general formula which is easily checked.

It also has an added feature, the elucidation of which closes out the discussion; that one can guess a first-order recurrence relation governing the enumeration but without being able to justify it. This open-ended element is something the student has already experienced in connection with the last first-order recurrence treated in this Topic.

By contrast with the treatment of the previous problem, the different stages one goes through when formulating and solving this second-order recurrence are clearly sign-posted and are synopsised in the following list:

1. Conjecturing the Recurrence: this stage involves exploration, pattern recognition, active participation and (one would hope) enjoyment.

2. Using the Recurrence: prior to justifying a recurrence one often simply assumes its truth and uses it to generate values from the initial condition to see if they accord with those obtained by direct enumeration and also to generate a few further values.

3. Justifying the Recurrence: is the 'thinking man's part' of the whole undertaking. While it is often the most difficult, it can at the same time be the most rewarding for those who find that understanding is the ultimate satisfaction. It is also the area where the Communications standard can best be addressed.

4. Solving the Recurrence: at the moment this is the chief focus of HLC examination questions where the method of solution is taken as a given and is simply applied. While it is an essential element, it is sterile if not integrated with the other aspects of recurrence relations.

5. Checking the General Solution: A crucial step in any form of problem solving.

In conclusion, it is recommended that a number of the recurrences given in the following list be part of the discussion of second-order recurrence relations at senior cycle level.

Second-Order Recurrences:

1. Tiling the $2 \times n$ Hallway:

$$t(n) = t(n-1) + 2t(n-2) \quad ; \quad t(1) = 3,\ t(2) = 5$$

2. The Fibonacci Sequence:

$$f(n) = f(n-1) + f(n-2) \quad ; \quad f(1) = 1,\ f(1) = 1$$

3. The Pell Sequence:

$$p(n) = 2p(n-1) + p(n-2) \quad ; \quad p(0) = 1,\ p(1) = 1$$

4. Derangements:

$$d(n) = (n-1)[d(n-1) + d(n-2)] \quad ; \quad d(1) = 0,\ d(2) = 1$$

`Remarks:` The first two recurrences have been discussed in detail here and are treated extensively in the core.

The third recurrence concerns the Pell sequence and arises in connection with rational approximations to $\sqrt{2}$ and the A series of paper. The closed form for $p(n)$, obtained by solving the recurrence, is given by

$$p(n) = \frac{1}{2}\left\{(1+\sqrt{2})^n + (1-\sqrt{2})^n\right\}$$

and so is similar to that for the Fibonacci numbers $f(n)$, in that it is specified in terms of irrational quantities and yet yields an integer value for every natural number.

In the fourth recurrence, the quantity $d(n)$ gives the number of different ways that n distinct objects can be rearranged among themselves so that *no* object remains in its original place. Such rearrangements are termed 'derangements'. This recurrence differs from the three others listed in that it contains *variable* coefficients, a circumstance which causes it to be insoluble by the classical method. However, one can verify that

$$d(n) = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right]$$

does indeed satisfy the recurrence.

Perhaps a more enjoyable exercise involving a simple exploration is to use either the recurrence or the formula for $d(n)$ to determine $d(3)$, $d(4)$ and $d(5)$ and then to confirm the numbers obtained, by directly counting the different permutations of ABC, $ABCD$ and $ABCDE$ respectively, where no letter is in its lexicographic place.

6.3 The Newton-Raphson Algorithm

The purpose of this section is to discuss the rationale for the Topic on the Newton-Raphson algorithm and to indicate the substance of that core piece.

The last entry in the section `Functions and Calculus` of the HLC syllabus reads[29]

> Newton-Raphson method for finding approximate roots of cubic-equations.

Since the introduction of the new HLC Mathematics course in 1994 and the year 2003, the material relating to this entry has been examined on six separate occasions. The questions asked can be found in Appendix G and are the source of the following comments:

1. With the exception of 1995 all the equations are cubics as stated in the syllabus. The equation of 1995 is the quadratic $x^2 - 2 = 0$, whose positive solution is $\sqrt{2}$. From a higher and historical viewpoint this question can be seen as asking that the Newton-Raphson method be used to derive the ancient formula of Hero for approximating the positive square root of 2. The question then asks that this formula be applied with the initial approximation of $x_0 = 1$ to determine the next approximation to three decimal places.

   It may have been more attractive to follow the practice adopted in

[29]Department of Education, *The Leaving Certificate Mathematics Syllabus (Higher Level and Ordinary Level)* (Dublin: Government Publications Sales Office, n.d.), 10.

some of the questions of later years where the desired approximation was to be expressed in fractional form and, perhaps, to then ask the students to comment on the merits of the approximation. In this case, the second approximation $7/5$ is such that $(7/5)^2 - 2 = -(1/50)$. Since $1/50$ is 'quite close' to 0, one might regard the approximation $1 \cdot 4$ as 'reasonably good'.

It should be remarked that in all subsequent years the initial or first approximation is always denoted x_1 and not x_0 as in this example.

2. In all six questions candidates are given an initial approximation rather than having to search for one themselves. One can understand the reason for this from the point of view of correcting papers, but this practice just adds another unrealistic element to questions already lacking in explicit relevance. In practice this is an important and often contrary aspect of the Newton-Raphson method.

3. With the exception of 1995 students are expected to know the statement of the algorithm though they have never been asked to derive it.

4. With the possible exception of 1998, the questions are all the same, being no more than procedural.

5. There is not a single instance of the given cubic being set in a context.

Textbook Provision: Under the heading `The Newton-Raphson Method of Approximating Roots` one of the current HLC texts begins:[30]

> In Chapter 2 we saw how to find the roots of cubic equations, if one of the roots is an integer. We now require a method of finding roots of cubic equations, if **none** of the roots is an integer.

This is the sum total of the author's motivation of the topic, which to be fair, is no worse than the unelaborated entry of the syllabus to which it is catering. The author does not explicitly state at this juncture, nor subsequently, that if a cubic equation does not possess an integer root then it is generally more difficult to find its roots. After establishing the algorithm with the aid of a diagram along standard lines with, however, no comment on the very fruitful blending of coordinate-geometry and the calculus, the author presents two examples of the method applied to cubic equations plucked out of thin air.

`Example 1: Find a positive root of` $2x^3-3x^2-12x-10=0$, `correct to two significant figures.`

`Example 2: Find a negative root of` $2x^3-3x^2+7=0$ `to the nearest integer. Use two iterations of the Newton-Raphson method to find this root correct to two decimal places.`

`Prove that this is the only real root.`

The treatment of these examples is followed by fourteen context-free exercises. However, Q2(c) of the Revision exercises to be found immediately after

[30] O. Murphy, *Discovering Maths 4, Leaving Certificate Higher Level*, 342.

these exercises is given a context and is accompanied by a diagram. It reads:[31]

A milk carton has a square base. Its height is 20 cm greater than the side of the base. The volume of the carton is 222 cm^3.
If $x =$ the length of the side of the base, show that

$$x^3 + 20x^2 - 222 = 0$$

Show that the solution to this equation lies between 3 and 4.
Taking $x_1 = 3$ as the first approximation, find two more approximations correct to three decimal places.

While it is heartening to see such a question, one might like to see more of this kind among the exercises with at least one being treated within the exposition as a full-bodied example of mathematics in action.

Another of the texts begins its treatment of *Solving Cubic Equations Using The Newton-Raphson method* by discussing how one can locate the presence of roots by 'change of sign' and by 'explaining' graphically that a cubic equation has either one real root or three real roots. It then states (without proof or the aid of a diagram) the NR algorithm and illustrates it by finding an approximation of the root of the cubic equation $1 - 3x - x^3 = 0$ which lies in the interval $[0, 1]$. Unfortunately, it does not verify that the result found does indeed furnish an (approximate) solution of the given equation. This example is followed by another purely technical one, which in reality is no

[31] Ibid., 333.

more than an algebraic exercise. All of the seventeen exercises that follow are of the same technical ilk as the two examples discussed in the text.[32]

A third text begins in a similar manner to that just described by first dealing with the nature and approximate location of the roots of a cubic equation before treating the NR method. The rationale for the method is explained in a more lengthy manner than the two previous texts and is followed by a derivation of the algorithm which is then illustrated by two technical examples. The section concludes with sixteen technical exercises.[33]

The fourth text charts the same course as that of the treatment just described in relation to the roots of a cubic. It then states the NR algorithm, illustrates it by finding one root of the cubic $2x^3+x-15=0$, before devoting a page to its derivation. Then ten purely technical exercises are provided.[34]

It should be said that all four of the texts use a number of nicely drawn graphs to illustrate different points raised in the technical exposition.

In summary one can say that, with the exception of the one applied exercise alluded to in reference to the first text, all of the texts illustrate the

[32]G. Humphreys, *New Concise Maths 4 for Leaving Cert Level*, 179-183.
[33]O. D. Morris, *Texts and Tests 4, Mathematics Higher Level*, 323-330.
[34]A. Roantree, *Leaving Certificate Maths, Mathematics for Higher Level Students*, 290-295.

NR method solely by way of solving unmotivated context-free cubics.

One might add a criticism concerning a failure to reveal something of the history and culture of mathematics: nowhere is it stated in any of the texts that Italian mathematicians of the sixteenth century found a general formula for solving all cubic equations which is similar to the quadratic formula, in that it too requires no more than the four basic operations of arithmetic and the extraction of roots for its implementation. While this should be related as an historical fact, it is also essential to indicate that the striving after general all-purpose formulae is an intrinsic part of the 'culture' of mathematics. This done, one can state that the 'cubic formula' is, unfortunately, a computationally impractical method which is one reason why numerical methods, such as the NR algorithm, are actually the preferred choice in practice for solving cubics.

By way of elaboration the student might be told that the application of the 'Italian formula' often involves considerably more labour than that required for the quadratic formula. One fundamental reason concerns the case where the cubic equation has three real roots; in this instance it is known that the cubic formula cannot yield these solutions without recourse to the use of complex numbers. As Turnbull relates, "To mathematicians of the sixteenth and seventeenth centuries this feature was very mysterious: they spoke of the *irreducible case*."[35] Because of this need to go outside the

[35]H. W. Turnbull, *Theory of Equations* (London: Oliver & Boyd, 1944), 123.

domain of real numbers and the approximations inherent in the practical extraction of cubed roots, it can often be easier and quicker to simply opt at the outset for a numerical method which will yield an approximation of the desired solution of the cubic equation to any prescribed number of decimal places of accuracy. It is suggested that what has just been described should form part of the syllabus segment designed to show how a little of the history of mathematics can be used to appraise the student of significant mathematical results.

Neither is there any historical explanation as to why Newton and Raphson are associated with this algorithm. One can well imagine a bright inquisitive student wishing to know why there are two names associated with the method along with wanting to know the contribution of each person. The same student might also be curious as to what kind of problem motivated Newton to develop the method such as it was in his time. In fact, by way of providing a modicum of motivation for studying the example $x^3 - 2x - 5 = 0$, the teacher might relate that Newton considered this particular equation in 1671 (although his method of solution did not appear in print until 1736). In 1690 Joseph Raphson (1648-1715), while still a student at Cambridge, published his *Analysis aequationum universalis* in which he describes a method for approximating roots which is similar to that of Newton. The teacher might add that Newton's actual calculations of 1671, while not appearing to have quite the same form as those obtained from an application of the NR method, are, when examined, seen to be essentially the same as those produced by

this algorithm.[36]

Rationale and Nature of the Topic: By contrast with the provision in the current texts and the nature of the questions being asked at HLC level, the first purpose of the 'NR Topic' is to reveal a number of contexts in which cubic equations arise.

As stated in the core piece, one obvious place to seek contexts for cubic equations is in connection with problems relating to volume because of the *three* dimensional nature of space. One such problem (already suggested in the core Topic on Quadratics) relating to `Constructing an Open-Box` acts as the introductory example. A simple analysis of the problem leads quite naturally to the cubic equation $x^3 - 14x^2 + 49x - 49 = 0$.

At this stage, the core treatment recommends a course of action which may well be the practice of some or all teachers in the classroom, but which is not explicitly spelt out in any of the texts in the manner to be described presently by way of guiding the teacher. Specifically it is suggested that once the teacher has one cubic, such as the one just revealed, that both the teacher and students spend some time exploring how one might find the roots of this equation, in particular, a root which is germaine to the question being asked. In this regard the teacher might use the context of the problem in conjunction with the technique of locating the presence of roots by 'change of sign', to determine the integers between which this root resides.

[36]E. Hairer& G. Wanner, *Analysis by its History* (USA: Springer Verlag, 1996), 95.

This important exercise reveals to the student that finding the roots of this particular cubic is not a simple matter, certainly not as straightforward as would be the case if the equation were one degree lower, that is, a quadratic equation, when the general quadratic formula comes to the rescue. In this way the teacher can have his students appreciate that even though a cubic equation is just one degree higher than that of a quadratic, the general task of solving such an equation is considerably harder. Should a student ask if a general formula, such as the quadratic formula, exists for solving all cubics, then the teacher can avail of this opportunity to discuss, along lines indicated above, that while there is such a formula it is not considered hugely practical for computational purposes although its discovery was a milestone in the history of mathematics. He then may add that, fortunately, there is a method, known as the Newton-Raphson algorithm, by which cubics (and other equations) can be solved numerically. This will be derived once a number of other equations, equally in need of numerical solution, have been obtained from the consideration of problems in context.

With this exploration/discussion behind him, the teacher will have fully revealed the two-fold purpose of this example which is to show the student

(i) why one needs to be able to solve a cubic equation

and

(ii) that finding the desired root is a task best undertaken using a numerical method.

The exercise which follows this particular example provides another context where a cubic equation arises in relation to another problem concerning volume. The first part of the question is an optimisation problem solvable using calculus which, besides being asked for its own sake, is used to show that the second part of the problem (giving rise to the cubic equation) does indeed have a solution.

The next example presented shows how cubic equations arise in connection with the evaluation of the trigonometric functions using the triple-angle formula. Specifically, it shows how to determine the values of the trigonometric functions for the angle $20°$. While many students may find it hard to regard this particular problem as a practical concern in today's world when the values of the trigonometric functions are readily available, either in table form or by immediate recourse to a calculator, the teacher might point out that such a problem was of central importance in former times when navigators and others were in dire need of accurate values for the trigonometric ratios. This general point is also made in the Topic on Quadratics where the double-angle formula is used to derive quadratics whose roots also yield the values of trigonometric ratios. One might add at this juncture, both for the teacher and the student, that by combining the double- and triple-angle formulae the early mathematicians were able to obtain many other values of the trigonometric ratios. Elaborating on how exactly this is achieved makes for a valuable student exercise testing the ability to communicate mathematical ideas in writing as opposed to performing specific calculations. It has been

mentioned a number of times that this valuable skill is not examined in any meaningful way by the current tests.

The next problem revealing a cubic equation arising in a context concerns the precise calibration of a dipstick for a spherical tank. It is one which most students grant is 'reasonably sensible'. The example confines itself to finding only one single calibration mark, that corresponding to a one-quarter-full tank, as a subsequent problem treats a similar calibration problem in complete generality. It requires either stating or deriving a result with the aid of integral calculus relating to the volume of a spherical cap. As stated in the core treatment "since the recommendation of this work is to present a small number of full-bodied problems in the classroom rather than a host of unmotivated and context-free fragments, the result stated is derived here as an 'application' of the integral calculus." Once this has been done, the student is then shown how the calibration problem is equivalent to solving a cubic equation. Besides displaying another instance of the need to solve a cubic equation, a second virtue of the particular problem is that it also affords the teacher a fresh context in which to show the student how natural it is to formulate such an equation in terms of a dimensionless variable.

This discussion on producing cubics in context concludes by briefly describing three other problems, related to the extraction of cubed roots, the geometrical construction of a regular heptagon and a chemical process, each of whose solutions requires the solving of a corresponding cubic equation.

The second intention of this Topic is to broaden the application of the NR algorithm to equations other than cubics which arise in the study of applied problems. This is begun with a short subsection outlining three problems whose solutions require the solving of a fourth degree polynomial equation or 'quartic'. Its sole purpose is simply to impress upon the student that polynomial equations other than cubics do arise and are also best solved using a numerical method such as the NR algorithm. It is also indicated that the final part of the Topic considers the problem of extracting square-roots, so that all-in-all the student sees the NR algorithm being applied to polynomial equations of degrees two, three and four.

Then the discussion proceeds to show that there are applied problems whose analysis lead to equations other than polynomial equations; non-polynomial or transcendental equations as they are termed. It is explained how it is often the case that the only way of solving these equations is using a root-finding algorithm such as the NR method. Problems relating to circular regions almost invariably lead to equations involving trigonometric functions which, more often than not, require a numerical method for their solution. The first problem of this type presented involves dissecting a circle in a given ratio, and is an exercise which might be of practical concern to a machinist or a carpenter. As stated in the core treatment, "Despite its apparent simplicity this is a problem which requires a degree of skill to solve. It serves as an ideal example for the teacher to show how to 'mathematise'

a problem", a term already described at the end of chapter 4.[37]

This example is followed by a description of a problem which has been doing the rounds for generations under the guise of a puzzle, which in fact it is not, since it cannot be solved by elementary means as one would expect, but requires the formulation and solution of a trigonometric equation. The relevant equation is presented to the student as the subsequent answer may be of some interest. The final offering is an optimisation problem which is easy to formulate and whose solution also involves solving a trigonometric equation only amenable to solution via numerical methods. Thus in all the student is shown three transcendental equations. These, and those derived earlier, constitute the essence of the first part of the Topic devoted to motivating.

Stage II: The second portion of the Topic is concerned with theory which, in this case, means the derivation of the NR algorithm. Since this is done in each of the texts, the treatment here has little that is new to offer outside of recommending a greater emphasis on the key idea involved and the use of a slightly more compact notation. The reason for recommending that stress be placed on the derivation of the NR algorithm is because the actual derivation itself is a showpiece of mathematical thinking. It reveals to the young student how a powerful tool, such as this one, is forged by interrelating concepts from two different areas of mathematics. In this instance, the areas are coordinate geometry and differential calculus and the blend is that of

[37]See pages 193-194.

equating the expression for the slope of a tangent line as given by coordinate geometry with that provided by calculus. The result is an efficient root-finding algorithm much used in modern numerical analysis. To ensure that this important derivation be taught in the classroom, it is suggested that it occasionally appear as part of a full-bodied examination question built around the NR algorithm.

In addition to expounding on the efficacy of a more compact notation, the treatment also provdes a little history of the origins of the algorithm by way of explaining why it subsequently became so-named.

Stage III: This final section begins by solving a number of the equations already derived in the first section where the task was to show how these equations arise in context. Since the current textbooks already provide a number of examples of the NR method solving context-free cubics, the treatment here solves only the cubic equation connected with the one-quarter-full spherical tank because of its intrinsic interest. This done, a number of the transcendental equations are solved before the problem which is considered the centrepiece of this final part on applications is discussed at length.

It will be seen from the treatment of the problems in this section that considerable emphasis is placed on showing (the student) how to acquire the essential skill of determining for oneself an initial approximation to the desired root of the equation. This contrasts with current practice where, as

already mentioned, an initial estimate is often provided. The task of having to find how to kick-start the algorithm externally highlights the somewhat negative feature of the method, which is that it is not self-starting. This particular drawback can have implications for the practical implementation of the NR method and is dealt with at some length in the primary substantial example of this section. Since the locating of roots by the change of sign method is illustrated in the texts, the exposition here describes two other approaches; where one either uses a physical understanding of the problem, or a graphical interpretation of the governing equation. The examples reveal how fruitful each of these methods can be in certain circumstances.

Although it is not done here, any discussion of the NR algorithm should, for the sake of completeness, mention, if not illustrate by concrete examples, that not alone must care be exercised in choosing an initial approximation (so as to guarantee convergence to the root being sought), but that on occassion the method may not work.

After revealing the power of the NR method on some of the 'original' equations, the algorithm is harnessed to find the positive square root of a positive real number because it yields the ancient fomula of Hero discussed in the Topic on Quadratics. As such it is an example providing another instance of an inter-relationship between different areas of mathematics. This discussion has an aside for the teacher explaining why the NR algorithm converges quadratically, the characteristic which explains its great efficiency.

This is followed by showing how the NR method can be used to find

the *reciprocal* of a non-zero number A, to any degree of accuracy "without using any divisions" – a concept which many students find intriguing. The fact that this can be done is of some importance in practical computation where multiplications require considerably less time to execute than divisions because it influences the design of circuits which implement such operations. The problem also shows how a different version of an equation can prove more fruitful than another in relation to the iteration it produces.

The final problem presented is a gem among gems both for its intrinsic interest and its broad educational value. Its rather full discussion forms the remaining portion of the Topic. It concerns the calibration of a dipstick for a cylindrical tank whose principal axis is parallel to the horizontal. The student is reminded that central heating tanks and the diesel tanks on lorries can be of this type, and that it is very useful to have a dipstick (or equivalent instrument) which, when dipped into the tank through an aperture at its top, gives a reading indicating the *volume* of liquid remaining in the tank.

Students have little difficulty in understanding this problem and regard it as highly practical. The core treatment reveals that the solving of this realistic problem demonstrates very forcefully the power of mathematical modelling implemented with the aid of a computer. While this full-bodied problem has many different aspects, all of which are accessible and highly instructional for students, its analysis involves no more mathematically than knowing

- radian measure
- the formula for the area of a sector of a circle
- how to express the area and height of a triangle in terms of the elementary trigonometric functions.

Thus one of its lessons is that of showing the student how basic knowledge is essential in solving a practical problem.

Once the problem has been described and understood, it is suggested that the teacher begin his discussion of it by first asking for physical solutions which use a minimum of mathematics – a 'farm-yard' approach which would be satisfactory within certain 'liberal' tolerances. He then might ask for reasons or scenarios where higher precision might be required and have the students appreciate that such ambitions are best achieved via a mathematical treatment. Not all will agree but most will want to see how it is possible to calibrate a stick for such a tank giving the corresponding marks for volumes expressed to any desired degree of accuracy, perhaps multiples of one-hundredth part of full volume or one-thousandth part or even smaller gradations.

During this exchange of ideas the teacher should also ensure that his students see for themselves that the length of the tank is not fundamental to the problem, and that its true nature concerns no more than the geometry of a circle. Furthermore, he should arrange matters so that some, if not all,

of his students make the crucial time-saving observation which is that if one can find calibration marks corresponding to volumes less than one-half a full tank then one can automatically calibrate for the complementary volume. This simple observation on how to get two calibrations for the work required for one is sure to prove gratifying.

Before embarking on an analysis of the problem the teacher should emphasise the importance of the preliminary exchange of general ideas which has just taken place by pointing out to his students that while they may not yet have used either arithmetic or agebra explicitly, their insights and observations constitute genuine mathematical thinking – an essential skill to be valued and not to be dismissed as 'just talk'.

The concrete examination of the problem begins with the specific question: "If the tank is a quarter full, what is the corresponding height?" This height may be expressed in terms of the radius or as a fraction of the diameter – a point which arises later in the detail of the solution. The teacher should ask each student to make an educated guess based on their physical intuition as to what this height is, with a promise to compare their estimates with the true height once it has been calculated. He might remark that this kind of educated guessing is part and parcel of any scientific work and that such estimates serve as a check on the actual value obtained, since having some appreciation of the correct magnitude saves one from accepting unthinkingly a value which is false. He might also mention that tackling a specific problem often points the way to tackle a problem in complete generality, and that one

should not be overly worried if one cannot immediately see how to solve a general problem in its entirety. The lesson is not to rush at a problem but to take one's time to unravel it bit by bit, and according as one's understanding of it grows, so too will the ideas on how to subdue it.

It is at this stage that the mathematisation begins and it is where the student has another opportunity to practice the important art of imposing a mathematical framework on a problem. This start-up step where the teacher must lead the students to introduce a variable (or parameter) to express the varying volume and height should be taken patiently even if it requires the same type of formulation skill taught in some of the earlier examples relating to the circle. This analysis leads to another 'trigonometric equation' which is not amenable to solution using the familiar trigonometric identities. Consequently it is another example of an equation requiring a numerical method, such as the NR algorithm, to find its solution and one which is crucial to providing an accurate calibration. An historical aside informs the student that Newton encountered a similar equation in 1687 in relation to a problem on planetary motion.

Rather than rushing headlong into solving the equation just obtained, the treatment urges that both teacher and students 'talk about the equation', that is, examine it to see if has any insights to offer. The important notion that an equation can be very informative even when unsolved is rarely dwelt upon by the texts where the focus is on finding its solutions. In this particu-

lar case, the equation reveals that the analysis of the underlying problem is independent of both the radius and length of the actual tank under consideration – a striking feature which the student is invited to comment on by way of describing possible implications. This exercise in simply reflecting on what the equation has to say is another of the valuable lessons this problem has to teach.

With the equation in place and the need for an iterative procedure made clear, the treatment derives the relevant iteration based on the NR method before tackling the problem of choosing an appropriate initial approximation so necessary to start the algorithm. Here the student is challenged to think about different ways to make an educated guess at the value of the desired root. As the discussion then remarks, "One way is to return to the underlying practical problem and examine it while another is to think about the equation $f(x) = 0$, or an equivalent form of it, from a graphical perspective. Whichever approach is taken, it calls for active mathematical thinking."

In this example the physical approach is perhaps the more appealing route, because in seeking an estimate as to the calibration mark corresponding to a quarter-full tank, almost all students will note that the height of the liquid will be more than half the radius of the tank and may estimate the height at about 60% of the radius. With this observation they readily acquire an initial approximation, the worth of which can be gauged by simply checking the equation. Thus with no more than a 'common sense' approach based on an understanding of the problem the student is able to activate

the algorithm and obtain the long-sought for calibration mark. Then the students can compare their initial guesses with the 'true' calibration mark obtained by mathematical analysis.

However, care is taken to show the students a graphical determination of the initial approximation. This is based on a simple but clever way of re-formulating the governing equation so that it can be interpreted as asking where two easily drawn graphs intersect. Having an alternative method such as this one at hand to provide an initial estimate can prove invaluable, particularly in situations where the physics of the problem may not be that evident.

At this juncture, mention is made of how the NR algorithm is implemented in practice using any of the powerful mathematical packages now available. The student is informed that it requires no more than a single line of code, an example of which is shown expressed in *Mathematica* along with the first six successive approximations to the root of the equation produced in under one-tenth of a second. For students who have gone through the labour of working out successive approximations by hand, this production of the same output in a fraction of a second in response to one single command impresses upon them how truly powerful a mathematical package can be. A number of other simple *Mathematica* commands are elaborated upon to show what they can achieve. In a framework which accommodates the use of mathematical programming as part of the mathematics programme, a class could be devoted to implementing the NR algorithm in this manner for this

particular calibration. Such an interaction with technology might well have some students keen to know how to have the computer effect a complete calibration.

How in fact this is done, constitutes the second and final stage of this calibration problem and is treated under the heading "The Need for a Computer". Having dealt with the specific case of determining the calibration mark corresponding to a quarter-full tank, it is natural to want to tackle, or at least learn in theory how to handle, the general problem of determining a complete set of calibration marks. After all, the job isn't finished until the stick is fully calibrated. Most students will grant that this is a genuinely realistic expectation and one worthy of achievement. However, such an ambition brings its own problems, both by dint of labour and mathematically.

Since each gradation corresponds to a different volume, it becomes clear that the Newton-Raphson algorithm must be implemented many times with slight modifications at each execution. The student will now appreciate that for practical purposes one must enlist the help of a computer. Whereas one might countenance implementing the NR algorithm by hand for one case, it is obviously out of the question to consider doing so fifty times. In any event, why would one make such an attempt when there are computers available which carry out such repetitive tasks quite easily, provided they are given the correct instructions.

Enlisting the help of the computer brings with it a 'technical mathematical problem' which threatens the whole endeavour. The problem is the

practicality of how to feed the the NR algorithm appropriate initial approximations at each calibration stage. This difficulty highlights in no uncetain way for students the implications of the NR method not being self-starting. The details are elaborated upon in the treatment where it is recounted that those students engaged by the problem invariably come up with a 'commonsense' suggestion which removes all obstacles and allows for a complete calibration. Again, it has to be stressed to the students that the proposed solution reveals mathematical thought of a high level even if it doesn't involve actual calculation.

With an understanding of how to see the project through, the general calibration equation is formulated and involves a *paramater* indicating the gradation stage. The teacher should convince the student that this general equation is no more complicated than the previous equation simply because it contains a parameter. This can be done by pointing out that once a value is specified for the parameter, the resulting equation is similar to that corresponding to a one-quarter full tank. Thus another important lesson taught by this general problem is that of showing the need for a parameter in a completely natural manner.

The single line of *Mathematica* code which allows the NR method to solve the general equation as the parameter runs through its domain of values is presented followed by the calibration table it produces. In this way, the student sees a problem of substance solved in complete generality by means

of mathematics ably assisted by modern technology – a truly educational lesson.

The discussion concludes by synopsising the key stages in the overall process by which the general problem is solved. This entails translating the given practical problem via analysis to a mathematical one which, in turn, requires for its effective use the assistance of a computer. Then the mathematical output produced by this programme is interpreted in the context of the original problem to furnish the solution sought. As a consequence this particular problem offers a first-rate illustration of the *mathematisation cycle* described earlier in connection with the PISA03 perspective on the process of solving real-life problems.[38] It can be seen that each of the five steps of this cycle are more than amply revealed in the treatment of the calibration problem.

Conclusion: The over-arching difference between the treatment of the NR algorithm to be found here and current practice is the emphasis placed on discussing a number of substantial examples and exploiting them, whenever circumstances allow, to teach much more than just the procedural implementation of an algorithm. Even if none of the suggestions made here were ever to be incorporated in a future framework, it would still, in the opinion of this writer, be preferable to cover but *one* cubic equation arising from the study of an applied problem rather than to persist with the drill of solving a host

[38]See p. 196.

of spurious examples.

However, should it be the case that the NR algorithm were not to be granted enough space to treat it in the comprehensive manner recommended here, then it is strongly suggested that the calibration problem be made the primary application of the method. This is because of the manner in which the example affords a genuine illustration of how mathematical analysis and computer implementation dovetail and so make it of great educational value. Furthermore, the tackling and solving of this problem in its full generality, allows the student to participate in a substantial undertaking which results in complete closure.

Chapter 7

Recreational Mathematics

One of the envisaged scenarios described under the heading 'Vision for School Mathematics' in the *Principles and Standards for School Mathematics* published by the US National Council of Teachers of Mathematics in 2000, reads:[1]

> Teachers help students make, refine, and explore conjectures on the basis of evidence and use a variety of reasoning and proof techniques to confirm or disprove those conjectures.

Yet, nowhere throughout this extensive document is there any explicit reference to 'recreational mathematics' as a field of mathematical activity which could be harnessed in the endeavour described in the above quote. Of course, this is not to say that the drafters of PSSM do not forsee puzzles, games, tricks and the like being used whenever instruction allows them to be appropriately called upon. However, outside of one reference to deciding whether a certain game of chance is fair, the document places no explicit

[1]National Council of Teachers of Mathematics (NCTM), *Principles and Standards for School Mathematics.* (USA: NCTM, April, 2000, x.

emphasis on analysing particular puzzles, games and tricks for the purposes of teaching school mathematics.

Likewise, the 2005 report *International Trends in Post-Primary Education* commissioned by the Irish National Council for Curriculum and Assessment does not explicitly mention 'recreational mathematics' nor does it refer to the use of puzzles and games in any context whatsoever.[2]

On the other hand, while the discussion paper *Review of Mathematics in Post-Primary Education* reports (as already quoted): "Children found mathematical games, puzzles and interesting problems a good motivational influence in their mathematics learning", it does not make any recommendation that RM be used as an explicit means of making contact with secondary school mathematics.[3] However, in relation to 'investigations' it relates that "in England & Wales and Northern Ireland, emphasis was placed in the 1990s on '*investigation*' more than on problem solving" but notes, some pages later, with respect to Ireland, that "in the absence of course work, there is little opportunity or encouragement for students or teachers to engage in a more extended investigation of any one area of mathematics."[4]

Thus it would appear, as of 2005, that there is no realisation or formal recognition by US or Irish educators of recreational mathematics as a possible area of mathematical activity in its own right. Perhaps, the vision of

[2]Paul F. Conway & Finbarr C. Sloane, *International Trends in Post-Primary Education.* (Dublin: NCCA, 2005)

[3]National Council for Curriculum and Assessment (NCCA), *Review of Mathematics in Post-Primary Education: Discussion Paper.* (Dublin: NCCA, October, 2005), 3.

[4]Ibid., 5 & 12.

RM and the extent of the coverage given in what follows may induce educators to view RM (as it relates to school mathematics) as a distinct area of school mathematical activity which is rich both in terms of motivational and mathematical content.

The purpose of this chapter is to describe some areas of recreational mathematics which might find a place in a future three-year senior cycle programme. The rationale for the inclusion of the suggested items is also discussed in relation to the fully-developed examples to be found in the core section.

Martin Gardner has long advocated that recreational mathematics (RM) should be incorporated into the standard curriculum "as a way to interest young students in the wonders of mathematics." In his view, "... mathematics is considered recreational if it has a playful aspect that can be understood and appreciated by non-mathematicians." He expands by saying that "recreational mathematics includes elementary problems with elegant, and at times surprising, solutions. It also encompasses mind-bending paradoxes, ingenious games, bewildering magic tricks and topological curiosities such as Möbius bands and Klein bottles."[5]

As recently as 1998, Gardner complained with regard to the proposal that RM be used as part of the mathematics' curricula of the U.S. that, "So far, movement in this direction has been glacial." While it is still the case

[5]M. Gardner, "A Quarter Century of Recreational Mathematics" *Scientific American* (August 1998): 68-75.

that there is no RM element in the Irish curriculum by which a student can acquire credit towards his overall Leaving Certificate result, there has been a recognition by the DES for some time now that part of the transition year may be used for exploratory activities.

RM in the Irish Curriculum: A pilot study carried out in 1994 on the use of 'Investigational Mathematics' in Transition Year has already been referred to in this work on a number of occasions.[6] In the abstract to this work the author writes:

> Doing Investigational Mathematics is of benefit to students both intellectually and socially. Working in an investigational mode can create difficulties for some, but not all, teachers; these difficulties can be overcome with experience and the provision of good resource materials. Finally, because of the creative and personal nature of this work it has a definite place as part of a Mathematics programme in Transition year.

Mc Namara reports that Investigational Mathematics was introduced into the curriculum of the United Kingdom in 1988 "in an effort to encourage activities within the classroom which would develop mathematical thinking." However, such 'coursework' as it became known was not made compulsory until 1991. Under the heading `Students and Investigations` she elaborated in depth upon the benefits of 'Mathematical Investigations' (MIs) as gleaned from her own investigation and her study of the English experience. These may be summarised as follows:

[6] A. Mc Namara, "Investigational Mathematics in Transition Year – A Pilot Project", (M. Ed. diss., Dublin University, 1994)

1. Many students enjoy MIs.

2. MIs encourage students to communicate mathematical ideas. As Jennings notes:[7]

> Communication has been an important part of doing mathematics. For some pupils the experience (doing coursework) has been one of motivation and enjoyment, ... The activities have been very enjoyable and the classroom atmosphere has been very relaxed. It has given some pupils a sense of achievement.

3. According to Tanner, "The major affective change, which is arguably the most important effect that introduction of investigations has had, is to develop a belief, in many students, that they can solve problems which initially appear beyond them."[8]

4. One GCSE student found MIs an "incentive to work harder, a better way of teaching", which gave him a "far better understanding" and "a good platform to build on" for his future.

However, as Mc Namara is careful to point out in relation to the GCSE coursework, the mode of teaching mathematics via MIs was not fully embraced initially. She notes that from 1988 to 1990 teachers preferred, by a margin of 10:1, syllabi without coursework so long as the choice remained open. She also states that 'right wing elements' in the Schools Examination and Assement Council saw to it that coursework carried only 20% as

[7]Jennings, "A review of Coursework in Mathematics at GCSE", *Teaching Mathematics and its Applications* 11, no. 2 (1992): 64-65. as cited in Mc Namara.

[8]Tanner, in *Introducing Mathematics* as cited by Mc Namara.

opposed to the recommended 40%. Over time it was found that because of the influence of assessment, the potential of MIs as a means of developing mathematical thinking is not always fully realised — "the awarding of too little marks is stifling the enterprise."

She also reports on the concerns of a number of commentators about the level of real mathematical content generated by MIs and the nature of the mathematics involved. As early as 1987, she relates that Coldman and Sheperd were very critical of the GCSE syllabus, considering that it had reduced mathematics to an "empirical science" resulting in the loss of rigour normally associated with the subject.[9] She remarks that Wolf noted that while the National Curriculum advocated the importance of practical mathematics, this area often generates little mathematical content. Wolf suggested that the solution is to find better investigations.[10] Mc Namara also relates that Thomas worried that students generalise from a very small amount of data, believe that generalistion is proof, but most gravely, do not have the abilities or skills to construct a rigorous proof. Thomas points out that this whole area of mathematics, proof, is almost totally ignored and students arrive at third level "with no concept of what is meant by a proof'. This, Thomas claims, is a great disadvantage to those entering the field of computers where "[t]he ability to produce a precise rigorous proof

[9]C. Coldman & K. Sheperd, "Mathematics", *GCSE: An Examination*, edited by Jonna North (London: Claridge Press with Educational Research Trust, 1987), 63-83. as cited by Mc Namara.

[10]A. Wolf, "Testing Investigations" in *Mathematics Versus the National Curriculum*, editors. Paul Dowling and Richard Noss (London: Falmer Press, 1990), 137.

of a mathematical result is exactly analogous to working with a computer." Investigations should not be seen as a means of avoiding proof but rather as a way of introducing students to the ideas and concepts that they cover. Much will depend on teachers and their ability to make students aware of the necessity of proof.[11]

Even if RM has yet no official sanction as part of the Irish senior cycle syllabus, many teachers are using the transition year in the manner envisaged by Mc Namara. As has already been reported here, one teacher uses a five-module mathematics syllabus in this year which is recreational in character and "which the students really enjoy".[12]

Use of RM in this Work: While it is accepted that mathematical investigations are a worthy means by which people of all ages learn mathematics through exploration and discovery, they are perhaps best undertaken at those stages in a school programme when the time needed for their proper pursuit is not an issue.

Thus while MIs can find a place in a transition year, the inclusion of an MI component as part of the final two years of the senior cycle programme is likely to be regarded by most as neither possible nor wise. In light of this and the misgivings regarding genuine mathematical content outlined in the previous section, the focus of the RM used in this work is as a vehicle

[11]G. Thomas, "Some Dangers in the Use of Investigations in Mathematics Teaching", *Teaching Mathematics and its Applications* 11, no. 2 (1992) as cited by Mc Namara.
[12]See 126.

by which the teacher makes contact with mainstream school mathematics while simultaneously teaching the invaluable skills associated with problem solving. Nevertheless, that said, it is believed that much of the work to be described here can and should be made the bases of MIs to be undertaken in the first year of a three-year senior cycle programme.

Puzzle or Problem? In the discussion which follows the words 'puzzle' and 'problem' are often used interchangeably. While one of these words may be a more appropriate description than another of a particular example referred to in this work, one criterion which applies throughout to each of the examples presented is that its solution should be accessible with no more than a knowledge of *basic arithmetic* assisted by clear, and at times clever, reasoning. Any task which requires more than these skills is classified as a problem.

For some, the statement or solution of a puzzle should also contain some element of surprise. Wittgenstein writes:[13]

> Think of mathematical puzzles. They are framed because they surprise: that is their whole sense.

While this yardstick may not strictly apply to all of the offerings here, it certainly is the case that many have this feature either in their wording or the ingenious manner by which they are solved. Those problems which combine

[13]Ludwig Wittgenstein, *Remarks on the Foundations of Mathematics* (England: Blackwell Publishers, Oxford, 1978), 112.

accessibility with surprise are the ones most likely to appeal to students and to make a lasting impression.

7.1 Magic Rectangles

The first RM piece to be found in the core concerns a family of Magic Rectangles. It is one of the many themes chosen to exemplify the assertion that the playful thinking associated with a puzzle can very often lead to real mathematics. By being easy to state and solve, it is an ideal candidate with which to illustrate that appropriately chosen recreational problems can be made to lead to interesting and useful mathematics in a natural but informal manner.

The puzzle posed in its first incarnation is at a level where one would expect it to be solvable by ten-year-olds. In fact, the earlier portions of the exposition were written as if this were the case. Throughout, the discussion addresses the student directly, save on the odd occasion where it makes suggestions to a would-be teacher as to how some aspect of the material should be approached. By contrast with the passive voice adopted in the core pieces relating to Topics, heavy use is made of the first person plural.

There is a natural progression of level of difficulty within the treatment; the earlier sections could be used to teach children in the first and second years of secondary school and even earlier, while the first six sections should be well within the compass of Junior certificate students. With the exception of the final section, the whole exposition should be accessible to transition and fifth-year students. The last segment which completes the whole investigation

could serve as a project for a final-year student, particularly one adept at programming.

It is clear from a reading of the entire piece that it is replete with mathematical ideas and contains much actual mathematical content. Furthermore, it meets the concerns of those who worry that MIs do not teach students how to prove results. The essence of the first-half of the exposition is to show how a key idea gleaned from the study of a simple puzzle is pivotal in deriving in a natural but rigorous fashion a central result from the senior-cycle programme. This is that

$$1 + 2 + 3 + \cdots + (n-2) + (n-1) + n = \frac{n(n+1)}{2} \quad ; \quad (n \in \mathbb{N})$$

The second portion of the presentation is concerned with how this result finds 'application' in numerous different scenarios as well as playing a major role in providing a completely general result. Among the applications discussed are

1. The Twelve Barrels Problem – a puzzle whose solution uses the idea of *double-sequencing.*

2. The False Minting Machine – a variation of the previous puzzle which occurs in quality control and which exploits the technique of double-sequencing.

3. Games in a Chess Tournament – scheduling the games.

4. Telephone Lines — Connecting houses to one another

5. Handshakes — discusses the isomorphism of problems

6. Regions in the Plane — a geometrical problem which uses recursion

7. Rolling a Die – discusses the probabilistic notion of 'average value'

8. Hotel Keys — a further exploration of average value along with worst case scenario

9. Triangular Numbers — pattern recognition and proof

Those students who take the full journey are aware by its end that they have reached a point where it is possible to survey the problem from its humble beginning to its final destination.

A synopsis of the different aspects and lessons that the puzzle and its extensions provide, along with various mathematical questions and problems its study suggests, close out the discussion.

Synopsis: The study of this problem

1. first and foremost provides enjoyment and fun

2. allows younger children to practice simple addition in a playful way

3. requires experimentation and active thinking

4. cultivates self-confidence because it is easily solved by trial and error

5. is an ideal vehicle for fostering interaction between pupil and teacher

6. contains a key idea which has far reaching mathematical consequences

7. suggests further puzzles and mathematical problems

8. arouses curiosity

9. teaches generalisation.

In addition the treatment suggests various mathematical questions and problems which allow the teacher, or as the case may be, student to

1. discuss existence and uniqueness, two themes that pervade mathematical investigations

2. learn about *permutations* informally

3. introduce factorial notation

4. show a problem in different guises

5. discuss the notion of being unique 'up to a permutation'

6. pose a new problem based on an observation

7. prove that some ambitions may be unachievable

8. distinguish between a necessary condition and a sufficient one

9. discover in specific cases a simple rule for summing the first n numbers when n is an even natural number

10. surmise via examples that the rule may be generalised to include all cases

11. distinguish between a conjecture and a proof

12. see the limitations of a certain technique

13. teach a little mathematical history which should encourage every young person

14. reveal how simple and elegant mathematical 'ingenuity' can be

15. appreciate the power of a more general technique

16. show the need to introduce the letter n to stand for a natural number in order to achieve generality

17. introduce in a natural manner the important mathematical problem of adding up the first n natural numbers when n is *any* natural number

18. reveal the power of simple algebra

19. illustrate the power of the formula for the sum of the first n natural numbers by means of a number of attractive puzzle-like problems. The solutions of some of these problems have immediate and easily understood practical applications. They also show

    (a) how the visual aspect of a well-conceived diagram can facilitate clear thinking

(b) the use of combinatorial reasoning.

(c) that apparently different problems can be seen to be intrinsically the same once the correct correspondence is identified.

(d) the need to be able to reason probabilistically

(e) the importance of the average case.

(f) how geometrical patterns reveal mathematical facts visually

(g) how to sum a general arithmetic progression.

20. the tremendous power of mathematical reasoning to obtain a completely general result with the minimum of mathematical machinery

Further Explorations: The treatment here confines its attention to the case of $2 \times 2n$ rectangles since this has more than enough to offer. However, students often ask about the possibility of constructing 'larger' magic rectangles, for example those of dimensions $m \times n$, where m and n are each natural numbers greater than two. The general mathematical result is that such rectangles are indeed constructible whenever m and n have the same parity but not otherwise. In this author's experience the caution that such rectangles may be difficult to construct by trial and error does not deter in the least some adventurous spirits from searching for and finding such rectangles whose dimensions can be anywhere up to and including 5×7. It is a simple but useful exercise in mathematical programming for a student to write code which tests a candidate array for 'magicality' or otherwise.

When $m = n$ the rectangular array is a square, but to be deemed magical it must also satisfy the extra criteria that the two 'diagonal sums' also match the row- and column-sums. The next recreational theme to be discussed concerns the 'smallest' case of such squares; that of 3×3 magic squares constructed from the first nine natural numbers. This charming puzzle is believed to have originated in Ancient China around 2000 B.C. when one solution became known as the Lo Shu magic square because of its connection with an ancient legend of the same name.

7.2 Three by Three Magic Squares

Many numerically-based puzzles are ideal for teaching sequential analysis; the process whereby one studies the facts of a case as they are presented. By thinking about each one logically, taking care to find all the possible answers at each stage, and resisting the temptation to be distracted, an investigator can often, after a sequence of such steps, get to the heart of a problem quite quickly. As he or she continues to probe, lines of enquiry suggest themselves which, when pursued logically and methodically, allow him eventually to unravel the problem and reveal its solution. According to the NCTM this form of "systematic reasoning is a defining feature of mathematics."[14]

The three by three magic square puzzle, when harnessed properly, provides the teacher with a miniature example of a problem which allows him to

[14]National Council of Teachers of Mathematics, *Principles and Standards for School Mathematics* (USA: April 2000), 56.

instruct his students in this powerful mode of thinking. Because of its perceived pedagogical importance, the exposition of this problem to be found in the core is also accompanied by a comprehensive PowerPoint presentation.

While the core treatment tackles this puzzle in four different ways, it begins with a sequential investigation. Having been given the puzzle, the student is initially allowed simply to use 'trial and error' to try find a solution. This preliminary stage is perhaps the most natural way to become familiar with what is required and is a warm-up procedure adopted by most investigators.

However, in this example this approach can often prove frustrating unless a certain 'correct number' is found. In fact, such a negative outcome best suits the teacher's purposes in this case. In order that the study of this problem can be of maximum benefit to the student, it is preferable that he or she does not find a solution by trial and error. Besides highlighting the limitations of such a hit and miss strategy, failure in this regard allows the teacher to introduce the main theme which is that it is preferable for a would-be-solver to 'reason things out' almost from the very start.

This is what is done in this puzzle to great effect. It is explained that after many attempts to place the nine numbers in the nine cells of the 3×3 square, most students (prompted by their geometrical intuition) begin to realise that the central cell holds the key to solving the puzzle. It is at this stage that they pause to wonder what number (or numbers) can be placed there. Then either on their own, or with guidance from the teacher, they soon reason out

that there is only one possibility. The PowerPoint presentation devotes quite a number of slides to detailing the process by which they might do this.

The student is then told to take stock of what has been achieved; using no more than a little logic he or she now knows with certainty that the only chance of success is to place the middle number 5 in the central cell. It is impressed upon the student that this signals a major advance and that a valuable lesson may be learned which is that:

a little thinking can save a lot of computing

At this juncture the treatment also avails of the opportunity to distinguish between a *necessary* and a *sufficient* condition; points of logic on which students can be hazy. It is explained that while it is necessary to place the number 5 in the central cell to have any hope of success, doing this does not of itself guarantee success; it still remains to be seen if indeed it will be sufficient.

Before continuing the student is advised: "Rather than resume the trial and error approach, try to make further progress by doing some more thinking. Think of it as smart detective work." In the presentation the tack taken is to investigate whether there are restrictions on where the largest number may be placed. Again it is reasoned that the number 9 cannot be placed in a corner cell. The student is then reminded that two major facts are now known, obtained using no more than simple reasoning and the minimum of computation, and that as a consequence 'victory is in sight'. Again the stu-

dent is urged to resist any temptation to stop thinking and to do a little more detective work because it is clearly proving very worthwhile.

After opting on a central-side cell for the number 9, a short argument reveals where the numbers 4 and 2 are obliged to go. At this stage, where all the remaining numbers go is forced, and the square 'completes itself'.

The Best Is Yet To Come: Were one to go no further with the treatment of this puzzle, it would still be a wonderful lesson for students in how sometimes 'things can be reasoned out'. It shows the importance of standing back from a problem so as to think about it rather than rushing at it. While it may be great fun just to dive in and begin fiddling around with numbers; to make real inroads on a particular problem it is better to stop and reflect. It can be fantastic to see what pure thought alone can achieve.

However, as the student is told, the best is yet to come precisely as a result of having thought the problem through from beginning to end, without ever running off at any stage to finish it by plugging in numbers until a combination which works is found. By using such a trial-and-error approach the opportunity of seeing deeper into the problem would be missed. The student is then shown that with a little more thinking it is possible to say exactly how many three-by-three magic squares there are in all. Almost without realising it, the student finds that he or she is in a position to *prove* the 'little theorem' which says that there are exactly eight three-by-three magic squares – a wonderful result which comes as an unexpected bonus but

a just reward.

As already mentioned, Thomas expressed grave reservations that MIs were not being used to teach anything about proof.[15] As this example shows, this need not be the case; here the whole investigation can be seen as one elaborate exercise in how natural it can be to arrive at 'proof'.

In fact, a simple combinatorial argument explains why there are exactly eight squares. It is at this stage the treatment reveals that these eight magic squares are strictly speaking not independent of each other, since each can be obtained from any other one by either a rotation or a reflection. The PowerPoint presentation illustrates this point dynamically. Thus the student learns indirectly, but in a completely natural way, the fundamental geometrical fact that a square has eight symmetries. A short PowerPoint presentation has also been designed so that a teacher can exploit this segment on symmetries to show his students in a visually accessible manner that the composition of transformations is a *non-commutative* operation; a significant lesson in itself.

The discussion of this particular investigation concludes with a comparison of the abilities of humans and machines; in so doing further emphasising the power of pure thought. The student is told that the 3×3 puzzle exemplifies how, with reasoning, a multitude of possibilities can be narrowed down to just a few candidates. This problem could have been tackled at the outset

[15] See page 390.

by having a computer program generate all 362,880 possible 3×3 squares, test the eight row, column and diagonal sums of each square and select only those for which all eight sums are 15. Although this enormous calculation would indeed give the eight solutions obtained, it could never explain why only eight are found. This can only be answered by a human being thinking about the problem and experiencing the thrill of understanding why things are as they are.

7.2.1 Other Approaches

Often, having discovered the solution to a problem an investigator examines the answers obtained in greater depth. In so doing, he may come across clues which point to other ways in which the original problem can be attacked. The theme of discovering new approaches to solving a problem based on the hindsights provided by a previous solution is one which Freudenthal refers to as the 'didactical inversion'.[16] The second treatment of the 3×3 magic square illustrates this process to some extent.

On examination of each of the eight solutions obtained, it might be observed that in each case the 'corner' numbers are always even while the other cells contain the five odd numbers with the number 5 always occupying the central cell. Armed with these simple but important observations one might search for ways of showing why each of these facts have to be a logical necessity. The second portion of the written treatment (as well as the latter half

[16]Hans Freudenthal, *Mathematics as an Educational Task* (Dordrecht-Holland: D. Reidel Publishing Company), 1973. v.

of the corresponding PowerPoint presentation) does exactly this. It begins with the two observations:

1. there are exactly four *even* numbers, namely $2, 4, 6, 8$, among the first nine natural numbers 1 to 9.

2. if a sum of 15 is to be obtained by adding three natural numbers then one of the following combinations:

   (a) even, even, odd

   (b) odd, odd, odd

   describes the *parities* of the three numbers in question.

and informs the student that, while these observations may not have been noticed, of themselves they are enough to solve the original puzzle in a fresh manner. This is then done through a series of deductions, which when combined lead to a magic square. Furthermore, because this path also adheres to strict logic, it is obvious by the end of the investigation why there are exactly eight magic squares. Clearly the subsequent discussion follows the lines already described in the first treatment.

By way of summary the teacher is told that this second approach emulates the logico-deductive aspect of mathematics made famous by Euclid's *Elements*. Here the two key observations made at the outset play the role of axioms. Once elucidated, the solution (theorem) is then arrived at through a series of logical deductions.

Students should be invited to compare and contrast the two different approaches taken to solve this puzzle before being exposed to any new approaches to solving this problem.

The third approach which receives written description in this work (but without an accompanying PowerPoint presentation) is similar to the one just described. It initiates an attack on the problem by systematically listing all the different possible ways a triplet of three numbers can be chosen from the first nine natural numbers with sum 15. This arithmetic approach finds that there are exactly eight such triplets which, the exposition notes with interest, is the same as the number of magic conditions that must be met in order to construct a magic square.

Then an elementary statistical analysis is carried out by forming a *frequency table* which displays the numbers 1 through to 9 along with the number of times each appears in different combinations. This done, an attempt is made to correlate the information gathered with the geometry of the 3×3 square using the device of an 'incidence-array'. This proves successful both in finding all eight 3×3 magic squares and in explaining why there are only eight.

Under the heading 'The Algebraic Way', the final portion of the written exposition (along with its short accompanying PowerPoint presentation) adopts an algebraic approach from the outset and investigates the broader question of forming a 3×3 magic square with arbitrary sets of nine natural

numbers. The student is informed that afterwards any general results obtained will be applied to the particular case of the first nine natural numbers to see what they yield.

The student is then shown how abstract thought alone applied to the eight magic conditions (and implemented with no more than simple algebra) arrives at general criteria which apply to every three by three square. In particular, it is learned that in the context of constructing an arbitrary three by three square from nine natural numbers:

1. The magic sum is one third the sum of the given nine numbers.
2. The sum of each 'opposite-pair' is two-thirds of the magic sum.
3. The sum of the four corner numbers is four-thirds of the magic sum.
4. The sum of the four central-side numbers is four-thirds of the magic sum.
5. The central number is one-third of the magic sum and is the average of the nine numbers in the square.

After this 'tour de force' the student is then shown how to arrive at a completely 'general' 3×3 square which is described in terms of three of the possible nine numbers. The treatment then specialises to the case of the first nine natural numbers to obtain a 3×3 with 5 in the central cell and each of the other cells specified in terms of two parameters. The discussion finishes at this point by leaving the student the exercise of completing the

construction of the square. As it turns out this is quite a challenging task which has many lessons to teach the inquisitive student. For reasons of space, these and other lessons are not elaborated upon.

`Summary`: The strength of this section on "Three by Three Magic Squares" is its ability to show the power of logical thinking. The central feature of each of the first three of the four treatments of the 3×3 puzzle is the manner in which they reveal to the student the power of simple reasoning to reach far-reaching conclusions. In this instance, this means the finding of the eight 3×3 magic squares along with knowing with certainity (*i.e.* being able to prove) that these are the only possibilities. Thus the student learns that above all else, it is only by thinking things through in a logical and thorough manner that one can hope to achieve the satisfaction of overall understanding.

Mathematical Content: While each of the approaches adopted teaches a number of valuable techniques, the technical mathematical content throughout is elementary in the extreme, with the possible exception of the final segment showing the power of algebra to arrive at general conclusions.

In order to have the exposition make contact with some technical mathematics the teacher needs to have the students consider the possibility of constructing arbitrary $n \times n$ squares for $n \geq 4$. This immediately prompts the question as to what the magic sum, $M = M(n)$, would be in such a case.

Having seen how $M(3)$ is obtained in the case of a 3×3 magic square, the students between them should have little trouble in coming up with the

following equation:

$$nM(n) = 1 + 2 + 3 + \cdots + (n^2 - 2) + (n^2 - 1) + n^2$$

Since the sum displayed here is that of the first n^2 natural numbers, the students will now have found a fresh application for the summation formula found in their study of the RM piece on $2 \times n$ Magic Rectangles. Using this result gives

$$\begin{aligned} nM(n) &= 1 + 2 + 3 + \cdots + (n^2 - 2) + (n^2 - 1) + n^2 \\ &= \frac{n^2(n^2 + 1)}{2} \\ \Rightarrow M(n) &= \frac{n(n^2 + 1)}{2} \end{aligned}$$

Thus they have a general formula for what the magic sum must be for *each* value of n. The teacher should take care to stress the importance of a general result such as this.

Immediately afterwards he should have the students check that it returns the right answer in the case $n = 3$. It gives

$$M(3) = \frac{3(3^2 + 1)}{2} = 15$$

as it should.

Setting $n = 4$ gives

$$M(4) = \frac{4(4^2 + 1)}{2} = 34$$

Thus if a magic 4×4 square does exist, it must have a magic sum of 34. However, while knowing this fact is undoubtedly important, on its own it is not enough to prove that such a square exists. The teacher might like to inform his students that such squares exist in abundance and challenge them to construct one. An accompanying presentation shows one way of doing this.

Extensions: Having devoted so much time to the discussion of one particular magic square, it is only natural to close out such an account by simply mentioning (without proof) results relating to:

1. the construction of a 4×4 magic square and, in particular, the 1514 square of Albrecht Dürer

2. Euler's construction of an 8×8 magic square and its connection with the knight's tour from chess

3. the Italian (also Siamese) method of constructing magic squares of odd dimension

4. pan and perfect magic squares

Some of these points are illustrated in a number of separate presentations accompanying the main presentation. The full selection of PowerPoint presentations relating to this RM segment on Magic Squares is labelled as shown in the following list:

1. **01** The Three by Three Magic Square – the main presentation accompanying the written core piece.

2. **02** Non-Commutativity – reveals visually that the composition of transformations is not commutative in general.

3. **03** The 3×3 Magic Square – accompanies the section dealing with 'The Algebraic Approach'.

4. **04** Four by Four – demonstrates a general procedure for constructing four by four magic square. It also displays Albrecht Dürer's celebrated example to be found in his woodcut *Melancholia* of 1514.

5. **05** Beyond Three by Three Magic Squares – describes the ingenious Siamese method for constructing a magic square of any 'odd' dimension.

These are to be found on the accomanying CD in the subfolder Magic Squares of the Recreational Mathematics folder. As can be seen, while given in the context of recreational mathematics, many also address both the history and culture of mathematics.

Related Problems: The core section relating to Magic Squares also contains a number of slides on the `Magic Triangle` problem, which might be regarded as a close relative of the 3×3 magic square puzzle. However, it has a number of features unique to itself which make it valuable. Firstly, unlike the 3×3 magic square puzzle, it has more than one possible magic sum. Secondly, it provides the teacher with another opportunity to reveal the power of

algebraic inequalities and thirdly it introduces the notion of 'complementary solutions'. It might make for a valuable homework assignment to be written up fully.

Furthermore, this section contains a written description and solution of the `Eight Numbers on the Cross` puzzle which is accompanied by a short PowerPoint presentation entitled "Eight in Eight". This puzzle is included here because its analysis, while being simpler than that relating to the three by three magic square, is similar to it, involving a blend of geometry and elementary arithmetic. It also allows the teacher to make contact with the fact that a rectangle possesses only four of the eight symmetries enjoyed by the square. It makes for a good exercise, re-inforcing the technique of sequential analysis.

7.3 Sequential Analysis

The next problem to be briefly discussed here 'requires only simple arithmetic but the thought processes of a detective'. It is another exemplar of the power of sequential analysis. While the three by three magic square problem may justifiably be classed as a puzzle on the grounds that its element of surprise resides in the fact that solutions to it exist at all, the `What Are Their Ages?` problem merits the same categorisation in that its very statement is truly surprising.

This charming and engaging problem, which is accompanied by a PowerPoint presentation, illustrates most forcibly the different steps in the problem

solving process which Burton lists as:[17]

1. Entry
2. Attack
3. Review
4. Extension

According to this author, 'Entry' is where the solver first tries to understand what the problem is about and clarify what must be done. This feature is certainly highlighted by this particular problem, as much of the initial difficulty lies in believing that the problem posed is indeed solvable based on the information supplied.

The second stage of 'Attack' is where the solver, in seeking a resolution, may get stuck once or many times and is required to use organising questions and procedures to help move forward. This is amply portrayed in this instance because the investigator is forced to find a systematic way of extracting the maximum amount of information conveyed by the first clue before even considering any further course of action. Specifically, this is done by listing the eight ways in which the number 36 can be written as the product of three natural numbers. When this task is completed, the solver may need, as Burton suggests, to return to the Entry phase to reassess the direction to be taken. This leads to a similar attack being made on the second clue after

[17]L. Burton, *Thinking Things Through – Problem Solving In Mathematics* (United Kingdom: Basil Blackwell, 1985), 24-25.

which one revisits the Entry stage one last time to adjust one's bearings. Deciphering the final clue calls for an insight which many are fortunate to be granted quite quickly while others struggle a little longer.

With the puzzle unravelled, the third stage of 'Review' is entered upon. This, according to Burton, is where one examines the essence of the resolution obtained to see if it is adequate. It is also the stage where, in practice, one should write up solution so that it can be understood by others. This involves useful checking and exercises communication skills. It also suggests further questions.

The final stage of 'Extension' is where the resolution of a problem can be made to lead on to further problems. This is done in the presentation accompanying the written treatment of this puzzle where, as Burton notes, it helps reinforce significant features of the resolution of the original problem.[18]

By way of exercises in this form of problem solving, the teacher might like to avail of the recreational puzzles apearing on the following list:

1. `The 100 Metre Dash`: The solution to this numerical puzzle, which is arrived at using no more than simple arithmetic applied in a linear fashion, comes as a surprise to many since it proves to be at odds with what is regarded as the 'common-sense' answer. As such it serves as an ideal lesson showing that so-called common-sense can be misleading and that one should trust to logic over intuition. It is presented in the

[18]The written treatment in the core follows that of the `The Insurance Man Puzzlein`: S. Flannery S. & D. Flannery, *In Code - A Mathematical Journey*, (USA: Workman Publishing, 2001), 13-16.

core as a PowerPoint presentation which concludes with two related problems which allow the teacher show the need and efficacy of algebra.

2. `Archibald's House Number`: This is another problem which requires no more that crisp logic and simple arithmetic applied in a sequential manner. It could be used as a very instructive example in how a problem can be solved simply using a case-by-case analysis.

3. `The Prison Wardens`: This cleverly designed recreational problem reveals a special property enjoyed by the 'perfect squares', and so it may be regarded as touching on number theory. However, it requires no more than clear logic and the use of simple arithmetic for its solution.

The solutions to the last three problems, which are outlined in some detail rather than fully solved, can be found in the core section relating to this particular theme in the subfolder Sequential Analysis of the Recreational Mathematics folder.

When a student has been exposed to a number of these puzzles best tackled by a sequential approach, he gradually learns how to turn an apparently difficult problem into a sequence of simpler ones, and in so doing acquires the ability to analyse a problem. Later, a number of problems are discussed which are not as amenable to this method of solution; requiring a form of thinking often described as 'lateral'.

7.4 The Chessboard

The chessboard is likely to be familiar to every senior cycle student even if most may not be able to actually play the game of chess. This section indicates a number of ways the teacher can use the chessboard to make contact with school mathematics in, what is hoped is, an interesting and engaging manner. The mathematical themes discussed initially are illustrated in an accompanying PowerPoint presentation entitled 'The Chessboard' which resides in a subfolder of the same name in the core folder on Recreational Mathematics. In addition to containing a number of other items (mentioned in due course), this subfolder contains a presentation 'Chess Notation and Terminology' which clarifies for the student the notion of rank and file. It also introduces him to the algebraic notation employed by the great majority of chess players throughout the world. Describing this system of notation could be viewed by the teacher as a precursor to the introduction of a similar double index scheme which the student will encounter later in his study of matrices.

The presentation 'The Chessboard - A Source of Many Questions' begins by asking: 'How many squares are there on a chessboard?' and has the student gradually understand that the answer might properly be given as

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2$$

rather than the expected 64 that one might first imagine is the 'obvious' answer. The manner in which the more comprehensive answer – as the sum

of the first eight squares – is obtained is a valuable exercise in a counting technique which the presentation is able to make clear because of its dynamic ability. While the student can compute this answer to be 204 in this particular case, when the teacher generalises the original question to that of an $n \times n$ chessboard, the student speculates that the answer is now given by

$$1^2 + 2^2 + 3^2 + \cdots (n-2)^2 + (n-1)^2 + n^2$$

but realises that arithmetic methods are no longer of any avail.

An important juncture has been reached where the teacher has revealed a 'situation' where the sum of the first n squares has been encountered in a context. Furthermore, in the passage from the particular to the general, the teacher can make it clear that arithmetic must give way to algebra, since the immediate 'mathematical' problem to be considered is whether one can find a 'closed formula' for

$$\sigma_2(n) := 1^2 + 2^2 + 3^2 + \cdots (n-2)^2 + (n-1)^2 + n^2$$

for each $n \in \mathbb{N}$.

Thus it can be seen how a question about the chessboard allows the teacher to make contact with a mathematical problem which is part of the general mathematics syllabus. Having motivated the abstract problem in a 'natural' manner, the next class or two might be devoted to the algebraic derivation of the closed formula (based on the difference of two cubes and telescoping) to arrive at the formula

$$\sigma_2(n) = \frac{n(n+1)(2n+1)}{6}$$

– first obtained around the tenth century by Arabian mathematicians. Alternatively, rather than break off the discussion, the teacher might content himself with stating the result for the moment with a promise to give its ingenious derivation at a later date.

Whatever course of action is decided upon, the teacher should take care to herald the result as a magnificent achievement and point out that one is not always so fortunate to achieve a closed formula for a sum. He or she might, as the presentation does, note that there is no known closed formula for

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{n}$$

– a sum which arises in probability.[19]

Before proceeding with the presentation the teacher should have the students check that the formula returns the answer 204 in the case $n = 8$, and also have them examine a number of other values of n including the cases $n = 1$ and 2. $\sigma_2(19) = 2470$ different squares.

The second major question posed in the presentation follows on naturally from the original. It asks for the total number of rectangles on a chessboard taking care to define exactly what is meant by the question. Whereas this might strike one as a quite a difficult question to answer, as indeed it is if

[19]In connection with the Coupon Collection problem as will be mentioned later.

tackled from the wrong perspective, the manner in which it is answered is a combinatorial vignette which the dynamic element of the PowerPoint presentation is able to make wonderfully clear. When the question is generalised to the total number of rectangles on a $n \times n$ chessboard, the answer is given by

$$\binom{n+1}{2}^2$$

– a result the teacher might point out could be written as $[\sigma_1(n)]^2 = \sigma_3(n)$.

Now an obvious exercise, requiring algebraic technique, is for the student to use the general answers to the first two questions asked to determine an expression for $r(n)$ – the total number of rectangles (which are not squares) on a $n \times n$ chessboard. This activity reveals the difference $\sigma_3(n) - \sigma_2(n)$ in a meaningful context.

At this point, the presentation takes a geometrical turn by asking how far a King standing on a corner square (say, a1 the bottom-left square) is from each of the other squares on the chessboard; meaning how many moves would it take him to arrive at each of these squares. The answer to this question has many facets. Firstly, it reveals to the student a 'metric' which differs from the usual Euclidean one. The student is asked to observe that while the bottom-right (h1) square is physically closer to the bottom-left (a1) square than is the top-right (h8) square, to the chess player (subject to the rule of chess which says that a king must move one sqaure at a time) both of these squares are actually at the same 'distance'. This metric is often

labelled the taxicab metric and arises in other more 'practical' contexts in relation to codewords.[20] As the presentation notes, "To a king in a corner, the chessboard is like the quadrant of a circle'. The lesson that there can be 'different concepts of distance depending on context' is of itself a very valuable one.

When each square is classified in terms of 'its distance' from the given corner square and a frequency table constructed giving the different distances along with the number of squares at each distance, the teacher can reveal the total of the frequencies is the sum $1+3+5+7+9+11+13+15$ – the sum of the first 8 odd numbers, which must equal 8^2 as this is the total number of unit-squares on the chessboard. With the observation that

$$1+3+5+7+9+11+13+15=8^2$$

the teacher can take time out to have the students generalise the result to the case of an $n \times n$ chessboard. In this way he can have them discover that

$$1+3+5+\cdots+(2n-3)+(2n-1)=n^2$$

for each $n \in \mathbb{N}$. This result says that the 'sum of the first n odd consecutive numbers is a square'. The teacher might ask the students to recall how this result can be deduced from Gauss's formula discussed under Magic Rectangles.[21]

[20]Referred to later in connection with error-control coding.

[21]One can also use the Principle of Mathematical Induction or general results about arithmetic series provided they have been covered by this stage.

This is the extent of the material covered in the presentation 'The Chessboard'. However, both the presentation and the current use of the chessboard can be extended. By relating the mediæval Arabic tale which says that after Sissah ibn Dahir invented the game of chess, he asked as a reward from his king that he be given 1 grain of wheat for the first square, 2 grains for the second, 4 grains for the third and so on, the student witnesses the finite geometric series

$$1 + 2 + 2^2 + 2^3 + \cdots + 2^{62} + 2^{63}$$

giving the total number of grains, arising in a context.[22] The teacher may now reveal the simple but ingenious way by which the sum of this series is expressed in 'closed' form as $2^{64}-1$; (a number equal to $18,446,744,073,709,551,615$). With no more than a little algebraic dexterity, the same technique shows that

$$1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-2} + 2^{n-1} = 2^n - 1$$

for each $n \in \mathbb{N}$. The one final application of the same method gives

$$a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}$$

for $r \neq 1$ and each $n \in \mathbb{N}$.

The teacher has now provided his students with the sum of an arbitrary finite geometric series – a powerful result which finds almost ubiquitous application throughout the realm of computation.

[22]This series also arises in connection with the Tower of Hanoi puzzle as treated in the Topic on Difference Equations.

Another useful question in relation to the chessboard asks for the number of different 'shortest' routes the king can take in travelling from the a1 corner to one diagonally opposite at h8. This is a combinatorial problem which has obvious practical links to 'routing problems' such as the following:

> How many different ways are there to travel from A to B shown in the 3×4 grid if one must always proceed eastwards or northwards at every junction?

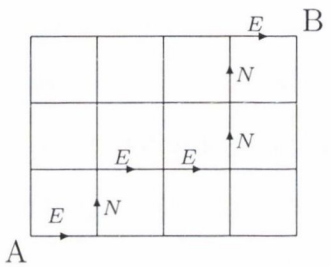

Figure 7.1: Shortest routes.

The teacher can solve this latter problem quite elegantly by noting that the route shown in the illustration can be recorded succinctly as $ENEENNE$. Then the insight which observes that since the grid is a 3×4 one, any such record of a shortest route must perforce consist of four Es and three Ns, reduces the geometrical problem to a purely combinatorial one. Since the total number of distinct seven character 'strings' consisting of four Es and three Ns is given by

$$\binom{4+3}{3} = \binom{7}{3}$$

the answer to the question posed is 56.

By analogy, the student should have little difficulty in explaining why

$$\binom{7+7}{7} = \binom{14}{7} = 3432$$

is the answer to the corresponding chessboard problem. By extension the same reasoning shows that the total number of shortest routes from the bottom-left corner to the upper-right corner of an $m \times n$ grid is given by

$$\binom{m+n}{m} = \binom{m+n}{n}$$

– a significant mathematical result with practical implications.

An Exploration: It is a simple matter to determine that there are 28 border-squares on the ordinary chessboard which when removed leave an internal 6×6 square. Likewise, it is an easy exercise to show that the corresponding number of border-squares for an $n \times n$ chessboard is $4(n-1) = 4n-4$ and that the remaining internal square is of dimensions $(n-2) \times (n-2)$. A simple exploration is to determine for which values of n, the set of border-squares can themselves form a square so that the original $n \times n$ square is *decomposed* as the sum of two (smaller) squares. The student should quickly discover that the 5×5 square has $4(5-1) = 4 \times 4$ border-squares and an internal 3×3 square, all of which accords with the fact that

$$3^2 + 4^2 = 5^2$$

– a numerical relationship normally associated with Pythagoras's theorem. In this way, the interested student will find that the suggested exploration links with the problem of finding what are known as 'Pythagorean triplets'.

7.4.1 The Chessboard and Information Theory

In this work the chessboard is also used to introduce the student to a simple but fundamental concept only clearly formulated as recently as 1948 by the American polymath Claude Shannon. The core section contains a written piece entitled 'Information Theory Game'. It describes in some detail a 'questions and answers' game associated with the chessboard called `Where's the Queen?`, whose purpose is to allow the teacher quantify the notion of 'information'. The mathematical prerequisites are elementary, involving no more than an understanding of what is meant by 'equally likely', the use of function notation and a knowledge of the laws of logarithms.

In fact, one of the great virtues of the discussion is the manner in which it provides the teacher with an opportunity to exhibit the efficacy of the logarithmic function in a context different from its normal role as a computational tool. The log function is shown as central to quantifying the notion of information in a precise manner. The importance of the logarithm function is something which, perhaps, has been better appreciated by physicists since the end of the nineteenth century through its connection with the thermodynamic concept of entropy.

The written exposition has an accompanying PowerPoint presentation –

'Basics of Information Theory' – which illustrates dynamically much of what can only be described verbally. However, the presentation is not simply a visual implementation of the writtem description; side by side with the notion of information it also treats such related concepts as uncertainty and entropy. Needless to say, it is expected that teachers would use it as a teaching-aid to present some of the key ideas at whatever time they are deemed to be most effective. The core section relating to this this theme also contains a PowerPoint presentation entitled 'Log Three to the Base Two' which explains slowly and carefully why the amount of information conveyed by a balance pan can be measured as $\log_2 3$. It does this by discussing a number of weighing problems which have traditionally been classified as the fare of recreational mathematics.

7.5 Games

Leibniz wrote:[23]

> So also the games in themselves merit to be studied and if some penetrating mathematician meditated upon them he would find many important results, for man has never shown more ingenuity than in his plays.

The Tower of Hanoi game has already been discussed in the section on Difference Equations where it is shown to lead to a genuine mathematical question relating to a first order recurrence. It could equally well be treated as a recreational game which reveals how a geometric series arises. However,

[23]Letter to Remond de Montmort, July 15, 1715.

the main purpose of this section is devoted to discussing how the board game Mastermind can be made to teach logical thinking in an attractive and stimulating manner while also prompting many mathematical questions in the spirit of the quote just given.

7.5.1 The Game of Mastermind

The board game Mastermind is one whereby students can learn logical reasoning in an enjoyable and concrete manner. It is also an ideal vehicle for collective participation within the classroom. One way of achieving this mutual interaction is for the teacher to display a number of rows of a game (appropriately scored) which, between them, contain enough information to allow the hidden configuration to be deduced on the next attempt. The task then is for the pupils to reason out together, under the possible guidance of the teacher, the four colours of the concealed pegs. The core material relating to this particular game contains a written account of such a problem and a presentation created in Flash which shows exactly how this can be done. Besides giving a tutorial on how to score a game, the presentation displays a problem of the type just described and dynamically steps through each of the different reasoning stages necessary to demonstrate convincingly that there is only one possible configuration that 'fits all the available facts'.

Another class on the same theme might have the students work privately on their own individual boards and have the teacher verify their suggested answer. This pitting of students against themselves is sure to introduce a

competitive edge into proceedings and guarantee total concentration on the task at hand.

A teacher can also engender an element of competition by having his students break up into groups of two, with one student designated as 'a scorer' and the other 'a solver'. Then he provides each of the 'scorers' with the same initial configuration and the task for the solvers is to try to determine the given configuration as expeditiously as possible. This exercise will have some students realise the influence the element of choice can have in this game. It can happen that a player who is strictly and comprehensively logical at each stage takes more rows to determine the hidden configuration than another more fortunate in his initial choices. This is a valuable lesson in its own right and forms part of the many mathematical exercises related to this game.

But the game has more to offer than merely teaching deductive skills. It prompts all sorts of questions in a natural way which the teacher can put to his students after they have enjoyed and mastered the game thoroughly. The core section also contains a written discussion of this aspect of the game which consists of two separate batches of exercises with answers provided. These are mainly combinatorial and probabilistic in nature. For example, one such exercise from the first set asks:

> What is the probability of initially choosing all four of the colours correctly?

which is a question the students might well pose of their own accord. If not, they certainly won't mind it being asked on their behalf and will be interested,

but not surprised, to be told that the odds against initially choosing all four of the colours correctly are 69 : 1, while the odds against picking the four right colours *and* placing them in their correct positions are as much as 1679 : 1. This exercise alone teaches combinatorial reasoning and the concept of 'joint events' along with the 'multiplication of probabilities in the case of independent events'.

After determining the probabilites relating to the choosing of each of the possible number of correct colours, the student is asked to note that the analysis has established the identity:

$$\binom{4}{0}\times\binom{4}{4}+\binom{4}{1}\times\binom{4}{3}+\binom{4}{2}\times\binom{4}{2}+\binom{4}{3}\times\binom{4}{1}+\binom{4}{4}\times\binom{4}{0}=\binom{8}{4}$$

When expressed more compactly in summation notation as

$$\sum_{k=0}^{4}\binom{4}{k}\binom{4}{4-k}=\binom{8}{4}$$

the student is then told, in passing, that this a particular case of the general identity:

$$\sum_{k=0}^{m}\binom{m}{k}\binom{n-m}{n-k}=\binom{m+n}{m}=\binom{m+n}{n}$$

– a result of central importance in relation to the *hypergeometric* probability distribution but which the student might note connects with the problem of shortest routes in an $m \times n$ grid discussed in connection with the chessboard.

The second batch of questions relate to counting the different number of ways one can achieve the various scoring configurations. The questions asked

are combinatorial in character and in answering them the student gets to see the (sometimes difficult) art of enumeration. The results obtained are then summarised in a number of highly informative tables, the design of which impart their own lessons to the student. There follows a short discussion on total and partial derangements which is linked to the enumeration process.

Because the formulation of difference equations has received considerable treatment in this work, a second-order recurrence concerned with counting (total) derangements given by

$$d(n) = (n-1)[d(n-1) + d(n-2)] \quad ; \quad d(0) = 1,\ d(1) = 0$$

is derived and illustrated. However, it is not envisaged that the student be expected to derive this difference equation, rather that he be able to use it as demonstrated (in the core piece).

It is quite easy to set reasonably straightforward examination questions based on this game. One option is to display a problem (such as the one given in the presentation) rendered by means of a coloured diagram and have the student describe the chain of reasoning leading to the hidden configuration. This substantial task addresses the NCTM's Communication Standard in that it obliges the student to write well-structured and sensible sentences of explanation. A second possibility is to ask combinatorial questions related to the game (of the type discussed in the core piece) while a third option is to blend a purely logical question with an enumerative one.

One beneficial effect the playing of this game might well have is that some

of those students who have not excelled technically realise that they are quite capable of reasoning clearly. If such were to prove the case, then the gain of confidence in their own ability might engender a more positive attitude to the subject of mathematics as a whole.

Other Games: While the ranking of the different hands in the game of Poker might warrant study as a recreational theme, the treatment given to this task in this work is that of an application of combinatorial mathematics and as such is taken up in the next chapter dealing with applications. There, different aspects of the game itself also receive mention as a tool by which the student can be shown the efficacy of probabilistic reasoning.

The core section relating to games contains two other presentations, both of them short. One called 'Tic-Tac-Toe' examines the popular children's game of nought-and-crosses (also known as tic-tac-toe). The treatment is far from complete but does indicate a strategy by which the second player can always draw the game. The other very short presentation is on 'The Game of Fifteen' – a game which requires for its successful play both an understanding of the three by three magic square puzzle and the game of tic-tac-toe.[24] Hence its appearance at this juncture.

The popular game of Nim could be used as a recreational means of introducing students to binary numbers since an analysis based on elementary binary concepts shows that the second player can always force a win. The

[24]J. Mason with L. Burton & K. Stacey, *Thinking Mathematically* (US: Prentice Hall, 1985), 79.

fact that mathematical reasoning can indicate the best strategy to adopt in certain circumstances, even if they be no more than games, can serve to convince students of the genuine ability of the subject to inform decision making.

7.6 RM and Modular Arithmetic

The mathematician G.H. Hardy once wrote:[25]

> The elementary theory of numbers should be one of the very best subjects for early mathematical instruction. It demands very little previous knowledge, its subject matter is tangible and familiar; the processes of reasoning which it employs are simple, general and few; and it is unique among the mathematical sciences in its appeal to human curiosity.

This section describes a number of problems, many of which might best be considered recreational in nature, and all of which require no more than the simplest elements of number theory for their solution. The main purpose for their selection is as a means of introducing the student to the 'mod' notation which is universally used in connection with arithmetic that is 'modular' in character. As will be discussed in the next chapter, this arithmetic is used in cryptography and error-control coding while the notation itself is increasingly being used outside its traditional sphere of pure number theory.

A written core piece entitled 'Modular Arithmetic' (which is accompanied by a short PowerPoint presentation of the same name) treats extensively a number of problems whose aim is to expose the student to this form of

[25]G. H. Hardy, "An Introduction to the Theory of Numbers", *Bulletin of the American Mathematical Society*, 35 (1929): 818.

arithmetic in an enjoyable but meaningful manner. As some of the problems make clear, most people use this type of arithmetic on a daily basis in some form or other, mostly in connection with measurements of time where it is often referred to as 'clock arithmetic'.

The first problem asks, "Which day of the week will it be 100 days from now if today is Sunday". In solving it, the student is made to realise the special role played by the number 7 as the 'measure' of the number of days in a week. Since the word *modulus* is a Latin word for 'measure', calculations involving weekdays invariably feature division by the number 7 which is referred to as the underlying modulus. Furthermore, because it is only the remainder 2, left over when 100 is divided by the modulus 7, that determines the day of the weekday, the student is told that it is important to be able to convey the information in such a calculation succinctly. One way of achieving this is to simply write that

$$100 \bmod 7 = 2$$

– a notation which concisely captures all the relevant numerical information.

After elaborating on the concept by way of an analogy involving a seven-day clock and presenting a number of other calculations 'modulo seven', the student is then exposed to calculations involving other moduli connected with angle measurement and twenty-four hour clocks, to name but two. This exercise is followed by showing how an old puzzle is modular in nature before introducing the student (without explanation) to an ingenious parlour room

trick based on the exotically named Chines Remainder Theorem. By way of further familiarising the student with modular calculations, the student is then shown a special property of the number 9 (which was much used by accountants in the pre-calculator days for the purposes of cross-checking calculations). The student also learns that 'The Rule of Nines' – whereby a whole number is divisible by nine if, and only if, the sum of its digits is divisible by nine – is a special case of this property.

Up to this point the problems considered in the core treatment involve no more than elementary arithmetic appearing in new clothing as modular arithmetic calculations. However, the next problem considered is a substantial one relating to the calendar which asks: "Which months, if any, begin on the same weekday as 1 January?" With this `The First of Every Month` problem, it is the technique by which it is solved that is of particular value to the student. At the end of the analysis the student is provided with a comprehensive scheme which works for any year including leap years. Specifically, given the weekday on which 1 January of the year falls, the student can say which other months start on this day also, as well as give the starting weekdays of each of the other months. As an extra bonus, the solution reveals that it is simply impossible to avoid at least one 'Friday the Thirteenth' in a year and that in some years there are as many as three such days.

The core treatment concludes with a discussion of two problems, both of which make use of the 'mod' notation. The first asks the student to guess `A Fast Rule for Multiplying a Two-Digit Number by Eleven` based on an

array of calculations given to him and then to express this rule firstly in words and afterwards more succinctly using mathematical notation. It is this latter task which is greatly facilitated by modular notation. The second problem `Ones and Zeros` shows that every natural number has a multiple which is a string of 1s followed by a string of 0s. The manner by which this particular result is established makes use of the *The Pigeonhole Principle* – a counting technique of great power which receives further application in a later section of this chapter devoted to lateral type problems.

By way of summary, it is hoped that by introducing the student to the 'mod' notation by means of a number of easily understood recreational problems, he will become familiar enough with it so as to understand its use in applied contexts, some of which are revealed in the next chapter.

7.7 RM Revealing the Power of Algebra

While the HLC does teach (and examine) a great deal of algebra, it is usually employed as a technical tool as part of a larger problem rather than as a means of explanation in itself. This chapter on RM has already revealed the explanative power of simple algebra both in its discussion of magic rectangles and three-by-three magic squares. This section indicates a number of other recreational contexts which the teacher might exploit as a means of demonstrating the worth of algebra. It begins by considering one or two simple puzzles and tricks that appeal to ten-year-olds and upwards before discussing a number of more advanced problems appropriate to senior cycle

students.

While the puzzles and tricks that appeal to pre-teenagers provide an ideal means for a teacher to introduce young students to the elements of algebra in an engaging manner, the emphasis here is on harnessing these same problems to convince more senior students of the power of algebra. This to be done by the simple expedient of requiring that they, the students, explain the tricks for themselves rather than the teacher doing so.

By way of illustration with regard to a puzzle, the teacher might inform the students that some children of a certain age never seem to tire of the problems which, to give an example, ask: "What age am I now if in 10 years time I'll be twice as old as I was 6 years ago?" The task for the more senior student might then be to state and solve this problem in general and then to comment. In this case, the teacher should expect the student to frame the general question along the lines: "What age, A, am I now if in F years time I'll be twice as old as I was P years ago?" and to use algebra to show that the general answer is $A = F + 2P$. More importantly, the teacher should expect the student to point out that knowing and understanding the general form of the question along with the formula for its answer allows one (often a parent) to pose any number of these puzzles without undue effort. In fact, it can happen that after solving a number of these problems, young solvers empirically discover the general method of solution but are at a loss to formally explain the 'algorithm' by which they arrive at the solution. Besides being able to explain why the method works, the senior student can, because

of the algebraic analysis, extend the original general puzzle to one where the multiplier is 3 rather than 2 and so offer the young puzzle enthusiast fresh challenges. Thus the senior student comes to realise the ability of a general algebraic approach to both explain and, in this case at least, extend a puzzle.

Similarly, by way of demonstrating the ability of algebra to explain tricks, the teacher might first perform `The Matchbox Trick` for the students and afterwards set them the task of deciphering its workings. This trick is one where the 'magician' asks the owner of a box of matches to count the number of matches in the box. This done, the matchbox owner is then requested to sum the digits of his answer and to remove that number of matches from the box. Then the magician requests a quick glance at the matches remaining in the box. This is to be done by opening it quickly and just as quickly closing it. Whereupon, to the amazement of all, the magician announces the correct number of matches remaining in the box.

The challenge for the student is to say why the magician always 'knows' the number of matches remaining in the box. An understanding of the place system combined with simple algebra is all that is needed to supply the short explanation which runs as follows: if the original number of matches in the box is denoted by xy, then the box actually contains $10x + y$ matches. When the owner removes $x + y$ matches, $9x$ matches remain. Thus the magician, versed in mathematics, knows that the number of matches remaining is a multiple of nine. The glance is to determine by a quick assessment of the volume of the matches in the box whether this number is $9, 18, 27$ or 36.

Some adroit performers dispense with the glance and simply rely on rattling the box.

These two examples serve only to give a flavour of the ability of algebra to explain a puzzle or demystify the workings of a trick. Puzzles and tricks which can be explained with equal ease abound and should be used by the teacher as vehicles by which the student can come to appreciate the efficacy of this 'arithmetic with letters'.

The core segment relating to this section contains a treatment of four problems, each at the level of the senior cycle and which demonstrate in different ways RM revealing the power of algebra. A short synopsis of the lessons each has to teach is now given.

What Are Our Ages? The difficulty with this recreational puzzle lies in disentangling its rather convoluted statement to determine what exactly is being said. This is done by replacing rather lengthy phrases by more succinct symbols. The resulting compression leads eventually to clarity and the subsequent solving of the question asked. The lesson this puzzle teaches could be described as an illustration in decipherment admirably achieved by algebra.

Sum to One Hundred: This problem asks if it possible to form an addition where each of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 appear once and once only to get a total of 100.

Once what exactly is meant by an 'addition' is clarified, the most fre-

quent initial reaction of students is one of optimism whereby they assume the objective is achievable. They then begin to search for a solution using trial-and-error and soon find an addition which comes tantalizingly close to the right answer. However, try as they might, they find on further investigation that success proves elusive. After many gallant but unsuccessful attempts, many begin to suspect that the task is impossible. Though this suspicion may smack of pessimism, such a crucial realisation is one which investigators must often contemplate when trying to solve real-world problems: is the problem, in fact, solvable? At this point, the true nature of this problem is evident: how can one prove that the given objective is, in fact, unattainable.

It is at this stage students learn that arithmetic must give way to algebra so that the problem can be treated in general. Using no more than an understanding of what is entailed by the place system, the teacher can help the students to convincingly explain why the requested total is unachievable. In this way, this RM problem reveals the power of algebra to establish a general result, and as such teaches a most valuable lesson.

However, rather than consider the problem closed, an extension to it asks if the task is possible using a two-tiered addition. The second portion of the treatment is devoted to examining this variation of the original question. In answering it, the key role played by algebraic reasoning is once again reinforced. Guided by no more than simple observations relating to the divisibility properties of natural numbers, the investigation draws to a

successful and substantial conclusion by enumerating all the different ways the second task can be achieved.

One Before and After: This problem asks:

> Is there a five-digit number having the property that if a 1 is placed after it, it is three times as large as with a 1 before it?

This is a problem which many pupils as young as twelve find engaging and eventually solve. It is included here because it is another wonderful example by which to reveal to slightly older pupils the power of algebra. The core treatment deliberately devotes its first pages to showing how one can, by a case-by-case analysis, discover that this problem has only the solution 42857. It is then explained to the student that in the jargon of mathematics *existence* and *uniqueness* has been established, meaning that, a solution to the problem has been shown to exist and, furthermore, this solution has been shown to be the only one.

While this initial approach is a lenghty one, it teaches a number of lessons of its own regarding the place system and the conducting of a systematic investigation. It also demonstrates that the problem has no solution for a number with *less* than five digits, which is an extra bonus. However, its true purpose is one of comparison; to emphasise how much more expeditiously the problem can be solved using simple algebra. The 'slick' algebraic solution also depends heavily on the student having a keen understanding of the place system.

Under the heading 'Making the most of a simple puzzle' the original

problem is then examined to see if it can be generalised. This exploration involves ideas from each of the earlier investigations and leads to interesting results in an elementary manner. In addition it also provides another instance of the 'mod' notation being used in a completely natural manner.

Should the core treatment be covered in full then the following are some of the lessons the student may learn from the study of this particular recreational problem:

1. the joy of exploration
2. the value of systematic thinking
3. the concept of existence and uniqueness
4. the importance of understanding the 'place system'
5. an appreciation for the power of algebra
6. the idea of a permutation
7. how a simple puzzle can lead to genuine mathematics
8. the all important realisation that sometimes things work, sometimes they don't; and that discovering the reasons why leads to further understanding
9. how to pose imaginative new questions and discover fresh avenues of attack
10. the efficacy of the 'mod' notation

Always 1089: There are two aspects to this short core piece; one algorithmic, the other algebraic. The first stage is understanding the mechanism of an attractive number trick by way of implementing the algorithmic procedure necessary for its execution. This requires that the student can read a set of instructions and carry out the simple arithmetic associated with each step. The second stage is one where the student is asked to explain why the trick always produces the number 1089. This requires the use of algebra and is another excellent test of a student's understanding of the place system.

7.8 RM and Geometry

Recreational mathematics is an ideal way to reveal to students geometrical facts which are not part and parcel of their curriculum and yet have much to teach. Currently the focus of school geometry is almost exclusively on stating and proving results from planar geometry treated in the tradition of Euclid. However, as already seen in this chapter, the discussion of a recreational puzzle concerned with constructing a three-by-three magic square can be exploited to reveal the important geometrical fact that a square has eight symmetries. As already stated, this fact is illustrated dynamically in the PowerPoint presentation related to the core discussion of three-by-three magic squares.

It is evident that software packages such as PowerPoint prove themselves of particular assistance in the conveying of geometrical facts because, by its very nature, geometric material is ideal for visual display. This section is

accompanied by three such presentations one of which, entitled 'The Symmetries of a Square', gives a slightly different demonstration than that to be found in the Magic Squares segment.

Another, of a more elementary nature, is a short one named 'Geometric Representation of Algebraic Identities'. It illustrates a number of algebraic identities geometrically. Many students find it much easier to remember algebraic formulae relating to, for example, *the square of a sum* or *the square of a difference* once they have seen these represented pictorially. The twelve slides give geometric illustrations (and at times dynamic derivations) of results connected with

1. the Commutative property: $ab = ba$
2. the Distributive property: $a(b + c) = ab + ac$
3. `The Square of a Sum` formula:

$$(a + b)^2 = a^2 + 2ab + b^2$$

   (The algebraic derivation of this result based on the different arithmetic properties is also given.)

4. the area formula for a right-angled triangle.
5. Pythagoras's theorem
6. `The Square of a Difference` formula:

$$(a - b)^2 = a^2 - 2ab + b^2$$

The presentation 'The Thirteen Convex Tangram Polygons' is concerned with one specific aspect of tangrams linked to a theorem only first proved in 1942 by the Chinese mathematicians Fu Traing and Chuah Chih. This result says that the seven tangram pieces can form exactly 13 different convex polygons which consist of

- one triangle
- six quadrilaterals
- two pentagons
- four hexagons

The task for the students is to construct over a period time all thirteen of these regions.

The presentation might be used initially to introduce the student to the seven tangram pieces, to explain what is meant by a convex region and to show the thirteen achievable convex regions. Subsequently, it might be used to show the students how the different regions can be constructed in a systematic manner.

This 'convex tangrams' exploration (which could also be a class competition to see who can form them all) should ensure that students clearly understand the term convexity and other terms used to describe the different regions. It can also be used by the teacher to reinforce such geometric notions as congruence and similarity as well as introduce the concept of 'invariance of area'.

7.8.1 The Third Dimension

The HLC considers three-dimensional problems if they are solvable by two dimensional means. This section concludes with a short discussion of a number of problems/puzzles which might receive consideration in an RM module with the purpose of cultivating greater spatial awareness among students. As will be seen, these problems teach other valuable lessons.

Dividing A Cake: Many people are completely stumped by the following 'practical' problem:

Divide a cake in eight equal pieces using only three cuts.

Students find this puzzle quite engaging because they have no difficulty whatsoever in cutting a cake into four equal pieces but some are completely at a loss when it comes to making that third cut which produces eight equal pieces. Yet the simple solution lies in the third dimension.

Specifically, one cut along a diameter of the top of the cake produces two equal pieces. A second cut again at the top of the cake at right angles to the first cut produces four equal pieces. What proves elusive, even after many attempts, is where to make that third cut. The idea is to move away from the top of the cake to a point halfway down its side. Then cut across the middle parallel to the top of the cake (and so at right angles to the previous two cuts) to obtain those eight equal pieces.

`Variations:` A puzzle solved just as simply but which often proves difficult is the `Four Equilateral Triangles` matchstick puzzle which asks for four equilateral triangles to be constructed with six matches. Likewise, the difficulty connected with solving `The Nine Coins` puzzle dissolves once the solver has the happy idea of thinking in the third dimension.[26]

Rope Around the Earth: A number of geometric puzzles serve to indicate that one's intuition can be misleading. A celebrated example which illustrates this point very forcibly is `The Rope Around The Earth` puzzle:

> Imagine a rope tied around the Earth's equator like a ring on a person's finger. Now imagine lifting off this rope (do not ask me how) and suspending it instead like a fence one metre from the earth's surface. Don't worry about whether it will fall down, just work out how much longer you'll have to make the rope to form this bigger circle the whole way around the planet.

The answer is a mere $6{\cdot}28$ extra metres of rope. Many adults find this answer so astounding that, even when it is explained to them, they can't believe it. Yet, no more than elementary arithmetic is all that is needed to supply this 'non-intuitive' fact. The original length of the rope around the equator is $2\pi R$ where R is the earth's radius while the length of the longer circle of rope is $2\pi(R+1)$. When the longer circumference is taken away from the original, one gets 2π which comes to $6 \cdot 28$ extra metres of rope.

A variation which proves equally non-intuitive asks for the gap which would result by adding just one extra metre to the original rope. It comes to

[26]Ivan Moscovitch, *1001 Play Thinks* (US: Workman Publishing, New York, 2002)

almost 16 cm, the width of a book. The teacher should have the student carry out the same calculations using a football as the sphere instead of the Earth. The important lesson here is that the result is the same – it is independent of the original sphere and only depends on the length being added.

Where Might I Be? This is a problem also involving the Earth which also has its own surprises:

> Where might I be on the surface of the earth if, after walking a mile south, followed by walking a mile west and then a mile north, I end up at my starting point?

After a little thought many solvers see that the North Pole is one such place. However, what comes as a complete surprise is the revelation that there are other places on the earth where one could be. Attempting to find out where these are is the extended and truly challenging puzzle.

One path to a solution is to consider that line of latitude which is exactly one mile in circumference and lies just a "little north of" the south pole. Any point a mile above this line is a solution to the problem, as indeed is any point a mile above a parallel of latitude half a mile long, and so on.

While these last two examples may have the salutary effect of having students take care not to rely overly on their intuition but to trust instead to logical reasoning, it is recommended, by way of concluding any discussion of recreational geometrical problems, that the teacher demonstrate the one-sided nature of the surface of the "Möbius Strip'. This exercise is sure to convince the students even further that the world is far from being as simple as it might first appear.

7.9 Algorithmic Problems

W. L. Schaaf notes with reference to the recreational puzzles which have been handed down through successive generations that, whether dressed in new clothes or not, the problems in general are of two kinds: those involving

- the manipulation of objects
- computation.

He adds that:[27]

> The first require little or no mathematical skill, merely general intelligence and ingenuity, as for example, so called decanting and difficult crossing problems.

The first problems to be discussed in this section are of this type and are considered worthy objects of study because the algorithmic procedures by which they are solved teach students the invaluable skill of organisation so important for communication. One of the crossing problems treated in the core was once used by the Microsoft corporation as part of its interviewing process. The mere act of designing a visual scheme to describe and effect a solution is an essential aspect of any form of problem-solving, be it mathematical or otherwise. As Krantz observes:[28]

> It is impossible to overestimate the importance of visual aids. They can help organize data in a fashion that mere scribblings cannot.

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[27] *Encyclopædia Britannica*, 15th ed., s.v. "Number Games and Mathematical Recreations".

[28] Steven Krantz, *Techniques of Problem Solving* (USA: AMS, 1991), 178.

However, it is important to note that some of these problems prompt questions whose answers require mathematical understanding, the very first example considered being a case in point.

The core treatment follows the order indicated in the Schaaf quote by describing and simply presenting the solutions to a number of decanting problems before treating a sample of crossing problems, some 'difficult' and other less so, in the same manner. While many of these puzzles can be given to children as young as ten, their purpose here is as vehicles by which the senior cycle student can learn different ways to organise solutions (in table format) to problems which are procedural rather than numerical in nature.

The Hailstone Algorithm: This easy-to-execute procedure which generates what are known as hailstone numbers is presented for a number of reasons, one of which is its intrinsic interest while another is that it introduces the student to an open-ended problem. The core treatment explains that the first person known to have discussed the hailstone numbers generated by this procedure (though not necessarily under this name) was the mathematician Lothar Collatz while he was a student in the 1930s. The student is also told that with the advent of PCs there was a resurrgence of interest in the algorithm which sparked off a lot of recreational computer programming.

The written treatment is accompanied by a short PowerPoint presentation designed to portray the dynamic element of this algorithm. The discussion

of this piece explains how the algorithm got its name while also relating that it should not be called an algorithm, since, strictly speaking, an algorithm must always terminate after a finite number of steps. The student is informed that a procedure which has all of the characteristics of an algorithm except that it possibly lacks *finiteness* may be called a 'computational method'. Consequently, the Hailstone algorithm should be classified as a computational method as it is still an open question as to whether or not it always terminates.

The treatment discusses the basic notion of an algorithm in greater depth than done heretofore by explaining some of the associated terminology used by computer scientists. The piece concludes by describing three algorithms which the student is urged to investigate; one concerned with `Generating Rational Approxiations to` $\sqrt{2}$, a second based on an historical incident of the first century A.D. first described by the historian Josephus and now known as the `Josephus algorithm` while the third entitled `A Fibonacci Fraction Algorithm` makes contact with the Fibonacci sequence already alluded to in this work.

The Twelve Balls Problem: The core segment relating to algorithmic problems contains both a written accout of this famous problem entitled "The Twelve Balls Problem" and an accompanying presentation created in Flash. The primary purpose for including this celebrated weighing problem is to show the students an ingenious solution based on a use of *ternary*

numbers. It is not to be expected that senior cycle students reproduce the solution, rather that they be able to implement the given algorithm and, more importantly, appreciate it as a significant use of a number base other than the customary denary system. The treatment could be made the basis of a very worthy project.

Since the execution of card tricks is of an algorithmic nature this section concludes with a subsection devoted to explaining how a number of card tricks work. By contrast with the decanting and crossing problems whose solutions are described by means of tables, the unravelling of the workings of many card tricks requires no more than an adroit use of simple algebra.

7.9.1 Card Tricks

The blurb announcing a Mathematical Association of America minicourse entitled "Introduction to Mathematical Card Tricks" remarks:[29]

> Card tricks liven up any gathering – including mathematics classes – and can help to convince people that math is fun and that there is a rational explanation to some seemingly impossible events.

Because anything which smacks of magic appeals to most young people, such tricks can be an ideal way of engaging the interest so necessary for the teaching and learning of mathematical ideas and methods.

As this section describes, many such tricks are easily explained using no more than simple algebra. The first few tricks treated in detail in the core

[29]Mathematical Association of America, "Introduction to Mathematical Card Tricks", *Focus Magazine* 23, no. 7 (October, 2003): 7.

segment are, for the most part, of this type. They are taken from Blum's *Math: Tricks, Puzzles & Games* and are described mainly using Blum's own person-to-person style.[30] However, the explanations supplied are considerably more elaborate than those given by Blum, who is often content with mere statements of fact rather than genuine elucidations.

The following is a list of the tricks (due to Blum) treated in the core along with a short description of the various lessons each has to teach:

1. The `Mind Reader` trick is perhaps the simplest of all the different ones given by Blum. From a pedagogical point of view, its lesson is one of showing the student how to transmit information secretly using a *code*.

2. The `Last Card` is an easily executed one based on the fact that dealing cards in reverse is arithmetically equivalent to reversing a sequence. The core explains the trick using no more than a single unknown. Afterwards, the student is shown how arithmetic sequences are reversed and how to use this knowledge to extend the trick. The trick reveals the simple use of algebra along with an arithmetic device that the student might not otherwise learn off but which proves most useful in having a computer program print a given (consecutive) sequence in reverse. The `Tree of Clubs` trick is a simpler one based on the same property.

3. The `Selected Card` trick is based on the special property of the number nine already discussed in the section dealing with modular arith-

[30]Raymond Blum, *Math: Tricks, Puzzles & Games* (US: Sterling Publishing Co., Inc., New York, 1994)

metic. As such it can be used to reinforce this property in an engaging manner. The explanation given in the core does not refer to this property but simply uses basic algebra to explain the trick *ab initio*.

4. Blum's explanation of what he calls the `Come Forth` trick is an arithmetic one which runs to no more than three lines and is one a student may find difficult to decipher. Because what one does in performing the trick can vary from one execution to another, the analysis given in the core uses simple algebra rather than arithmetic. While doing this makes it possible to explain the mechanism of the trick clearly in no more than a paragraph, the treatment also gives a stage-by-stage breakdown of the trick so that a student sees the algebra clarifying the function of each step in the sequence of steps that constitutes the overall card trick.

5. The `Seven To Eleven` trick is also best explained using algebra. The treatment shows the student how to organise the information given in table form; a visual strategy which proves of great assistance in understanding the workings of the trick. It also proves fruitful in indicating to the student how to generalise the trick so as to lend it greater variety. The case study of this simple trick, in addition to showing the analytic power of algebra, also teaches the student how effective a well-conceived table can be in lending clarity to an investigation.

The core also treats two other tricks in some detail. These are `The Twenty-One Card Trick` and `The Best Card Trick Ever`.

The Twenty-One-Card Trick: The first part of the exposition relating to this easy-to-execute trick simply teaches the student how the trick is performed. This is best re-inforced by having the students break into pairs to practice the workings of the trick a number of times for themselves. Once they can perform it effectively, their task is to explain why it works.

The solution presented examines the mechanism by which a certain selected card is isolated. This is done by carefully tracing the journey of the card chosen from the beginning of the trick to its end. The trick is at bottom a 'squeezing-algorithm' which forces the card into a unique (fixed) position – a technique that permeates topology.

Many young students may not be inclined to regard the analysis given as mathematical, simply because neither arithmetic nor algebra is used explicitly. However, this is a misconception based on the mistaken belief that mathematics is solely concerned with numbers or geometrical figures. Thus, it is important to impress on those who follow the explanation that it also passes for genuine mathematical reasoning, being the application of logic to the study of an algorithmic process. Should they wish for a 'more arithmetic' treatment, the students might like to present the solution as a series of inequalities, which eventually place the number being sought between two identical numbers.

The Best Card Trick Ever: This is a card trick which scores highly under the headings of fun and enjoyment but more so for the many genuinely mathematical lessons it teaches. It is one which some students find fascinating to the point of wanting to know every detail of how and why it works. Because of its importance, the core treatment contains an extensive written commentary which is accompanied by a PowerPoint presentation. Both the written account and the presentation are based on the 'reader-friendly' article "The Best Card Trick Ever" written by Michael Kleber.[31]

The problem is a fine example of the use of simple but very pure mathematical ideas involving concepts from information theory, the pigeon-hole principle, permutations, a different concept of distance and coding.

More elaborately, some of the many lessons both the description and explanation of the trick have to offer are described as follows:

1. the basic idea behind a code where information is being passed secretly, in this case, via the ordering of the cards. The treatment reinforces the use of the information measure already discussed in connection with the chessboard game `Where's the Queen?`

2. the pigeonhole principle, invoked in the context of this trick through the simple but crucial observation that among five cards there must always be at least two of the same suit. The teacher may decide to elaborate by stating this principle in the abstract: "If $k + 1$ or more pigeons

[31]Michael Kleber, "The Best Card Trick Ever" *The Mathematical Intelligencer* 24, no. 1 (Winter, 2002): 9-11.

are placed into k pigeonholes, then *at least one* pigeonhole will contain two or more pigeons" and illustrate it by noting that if 10 pigeons are to be placed into 9 pigeonholes, then one pigeonhole must contain at least two pigeons. This principle has already been referred to in one instance in this work and receives further mention in a number of other contexts. It is also known as Dirichlet's drawer principle or the shoebox principle.

3. the use of a different metric or 'distance' on a clock face which proves central to understanding why the trick works. This aspect of the problem links with the modular arithmetic of an earlier section.

4. the effective use permutations as a coding mechanism to transmit information. In passing it may be remarked that one nine year old, who was totally taken by the trick, had no trouble at all after a little help in appreciating that three cards can be arranged in six different ways and was thrilled to see the clever way these permutations are used to represent the numbers 1 to 6. His understanding of how the trick works was greatly re-inforced when he explained its workings to an older brother.

In the final section of this chapter the mathematics takes a back seat to the logic of the problem; yet the different modes of reasoning to solve such problems are regarded as deeply mathematical. Again, the truly essential lessons to be taught are the means by which the problems are solved as opposed to the conclusions reached.

7.10 Logic Problems

In this last section a number of problems, not easily categorised under the previous headings, receive mention for the special lessons they can teach. The first few puzzles (relating to deciphering information presented in the form of true and false statements) are included because the systematic manner in which they can be solved provide additional examples of the explanatory power of a well-conceived tabular scheme; a theme already touched upon with regard to certain types of algorithmic puzzles. The lesson is one of displaying the power of an organised approach to solve a problem, which otherwise might prove too intricate to unravel. This illustrates the point made by Schaaf who writes: "A problem regarded in medieval times as very difficult may prove to be quite simple when attacked by the mathematical methods of today."[32]

The core treatment begins by presenting one such 'True or False' problem which is then solved in a sequential manner. The solution is a short one requiring no more than two paragraphs. However, as a question it proves an excellent one promoting both the ability to think logically and the equally important skill of being able express oneself in like manner, concisely and accurately, in written form.

The second problem discussed is slightly more involved as it uses a greater number of different statements and allows for two possible premises. It is

[32] *Encyclopædia Britannica*, 15th ed., s.v. "Number Games and Mathematical Recreations".

first solved in a traditional manner on a case-by-case basis which entails a considerable amount of exposition. Then, mindful of Krantz's advice that it is impossible to overestimate the importance of visual aids, the student is shown how a table cleverly captures the implications of each of the statements based on the different premises. This leads immediately to the previous answers but without the need of extensive written explanations.

The third problem is a substantial one which also uses a two-by-two array as an aid to deciphering the information content of a considerable amount of data supplied in the form of eight statements. The solution of the puzzle begins by ruling out impossible connections between different elements of the array and proceeds by systematically updating the table as the information inherent in the data is uncovered and recorded. Once the table has been completed it proves a simple matter to draw inferences from it and to quickly answer the questions posed. While Krantz remarks that the problem has been solved using 'straight logic', the huge lesson it teaches is that of demonstrating how the reasoning scheme employed is highly efficient at extracting the desired information from what might otherwise be judged an overwhelming amount of data.

By way of a final word in relation to the use of 'True or False' problems as teaching aids, it should be pointed out that the examples treated by Smullyan in the second chapter of his classic book *The Lady or the Tiger?* must be deemed exemplary.[33] They teach logical reasoning in a charming manner

[33]Raymond Smullyan, *The Lady or the Tiger?* (US: Pelican Books, 1983)

and make for ideal examination questions. Besides being easy to set, such questions are not overly difficult to answer and have the virtue of addressing the Communications Standard.

The remaining puzzles discussed in this section are often referred to as lateral ones, since the form of thinking required to solve them is thought to differ from the vertical (or sequential) thinking so useful for unravelling other types of problems, samples of which have already received detailed attention in this work. While it may be that the ability to think 'laterally' calls on a different side of the brain than that utilised by other forms of reasoning, a newspaper article entitled "What Exactly is Lateral Thinking?" quotes Edward de Bono as stating:[34]

> The technical definition is that self-organising systems such as the human brain makes patterns. Patterning systems are asymmetric, which means that the route from A to B is not the same as the route from B to A. That gives rise to humour and creativity. Lateral thinking consists of processes or techniques for moving across patterns instead of moving along them.

The different problems treated in the core might be deemed lateral as the solution to each requires an effective use of an 'imaginative leap'. According to the nineteenth century mathematical logician Augustus de Morgan: "The moving power of mathematical invention is not reasoning but imagination."[35]

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[34]Edwatd de Bono, "What Exactly is Lateral Thinking?" *Irish Independent*, March 3, 2005): 16.

[35]R. E. Morritz, *Memorabilia Mathematica* (USA: Mathematical Association of America, 1993), 31.

Two of the problems are accompanied by short PowerPoint presentations.

The following list gives a short explanation of the different lessons each has to teach.

1. `The Three Light Switches` problem combines a clever use of logic and simple physical understanding, yet its solution proves to be a genuinely useful one in that it saves time.

2. `The Commuter` puzzle appears at first sight to lack vital information which a little thought subsequently reveals is not necessary. The economic manner in which the solution is arrived at highlights the importance of looking at a problem from the right vantage point; in this case from a temporal perspective rather than a spatial one.

3. `The Two Trains` puzzle is an invaluable one which teaches the value of insight over technique. It also allows the teacher to inform the student that one of the attributes of puzzle-solving is that it is not always the 'professional' who first finds the solution to a puzzle, or, if finding a solution, discovers the most elegant and imaginative path to its unveiling. In the account of this puzzle the professional is the legendary John von Neumann who solves the given problem by formulating and summing (with lightening speed) a geometric series. However, the instructive element of this probem is that by looking at it differently one can deduce the answer without recourse to 'technical mathematics'.

4. `The Buddhist Monk` is a classical problem taken from Koestler's *The Act of Creation.*[36] The solution given in the written core piece described by Koestler as brilliant is due to a young woman with no scientific training. This fact alone can serve as a lesson to the student that lacking the 'vertical' training of the academic, the amateur by necessity must be more creative and is often rewarded by finding the shortest and most beautiful route. This particular problem has an accompanying PowerPoint presentation which also provides a 'mathematical solution' presented in an effective visual manner before describing the young woman's solution. The mathematical and the purely logical solution both involve the lateral idea of a temporal superposition.

5. The `Friends and Strangers` puzzle is another of those problems where one appears to have no numerical facts on which to begin an attack, yet the problem is solvable. The clever but visual scheme by which the problem is solved is another application of the pigeonhole principle. The student is told that this puzzle is an example of an *existence* problem where one can show indisputably that something exists but one may not be able to point it out explicitly. For example, one can be absolutely certain there are two people in Dublin city with the same number of hairs on their heads but unable to produce the two people in question.

[36]Arthur Koestler, *The Act of Creation* (UK: Picador, 3rd ed. 1978), 183-184.

6. `The Russian Postal System Puzzle` is offered as an exercise whose answer is given in an accompanying short PowerPoint presentation. The lesson this puzzle teaches is how one can achieve secure communication over an insecure channel.

Sudoku: Since the solving of a Sudoku puzzle, unlike the 3×3 magic square problem, does not involve an arithmetic (or other mathematical) element, it must be considered to lie firmly within the domain of logic. However, within this domain, it has much to teach the would-be-learner of mathematical thinking since the logical processes it employs such as:[37]

1. scanning
2. cross-hatching
3. searching for the lone number
4. mark up
5. twins
6. matching pairs

and needed for its successful solution are worthy examples of the many useful procedures employed in any form of sequential analysis.

[37]Michael Mepham, *Sudoku*, (UK: Pan Books in association with *The Daily Telegraph*, 2005), vii-xiv.

Furthermore, the depiction of the full solution-path to such a puzzle (graded as moderately difficult) is ideal material for a slide-show presentation by which young learners can collectively participate in the art of logical reasoning over the span of a single class period.

7.11 Summary

The ten sections of this chapter indicate a variety of ways by which recreational material can be harnessed by the teacher in an engaging manner to motivate and make contact with mainstream school mathematics. The great bulk of the proposals made are underpinned by explicit and extensive exposition in the form of written pieces and presentations residing in the core folder relating to this chapter.

In what has been covered, the student has been exposed to many different types of problem which might be grouped under the general headings:

1. numerical – requiring no more than simple arithmetic but often a deal of persistence by way of sequential analysis

2. combinatorial – as arise in connection with games; evidenced here in particular in relation to the chessboard and the game of Mastermind

3. spatial – by which the student can acquire a greater sense of the three-dimensional world

4. logical – problems often requiring a substantial use of imaginative reasoning with little or no call for numerical calculation

Leaving to one side the host of general lessons taught by this material (which have been signposted at various junctures throughout the commentary), the significant points of contact made with mathematical methods and results are synopsised as follows:

1. summation formulae relating to the sum of the first n natural numbers, the sum of the first n odd numbers, the sum of the first n squares and the sum of an arbitrary geometric series
2. the use of pemutations
3. combinatorial reasoning
4. conjecture and proof; (a strong emphasis is placed on establishing what's possible along with showing in an equally convincing manner what is not possible. This is done to counter fears that the treatment of RM tends to be shallow.)
5. 'mod' notation and modular arithmetic
6. concept of a different metric
7. the power of algebra to reveal and explain
8. elements of probability
9. use of the binary and ternary number bases
10. information theory and the central role of the logarithmic function

11. the multiplication and pigeonhole principles

12. the use of tabular schemes.

Chapter 8

History and Culture of Mathematics

Victor Katz, the editor of *Using History to Teach Mathematics: An International Perspective* states:[1]

> Throughout the twentieth century, mathematics educators at various levels have argued that the history of mathematics is a marvelous resource for motivating and exciting students studying mathematics.

For Katz and the other contributors to this volume, the use of mathematics history in the teaching of mathematics is 'an idea whose time has come'.[2] In fact, somewhat later in the year 2000, the major ICMI study *History in Mathematics Education* appeared.[3] It poses the question: "*Does* history of mathematics have a role in mathematics education?"[4] By way of partial reply

[1]Victor Katz (ed.), *Using History to Teach Mathematics: An International Perspective.* (Washington DC, Mathematical Association of America, 2000), vii.

[2]Ibid.

[3]John Fauvel & John van Maanen, *History in Mathematics Education: The ICMI Study* (Dordrecht: Kluwer Academic Publishers, 2000)

[4]Ibid., xiii.

it states: "The ICMI study is posited on the experience of many mathematics teachers across the world that its history makes a difference: that having history of mathematics as a resource for the teacher is beneficial."[5] The eleven chapters of the Study co-written by numerous educators from countries world-wide are devoted to substantiatong this response.

Kleiner, writing in 2001, asks: "What does history have to offer the student and teacher of mathematics?"[6] His answer accords with the viewpoint of Katz *et al.* just quoted when he states: "First and foremost motivation. History points to the sources of the subject, hence to some of its cenral notions. It considers the context in which the originator of an idea was working in order to bring to the fore the 'burning problem' which he or she was striving to solve."[7] Elaborating on this perspective, he quotes Polya:[8]

> Having understood how the human race has acquired the knowledge of certain facts or concepts we are in a better position to judge how [students] should acquire such knowledge.

Kleiner remarks that this is "but one version of the 'genetic principle' in mathematics education" and adds, "as Polya notes one should view it as a guide to, not a substitute for, judgment."[9]

Later in 2001, Gulikers & Blom published a report of a survey they had undertaken of the recent literature on the use and value of history in geomet-

[5]Ibid., xvii.

[6]Israel Kleiner, "History of the Infinitely Small and the Infinitely Large in Calculus." *Educational Studies in Mathematics* 48, nos. 1-3 (2001): 137.

[7]Ibid.

[8]Ibid., 138.

[9]Ibid., 138.

rical education.[10] They indicate some of the research carried out in relation to teacher interest in the use of the history of mathematics:[11]

> Fraser and Koop (1978) and Philippou and Christou (1998) reveal that teachers are interested in the history of mathematics but at the same time, are not well resourced to use such material in their own teaching. A pilot study carried out in 41 secondary schools in Hong Kong presents similar results in Lit *et al.* (1999)

The major obstacle they note to teacher use of the history of mathematics is that "most teachers do not have enough historical expertise."[12] They also comment that "a gap exists between historians writing 'general' articles, and teachers, writing 'practical' articles."[13]

One of the purposes of this chapter is to indicate how a teacher might draw on the history of mathematics to motivate teaching and learning in the classroom in a straightforward manner without the need to undertake an in-depth study of that history.

A second aim of this chapter is to make known the essential role aspects of the culture of mathematics can be made to play in the mathematical *education* of young students. As Bishop writes:[14]

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[10]Iris Gulikers & Klaske Blom, "'A Historical Angle', a survey of recent literature on the use and value of history in geometrical education." *Educational Studies in Mathematics* 49, no. 2 (2001): 223-258.

[11]Ibid., 224.

[12]230.

[13]241.

[14]Alan Bishop, *Mathematical Enculturation: A cultural perspective on mathematics education.* (Dordrecht: Kluwer Academic Publishers, 1988), 3.

> *Educating* people mathematically consists of much more than just teaching them some mathematics. [...] It requires a fundamental awareness of the *values* which underlie mathematics and a recognition of the complexity of educating children about those values. It is not enough merely to teach them mathematics, we need to educate them *about* mathematics, to educate them *through* mathematics, and to educate them *with* mathematics.

The treatment which follows seeks to address the precepts referred to in this extract by revealing (in some measure) that:[15]

> Mathematics is one of the greatest cultural and intellectual achievements of human-kind, and citizens should develop an appreciation and understanding of that achievement, including its aesthetic and even recreational aspects.

8.1 History of Mathematics

Outside of its intrinsic interest, the history of mathematics provides the teacher with a rich source of examples, both pure and applied which can be harnessed to design questions which reveal mathematical results from an historical perspective while simultaneously testing such knowledge and technique at a level appropriate to senior cycle. The subsequent discussion (as it pertains to history) is divided into three subsections:

- Illustrations from History
- An Itemised History Time-Line

[15]National Council of Teachers of Mathematics, *Principles and Standards for School Mathematics*, 4.

- History Projects

each of which is now described in the order indicated by this listing.

8.1.1 Illustrations from History

The core segment supporting this section shows different aspects of historical mathematics being used to make contact with school mathematics in a substantial manner while informing the student of engaging and important facts. The exposition takes the form of a series of questions posed in an historical context and accompanied by fully worked out solutions which reveal the elementary mathematics being used. It is presumed that if these questions were to be used for examination purposes then they would form part of a suite of historical problems to be covered in the classroom.

There have been a number of uses of historical sources in this work to date with two such being illustrated in the Topic on Quadratics: the derivation of Hero's formula for extracting square-roots dating from the first century and al-Khowârizmî's method of completing the square from the ninth century. In addition, the derivation in 1785 by the eight-year-old Gauss of the sum of the first n natural numbers is treated in the Topic on Magic Rectangles as well as elsewhere. The following list gives the title assigned to each of the historical cameos along with a brief commentary on each which are to be found either in the core piece "Illustrations from History" or as PowerPoint presentations. They are arranged in chronological order.

`The Six Brothers` is an "inheritance" problem which appears on the ancient Babylonian tablet YBC 4608. It asks that a right-angled triangular field (of specified dimensions) be divided among 6 brothers by drawing equidistant lines perpendicular to the base of the triangle. The task for the student is to determine the relative sizes of the different allotments in a manner which the Ancient Babylonians could understand.

The solution given is a simple but clever geometrical one which clearly reveals that the proposed method of division assigns the brothers allotments which are in *arithmetic progression*. The student is then asked to find the exact area of each of the allotments. He is also given the exercise of explaining why the vertical boundaries of the physical allotments are also in arithmetic progression. While the solution of the problem requires no more than an elementary knowledge of geometry and arithmetic, the manner by which it is solved shows how a little ingenuity can reduce what might otherwise be regarded as a tricky problem to a simple one.

`Egyptian Multiplication` is problem 32 from the Rhind Papyrus which displays an ancient Egyptian method of calculating 19×15 and asks that the student to decipher its workings. It is quite a substantial problem which ends up surmising that the Ancient Egyptians knew about binary numbers (in some form) since the technique employed is essentially a binary one. As Anglin and Lambek remark: "By doubling and adding, the Egyptians were able to multiply any two natural numbers - without having to memorize

multiplication tables!"[16]

The student is also informed that the multiplication algorithm used is also known as 'peasant multiplication' which was practiced by Russian serfs as late as the nineteenth century. It is also remarked that the algorithm has certain similarities with the Fast Exponentiation Algorithm (FEA) which is based on *repeated squaring*; it is much used in cryptography.

This piece is accompanied by a short PowerPoint presentation of the same name which also illustrates the peasant multiplication algorithm. This latter theme is also the subject of a written piece which, along with the presentation, resides in the core in the History folder. Between them they provide an attractive avenue by which to introduce students to binary representations.

`The Truncated Pyramid` is the title of a short PowerPoint presentation based on a problem from The Moscow papyrus (1850 B.C.) which deals with the computation of the volume of a truncated pyramid. After presenting the example described by the ancient scribe, the student is informed that the general formula for the volume of a frustum must have been known to the Egyptians even though its derivation depends on the methods of modern calculus. This formula is then given and is followed by a speculative discussion as to how this result may have been obtained. The treatment involves nothing more than a use of similar triangles.

[16]W. S. Anglin & J. Lambek, *The Heritage of Thales* (USA: Springer 1995), 8.

`The Great Pyramid at Giza` tells how Thales of Miletus (624 B.C.- 546 B.C.) calculated the height of the Great Pyramid at Giza using no more than the proposition that two similar triangles (*i.e.* with identical angles) have their sides in proportion. The problem reveals a practical application of simple geometrical principles. This example and the next one, also involving Thales, are meant to show the power of Greek geometry to calculate distances.

`Triangulation` is another problem considered by Thales concerned with measuring the distance of a remote object such as a ship out at sea. It introduces the student to the idea of *triangulation*, a system of measurement which finds application in global positioning systems. The student is shown Thales's method of solution which again is based on a use of similar triangles but requires the uses of a carefully drawn diagram. It is then mentioned how problems such as this one can be solved more effectively using trigonometry to obtain a formula which allows the distance to the ship to be calculated from a knowledge of a baseline distance and two angles. While this general method avoids the inaccuracies that might arise as a result of a poor drawing, it relies on the availability of set of trigonometric tables.

This observation gives the teacher the opportunity to mention that the construction of such a table is no mean task; he can invite students to think about how one might go about furnishing such a table using one's own mathematical knowledge. A little thought reveals that this is by no means as elementary as one might first suppose.

The trigonometric treatment of this problem is covered in a separate core piece entitled "Two Tales about Thales" based on accounts by Webb.[17] This particular problem can also be solved more effectively using the cotan formula referred to in the Applications chapter in connection with surveying.

However, this particular core piece also deals with the vitally important issue of accurate measurement, an issue which is barely touched on in the HLC program. It is shown by means of an elementary example that an error of just half of a degree can cause an error of several tens of metres in a calculated distance. It is very important for students to realise that an inability to measure a quantity as accurately as one might like, can lead to an uncertainty about the true value of a related quantity. This is a lesson that anyone involved with measurement (such as scientists and engineers) must fully appreciate.

`The Shoe Maker's Knife` is an elementary problem in geometry and algebra relating to the arbelos which, because of its shape, is also called the 'shoe maker's knife'. The student is shown how to draw the arbelos and is then asked how Archimedes found the formula for its area. A suggested variation is to ask:

> Describe, with the aid of a sketch, what is meant by an arbelos. Derive Archimedes's formula for its area.

In addition to having to perform an elementary mathematical calculation,

[17]Stephen Webb, *Measuring the Universe, The Cosmological Distance Ladder* (UK: Praxis Publishers, 2000), 8-17.

this also tests the student's ability to *communicate* a description.

`What Age was Diophantus when he Died?` might be described as an historical word problem which requires the use of simple algebra for its solution. The harder portion of the problem is in deciphering the information given to formulate a linear equation whose solution reveals the age at which this famous third century Alexandrian mathematician died.

`Carpenter's Squares` is a short PowerPoint presentation describing al-Kaski's geometric demonstration in 1010 A.D. of the surprising result that the square of the sum of the first n natural numbers is the same as the sum of the first n natural cubes, *i.e.*, that

$$\sigma_1(n)^2 = \sigma_3(n)$$

or more elaborately, that

$$[1 + 2 + \cdots (n-1) + n]^2 = 1^3 + 2^3 + \cdots + (n-1)^3 + n^3$$

The highly visual proof is based on the use of gnomons or carpenter's squares. The student is told (and gets to see) that in geometry, a gnomon is a region which, when added to a polygon, produces a new polygon that is geometrically similar to the original one.

While this highly intriguing result may not be part of the current school curriculum, the closed formulae specifying both $\sigma_1(n)$ and $\sigma_3(n)$ are.

`Jordanus de Nemore` is the title of an historical cameo which relates how this mediaeval mathematician used a rudimentary algebraic symbolism around 1220.[18] This is long before its use by later European mathematicians such as François Viète (1540-1603) who is normally credited with the invention of algebra through his use of letters in calculations. The core piece describes the four general problems de Nemore considered along with the particular examples he chose for purposes of illustration.

The student's task is to use modern algebraic notation and methods to solve these problems. The solutions reveal a number of systems of simultaneous systems of equations, of the type which feature regularly on HLC papers. With x and y denoting the unknowns and a and b the 'givens' in each system, these are as shown:

$$\begin{array}{rcl} x+y & = & a \\ x-y & = & b \end{array} \quad ; \quad \begin{array}{rcl} x+y & = & a \\ xy & = & b \end{array} \quad ; \quad \begin{array}{rcl} x^2+y^2 & = & a \\ x^2-y^2 & = & b \end{array}$$

The latter two systems require a knowledge on the student's part of the quadratic formula – a formula which was not written down in general form until Michael Stifel did so in 1544. The core treatment takes care to also give de Nemore's own descriptions of his methods which show clearly that he anticipated much of what was revealed some three hundred years later.

`A Mistaken Claim` concerns an assertion made in 1494 by Luca Pacioli to the effect that the number

$$M = 1 + 2 + 2^2 + 2^3 + 2^4 + \cdots + 2^{26}$$

[18] John Fauvel & Jeremy Gray, *The History of Mathematics: A Reader*, (UK: Macmillan Press, London, 1988), 240-243.

is a *prime* number. The student proves that this is not so by using his modern knowledge of the sum of a finite geometric series to show that $M = 2^{27} - 1$. He then proceeds to use the factorisation

$$x^3 - 1 = (x - 1)(x^2 + x + 1)$$

with $x = 2^9$, to reveal that $M = (2^9 - 1)(2^{18} + 2^9 + 1)$ – a result which reveals that M is a product of two non-trivial natural numbers and so is composite.

Here the school mathematics used is the sum of a geometric series along with a factorisation familiar to current HLC students. However, as the core commentary notes, since this demonstration may make use of mathematical techniques which were unknown to Friar Pacioli, one might ask if it is possible to disprove his claim using no more than the mathematics of his day. The exposition then reformulates the original question to indicate that this is possible and presents the required solution which it avers is another example of a mathematical vignette.

The treatment finishes by pointing to a very strong generalisation suggested by the different refutations given.

`The Sum of Two Squares` problem relates to Pierre Fermat's theorem that each prime number of the form $4n + 1$ can be written as the sum of two squares. The student is asked to show this for the first two primes of this form; to verify the identity

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (bc + ad)^2$$

and then to use this result to express the product of the two primes as the sum of two squares also. Furthermore, in order to emphasise the domain of application of Fermat's result, the student is asked to show that it does not extend, in general, to prime numbers of the form $4n + 3$.

While the solution of this problem requires no more than a basic knowledge of algebra, it does test whether a student can interpret a theorem and also furnish a counterexample.

`An Elusive Result` begins by relating that Jacob Bernoulli (1654-1705) and his brother Johann Bernoulli (1667-1748) long sought in vain to find the sum of the series[19]

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

which their compatriot Leonhard Euler (1707-1783), showed in 1734, sums to $\frac{\pi^2}{6}$.

The question the student is asked is to use this fact to deduce that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots = \frac{\pi^2}{8}$$

Besides revealing a famous sum which arises in the study of probability, this problem is a simple exercise in manipulating (absolutely convergent) series. The solution shows how this is done efficaciously using summation notation. A related exercise gives the sum of the reciprocals of the fourth powers in terms of π.

[19]H. D. Ebbinghaus *et al.* (eds.), *Numbers*, (US: Springer Verlag, 1991), 147.

The exposition relates that there is no known closed formula for the finite sum of the reciprocals of the third powers but that this sum is known as Apèry's constant since that mathematician proved its irrationality.

`Three Historical Figures` is a substantial problem linking three mathematicians. It begins by revealing that the great German mathematician Gottfried Leibniz (1646-1716) falsely asserted in 1702 that the quartic polynomial x^4+1 could not be written as a product of two quadratic polynomials with real coefficients. The student's first task is to demonstrate otherwise. As the exposition explains, the insight required to find a factorisation of x^4+1 expeditiously is to adjust the expression so that it can be written as a difference of two squares. Thus this stage of the problem constitutes a technical exercise in a clever use of the difference of two squares formula $a^2-b^2=(a-b)(a+b)$.

However, the problem is then given greater substance by requiring that the factorisation yield *Aurifeuille's identity* which was first written down by the French mathematician Aurifeuille in 1879. This again involves no more than algebraic insight implemented with skill. Finally the problem is rounded off by jumping back in time ten years to inform the student that the Parisian mathematician Fortuné Landry worked on numbers of the form 2^n+1, and that in 1869 he demonstrated, after great labour, that the eighteen-digit number $2^{58}+1$ is composite. The student's task is to explain how Aurifeuille's identity renders the factorisation of $2^{58}+1$ routine. As

the commentary remarks, "This story illustrates the power of theory over practice – or why theory is useful."

The problem is accompanied by a related exercise which asks that the factorisation be used to find the four non-real roots of the quartic using the quadratic formula. The student is to confirm the results obtained by a direct use of complex numbers.

`President Garfield` is a derivation of Pythagoras's theorem dating from 1881 and due to the 20th US President James Garfield (1831-1881). The student is shown a diagram similar to the one Garfield drew and asked to give Garfield's proof which requires no more than knowing the formula for the area of a right-angle triangle and that of the area of a trapezium. This demonstration could be considered a geometrical vignette.

`The Pentagon` describes a construction of a regular pentagon given by the famous puzzlist H.E. Dudeney who was also an accomplished amateur mathematician.[20] The student's task is to explain why the construction works. This requires interpreting the instructions given along with supplying their rationale. This involves a straightforward application of Pythagoras's theorem but quite a sophisticated understanding of trigonometry by way of knowing that the result (established in the Topic on Quadratics)

$$\cos 72^\circ = \frac{\sqrt{5}-1}{4}$$

[20]H. E. Dudeney, *Amusements in Mathematics* (UK: Dover, 1970), 37-38.

is crucial to such a demonstration.

8.1.2 An Itemised History Time-Line

It has already been surmised (based on a number of small surveys) that, in general, students' present understanding of mathematical history is either non-existent or at best quite sketchy. It would appear that outside of knowing a little about the Greek mathematicians Archimedes and Pythagoras, most are aware of few other individual mathematicians, either by way of name or what they might have accomplished. While the rare student might venture the opinion that Isaac Newton had something to do with the invention of calculus, very few will know of Gottfried Leibniz's independent contribution to the discovery of this branch of mathematics; despite the fact that it is this polymath's notation that is still in use to this day.

The core piece relating to this segment begins by presenting a table due to Davis and Hersh which indicates the engagement of different cultures with the development of mathematics over seven different eras.[21] This table is expanded upon by presenting the PISA 2003 categorisation of the 'mathematical eras'; a classification which has already received mention in this work.[22]

There then follows an itemised history time-line which recounts (using the present tense) various mathematical facts along with the estimated date of their discovery or invention. By contrast with the 'illustrations' of the

[21] Philip Davis & Reuben Hersh, *The Mathematical Experience*, 9.
[22] *The PISA 2003 Assessment Framework* (Paris: OECD, 2004), 35.

previous section or the detailed accounts of historians, most of these pieces are brief ones content with describing the mathematical fact in question and the person connected with it. The items appearing in this compilation (which makes no claim to being comprehensive) are chosen on the basis that they can be made to touch on school mathematics; something which is explicitly evidenced in the illustration section. The purpose of this time-line is two-fold: firstly by simply reading the account in full, a teacher or student can gain a more detailed glimpse of the development of mathematics than is possible from the two classifications just mentioned and secondly it can be used by a teacher (or examiners) to design questions with an historical theme. With regard to this latter point, a number of the entries are elaborated upon to indicate how this might be done; some actually contain a suggested question. It is to expected that teachers could make versatile use of the material to design questions of their own.

Those entries which concern such towering figures as Archimedes and Newton are of necessity considerably longer than those devoted to some of the lesser luminaries of mathematic's past but the particulars described are maintained at the level of senior cycle mathematics. Furthermore, whenever the discoveries of a mathematician, such as Fermat, proves especially rich in accessible material which students can easily grasp and appreciate, the exposition also becomes that little bit more expansive. On the other hand, the number of items listed for the twentieth century is quite small, being directly proportional to the fewer points of contact that can be made between

the mathematical findings of that century and the classroom. However, a number of themes do receive mention, with those relating to information theory, coding and cryptography having already been given special attention in this work.

While the historical account abounds with facts, it also teaches many elementary but telling lessons which can prove encouraging to students. As Lancelot Hogben writes in *Mathematics for the Million*: "It is very comforting to recall the curious mistakes which some of the most eminent mathematicians of the seventeenth century made ..."[23] It has already been mentioned by way of remarking on Leibniz's temporary mistake (with regard to the product rule of calculus which now bears his name) that knowing how even the great have erred can cause a student to adopt a more robust attitude; being careful in one's work but without condemning oneself too harshly for making an odd mistake. Students may derive further comfort on learning that the renowned French mathematician Hermite failed as a senior secondary student to derive the general formula for the sum of a geometric series although this was covered in his school book on Euclid's *Elements*. It may also be heartening to learn that, from the time of Pascal and Fermat onwards, almost every great mathematician has at some time or other been the source of a false conjecture or held a mistaken belief on a matter of probability.

More positively, the same student may like to know that Napoleon was of the firm opinion that: "The advancement and perfection of mathematics

[23]Lancelot Hogben, *Mathematics for the Million* (USA: W. W. Norton, 1993), 420.

are intimately connected with the prosperity of the state", while Abraham Lincoln cherished the work of Euclid for its logical development which he maintained helped make him a better advocate (as a barrister-at-law).

The Troubled History of Complex Numbers: At its conclusion the core piece presents a number themes compiled from extracts from the time-line. Remarking that the invention of printing in the fifteenth century led to the standardisation of mathematical notation one short table gives the dates when each of the arithmetical symbols first appeared. Another time-line traces the history of Linear Algebra while a final segment discusses "The Troubled History of Complex Numbers" where it is noted by R. Remmert that:[24]

> It is almost impossible for anyone today who already hears at school about $i = \sqrt{-1}$ being a solution of $x^2 + 1 = 0$ to understand the difficulties the complex (that is, imaginary) numbers presented to mathematicians and physicists in former times.

This piece outlines how tortuous the path can be to eventual acceptance of ideas which are initially regarded as quite foreign; it is a lesson in the progress of civilisation. It concludes by remarking that today the educational ambition is to have school students realise both the humour and sense in the quip: Imaginary numbers are real! Complex numbers are simple!

Notation: As the celebrated number-theorist Barry Mazur states:[25]

[24]H. D. Ebbinghaus *et al.* (eds.), *Numbers*, (US: Springer Verlag, 1991), 56.
[25]Barry Mazur, *Imagining Numbers* (UK: Penguin Books, 2004), 163.

> Notation, ..., is enormously important in mathematics. A seemingly modest change of notation may suggest a radical shift in viewpoint. The new notation may ask new questions.

To support this view one need only think of Arthur Cayley's introduction of matrix notation. When this renowned English mathematician first had the happy idea around 1855 of writing a system of simultaneous linear equations more economically in matrix form, he opened up a whole new world – one where the geometric nature of such equations became evident. When viewed in their 'true' setting, they are seen to be independent of the source that generates them. Within decades, matrix analysis found concrete application in Heisenberg's version of quantum mechanics. This example alone is a striking case of a serendipitous choice of notation having unforseen but far reaching beneficial effects internal to mathematics and external to it.

It is widely believed that the symbolism and the concise forms of expression that have developed within the subject of mathematics imbue it with much of its power. A separate core piece begins by giving a short and very incomplete time-line which indicates the approximate date when a word or notation (now in common use) is believed to have been introduced for the first time. This is followed by indicating a number of more recent notations that might profitably be incorporated into the school syllabus by way of modernising it in this regard.

In relation to the `Number Systems` the core piece comments that while students may be familiar with the use of the notation $\mathbb{N}$ for the set of natural numbers and, perhaps, $\mathbb{R}$ for the field of real numbers, they may not be clear

about all of the entries in the following chain which shows how the different number systems nest within each other:

$$\mathbb{N} \subset \mathbb{N}_0 \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

Likewise, neither they nor their teacher may know why certain symbols appearing in this chain were chosen. For example, it is quite possible that a teacher may be unaware that the symbol $\mathbb{Z}$ is used to denote the ring of integers because z is the initial of the German word 'zahl' standing for a whole number.

Every student should know the nature of this chain as part of their general mathematical education as it gives a compact overview of a significant vein in the mathematical landscape. Should someone enquire as to why the chain 'ends' as it does, then the teacher may avail of this opportunity to mention the fundamental theorem of algebra (first proved by the nineteen-year-old Gauss in his doctoral thesis of 1796).

The core treatment then suggests that a number of abbreviations and symbols used by the mathematical and scientific fraternities be given some exposure in the senior cycle classroom because the brevity which they achieve can be very useful at those times when it is important to present mathematical information in a concise and uncluttered manner. A number of items relating to the use of logical operators, the assignment convention $:=$, open and closed intervals of the real line, the designation of left- and right-hand limits at a point of continuity of a function, the greatest common divisor and

logarithms to different bases are indicated and illustrated by an example.

The segment concludes with three major recommendations. The first relates to summation notation. While such notation is familiar, if not overly used at LC level, it is suggested that the 'small sigma' notation be used (as is done at various junctures in this work) to denote the following series which arise so often in different contexts:

$$\begin{aligned}
\sigma_1(n) &:= 1+2+3+\cdots+(n-2)+(n-1)+n &&= \sum_{k=1}^{n} k \\
\sigma_2(n) &:= 1^2+2^2+3^2+\cdots+(n-2)^2+(n-1)^2+n^2 &&= \sum_{k=1}^{n} k^2 \\
\sigma_3(n) &:= 1^3+2^3+3^3+\cdots+(n-2)^3+(n-1)^3+n^3 &&= \sum_{k=1}^{n} k^3
\end{aligned}$$

where $n \in \mathbb{N}$. These particular 'sums' are part of the current HLC syllabus.

The efficacy of this notation soon makes itself felt in expressions such as the following:

$$\Big\{\sigma_1(n)\Big\}_{n=1}^{\infty} = \{1, 3, 6, 10, 15, \ldots\}$$

– the sequence of 'triangular numbers', or the famous equation

$$\sigma_1(n)^2 = \sigma_3(n)$$

or the result

$$\sigma_1(n)^2 - \sigma_1(n-1)^2 = n^3$$

which says that 'every cube can be expressed as the difference of two squares' while specifying the squares explicitly.

The second recommendation relates to the floor and ceiling $\lfloor\ \rfloor$ and $\lceil\ \rceil$ functions introduced in 1962 by K. E. Iverson. The core piece gives a number of examples where this notation proves useful in various contexts. Some of the examples listed and illustrated are as follows:

1. For each $m \in \mathbb{N}$, the least natural number n for which $\sigma_1(n) \geq m$ is given by

$$n = \left\lfloor \sqrt{2m} + \frac{1}{2} \right\rfloor$$

while

$$n = \left\lfloor \sqrt{2m} - \frac{1}{2} \right\rfloor$$

gives the greatest natural number n for which $\sigma_1(n) < m$.

The first of these formulae is established in the Topic on Quadratics where it arises in connection with an applied problem relating to computer networks. It has already been remarked that the derivation to be found there is ranked among one of the many mathematical vignettes presented in this work.

2. The number of digits in the denary representation of a number n is given by either of the expressions:

$$\lfloor \log_{10} n + 1 \rfloor \quad \text{or} \quad \lceil \log_{10}(n+1) \rceil$$

Establishing both of these formulae is a simple but very valuable exercise which reveals another important context for the logarithm function.

From time to time, there comes an announcement on the Internet and in the newspapers of a new contender for the title of 'the largest known prime'. As of the time of writing, the record holder, found on 26 February 2005, is the 42nd Mersenne prime

$$M(25964952) = 2^{25964952} - 1$$

Currently, very few students would be able to determine the exact number of decimal digits in the representation of this 'megaprime'. Fortunately, the latter of the two formulae just displayed is ideal for the task in this particular instance. Invoking no more than one of the logarithm laws, a straightforward calculation reveals that this prime number has exactly $7,816,230$ digits in its denary representation. It is suggested that the teacher stress that this gigantic number does not have a single proper factor.

3. The least number of years required for an investment of A to double if interest is compounded at the annual (decimal) rate r is given by

$$\left\lceil \frac{\log 2}{\log(1+r)} \right\rceil$$

where the base of the logarithm can be of one's own choosing. Establishing this formula is another valuable exercise for the student.

4. If $D(n)$ is the number of digits used in representing *all* the natural numbers up to and including n then

$$D(n) = m(n+1) - \left(\frac{10^m - 1}{9}\right) \quad ; \quad m = \lceil \log_{10}(n+1) \rceil$$

This formula receives mention in the chapter on Applications.

The third major recommendation relates to the mod notation. The reader is reminded that to date there has been a number of illustrations of the $m \bmod n$ notation (standing for the remainder of m upon division by n) given in recreational contexts.

However, the core piece then presents a scenario where this notation is required in an applied context. This relates to a swimmer who swims at a constant rate of 2 metres per second up and down a lane of a 50 metre pool, starting from one end of the lane. It is an important practical exercise for the student to show that if $f(t)$ specifies the distance of the swimmer from the starting end of the lane at time t (measured in seconds) then $f(t)$ is defined *piecewise* over $[0, 50]$ as follows:

$$f(t) = \begin{cases} 2t & ; \quad 0 \leq t \leq 25 \\ 100 - 2t & ; \quad 25 \leq t \leq 50 \end{cases}$$

with

$$f(t + 50n) = f(t)$$

for each $n \in \mathbb{N}$.

The core piece shows how this is done and comments on how this particular example serves as a powerful illustration of the efficacy of the function notation. It is then pointed out that thus far, the student has been shown a periodic function, other than the standard trigonometric ones, occurring in an easily understood and credible context. The student is asked to draw a graph of this particular function to obtain a waveform consisting of a train of isosceles triangles of duration 50 seconds peaking at their centres. Thus he sees a periodic triangular waveform arise in a practical context. It is suggested that the teacher might inform him that waveforms of this exact type also arise in connection with electrical signals.

It is at this stage the student is given the task of explaining how a computer can use the information derived to tell where the swimmer is at times such as $t = 80$ or $t = 160$. Specifically, how does he get the computer to apply the correct piece of the given formula for $f(t)$ to produce the correct values of $f(80)$ and $f(160)$ respectively? It is surmised that the student will soon realise that the answer lies in first determining the remainder when each of these times is divided by (the modulus) 50, and that for any t,

$$f(t) = f(t \bmod 50)$$

Since he knows how the latter expression is calculated, he has solved his problem. By way of summary it may be said that in addition to studying an applied problem revealing a periodic function, the student is shown a realistic

situation where the 'mod' notation is crucial.

This particular example is followed by the `New Years Day Algorithm` which is no more than a formula involving both the floor and mod notations whereby given any year one can calculate the day of the week on which the 1 January fell or will fall.

The discussion of this piece brings to a close this subsection connected with the history time-line. The final section relating to history projects marks the end of the first portion of this chapter relating to the explicit use of the history of mathematics as a teaching and learning resource.

8.1.3 History Projects

The time-line described in the previous section points to a number of suitable themes for the subject of larger historical investigations which might serve as student projects. The following short list alone indicates a number of topic headings which a student, under guidance, might research and report on as part of an overall examination portfolio. Much of what appears on this list could also find a place in classroom teaching by way of short Topics.

1. `An Historical Time-Line:` to give a time-line of the different discoveries in mathematics down through the ages; in particular those that can be described at the level of school mathematics. While this is effectively done in the core piece, there is plenty of room for enlargement and elaboration.

2. `A Geographical Map`: to show the shifting centres of mathematical developments throughout the ages along with describing aspects of the mathematics involved.

3. `Men and Women of Mathematics`: to give an account of some of the men and women who have contributed to mathematics along with a description of their more accessible results. This to include study of the records of their thoughts and possible indications of how they made their discoveries. Déscartes is one mathematician who wrote about his methods as did the two other French mathematicians Henri Poincaré and Jacques Hadamard. Likewise George Polya and Imre Lakatos have also written extensively on 'mathematical discovery'.

4. `A Legacy of the Greeks`: to discuss the role of proof in mathematics – direct and indirect modes of reasoning – and perhaps to report on Gödel's results in relation to the nature of truth and proof.

5. `Rational versus Irrational`: to discuss the new and necessary presence that was $\sqrt{2}$. To indicate just how seriously doctrines were taken as evidenced by Hippasus being cast overboard to drown for revealing that the cherished belief that 'All is number' was in fact false. (Discussed in the Story of Root Two section.)

6. `Fibonacci Numbers`: Their origins, where they arise and their connection with the golden rectangle.

7. `Logarithms`: their historical origins and what they achieved; to include a discussion of aspects that are now obsolete and those which are still essential. What does the school-goer still need to know? How important are the different bases and which find the most use and why?

8. `Mathematical Notation`: the importance and benefits of good notation. One might furnish examples from history relating to:

   (a) Leonardo de Pisa introduction of the place system to Europe – a paramount example.

   (b) Leibniz's happy choice of notation for the calculus by contrast with that of Newton and the resulting effects of each.

   (c) Gauss's congruence arithmetic and its central importance to applications in cryptography and error-control coding.

   (d) Cayley's introduction of matrix notation and its subsequent impact within mathematics and external to it. (Discussed in the section dealing with Modernisation.)

9. `The Number Chain`: to give a brief description of the evolution of each subset in the chain

$$\mathbb{N} \subset \mathbb{N}_0 \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

10. `The Early History of Probability`: beginning with Pascal and Fermat and to include mention of Jakob Bernoulli's 'Law of Large Numbers' and the influence of Laplace.

11. `Famous Numbers`: the definitions and appearance of each of the numbers:

$$\pi \quad ; \quad \sqrt{2} \quad ; \quad \phi \quad ; \quad e$$

For example, in relation to the history of π one might note that it was only in 1706 that one W. Jones introduced the letter π for the circle ratio as the Greek counterpart of the initial p in the word periphery. However, an Egyptian approximation from around 1600 B.C. gives

$$\pi \approx \frac{256}{81} = 3 \cdot 16...$$

while Archimedes showed, by considering 96-sided polygons inscribing and circumscribing a circle, that

$$3\frac{10}{71} < \pi < 3\frac{10}{70}$$

In addition several Chinese mathematicians devoted their attention to evaluating π:

- Zhang Heng (78-139 AD): $\pi \approx \sqrt{10}$.
- Liu Hui (c. 260 AD): Calculated the areas of regular polygons with 96 and 192 sides to find that $3 \cdot 1410 < \pi < 3 \cdot 1427$.
- Zu Changzhi (429-500): Calculated the areas of regular polygons with 12228 and 24576 sides to find that

$$3 \cdot 1415926 < \pi < 3 \cdot 1415927$$

He also found the celebrated rational approximation:

$$\pi \approx \frac{355}{113}$$

12. `Men and Women of Modern Mathematics`: to give a broad overview of the current frontiers of mathematics and those involved. What do people do in mathematics right now and who is involved. The opportunity to teach what is happening currently in mathematics should be seized to talk a little about the men and women who make careers in it or in related disciplines. It might also be worthwhile to profile a number of top-flight mathematicians of the present, particularly younger ones such as Andrew Wiles of Fermat fame. Pictures show at a glance how young some of the current giants are. A number of videos are available which present great twentieth century mathematicians such as Paul Erdős giving elementary lectures on various mathematical topics. Seeing the humanity of these modern masters might be just what is needed to inspire students to study mathematics with a stronger sense of purpose.

8.2 The Culture of Mathematics

While the term culture is a broad one embracing many aspects, its interpretation as "the predominating attitudes and behaviour that characterise the functioning of a group or organisation" suffices for the purposes of the present discussion aimed at revealing (both for the teacher and student) the beliefs,

values, modes of expression and products of those who involve themselves with the discipline of mathematics.[26]

A number of themes are expounded upon in this the second portion of the chapter. They have been segregated as much as is possible to appear under the following topic headings:

About Proof: Central to the culture of mathematics is the almost sacrosanct belief in the necessity for proof. This segment informs the student about the nature of proof, why it is so highly valued while all the time taking care to make numerous contacts with school mathematics.

Mathematical Literature: The term literate is used here in the sense of being well-informed through reading about the general nature of the mathematical endeavour. Since this theme has been discussed in broad terms in the Introductory chapter where a number of specific recommendations are made, this segment discusses a specific Socratic dialogue designed to teach multiple lessons. This story is to be found in the core folder relating to this chapter.[27]

Using Mathematical Humour: Just as it is accepted that different cultures may have a sense of humour that is peculiar to them alone, it can also

[26] *American Heritage Dictionary of the English Language (fourth ed.)* (US: Houghton Mifflin & Co., 2000)

[27] It is also available in book form: Flannery D., *The Square Root of Two: A Dialogue Concerning a Number and a Sequence* (USA: Copernicus Books, 2006)

happen that different disciplines, be they scientific or otherwise, acquire a brand of humour which is intimately connected to the nature and methods of the discipline in question. In this regard mathematics is no exception as this segment makes clear while demonstrating the various ways this particular humour can be exploited to help teach a subject which some regard as 'dry'.

Interesting and Intriguing Results: By contrast with the topic 'About Proof' whose main concern is with all that that notion entails, this final section advocates presenting aspects of mathematics without the necessity of proving every assertion. Its general thrust is that while proof is valued and should be sought whenever possible, it should not act as an impediment which prevents a teacher from revealing and discussing important results. While the subjectivity of this section's heading receives some mention, the purpose of its catch-all title is to allow the section house a number of sub-topic areas, each of which is replete with ideas and methods by which the teacher can engage the interests of his students. Its main contention is that every student at senior cycle should be exposed to a number of the 'big ideas and concepts' of mathematics, irrespective of whether or not that student intends to pursue mathematics or related subjects at a future date.

Each of these themes is now discussed at greater length in the order in which they are listed here.

8.2.1 About Proof

While the central purpose of the core piece devoted to this theme is to convince the student of the need for proof, it also reveals the ingenuity and elegance associated with this endeavour. Furthermore, in addition to clarifying such important notions as inductive and deductive reasoning, it enlightens the student on the art of conjecturing and the devastating effect of counterexamples – those unwelcome 'ugly facts' which, as Thomas Henry Huxley noted, destroy otherwise 'beautiful theories'.[28]

The core treatment begins with a discussion of a topic entitled "Conjectures and the Dangers of Induction" designed to reveal the pitfalls associated with inductive reasoning. It is explained that this form of reasoning is where one seeks to draw a general conclusion from the evidence provided by a number of particular cases. The student is informed that no matter how large the number of specific instances of an assertion known to be true is, it is not valid to infer that it is true in general based on this evidence alone. An example is presented which reveals this in no uncertain terms: a proposition which in fact is true in the first

$$12,055,735,790,331,359,447,442,538,766$$

cases but which is not true in the next instance, a circumstance which means that the proposition cannot be regarded as an expression of a general truth

[28]Based on: "Science is organised common sense where many a beautiful theory is killed by an ugly fact."

or theorem. The student is told that general statements which are not substantiated by proof are properly described as 'conjectures' and retain their status as 'an educated guess' unless disproved by a single counterexample or placed on a firm footing by a well-founded argument or proof.

Six examples of conjectures drawn from different eras, some of which emanated from great mathematicians, are presented and shown to be false by a counterexample which, in some cases, was many centuries in the finding or required the power of a computer to unearth. The mathematics involved in each is completely accessible at the level of senior cycle and exposes the pupil to a wide variety of different facts and ideas from the history of the subject which could be used as questions or made the bases of other ones. But above all, these examples between them convincingly demonstrate that arguing from the particular to the general is simply not a valid form of reasoning.

The student is then informed of a number of 'open conjectures', meaning ones that have neither been proved true in general nor shown to be false in a specific case. Elementary number theory abounds with such conjectures whose statements are easily understood but which have defied every effort made over the centuries to pronounce on them one way or the other. As the Hungarian mathematician Pál Erdős is alleged to have remarked, "Babies can ask questions about numbers that grown men can't answer." Four such conjectures are presented, with one used to convince the student in a humourous manner that proving something is true in an *infinite* number of

cases may not be the same thing as proving it is true in *all* cases.

Having dealt with conjectures shown to be false and revealing others which have neither been disproved nor been elevated to the status of a theorem (having been substantiated by proof based on deductive reasoning), the treatment under the heading 'The Need for Proof' discusses what in fact is meant by a theorem. The author Simon Singh's excellent discussion of this notion in his celebrated book *Fermat's Last Theorem* is quoted from heavily as it explains with great clarity both the universality and permanence of proof.[29] Incidentally, it might be pointed out to students (as is done in the core piece of the Itemised History Time-Line) that Fermat's last theorem is an example of an assertion which was no more than a conjecture for over three hundred years until it was finally shown to be a general truth by Andrew Wiles and others in 1995.

Singh makes the point in relation to Pythagoras's theorem that while it was actually used by the Chinese and Babylonians one thousand years before Pythagoras, these cultures did not know that the theorem was true for every right-angled triangle nor had they a way of showing that it is true for all right-angled triangles. The reason for Pythagoras' claim to the theorem is that it was he who first demonstrated its universal truth. Singh then goes on to expand on what is meant by a theorem, which Krantz defines simply as:[30]

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[29]Simon Singh, *Fermat's Last Theorem* (United Kingdom: Fourth Estate, London, 1999)
[30]Review of a *A New Kind of Science* in *Bulletin of the AMS* 40, no. 1, 144.

> carefully formulated statements of assertions, supported by highly structured and tightly knit reasoning.

After remarking that the concept of mathematical proof is far more powerful and rigorous than the concept of proof casually used in our everyday language, or even as understood by physicists or chemists, Singh goes on to describe *deductive* reasoning, which might be briefly summarised as arguing from generally accepted principles to particular truths; a form of reasoning the teacher should point out is the opposite of inductive reasoning.

The core exposition then discusses the legacy of proof bequeathed to posterity by the Ancient Greeks, who between the 6th and 5th centuries B.C. focused their interest on justification of propositions by proofs and for whom proof came to be regarded as the only means of establishing truth. Bashmakova and Smirnova make a highly telling observation in relation to the Greeks' discovery of new, previously unknown propositions:[31]

> A striking instance of such a discovery is the proposition about the incommensurability of the side and the diagonal of a square. We note that not only could this proposition not have been found intuitively but that it contradicts all accumulated human experience which tells us that there is a common measure for all magnitudes, *i.e.*, that all magnitudes are commensurable. One had to command highly developed abstract thinking to prefer a result obtained by proof to experience and intuition.

This work on numerous occasions stresses the importance of teaching the student the civilising lesson: "to prefer a result obtained by proof to experience and intuition."

[31] Isabella Bashmakova & Galina Smirnova, *The Beginnings & Evolution of Algebra* (US: MAA 2000)

The method of proof, known as *reductio ad absurdum*, by which the Greeks established the *irrationality of* $\sqrt{2}$ and the *infinitude of the primes* is believed to be amongst the greatest testaments to the power of human thought. The mathematical content of these two wonderful proofs, in terms of calculation, is absolutely elementary but the indirect reasoning used to establish their truth is spectacular. As such they are as important and enriching a legacy as any great poem or play.

Because these ancient demonstrations are quite simple to follow, involving short but ingenious 'proofs by contradiction', there is no reason why second level students should not be shown them and asked to reproduce them as they would the lines of a poem or the text of a scene. Above all they should appreciate that, the 'dawning' on someone that it might be *impossible* to express $\sqrt{2}$ as a rational number or that the prime numbers are infinite in extent, were rightly viewed in their day as thoughts of great imagination. The irrationality of $\sqrt{2}$ is discussed under "Mathematical Literature' where the idea of using a Socratic dialogue to teach mathematics as one uses a play to teach English is explored in detail.

Christiansen, Howson & Otte are unequivocal about the role of proof in mathematical education saying:[32]

> The question should not be *if* there should be proofs in school mathematics ... rather *how* to have examples of proofs, *i.e.*, hypothetical-deductive thinking, at all level of school mathematics.

[32]B. Christiansen, A. G. Howson & M. Otte, *Perspectives in Mathematics Education* (Dordrecht: D. Reidel Publishing Company, 1985), 66.

They observe with disappointment that: "Indeed, in today's teaching of mathematics the idea of proof occurs only very rarely, somehow as a highlight for the most gifted students". They assert that the ability to argue, to reduce assertions to evident propositions or to already sufficiently checked ones, is of importance for every occupation and they stress that *the idea of proof is not restricted to the realm of formalised mathematics* but can be the object of learning experience throughout any kind of mathematical education.[33] While Davis and Hersh note that:[34]

> Euclidean geometry is the first example of a formalized deductive system and has become a model for all such systems. Geometry has been the great practice field for logical thinking, and the study of geometry has been held (rightly or wrongly) to provide the student with a basic training in such thinking.

this work has already illustrated by numerous concrete examples (drawn from recreational mathematics) that the idea of proof *need* not be restricted to the realm of formalised mathematics.

With the need for proof firmly established and its value as a teaching tool elaborated upon, the core piece treats different aspects of proof by way of examples. Any one of these subsections might be viewed as a rich source of possible questions at the level of school mathematics that illustrate the notion of proof from a particular perspective. The first short piece is concerned with having students understand what is meant by an "if and only if" statement

[33]Ibid., 67., emphasis added.

[34]Philip J. Davis & Reuben Hersh, *The Mathematical Experience* (USA, Penguin Books, 1981), 7.

or, equivalently, a "necessary and sufficient' condition. The exposition begins with a discussion of the assertions: "A number is even if and only if it is divisible by 2" and

$$m^2 \text{ odd} \iff m \text{ odd} \quad ; \quad m \in \mathbb{N}$$

This is followed by brief discussions of a number of more substantial but easily understood assertions of the same nature relating to `Mersenne primes`, `Magic Rectangles` and `The Rule of Nines`, all with the purpose of having the student appreciate the difference between a necessary condition and a sufficient one. This segment concludes with a short remark on terminology illustrated by a number of 'one line' statements.

Under "An Acceptable Proof" the student is shown a simple proposition and a purported proof of same and is asked to explain if the putative solution is indeed one. As the commentary notes, the answer given is not a proof since it begins by assuming the truth of the assertion and then showing through a chain of inferences that this assumption leads to a statement which is clearly true. Regrettably many HLC students have been led to believe that this form of argumentation passes for proof. However, it is pointed out to the student that, strictly speaking, to prove the given assertion one should now see if the recently obtained chain of inferences can be reversed. If this can be done, then the assertion is true since it follows from a known truth through a chain of logical reasoning.

The main purpose of the `True or False?` suite of six questions is to

illustrate the importance of exercising care in mathematical arguments. The questions, while elementary in appearance, test the students's ability to analyse an argument for hidden unwarranted assumptions. Short answers with accompanying comments are given in each case.

The first example teaches the simple but powerful lesson that great care must be exercised when dealing with inequalities. The second example presents a formula which is shown to be dimensionally wrong. The student is told that wherever possible it is a very useful check to carry out a dimensionless analysis on a formula. No matter how cleverly it may have been derived, if its dimensions don't accord with the underlying physics of the problem being investigated, then one knows that the formula simply cannot be right. Thus one is saved from wasting time using it and is encouraged to re-examine the derivation in the hope of finding a fault in the reasoning which may, if one is lucky, be corrected.

These examples are followed by an 'infamous' argument purporting to prove by algebra that $2 = 1$. It is the experience of many teachers that students find this example intriguing. It also acts for them as a powerfully strong warning that one must exercise caution when cancelling or dividing. The next offering is of the same type and asks for the flaw in an argument 'showing' that $1 = -1$. The student is told that the mistake is in assuming that

$$\sqrt{ab} = \sqrt{a} \times \sqrt{b}$$

is a general law, which it is not. It is remarked that this example highlights the importance of knowing the exact circumstances under which a rule applies. It serves to impress on students why part of the "culture of mathematics" is an insistence on a precise statement accompanied by a proof – a theorem. They come to understand that it is theorems which protect one from making false application and lapsing into absurdity.

The student is then shown a method for finding the sum of the infinite geometric series which on the surface seems irreproachable until it is pointed out that its assertion leads to patently absurd results. This subtle problem illustrates the danger of hidden assumptions. Just as one must exercise care when dealing with quantities which may become 0, one must be equally vigilant that quantities under consideration do not become infinite. For example, while $2 \times \infty$ and $3 \times \infty$ might both be regarded as being infinite, to judge them as 'equal' and so feel entitled to cancel the common '∞' leads to the absurdity $2 = 3$. The example serves as an excellent introductory lesson to the notion of convergence. The final example is another one where the mistake made is quite hard to detect; it provides yet another forceful illustration of the care that must be taken when dealing with inequalities.

It is important for students to understand that while most theorems make positive assertions about what is true or possible in general, some of the more important and famous results in the history of mathematics are about showing that certain goals are *impossible* to achieve. To this end the student is presented with the following short list of "The Impossible":

1. It is impossible to express $\sqrt{2}$ as a rational number.
2. It is impossible to 'double the cube'.
3. It is impossible to 'square the circle'.
4. It is impossible to trisect an angle with compass and straight edge alone.
5. It is impossible to deduce Euclid's parallel postulate from his other axioms.
6. It is impossible to find the centre of a circle with straight edge alone.
7. It is impossible to solve the general quintic equation.
8. It is impossible to create a machine which would tell for every statement whether it is true or false.

While the teacher will undoubtedly need to expand upon these results, they allow the young mind to gain a glimpse of the subtle nature of the physical and mental worlds he shares with others.

The final portion of this section dealing with proof talks about a separate core piece entitled "Mathematical Vignettes". At least four such have been described in this work to date; the first is the ingenious derivation of the square-root algorithm attributed to Hero which is discussed in the core Topic on Quadratics; the second, from that same section, is the manner by which the first value of n making $\sigma_1(n) \geq m$ is expressed in terms of m using

the floor function; the third is the clever combinatorial argument used to count the number of rectangles present on the chessboard discussed in the chapter on Recreational Mathematics while the fourth, also from that chapter, is Gauss's method of adding up the first one hundred natural numbers; a mathematical argument which Hilton claims achieves 'a transcendental beauty and elegance'.[35] Speaking of elegance, Trigg says, "an elegant solution is generally considered to be one characterised by clarity, conciseness, logic and surprise."[36]

The core piece presents a number of other 'mathematical gems' and indicates others which might form a suite of solved-problems which reveal to the student the ingenuity and elegance of mathematics.

List of Mathematical Vignettes: The following is a list of possible mathematical vignettes with an indication of their content.

1. The derivation of Hero's formula for extracting square-roots (covered in Topic on Quadratics)

2. The demonstration that $n = \lfloor \sqrt{2m} + 1/2 \rfloor$ is the first value of n for which $\sigma_1(n) \geq m$ (covered in Topic on Quadratics)

3. The evaluation of $\cos 18°$ (covered in Topic on Quadratics)

[35]Peter Hilton, "The Joy of Mathematics: A Mary P. Dolciani Lecture", *College Mathematics Journal* 23, no. 4 (September 1992): 276.

[36]Charles W. Trigg, *Mathematical Quickies* (UK: Dover, 1985)

4. Gauss's method of adding up the first n natural numbers (covered in RM)

5. The combinatorial argument used to count the number of rectangles present on an $n \times n$ grid (covered in RM)

6. The combinatorial argument used to count the number of shortest routes present on an $m \times n$ grid (covered in RM)

7. Expressing ϕ^n as a linear combination of ϕ and 1, *i.e.*, as a linear polynomial in ϕ; connection with the Fibonacci sequence via recurrence relations.

8. Proof of the irrationality of $\sqrt{2}$ (covered in the $\sqrt{2}$ story)

9. Proof that an irrational raised to the power of an irrational may be rational (covered in Mathematical Vignettes)

10. Bounding Apery's constant (covered in Mathematical Vignettes)

11. The derivation of two widely used anti-derivatives by means of Euler's formula (covered in this Mathematical Vignettes)

12. Proof of the divergence of the harmonic series (covered in Mathematical Vignettes)

13. The elementary refutation of Luca Pacioli's 1494 claim that the natural number $1 + 2 + 2^2 + 2^3 + 2^4 + \cdots + 2^{26}$ is prime (covered in Illustrations from History)

14. The factorisation of $x^4 + 1$ leading to *Aurifeuille's identity* and a proof of the compositeness of $2^{58} + 1$ (covered in Illustrations from History)

15. President Garfield's derivation of the theorem of Pythagoras (covered in Illustrations from History)

16. The demonstration that 'Fermat-five $F(5)$' is composite via $641|2^{32} + 1$ (covered in History Time-Line)

17. The demonstration that $341|2^{340} - 1$ (covered in History Time-Line)

18. The geometrical solution of the pump house problem (described in the Calculus section)

19. Euclid's proof of the infinitude of primes

20. The Peter and Paul problem from probability (covered in The Pleasures of Probability)

21. John Conway's proof of the countability of $\mathbb{Q}$ using base-11.

22. Euler's proof that there can be no tour of the bridges of Königsberg that crosses each bridge exactly once (covered in History and Mathematical Culture)

8.2.2 Mathematical Literature

As already indicated, the present section explores the idea of using a particular mathematical story as a means of teaching mathematics. In searching for a story with a unifying theme and an exposition that a young student would be able to follow under the guidance of a teacher, this writer came to realise that there isn't anything which is quite suitable. Most mathematical magazines which carry articles that might otherwise interest young secondary school students are, unfortunately, not accessible because of the level of mathematics assumed as a prerequisite. Furthermore, there appears to be very few books aimed at popularising mathematics which actually allow an intelligent reader to follow the exposition from start to finish, something which must cause frustration. There are various reasons why this is the case, one being that the author, overcome by a natural desire to reveal wonderful mathematical facts, often tolerates a less than self-contained treatment. One outcome is that the reader ends up knowing a lot but without really understanding much. A great deal of mathematical exposition has the appearance of being a collection of fragmentary tales which start out enthusiastically enough but fall short of achieving anything substantial.

As a result of not being able to find an appropriate mathematical tale which avoided these shortcomings and in order that the discussion of this section be centred on something specific and concrete, a Socratic dialogue based on the irrational number $\sqrt{2}$ and a celebrated sequence of rational approximations to it is included in the core section.

The tale, entitled *The Square Root of Two: A Dialogue Concerning a Number and a Sequence*, aims to teach by telling a mathematical tale at a slow pace. It is intended primarily to place mathematical ideas and modes of thought into the minds of senior cycle students. One of its first objectives is to help the reader appreciate the fact that the concept of number is more subtle than one might first imagine. It shows by way of an ancient Indian demonstration, which requires no more than the most primitive of geometrical notions, that there are numbers other than the ones used in the normal concerns of everyday life. It is a lesson in how a civilisation evolves in its understanding of both abstract and concrete concepts. Another theme is to explain how natural it is to observe patterns and to want to know if they persist indefinitely. Consequently emphasis is placed on the importance of exploration in discovering mathematical results for oneself.

The dialogue device is used so that all sorts of simple points can be raised and dealt with by way of questions and answers in as broad a manner as possible. The discussion between the two curious characters is deliberately expansive so as to allow the reader absorb the material as he reads. The exposition avoids using mathematics for its own sake preferring words whenever they will do the job even though they require more space. When mathematics is used, nothing beyond school algebra of the simplest kind is called on, but in ways which show clearly the need of this branch of mathematics. While the algebra used is simple, it is often clever, revealing that a few tools handled with skill can achieve a great deal. The hope is that young

imaginative readers can appreciate from the story this aspect of algebra – its power to prove things in general. Wherever appropriate, time out is taken (by way of a few lines of narrative) to explain or to remind the reader of some little rule or technique, be it arithmetic or algebraic, that he may have forgotten.

The story is laced with small puzzles of the type that mathematicians encounter when investigating a problem. Its proofs, some of which are highly significant, are portrayed as nothing more than the solutions to these puzzles and are discussed at some length by the enquiring duo. As such one of the story's most important attributes is that it conveys a sense of how mathematicians think, how it is they make their conjectures and how they place them on a firm footing afterwards, often with no more than a simple use of algebra. It reveals the process of discovery in mathematics, how one observation leads to another and so exemplifies the evolutionary manner in which new mathematical results are built on old ones.

A mathematician colleague and co-author of two books on differential geometry after reading an early draft of the story wrote:

> Mathematical experimentation, forming conjectures, checking special cases and finally using mathematical techniques to prove the conjecture is a process which is illustrated over and over. This is what mathematicians do and the book conveys this so well in addition to demonstrating the excitement and satisfaction mathematicians experience.

An accompanying PowerPoint presentation covering some aspects of the story is also to be found in the core segment related to this section.

8.2.3 Mathematical Humour

William Juraschek states:[37]

> Humor also has a place in the mathematics classroom. Jokes, limericks, puns, cartoons, and so on as well as spontaneous humor can help establish rapport, ease tension, and facilitate learning.

He observes that "although research on the specific effectiveness of humour in education settings is generally inconclusive", he avers that certain positive inferences can be drawn, one being that an apt use of humour relevant to the understanding of a paricular concept being studied can be a powerful reinforcer of that concept.[38] One can well imagine the English number-theorist Littlewood (1885-1977) agreeing with this assertion for he once wrote:[39]

> A good mathematical joke is better, and better mathematics, than a dozen mediocre papers.

This short section attempts to illustrate the wisdom in this humourous quip by discussing how examples of mathematical humour and faux-pas might be harnessed to enhance the teaching of mathematics in a light-hearted but meaningful manner. As a first illustration consider the joke which runs:

"Did you hear about the fellow with his head in the oven and feet in the fridge?"

"No."

"On average, he felt OK."

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[37]William Juraschek, "Humor", in *Encyclopaedia of Mathematics Education*, ed. Louise S. Grinstein & Sally Lipsey, 334-335. New York: RoutledgeFalmer, 2001.

[38]Ibid.

[39]Béla Bollobás, (ed.) Littlewood's miscellany, (UK: Cambridge University Press, 1990), 24.

Almost everyone who hears this tale laughs at the idea that anyone in this man's circumstances could feel in any sense okay. As such, it is an ideal vehicle for a teacher to discuss the limitations of one statistical measure to describe data adequately. It affords a humourous explanation as to why one should also supply a measure of spread along with a measure of location (given in this case by the average). It is much more likely to be remembered by a pupil than any of the five or six sentences a teacher may use to explain the importance of a measure of dispersion. Another joke which can be harnessed to reinforce the point tells of 'the fellow who drowned while attempting to cross a river whose average depth was 3 feet.'

As another illustration relating to the arithmetic mean one might frame a question along the following lines regarding a faux-pas of a former English Prime Minister.

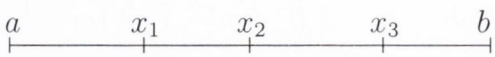
`Question:` Explain what is wrong with the statement attributed to Harold Wilson which goes:

> The ambition of the present Labour government is that every worker in the country will have a greater than average income.

The ability to write an answer to this question is as good a test of statistical understanding, if not better, as correctly answering the following question asked on a HLC paper.[40] In fact, the teacher might profitably use the Wilson story to give a context to this rather abstract problem.

[40] 7(c)(i) Section A, Paper II, 1996.

Real numbers x_1, x_2 and x_3 are each greater than a and less than b as shown on the number line:



Prove that $a < \bar{x} < b$ where $\bar{x}$ is the mean of x_1, x_2 and x_3.

Figure 8.1: Past examination question.

The following list indicates a number of other examples of mathematical humour which a teacher might exploit at various junctures in the school curriculum to emphasise the point indicated in the title of the example. Some are given without comment where it is felt that their intent is clear. However, in other cases a number of lines of elaboration are given along with a possible question that a teacher might pose to reinforce the lesson being taught.

`The Associative Property`: The associative property of arithmetic says that for any numbers a, b and c it is always the case that

$$(ab)c = a(bc)$$

- an important law which guarantees that the same numerical result is obtained whether one first calculates the product ab and then multiplies the answer obtained by c or one first calculates the product bc and then

multiplies this answer by a. However, in language a phrase such as a

a rich farmer's daughter

can in fact have two entirely different meanings depending on where one (mentally) places the brackets.

Question: Comment on the differences between the following:

a (rich farmer)'s daughter ; a rich (farmer's daughter)

All things being equal in matters of the heart which of these options would you choose and why?

`Binary:` There are 10 types of people, those who know binary and those who don't.

`Symmetry:` "Like the ski resort full of girls hunting for husbands and husbands hunting for girls, the situation is not as symmetrical as it might seem." Alan Lindsay Mackay

`A Faulty Use of Probability:` The probability of two independent events is obtained by multiplying the probabilities of the individual events. If these individual probabilities are 'small' then the probability of the joint event is a good deal smaller still.

Question: What would you say to a person who gave this as an argument for carrying a bomb onto an airplane?

`A Surprisingly Easy Task:` List *all* of the prime numbers ending in a 2 or a 5.

`Infinity Need Not Mean All:` Goldbach's Conjecture contends that:

> EVERY EVEN NATURAL NUMBER GREATER THAN TWO CAN BE WRITTEN AS THE SUM OF TWO PRIMES.

This conjecture has been verified by computer search for millions upon millions of even numbers without once encountering a counterexample. Here is one striking illustration:

$$2^{100} = 479 + 1,267,650,600,228,229,401,496,703,204,897$$

However, one mathematician asserts that he can prove this conjecture for an infinite number of even numbers and adds that his proof is 'a one liner'.[41] This marvel of brevity reads:

> The even number $2p$ is equal to $p + p$ for each prime p.

He is right, of course, since Euclid proved over two thousand years ago that there is an infinite number of primes. This is a humourous illustration of the difference between proving something is true in an infinite number of cases and proving it is true in *all* cases.

[41] Told to the author by Des Mac Hale, associate professor of mathematics, NUI, University College, Cork.

Mathematical Verses: This section contains a number of humourous mathematical verses which touch on

1. number theory via primes, twin primes and modular arithmetic
2. the probability of rare events
3. the notions of conjecture, proof and counterexample
4. the nature of number and approximations
5. history via the famous mathematicians mentioned
6. cryptography – the science of secrecy.

As such, they might be used by a teacher to introduce or reinforce a discussion concerned with the above concepts.

Twin Primes: The lawyer and mathematician Peter Rosenthal wrote:[42]

> The infinitude of primes,
> is the subject of plenty of rhymes,
> But we can't begin
> to prove there's a twin
> an infinite number of times.

– five lines which make mention of primes, the infinitude of same, twin-primes and the still unproven conjecture that there is an infinity of these twin-prime pairs.

[42]Peter Rosenthal, Professor of Mathematics, University of Toronto.

An Infinity of Primes: The US mathematician Nathan Fine (1916-1994) penned the following:[43]

> Chebyshev said and I'll say it again.
> "There is always a prime between n and 2n."

– a verse which is an ideal companion to the one just discussed as it also pertains to number theory. It states a very deep but easily understood result for which the famous Hungarian mathematician Pàl Erdős provided a much admired proof when he was a young man, although his was not the first proof of this celebrated proposition. Getting to understand what is being asserted is an enjoyable exercise for young students. Even verifying whether or not the statement is true for the first case $n = 1$ is sure to lead to a lively linguistic wrangle concerning the interpretation of the word 'between'. A teacher might in conjunction with any such discussion ask the disputants for their understanding of a statement such as, "Only children between the ages of nine and ten will be admitted."

Modular Arithmetic: The following limerick, which also concerns number theory, gives a humourous slant to modular arithmetic – a topic which has been already discussed in this work in relation to recreational problems:[44]

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[43] As quoted in B. Schechter, *My Brain is Open: The Mathematical Journey of Paul Erdős.* (New York: Simon & Schuster, 1998), 62.

[44] Anonymous, quoted in Martin Gardner, "Gauss's congruence theory was mod as early as 1801", *Scientific American* (February 1981): 14.

There was a young fellow named Ben
Who could only count modulo ten.
He said "When I go
Past my little toe
I shall have to start all over again."

Probability of Rare Events: The famous English physicist Arthur Eddington wrote the following limerick:[45]

There once was a hairy baboon
Who always breathed down a bassoon
For he said "It appears
That in millions of years
I shall certainly hit on a tune."

– a verse which touches on the notion that events with an extremely small probability of occurrence will be observed given enough time or trials.

Conjecture and Counterexample: The surname of the mathematician Pàl Erdős which appears in the following limerick is pronounced 'air dish':[46]

A conjecture both deep and profound,
is whether a circle is round.
In a paper of Erdős
written in Kurdish
a counterexample is found.

This humourous verse, stronger on rhyme than mathematical content, uses the terms conjecture and counterexample. Thus it allows the teacher to discuss both of these important notions. With regard to the latter, the

[45]Arthur S. Eddington, New Pathways in Science, (UK: Cambridge University Press, 1939.)

[46]Béla Bollobás, "A Life of Mathematics: Paul Erdős (1913-1996)." *Focus Newsletter* 16, no. 6 (1996): 1. Also quoted in Paul Hoffman, *The Man Who Loved Only Numbers* (UK: Fourth Estate, London, 1998) 7.

teacher might pose a number of would-be propositions which the students can dispose of quite quickly by means of counterexamples.

The Number Π: Harvey L. Carter wrote:[47]

'Tis a project of mine,
a new value of pi to assign.
I would fix it at 3
for it is simpler you see
than 3 point 14159.

This amusing verse has actual echoes from nineteenth century American history when a ludicrous proposal of the kind mentioned in these lines was laid before the Indiana legislature for its consideration. It is related that the proposal would have been enshrined in state law were it not for the timely intervention of a mathematics professor who pointed to the absurdity of humankind trying to impose its will on nature (who had decreed that things should be otherwise).

It should be pointed out that the number mentioned in the last line is an 'excellent' decimal approximation to π, an observation which might call attention to the various efforts made throughout history to obtain fractional approximations to π. The two most distinguished are perhaps that of Archimedes', 22/7, dating from before 200 B.C. and Zu Changzhi's, 335/113, a 5th century A.D. approximation accurate to six decimal places.[48] There are numerous series expansions, including the Gregory-Leibniz series, which

[47] As quoted in W. S. Baring-Gould, *The Lure of the Limerick* (US: Panther, 1970)
[48] Both these fractions can be obtained from the continued fraction expansion of π.

link π in a non-geometric way with rational numbers. The irrational and transcendental nature of π could also be of interest to higher level students.

On a more modern note, students might like to know that π is now known to over 2 billion decimal places, an achievement having no great impact on the physical sciences but which is used to benchmark new models of computer.

Number Theory Finds Application: The verse written by Elaine Sullivan which reads:[49]

> A princess was the Theory of Number,
> whom no practical use did encumber,
> 'til Cryptography the prince
> ('tis all secrecy since)
> did kiss her awake from her slumber.

alludes to the fact that the subject of number theory lay strictly within the province of pure mathematics for centuries until it was realised that it had numerous applications to cryptography.

8.2.4 Interesting and Intriguing Results

As evidenced by the findings of the surveys discussed earlier, most students leave school after the LC with little knowledge or understanding of matters mathematical in any sense which is broader than that presented to them within the narrow confines of the school curriculum. Outside of having the vaguest notions of where the subject is applied, many students are unaware of general mathematical facts relating to the world around them and their own

[49]S. Flannery with D. Flannery, *In Code - A Mathematical Journey*, (USA: Workman Publishing, 2001), as cited on p. 46.

interests. For example, with regard to the external world, how many could explain why it is that one flies over Iceland when travelling from Shannon to Chicago or why the map of the London Underground is so useful even though it clearly doesn't remotely represent any physical reality. Likewise, in terms of possible personal interest, who can say with confidence what chance they have of winning the National Lottery with the purchase of one ticket, or how it is one calculates the correct odds in a gambling 'double'. While it is certainly true that one can get through life quite successfully without knowing the answers to any of these questions, it is no burden to know them and it may even be one that one is more positive in attitude as a result of knowing these, or similar, facts.

This section discusses and advocates a policy whereby the teacher is granted the freedom to present mathematical results and ideas aimed at enlarging students' vision of the mathematical landscape so that they may be culturally enriched by knowing more about the 'beauty' of mathematics itself and the many 'interesting and intriguing' insights the discipline reveals about the external world and the internal one of the human mind. In so doing, the exposition will present numerous cases where this exercise makes contact with themes which can be used as part of school mathematics.

Regarding the use of subjective words such as 'beauty' or equally subjective phrases such as 'interesting and intriguing', one can only agree with Richard Bellman, a twentieth century mathematician renowned for his bril-

liant application of mathematics to 'real-world problems', when he remarks:[50]

> It is impossible to describe the beauty of mathematics, just as it is impossible to describe the appeal of music. Just as different people prefer different types of music, different mathematicians prefer different parts of mathematics. De gustivus non disputandum.

Perhaps, a more important question than that relating to the subjectivity of aesthetic judgments, is that which asks how one is to be guided in presenting 'interesting and intriguing' results and with what level of rigour. The first part of this question might be simply answered by saying that if such a result shows mathematics being used in a way that is informative while remaining at the level of senior cycle, then it should be deemed admissible and indeed welcomed.

With regard to the latter portion of the question, the suggestion is that there be a flexible approach which embraces the notion of revealing 'mathematics without total explanation'. This recommendation is not new and was mooted by the French mathematician Jean Dieudonné as early as 1959 in connection with the 'New Math'. More recently, his compatriot fellow mathematician and Field medalist Jean Pierre Serre lends his support when he advocates:[51]

> And one should also feel free to state theorems without proving them ...

[50]Richard Bellman, *The Eye of the Hurricane* (US: World Scientific Publishing Co., 1984), 34. Vandyck gives the Latin quote as *De gustis et coloribus non disputandum* which translates as "About flavours and colours it should not be argued."

[51]C. T. Chong & Y. K. Leong "A Mathematical Conversation with Jean Pierre Serre", *Mathematical Intelligencer*(December, 2003): 36.

Serre continues, "As for high school students, the main point is to make them understand that mathematics *exists*, that it is not dead (they have a tendency to believe that only physics, or biology, has open questions). The defect in the traditional way of teaching mathematics is that the teacher never mentions these questions. It is a pity." He notes that there are many such questions, and instances a number of examples from number theory "that teenagers could very well understand". In fact, all of the examples he mentions receive an airing in the section "About Proof".

To insist on always having to prove assertions is to shut off from students a vast domain of mathematics which has much to teach and to deny them access to some of the wonder and excitement of the subject.[52] The over-riding objective with presenting mathematical results in this manner should be to expect that what is being related is understood and appreciated to an extent that a student can later report on the material in a sensible fashion. Students would also benefit from being told that often researchers neither find it necessary, nor expect, to prove a given result for themselves once they are assured that a well-vetted proof exists; that this principle guides their approach in relation to the reading of papers and books written by other experts and they merely try to assimilate the relevant results so as to use them to advance their own endeavours.

A policy whereby students can be told results without expecting them to

[52]There are areas of the current HLC where important results relating to the normal distribution and Maclaurin series, to name but two, are taken as given.

be established with full rigour allows a wider variety of mathematical material to be presented in the classroom. But it also affords greater scope for the setting of questions: in addition to having students recall and describe the different facts related to them, one might also expect that they be able to answer straightforward mathematical questions connected with the material. The next few subsections indicate how this might be done in a number of areas beginning with elementary probability theory which is a field perhaps unrivaled for captivating and curious problems.

8.2.5 The Pleasures of Probability

Freudenthal states:[53]

> To explain to people what mathematics really means, one finds the most convincing examples in probability. I would like to do this using the questions in probability history started with (except for unavoidable forerunners) – the two problems which are said to have been posed by Chevalier de Méré to Pascal. ...
>
> I like the two de Méré problems, not only because they are mathematical jewels and highlights in the history of mathematics, but even more because they illuminate profound and characteristic features of teaching and learning mathematics.

The correspondence which arose between the 17th century Parisian mathematician Blaise Pascal (1623-1662) and his fellow countryman, the Toulouse mathematician Pierre Fermat (1601-1665), concerning these two particular problems is acknowledged as the starting point of the subject of probability as a genuine mathematical area of endeavour.

[53]Hans Freudenthal, *Mathematics as an Educational Task*, 583.

Any study devoted to revealing the relevance and pleasure to be had from probability could do much worse than to discuss both of these problems, but in particular the second of the duo known as the "problème des partis". This 'problem of points' is one which can be clothed in many different modern garbs that might appeal to students; one such being in the context of a snooker contest in it final stage where the leading contender needs to win but one more frame before an opponent can win both of the frames needed to ensure his victory. In assessing the relative chances of either player winning (based on the assumption of equal ability), the untutored mind can draw plausible but erroneous conclusions as did the Chevalier and his protagonist some four hundred years ago. The manner in which Pascal analyses the original problem is a lesson in clear and simple methodical thinking of which any decision maker would be justly proud.

A complaint made earlier about the type of probability question being set at present on the HLC course is that there is "a lack of emphasis on decision making". The discussion of the 2003 HLC papers indicates that while the questions asked touch somewhat on the personal and social lives of those taking the examination and demand clear thinking to answer them correctly, many have the appearance of being contrived. It is suggested that these artificial questions would be much stronger if given in a more creditable context requiring the same clarity of analysis. This attribute of probability to inform decision making in the presence of chance is illustrated in the first of the two sections which follow. In keeping with the historical roots of the

subject, it deals with games of chance where often the ability to make an informed decision is vital. The second section treats a number of problems that teach many lessons, one of the more important of which is that intuition is not always a safe guide.

A future syllabus, with both of these elements accommodated in its probability segment, would prove to be a more substantial and invigorating one, capable of addressing the "misunderstanding of probability" which Stephen Jay Gould claims "may be the greatest of all impediments to scientific literacy."[54]

Games of Chance: While ethical questions might arise in connection with analysing gambling problems with secondary school students, there is little doubt that many young adults are extremely keen to be educated in this regard and are never more attentive in class than when matters relating to chance are up for discussion. Whatever about its intrinsic relevance the question, "How much is a 5:1 double worth?" may be a far more engaging to a student than one asking for the cost of a mortgage. Those 'hard-chaws' who never take an interest in other aspects of mathematics may suddenly rouse themselves to lend a careful ear to every word that is being said in relation to assessing chances and making correct decisions as a result.

Such problems provide a very concrete and accessible area for teaching as well as being a source of excitement for many. To place any mention of

[54]John Haigh, *tAkiNg cHanCes: Winning with Probability*, (UK: Oxford University Press, 1999), epigraph.

them beyond the pale for a teacher is to deny the subject's beginnings and to overly restrict this educator. Often, there is no better way to remove the obfuscation in the thinking habits of a young person than to produce a pair of dice or to employ a deck of cards. Many gambling problems mirror real ones relating to the inherent unpredictability of those aspects of life whose study is deemed respectable for those actuaries and stock brokers who regard the intelligent assessment of risk as being of central concern.

Any course on probability should introduce students to the concept of 'odds' as used by the gambling fraternity and show them how the probability rules can be used to calculate these with ease.[55] To justify this recommendation one need think of nothing more sinister than the games of chance on offer in amusement arcades at seaside resorts or in the fairground on the day of a festival and which prove so hugely attractive to most young people, as well as a considerable number of adults. Yet, like their more sophisticated counterparts to be found in casinos, all of these games are 'unfair' in the mathematical sense that the odds offered the punter are never in accordance with the true mathematical odds. These odds are always such that the 'house' has an edge, known as the 'house's percentage', which ensures that in the long run the customer is the loser.

For this reason alone, it is incumbent on any educational system to inform the young people entrusted to its care of facts such as these and to equip

[55]It is worth noting that a discussion of 'odds' is an ideal way for young pupils to learn about the relative sizes of fractions. Determining whether an offer of odds at $5:4$ is a better proposition than odds at $7:6$ amounts to asking if $5/4 > 7/6$.

them with the ability to assess the level of inequity involved. Thus at the very least, they can make informed decisions as to whether or not they wish to indulge in such biased games. It is strongly recommended that students be shown how to use the elementary (classical) theory of probability to assess the proper odds associated with dice games, fairground games, roulette wheels, the National Lottery and card games such as poker. The next two subsections indicate how the teacher might use the game of poker and the National Lottery as means of revealing the truly educational value of probability. It should also be recalled that a number of important examples of probabilistic reasoning have already been given in connection with the game of Mastermind.

`The Game of Poker:` This game is ideal for illustrating how probability informs decision making. Most young people either know the game, or can be taught its fundamentals quite quickly, and relate to it quite naturally. It is educational, without any hint of corruption, just to know why the various hands are ranked as they are and might be regarded as some, as a legitimate part of one's culture. A core presentation rendered in Flash corresponding to this segment does just this.

One of its tutorials teaches the student the nature of the deck and explains such terminology as suits, ranks and honour cards. A second tutorial takes the student through the different poker hands beginning with royal flushes, straight flushes, four of a kind, houses, blues, straights, trips, two-pairs, pairs and ending with 'bust' hands. This segment of the presentation might

prove useful in the primary classroom to teach concepts relating to numerical sequences and pattern recognition. The heart of the presentation deals with the mathematics of the poker rankings. It might be regarded as a substantial and valuable class in the art of combinatorial reasoning. It guides the student through each of the enumerative arguments needed to count the total number of different ways each of the poker hands can arise, remarking that the rarer the hand, the higher its rank. Since the total number of possible poker hands is also given as $2,598,960$, the student can use the figures obtained to determine the *ab initio* probability of each hand being dealt.

But the more important aspect of the game is the manner by which it can be made teach the lesson that a knowledge of probability helps one to be a better player. The teacher can use the 'buying' stage of the game to inform his students of the important principle, known and adhered to by professional players worldwide, that one does not continue to invest in a poker hand unless one has a reasonable expectation that the 'pot odds' at least match the mathematical odds of 'filling' the hand. The constant violation of this simple rule by amateur players is one of the reasons they lose in the long run while their more informed playing companions grind out a steady profit without having undue recourse to bluffing. It should be clear that exactly the same principle should be observed in matters relating to real-world problems where an element of chance prevails.

`The National Lottery:` Lotteries were first introduced in Italy in the nineteenth century as a means by which governments could raise money. One astute member of the parliament asked to endorse the establishment of this lottery was keenly aware that in so doing, those charged with ruling the country were attempting to exploit the impulse of its citizendry to gamble and their ignorance of the true odds of winning such a lottery. He famously described such lotteries as 'a tax on idiocy'. A teacher might well use this historical cameo as a means of having his students understand that the buying of a lottery ticket is not a fair contract between him and the Lottery and that the purchase of such a ticket should be for the sense of involvement it engenders along with whatever dim prospect it affords one of actually winning. The mathematics involved in assessing one's chances of winning involve nothing more than simple combinatorics as revealed in an engaging article of Boland written at the level of school mathematics.[56]

This section on using games of chance to teach elementary probability concludes with a discussion of two examples elegantly solved by the application of a powerful principle. The second of these also informs the student how to fix fair odds in accordance with correct probabilities.

`The Symmetry Principle:` A number of problems concerned with the assessment of chance in relation to games can be solved very ingeniously using

[56]Philip J. Boland, "Topics in Probability Theory", *IMTA Newsletter* 87 (April, 1995): 49.

the principle of symmetry which proves so useful in solving similar 'real-world' problems. Two examples, which might rightly be considered vignettes, illustrate the power of this technique quite convincingly. The first states:

> Two cards are drawn from a deck of fifty-two cards. What is the probability that the second card drawn is higher in rank than the first?

This is an ideal problem by which a teacher can emphasise the difference between 'unconditional' and 'conditional' probabilities. In the absence of any knowledge of the first card drawn, the probability that the second card drawn is higher in rank than the first is exactly the same as the probability that the first card drawn is higher in rank than the second. In other words, by *symmetry* both probabilities are equal.

These two possibilities, along with the possibility that both cards are the same, constitute the complete set of possibilities and so have a combined probability of 1. Since the probability that the second card is the same as the first is given by

$$\frac{3}{51} = \frac{1}{17}$$

the probability that the second card drawn is higher in rank than the first is 8/17.

Students come to fully appreciate the elegant nature of this solution when they enumerate all the different ways this particular event can occur, calculate the individual probabilities in each case and then sum the results to obtain, by this more circuitous route, the answer found previously. This

exercise alone convinces the student that the symmetry principle can circumvent tedious enumerations.

The second example is an important one dealing with the inequity that often arises in two-person games caused by the fact that one player must go first. It states:

> Peter and Paul take turns at tossing a coin. The first to get a head wins. If Peter goes first, how much better are his chances of winning than those of Paul's?

Whereas this question (which will be answered momentarily) asks for an assessment of the chances of each contestant winning, the problem can be imbued with a greater relevance by asking that the answer obtained be used to determine the odds which makes the playing of such a game fair for both adversaries.

The elegant solution to this problem rests on the observation that if Peter loses the first toss, Paul is then in Peter's original position. The symmetry that results between Peter's original unconditional probability of winning and that of Paul's probability of winning (conditioned on Peter losing his first toss) is exploited as follows:

The probability of getting a head on an individual toss is 1/2. If Peter's probability of winning the game is p then that of Paul's is $1 - p$. However, if Peter loses on the first trial, then Paul's (conditional) probability of now winning the game is also p, as he is now in Peter's initial position. Thus

$$1 - p = \frac{1}{2} \times p \Rightarrow p = \frac{2}{3}$$

Thus Peter's chances of winning are 1/3 greater than Paul's.

Hence, for the game to be fair, Peter should match every unit wagered by Paul with two of his own.

A straightforward assault on this problem must take into account an infinite number of possibilities and involves the summing of an infinite geometric series whose sum leads to exactly the same results. However, in addition to highlighting the force of the solution obtained by an invocation of the symmetry principle, this method also displays another context where an infinite geometric series is encountered and as such is of value in its own right.

This particular problem is another example which teaches the student the important lesson that in order for games to be fair, the odds must be fixed in accordance with the actual probabilities.

Any discussion of gambling should contain a caution. It is of the utmost importance that the young student be made understand that while a knowledge of probability may be indispensable in making informed decisions, of itself it can never provide a complete picture. Whenever the human condition is also a consideration, a gambler would do well to recall the following words of Bentham:[57]

> Though the chances, so far as relates to money, are equal, in regard to pleasure, they are always unfavourable. I have a thousand pounds. The stake is five hundred. If I lose, my fortune is diminished one half; if I gain it is increased only by a third. Suppose the stake to be a thousand pounds. If I gain my happiness is not doubled with my fortune; if I lose my happiness is destroyed; I am reduced to indigence.

[57] Jeremy Bentham, *The Theory of Legislation* (England: 1790)

Remark: While this section has attempted to make a robust case for the discussion of some gambling problems within Irish senior cycle mathematics classrooms, it is hoped that the considerations given here would not be construed as an inducement to young people to gamble. Furthermore, a teacher would need to ensure that such a course of action is by no means repugnant to the cultural or religious beliefs of any particular pupil. Should this not be the case, then it is recommended that one forgo the treatment of such matters.

Probability and Intuition: The teacher who might prefer to avoid any explicit reference to gambling problems has still much at his disposal by way of illustrating the wonders of probability through intriguing examples that above all confound the intuition. Schultz and Leonard remark:[58]

> Unlike such traditional subjects as algebra and geometry, probability and statistics often engender a feeling of emotional or intuitive distrust. [...] Such conflicts with human intuition can bring a sense of challenge and excitement to the mathematics classroom.

This section discusses three examples that illustrate for students the notion that 'although intuition is a powerful tool in mathematics, it cannot be relied on exclusively'.[59] While these examples strike many as counter intuitive they lend themselves to playful experimentation and prove hugely

[58]Harris S. Schultz & Bill Leonard, "Probability and Intuition", *Mathematics Teacher* (January 1989): 52-53.
[59]Ibid., 53.

informative. It is recommended that students be exposed to at least two of the problems on the following list:

1. The Birthday problem
2. The Car and Goats TV Game Show.
3. Effron's non-transitive dice
4. The Town of Twosville
5. The Hat Check Problem (alias the Two Decks Problem)
6. Paradoxes in Averages

`The Birthday Problem:` This problem conceived by Richard von Mises in 1939 asks for the probability that *at least* two people in a random group of people (such as one might find at a party) share the same birthday (meaning that they were born on the same day of the same month).

The minimum number which must be in a group for it to be 50% probable comes as a complete surprise to most students. As one teacher put it, "when you tell them that the answer is about 24, the class is flabbergasted to the point of disbelief". When students are told that for a group of 40 people the probability that at least two have the same birthday is $0 \cdot 891$, they are still incredulous, a circumstance the wise teacher can exploit by having his class of students cooperate in a simulation. This simple experiment asks that between them they write down the birthdays of 40 people. This is

normally done by each jotting down his own birthday along with that of a brother or sister. Then a systematic check by the teacher invariably results in a match. This simulated experiment verifying the theoretical result often proves a greater demonstration for the student of the power of mathematics than the actual argument by which the result is established.

The proof of this counter-intuitive result is well within the compass of most students and illustrates the usefulness of what Boland calls 'the complements rule', by which the probability of an event is obtained by first deriving the probability of its complement.[60]

`The Car and Goats TV Game Show`: This example concerns the game show "Let's Make a Deal' which was shown regularly on American TV in the autumn of 1990. It runs as follows:

> A car is hidden behind one of three doors, while each of the other two doors conceals a goat. Lucky you, who gets to guess where the car is, choose door 1. Monty, the game show host, who knows where the car is, then opens one of the other two doors to reveal a goat. He asks you whether or not you wish to switch from your original choice of door to the other closed door. Should you switch?

[60]Philip J. Boland, "Topics in Probability Theory", *IMTA Newsletter* 87 (April, 1995): 50.

An answer to this question given by Marilyn vos Savant caused a flurry of debate across America in the autumn of 1990.[61] Most students think the answer is obviously 1/2, a belief shared by most of the lay participants and a high proportion of the academic participants in the 1990 debate. However, just as with the previous example, it is possible for students to perform a simulation which might help them to think clearly about this subtle problem.

One method is for a student to get another to help him simulate the game show using three mugs (to act as the doors) and a matchstick (to act as the car). The student closes his eyes while his helper hides the matchstick under one of the mugs at random. Then he opens his eyes and chooses a mug. At this juncture the helper reveals a mug with nothing under it. The student should play this game many times and count how often he wins by not switching. Then he should perform a second series of games and count how often he wins when he adopts a switching policy. The experimental results obtained in this manner are sure to be compelling and may cause a student originally in error to detect the fault in his reasoning. In any event, all of the students should be keen to have the correct answer explained to them in as simple a manner as possible.

The following explanation of the answer "Yes, you should switch" is due to Erich Neuwirth: "Imagine two players, the first one always staying with the selected door and the second one always switching. Then, in each game, exactly one of them wins. Since the winning probability for the strategy

[61] Leonard Gillman, "The Car-and-Goats FIASCO", *Focus* (June-July, 1991): 8.

'Don't switch' is 1/3, the winning probability for the second one is 2/3 and therefore switching is the way to go."[62]

The final offering of this section shows that the transitivity of equality and inequality familiar in arithmetic and geometry do not necessarily apply in probabilistic situations.

`Effron's Nontransitive Dice:` Transitivity is that principle which says, to give an example, that if A is taller than B and B is taller than C, then it follows that A is taller than C. While most would regard this as obvious, it is also understood that this transitive property depends very much on the relationship under consideration, so that while it holds for 'being taller than', it doesn't hold for teams 'beating' each other. If team A beat team B and team B beat team C, one does not conclude that team A will beat team C when they meet, even when that possibility is regarded as being probable. In this latter case, the relation 'beating' is understood to be a non-transitive one.

However, there are more subtle scenarios where this transitivity principle is applied almost instinctively and the example to be presented proves a case in point for many. It is another of the many examples alluded to in this work whose chief purpose is to point out that an unthinking 'application' of intuition, however natural it may seem, can sometimes lead to error. The

[62]Eric Neuwirth, "The Monty Hall Problem Put to Rest", *The College Mathematics Journal* 30, no. 5 (November, 1999): 369.

following diagram:

A	0	0	4	4	4	4
B	3	3	3	3	3	3
C	2	2	2	2	7	7
D	1	1	1	5	5	5

Figure 8.2: Effron's nontransitive dice.

describes four unusual dice A, B, C and D, often called the Effron dice after their creator Bradley Effron. Now, if

$$>:= \text{“is more likely to win”}$$

then one may write that $A > B$ because when these dice are cast, die A will show a higher score than B with probability 2/3. Similarly, it is true that

$$B > C$$

and that

$$C > D$$

However, and rather amazingly, knowing the truth of these relationships, it is *not* correct to infer that

$$A > B > C > D$$

and to *conclude* that

$$A > D$$

Why not? Because, perhaps surprisingly,

$$D > A$$

The false conclusion that $A > D$ is obtained as a result of assuming that the relation, $>:=$ "is more likely to win" is a transitive one which, as this example shows clearly, it is not. It is for this reason that the dice are said to be non-transitive.

Between them, these three problems reveal in no uncertain terms for the student the particular care that must be exercised when drawing inferences.

8.2.6 The Nature of the Infinite

Almost as soon as children learn the counting numbers $1, 2, 3, \ldots$ they encounter the notion of a set without end through the realisation that there can be no largest natural number. While this glimpse of the existence of the infinite enlarges their mental world, few dwell on the concept to an extent which might cause them to be perplexed. Such is the case also with most adults, with the exception of a few bold minds through history who have pondered the nature of the infinite enough to appreciate that such sets hold their own mysteries. The short core piece dealing with this 'big idea' of mathematics begins by relating that Galileo observed that when each of the even numbers

$$2,\ 4,\ 6,\ 8,\ 10,\ 12,\ 14,\ 16,\ 18,\ 20,\ldots$$

is divided by 2 in turn, the complete sequence of natural numbers

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ldots$$

is obtained; a circumstance he found a little surprising. At this juncture the student is invited to venture an opinion as to why this celebrated thinker from the past may have experienced this feeling.

The exposition goes on to explain that the reason that Galileo found the answer surprising is because it says that there are as many even natural numbers as there are natural numbers. This is because his scheme shows clearly that *every* natural number has a unique even number corresponding to it. But Galileo wondered how this could be when it is also the case that the set of even natural numbers is a *proper* subset of the set of natural numbers?

The lesson (not easily grasped) is that reasoning valid in relation to finite sets may not apply to infinite sets. In this case, an infinite team wearing jersies with numbers given by the first infinite sequence can mark (man for man) a team wearing jersies with numbers given by the second infinite sequence!

A teacher can rest assured that by this juncture the discussion amongst his students will be a lively one with many adopting an attitude best described as dismissive derision with others, it is hoped, remaining more open-minded and receptive to new ideas. Whatever the nature of the ensuing debate,

the teacher can use it to hammer home the notion of counting as a one-to-one mapping between sets, a concept central also to finite mathematics in relation to the inversion of functions such as those used for enciphering and deciphering information.

However, rather than explore the subtle if intriguing questions connected with the infinite in the ground-breaking manner of George Cantor, the core treatment presents the student with a number of questions relating to scenarios which reveal other 'strange' aspects of the infinite. Most of the questions posed are easily answered, although there are one or two very natural questions which even a teacher might have difficulty answering. With regard to these particular questions, classified as `Harder Questions`, it is pointed out that there is no expectation that the student or teacher answer them; their purpose is to show that a simple question may not have a simple answer. In some cases the easier questions are furnished with answers which clearly reveal school mathematics addressing relevant problems.

The first of the final two offerings in this core segment shows a region consisting of an infinite number of rectangles. The task is to show that the combined area of the rectangles is finite but that their overall perimeter is infinite. The answer reveals the area as the sum of a convergent infinite geometric series while a simple observation establishes the infinite extent of the boundary. The second example, relates to the Celestial Horn which is an infinite trumpet that has a *finite* volume but an infinite surface area. "So even though you can fill it with paint, you couldn't hope to paint its surface

even with a coat (of paint) a micron thin!"[63]

It an exercise in calculus relating to the rectangular hyperbola $xy = 1$. However, as it involves integrals over infinite intervals which are not treated as part of the current HLC course, its solution is not given.

8.2.7 Complexity

The basic purpose of this section relating to 'Big Ideas' is to teach the simple but fundamental lesson that:

> knowing in theory how to solve a problem
> need not mean that the problem can be solved.

`The Travelling Salesman` problem, introduced in the 1930s by the Viennese mathematician Karl Menger, is a prime example of those very-simple-to-state problems found throughout mathematics which defy solution. The fact that, even with the use of powerful computer, its general solution still proves intractable is both startling and vitally revealing, juxtapositioning as it does simplicity and complexity.

Its very accessibility as an example makes it a wonderful illustrator for students of the all important and practical concept of a calculation being *computationally prohibitive*. By way of illustration a teacher might discuss the case of such a salesman having to visit ten towns. Since theoretically the salesman can visit the towns in any order he chooses, the actual number of possible routes is given by 10! In terms of touching on school mathematics this

[63]Frank Swetz *et al.*, *Learn from the Masters* (USA: Mathematical Association of America, 1995), 87.

problem affords the teacher a concrete situation where permutations arise. Although the geographical locations of the towns in question will clearly allow a human investigator eliminate a large number of these permutations from consideration, the same may not be appropriate for a computer assigned to do the same task.

While students may have some idea that factorial numbers 'grow very quickly', they might not be as aware of the practical implications of such a fact. One way the teacher might counteract such vagueness is to have his students explain why there are exactly 10! seconds in six weeks. This rather unsuspected numerical coincidence can then be used to drive home the notion of a calculation being infeasible or bordering on being so. In the context of the ten-towns problem, this observation shows that a computer which takes one second to check a possible route for its overall distance would, in fact, require six weeks to decide on which routes are optimal. This concrete illustration of the actual time it can take to execute a computational task gives the student some understanding of what is meant by an otherwise acceptable solution being deemed computationally infeasible.

There is a possibility that a teacher will have some student argue, quite rightly perhaps, that a computer could carry out the indicated individual calculation in a fraction of a second; a circumstance the teacher can exploit to advantage by having the class do a comparative time analysis based on different execution speeds. Performing a thousand times faster, the machine still requires a little under an hour to effect the computation. The teacher

can then change the goal-posts and ask what would happen in the case of a one hundred such towns, in order for the pupils to see that at this level even the computational might of the computer is of no avail.

Alternatively, the teacher may switch tack altogether and present a different scenario which illustrates computational infeasibility in a more tangible manner. He can have his student observe that the number of different ways one can photograph 10 people (fashion models, say) is also given by is $10!$ and ask for a time estimate of how long it would require to take all $3,628,800$ such photographs. In this case, the students will see that one could not possibly organise for a different photograph to be taken every second so that such a photo-shoot would require far in excess of six weeks and that without taking a single break. A practical exercise associated with such a task is how one would organise the different shots to be taken in as efficient a manner as possible.

`Factorisaton:` The same lesson contrasting the apparent simplicity of a suggested theoretical solution with the actual practical complexities involved in the implementation of such a scheme can be illustrated most forcibly in relation to the factorisation of an arbitrary natural number. A common misconception is that it is a simple matter to decide whether a number is prime or composite using the trial-division algorithm whereby one either finds a proper factor of a number, or failing this, pronounces the number to be

prime. However, as the number-theorist Carl Pomerance notes:[64]

> The main drawback of the trial division algorithm is that it takes too long if a number has *no small prime factors*. For example, say we use the trial division algorithm on a computer that can do one million trial divisions per second to determine if a given number is prime or composite. If the number is a prime near 10^{40} (a number with approximately 40 digits), the running time will be about one million years while if it's a prime near 10^{50} (a number with approximately 50 digits), the age of the universe would not suffice.

That's pretty bleak. Although in theory it is simple to decide whether or not a natural number is prime, this method is not practical. The problem of factorisation is believed from centuries of experience to be intractable; a belief rather than a certitude which however underpins much of public key cryptography.

Any discussion of the complexity of factorisation requires that a teacher talk a little about divisibility and the notion of composite and prime numbers, all items which are immediately accessible yet central to a student's general mathematical education.

`Cellular Automata:` The core segment relating to the concept of complexity also contains a PowerPoint presentation on cellular automata. Here the central theme is that "simple computer programs can produce complex behaviour".[65] Commenting on the "the generation of ostensibly random be-

[64]Carl Pomerance, "Recent Developments in Primality Testing", *Mathematical Intelligencer*, 3 (1981): 97-105.

[65]Stephen Wolfram, *A New Kind of Science* (US: Wolfram, 2002), 2.

haviour from simple, deterministic rules", Krantz remarks:[66]

> The point here is elementary but incisive: we begin with a single black square in the grid and a very simple rule of logic for replication. The result is a fantastically complicated organism of squares that one could never have predicted nor analysed.

The presentation illustrates all of the different cellular automata that can be created in this manner and while Krantz rightly remarks that:[67]

> Pictures of cellular automata are all well and good ... and they certainly illustrate that a cellular automaton can create a baffling complexity starting from very little, I am not yet convinced that they tell us much of anything about science.

the purpose here is to indicate a number of points of contact with school mathematics that a teacher may care to exploit while sharing with the student a most powerful illustration of the concept of complexity.

`Fractals`: As another illustrator of the complexity that can arise from the repeated application of a simple rule, there is also a PowerPoint presentation in the core showing the recursive generation of different fractal or self-similar images. While the thirty or so slides are uncommented, the student will see either by its title or the picture being created before his eyes on the slide that a number of these images resemble very closely actual objects in the physical world. Noteworthy examples are the *Von Koch Snowflake* and the *Barnsley Fern.*

[66]Steven Krantz, Review of "A New Kind of Science" by Stephen Wolfram, *SIAM News*, 146.

[67]Ibid.

If desired, the teacher can use the fractal theme to touch on school mathematics. For example, in relation to the Von Koch snowflake the student can show, with no more than an application of geometric series, that while the area of the snowflake is finite, its perimeter is infinite.

8.2.8 Topological Ideas

The purpose of this core piece is to reveal some of the accessible ideas of two-dimensional topology which every student should learn about simply as a matter of mathematical literacy. One such notion has already been alluded to in relation to the standard map of the London Underground designed in 1931 by Henry Beck, a twenty-nine-year-old temporary draughtsman for the London Underground system. This map, Devlin avers:[68]

> is widely regarded as one of the best ever drawn, and several attempts to improve it have all met with failure.

As already touched on:[69]

> Yet how many who use the map realize that it illustrates the immense power of topology. It does so because in every respect save two, the map is completely inaccurate. It is not drawn to scale and as a result, the distances are all wrong. What's more, those neatly drawn straight lines representing the trains' paths bear little resemblance to the actual subway lines, in which the trains threaten to throw standing passengers to the floor as they twist and turn their way beneath the London streets. And just because a stretch of track is shown running north-south, it doesn't mean the real line does the same – it could even run almost east-west.

[68]Keith Devlin, *The Millennium Problems – The Seven Greatest Unsolved Mathematical Puzzles of Our Time* (UK: Granta Books, 2002), 161-162.
[69]Ibid.

While these observations may never occur to one, it is clear from a reading of this account that the features alluded to are indeed not relevant ones. As Devlin explains:

> One of the two things that the map does get right is that if a station is shown to the North of the River Thames, then the real station is indeed north of the Thames, and likewise for south. The other accurate feature of the map is the way it depicts the network: the order in which the stations lie on each line and the places (stations) where any two lines intersect. This property is really the only item of information an Underground traveller needs to know from the map – where to get on and off and where to change lines. The Underground map works by being completely accurate in its depiction of the one thing travellers need to know how to use the system, and sacrificing all other details in favor of a clear and accurate design.

This same author gives another example of a topological map, similar to the Underground map – a wiring diagram for a supermarket refrigeration unit. It shows how the different components should be wired together, but does not specify the exact layout of the components or the lengths of the wires and the paths they take. "Both of these examples illustrate the essence of two-dimensional topology. If the Underground map were printed on a perfectly elastic sheet of rubber, it could be stretched and compressed so that every detail was correct, giving a standard, geographically accurate map, drawn to scale with every stretch of line correctly oriented to the compass bearings. This stretching would not affect the way the lines connect the various stations. The reason, in mathematical terms, is that the configuration of a network (defined as a collection of points connected by various lines) is a

topological property. Put simply, networks are topological objects. You can twist or stretch any of these connecting lines in a network without changing the overall configuration. To change the network, you must either break a connection or add a new one.

Likewise, a diagram illustrating the route of a travelling salesman is of a similar type. The relevant details are which towns are connected to each other and the distances between towns. All other information is superfluous.

These examples alone give the student a flavour of two-dimensional topology or rubber-sheet geometry as is sometimes known.

The core piece then describes 'the first topological theorem' which is Euler's solution in 1735 of the Königsberg bridges problem. The student is informed that Euler solved this problem by realising that the exact layout of the islands and bridges is irrelevant, that what is important is the way the bridges connect, that is to say, the network formed by the bridges. In other words the problem is one of topology, not geometry. The problem remains the same if the island bridge network is drawn more simply. This is then done and followed by a description of the simple insight (a fundamental one taught in all introductory courses in graph theory) which resolves the problem.

The core treatment concludes by relating how simply Euler showed that the formula

$$V + F - E = 1$$

always holds true for any network drawn on a *flat* surface, where V is the total number of vertices (or nodes), E is the total number of edges (or connecting lines), and F is the total number of "faces' (*i.e.* regions enclosed by three or more edges). Verifying this formula for different planar networks is an exercise which requires no more than simple enumeration but is one which is likely to reinforce a student's appreciation of this topological characteristic.

Because a similar formula holds for polyhedra the final core piece discussed here under the umbrella of interesting and intriguing ideas deals with the five Platonic Solids.

8.2.9 The Platonic Solids

The first purpose of this core piece is to acquaint the student with the fact that there are only five regular solids and to have him verify that Euler's celebrated 'polyhedron formula':

$$\#(\text{vertices}) - \#(\text{edges}) + \#(\text{faces}) = 2$$

or

$$V + F = E + 2$$

holds in each case. The student is also asked to describe the nature of the faces appearing on each of the regular solids.

The model solution is given in the form of a table which lists the five regular solids in ascending order according to the number of faces (each of

which are described) along with an enumeration of V, F and E and a check that $V + F - E = 2$ is true in each case.

This is followed by a discussion of each of the solids which makes various combinatorial observations. This exercise serves as a warm up for an exploration which investigates, if using algebra and combinatorics, one can establish "necessary conditions" on the number n of possible faces a regular polyhedron can have and on the number of sides s each of these faces may have, given that Euler's formula is true in general.

The student is then stepped through an algebraic analysis which reveals, using no more than elementary reasoning, that there can be at *most* five possible regular polyhedra. It indicates in each case the number of faces and their shapes. However, it is stressed that it is important that the student understand that this does not establish the existence of such solids. By way of elaboration, the first result says that if there is to be a four-sided regular polyhedron then it must be built from four equilateral triangles. However, this does not imply that these four triangles can be arranged in space so that they do form what is known as a tetrahedron.

Chapter 9

Applications

The first paragraph of the Introduction to the *Aims and Objectives of Leaving Certificate Mathematics* reads:[1]

> Mathematics is a wide-ranging subject with many aspects. On the one hand, in its manifestations in terms of counting, measurement, pattern and geometry it permeates the natural and constructed world around us, and provides the basic language and techniques for handling many aspects of everyday and scientific life. On the other hand, it deals with abstractions, logical arguments, and fundamental ideas of truth and beauty, and so is an intellectual discipline and a source of aesthethic satisfaction. These features have caused it to be given names such as "the queen and servant of sciences." Its role in education reflects this dual nature: it is both practical and theoretical – geared to applications and of intrinsic interest – with the two elements firmly interlinked.

The duality of the educational enterprise alluded to in this extract has been illustrated at length in this work in the three preceeding chapters. One

[1]Department of Education, *The Leaving Certificate Mathematics Syllabus (Higher Level and Ordinary Level)* (Dublin: Government Publications Sales Office, n.d.), 1.

might confidently assert that the chapter History and Culture of Mathematics is concerned to a very large extent with "abstractions, logical arguments, and fundamental ideas of truth and beauty" and gives numerous examples by which all of these facets of mathematics can be revealed to the student in an engaging manner. Furthermore, the RM chapter illustrates quite forcefully how recreational mathematics can contribute substantially to each of these objectives while making it clear that the learning of mathematics can be a pleasurable pursuit possessed of a sense of fun and capable of being truly enjoyed. However, these two chapters and the one on Topics never lose sight of the responsibility to indicate at every possible opportunity the complementary aspect of mathematics mentioned in the above quote; that of showing the relevance of mathematics through its many applications.

The LC *Aims and Objectives* is unequivocal about the importance of applications, devoting a number of paragraphs to discussing this theme whose ultimate objective is to have students "be able to use mathematics and see it as a powerful tool with wide-ranging areas of applicability." It is added that:

> Where possible, the applications should be such that they can be made clear to the students (now, rather than in some undefined future), and hence ideally addressable to some extent within the course.

However, as stated in the Introduction, these laudable aims are not being in any way helped by the manner in which the current LC progamme in mathematics has been examined over the last ten years. The chapter "Es-

tablishing the Case" points out that no more than one-fortieth of Paper I of the 2003 HLC paper is concerned with an application, with matters not being much better in Paper II.

It might be added in passing that according to a recent OECD report similar problems pertain at Junior Certificate level.[2] In December 2004, the first results of the PISA 2003 examinations *Learning for Tomorrow's World* were published. According to the education correspondent Flynn, this OECD study found that "Irish 15-year-olds are performing at only average levels in maths compared to their counterparts ..." and that "Ireland came 17th in mathematics among the 29 OECD states which is significantly lower than top-performing countries such as Finland, Belgium and Switzerland."[3] The same correspondent recounts in an accompanying editorial that:[4]

> The PISA study makes a strong case for a more practical approach in the subject. It points out, for example, that there is a considerable mismatch between the concepts it tests and those found in the Junior Certificate exam. It makes a stronger case for greater emphasis on real-life applications in maths classes which could lead to increased interest and motivation amongst students – and higher standards.

While the meaning of the word 'application' is taken as understood throughout the *Aims and Objectives*, it might be as well to elaborate a little on the notion before describing the purpose and content of this chapter.

[2]OECD, *Learning for Tomorrow's World – First Results from PISA 2003* (Paris: 2004)

[3]Seán Flynn, "Low Irish maths standards criticised", *The Irish Times*, December 7, 2004, 8.

[4]Ibid., 19.

Writing in 1999 Pollak defines 'applications for the classroom' as "components of the curriculum consisting of real-world problems solved by mathematics."[5] and goes on to elaborate in detail what is meant by a real-world problem. Much of this discussion mirrors what has already been said with regard to the PISA definition of an extra-mathematical application and the process of mathematisation. Pollak says that the forces that affect the teaching of applications are the increasing mathematisation of other fields with the subsequent increasing mathematical needs of many jobs. He notes that applications are now recognised as an integral part of school mathematics and emphasises their ability to motivate students.

Pollak also details some of the factors that impede the teaching of applications: one is a belief by some mathematics educators in the purity of the subject, another is a possible ignorance and even fear of other disciplines on the part of teachers; while a third is the extra time that such teaching requires along with the fact that some parents, as well as educators, believe that mathematics should be taught the same way they learned it. With regard to this latter point he adds that parents and educators resist the influence of technology, of newly important subjects in the mathematical sciences and the inclusion of applications because they are different from their own experience. However, he observes that economic competitiveness often proves to be the key answer to such objections.

[5]Henry O. Pollak, "Applications for the Classroom, Overview", in *Encyclopaedia of Mathematics Education*, ed. Louise S. Grinstein & Sally Lipsey, 33-37. (New York: RoutledgeFalmer, 2001), 31.

Much of this work is predicated on the tenet that the teaching of applications is central to students' motivation and that it should be an integral part of their mathematical education. It is believed that to really convince students of the efficacy of mathematics, one needs to actually show them in the classroom, using well-chosen examples, many genuine applications within the compass of their knowledge.

The purpose of this chapter is to outline a number of applications (not given heretofore) by which current school mathematics can be made to show more relevance than it does at present. In addition, it is proposed to discuss new areas of application which could also be treated in the classroom without any increase in the level of mathematics needed. Those existing syllabus segments which can be strengthened in a number of ways are considered first; then the rationale and content of the new areas is described. In some cases corresponding core material is provided.

9.1 Geometry/Trigonometry

This topic is one where there is great scope for enriching the present provision with genuinely applied and engaging problems. Two examples have already been described (and illustrated in the core) in connection with Thales. These involve a use of similar triangles to determine the height of a building and the use of triangulation to fix the location of a remote object.

The converse of Pythagoras's theorem is another result known to the ancients which can be used to square off a building, something which is

of some practical importance when it comes time to fix the roof or hang wallpaper. Yet this particular application does not appear to be receiving exposure by way of examination nor is the famous formula of Hero which states that the area A of a triangle is given by

$$A = \sqrt{s(s-a)(s-b)(s-c)} \quad ; \quad 2s = a+b+c$$

That this formula is both remarkable and useful should be fully impressed on students. It allows one to calculate the area of a triangle from a measurement of the lengths of its sides alone. The significance of this is that it obviates the need for angle measurements. This student will realise immediately that the task of measuring lengths accurately is a far less troublesome one than that of determining angles to the same precision. It is also clear by extension that the area of any polygonal region can be ascertained through the measurement of lengths alone, a circumstance well-known to surveyors.

Mac Hale wrote in an IMTA article of 1994 that "Elementary problems in surveying are an excellent motivation for pupils beginning trigonometry."[6] The short article in question considered the following problem:

> To find the height and distance of an inaccessible object on a horizontal plane. An observer notices that the angle of elevation of a distant tower is α, but on travelling a distance x towards the tower along a horizontal plane, notes that the angle of elevation is now β. Required to find the height of the tower and the observer's original distance from the base of the tower.

[6]Des Mac Hale, "Simplifying a Trigonometric Formula", *IMTA Newsletter* 91 (April, 1996)

This particular problem can be made very appealing to students by a teacher who elaborates on its practical aspect: where with one distance measurement and two angle measurements one can determine another distance (or height) which might otherwise be awkward to ascertain. A core piece entitled "Surveying Problems" provides a number of illustrations which will be elaborated upon presently.

The exposition begins by giving Mc Hale's discussion of the problem quoted where he first presents a solution due to Hall & Knight which uses the sine rule. He follows this by deriving two formulae giving the height h of the inaccessible object and its distance d from an observer respectively using nothing more than the definitions of tan and co-tan. He remarks that these formulae are computationally more efficient than the those given by Hall & Knight. Specifically, he shows that

$$h = \frac{x}{\cot\alpha - \cot\beta}$$

and that

$$d = \frac{x\cot\alpha}{\cot\alpha - \cot\beta}$$

The derivations of these formulae make for a very valuable exercise in trigonometry because they find immediate application.

In fact, the core piece applies them to a number of problems, two of which are concerned with `Determining the Altitude of a Plane` and the `Distance of a Boat from Shore`. Students will readily grant that these are problems worthy of some attention. A third application, is also related

to finding the altitude of a plane but where the points of observation lie on different sides of the line of altitude. The student is informed that it is a worthwhile exercise to show that in this case the relevant formula for the altitude h is given by

$$h = \frac{x}{\cot\alpha + \cot\beta}$$

After solving this particular problem, another one concerned with the locating of a fire is discussed and is again an example showing the relevance of mathematics.

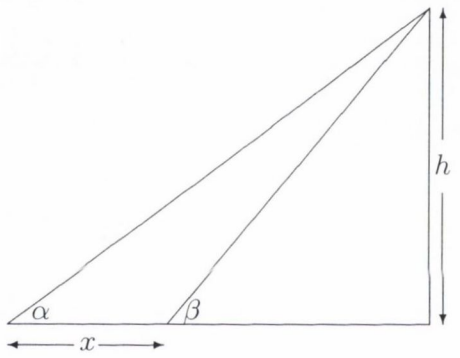
`HLC questions:` It is clear that problems such as these are visibly applied and are likely to engage students. Yet the HLC examination has a very poor record in this regard. In 1994 it carried a question which is a particular case of the general problem discussed in the Mac Hale article. Another question from a sample paper of 1994 asks for a general formula which is easily shown to be a first cousin of the one obtained by Mac Hale. However, similar such questions have not been asked in the intervening years. Q5(c)of paper II, 1994 reads (without an accompanying diagram):

> A vertical flagpole stands on horizontal ground. The angle of elevation of the top of the pole from a certain point on the ground is θ. From a point on the ground 10 metres closer to the pole the angle of elevation is β. Show that the height of the pole is
>
> $$\frac{10\sin\theta\sin\beta}{\sin(\beta-\theta)}$$

However, the solver is not asked to apply this solution to a particular problem or to comment on its possible usefulness. A possible extension to it might ask "Describe a possible application of this result." [This problem is solved in the core article.]

Q4(c)of Sample Paper B, II, 1994 (with an accompanying diagram) reads:

> The angles of elevation of the top of a tower h metres from two points on level ground which are x metres apart are α and β as shown:



> Prove
>
> $$h = \frac{x \tan\alpha \tan\beta}{\tan\beta - \tan\alpha}$$
>
> Hence calculate the height of the tower in the form $a\sqrt{b}$ when $\alpha = 30°$, $\beta = 60°$ and $x = 20$m.

Figure 9.1: Sample examination question.

This problem is superior to the previous one in that it shows the theoretical result being used. [This problem is also solved in the core article.]

By way of giving further practical examples of geometry and trigonometry, one could also relate how Eratosthenes measured the radius of the Earth using similar triangles or discuss the problem of how far one can see from the crow's nest on a ship or variations thereof. As Ralph Kahn of the US Jet Propulsion Laboratory remarks: "Fire towers are built so that firefighters in forested regions can scan large areas for potential trouble. Upper-storey and penthouses apartments rent for the highest prices because of the terrific view."[7]

Attention is now turned to the other aspect of trigonometry embodied in the waveform nature of the sine and cosine functions. While the LC syllabus does acknowledge this feature of the functions and requires that the student be instructed in this regard there is little support given by way of examination questions with the result that many students are barely aware of the sinusoidal nature of the graphs, much less have any knowledge of such concepts as periodicity and frequency. Yet to realise that this is hugely important, one need only think of how electricity is supplied to households.

The waveform aspect of the trigonometric functions receive some attention in the discussion to follow which treats the issue of whether a Leaving Certificate question should continue to be set in a context-free or 'pure' fashion when it is possible to set the same question in an applied context without altering its underlying mathematical context.

[7]Ralph Kahn, "How Far Can You See?" *The PUMAS Collection* (US: 1996)

Pure versus Applied: Consider the following problem:

Alternating voltages are given by

$$v_1(t) = 60\cos 2t \quad ; \quad v_2(t) = 40\sin 2t$$

where t is measured in seconds.
Find the maximum value attained by $v(t) = v_1(t) + v_2(t)$ and state when this maximum value is first reached.
Find also when $v(t)$ first reaches 65 volts.

This problem would almost certainly be deemed inappropriate for a current HLC mathematics paper because of the use of terms or concepts which are not strictly in the vocabulary of mathematics but are external to it, coming as they do, in this case, from the vocabulary of physics and/or electrical-principles. One view is that a student unfamiliar with the term alternating voltage may become unsettled and not see that the question is one which requires only mathematics for its solution. When this overly-cautious and timid view prevails, the above 'applied' question is often rendered in a sanitised form mathematically pure enough so as to furrow no brow. A typical re-working to conform with the current style of HLC questions might ask:

Express

$$60\cos 2t + 40\sin 2t$$

in the form

$$A\cos(2t - \phi)$$

where A and ϕ are constants.
Use the result obtained to find the maximum value attained by $60\cos 2t + 40\sin 2t$ and determine when this maximum value is first reached.
Find also when $60\cos 2t + 40\sin 2t$ first attains a value of 65.

This question, cleansed as it is of 'external' terminology, is not a faithful rendering of the original problem in relation to its mathematical content. It is instead, a 'stepped' version because it suggests a method of tackling the problem. As a technical mathematical question this one has obvious merit, requiring the use of a compound-angle formula from trigonometry for its solution. But, as posed, that is about as far as it goes. It is still only the skeleton of the full-bodied 'applied' problem from which it is distilled – a lifeless imitation of a real problem. The result of emasculating the original practical problem by transforming it into one which is context-free is to make it virtually meaningless except as a question examining mathematical technique. *There is no indication of why one might want to carry out such a calculation.*

It might be argued that the question could be partially redeemed were there to be a rider asking where such a calculation arises in practice. However, to answer this, a student would have had to be given a context in his instruction at some time or other. If this is the case, then why not ask the question in a context in the first place?

As stated in its original applied form the practical nature of the problem is evident even if one is not familiar with the physical context. Furthermore, despite its practical appearance, it is clear that in essence the question is entirely a mathematical one, requiring no knowledge of physics either for its understanding or solution.

As already said, in order to solve it, use is made of an appropriate compound-angle formulae to convert the sum of the two 'sinusoidal waveforms at the same frequency' into a single waveform in amplitude-phase form. Thus this problem shows a student how one of the compound-angle formulas finds application in a fundamental way to a physical problem.

In essence this problem is about adding two sinusoidal waveforms (of the same frequency) to obtain a single waveform with the same frequency as its two components. It is important to stress this aspect and to do all in one's power to make it visually clear. This is a situation where a mathematical package proves invaluable to the teacher in illustrating what would simply be too laborious and time consuming to be undertaken manually.

`Visualisation`: The discussion of this problem is greatly enhanced by a use of a graphics package. Initially, one might show graphs of the two individual sinusoidal waveforms on separate graphs and comment on their common frequency but different amplitudes:

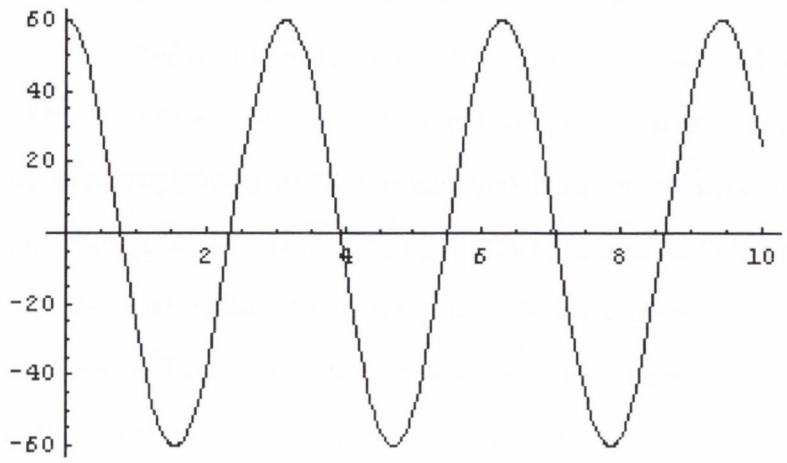


Figure 9.2: Cosine waveform.

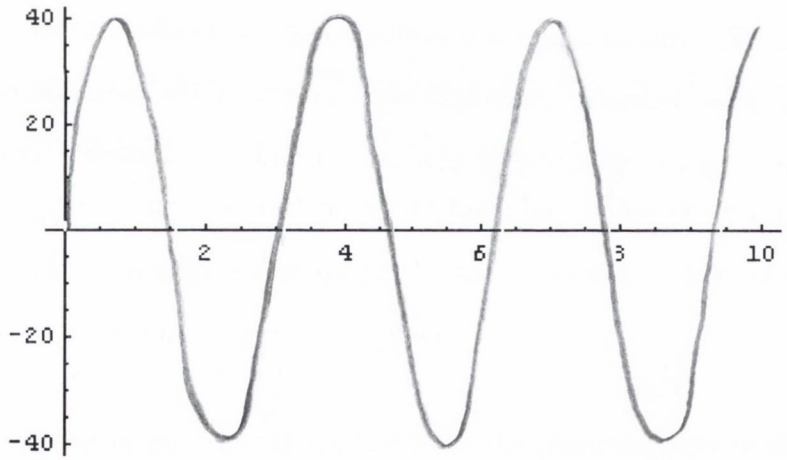


Figure 9.3: Sine waveform.

Then one might show both waveforms on the same set of axes where, not alone will their different amplitudes be more evident, but where their relative

displacement or 'phase difference' will also be striking. Here are the two waveforms plotted on the same set of axes:

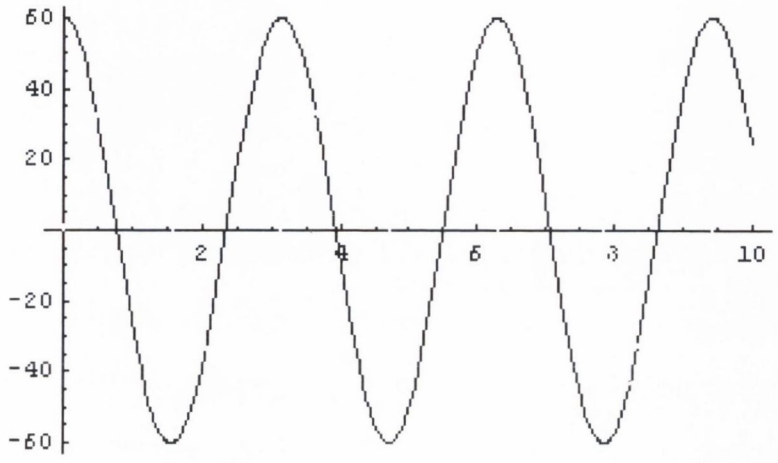


Figure 9.4: The two waveforms.

This done, show a graph of their combination plotted as one single resultant sinusoidal waveform:

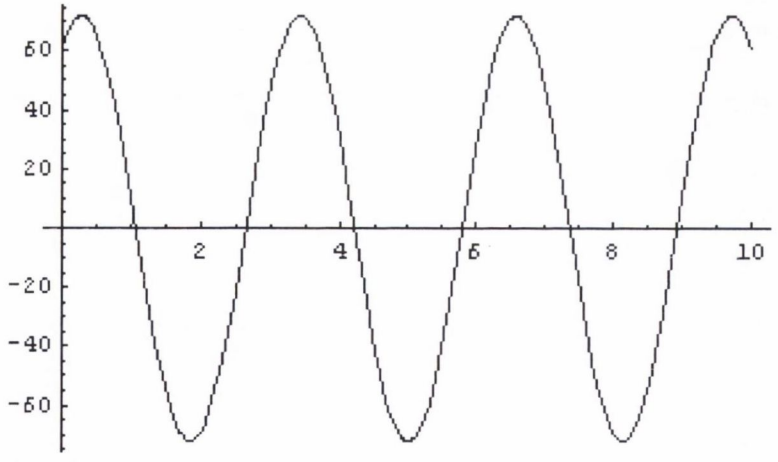


Figure 9.5: The resultant waveform.

The student might then be asked to confirm that the frequency of the resultant waveform is exactly the same as each of its component waveforms, as the mathematical solution says it should be. When the student examines the amplitude of the resultant waveform and finds that it matches that predicted by the mathematical solution obtained with pen and paper, a lesson about the power of mathematical reasoning has been assimilated. Many students find it interesting if not surprising that the amplitude of the resultant waveform is *never* as great as the sum of the amplitudes of the individual (out of phase) components. They should be challenged to explain physically why this is the case. As an aid they might be shown the following diagram which shows the two component waveforms and their resultant:

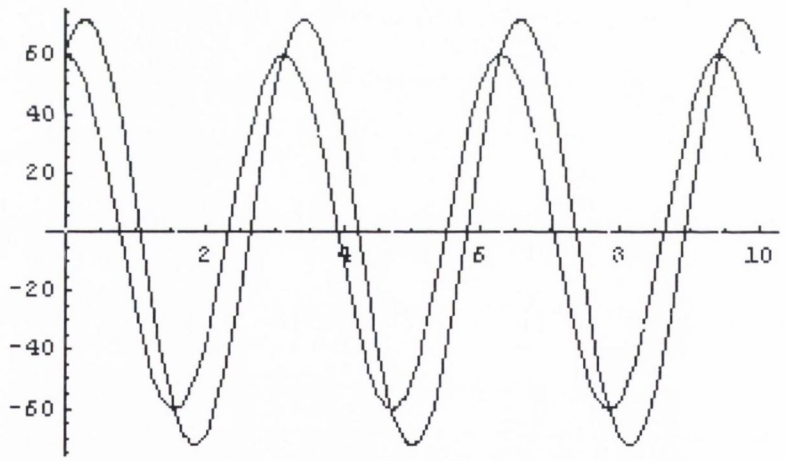


Figure 9.6: Component waveforms and their resultant.

Illustrating both the components and resultant waveform on the same graph reinforces the predictive power of the pen-and-paper mathematical solution.

To further emphasise the phase difference between the resultant and the reference component waveform – the cosine component in this example – display the graph of the cosine component and the graph of the resultant:

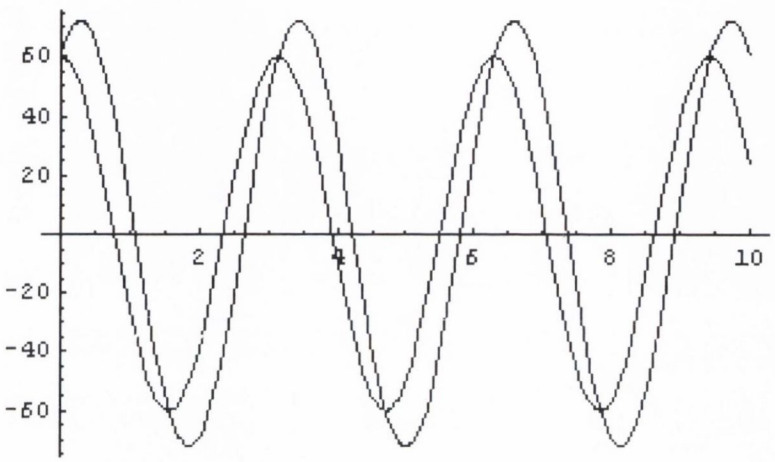


Figure 9.7: Cosine waveform and the resultant waveform.

The student should try to see if the phase difference he observes graphically matches the phase term appearing in the amplitude-phase formula for the resultant.

Though the terminology used here in relation to waveforms is physical, it is readily understandable to any student familiar with the graphs of the cosine and sine functions. Moreover, its very use reinforces the realism of the problem. Illustrate enough examples of this type of combination of waveforms in the manner just outlined and in time a student will no longer see a mathematical expression of the form

$$A\cos(\omega t - \phi)$$

as a mere compact mathematical formula but as a sinusoidal waveform.

While the rather full treatment given here requires considerable time by way of preparation and presentation, it results in a much more interesting and invigorating discussion than the solving of the purely mathematical problem which fails to make contact with reality.

`How Much Effort?` How much extra effort does it require on the part of teachers, syllabus writers and others involved in the educational system to allow the sanitised version be set as posed originally?

The differences between the two versions of the question are substantial. The applied version of the underlying mathematical question is far from being a spurious application of the type one encounters when certain authors attempt to set elementary mathematical problems in some form of real-life context. As one can see from its statement, it asks, as one might in practice, that certain facts be ascertained about a real quantity – the resulting voltage. Its statement makes use of the function notation to denote the component voltages and the resultant voltage. The importance this alone attaches to the quantities involved is missing from the pure version. More significantly, as worded the mathematical version *indicates the approach to be taken towards a solution.* This adds to its artificiality by being in stark contrast to a real problem which rarely comes with hints as to its solution.

Though the applied version has the merit of being a good deal more realistic than its pure counterpart, the absence of any suggestion as to how one is to obtain the required answers makes it undoubtedly harder as an exami-

nation question. However, it is suggested that realism shouldn't be sacrificed for the sake of an ease which conceals the very purpose of the calculation. After all, in defence of maintaining realistic questions on examinations which hope to accommodate these superior type of questions, one can argue that the discussion of many such questions by the teacher in his instruction should be an adequate safeguard against charges of unfamiliarity and surprise.

A core piece entitled 'Applied Trigonometric Problems' gives a number of other examples of purely trigonometric problems set in applied contexts.

9.2 Counting

The sample problems presented in the core segment "Applied Counting Problems" are about counting, a skill that is vitally important in all manner of contexts. The piece is presented as a set of questions with accompanying solutions, a feature which means that they can be used as ready-made examination questions.

The first problem begins by asking the student to state *The Multiplication Principle* and then expects that it be used to determine the number of cars that can be registered using two distinct letters of the alphabet followed by a non-zero four-digit number. The student is then asked to compare this with a scheme which uses three distinct letters of the alphabet followed by a non-zero three-digit number. Clearly this easily understood problem is a practical one which reveals how important it is to be able to count.

The second problem is concerned with shortest routes, a theme which re-

ceives mention in the RM piece relating to the chessboard. A third problem counts the number of different ways certain poker hands can occur and is important because it reveals how essential it is to have a basic knowledge of combinatorics. The remaining problems deal with the number of digits/bits needed to represent numbers and sets of numbers in denary (binary) form. The ability to make such enumerations can prove vital in assessing storage requirements for computers or the times to execute certain printing tasks. They connect with a number of problems dealt with in the segment on "Sequences and Series".

9.3 Sequences and Series

The core piece entitled "Applied Sequences and Series Problems" is also in the form of a set of questions, each accompanied by a solution. It begins by presenting two problems connected with salaries and investments which require no more than a knowledge of arithmetic series for their solution. Their main purpose is to reveal how such series arise in applied contexts and why it is necessary to be appraised of the simple theory that effects their solution. It has already been pointed out in the analysis of the 2003 HLC papers that neither paper contains a question touching on financial matters relating to investments, interest or the like. This is a serious case of a missed opportunity to treat manifestly applied problems which relate to the 'personal situation', where as a rule it so much more difficult to show the relevance of mathematics than it is to the 'scientific situation'.

These examples are followed by ones relating to geometric sequences and series. The first concerns a drilling process and is one where an infinite geometric series makes an appearance in connection with determining what is the maximum achievable depth the process can attain. It comes as a surprise to some students that one cannot drill to an arbitrary depth but that there is a limit on what is achievable. A similar problem is concerned with removing air from a cylinder using a vacuum pump while another studies the oscillations of a pendulum. A fourth problem deals with the important issue of determining what is realistically achievable as opposed to what is theoretically possible. It might be gainfully used as a humorous piece to distinguish between the (imputed) different approaches of mathematicians and engineers to problems, where the former allegedly make no attempt to achieve what they calculate to be unattainable while the latter pragmatically declare that they can get close enough for all practical purposes – a distinction which serves as an important lesson in itself.

Having considered problems involving both arithmetic and geometric series, the core piece turns to a theoretical problem relating to arithmetic-geometric series. It concerns a purely technical problem which has appeared on a number of recent HLC papers. In a slightly embellished form (for notational convenience), it asks that the arithmetic-geometric series

$$S(n,x) := 1 + 2x + 3x^2 + 4x^3 + \cdots + (n-1)x^{n-2} + nx^{n-1}$$

be expressed in a closed form, valid for all $x \neq 1 \in \mathbb{R}$.

Unfortunately, these HLC questions neither indicate where this series arises nor why knowing its sum in closed form is worthwhile; yet it is easy to show the student how this series arises in a context. One such is connected with determining the total number of digits required to represent all the natural numbers having *at most* n digits in their decimal representation. As a particular application, one might ask (as is done in the core) how many digits are required to print the first million natural numbers or how much space is needed to store these numbers on a floppy disk. (Perhaps a little more engagingly, the teacher might ask that the general formula be applied to determine how many brass door-digits must be purchased so as to consecutively number the houses in an estate of a hundred houses.)

To begin answering this question the student must explain why the total number of digits required to represent all the natural numbers having *exactly* k digits in their decimal representation is given by $9k.10^{k-1}$. This is a simple but important problem in counting, covered in the segment 'Counting' and repeated here for convenience. It then follows that the total number of digits required to represent all the natural numbers having *at most* n digits in their decimal representation is given by

$$9[1 + 2.10 + 3.10^2 + 4.10^3 + \cdots + m.10^{n-1}] = 9S(n, 10)$$

using the notational abbreviation introduced above.

Thus at a stroke, the teacher shows the student the above series arising in a context with $x = 10$. As a result, the same student can appreciate

why one would want to sum the series (in this particular case at least). However, as the teacher should now mention, the same series (with $x = 2$) is encountered if one asks for the total number of bits required to represent all the natural numbers having at most n bits in their *binary* representation. At this juncture, the student may also see that asking the same question for other bases yields (multiples of) the same series with the value of x being specified by the base.

Consequently, the teacher demonstrates how the general form of the series incorporates a multitude of particular problems. It is only at this stage that he should embark upon the technical exercise of deriving (either by elementary means or by calculus methods) the closed form for $S(n, x)$. Then the student can check that the formula so obtained does indeed yield the right answer for small values of n and for the bases ten and two, say.

This whole exercise is a substantial one, motivated by a practical problem which reveals the need for and power of *technical* mathematical know how. Its fuller treatment along with a number of other applied problems relating to denary and binary representations are discussed in the core. For example, one such shows that it takes almost 19 million bits to represent all the natural numbers less than a million, a practical result which the student might like to express in terms of kilobytes or gigabytes to quantify the amount of space required to store this amount of data on a computer disk.

A later section indicates how a knowledge of the basic theory surrounding arithmetic sequences is essential for the execution of certain mathematical

programming tasks.

9.4 Applied Probability

Probability is a topic which need not defend itself on grounds of utility. It enlightens our decisions in situations of uncertainty, it debunks misconceptions about coincidences and warns against an over-reliance on intuition. It is used by the medical and legal professions, insurance brokers, financial institutions, communication engineers and ubiquitous legions of gamblers. With the possible exception of financial mathematics, probability is, perhaps, the topic which more than any other allows the teacher touch on applied problems (lying outside the scientific situation) which have a direct bearing on the personal lives of his or her students. As such, its study plays a pivotal role in demonstrating the relevance of mathematics to each and every member of the classroom.

The previous chapter, under the heading "The Pleasures of Probability", reveals the civilising effect of probabilistic reasoning by the manner in which it informs decision making and alerts one to the unreliability of intuition. In relation to gambling the discussion shows that an understanding of what constitutes a fair game can protect one from folly; that the ability to compute elementary probabilities can make one a better player in a game, and that while the purchase of a lottery ticket may be regarded from a mathematical viewpoint as "a tax on idiocy", it may be fully justified on the grounds of enjoyment.

The treatment in that chapter also shows how a number of problems (whose solutions appear counter-intuitive) reveal logical analysis as the only safe method by which inferences should be drawn. The student learns the valuable lesson that sometimes 'commonsense' can cause one to leap to a conclusion which a little reflection shows to be mistaken. Furthermore, it is demonstrated that an empirical simulation can often help clarify one's thinking about a tricky problem and point the way to its solution.

This section of this chapter concerns itself with probability problems which are visibly of an applied nature. According to Aldous, one of the reasons that textbooks don't succeed "as a first-and-only account of probability" is because they [8]

> ... focus on the mathematical structure (...) rather than the implications probability has for the real world.

By way of addressing 'the implications probability has for the real world' it is recommended that the current segment of the syllabus dealing with probability be widened to cover

- the analysis of risk
- the reliability of engineering systems
- quality control
- Bayes' theorem
- the standardisation of variables

[8]David Aldous, *American Mathematical Monthly* 105, no. 16: 575.

- the notion of an average

- the connection between the mean and standard deviation as given by Chebyshev's theorem.

Many of the proposals made here have been suggested before. Boland, writing in 1995, remarked that: "Much of the emphasis on probability theory in the Leaving Certificate syllabus is in the classical sense", *i.e.* dealing with problems of gambling and games of chance.[9] He added:

> I think it is important to indicate however that today probability theory is useful in many areas like genetics, financial risk theory, engineering reliability, quality control, medical trials, *etc.*

To reinforce his point in this regard he supplies two concrete examples; one relating to forensic probability and the other to the selection of a jury.

The core piece connected with this section is in the form of a set of exercises, which contains applied problems covering each of the aspects listed above. The examples relate to:

1. Random Number Generation.

2. Bayes theorem and medical testing. One such example highlights the dangers associated with testing from a whole population for relatively rare diseases such as AIDS. It is an example which is clearly relevant at a personal level. The method of solution makes a clever use of an

[9]Philip J. Boland, "Topics in Probability Theory", *IMTA Newsletter* 87 (April, 1995): 46.

array which is easily understood by the student and also obviates the need for invoking Bayes's theorem in a formal manner.

3. Reliability Block Diagrams (RBDs) are very convenient diagrams which facilitate the assessment of the reliability of a device consisting of multiple components interconnected to each other in various ways and where each has a 'small' chance of failing. Students have little difficulty in accepting the relevance of these particular examples, find them easy to solve and admire the manner in which apparent complication can be reduced to simplicity with no more than the application of the probability laws. One example is to show how, using multiple unreliable components of the same type in parallel, one can build a reliable component with the same function as each. It is particularly easy to design sensible questions in this field.

4. Quality Control. One example gives an application of the binomial distribution determining the average cost of a sampling procedure.

5. Reliable Communication. These examples show probability explaining how error-control schemes, such as the Hamming $[7, 4]$ code, actually improve the reliability of communication over 'noisy channels'.

6. Standardisation of Variables.[10]

[10]Brendan Guildea, "Fie, Fi, Fo and Fum", *IMTA Newsletter* 88 (September, 1995) is an engaging three-page story explaining the use of z-scores for the purposes of comparing data from different sets.

7. Chebyshev's theorem. This is one of the very few elementary results which demonstrates a tangible connection between the mean and standard deviation of a sample.

This segment on applied probability is brought to a close with a short discussion recommending that students learn the link between probabilities and averages.

What Happens On Average? At present the HLC segment on probability covers the principle by which a student can determine that the probability of rolling a two on the cast of a fair die is 1/6. However, it does not supply the information to answer the question:

> On average, how often must a die be cast in order for the number two to appear?

At present one would have to take a probability course at third level to learn the answer to this natural question, as indeed would one have to do for the question:

> On average, how many packets of cornflakes must be purchased before a child has a full set of the six different tokens on offer?

This particular question is a variant of the `Coupon Collector's Problem`.

"What happens on average?" or "How long does a game take on average?" or "How much will such a plan of action cost on average?" are all natural

questions which a student should be able to answer on completing secondary school.

In answer to the first question, one student surmised that it would take three throws on average. However, upon discussion he agreed that this answer wasn't plausible because, by symmetry, the same average value should hold for the other five numbers. However, this amounts to asking that all six numbers appear, on average, once every three throws, an expectation which is clearly absurd. By this observation he, and other classmates, inclined to the view that the average number of casts needed to see each number once is six. This indeed is the case, as follows from the fact that the expected value of a geometrically distributed random variable with parameter p is $1/p$. In plain language, this result states that, on average, the number of casts of a fair die needed to observe a particular number is the reciprocal of the probability of obtaining that number on the single throw of a die.

The previous chapter advocates the presenting of 'mathematics without total explanation' in situations where so doing allows the teacher to reveal the subject in more meaningful and engaging ways. The importance of this policy is nowhere more clearly revealed than in relation to the general result just enunciated. Invoking it without proof, allows each of the natural and relevant questions posed above to be answered.

With regard to the second question, it makes for a valuable classroom discussion to arrive at the answer

$$\begin{aligned}\left(1+\frac{6}{5}+\frac{6}{4}+\frac{6}{3}+\frac{6}{2}+\frac{6}{1}\right) &= 6\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}\right)\\ &= 6(2\cdot 44999...)\\ &= 14\cdot 69994\end{aligned}$$

This result says that, on average, it takes $14 \cdot 7$ purchases to obtain all six tokens (or equivalently it takes, on average, $14 \cdot 7$ throws of a balanced die to produce all six numbers).

This particular problem affords the teacher the opportunity of revealing 'an applied context' for the series

$$H(n) := 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-2}+\frac{1}{n-1}+\frac{1}{n}$$

because the general solution of the Coupon's Collector problem in the case of n coupons is given by $nH(n)$. Here $H(n)$ is the nth partial sum of the infinite harmonic series $1+\frac{1}{2}+\frac{1}{3}+\cdots$. As mentioned earlier, there is no known closed formula for $H(n)$ while the section on Mathematical Vignettes gives a 'modern' proof of the divergence of the harmonic series.

`Remark:` If one is willing to assume that the average value (of a geometrically distributed random variable with parameter p) is finite, then one can obtain it without having to introduce the notion of a random variable or having recourse to summing an infinite series. The proof which follows emulates the

solution of the 'Peter and Paul problem' presented in the previous chapter and is another example of a mathematical vignette:

> Let D be the average number of trials until the first success. (The hidden assumption is that this average number is finite.) The first success occurs on the first trial with probability p, in which case the experiment lasts exactly one trial. However, with probability $1 - p$, the first trial results in failure, in which case, it takes, on average, a further D trials for the first success to occur. Thus
>
> $$D = p\,D + (1 - p)(D + 1)$$
>
> – an equation which gives $D = 1/p$.

Perhaps one should present this 'proof' in the classroom so as to have the freedom to invoke so useful a result with impunity.

9.5 Calculus

There is perhaps no branch of mathematics that is so visibly applied as that of Calculus, particularly in relation to the physical sciences, yet it can be fairly said that most HLC students do not have any true awareness of its potency. It is true that most students with a C grade or higher in the HLC know the rules of differentiation and can differentiate (sometimes with the assistance of the mathematical tables) all manner of expressions involving polynomials, radicals and trigonometric functions. Likewise, they are competent at finding anti-derivatives using the standard techniques 'up to and

including' integration by parts. This is because (as a perusal of past HLC papers testify) the manner in which calculus is examined places most emphasis on the technical aspects, or the 'how' of the subject, with less stress on the 'why'.

One consequence of this bias manifests itself clearly at the first year of third level. Here it is found that many students fail to recognise the derivative in its first principles form when they encounter it in the formulation of simple first order differential equations.

Furthermore, while most students are aware that the derivative of a function in terms of the slope of a tangent line, few understand that in a time context it gives an instantaneous rate of change. A perusal of the textbooks reveals that normally no more than a single page is devoted to motivating the fundamental 'first principles definition' of the derivative and that this is done in purely geometrical terms.

The Derivative from First Principles: The core treatment contains an introduction to the derivative from first principles which is a very slowly paced development of this fundamental concept. It is offered as an alternative to the often too scant and hurried presentations to be found in many text books. Its central purpose is to show why the derivative of a function is defined as it is, and thereby, it is hoped, remove the mystery that can often otherwise be experienced by good and poor students alike.

The temptation to simply outline ideas without substantiating how they

should be acted upon was resisted, in the belief that such a course of (in)action smacks too much of theory and too little of practice. This writer is not aware of a comparable introduction which elaborates at such length on one possible method of introducing students to the notion of a limit which underpins the modern form of the calculus in both its aspects. As Stewart notes:[11]

> It is the idea of a limit that sets calculus apart from other areas of mathematics.

The treatment is based on the assumption that the physical notion of speed is one with which every teenager is familiar. The discussion begins with a description of the formula governing the distance covered by a ball dropping from a height above the earth's surface in the absence of air resistance. After acquainting the student with how to use the 'distance function' to calculate distances descended in given intervals of time, the student is led through further calculations to the conclusion that the speed of the dropping ball must be continually changing. This is done by calculating the distances travelled by the ball over time intervals of equal duration, and showing that these vary, depending on the location of these time intervals. The device of imagining a speedometer attached to the dropping ball as it descends is then used to give credence to the notion that the ball has a speed at any *instant* in time.

Once this is done, the stage is set for posing the general problem which is that of determining the *instantaneous speed* of the ball at any specified time. The central difficulty of always requiring two distinct moments in time

[11]R. Stewart, *Calculus, 3rd edition*, (USA: Brooks/Cole, 1995), 45.

in order to compute an average speed is set forth quite prominently and is used to motivate the fundamental notion of a limit.

Throughout, the presentation always proceeds from a consideration of particular (easily understood) problems to more general ones. In this way, the student sees the limitations of arithmetic which must give way to algebra when generality is required. The culmination of the whole discourse, indeed its main objective, is a natural arrival at the 'definition from first principles' of the derivative of a function.

The amount of material appearing on each page is kept to a minimum and always strives to avoid introducing more than one new idea at a time. At appropriate junctures a number of very simple but reinforcing exercises are posed which students of even the most moderate ability will solve easily. The idea is to engender a sense that the general problem under consideration is not just easy to understand but should be solved equally simply.

The mathematics used is elementary (Junior Certificate level) but

1. reinforces the function notation
2. reveals the power of algebra
3. shows the need for a 'limit'
4. highlights the pitfalls of dividing by zero

It has been this writer's experience and that of other teachers, that motivating the concept of a derivative using the familiar notion of speed, is one

which many students find natural. They understand the practical nature of the problem from the outset and even appreciate the subtleties connected with it. They get to see the 'why of the problem' before being led carefully to the 'how of the solution' which, most would agree, is 'doing things' in the right order. This way gives a teacher a much better chance of inducing his students to endure a certain amount of tedium when the time comes for them to master the various rules of differentiation; rules which allow derivatives to be obtained more readily than from first principles. Once they know that there is a good reason for learning these technical skills, these same students are likely to be less resentful than those who may be at a loss as to the efficacy of what they are learning.

It is hoped that what is written might be used as a guide by some teachers and that the expansive nature of the exposition lends itself to private reading so that any well-motivated student might assimilate its contents over a number of weeks.

Other Shortcomings: Another consequence of the too little stress on concepts is that students have little understanding of an integral as the limit of a sum. For example, when they encounter a sum of the type

$$\sum_{i=1}^{n} f(x_i)\Delta x_i$$

arising, for example, as an approximation to the mass of a rod of unit-cross-section and variable density lying along the $x-$axis between the points a and b, they often do not know that

$$\lim_{\substack{n \to \infty \\ \Delta x_i \to 0}} \sum_{i=1}^{n} f(x_i)\Delta x_i = \int_a^b f(x)dx$$

i.e., they do not understand an integral as the limiting value of a *Riemann* sum. This aspect of integration is needed for understanding why a formula such

$$\frac{1}{b-a}\int_a^b f(t)\,dt \tag{9.1}$$

gives the average-value of $f(t)$ on $[a, b]$ – a result which does not appear to be on the current HLC syllabus.

This is another example of a 'missed opportunity' since this result is of practical significance. One case in point being when $f(t)$ specifies the temperature at the instant t during the day, as then (9.1) with a and b appropriately chosen yields the *average* temperature over the time span $[a, b]$. The integral in (9.1) can often be determined by inspection from the graph of $f(t)$ on interpreting the integral as measuring the signed area subtended by the curve of $f(t)$ and the horizontal axis over the given interval; an observation which is regularly used by engineers in determining the d.c. term in a Fourier series expansion of a periodic waveform.

Furthermore, few students, if any, have a genuine understanding of the Fundamental Theorem of Calculus (FTC). Most will perform the calculation

$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b \quad \text{where} \quad F'(x) = f(x)$$

mechanically but are incapable of explaining why the integral can be evaluated in this manner and really do not appreciate how this result is truly fundamental in the sense that it achieves the exact evaluation of (the limiting value of) a Riemann sum without any need to perform the indicated summation. An historical comparison between the effort Archimedes had to expend to determine the quadrature of a parabola and that involved in the one line solution of Newton and Leibniz illustrates in no uncertain terms why this result is so named.

In fact, a valuable exercise, reinforcing the power of the FTC but also revealing an intra-mathematical application of the formula

$$\sigma_2(n) = 1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6}$$

is to apply the method of Eudoxus to

$$\int_0^1 x^2\,dx$$

to get that

$$\frac{1}{n^3}\,\sigma_2(n-1) < \int_0^1 x^2\,dx < \frac{1}{n^3}\,\sigma_2(n)$$

On determining the limiting values of both these Riemann sums, one finds that

$$\int_0^1 x^2\,dx = \frac{1}{3}$$

exactly without the use of the FTC.

This substantial exercise involving Riemann sums and requiring, as it does, considerable effort gives the student some inkling of the computational power of the FTC.

The teacher might inform his students that Archimedes apparently knew the value of the integral to be approximately $0 \cdot 333$. Had he known the closed form for $\sigma_2(n)$, he would have found the exact quadrature. As already mentioned, the formula for $\sigma_2(n)$ was derived by the eleventh century mathematician al-Karkhi of Bagdad. (This derivation is the subject of a PowerPoint presentation to be found in the core.)

Thus it is suggested here that there should be a greater stress on the link between the two processes of calculus as encapsulated in the FTC so that students come to understand the true significance of the result.

By way of concluding this discussion of 'shortcomings' one would have to say that students' understanding of the number e and the exponential and logarithmic functions can also be decidedly shaky. However, this writer finds it hard to be overly critical on this issue as it is no easy matter to know how best to introduce these functions (if at all) at senior cycle level in any way which might be deemed satisfactory. In the preamble to an NCCA article discussing different approaches to finding the *derivatives* of the exponential and logarithmic functions one reads that "there are drawbacks to all of these".[12] The writer referring to the particular approach adapted

[12]NCCA, "An Approach to the Derivatives of the Exponential and Logarithmic Func-

in this article states that it "... is not specified in the syllabus and is not therefore examinable material. The intention is rather to show the student that the differentiation of e^x may be approached in the same manner as that of $\sqrt{x}$ or $\sin x$."[13] One could well understand students finding the derivation quite involved.

However, whatever approach is taken to finding the derivatives of these functions, it is strongly recommended that any introduction to the number e involve a discussion of Bernouilli's problem of 'continuous compounding' so that the student is told that

$$e := \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

– a definition which connects this number with 'growth'.

The final chapter has more to say on exponential and logarithmic function as they relate to the current LC option "Further Calculus and Series".

Whereas the discussion heretofore has focused mainly on the issue of students acquiring a firm grasp of the concepts underpinning the calculus, the question as to what applications should be taught is now addressed very briefly. Essentially the recommendation is that there be a greater emphasis on applications than is the case currently to include

1. max-min problems,

2. related rates,

tions", *Notes on the Leaving Certificate Mathematics Courses* (Dublin: n.d.), 91.

[13]Ibid., 91.

3. averages

4. volumes of revolution

This suggestion is not a radical one nor is it new since, from an historical perspective, it can be interpreted as advocating a return to a previous policy where applications were given a much greater coverage at senior cycle. The rationale for such a proposal rests simply on the fact that doing so further enables the teacher to show the relevance of mathematics very convincingly, particularly as it relates to scientific problems. This work has already alluded to the power of the calculus in relation to projectile motion (in the Topic on Quadratics) and the NR algorithm. Teachers do not need to search for 'genuine' applications of calculus since these abound in the many excellent Calculus books available as well as in those textbooks relating to previous Leaving Certificate programmes.

What this work provides with regard to applications of the calculus is a core piece on "Bézier Curves". It also includes a piece entitled "Two Max-Min problems solved without Calculus." The purpose of each of these pieces is now elaborated on briefly.

`Bézier Curves:` Specifically, a quadratic Bézier curve with control points P_1, P_2, P_3 is a curve from P_1 to P_3 such that the line from P_1 to P_2 is tangent to the curve at P_1 and the line from P_3 to P_2 is tangent to the curve at P_3. For two curves (or a curve and a straight line) to join smoothly, they must

have the same tangent at the point where they meet. Bézier control points therefore provide a convenient way of specifying a curve. In order to be able to determine the control point for a quadratic Bézier curve one may need to know

1. how to express the equation of the desired curve as a quadratic polynomial
2. use differential calculus to find the tangent lines
3. formulate and solve an appropriate equation to find the intersection coordinates of the tangent lines

The solution of the two problems treated in the core, requires all of the skills described along with a use of elementary integration. Consequently, they constitute a substantial 'across-topics' question with the evident purpose of enabling one to use a graphics package to draw the relevant curves.

`A Salutary Lesson:` After a while spent solving max-min problems in the fashion prescribed by the differential calculus, it can be an automatic reaction on encountering a fresh optimisation problem to launch an attack on it using this approach. The core treatment connected with this section contains a discussion of one applied problem which illustrates forcibly how this can at times be folly. The accompanying exercise reinforces the point; it is another practical problem (regarding where *best* to locate a pump house on the side of a river) which immediately strikes one as requiring calculus for its solution

but which can be solved in a line using no more than elementary geometry. As the introduction to the piece says, these two examples are meant as a salutary lesson to the student "to think how best to go about solving a problem before launching an attack on using the first method that strikes one; it is about choosing the proper perspective."

Besides displaying two applied problems, the piece demonstrates that sometimes problems which might appear to require an 'advanced' technique for their solution can be solved with less sophisticated knowledge. It shows the students the importance of reasoning and ideas. As the student is told, according to Gilbert Strang, "A mathematician puts:[14]

formulas ahead of numbers,
and reasoning ahead of formulae
and ideas ahead of reasoning."

9.6 New Topic Areas

This work has already indicated at least two new topic areas: Threshold Schemes and Information Theory. The first is described in the Topic on Quadratics as a very recently discovered application of polynomials to the practical problem of 'sharing a secret'. The second is covered in the RM chapter where a game connected with the chessboard is used to quantify the concept of information which proves of central importance in the study of

[14]Gilbert Strang, *Calculus* (US: Wellesley-Cambridge Press, 1991) 61.

what can be communicated in the electronic world where uncertainty may prevail.

While both of these areas of study are immediately accessible using current school mathematics the new topics to be described presently require a familiarity with 'modular arithmetic' or 'finite arithmetic' as it is often termed.

This arithmetic has received a number of mentions to date. A complete section of the RM chapter is devoted to showing how it might be introduced to students through the consideration of various recreational tricks and games. In a later section dealing with the importance of mathematical notation, a more practical need for its use is revealed in a situation underpinned by periodicity.

One field of application where finite arithmetic is the 'lingua franca' is that of cryptography. Finite arithmetic is also used in check-digit schemes and in error-control coding. It is proposed to give a short description of what each of these topics entails and in so doing to indicate the relevant core material written to provide concrete illustration.

Check schemes: A core piece entitled "The International System of Book Numbering" describes the purpose and implementation of what are known as ISBNs. By way of elaboration, every modern book carries a 10-digit number on its back cover, normally near the bottom right-hand corner, such 1 86197 222 9. This number is this book's ISBN (International Standard

Book Number). Its first nine digits are 'information digits' because they code information about the book; in this case the first digit 1 identifies the book as being published in an English-speaking country, the next eight digits identify among other things the publishing house. The tenth and final digit is known as a *check* digit and though, in general, it is redundant in the information sense, it is very useful in checking whether or not a purported ISBN is a valid number.

Apparently the two types of error most commonly committed by humans when transcribing long strings of digits are a single transcription error in any position, and an error due to the transposition of any two digits. If an error of these types occurs during the transcription, then the tenth digit calculated by the recipient of this supposed ISBN will not match the tenth digit of the submitted number. Thus a bookseller, for example, need not waste time searching for a book using an invalid ISBN.

The core piece describes how this check digit is calculated, without explaining why computing it in this way provides a 'flag' for the presence of one of the errors just mentioned. However, providing such an explanation on behalf of the teacher reinforces the student's understanding of modular arithmetic and allows him acquire a greater appreciation of this particular application of mathematics.

This specific discussion of the ISBN scheme makes the general idea behind all such schemes clear: they allow the 'receiver' to check independently of the 'sender' and therefore allow him a certain measure of information. Their

importance for the student is that they show mathematics in action in a totally accessible manner.

Error-Control Coding: Check schemes *detect* the presence of errors and so allow for certain forms of action to be initiated, such as asking for a re-transmission of data, However, it is far more valuable to have a scheme which on detecting an error, attempts to make an intelligent guess as to what this error might be and to correct accordingly. While check schemes were used by electronic engineers as early as 1929, it was not until 1948 that Richard Hamming and others conceived of error-correcting codes as they are now known.

It is recommended here that such codes, in particular the Hamming [7,4] code, be studied at senior cycle because of the powerful manner in which they reveal the efficacy of school mathematics. To understand the design of such a code requires no more than an elementary knowledge of binary arithmetic, that is, arithmetic modulo 2, while simple probability is all that is needed to explain quantitatively why such codes increase the reliability of communication. But taken as a whole, the discussion of this code provides the teacher with a blue ribbon extra-mathematical application which reveals how cleverly mathematics can solve a practical problem.

The recommendation that such a code be taught at secondary level is not a new one. It was suggested by Paul Igodt as early as 1992 in an article

"The Power of Examples in Mathematics"[15] Later in 2000, Fitzpatrick and Kingston made the same suggestion in an IMTA article "Error Correcting Codes and Cryptography"[16] Both articles discuss different codes, their uses in transmitting information along with their error detection and error correction capabilities and might serve as a starting point for a teacher to learn about such codes.

The core section accompanying this piece contains a Flash presentation which begins by describing a situation where one might use a $[7, 4]$ Hamming code, sketches the construction of the code and shows it correcting single errors.

Cryptography: The final recommendation in relation to new topics is that of using elementary cryptography because it is self-evidently applied. A teacher can explain to the student that cryptography is the study of methods of sending messages in disguised form so that only those for whom they are intended can remove the mask and read their contents. In former times it was used mainly by the military and diplomatic services, but with the advent of electronic communication it is increasingly being used by financial institutions to ensure privacy in transactions between client and customer.

In order to illustrate this concept he might use the Cæsar Cipher as an

[15]Paul Igodt, "From basic number theory to elementary coding theory. The power of examples in Mathematics.", *IMTA Newsletter* 92 & 93 (April & October, 1997): 5-16. & 44-63.

[16]P. Fitzpatrick & J. Kingston, "Error Correcting Codes and Cryptography", *IMTA Newsletter* 97 (April, 2000): 45-58.

introduction to the ideas of enciphering and deciphering. The teacher can explain that while the Cæsar cipher is a cryptosystem which can be thought of as applying solely to alphabetic characters, and can be implemented using tables or wheels, or even on a computer *without* ever having to use numbers, it is preferable, for at least two reasons, if at their core cryptosystems use numbers. The first reason is that computers are ideally suited to handling numbers, even huge ones, with great speed. The second is that there is a large body of mathematical functions available which can transform numbers into other numbers in complicated ways. This makes it possible to design secure but efficient systems.

Mathematising this particular cipher so as to achieve its numerical implement ation reveals in very clear terms why modular arithmetic is so essential. But this task is also ideal for reinforcing the importance of

a 1:1 function

an inverse function.

As already mentioned in relation to mathematical programming, implementing a simple cryptosystem such as this so that it runs on a machine can be one of the better ways of having a student deepen his understanding of mathematics. An enthusiastic student might want to try his hand at more sophisticated encryption based on more complicated functions and block enciphering.

9.7 Error Analysis

By way of bringing this chapter on Applications to a close it is vital to point to the importance of students having some understanding of the highly practical skill of being able to analyse the possible effects of errors in measurement. A student may well think that 99% accuracy in a measurement is very good until it is pointed out to him that such an approximation in relation to time measurement represents a discrepancy of 36 seconds per hour and $14 \cdot 4$ minutes per day. While an accuracy of $99 \cdot 9\%$ might seem irreproachable, in terms of time measurement it still amounts to a loss or gain of $1 \cdot 44$ minutes per day.

Whenever the input to a mathematical model designed to solve a real-world problem depends on quantities which have been measured externally then consideration must be given to the possible effects errors in such measurements might have. That this is a practical consideration of some importance is hinted at by the examples just given and has already been illustrated for the student in the core discussion relating to triangulation where the effect of a (relatively) small error in measurement is shown to have serious implications.

Thus it is recommended that there be a much greater emphasis on error analysis than there is at the present and that the treatment of this topic involve a discussion of the distinction between accuracy and precision. By way of indicating what one might do in this regard there is an accompanying core

piece, "Error Analysis", which gives a number of fully worked out problems treating different aspects of this enterprise.

By way of searching for further 'applications' of mathematics accessible to students at senior cycle level one could extend the topic headings beyond those given here to include elementary illustrations from the following areas:

1. Transformation of Functions
2. Boolean Algebra and Switching Circuits
3. Data Compression
4. The Mathematics of Equity
5. Scheduling Problems

Chapter 10

Use of Technology

In the discussion of the NCTM's 2000 publication *Principles and Standards for School Mathematics* given in Chapter 4, it is pointed out that the elevation of Technology to that of a Principle indicates the importance American educators have come to attach to the use of technology as an educational tool. This principle is given as:[1]

> Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning.

The 1982 Cockcroft report had forecast that the increasing availability of computers in school classrooms "should change what is taught or the relative stress which is placed on different topics within the mathematics syllabus" and in alluding to the receptive attitudes of young learners, advised that "their ready acceptance of technological innovation should be fully ex-

[1]National Council of Teachers of Mathematics, *Principles and Standards for School Mathematics*, 16.

ploited."[2]

As of the early 1990s, Apostol and Blinn noted that it was the computer's capabilities "that brought mathematics and physics to life in many ways that could not be done at the chalkboard or in the textbook."[3] Remarking that "the visual representation of mathematical ideas, principles, or problems – has always played an important role in both the teaching and learning of mathematics", they list the following attributes of a computer animation:[4]

1. it grabs the viewer's visual attention and maintains the viewer's involvement
2. it capitalises on the viewer's visual intuition
3. it portrays a large quantity and diversity of information in a brief period of time
4. it takes advantage of the viewer's sophistication in 'reading visual' clues
5. it conveys mathematics in a rich cultural context

Mc Clintock notes that "computer technology provides a variety of tools for solving problems in creative, non-traditional ways, decreasing the prerequisite-dependency aspects of solving problems", and adds that "we can do much

[2]W. H. Cockcroft *et al.*, *Mathematics Counts: Report of the Committee into the Teaching of Mathematics in Schools.* (London: HMSO, sixth impression, 1985), 111, 119.

[3]Tom M. Apostol, & James F. Blinn. "Using Computer Animation to Teach Mathematics." *Conference Board of the Mathematical Sciences (CBMS): Mathematics and Education Reform 1990-1991* (1991): 13.

[4]Ibid.

that was not within our grasp earlier, and we can achieve deeper understanding of those still important goals that are retained."[5] For him, "it is clear that computer technology potentially contributes to a learning environment in which students can explore, conjecture, test, validate, do mathematical modelling and build understanding. This implies that introducing problems as a means of motivating and as a context for new mathematical ideas has an important place in mathematics."[6]

The heavy role played by visualisation as noted in Apostol and Blinn is echoed by Arcavi in 2003 when he writes:[7]

> Nowadays, the centrality of visualiztion in learning and doing mathematics seems to have become widely acknowledged. Visualization is no longer related to illustrative purposes only, but is also being recognized as a key component of reasoning (deeply engaging with the conceptual and not the merely perceptual), problem solving, and even proving.

However this author acknowledges that certain parties devalue visualisation based on a cultural belief that 'this is not mathematics' and observes that such forms of presenting mathematics can be rejected by students (and possibly teachers as well) on the grounds of being too 'slippery', 'risky' or 'inaccurate'. He is also careful to point out that the cognitive demands of visualisation can be high since what is revealed needs to be capable of flexible and competent translation back and forth between the visual and analytic

[5]Edwin Mc Clintock, "Computers" in *Encyclopaedia of Mathematics Education*, ed. Louise S. Grinstein & Sally Lipsey, 139-141. New York: RoutledgeFalmer, 2001.

[6]Ibid.

[7]Abraham Arcavi, "The Role of Visual Representations in the Learning of Mathematics." *Educational Studies in Mathematics* 52, no. 3 (2003): 215-239.

representations of the same situation, a process which he asserts is the core of understanding mathematics.

The Introductory chapter draws attention to the scant use of technology in the Irish mathematics classroom at present and makes a number of recommendations in this regard. These recommendations relate to exploiting the Internet as a teaching and learning resource, the use of software packages (such as PowerPoint) for the presentation of mathematical material in the classroom and the harnessing of mathematical packages both for their computational and graphics abilities as well as the manner in which they can be made to teach clear thinking via mathematical programming. It is the purpose of this short chapter to revisit each of these themes and to elaborate to various extents on the attributes of each.

10.1 The Internet as a Resource

As mentioned in the Introduction, the Internet can be of great assistance to teachers in providing course material via individual web-pages and notes residing on Open Course Ware (OCW) along with supplying exercise sheets. For example, *The World of Mathematics* site maintained by *Mathematica*'s Eric Weinstein is a repository of a vast amount of technical and historical information on mathematics, ranging from the elementary to the exotic. A teacher or indeed pupil need never want for a definition, an example and possibly an area of application.

Those altruistic individuals and reputable academic institutions (such as

Stanford and MIT) who generously share their knowledge, also allow free access to a number of Java applets designed to interactively help reinforce a student's understanding of certain basic concepts. To instance but one example: there are a number of web sites which provide such a facility in relation to the transformation of functions – a topic heavily reliant on the ability to render good graphs. One such site allows the user to specify a 'parent' function $f(t)$ and will then sketch those functions obtained from this parent via reflections, shiftings (translations), scalings (dilations) and combinations thereof once the user specifies the desired function appropriately in terms of $f(t)$. In the other direction, the user can ask to be shown a parent function and a related function obtained from it by a transformation specified in written form and set himself the task of describing the result in functional form. Upon entering his answer, he is informed as to whether he is correct or not. It is clear that an aid such as this one is invaluable to a student keen to deepen his understanding of concepts and techniques being taught in the classroom.

10.2 Presentations/Slide shows

It is not stretching a point to say that a presentation is an ideal way to impart mathematical knowledge. The visual aspect alone ('a visual feast' to quote one teacher) engages one's audience right from the start, something a good teacher constantly strives to do. The prepared slides allow one to build up an argument little by little, and if necessary, to backtrack with equal

ease through some or all of them as often as may be required. But perhaps the over-riding attribute, as it pertains to the teaching of mathematics, that makes these presentations such a powerful pedagogical tool is their dynamic element. This is something a teacher cannot hope to achieve with talk and chalk or slides prepared for an over-head projector.

While such presentations require a considerable amount of time and effort to design, the result is normally well worth it. Once a presentation has been given shape, it can be used anywhere facilities allow. It can be enhanced on a continual basis as one acquires fresh insights and/or receives comments, corrections and ideas from those who view it. It can be copied and modified to suit different audiences. It can also be made available to others, such as students, to view at their leisure and as often as desired.

To design a significant presentation on a mathematical theme requires a certain amount of mastery, such as one acquires from years of teaching and thinking about mathematics. However, obliging students to design their own presentations can prove an exciting and enjoyable way to get them thinking about mathematics. Such presentations could be undertaken or assigned as projects. The work done, under guidance, is almost sure to give them a thorough knowledge of the mathematical topic under consideration – the very effort of deciding on the content for individual slides makes them dissect and ultimately understand the mathematical argument being presented – and results in a very clear understanding of the sequence of logical steps essential to elucidating the theme effectively. Those with a creative flair

might contribute much on their own behalf by way of clever ideas and tricks in the design of slides.

As has already been indicated on numerous occasions heretofore, the core section relating to this work contains a number of presentations prepared with PowerPoint and a smaller number with the multi-media program Flash. It is believed that they indicate clearly how this particular use of technology can assist with the teaching and learning of mathematics. The nature and rationale for each of these presentations has already been outlined at those junctures where it was most appropriate to do so.

However, this chapter has been assigned its own presentation, as an historical tribute to Oliver Byrne, who as early as 1847 advocated the use of colour in proving propositions from Euclid. This radical pedagogical idea appears on the cover of one of the issues of the *Notices* of the American Mathematical Society with the caption "A nineteenth-century window on Euclid". It receives further elaboration in the same magazine.[8] Were this imaginative educator to have had a package such as PowerPoint available to him at the time, one can only wonder as to how he might have been inspired. While remaining faithful to the colours chosen by Byrne, the three slide presentation endows his static proof (of Euclid's Proposition 32, Book I) with a dynamic element which one can be certain he would have approved of while also fully appreciating its great potential as a teaching aid. This presentation

[8]Bill Cassellman, "A nineteenth-century window on Euclid", *Notices of the American Mathematical Society* 49, no. 5 (May 2002): 565.

may be seen as a showpiece exemplifying the positive features attributed to presentations in the above paragraphs.

If the future framework accepts that pre-prepared presentations should be part of the new curriculum, then the collection accompanying this thesis would provide teachers with a ready supply of slide-shows that could be immediately brought to the classroom to assist and enrich their teaching. Over the years a suite of presentations could be built up by adding contributions from different teachers and lodging these in the public domain.

10.3 Mathematical Packages

The recent emergence of extremely powerful mathematical packages presents educators with a discontinuity in the educational enterprise in that, all of a sudden, they offer the teacher radical possibilities to enrich classroom teaching. One might predict confidently that over time their usage will have an overwhelming effect on mathematical instruction.

What a mathematical package may lack by way of presenting mathematical information in the cinematic style of a PowerPoint presentation, it makes up for by its ability to carry out large-scale arithmetic calculations with lightening speed and the facility with which it implements all manner of procedures relating to algebra and calculus. Whatever about the logistics of teaching such packages in the classroom there is little doubt that they can be of great help with standard school mathematics in areas such as

Arithmetic including that of Complex Numbers

Simplification and Evaluation of Formulae

Factorisation

Solving of Equations both exactly and numerically

Graphics in both two and three dimensions

Differentiation and Integration

Matrix Algebra

Modular Arithmetic.

Any one of these packages allows a teacher to present a class on a particular topic in such a way that the material (prepared in advance) appears before the pupils' eyes as if it were being created on the spot. Furthermore, the power of the computer to execute calculations or render graphics far beyond anything one could hope to achieve at a board, enables the teacher to imbue his examples with a greater sense of realism likely to make them more attractive to students.

The power of mathematical packages to graph the most complicated function at will is of special significance. It allows either the teacher or student to investigate many possibilities in real time and so stimulate great creativity. Beginning with the graph a function (of compact support, say), a teacher/student can by means of reflections, translations, dilations or combinations of these transformations generate the graphs of related functions –

'cousins by transformation'. In addition, by plotting the graphs of a function along with a selection of these cousins by transformation one can obtain all manner of graphs, such as periodic waveforms that arise in signal analysis, to give but one example. Having the computer respond to their commands by displaying the corresponding graphs instantly, truly engages the student, so much so that many exhibit a ready ingenuity at generating quite complicated configurations using one basic function and cousin functions obtained by transformations. All of which ensures that such students arrive at a very good understanding of the transformation of functions in an enjoyable manner.

While it can be said without exaggeration that almost all students love to 'crunch' numbers if only with a calculator, the interactive element involved in using a mathematical package on a computer can prove even more appealing, be it only at the simplest level of input and output. Whether or not they understand what they are doing, they take delight in something as simple as seeing the decimal digits of π printed to a hundred or a thousand decimal places. Enliven this task by asking if such expansions are approximately normal and one may then find that they are eager to know what 'normal' means and keener still to learn how they themselves might, through programming, test this property.

10.4 Mathematical Programming

Simple mathematical programming assignments (requiring knowledge of no more than a handful of commands) are a way for young students to appreciate the importance of knowing elementary but significant mathematics. When learning how to write programs to perform a task, which a human knows how to do almost without reflection, learners soon become convinced of the need for a sound understanding of the mathematics underpinning the problem.

For example, the request to write the first one hundred natural numbers out in succession in rows consisting of ten numbers each (as one might do on a page) will be executed effortlessly by any sixteen-year-old student, but how does the same student get a computer program to do this? One needs to know how to 'talk' to the computer, and this in a logical manner which is best assisted by the language of mathematics. In this way, programming aids mathematical thinking very naturally, often very convincingly, by forcing the student to think in an ordered fashion. Although this may cause frustration from time to time, a student is less likely to be offended by the impersonal demand of a computer program for rigour than he may be by a similar insistence from a teacher. In the first instance the student knows that, "It won't work if I don't get it exactly right", while in the second case he may venture a petulant, "You know what I mean" and expect not to be pressed for greater precision.

The need for programming is manifestly more visible in some areas than

others. It is immediately obvious that it is essential in cryptography where one needs to encipher and decipher. This topic is an ideal hunting ground for programming projects which can range from the very simple to the sophisticated but all of which call on an accurate use of mathematical knowledge. They illustrate in concrete detail the general point that, in addition to knowing some mathematics, one needs to be able to implement this knowledge in a strictly logical manner so as get the computer to do one's bidding. As Bellman wryly observes "what we don't understand a computer cannot do."[9]

10.4.1 Concrete Illustration

This purpose of the next few paragraphs is to indicate by means of a number of explicit examples how mathematical programming tasks can be made to reinforce aspects of school mathematics in a very concrete manner. While the discussion is confined to a very narrow portion of the mathematical syllabus it is illustrative of what could be attempted across the full spectrum; an undertaking sure to prove to be a study of great value.

The printing of lists and tables requires a basic knowledge of sequences. Knowing this and given a task connected with programming, a pupil might learn this material with more readiness. The following simple problem is an illustration chosen because it demands very little knowledge of the *Mathematica* programming language other than the **Table** 'command' used to generate arrays.

[9]Richard Bellman, *Eye of the Hurricane* (U.S.A.: World Scientific Publishing Co., 1984)

`Task`: Write code which uses the **Table** command to print the following ten numbers in the order shown:

$$\{1,\ 11,\ 21,\ 31,\ 41,\ 51,\ 61,\ 71,\ 81,\ 91\ \}$$

To be able to do this, the student must know that the general term (of this finite sequence) is given by

$$a + (i-1)d = 1 + (i-1)10 = 10i - 9$$

with the index i running from 1 to 10. Then the corresponding piece of code is

$$\mathbf{Table[10i - 9,\ \{i, 1, 10\}]}$$

While this might strike some as being overly simple, it can prove enormously encouraging and stimulating for the student to see the computer implement his instruction via this line of code to produce the above sequence on the screen. Trite as it may seem, it is an example of mathematics in action and, as such, is of tremendous importance as motivation.

A related question not actually requiring a computer, but still a valuable one, is to ask the student to write down the output produced by the above code. One might also ask if this output differs from that produced by $\mathbf{Table[10i + 1,\ \{i, 0, 9\}]}$.

The next example draws on the one just discussed. Here the challenge is to print out the first one hundred natural numbers in a ten by ten square array,

a task already mentioned as one any student would have little difficulty doing by hand even if the array were not explicitly shown as it is in this exercise. However, in order to instruct a computer to execute the task, one must know about double indexing and sequences. As such, it is a testing problem.

`Task`: Write code to print the following array:

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Table 10.1: Ten by ten array of the first 100 natural numbers.

Most students will readily acknowledge that it important to be able to instruct a computer to print an array such as this; otherwise how can such machines be considered to be of any real use? However, a few moments of reflection should make it clear to the student that to get the machine to do one's bidding in this regard, it has to be instructed in mathematical terms. Hence, if nothing else, the student sees an explicit need for mathematical knowledge. The teacher would be well advised to exercise great patience in relation to this problem so as to allow the student on his own, or in concert with others, to overcome whatever mathematical difficulties he may initially encounter while pondering this problem. The desire to have the computer

print this array on his screen should prove sufficient spur for him to give it his full concentration. The lessons learned in this manner are sure to prove invaluable. The following code does the trick:

Table[10i+j, {i,0,9}, {j,1,10}]

A variation, which might serve as a written examination question connected with sequences and simple mathematical programming, is to ask for the output produced by the line of code just displayed.

The next task, while still well within the ambit of the school syllabus, is another test of sequences more involved than the two previous exercises. The task is to write code to print the array shown on top of the next page:

`Task`: Write code to print the following array:

1									
2	3								
4	5	6							
7	8	9	10						
11	12	13	14	15					
16	17	18	19	20	21				
22	23	24	25	26	27	28			
29	30	31	32	33	34	35	36		
37	38	39	40	41	42	43	44	45	
46	47	48	49	50	51	52	53	54	55

Table 10.2: Triangular array of the first 55 natural numbers.

A deal of thinking is required to achieve this. As such, the teacher might use this task to motivate a discussion in relation to the sequence of triangular numbers, whose successive terms are visible at the end of each row. The

following code does the trick in this case:

$$\textbf{Table}\left[j, \left\{j, \frac{(i-1)i}{2}+1, \frac{(i+1)i}{2}\right\}, \{i, 1, 10\}\right]$$

Deciphering why this is so might prove challenge enough. Alternatively, a written examination question might ask for the output produced by this line of code.

The final task is a much more challenging problem which a teacher might hold in reserve for the inevitable small body of students who will reveal themselves as gifted at this form of programming. Its mathematical content might be regarded as being at the level of pre-university scholarship mathematics which some may find very attractive. (See diagram overleaf.)

`Task`: Write code which will print the numbers in the following array:

1	3	6	10	15	21	28	36	45	55	...
2	5	9	14	20	27	35	44	54	...	...
4	8	13	19	26	34	43	53	...	...	...
7	12	18	25	33	42	52	...	...	...	...
11	17	24	32	41	51	...	...	...	...	...
16	23	31	40	50	...	...	...	...	...	...
22	30	39	49	...	...	...	...	...	...	...
29	38	48	...	...	...	...	...	...	...	...
37	47	...	...	...	...	...	...	...	...	...
46	...	...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...	...	...

Table 10.3: Second triangular array of the first 55 natural numbers.

The following code does the trick:

$$\textbf{Table}\left[\frac{\mathbf{j(j+1)}}{\mathbf{2}}+\mathbf{ij}+\frac{\mathbf{(i-1)i}}{\mathbf{2}}, \{\mathbf{i}, \mathbf{0}, \mathbf{9}\}, \{\mathbf{j}, \mathbf{1}, \mathbf{10}-\mathbf{i}\}\right]$$

An easier but related question which might act as a written examination question is to have students display the configuration giving the first five outputs generated by this code.

While these examples serve to illustrate the manner in which mathematical programming can be made to reveal the importance of mathematical theory while also teaching clear thinking, this activity can prove hugely enjoyable in exploring different aspects of mathematics. Armed with no more than a few 'commands' from a programming language one can pursue all manner of investigations. One area which abounds with easily understood and achievable tasks is that of number theory. The two 'sample questions' which follow give a flavour of what a student might hope to achieve in this regard were he willing to learn at most ten basic commands.

`Question:` What is the code

Select[Range[999901,999999], PrimeQ]]

meant to produce?

`Answer:` All the prime numbers in the range [999901,999999] or between 999,901 and 999,999 inclusive. It outputs

$$\{999907, 999917, 999931, 999953, 999959, 999961, 999979, 999983\}$$

Question: What is the code

Max[Select[Range[999901,999999], PrimeQ]]]

meant to produce?

Answer: The number 999983 which is the largest prime less than a million.

The core piece corresponding to this segment contains a list entitled 'Six Programming Tasks' which indicates a number of exercises a student might attempt in regard to blending mathematics and simple programming.

10.4.2 Overview of Mathematical Programming

Outside of the possibility of students being expected to program themselves, it is recommended that they be given some overview of the usefulness of this discipline to mathematicians, scientists and engineers. In this regard it is incumbent to make some reference to computer simulation, the importance of axiomatics and rules as well as remarking on the limitations of the computer as a tool for doing mathematics.

`Simulation`: It is important that students realise that one can solve certain types of mathematical problems with a computer that simply weren't achievable before its advent. Von Neumann and Ulam were the first to harness the computational might of such machines to solve problems by a procedure which Ulam dubbed *The Monte Carlo Method.* He noted that it is simply too difficult to analyse the game of Solitaire mathematically to determine the odds that any particular game will lead to a successful conclusion. Yet one

can write a program to simulate the playing of an individual game beginning with a random configuration of the deck. By having the program execute the playing of millions of such games in this fashion and recording the relative frequency of completion, one can (by invoking the law of large numbers) estimate the theoretical probability that an individual game will be completed. This is an example of an 'answer' obtained by computer simulation.

Another attractive example from the field of probability but closer to the classroom is `The Encounter Problem`. This proves ideal for illustrating in an accessible manner how a computer simulation can be made to solve a problem and so obviate the need for a mathematical solution. However, as it turns out in this case, the stimulating exercise of having to think about the exact manner by which such a program would perform this simulation causes the student to think about the problem in a way which makes the elegant mathematical solution (presented subsequently by the teacher) appear completely natural. In fact, whether the student realises it or not, the writing of the program cannot be effected without the ability to think mathematically.

Alluding to the unseen mathematics underpinning programs, de Pillis, under the caption "Mathematics – The Hidden Engine", writes:[10]

[10] John de Pillis, *777 Mathematical Conversation Starters* (US: MAA, 2000), 69.

> Dazzling effects in computer graphics are more the results of mathematical algorithms and less the product of the optical camera. Surface-generating algorithms, for example, often depend on classical variational methods and application of energy minimization. Nonetheless, the foundation of these effects – the mathematics that produce them – is generally ignored or unappreciated.

Perhaps, by way of concluding this short discussion on simulation, it should be mentioned that the ability to simulate a real-world situation can often save a vast amount of money, an observation on which the student might be asked to comment.

`The Importance of Axioms and Rules:` A discussion of mathematical programming is also ideal for having the student appreciate the importance of axioms and rules because, as the teacher can explain, these are exactly what a computer package needs to be supplied with so that it can perform such tasks as differentiation in a coherent and consistent manner. What the student may find tiresome to remember is what is vital to the computer, while those examples that the student might cherish are a matter of complete indifference to the machine. These observations alone afford the student a glimpse of the value of abstraction which is such a part of the culture of mathematics. The constant striving to isolate the essential from the inessential through the formulation of axioms and rules of operation that are quintessential is precisely what is needed when the computer is enlisted to aid in the mathematical enterprise.

`Limitations`: It has already been pointed out in the segment dealing with mathematical complexity that there are also tasks which can be done in theory but which are *computationally prohibitive* even with the availability of powerful computers. The example of `The Travelling Salesman` clearly shows that there are boundaries to what is feasible – an enlightening circumstance revealing to the student the limitations of what is realistically achievable. Another valuable lesson that the student should also be taught is that while a program can calculate and verify, it cannot prove things, as of yet, at any rate.

Summary: The more the teacher can take advantage of the beneficial features of technology described here in the classroom, the richer and more invigorating will be the mathematics revealed and learned. However, while the various technical tools should be appreciated for the tremendous assistance they provide, the student must never be allowed forget that in relation to mathematics there is no substitute for human thinking.

Part III

Chapter 11

Contribution/Implementation

The chapter introducing this work asserts that the current provision of mathematics education at senior cycle suffers from a number of shortcomings. These are described as a failure to show the relevance of the material being taught and an almost total neglect of the history and culture of the subject along with an ignoring of its recreational element. It is contended that the reason the current programme is not having its ambitions realised effectively is due to an overly narrow interpretation of the course material by way of texts and examinations. It is also maintained that there is a lack of substance/rationale in the current presentation of syllabus segments and a scant use of technology.

Collectively these circumstances determine the rationale and basis for the constructive element of this work which, with the exception of a minor contribution from Part I, constitutes all of Part II of this dissertation as well as the associated core element presented in the accompanying compact disc.

The chapter structure of Part II is designed in such a way as to address

each of the issues alluded to above in a particular order which (it is believed) lends the discussion maximum cohesion. The relevant chapters discuss in elaborate detail their rationale and, with the assistance of the corresponding core segments, provide concrete and extensive illustration of the points being made so that the intent of each is completely clear and achievable.

The remaining portion of this endeavour (consigned to the accompanying cd) is of still greater extent, containing as it does all of the concrete material written to support the various assertions made in the main body of the work. It consists of the explicit illustration of the three Topics described in chapter 6, all of the PowerPoint and Flash presentations mentioned at various junctures throughout the work, numerous other written pieces as well as different sets of sample examination questions.

This writer is conscious of the fact that this dissertation on its own is of considerable extent and should under different circumstances be a briefer work which would prove less taxing on those charged with reading and assessing its merits. However, a basic tenet which has driven much of the present undertaking is that one cannot hope to effect change, however gradual it may need to be in coming, in a significant and essential manner *without* spelling out in explicit concrete detail for educationalists and teachers that which one has in mind. The result of delivering on this promise in a realistic and tangible fashion accounts for the size of this work.

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The next section is dedicated to revisiting in some detail the specifics of the contribution made by this work and includes a review of both the type of reform proposed and the recommended pedagogical cycle. While it also touches on the issues of assessment and the possible utility of the materials presented in this work in relation to teacher education, it is in the section devoted to implementation that these two matters are discussed at appreciable length.

Contribution

The initial contribution made by this work might be regarded as the substantiation of the criticisms levelled at current provision as synopsised in the opening paragraph of this chapter. This was done at a number of junctures throughout the early chapters but in particular, it was carried out through a scrutiny of current examinations and texts in the chapter "Establishing the Case". While much of this argumentation was perforce negatively critical and so of little constructive value, the in-depth analyses of the 2003 HLC

test papers contained much commentary which indicated various ways by which those questions deemed overly-technical could be set within engaging and purposeful contexts (were classroom provision allowed to cater to them). This exercise of its own, validates, in some measure, a sub-theme of this work which is that "minor changes of emphasis and approach to existing syllabus material can be made to achieve major improvements in appreciation and understanding."[1]

The central assertion of this thesis is that a significant enhancement of the teaching and learning of mathematics at senior cycle can be achieved by a much greater dedication to the provision of context for the purposes of motivating the need for technical mathematical content along with the subsequent demonstration of the power and relevance of that theory. Furthermore, it is averred that this ambition is best attained by adherence to an instructional sequence where 'setting in context' precedes the exposition of theoretical content.

One way by which this work engaged with the 'context – content' component of this assertion was by specifying in concrete terms how the use of the 'applications domain' can be expanded on several fronts to reveal mathematics solving sense-making problems, some of which are quite substantial. However, two new arenas of mathematical activity were also identified; the as yet untapped (in terms of current of Irish provision) domains of recreational mathematics and the history/culture of mathematics. It was demonstrated

[1]See page 11.

in numerous ways (to be synopsied presently) how these two domains can be exploited to provide additional and rich contexts for making contact with technical (school) mathematics.

The central methodology adopted in this work represents a major paradigm shift from the current 'traditional' model (with its roots in the New Mathematics movement) to one which has close ties with what is styled Mathematics in Context. This pedagogical model was illustrated diagramatically as shown:

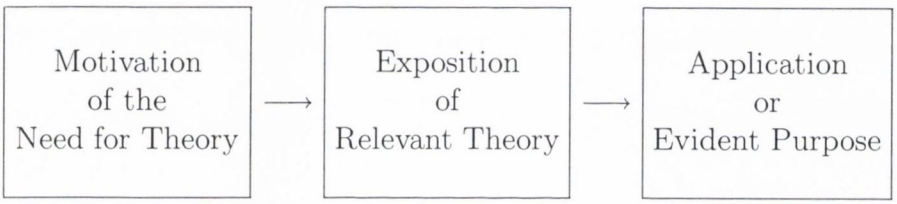


Figure 11.1: Instructional sequence guiding the design of a Topic.

This instructional sequence informs the design of a Topic which in this work translated into the writing of the complete set of notes a teacher or textbook writer might prepare to cater to an existing syllabus entry, or a newly proposed one, so that upon full delivery, the reason for the inclusion of the particular syllabus entry is made indisputably evident.

The three separate core-pieces written specifically to illustrate this notion of a Topic for teachers are a 268-page piece *On Quadratics*, a 67-page piece on *Difference Equations* and another of approximately the same length on *The Newton-Raphson Method.* In addition to showing how the instructional sequence informs the underlying layout of the material, each of these treat-

ments is laced througout with didactical observations, historical allusions and culminates in the provision of a number of full-bodied applications of mathematics.

While familiarising oneself with a Topic clearly demands much more of a teacher's time and understanding than that called for by current textbook treatments, it is believed that the effort required is rewarded by the acquisition of a more substantial and satisfactory overview of the topic at hand. Its study provides the teacher with a broad and deep knowledge of both the mathematical and pedagogical content by which subject matter can be better represented to support student learning.

The second recommendation made here with regard to methodology was that there should be a much greater use of technology in all its different manifestations combined in different ways to vary classroom presentation and so produce more powerful learning environments. Specifically, forty-one different slide-shows make clear in a very tangible manner how software programs such as PowerPoint and Flash can assist in the dynamic presentation of mathematical material. The many attributes of mathematical packages were also discussed at length but the most valuable proposal, perhaps in terms of novelty, was the advocacy of the use of mathematical programming as a means by which students cultivate clear thinking and learn of the manner in which elementary mathematics is crucial in instructing a computer.

The three sub-sections which follow give brief summaries of the general contribution made to the 'context-content' domains labelled as Applications,

Recreational Mathematics and the History/Culture of Mathematics.

Applications: This work devotes considerable effort to revealing how genuine applications of mathematics can be made to connect with the many existing syllabus segments which, as shown, are starved in this regard at the moment. It is hoped that their adoption in a future framework would counter in some way the 'irrelevancy' criticism that bedevils current provision and which surfaces time and time again to damage the already tarnished image of school-mathematics.

As already mentioned in relation to the critical analysis of the style of the current HLC examinations, it was shown on a small scale how some of those examination questions of a technical nature can be made (by extension and/or reframing) to link with an application. A much more substantial provision of applications, fully-motivated by context, was given in the extensive illustrations of the notion of a Topic. There, numerous examples solving transparently relevant problems address both 'mathematics for all' and 'mathematics for science' while revealing the different 'situations' in which these applications arise.

The chapter devoted to Applications, in addtion to being supported by ten written pieces and two presentations from the core, itemised in some detail new applications relating to the subject areas of geometry, trigonometry, counting, sequences and series, calculus and probabiliy. Furthermore, under the heading of 'new topic areas' this work described the many applica-

tions associated with check-schemes, error-control coding and cryptography which introduce new ideas enhancing and envigorating the provision without transcending its current technical mathematical level

However, in addition to furnishing future provision with a multitude of applications this work has placed before the reader a number of what it regards as substantial pieces of mathematics or 'grade I' applications, whose over-riding purpose for this writer is to reveal the relevance and potency of mathematics to students so that they may come to appreciate its ability to solve 'real' problems. It is strongly recommended that such show-pieces be always part of the mathematical provision at senior cycle irrespective of any consideration as to whether or not the student is expected to know or report on these applications.

Recreational Mathematics: As can be judged from the number of written pieces and presentations residing in the core (twenty-six and twenty-five respectively), along with the extensive discussion devoted to RM in chapter seven, a high perecentage of this work's overall endeavour is dedicated to arguing for a much more central positioning of RM in any future provision. In fact, it could be said that the manner in which this work has chosen to treat RM is one which, perhaps unintentionally, identifies it as a stand-alone domain in its own right; in much the same way that many educators have always regarded the 'history of mathematics'. This latter area is justifiably considered a well-defined subset of the universal set of mathematical activity

which is capable of informing the teaching of school mathematics. The extent of the material treated in this work along with the demonstration of its effectiveness as a rich source of motivation for making contact with technical mathematics at secondary school level, lends strong support to any claim RM might wish to make to be placed on a similar footing. While this is the stance adopted in this work, it should be noted that the classifying of RM as a distinct area of activity is not discernible in current educational literature with, perhaps, the exception of a small number of journals for whom it is their *raison d'être*.

The major use made of RM in this work is not the traditional one of viewing it as a source of activities geared towards self-learning by discovery, although this aspect is not neglected, but rather as a means by which judicious motivation can facilitate 'quick contact' with significant concepts and results from technical mathematics. Beyond demonstrating the numerous lessons RM teaches in relation to logical thinking and arithmeic, this work has shown explicitly how RM makes contact with combinatorics, algebra and geometry. At the highest technical level this is done in relation to arithmetic series, geometric series, and a-g series while the supplying of 'proofs' calling on aspects of arithmetic, algebra and geometry is demonstrated throughout.

History & Culture: The main thrust of the efforts to be found in this work relating to the history of mathematics is one of making it easy for teachers to avail of historical material without their having to undertake an

in-depth study of this history.

The simplest way it was envisaged that this could be accomplished was by the creation of an historical time-line which, while being in no way comprehensive, contains sufficient facts touching on most aspects of the school syllabus whereby teachers could impart many elementary but telling lessons that would prove encouraging to students. The expectation is that teachers consult this time-line to extract from it an historical cameo which would provide a possible context for a paricular topic. A number of the pieces appearing in this resource also indicate how one might design a substantial examination question based on an historical theme and making significant contact with technical mathematics.

In addition to acquainting teachers with some aspects of mathematics history of which they may not be aware, it is hoped that something like this time-line would be made the basis of a future syllabus segment catering to the history and culture of mathematics.

Should teachers wish for something more substantial than extracts from this time-line, then any one of the twelve self-contained `Illustrations from History` (residing in the core) can be made the basis of a full class with history supplying the motivation for the study of technical content. The coverage given to the role of its history in the teaching of mathematics also provides explicit indications of project areas suitable for student-portfolio work.

The second section of the chapter devoted to the history and culture of

mathematics is concerned with broadening the student's vision of the mathematical landscape so as to reveal more than the 'useful mathematics' which is so often regarded as occupying a central position within mathematics. It is about exposing some of the 'big' ideas of mathematics and the influence they have on human thought.

This was done by discussing numerous explicit examples, many of which are further supported by writings and presentations in the the core, relating to such notions as the art of conjecturing, the dangers of induction, the pitfalls of intuition, the need and importance of proof, complexity and computational feasibility and the subtle and often counter-intuitive nature of the infinite. One specific core piece illustrates through its use of mathematical vignettes the 'elegance' that many of the mathematical fraternity acknowledge pervades the discipline of mathematics. The examples given are accessible at the level of school matheamtics and offer a window to a number of significant ideas.

By way of delivering on the assertion that students should be given access to much more by way of mathematical literature aimed at conveying the beliefs, values and working methods of mathematicians, this work is accompanied by a Socratic dialogue running to over 250 pages which uses no more than algebra and whose central theme is that of illustrating the 'process of discovery'.

Perhaps, the most valuable contribution of this section will be seen as its treatment of probability in relation to its ability to civilise by informing

decision making, debunking myths and protecting one from folly.

One aspiration which has encouraged and driven the creation of the extensive written material and presentations accompanying this work was that much of this material might find impementation in some measure in the short term with the greater bulk of this contribution securing a place under the umbrella of a longer-term new framework based on MiC principles. How these materials might be used to assist teachers with the actual task of implementing the suggested innovations in as seamless a manner as possible is treated in the next section where a discussion of possible new forms of assessment is also to be found.

Implementation

The question of implementation is crucial to any reform; "the emerging view in the literature on educational change that planning for reform, without engagement at the outset with issues of implementation (particularly resourcing), ensures that the reform, however worthy, will in large measure fail."[2] While the vast majority of the recommendations made in this work are fully justified by way of rationale and extensively illustrated in a concrete fashion so as to assist with their implementation in the classroom, this effort of its own is not enough to see them readily incorporated into an educational

[2]National Council for Curriculum and Assessment (NCCA), *Developing Senior Cycle Education – Consultative Paper on Issues and Options* (Dublin: NCCA, December 2002) cited in Section 1.2.

programme of the near future.

There still remains much to be done by way of effecting the implementation of some or all of the proposals described here. Such a task is one which might be most effectively accomplished by those currently involved with changing the existing method of assessment – a single final examination at the end of a two-year cycle – with a more flexible one which envisages a form of continuous assessment over a three-year program. Alternatively, or side-by-side with such an effort, the job of designing a structure which would best accommodate the recommendations made here might serve as a valuable two-year study for an M. Ed. thesis.

However, while it is not the purpose of this work, nor within the expertise of this writer, to address these issues in any manner which might be regarded as comprehensive, it is necessary to say a few words about the implications of the proposals set forth and to make a number of suggestions as to how they might become part of a future framework.

Grafting to the Current Syllabus: Since the proposal that there be a more substantial approach to the teaching of mathematics (as exemplified by the notion of a Topic) is very much at odds with current teaching practice, the present programme is not seen as one suited to taking on board in a truly effective manner the full body of recommendations made here.

However, as already stated, 'minor changes of emphasis and approach to existing syllabus segments can be made to achieve major improvements in

appreciation and understanding'.[3] This work has substantiated this point over and over again by revealing how material treated heretofore in a purely technical and context-free manner can be greatly invigorated by being set in a context which students will grant is genuinely applied. In many cases this would involve no more than a teacher using material presented in the core to supplant an existing example by a corresponding applied one. The more this could be done across the spectrum of topics, the richer would classroom teaching become as a result. Of course, as has been stated heretofore, teachers could only do this with confidence were they assured that future examinations reflected the shift from context-free questions towards more relevant ones.

In addition, it is conceivable that a teacher could use much of the material presented in connection with the Topics to assist with the sequencing of an existing topic and lending it greater relevance, certainly as it might relate to say, quadratics and the Newton-Raphson method. It is expected that the historical references to be found in this material would be availed of, as might any PowerPoint presentation deemed useful, by way of motivation or elaboration.

Thus, as a first step towards implementing some of the recommendations made in this work, one might envisage the entire gamut of core material being mined to see which elements of it could, without too much adjustment, be seamlessly stitched into the current programme. Of course, this would

[3]See page 11.

have to be done with the guarantee that the examination would reflect the changes involved in adopting such a course of action. This writer cannot say whether teachers would be able to accommodate themselves to such proposals based solely on their existing mathematical knowledge and their ability to assimilate the core material or whether it would be necessary to also have in-house training to aid in this endeavour. Investigating this and related matters might make for a valuable study in itself.

Recreational Mathematics: While such a re-jigging of the existing programme would clearly be better than the status quo, it is at best a half measure which simply does not exploit a fraction of what is being proposed here. As a second step towards achieving a more full-bodied implementation, while still remaining close to the current programme, the transition year could become the first of a three-year cycle, a recommendation already implicit in Options III and IV of the NCCA's 2001 consultative paper.[4] One suggestion, very much in keeping with the present practice of some teachers, would be to devote part or all of this first year to the teaching of recreational mathematics (RM).

However, this is an undertaking which would have to be approached with care and planning to ensure that it achieve its full potential. Furthermore, a number of key issues would have to be addressed, relating to the amount of actual school mathematics to be taught, whether the material should be

[4]National Council for Curriculum and Assessment (NCCA), *Developing Senior Cycle Education – Consultative Paper on Issues and Options*, 61-62.

examined, and if so, in what form and whether marks obtained should go towards the overall accreditation of a student over the three year cycle. It may be best to attempt to answer those questions relating to examinations first as it involves a major decision: whether to adhere to the present system of a single terminal examination or to embrace the new assessment procedures as embodied by Option III or Option IV.

If a single final examination still proves to be the reality for some considerable time, then one course of action is simply not to examine this element in any formal manner whatsoever. Such a policy affords the teacher great freedom to present recreational material in his own fashion and perhaps to his own particular strengths. However, in light of what is written in the RM chapter, it is hoped the teacher would be guided by the material to be found there, in particular with reference to touching on standard school mathematics in a meaningful but enjoyable manner.

If, on the other hand, one inclines to the view that there should be some form of assessment at a national level relating to the teaching of RM, then this decision automatically demands that there be uniformity in the RM material presented in the classroom so as to allow for the setting of 'fair' questions on whatever written tests may be conceived.

In the case of a single final examination, such testing would have the disadvantage of taking place at a time remove of at least two years from when the student is first exposed to the material being examined – a logistical shortcoming which is one reason why serious consideration needs to be given

to alternative modes of assessment. Another is that a sole written test neither accommodates or acknowledges more flexible forms of learning such as project work which is particularly suited to RM.

While a single examination may have much to recommend it logistically from an administrative viewpoint, drawbacks such as these (and others which have already been discussed at length earlier in this work) strongly suggest that the time may have come to break with tradition and to consider more extensive means of testing a student's overall performance.

In relation to the teaching of RM, one might recommend that a one-year course built around RM (and possibly other constituents) be examined by means of a national examination at the end of the first year of the three year cycle. It would of course be imperative that there also be a mechanism for assessing RM project work carried out during that year. It is not the intention here to dwell on matters such as these, perhaps best left to others trained in this regard, except to mention that while this writer is aware (to some extent) of the financial and administrative implications such suggestions have, it is to be hoped that parsimonious considerations of cost should not prove an impediment to implementing innovations which, however radical they may first appear, can only improve the lot of the student and the teacher.

However, if one opts for the formal testing of RM then, irrespective of the possible modes by which it may be examined, it is strongly recommended that the RM chapter (possibly in conjunction with other sources of a similar nature) be made the basis of a study which would carefully consider

the actual recreational content to be taught, indicate the manner in which it should be sequenced and describe clearly the technical school mathematics to be assimilated by the end of such a course. This endeavour would be greatly informed if a number of pilot studies were carried out (perhaps in current transition year classrooms) to test the feasibility of proposed programmes. These investigations might pay particular attention to the methods of teaching, the time required to cover various elements and the level of interest evinced by the students in each of these elements. Furthermore, it would be essential to have some measure of the amount of technical mathematics acquired by the participating students.

Whereas considerations of time simply did not allow this writer to pursue questions of implementation as he would have liked, it is believed that the structure of the RM chapter and the accompanying material should prove of very definite assistance to anyone involved with the studies and investigations just described. To help further with the task of exploiting RM as a means of teaching mathematics, the core section accompanying this present discussion of implementation contains a piece entitled a 'A Fun Quiz'. This is a collection of short 'mathematical' questions (answers supplied) arranged in three categories: those carrying two marks for a correct answer, those carrying four marks and 'six markers'. They are for the most part simple arithmetic problems, amusing riddles or puzzles requiring little or no mathematical expertise other than logical thinking which are suitable for pupils of Junior Certificate level or higher. They test all sorts of skills and so make

for a concrete source of teaching material. One method of exploiting this piece to advantage would be to discuss its content in 'brainteaser' or 'mind-stretcher' sessions and to later use a number of the questions as part of a written examination or an oral quiz. A teacher might also simply use this suite of problems as a Christmas Quiz as part of the pre-holiday festivities, perhaps on the last day of term to competing teams of students. No matter what their level, most will find it both challenging and entertaining.

The answering of puzzles whose solution can be described in a paragraph or so addresses the communications standard and is as valid a measure of mathematical ability as the solving of an equation or the differentiating of a function. However, it would be important to require a reasonably high standard of answering where students would be penalised not alone for incorrect mathematical reporting but also for faulty spelling, grammar and punctuation.[5]

Such a quiz could be greatly expanded upon over time by contributions from teachers and students alike, thereby enriching its value as a teaching source and an examination instrument. Implementing such a quiz by means of a PowerPoint presentation might make for an attractive and valuable student project in RM.

The proposals made here in relation to the implementation of RM are based on a perception of it as a substantial stand-alone element of a larger

[5]One of the criticisms levelled at the PISA03 endeavour is that it tolerates faulty spelling, bad grammar and poor punctuation in the answering of mathematical questions.

syllabus, being taught in the first year of a three-year programme and being examined by means of a written paper and a project. In the event that the whole-hearted adoption of such a recommendation proves some time in coming, it is to be hoped that in the meantime the designers of syllabi and drafters of examination papers provide some means by which the teacher can avail of RM as an effective agency enhancing the teaching of mathematics at every stage of the three year programme.

Use of Technology/Mathematical Programming: Before touching on the implementation of some of the other major themes of this thesis, it may be appropriate to mention at this juncture that the PowerPoint and Flash presentations accompanying this work afford the teacher concrete demonstrations of just one particular means of seeing how technology (beyond the use of overhead projectors) can aid one's teaching in the classroom.

With regard to mathematical packages (possessed of a mathematics engine) tangible illustration could also have been provided in the section by showing how a class on a particular topic can be presented using such a package. While the positive features of this mode of presentation have been described, this work is not accompanied by any such demonstration because many expertly designed examples (prepared specifically for the classroom) are provided by the makers of these packages. While the prime focus of the major companies manufacturing such packages may not be on the classroom, each has a division dedicated to mathematical education which is actively

committed to improving their product so that it has greater impact on the classroom. As a consequence, the next ten years is almost certain to see a wealth of material emanate from these sources so that one can envisage the DES/NCCA facing a decision which will not be one of whether or not such packages should be part and parcel of the mathematical classroom, rather which one should be chosen as the 'best-of-breed' by which to achieve the necessary uniformity across the nation's classrooms.

Likewise, while the use of mathematical programming has been strongly advocated in this work and illustrated to some extent using the *Mathematica* language, it has not been felt necessary to supply actual programs since those needed to execute what is being proposed here consist of no more than one line of code (as exemplified in the relevant section). However, a worthwhile M. Ed. undertaking might be an investigation into the amount of school mathematics that can be taught effectively in this manner.

In general, and however long it may be in coming, there will be a progressive shift from traditional learning to technology based systems, not alone of the types just described but also to e-learning facilitated by web either at face-to-face level or at a distance. While all of these aids should never supplant the tried-and-tested methods offered by the teacher they may be invaluable in complementing them.

History and Culture of Mathematics: As recommended for the teaching and learning of RM, one could also propose that a stand-alone policy be

adopted towards the teaching of the material outlined in the chapter on the History and Culture (H&C) of Mathematics. Needless to say, while such an undertaking would require an amount of planning commensurate with that suggested for RM, it would almost surely prove to be very valuable and might even find itself implemented in the near rather than distant future.

However, since the H&C aspect lends itself to easier blending than does RM with the subject of mathematics as a whole, one could envisage, in the short term at least, H&C being treated as simply as an undifferentiated part of the syllabus. In this case, the expectation would be that the teacher make much greater reference to historical sources than at present, even to the extent of using an entire class in this regard. Some of the Topics material illustrates how this might be done while still ensuring that non-trivial school mathematics is being taught. Furthermore, it is quite a simple matter to set mathematical questions in an historical setting without any risk of their being spurious. An accompanying core piece contains a sample of such history questions, many of which are commented upon while some few are supplied with solutions (which make clear that the questions are neither superficial nor overly difficult). Furthermore, those items chosen for the core piece "Illustrations from History" could each serve as a substantial class lecture and a possible examination question.

Thus at the level, where the teacher makes reference to historical sources in the classroom and where the examinations assist in this endeavour, one could envisage the history material in the H&C chapter being harnessed

to invigorate much of what is being taught at present. However, a broader framework based on a three-year programme incorporating continuous assessment would be necessary if one is to make use of the many History projects (described in some detail earlier) as a means by which a student takes a more active part in his or her own learning.

While one could glibly suggest that much of the material designated as 'cultural' in the H&C chapter could also be grafted to the current programme, to do so properly would inevitably radically alter the nature of what one was hoping to preserve. However, given a willingness to consider possible extensions counterbalanced by cuts elsewhere (a proposal to be discussed later) one can see from the nature of the material that any of the four themes described could be treated over a number of classes and examined by substantial single-theme questions.

For example, on the theme of 'Mathematical Literature', which in this work is addressed by asking for the reading with understanding of a mathematical story, one or more questions relating to this task could appear in a section of the final examination paper. To aid in this endeavour and to indicate one way by which it might be carried out, the core contains a set of exercises (with accompanying solutions) amplifying many of the points made in the story. The exercises are designed to make the reader think and reflect further on the various lessons the tale attempts to teach.

Similarly, the 'About Proof' theme is a compact one, capable of being taught over a number of classes, whose material makes for obvious but edu-

cationally important examination questions. The 'Mathematical Vignettes' described in this work also provide an extra source for such classes and questions. While there are clearly far too many to be given in a single three-year programme, one might single out a number of them as examinable in a particular cycle.

Likewise the 'Mathematical Humour' section harbours no hidden difficulties as to how one might teach and examine it, although this latter task would need to be treated seriously.

The section 'Intriguing and Interesting Results' contains a number of subsections, some more wide-ranging than others. Of the shorter ones: 'The Nature of the Infinite', 'Complexity', 'Topological Ideas', it might be best to prescribe no more than one for a single cycle and to combine it with a selection from the many sub-themes treated in the important and more substantial topic 'The Pleasures of Probability'.

Applications: It has already been suggested that many of the applications described in this work could be gainfully used as examination questions to supplant context-free ones. While this suggestion is perhaps feasible only in relation to those applied problems which lend themselves to being answered in no more than a paragraph or two, it is strongly recommended that all applications, irrespective of their length, be set before the student in the classroom as part of a broader theme and not as unmotivated isolated elements. In fact, the greatest efforts of this work have been invested in

describing how a teacher might remain faithful to this principle through the use of Topics.

While the rationale and purpose of a Topic has received considerable elaboration along with extensive concrete illustration by means of three examples, the presenting of a Topic poses the most radical challenge to the teacher in terms of actual implementation. The reason for this is clear to different extents from the Topics described and rests on the fact that often for a Topic to be truly effective it is desirable that a student be algebraically adept and reasonably familiar with a number of standard mathematical topics, in particular trigonometry and calculus.

This is certainly true of the NR Topic and could also be said of the Topic "On Quadratics" were it the ambition to present it in its entirety as described in the core. Clearly, this latter possibility could be realistically entertained only if each of the requisite areas of knowledge were previously covered by the teacher and adequately assimilated by his students. In this case, one would envisage the full presentation of this Topic being kept to the latter portion of the three-year programme and that its treatment would entail an element of revision as well as extension. By its very nature the NR Topic would have to wait until the differential calculus had been covered, by which time, most teachers would have treated polynomials and trigonometry, the two topics which supply the bulk of the problems used to illustrate this powerful algorithm.

However, it is clear that the Topic "On Quadratics" can also be used as a

model by which a teacher should approach the 'ordinary topic' of quadratic polynomials and equations for the first time; this being done by treating only those items which are within the student's current knowledge. In this regard, as with other Topics (no matter how short), the only injunction is to adhere to a sequence where motivation through concrete examples is followed by the exposition of relevant theory, the usefulness of which is then demonstrated by the solving of the original problems along with new, and possibly more substantial, ones.

Projects: Wherever appropriate this work has indicated a number of areas where the teacher's job of trying to impart mathematical knowledge might be assisted by the student's undertaking a project on a recreational or historical theme. It is believed that these individual projects provide a vehicle for students to learn about a particular aspect of mathematics at their own pace and in a manner essentially determined by themselves. However, such projects would have to be more than a mere cataloguing of facts with the student expected to stand over anything that is capable of proof at senior cycle level. While the essential mathematical content should be at this level, students might be allowed, and even encouraged, to state (without proof) one or two results beyond the range of their course as long as they understand the import of what is being asserted. One might also allow a student to propose a project of his or her own provided it is well thought-out and substantial.

These projects might be enhanced by requiring that the student prepare

a PowerPoint presentation to be given in front of classmates permitted to ask questions afterwards. The mere act of reading the relevant material, writing it up and presenting it in PowerPoint form in front of a possibly critical audience should act as sufficient incentive for the student to acquire a deeper understanding of mathematical ideas than he might otherwise have.

Some students, perhaps formerly more reticent with regard to mathematics might enjoy being able to choose a project of their liking and working away on it in a leisurely fashion but always able to consult the teacher. Likewise, projects prove an excellent way of giving 'bright' students an opportunity to learn more mathematics. In particular, open-ended projects might be further developed into 'Young Scientist' projects for entry in the Mathematics/Science category.

Ideally one would like to compile a comprehensive list of topics where each is described in written form to such an extent that even the weakest students could grasp the essentials of what would be involved in undertaking such a project. After perusing such a list, the descriptions given should be complemented by the teacher-supervisor providing a brief verbal overview of each of the topics before students are obliged to choose. Again, ideally, one would hope that every student be fortunate enough to be assigned a topic that would appeal to her or him.

The chapter on the History & Culture of mathematics gives a list of twelve history projects along with a short explanation of each. It is clear that this could be extended with little effort. The core piece accompanying this

section contains another list describing a number of miscellaneous projects which might also be used to assist a teacher to form a general list of projects. Furthermore, it is to be expected that most teachers will have many ideas of their own as is evinced by another list given in the core which describes the different projects suggested by a small group of teachers at a meeting in University College Cork some number of years ago.

Any recommendation that projects become part of a student's overall portfolio of assessment is based on at least two assumptions: firstly that such projects be assigned marks and secondly that a teacher's knowledge of the requisite mathematics is extensive enough for him to be assured that he can guide and assess such work with confidence.

The second assumption may be very far from true and require extensive training on the teacher's part to turn it into a reality. For the first assumption to become a fact it is necessary to change from the current situation of a single final examination to a system of continuous assessment as described by the NCCA's Option III or Option IV. However, in July 2005 an article entitled "Hanafin backs two-stage Leaving exam" relates that: "Minister for Education Mary Hanafin has backed a radical reform of the Leaving Certificate that would see an earlier exam in each subject during the final school year as well as the the June exams. The report adds that: "The earlier exams would account for about 25 per cent of the total marks" and "Like the main exams in June, the earlier ones will be marked externally and not by

class teachers."[6] Thus it would appear that some form of school assessment is not to become a reality in the near future and at best there is to be a modest form of external assessment.

11.1 Conclusion

While the discussion to date has indicated a number of ways that many of the recommendations set out in this work, and substantiated through concrete illustration, might be made to enhance the present programme, it is abundantly clear from that same discussion that any such attempt is destined to be at best a partial measure. It is evident both from the nature and extent of the proposals that to have any hope of their achieving satisfactory implementation, one needs a three-year programme allied to a system of continuous assessment.

Furthermore, it is also clear from the exposition illustrating the notion of a Topic that there is a greater emphasis placed on the consideration of substantial problems than is the practice currently. To teach school mathematics in this manner requires a great deal of time if is to be done effectively. In the opinion of this writer it would be overly ambitious, even within the framework of a three-year programme, to expect to implement these proposals while still attempting to cover the broad range of topics currently covered on the HLC. Consequently, it is suggested that a great deal of the existing programme be dropped. It is for this reason that this work has not sought

[6]Seán Flynn, "Hanafin backs two-stage Leaving exam" *The Irish Times*, July 1, 2005)

to enhance such valuable topics as linear algebra, vectors and some of the further options, since it is believed that they can be safely left to third level.

Specifically, it is recommended that the option Further Probability be combined with the existing core topic Statistics and Probability to form a larger and more substantial Topic on Probability and that all the other options be left out. Since such topics as integration by parts and Maclaurin series would no longer be covered, the suggestion with regard to calculus is that the basic principles and concepts be taught to such an extent that students have a full appreciation of the Fundamental Theorem of Calculus (something they are decidedly lacking at present) and that there be a greater emphasis on applications to include max-min problems, related rates, averages and volumes of revolution.

While such a proposal is sure to be regarded as radical and cause some concern to those at third level, one might seriously question how much the current coverage of these topics at HLC is actually achieving. One need only consult the analysis given of these particular topics on the 2003 HLC papers in the chapter "Establishing the Case" to see that their treatment is subject to the same negative criticisms made of the other topics. Furthermore, it may be argued that these particular topics, more than the others, rightly belong at third level where they can be (and indeed are) taught in the first year.

Although it may be construed that one implication of this proposal is that those involved in third level education look after their own special interests

and forego taxing the secondary system with tasks which are encumbering the latter's mission, the true purpose of the proposal is to provide the secondary teacher with sufficient time to teach fewer topics to a greater depth. The effect of this time-freeing proposal should be that all secondary students are given a better mathematics education while the third level sector will benefit much more in the long run by receiving self-confident entrants skilled in the art of problem solving and possessed of an appreciation that mathematics is both relevant and aesthteic.

This proposal would necessitate the designing of a syllabus with a shorter range of topics with each syllabus segment containing at least one substantial element revealing the relevance of mathematics. The intention then should be to examine this material accordingly through full-bodied questions, some of which integrate different syllabus elements. While this work might serve as a blueprint for such an endeavour it is clear that an undertaking of this magnitude would require a team of experts working for a considerable number of years to effect its proper implementation.

Radical reform must be undertaken, sooner or later, for the sake of students, teachers and the subject of mathematics itself. This imperative has been the driving force of this work whose over-riding purpose is to assist in the endeavour of making the study of mathematics relevant, enjoyable and culturally enriching.

Illustration of a Topic

This section provides a concrete illustration of a Topic based on Difference Equations. It treats both first- and second-order difference equations. As has already been indicated, the material here is augmented by two PowerPoint presentations (relating to first-order recurrence relations) which are to be found on the CD accompanying this work. The rationale and nature of these presentations has been described in some detail in Part II.

The first problem discussed here is of genuine practical concern as it relates to mortgage repayments, and as such is an example of an extra-mathematical application. By belonging to the 'personal situation', rather than the more usual scientific domain, it is likely to be of greater appeal and value to a broader band of students. The manner in which it is solved illustrates the power of recursive thinking while the solution obtained provides the learner with information which is undeniably useful.

It is recommended that the first-order recurrence relation relating to compound interest be treated as a precursor to this problem because of its obvious financial connection. It would also be most useful if, before tackling this problem in the classroom, a number of other first-order recurrence relations

(such as those listed) were discussed quite fully, so that by the time students come to study this problem, they will find the formulation of the pertinent recurrence relation almost self evident. On the understanding that students will have been exposed to a great deal of the general terminology used, the exposition here, while being self-contained, is not as expansive as it is in the two PowerPoint presentations. However, the account is still extensive and, it is hoped, conveys a little of the excitement and interest this problem can engender.

The Mortgage Problem

When a loan is raised to purchase a house or apartment it is often called a mortgage. The word mortgage is related to the word 'mort', meaning death in French. In the context of paying off house loans, it is used in the sense of putting the loan to death – 'amortising' it.

At today's prices a young couple purchasing a house for the first time might need a loan anywhere between €100,000 and €200,000. If, as is typical, this loan is to be paid back over a period of 20 years at the prevailing rate of interest, then the over-riding concern to would-be borrowers is the amount of the fixed monthly repayments needed to do this.

It is easy to convince students that rather than tackle a specific problem such as this one, it is preferable, in fact essential, to consider the problem in general, since the solution to the general problem can be applied to any particular case. Students will also appreciate that while such an approach

achieves the flexibility necessary to cover all cases, it comes at the cost of having to use algebra rather than arithmetic.

`The General Case:` A sum of money of €A is borrowed as a mortgage from a building society. The normal practice is to amortise this loan over an agreed number of years, m, say, by making fixed monthly or annual repayments. The problem to be investigated is:

> What should the amount €p of these repayment installments be if, for the duration of the loan, interest is compounded annually at a fixed rate r?

Here r is expressed as a decimal so that an interest rate of 8% means that $r = 0 \cdot 08$.

The case of amortising the loan over m years making fixed *annual* payments of €p is considered first. Once this is done, it requires only a little modification to treat the case of monthly repayments.

Formulation of the Recurrence Relation: If €A is borrowed on 1 January of this year, say, then the first repayment of €p must be made on 1 January of next year, with subsequent payments being made on 1 January of each successive year until the loan is paid in full after exactly m years.
To begin thinking about the problem one might simply ask:

"How much is owed to the building society
immediately after making the first repayment of €p?"

On 31 December of the year the €A was borrowed the building society is owed

$$A + rA = (1 + r)A$$

Why? Because this is how much money their €A would make if, instead of loaning it out, they had invested it in a bank at an interest rate of r – all of which seems no more than fair. So immediately after the first payment of €p is made on the morning of 1 January of next year, the bank is owed

$$(1 + r)A - p$$

For example, if €$120,000$ is borrowed at 5% and fixed annual repayments of €$9,629 \cdot 11$ are being made, then immediately after the first repayment the building society is owed

$$(1 \cdot 05)120,000 - 9,629 \cdot 11 = 126,000 - 9,629 \cdot 11 = 116,370 \cdot 89$$

Students may find this a little depressing as the amount owed has only been reduced by a paltry €$3,629 \cdot 11$ even though the building society has been given close on ten thousand euro. In fact, most are startled by the stark realism of a calculation such as this, and often wonder despairingly at this juncture if such a loan can ever be paid off. For some this is a moment of deep realisation concerning the cost of borrowing money, a circumstance

which may be regarded as educational in itself.

When one proceeds to determine how much is owed immediately after the second repayment is made, it is observed that the algebra becomes a little unwieldly. At this stage a teacher might explain slowly and carefully why the amount owed is given by

$$[(1+r)A - p] + r[(1+r)A - p] - p$$

and simplify this expression to get

$$(1+r)[(1+r)A - p] - p$$

which may be written as

$$(1+r)^2A - (1+r)p - p$$

Expressed in this form, it is clear that a pattern is beginning to emerge which becomes even more discernible if one undertakes the algebra required to work out how much is owed immediately after a third repayment has been made. However, it is preferable and more enlightening not to proceed in this direction but to introduce some labour saving measures.

One way of doing this is to employ function notation which, as will be seen, has the effect of facilitating the reasoning and of avoiding excessive algebraic manipulation. Thus another pivotal learning stage in the discussion of this problem has been reached.

Let the shorthand notation $a(1)$ be used to stand for the amount owed to the building society immediately after the first repayment of p has been made. The letter a is the initial of the word amount and $a(1)$ is the amount owing immediately after 1 repayment. Similarly $a(2)$ stands for the amount owed the same society immediately after 2 repayments of €p have been made. It has already been shown that

$$a(1) = (1 + r)A - p$$

and that

$$a(2) = (1 + r)^2 A - (1 + r)p - p$$

Now it is investigated how $a(2)$ is related to $a(1)$.

The $a(1)$ euro owed on 1 January immediately after the first repayment becomes

$$a(1) + ra(1) = (1 + r)a(1)$$

by 31 December of that same year. As soon as the second repayment of p euro is made on the following day, 1 January, the amount owed is

$$(1 + r)a(1) - p$$

Since this is the amount owed immediately after 2 repayments, it is $a(2)$. Hence, the connection between $a(1)$ and $a(2)$ is given by the equation

$$a(2) = (1 + r)a(1) - p$$

One can verify immediately that this relation gives the explicit amount already obtained for $a(2)$. Substituting $(1+r)A-p$ for $a(1)$ in the expression gives

$$a(2) = (1+r)[(1+r)A - p] - p$$

Expanding the r.h.s. gives

$$a(2) = (1+r)^2 A - (1+r)p - p$$

as before.

It is an easy generalisation of the notation introduced to write $a(n)$ for the amount owing immediately after n repayments of p euro have been made. Here the amount a owing is a function of n, the number of repayments made.

It becomes clear that the problem being studied is one of determining an explicit expression for $a(n)$. Such a formula, if it exists, would give the amount owed to the building society immediately after any specified number of repayments have been made. Furthermore, it would enable one to work out the size of the repayment p required to amortise a loan over m years, where m may be any number of years.

It is now shown how this is done by describing how the amount $a(n)$ owed immediately after n repayments is related to the amount $a(n-1)$ owed immediately after $n-1$ repayments have been made. By using exactly the same reasoning connecting $a(2)$ to $a(1)$ one can say that

$$a(n) = (1+r)a(n-1) - p$$

for any value of n greater than or equal to 2.

This equation gives a concise description of the fundamental mechanism whereby $a(n-1)$ becomes $a(n)$. It incorporates both the effects of interest accruing on the outstanding balance $a(n-1)$ as of year $n-1$, and the reduction due to the current repayment of p euro.

Since

$$a(1) = (1+r)A - p$$

setting $a(0) = A$, makes this last equation read

$$a(1) = (1+r)a(0) - p$$

which is the general equation with $n = 1$. If one interprets $a(0)$ as the amount of money owing immediately after 0 repayments have been made, then this is simply the initial capital borrowed, A. Hence setting $a(0) = A$ is consistent with the nature of the problem. Thus one may write that

$$a(n) = (1+r)a(n-1) - p$$

for $n = 1, 2, 3, \ldots$ subject to the *initial condition* $a(0) = A$.

This essential information regarding $a(n)$ for $n = 1, 2, 3, \ldots$ and the initial condition $a(0) = A$ may be summarised in the following display:

$$a(n) = (1+r)\,a(n-1) - p \quad ; \quad a(0) = A, \quad \text{for} \quad n = 1, 2, 3, \ldots$$

This is an example of what is known as a *recurrence relation* or a *recurrence* for short. The word 'relation' is used because the first equation relates how the quantity $a(n)$ can be obtained from $a(n-1)$, while the word 'recurrence' is used because by repeatedly using this same general equation, successive values of $a(n)$ can be generated starting from the value $n = 0$ and incrementing n by 1 after each use.

The strategy whereby a problem is linked to a similar but smaller instance of the same problem constitutes what is known as the *recursive* method, often simply referred to as *recursion*.

With the recurrence relation in place the formulation stage of this problem can be considered complete.

Solving the Recurrence: To begin, write R for $1+r$ so that the recurrence relation 'governing' this problem may be expressed as

$$a(n) = Ra(n-1) - p \quad ; \quad a(0) = A$$

In order to gain some understanding of what this equation is saying, assign n a 'small' value, such as 3, for example. Doing this gives

$$a(3) = Ra(2) - p$$

Now set $n = 2$ in the general equation of the recurrence to express $a(2)$ in terms of $a(1)$ to get that

$$a(2) = Ra(1) - p$$

Replacing the $a(2)$ in $a(3) = Ra(2) - p$ by $Ra(1) - p$ gives

$$a(3) = R[Ra(1) - p] - p = R^2a(1) - Rp - p$$

– an expression which relates $a(3)$ to $a(1)$. If now $Ra(0) - p$ is substituted for $a(1)$ in this last expression, one gets that

$$a(3) = R^3a(0) - R^2p - Rp - p$$

Then setting $a(0) = A$ gives

$$a(3) = R^3A - p(1 + R + R^2)$$

which is an explicit expression for $a(3)$ in terms of the quantities A, R and p.

Recognising that $1 + R + R^2$ is a geometric series of three terms with initial term 1 and common ratio R, one may write that

$$1 + R + R^2 = \frac{R^3 - 1}{R - 1}$$

since $R - 1 = r \neq 0$. Replacing R by $1 + r$ gives

$$a(3) = (1 + r)^3 A - p\left[\frac{(1 + r)^3}{r}\right]$$

as the amount owing immediately after three annual repayment of p euros have been made.

When all of the above steps are displayed in one array of calculations, it looks as follows:

$$\begin{aligned}
a(3) &= Ra(2) - p \\
&= R[Ra(1) - p] - p \\
&= R^2 a(1) - Rp - p \\
&= R^2[Ra(0) - p] - Rp - p \\
&= R^3 a(0) - R^2 p - Rp - p \\
&= R^3 A - p(1 + R + R^2) \\
&= R^3 A - p\left[\frac{R^3 - 1}{R - 1}\right] \\
\\
\Rightarrow a(3) &= (1 + r)^3 A - p\left[\frac{(1 + r)^3 - 1}{r}\right]
\end{aligned}$$

All of the above steps are displayed once again but this time with comments:

$$
\begin{aligned}
a(3) &= Ra(2) - p \\
&= R[Ra(1) - p] - p && \text{(a recursive call)} \\
&= R^2 a(1) - Rp - p && \text{(tidying)} \\
&= R^2[Ra(0) - p] - Rp - p && \text{(a recursive call)} \\
&= R^3 a(0) - R^2 p - Rp - p && \text{(tidying)} \\
&= R^3 A - p(1 + R + R^2) && \text{(because} \quad a(0) = A\text{)} \\
&= R^3 A - p\left[\frac{R^3 - 1}{R - 1}\right] && \text{(using formula)} \\
\Rightarrow a(3) &= (1 + r)^3 A - p\left[\frac{(1 + r)^3 - 1}{r}\right] && \text{(because} \quad R = 1 + r\text{)}
\end{aligned}
$$

This specific example based on $n = 3$ might lead one to conjecture that

$$a(n) = (1 + r)^n A - p\left[\frac{(1 + r)^n - 1}{r}\right]$$

is the general formula for $a(n)$.

One can verify if this is the case simply by checking whether or not this formula for $a(n)$ satisfies both the difference equation and the initial condition.

Checking the initial condition is the easier task. Setting $n = 0$ into the conjectured formula gives

$$\begin{aligned} a(0) &= (1+r)^0 A - p\left[\frac{(1+r)^0 - 1}{r}\right] \\ &= A - p\left[\frac{1-1}{r}\right] \\ \Rightarrow a(0) &= A \end{aligned}$$

which is correct.

On the other hand, the calculations

$$\begin{aligned} (1+r)a(n-1) - p &= (1+r)\left\{(1+r)^{n-1}A - p\left[\frac{(1+r)^{n-1}-1}{r}\right]\right\} - p \\ &= (1+r)^n A - p\left[\frac{(1+r)^n - (1+r)}{r}\right] - p \\ &= (1+r)^n A - p\left[\frac{(1+r)^n - 1}{r}\right] - p\left(\frac{-r}{r}\right) - p \\ &= (1+r)^n A - p\left[\frac{(1+r)^n - 1}{r}\right] \\ &= a(n) \end{aligned}$$

verify that

$$a(n) = (1+r)a(n-1) - p \quad ; \quad (n \in \mathbb{N})$$

It should be stressed for the student that in the foregoing the correct formula for $a(n)$ (valid for $n = 1, 2, 3, \ldots$) was guessed at from an examination of small cases; subsequently it is verified because the governing recurrence relation has been firmly established. As already mentioned, guessing general answers from particular cases is a much valued skill in any form of investigative work as it often proves to be the fastest route to an overall solution.

However, the formula for $a(n)$ can be obtained directly by emulating the 'recursive' argument which expressed $a(3)$ in terms of $A, 1 + r$ and p. This method of deriving a closed formula is known as 'downward recursion'.

`Direct Derivation using Recursion:` The steps by which one can obtain a general formula for $a(n)$ from the recurrence relation are displayed, accompanied by comments, in the array of calculations (shown opposite):

$$
\begin{aligned}
a(n) &= Ra(n-1) - p \\
&= R[Ra(n-2) - p] - p && \text{(a recursive call)} \\
&= R^2 a(n-2) - Rp - p && \text{(tidying)} \\
&= R^2[Ra(n-3) - p] - Rp - p && \text{(a recursive call)} \\
&= R^3 a(n-3) - R^2 p - Rp - p && \text{(tidying)} \\
&= R[Ra(n-4) - p] - R^2 p - Rp - p && \text{(a recursive call)} \\
&= R^4 a(n-4) - R^3 p - R^2 p - Rp - p && \text{(tidying)} \\
&= \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots && \text{(inductive leap)} \\
&= R^n a(0) - R^{n-1} p - \cdots - R^2 p - Rp - p && \text{(tidying)} \\
&= R^n A - p(1 + R + R^2 + \cdots + R^{n-1}) && \text{(because} \quad a(0) = A\text{)} \\
&= R^n A - p\left[\frac{R^n - 1}{R - 1}\right] && \text{(using formula)} \\
&= (1+r)^n A - p\left[\frac{(1+r)^n - 1}{r}\right] && \text{(because} \quad R = 1 + r\text{)}
\end{aligned}
$$

Hence, as conjectured and verified,

$$a(n) = (1+r)^n A - p\left[\frac{(1+r)^n - 1}{r}\right] \quad ; \quad (n \in \mathbb{N})$$

This general formula for $a(n)$ allows one to obtain a formula for the fixed repayment p almost immediately, as is now demonstrated.

`Determining the Fixed Repayment`: If the initial loan A is to be amortised with m annual repayments, then

$$a(m) = 0$$

since this represents the fact that there is nothing owed to the building society once the mth repayment has been made. This equation allows p to be determined in terms of the principal A, the fixed rate of interest r and the loan-duration m as follows:

$$\begin{aligned} & & a(m) &= 0 \\ \Rightarrow & (1+r)^m A - p\left[\frac{(1+r)^m - 1}{r}\right] &= 0 \\ \Rightarrow & p\left[\frac{(1+r)^m - 1}{r}\right] &= (1+r)^m A \\ \Rightarrow & p\left[\frac{1-(1+r)^{-m}}{r}\right] &= A \end{aligned}$$

Thus

$$\boxed{p = A\left[\frac{r}{1-(1+r)^{-m}}\right]}$$

gives the amount of the fixed repayments in terms of the principal A, the fixed rate of interest r and the duration of the loan m.

`Illustrative Example:` If one borrows a sum of €160,000 to be paid back over twenty years with interest compounded annually at an interest rate of $r = 0 \cdot 05$, then this repayment-formula says that the fixed annual repayment is given in this case by

$$p = 160000\left[\frac{0 \cdot 05}{1-(1 \cdot 05)^{-20}}\right]$$

$$\Rightarrow p = 12838 \cdot 81$$

– a considerable annual repayment on a loan which might be judged moderate in terms of today's prices.

It is revealing at this stage to multiply this annual repayment by the duration of the loan to get the total cost of the loan (exclusive of any tax breaks) as

$$(12838 \cdot 81) \times 20 = 256,776 \cdot 2$$

- an amount which often has students reeling in shock upon realising that such a loan appears to cost nearly a hundred thousand euro.

The table of repayments relating to this loan are presented shortly and often prove equally eye-opening for anyone who may never have thought about the 'cost of borrowing'. Before doing so the total cost of monthly repayments is investigated.

`Monthly Repayments:` In practice, most borrowers make monthly repayments to the lending institutions. In this case it is clear that an m-year loan translates into a $12m$-month loan. But what should the corresponding rate of interest be? Because there are twelve months in a year it is taken as $r/12$ where r is the annual rate.

Students are keen to see how this scheme of repayments differs from the one based on annual repayments. Using the same figures as above, where one borrows a sum of €160000 to be paid back over twenty years with interest compounded annually at an interest rate of $r = 0 \cdot 05$, the repayment-formula gives the fixed monthly repayment

$$p = 160000\left[\frac{0 \cdot 004167}{1 - (1 \cdot 004167)^{-240}}\right]$$

$$\Rightarrow p = 1055 \cdot 93$$

Multiplying this monthly repayment by the monthly duration of the loan gives the total cost of the loan (exclusive of any tax breaks) as

$$(1055 \cdot 93) \times 240 = 253,423 \cdot 00$$

which the student will note is not significantly cheaper than a loan amortised on an annual basis.

Using the Recurrence Relation: Once the fixed repayment is known, one can use the recurrence relation to generate the successive outstanding balances remaining on a loan. Of course, it is not necessary to do so since one possesses a formula capable of calculating $a(n)$ for each value of n. Either way, one has two choices for making out a table showing how much is owed on a mortgage at the end of each year during the term of the loan.

`Mathematical Theory and Computer Implementation:` In practice, such a table is compiled with the use of a package such as Excel implemented on a computer. It is by using such a package to numerically investigate a wide range of possibilities, which would be too time consuming to do manually, that students come to appreciate the power of technology. For example, they can examine the repayments on different sized loans, at different rates of interest, and over varying loan-durations. One result of such computer-aided explorations should be that the student comes to fully appreciate that it is the mathematical theory which empowers one to carry out all these informative calculations, ably assisted of course, by the computer.

`Table of Annual Mortgage Repayments:` It was already shown that the fixed annual repayment on an initial borrowing of€$160,000$ to be paid back over twenty years where interest is compounded annually at a rate of $r = 0{\cdot}05$ is €$12838 \cdot 81$. The following table shows both the outstanding balance and reduction of principal at the end of each year during the term of the loan:

Year	Outstanding Balance	Reduction of Principal
0	160000.00	00.00
1	155161.19	4838.81
2	150080.43	9919.57
3	144745.64	15254.36
4	139144.11	20855.89
5	133262.50	26737.50
6	127086.81	32913.19
7	120602.34	39397.66
8	113793.64	46206.36
9	106644.51	53355.49
10	99137.92	60862.08
11	91256.00	68744.00
12	82979.99	77020.01
13	74290.17	85709.83
14	65165.87	94834.13
15	55585.35	104414.65
16	45525.80	114474.20
17	34963.27	125036.72
18	23872.62	136127.38
19	12227.44	147772.56
20	00.00	160000.00

Again this table may hold a number of surprises for the student. An oft-made observation is that after ten years of repayments amounting to $€128,388 \cdot 10$, a borrower still owes close on a hundred thousand euro. Explaining why this is so can make for a valuable classroom discussion.

A Counting Problem

The problem now to be discussed concerns 'codewords' composed of characters from the *ternary* alphabet $\{0, 1, 2\}$, that is, made using the three symbols 0, 1 or 2 and no others.

For example, the codewords that can be formed using one and only one symbol are given by the three codewords

$$0 \;;\; \mathbf{1} \;;\; \mathbf{2}$$

These represent all the codewords of 'length' one. The reason some are rendered in bold typeface will be made clear later.

Using two symbols one can form $3 \times 3 = 3^2$ codewords of length two. These nine codewords of length two are as shown:

$$\begin{array}{ccc} \mathbf{00} & 01 & 02 \\ 10 & \mathbf{11} & \mathbf{12} \\ 20 & \mathbf{21} & \mathbf{22} \end{array}$$

Similarly, the number of different codewords of length three which can be constructed in this manner is

$$3 \times 3 \times 3 = 3^3$$

which is a calculation the student might like to explain using the *multiplication principle.*

The twenty-seven codewords of length three are as shown:

000	**001**	**002**
010	011	012
020	021	022
100	101	102
110	**111**	**112**
120	**121**	**122**
200	201	202
210	**211**	**212**
220	**221**	**222**

By this stage the student will have little difficulty in seeing that there are $3^4 = 81$ codewords of length 4, and $3^5 = 243$ of length 5 and that, in general, there are 3^n codewords of length n for each positive integer n.

It may also be the case that by this stage the class's curiosity is aroused to such an extent that someone will enquire as to why some of the entries in the above arrays appear in bold face while others don't. If so the teacher will have been granted the opportunity to introduce the real purpose of his deliberations to date. He can explain, or better still have the students see for themselves, that each of the highlighted entries contains an *even* number of 0s (where zero is regarded as an even number).

For example, of the three codewords of length 1, exactly two of them (namely **1** and **2**) have an even number of 0s; here none is counted as zero 0s and is regarded as even. Of the nine codewords of length 2, the five codewords

$$\mathbf{00} \quad ; \quad \mathbf{11} \quad ; \quad \mathbf{12} \quad ; \quad \mathbf{21} \quad ; \quad \mathbf{22}$$

have an even number of 0s, four having zero 0s and one of them having two 0s. The student can count for himself that of the twenty-seven codewords of length 3, exactly fourteen of them have an even number of 0s.

At this juncture some students will probably anticipate the task ahead of them which is:

> To determine the number of codewords of length n in which the character 0 appears an *even* number of times.

Previous experience should now have them introducing some function notation along the lines of letting $a(n)$ denote this number. Then they can immediately write that

$$a(1) = 2 \quad ; \quad a(2) = 5 \quad ; \quad a(3) = 14$$

By a direct if tedious enumeration of the codewords of length 4, it is found that $a(4) = 41$ so that the sequence of $a(n)$ values has the leading terms

$$1,\ 5,\ 14,\ 41,\ \ldots$$

It will not be long before some student will conjecture that 'the rule of formation' governing this sequence is that the next term is obtained by multiplying its predecessor by three and subtracting one. When this conjecture

is expressed formally it becomes the recurrence relation

$$a(n) = 3a(n-1) - 1 \quad ; \quad a(1) = 2$$

for $n \geq 2$.

In this writer's experience most students have difficulty in finding an argument which validates this conjecture.

Thus it comes as a surprise and a very salutory lesson when the teacher establishes a different recurrence relation in a relatively straightforward manner as follows:

- The total number of codewords of length $n-1$ is 3^{n-1} since there are three choices for each character.

- Of these, there are $3^{n-1} - a(n-1)$ codewords with an *odd* number of 0s since $a(n-1)$ is, by definition, the number of codewords of length $n-1$ in which the character 0 appears an *even* number of times.
  By adding a 0 to each of these codewords one obtains $3^{n-1} - a(n-1)$ codewords of length n with an even number of 0s.
  By adding a 1 or a 2 to each of the $a(n-1)$ codewords with an even number of 0s, one obtains a further $2a(n-1)$ codewords of length n.

- Since these are the only two ways in which one can have a codeword of length n with an even number of 0s, there are in all

$$[3^{n-1} - a(n-1)] + 2a(n-1)$$

codewords of length n with an even number of 0s.

- Hence

$$a(n) = a(n-1) + 3^{n-1}$$

for $n \geq 2 \in \mathbb{N}$.

- It has already been shown that of the three codewords of length one, two have an even number of 0s. Hence $a(1) = 2$.

- Thus a recurrence governing $a(n)$ is given by

$$a(n) = a(n-1) + 3^{n-1} \quad ; \quad a(1) = 2$$

Students can verify immediately that setting $n = 2, 3$ and 4 in turn, in the general equation of this recurrence gives

$$1,\ 5,\ 14,\ 41,\ \ldots$$

as above.

Those students who have genuinely struggled to justify their own recurrence relation are usually quite keen to derive the solution of the 'teacher's recurrence' and to see if the solution fits their own equation. The downward recursion argument calls on knowledge of the sum of a finite geometric series and runs as follows:

$$\begin{aligned}
a(n) &= a(n-1) + 3^{n-1} \\
&= a(n-2) + 3^{n-1} + 3^n \\
&= a(n-3) + 3^{n-2} + 3^{n-1} + 3^n \\
&= \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&= a(1) + 3^1 + 3^2 + 3^3 + \cdots + 3^{n-2} + 3^{n-1} + 3^n \\
&= 1 + [1 + 3 + 3^2 + 3^3 + \cdots + 3^{n-2} + 3^{n-1} + 3^n] \\
&= 1 + \frac{3^n - 1}{3 - 1}
\end{aligned}$$

giving

$$a(n) = \frac{3^n + 1}{2}$$

for each $n \in \mathbb{N}$.

Even though this formula generates the first four values given in the above sequence, this alone does not guarantee that this putative solution is valid for

all n. At this stage it is imperative that the student check that the formula arrived at recursively does indeed satisfy the established recurrence

$$a(n) = a(n-1) + 3^{n-1} \quad ; \quad a(1) = 2$$

`Check`: Setting $n = 1$ in the right-hand-side of the expression gives $a(1) = 2$ which is correct while

$$\begin{aligned} a(n-1) + 3^{n-1} &= \frac{3^{n-1}+1}{2} + 3^{n-1} \\ &= \frac{3^{n-1}+1+2.3^{n-1}}{2} \\ &= \frac{3.3^{n-1}+1}{2} \\ &= \frac{3^n+1}{2} \\ &= a(n) \\ \Rightarrow a(n) &= a(n-1) + 3^{n-1} \end{aligned}$$

Thus one can be sure that

$$a(n) = \frac{3^n+1}{2}$$

is the solution of the recurrence

$$a(n) = a(n-1) + 3^{n-1} \quad ; \quad a(1) = 2$$

But does it also satisfy the 'students' recurrence'

$$a(n) = 3a(n-1) + 1 \quad ; \quad a(1) = 2 \ ?$$

The initial condition has already been verified so it is only the general equation that needs to be checked. The calculations

$$\begin{aligned} 3a(n-1) - 1 &= 3\left(\frac{3^{n-1}+1}{2}\right) - 1 \\ &= \frac{3^n + 3}{2} - \frac{2}{2} \\ &= \frac{3^n + 3 - 2}{2} \\ &= \frac{3^n + 1}{2} \\ &= a(n) \end{aligned}$$

show that

$$a(n) = \frac{3^n + 1}{2}$$

also solves the recurrence

$$a(n) = 3a(n-1) - 1 \quad ; \quad a(1) = 2$$

`Exercise:` The student should use the method of downward recursion to solve the recurrence relation

$$a(n) = 3a(n-1) - 1 \quad ; \quad a(1) = 2$$

in its own right. Having seen how the teacher's recurrence is solved, he should have little trouble re-deriving the solution.

Nevertheless, it remains an outstanding challenge for him to supply an argument (connected with the context of the problem) which places this recurrence on a firm footing.

It is rather significant that once this is done, the formula for $a(n)$ can be deduced almost immediately using no more than algebra and certainly without the labour associated with backward recursion. This painless derivation begins by equating this recurrence combined with that of the teacher to give

$$3a(n-1) - 1 = a(n-1) + 3^{n-1}$$

Thus

$$2a(n-1) = 3^{n-1} + 1$$

and

$$a(n-1) = \frac{3^{n-1} + 1}{2}$$

Incrementing $n-1$ by 1 gives

$$a(n) = \frac{3^n + 1}{2}$$

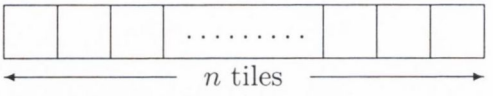
as already obtained!

Second-Order Difference Equations

Since a synopsis of the rationale and nature of the example about to be described has already been given, it is stated here without further ado.

Tiling a $1 \times n$ Hallway:

The diagram shows a $1 \times n$ hallway:



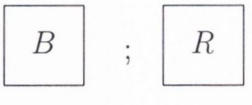
which is to be tiled using 1×1 red and blue tiles subject to the condition that a red tile is never followed by another red tile.

The task is to determine the number of different ways this can be done.

`Solution:` To begin let $a(n)$ be the number of ways that this $1 \times n$ strip can be tiled subject to the stipulated restriction and carry out an analysis of small cases beginning with the case $n = 1$. Since a 1×1 hallway:



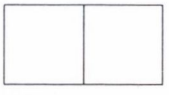
can be tiled with either a blue 1×1 tile or a red 1×1 tile as depicted:



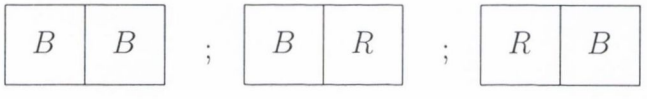
it follows that $a(1) = 1$.

Thus the student has obtained his first concrete value of $a(n)$ in the case $n = 1$ by simply enumerating the different possible tilings systematically.

The case $n = 2$ concerns the 1×2 hallway:



This can be tiled subject to the given restriction in the following three distinct ways



Hence $a(2) = 3$.

At this stage students proceed to find, by careful enumeration of the different possibilities, that

$$a(3) = 5 \quad ; \quad a(4) = 8 \quad ; \quad a(5) = 13$$

In so doing many will note that it appears that one can obtain the 'next' value of $a(n)$ by simply adding the two 'previous values', as exemplified by the calculation $13 = 8 + 5$. The teacher's immediate task then is to have his students express this observation in general terms, that is, in terms of n. Some students will have little trouble in providing the answer

$$a(n) = a(n-1) + a(n-2)$$

while others will need to be given time and further elaboration to grasp this abstract formulation in terms of the function $a(n)$.

Once this is accomplished the teacher might have the students discuss the status of this observation so that they come to realise that, as it stands, the observation constitutes no more than a conjecture or 'informed guess' as to the rule by which successive values of $a(n)$ can be calculated. The teacher can explain that *if* the rule is valid then its possession marks a huge advance and have his students explain why this is so. Some will see that by applying this relation over and over again, that is, by recursively incrementing the values of n, one can, in theory, count the number of different ways *any* $1 \times n$ hallway can be tiled subject to the prescribed restrictions.

The teacher might then mention that for the purposes of generating successive values of $a(n)$, the rule must be accompanied by the first two values of $a(n)$, namely $a(0)$ and $a(1)$, as without these two 'initial values' one cannot get started. Furthermore, in the interests of completeness the values of n for which the rule can be applied should be indicated. Mindful of all these considerations, it is best to express the conjecture made more informatively as

$$a(n) = a(n-1) + a(n-2) \quad ; \quad a(1) = 2,\ a(2) = 3$$

valid for $n = 2, 3, 4, \ldots$.

The teacher can now say that the overall expression is referred to as a *second-order* recurrence relation and, on the understanding that the phrase 'recurrence relation' is already known, explain that the qualifying phrase 'second order' is used because the current value is obtained from the *two* previous values. This clearly is an ideal juncture to mention in passing that the recurrence relations discussed prior to this are classified as *first-order* ones and to seek the students' opinions as to why this is so. The answer, that in each of these cases the current value depends only on the previous value, should be readily forthcoming. Although the phrase 'difference equation' has already been introduced in a natural way (in the discussion of the `Planar Regions` problem as an alternative for recurrence relation) the teacher may not choose to explain at this moment in time that second-order recurrence relations are also called second-order *difference equations* preferring, as is done here, to deal with this point of nomenclature later.

For the present, the less distracting and more appealing course may be to have students simply use the recurrence relation to calculate the first dozen values of $a(n)$ with an undertaking that they must later try to justify the equation. Doing this yields the leading terms of what might be termed the 'tiling sequence':

$$2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ 233,\ 377,\ \ldots$$

The teacher has now the opportunity to explain that this sequence is very similar to the 'Fibonacci' sequence

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ 233,\ 377,\ \ldots$$

which first appeared in print in 1202 in the book *Liber Abaci* published by Leonardo de Pisa (1180 - 1250). Some students like to be informed that this renowned medieval mathematician first encountered this latter sequence while studying a problem connected with rabbits and find the particular problem of Leonardo's also of some interest. By way of historical embellishment the teacher might add that Leonardo learned his mathematics in Algeria where his father, whose surname in Latin was Bonaccius, was a custom house official. Being the son of Bonaccius, or 'filius Bonacci', Leonardo was often referred to as 'Fibonacci' for short; hence the name given to the sequence.

By explaining the Indian system of numerals in his book, Leonardo became responsible for introducing the 'place system' into European mathematics, an event which proved a milestone in Western civilisation.[7]

Isomorphism of Problems: It is important that the students observe that there is no essential difference between the Fibonacci sequence and that connected with the tiling problem since, outside of two extra terms appearing at the start, successive terms of the Fibonacci sequence are formed by exactly

[7]Anglin W.S. & Lambek J., *The Heritage of Thales* (USA: Springer 1995), 122.

the same rule as the 'tiling' sequence. It is a useful exercise for the teacher to show students how this difference is captured by writing that

$$f(n) = f(n-1) + f(n-2) \quad ; \quad f(1) = 1,\ f(2) = 1$$

for $n = 2, 3, 4, \ldots$, where $f(n)$ stands for the nth Fibonacci number. Thus, the student sees that both sequences are generated by the same recurrence relation but from different initial values.

`A Related Applied Problem:` Before returning to the task of establishing the truth, or otherwise, of the recurrence relation used to obtain the tiling sequence, the teacher can connect the tiling problem with an extra mathematical one which provides an applied context for the foregoing discussion and, as such, may prove more engaging for students. The problem is this:

> Messages are transmitted over a channel using only two signals, A and B. If A takes one unit of time to transmit and B takes two units of time, and if a codeword is any sequence of As and Bs, what is the total number of possible codewords that can be transmitted in *exactly* n units of time?

If $c(n)$ denotes the total number of possible codewords that can be transmitted in *exactly* n units of time, then by carrying out a small case analysis, the student discovers by simple enumeration that

$$c(1) = 1 \quad ; \quad c(2) = 2 \quad ; \quad c(3) = 3 \quad ; \quad c(4) = 5$$

and will either suspect by this stage, or after some further investigation, that

$$c(n) = c(n-1) + c(n-2) \quad ; \quad c(1) = 2,\ c(2) = 3$$

for $n = 3, 4, \ldots$.

Thus the student sees the same recurrence relation arising in a third context but with a pair of initial conditions different from the two previous pairs associated with the tiling sequence and the Fibonacci sequence. This time recursion produces the sequence which begins

$$1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ 233,\ 377,\ \ldots$$

so that this 'codes' sequence is the Fibonacci sequence less its first term.

By way of a straightforward exercise the students might be given time out to determine the number of different codewords that can be transmitted in an interval of exactly thirty units of time. This question can be interpreted in two different ways, each of which can be answered by appropriately extending the number of elements in the above sequence.

A second 'applied' context which might be associated with this problem is discussed in the article of O'Loughlin's mentioned in the relevant discussion of this Topic appearing in Part II. In that author's words:[8]

> It deals with finding a way to measure the number of strings (collections) of n binary digits that do not contain the pattern 11.

[8]M. O'Loughlin, "Difference Equations", *IMTA Newsletter* 86, (1995): 15-16.

Using $s(n)$ where the author used s_n to denote the number of n-bit strings that do not contain the pattern 11, one can show, as O'Loughlin does, that

$$s(n) = s(n-1) + s(n-2) \quad ; \quad s(1) = 2,\ s(2) = 3$$

for $n = 3, 4, \ldots$.

Note that this is exactly the same second-order recurrence as the previous one relating to codewords.

The examples of the last few paragraphs reveal the isomorphism that can exist between apparently unrelated problems, in that each can be governed by the same underlying recurrence relation, albeit with different initial conditions.

The next order of business is to establish the validity of this recurrence relation in the context of the original motivating problem concerned with tiling. The manner in which this is done acts as a guide on how to demonstrate the truth of the recurrence in each of the other contexts given above.

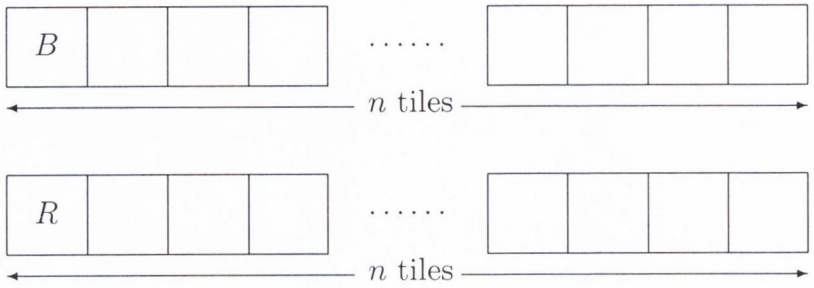
Proving the Conjecture: How might one establish the truth, or otherwise, of the conjecture that

$$a(n) = a(n-1) + a(n-2) \quad ; \quad a(1) = 2,\ a(2) = 3$$

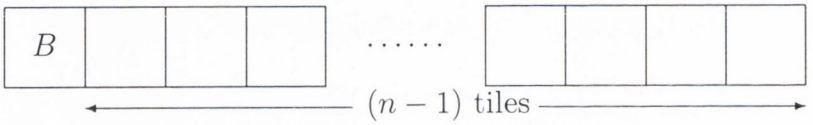
for $n = 2, 3, 4, \ldots$ for the tiling problem?

The answer lies in finding how to translate the restricting stipulation into mathematical expressions involving the counting function $a(\cdot)$.

A key observation is to note that the tiling in a $1 \times n$ hallway must begin with either a blue 1×1 tile or a red 1×1 tile.

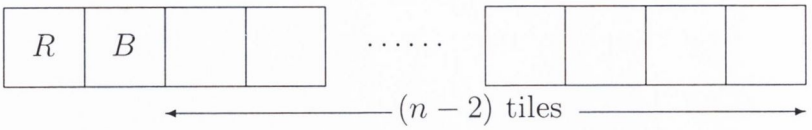


If the $1 \times n$ hallway begins with a blue tile as shown:



then the second tile in such a hallway can be either a blue or red tile. Consequently the number of $1 \times n$ hallways that start with a blue tile is $a(n-1)$ since this function counts the number of different ways a $1 \times n-1$ hallway can be tiled in the manner prescribed.

On the other hand, if the tiling of a $1 \times n$ hallway begins with a red tile then the second tile in such a hallway *must* be a blue tile since a red tile cannot follow a red tile. Thus one obtains the following diagram:



Since the remaining tiles constitute a $1 \times (n-2)$ hallway, the $1 \times n$ hallway can be completed in $a(n-2)$ ways.

Thus it has been shown that the total number of $1 \times n$ hallways beginning with a blue tile is $a(n-1)$ while the total number of $1 \times n$ hallways beginning with a red tile is $a(n-2)$. Since, between them, these constitute the total number of ways of tiling a $1 \times n$ hallway, it follows that

$$a(n) = a(n-1) + a(n-2)$$

for $n \geq 3$. Since the initial conditions are self-evidently true one may now say that the recurrence relation is true.

It should be noted that the proof just presented requires quite a deal of written explanation to reveal the mathematical ideas involved. Hence asking for its description provides an excellent means of testing the Communications Standard.

`Exercise`: By way of gaining a deeper understanding of this proof the student should prove the recurrence

$$c(n) = c(n-1) + c(n-2)$$

in the context of codewords containing As and Bs.

The following list describes two other contexts governed by the 'Fibonacci' recurrence relation. On the understanding that these problems would be discussed in class, they could be included as part of a bank of questions aimed at testing this famous recurrence in different guises over a number of years.

1. If $h(n)$ is the total number of distinct sequences of n tosses of a coin in which no two consecutive heads appear, explain why

$$h(n) = h(n-1) + h(n-2) \quad ; \quad h(1) = 2,\ h(2) = 3$$

Note that since $h(n)$ satisfies the same difference equation and the same initial conditions as those satisfied by $a(n)$ in the tiling problem, the student should be able to establish a direct isomorphism between the two problems. This is easily done by identifying a red tile with a head and a blue tile with a tail.

2. An elf has a stairs with n steps to climb. If he can cover either one step or two steps at a time, and if $s(n)$ is the number of distinct ways that he can ascend the stairs, then explain why

$$s(n) = s(n-1) + s(n-2) \quad ; \quad s(1) = 2,\ s(2) = 3$$

Since $s(n)$ satisfies the same difference equation and the same initial conditions as those satisfied by $a(n)$ of the tiling problem, it is an exercise for the student to establish a direct isomorphism between the two problems.

Nomenclature: The syllabus segment on Difference Equations offers no explanation as to why it so titled. Likewise, the texts define a difference equation in the manner indicated by this syllabus entry but make no reference as to why it is so called. While an effort has been made in this treatment to suggest why the general equation in one particular first-order recurrence relation (relating to the `Planar Regions` problem) could also be regarded as a first-order difference equation, this short section gives an historical explanation as to why the phrase 'difference equations' (with specified initial conditions) is used in reference to what have been consistently termed recurrence relations in this exposition up to now. However, this writer's preference would be to use the latter term because of the visible manner by which such equations are typically derived connects with recursive thinking.

In an early treatise dealing with sequences, the English mathematician George Boole introduced the operators E and Δ defined, respectively, by the relations

$$E[f(n)] = f(n+1) \quad \text{and} \quad \Delta[f(n)] = f(n+1) - f(n)$$

where $f(n)$ is regarded as the typical element of the sequence $\{f(n)\}_{n=0}^{\infty}$. Since the operator E has the effect of shifting along the sequence by one element it is termed the 'shift operator'. On the other hand, since $f(n+1) - f(n)$ is the difference in value between the typical term $f(n)$ in the sequence and its successor $f(n+1)$, the Δ operator is known as the 'difference operator'.

Furthermore, Boole, by introducing the identity operator $I[f(n)] = f(n)$,

and boldly applying the ordinary rules of algebra to these operators wrote that

$$\begin{aligned} \Delta[f(n)] &= f(n+1) - f(n) \\ &= E[f(n)] - f(n) \\ &= (E-1)[f(n)] \end{aligned}$$

He then said that

$$\Delta \equiv E - I \quad \text{or} \quad E \equiv \Delta + I$$

thereby giving operator-equations connecting the three operators E, I and Δ. With this notation the general equation

$$r(n) - r(n-1) = n$$

in the recurrence relation governing the `Planar Regions` problem can be written as

$$\Delta[r(n-1)] = n$$

Since this equation involves the 'difference' operator Δ it is known a 'difference' equation.

Similarly the equation

$$a(n) = 2a(n-1) + 1,$$

which is the general equation in the recurrence relation governing the `Tower of Hanoi` problem, can be written as

$$E[a(n-1)] = 2I[a(n-1)] + 1$$

This is equivalent to

$$(E - 2I)[a(n-1)] = 1$$

or

$$(\Delta - I)[a(n-1)] = 1$$

As this is an equation involving the difference operator to the first power, it is called a 'first-order' difference equation.

The operators E^2 and Δ^2 are defined 'by extension' in the following manner:

$$E^2[f(n)] = E\Big[E[f(n)]\Big] = E[f(n+1)] = f(n+2)$$

and

$$\begin{aligned}
\Delta^2[f(n)] &= \Delta\Big[\Delta[f(n)]\Big] \\
&= \Delta[f(n+1) - f(n)] \\
&= \Delta[f(n+1)] - \Delta[f(n)] \\
&= [f(n+2) - f(n+1)] - [f(n+1) - f(n)] \\
\Rightarrow \Delta^2[f(n)] &= f(n+2) - 2f(n+1) + f(n) \\
&= E^2[f(n)] - 2E[f(n)] + I[f(n)] \\
&= (E^2 - 2E + I)[f(n)]
\end{aligned}$$

Since the operators Δ^2 and the linear combination $E^2 - 2E + I$ has the same effect on a typical $f(n)$ one obtains the operator equivalence

$$\Delta^2 \equiv E^2 - 2E + I$$

It should be noted that this result can be obtained directly from the operator identity $\Delta \equiv E - I$. By treating this as an algebraic equation which can be squared in the ordinary way, one obtains operator equivalence just displayed. Similarly

$$\begin{aligned} E &\equiv \Delta + I \\ \Rightarrow E^2 &\equiv (\Delta + I)^2 \\ \Rightarrow E^2 &\equiv \Delta^2 + 2\Delta + I \end{aligned}$$

Armed with these second-order operators the equation

$$a(n) = a(n-1) + a(n-2)$$

can be written as

$$(E^2 - E - I)[a(n-2)] = 0$$

Substituting $\Delta + I$ for E and $\Delta^2 + 2\Delta + I$ for E^2, this equation becomes

$$(\Delta^2 + \Delta - I)[a(n-1)] = 0$$

Because this equation contains the difference operator Δ to the power of 2, it is often called a second-order difference equation.

Since the general equation in any second-order recurrence relation can be transformed, in the manner indicated, into an equation of this type, these equations are also known as second-order difference equations. With this detour on why a second-order recurrence relation is also referred to as a second-order difference equation with specified initial conditions completed, it is time to return to a discussion of why it is necessary to be able to solve these recurrences and the way in which this is done.

The Need for a General Solution: Having justified the recurrence relation (and made whatever mention he deems necessary to explain the term difference equation), the teacher should reactivate the discussion on how a recurrence relation is used to generate successive values of $a(n)$ recursively. This has already been done in the case of first-order recurrences where the limitations of recursively generating specific values were highlighted as a task that may be hugely labour intensive even with the assistance of a computer. However, in the case of first-order recurrences it was shown that the method of backward or downward recursion often leads to a closed form expression and thus solves the problem. When this procedure is emulated on the general equation of a second-order recurrence, such as the Fibonacci recurrence for example, it soon becomes clear that the technique of backward recursion leads nowhere and so is of no avail at the level of second-order (or indeed higher-order) difference equations.

In this way the student is made to realise to an even greater extent the

need for a theory by which it is possible to solve equations of this 'harder' type. One systematic procedure for solving equations of this order is the classical method (due to the Swiss mathematician Euler) which is simply presented as a given in the DES syllabus segment. Whereas one might like to see some form of motivation for the method, if it be the case that time is at a premium, then it is better to opt for the source of equations being revealed rather than their method of solution. In what follows the classical method of solution is taken as a given.

Returning to the Fibonacci recurrence

$$f(n) = f(n-1) + f(n-2) \quad ; \quad f(1) = 1,\ f(2) = 1$$

and applying the classical method of solution to the general equation leads immediately to the quadratic equation

$$x^2 - x - 1 = 0$$

It must be remarked that since this quadratic does not factorise over the rationals, any recurrence with this characteristic equation would not appear as a question on the current HLC examination where the practice is invariably to choose test questions containing recurrences whose characteristic equations split over the rationals. This despite the almost ubiquitous presence of this recurrence in all manner of different contexts and despite the fact that one can use the quadratic formula to determine the two distinct real roots of its quadratic to obtain its solution basis.

In fact, as discussed in this work in the Topic on Quadratics, the positive root of the quadratic is the golden ratio

$$\phi = \frac{1+\sqrt{5}}{2}$$

while the negative one, is the 'irrational-conjugate' of ϕ, given by

$$\bar{\phi} = \frac{1-\sqrt{5}}{2}$$

When the initial conditions are taken into account one obtains, for each $n \in \mathbb{N}$, that

$$\begin{aligned} f(n) &= \frac{1}{\sqrt{5}}\left(\phi^n - \bar{\phi}^n\right) \\ &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1+\sqrt{5}}{2}\right)^n\right] \end{aligned}$$

This is the celebrated *de Moivre-Binet* formula yielding the typical term of the Fibonacci sequence.

Students, and others, find it quite remarkable that this formula, specified as it is in terms of irrational quantities, yields an integer value for every natural number n. However, it should be stated that this expression for $f(n)$ is not ideally suited for computation and is often rendered in a form more amenable to this purpose.

`Other Solutions`: With this general formula for $f(n)$ in place it is a simple matter also to give the solutions of some of the other problems related to the Fibonacci recurrence in closed form by expressing the quantities in question in terms of $f(n)$.

For example, in the tiling problem it was noted that the tiling sequence is the Fibonacci sequence less its first two terms. Thus with a little reflection the student should see how to express the $a(n)$ of the tiling problem in terms of $f(n)$. Specifically, noting that

$$\begin{aligned} a(1) = 2 &= f(3) \\ a(2) = 3 &= f(4) \\ a(3) = 5 &= f(5) \\ a(4) = 8 &= f(6) \end{aligned}$$

the student should be confident enough to state that

$$a(n) = f(n+2)$$

for each $n \in \mathbb{N}$.

Thus he may with equal confidence remove all reference to $f(n)$ and assert that

$$a(n) = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{n+2} - \left(\frac{1+\sqrt{5}}{2} \right)^{n+2} \right]$$

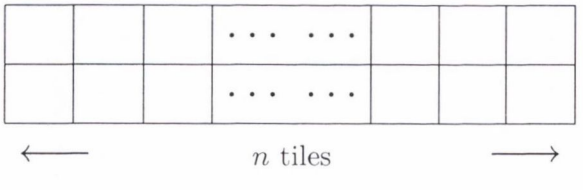
for each $n \in \mathbb{N}$.

`Exercise:` The student should express the solutions of the other Fibonacci-like problems in closed form either by linking them with the Fibonacci recurrence or solving them in their own right using the classical method.

The final problem presented here differs from the one just treated in that it involves a second-order recurrence relation whose associated characteristic equation is easily factorised over the rationals. As such it would accord with the intentions of the current syllabus and might be made the basis of a future examination question on the understanding that the specific example would be explicitly mentioned in the syllabus segment.

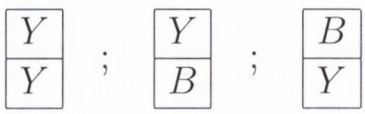
The $2 \times n$ Hallway:

It is desired to tile the $2 \times n$ hallway shown in the diagram using either yellow or brown 1×1 tiles. However, it is never permitted to have two brown tiles adjacent, either horizontally, vertically or diagonally.



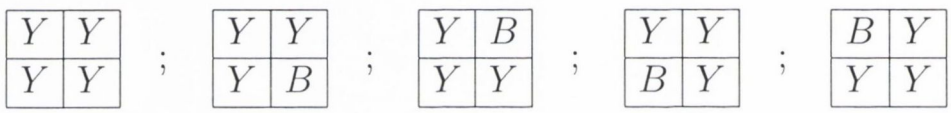
Determine the number of different ways this can be done.

`Conjecturing the Recurrence`: One might begin with the introduction of some notation. As in the previous problem, let $t(n)$ stand for the number of ways the $2 \times n$ hallway can be tiled subject to the given restriction. Then undertake an examination of small cases.
For $n = 1$ the diagrams:



show immediately that $t(1) = 3$.

For $n = 2$ the diagrams:



give $t(2) = 5$.

A careful enumeration of the case $n = 3$ leads to $t(3) = 11$ while, if undertaken, one finds that $t(4) = 21$. Thus the leading terms of the sequence, associated with the determination of a general expression for $t(n)$, are

$$3,\ 5,\ 11,\ 21,\ \ldots$$

It is often the case that by this stage some students discern the rule of formation which is that, apart from the two initial terms, each subsequent

term is obtained by adding the previous value to twice the one before it. Thus the conjectured recurrence relation is

$$t(n) = t(n-1) + 2t(n-2) \quad ; \quad t(1) = 3,\ t(2) = 5$$

valid for $n \geq 3$.

Since the typical value, $t(n)$, depends on both $t(n-1)$ and $t(n-2)$ the student has been shown another example of a 'second-order recurrence relation' or a 'second-order difference equation with two specified initial conditions' arising out of a specific context.

`Using the Recurrence`: Before setting about the task of justifying the recurrence, students should simply practice using it to generate the terms already found in the above sequence and then extend it by appending some more entries. Explicitly, beginning with the initial conditions $t(1) = 3$, $t(2) = 5$ and setting $n = 3$ gives

$$\begin{aligned} t(3) &= t(2) + 2t(1) \\ &= 5 + 2(3) \\ &= 11 \\ \Rightarrow t(4) &= t(3) + 2t(2) \\ &= 11 + 2(5) \\ &= 21 \end{aligned}$$

$$\begin{aligned}
\Rightarrow t(5) &= t(4) + 2t(3) \\
&= 21 + 2(11) \\
&= 43 \\
\Rightarrow t(6) &= t(5) + 2t(4) \\
&= 43 + 2(21) \\
&= 85 \\
\Rightarrow t(7) &= t(6) + 2t(5) \\
&= 85 + 2(43) \\
&= 171 \\
\Rightarrow t(8) &= t(7) + 2t(6) \\
&= 171 + 2(85) \\
&= 341
\end{aligned}$$

Thus one finds the first eight leading terms of the sequence to be

$$3,\ 5,\ 11,\ 21,\ 43,\ 85,\ 171,\ 341,\ \ldots$$

The last entry says that a 2×8 hallway can be tiled in 341 different ways subject to the given restriction.

A useful exercise for the student is to express this number as a fraction of the total number of ways this 2×8 hallway could be tiled with brown and yellow tiles if there were no restrictions.

`Justifying the Recurrence`: In general, this is the 'thinking man's part' of the whole undertaking. While it is often the most difficult, it can at the same time be the most rewarding for those who find that understanding is the ultimate satisfaction.

One way of having a student see why the conjectured recurrence relation holds is to have him think of himself as the tiler and enquire what are the consequences of how he starts out. For example, if as the tiler he decides to lay a brown tile first, then the first 2×2 portion of the hallway is constrained to look like one or other of the following two diagrams:

B	Y
Y	Y

;

Y	Y
B	Y

At this stage many students realise that the $2 \times (n-2)$ portion of the hallway remaining to be tiled constitutes a smaller example of the original problem because one is free to choose the next tile to be either a brown or a yellow tile. Thus the number of ways of completing the tiling, beginning with either of these configurations is

$$t(n-2) + t(n-2) = 2t(n-2)$$

where it is assumed that $n \geq 3$.

If a brown tile is not chosen as one of the first two tiles then the $2 \times n$ hallway begins with the configuration:



This time the $2 \times (n-1)$ portion of the hallway remaining to be tiled again constitutes a smaller example of the original problem because one is free to choose the next tile to be either a brown or a yellow tile. Thus the number of ways of completing the tiling, beginning with this configuration is $t(n-1)$, where it is assumed that $n \geq 2$.

Since these three configurations constitute the only ways the tiler can start out, it follows that

$$t(n) = t(n-1) + 2t(n-2)$$

for $n \geq 3$.

As this is the general equation of the conjectured recurrence, it may be considered as having been established at this juncture.

The manner in which this recurrence is established reveals very clearly the fundamental notion of recursion which has already been described as the technique whereby the solution to a problem depends on finding the solutions to smaller instances of the same problem.

`Solving the Recurrence:` Knowing that the recurrence is valid leads to the task of solving it. At the moment the chief focus of the current HLC examination is on this aspect alone where the method of solution is taken as a given and is simply applied. The procedure is to first write down the characteristic equation associated with the recurrence which, in this case, is the quadratic equation,

$$x^2 - x - 2 = 0$$

It will be observed that this quadratic factorises over the integer, a requirement considered necessary at the present time for a question to be deemed suitable for the HLC examination. The factorisation $(x-2)(x+1)$ reveals the roots of the quadratic to be 2 and -1 respectively. Hence the solution of the general equation in the recurrence is given by

$$t(n) = a\,2^n + b\,(-1)^n$$

for arbitrary constant a and b and $n \in \mathbb{N}$.

It is now that the initial conditions are invoked. In this case they yield the 2×2 system of simultaneous equations:

$$\begin{aligned} 2a - b &= 3 \\ 4a + b &= 5 \end{aligned}$$

Adding gives $6a = 8$ so that $a = 4/3$. Then $b = 5 - 4a$ gives $b = -1/3$, which is consistent with the second equation. Thus, the unique solution of

the system has been found. Hence

$$\begin{aligned} t(n) &= a\,2^n + b\,(-1)^n \\ &= \frac{4}{3}\,2^n + \frac{-1}{3}\,(-1)^n \\ \Rightarrow t(n) &= \frac{2^{n+2} + (-1)^{n+1}}{3} \end{aligned}$$

Before checking that this general formula satisfies the recurrence in all respects, students should simply use it to recover the leading terms of the sequence obtained previously. For example, the calculation for the sixth term runs as follows:

$$\begin{aligned} t(6) &= \frac{2^8 + (-1)^7}{3} \\ &= \frac{256 - 1}{3} \\ &= \frac{255}{3} \\ &= 85 \end{aligned}$$

in agreement with sixth entry of the result obtained previously using forward recursion.

`Checking the Solution:` If

$$t(n) = \frac{2^{n+2} + (-1)^{n+1}}{3}$$

then

$$t(n-1) = \frac{2^{n+1} + (-1)^{n}}{3} \quad \text{and} \quad t(n-2) = \frac{2^{n} + (-1)^{n-1}}{3}$$

Hence

$$\begin{aligned} 3[t(n-1) + 2t(n-2)] &= 2^{n+1} + (-1)^n + 2[2^n + (-1)^{n-1}] \\ &= 2^n[2+2] + (-1)^{n-1}[-1+2] \\ &= 4.2^n + (-1)^{n-1} \\ &= 2^{n+2} + (-1)^{n+1} \quad \text{because} \quad (-1)^{n-1} = (-1)^{n+1} \\ \Rightarrow t(n) &= \frac{2^{n+2} + (-1)^{n+1}}{3} \end{aligned}$$

Furthermore,

$$t(1) = \frac{2^{1+2} + (-1)^{1+1}}{3} = \frac{8+1}{3} = 3$$

with

$$t(2) = \frac{2^{2+2} + (-1)^{1+2}}{3} = \frac{16-1}{3} = 5$$

verifying the initial conditions, and the overall recurrence.

`A First-Order Recurrence:` From an examination of the first few values of the sequence

$$3,\ 5,\ 11,\ 21,\ 43,\ 85,\ 171,\ 341,\ \ldots$$

some students observe a different rule of formation than that already described and justified on physical grounds. They note that each term is twice the previous term plus or minus one depending on its position. Thus, for example, they note that

$$\begin{aligned} 5 &= 2(3) - 1 \\ 11 &= 2(5) + 1 \\ 21 &= 2(11) - 1 \\ 43 &= 2(21) + 1 \\ 85 &= 2(43) - 1 \\ 171 &= 2(85) - 1 \\ 341 &= 2(171) - 1 \end{aligned}$$

Expressing this observation mathematically affords the teacher an ideal opportunity to demonstrate the efficacy of the $(-1)^n$ notation, if he has not done so already at some other point in his instruction. He can explain that this conjecture may be expressed as the first-order recurrence

$$t(n) = t(n-1) + (-1)^{n+1} \quad ; \quad t(1) = 3$$

valid for $n \geq 2$.

The student should check that this recurrence does in fact generate the first half-dozen or so terms in the recursive sequence.

The fact that one has obtained a first-order recurrence relation for the problem means that, if its general validity could be substantiated, one could find the closed form for $t(n)$ by applying the method of downwards recursion to the recurrence. In fact, whether or not the students can come up with an explanation for this first-order recurrence, they should use this method to confirm the result obtained using the classical method.

By way of summary one can see that this example is rich in many respects. Initially it affords the student plenty of room for careful exploration leading to the recognition of a certain pattern which results in a conjectured recurrence. This is followed by using the recurrence and then the search for its proof. In addition, the problem has the virtue of being easily solvable by the classical method to yield an amenable general formula which is easily checked.

But, perhaps best of all, it has an open-ended element to it, which is that students invariably part company with the problem unable to justify the first-order recurrence they have observed. This 'unfinished business' feature gives them a genuine glimpse of the nature of the mathematical endeavour, which is that in solving one problem, another invariably raises its head for which a solution may not be readily forthcoming.

Mathematical Vignettes

A mathematical vignette has already been likened to "a short poetic aphorism which strives to give the essence of an argument in the most condensed and memorable form".[9] At least four such have been described in this work to date; the first is the ingenious derivation of the square-root algorithm attributed to Hero which is discussed in the core Topic on Quadratics; the second, from that same section, is the manner by which the first value of n making $\sigma_1(n) \geq m$ is expressed in terms of m using the floor function; the third is the clever combinatorial argument used to count the number of rectangles present on the chessboard discussed in the chapter on Recreational Mathemtaics while the fourth, also from that chapter, is Gauss's method of adding up the first one hundred natural numbers; a mathematical argument which Hilton claims achieves 'a transcendental beauty and elegance'.[10] Trigg says, "An elegant solution is generally considered to be one characterised by clarity, conciseness, logic and surprise."[11]

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[9]See 40.

[10]Hilton P., "The Joy of Mathematics: A Mary P. Dolciani Lecture" *The College Mathematics Journal* 23, no. 4 (September 1992): 276.

[11]Trigg Charles W., *Mathematical Quickies* (UK: Dover 1985)

Concerning the concept of elegance, the great French mathematician Poincaré writes:

> What is it indeed that gives us the feeling of elegance in a solution, in demonstration? It is the harmony of the diverse parts, their symmetry, their happy balance; in a word it is all that introduces order, all that gives unity, that permits us to see clearly and to comprehend at once both the ensemble and the details.

This short section presents some more of these 'mathematical gems' and indicates a number of others which might form a suite of solved-problems which reveal to the student the ingenuity and elegance of mathematics. As can be seen they are drawn from different topic areas. Strictly speaking, the first three examples might be regarded as being no more than exercises in technical virtuosity but the remaining offerings are genuine specimens.

1. `Using an Equation:` If

$$x + \frac{1}{x} = a$$

for $a \geq 2$, determine *without* computing x the value of

$$x^2 + \frac{1}{x^2}$$

`Solution:` Here the simple but clever idea is to 'square' the equation to get

$$x^2 + 2 + \frac{1}{x^2} = a^2$$

$$\Rightarrow x^2 + \frac{1}{x^2} = a^2 - 2$$

2. `Cubed-Roots of Unity:` If

$$z^2 + z + 1 = 0$$

show *without* computing z that $z \neq 1$ but that $z^3 = 1$.

Deduce, from geometrical considerations, the possible values of z.

`Solution:` It is easy to show that $z \neq 1$ since 1 is not a solution of the equation $z^2 + z + 1 = 0$.

However, to show that $z^3 = 1$ without solving for z requires the simple but clever idea of 'multiplying the equation through by z'. Doing this gives

$$\begin{aligned} z^3 + z^2 + z &= 0 \\ \Rightarrow z^3 &= -z^2 - z \\ \Rightarrow z^3 &= 1 \end{aligned}$$

since $z^2 + z + 1 = 0 \Rightarrow 1 = -z^2 - z$. Thus $z^3 = 1$ but $z \neq 1$.

Therefore the value of z is either the complex cubed-root of unity lying on the unit circle (centred at the origin of the complex plane) at an angle of 120° to the positive direction of the real-axis, or it is the complex-conjugate of this number (which lies on the unit circle at an angle of 240° to the positive direction of the real-axis).

This example shows how to use an equation to find information about its solutions without solving for them explicitly.

3. `An Arithmetic Progression:` Find five numbers in arithmetic progression which add to 10 while their squares add to 60.

`Solution:` The ingenuity here is in denoting the five numbers as

$$a - 2d, a - d, a, a + d, a + 2d$$

since then their sum is $5a$; a quantity which is indepndent of d and from which it immediately follows that $a = 2$.

Squaring each of the five terms and adding gives

$$\begin{aligned} 5a^2 + 10d^2 &= 60 \\ \Rightarrow d^2 &= 4 \\ \Rightarrow d = \pm 2 & \end{aligned}$$

Each value of d yields the five numbers (arranged in increasing order) as

$$-2,\ 0,\ 2,\ 4,\ 6$$

4. `Rational or Irrational:` Can an expression of the form

$$(\text{irrational})^{\text{irrational}} \qquad (\dagger)$$

ever be rational?

`Solution:` The answer is "Yes" as the following ingenious argument shows. It relies on knowing that the number $\sqrt{2}$ is irrational. Showing this is a vignette in itself which is treated in a separate section of this work.

The number $\sqrt{2}$ is irrational so the expression $\sqrt{2}^{\sqrt{2}}$ is of the form (†). If this expression is rational then we are done; if it is not then

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$$

is of the form (†).

But this expression simplifies to the rational number 2.

Thus, one way or the other, an expression of the form

$$(\text{irrational})^{\text{irrational}}$$

can be rational.

5. `Apery's Constant:` Show that

$$\sum_{n=1}^{\infty} \frac{1}{n^3} < \frac{29}{24}$$

`Solution:` The ingenuity displayed here is in finding and exploiting an inequality. Using no more than partial fractions the sum of an infinite series is obtained elegantly by means of telescoping. Specifically, since

$$n^3 > n(n-1)(n-2)$$

for $n \geq 3 \in \mathbb{N}$, it follows that

$$\frac{1}{n^3} < \frac{1}{(n-2)(n-1)n} = \frac{1}{2}\left(\frac{1}{n-2} - \frac{2}{n-1} + \frac{1}{n}\right)$$

Thus

$$\begin{aligned} \sum_{n=3}^{\infty} \frac{1}{n^3} &< \frac{1}{2}\left[\sum_{n=3}^{\infty}\left(\frac{1}{n-2} - \frac{1}{n-1}\right) - \sum_{n=3}^{\infty}\left(\frac{1}{n-1} - \frac{1}{n}\right)\right] \\ &= \frac{1}{2}\left[\left(1 - \frac{1}{2}\right) - \left(\frac{1}{2} - \frac{1}{3}\right)\right] = \frac{1}{6} \end{aligned}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^3} < 1 + \frac{1}{8} + \frac{1}{6} = \frac{29}{24} = 1 \cdot 2083333....$$

`Remark:` It is known that

$$\zeta(3) := \sum_{n=1}^{\infty} \frac{1}{n^3} = 1 \cdot 2020569031595942854 0....$$

– a number shown to *irrational* by the french mathematician Apery.

6. `Two for the Price of One:` Evaluate the integrals

$$\int e^{ax} \cos bx \, dx \quad ; \quad \int e^{ax} \sin bx \, dx$$

where a and b are non-zero real numbers.

`Solution:` Consider the sum

$$I = \int e^{ax} \cos bx \, dx + j \int e^{ax} \sin bx \, dx$$

Euler's celebrated formula

$$e^{j\theta} = \cos\theta + j \sin\theta$$

gives

$$\begin{aligned}
I &= \int e^{ax} \cos bx \, dx + j \int e^{ax} \sin bx \, dx \\
&= \int e^{ax} [\cos bx + j \sin bx] \, dx \\
&= \int e^{ax} [e^{jbx}] \, dx \\
&= \int e^{(a+jb)x} \, dx \\
&= \frac{e^{(a+jb)x}}{a+jb} \\
&= \frac{e^{(a+jb)x}}{a+jb} \left[\frac{a-jb}{a-jb}\right] \\
&= \frac{(a-jb)e^{ax}e^{jbx}}{a^2+b^2}
\end{aligned}$$

$$= \frac{e^{ax}}{a^2+b^2}(a-jb)[\cos bx + j\sin bx]$$

$$= \frac{e^{ax}}{a^2+b^2}[(a\cos bx + b\sin bx) + j(a\sin bx - b\cos bx]$$

where for ease of presentation constants of integration have been ignored. Comparing the real and imaginary parts of each side of the final equation gives

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\cos(bx) + b\sin(bx)] + C$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin(bx) - b\cos(bx)] + C$$

on appending a constant of integration C in each case.

`Remark:` The elegance to be found in this solution rests on Euler's formula. Combining the linearity of integration with knowledge of the antiderivative of e^{kx} and using complex numbers as mere scaffolding, it yields two significant real anti-derivatives almost effortlessly. To obtain either of these results in the realm of real numbers requires a *double* application of the integration by parts rule.

7. `Divergence of the Harmonic Series:` The following 'reductio ad absurdum' proof that the harmonic series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$$

diverges is due to the American mathematician Leonard Gillman who wrote it down at 3.00 a.m while on a train journey on Wednesday, 29 November, 1961.

`Solution:` Assume the series converges and let its sum be S. Then

$$\begin{aligned} S &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\cdots \\ &> \left(\frac{1}{2}+\frac{1}{2}\right)+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{6}+\frac{1}{6}\right)+\left(\frac{1}{8}+\frac{1}{8}\right)+\cdots \\ &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots \\ &= S \end{aligned}$$

$\Rightarrow S > S$ a contradiction!

This elegant solution should be compared with the standard one [due to the French bishop Nicolas Oreseme (1320-1382)] given in the textbooks.

`Remark:` Having given this proof the teacher might linger a while to reveal the 'slow divergence' of the harmonic series simply for interest's sake. If, for each $n \in \mathbb{N}$,

$$H(n) := 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}$$

then the following table gives the first value of n for which $H(n) > m$ for the indicated integer values of m:

m	n	m	n
1	1	5	83
2	4	6	227
3	11	7	616
4	?	8	?

The result

$$\sum_{k=1}^{10^6} \frac{1}{k} = 14 \cdot 39....$$

indicates that the sum of the first million terms of the harmonic series is below 15 while

$$\sum_{k=1}^{10^{100}} \frac{1}{k} = 231 \cdot ....$$

shows that even summing a googol of the leading terms of the harmonic series gets one just beyond 231.

Link with the log function:

$$H(n) = \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{\epsilon(n)}{120n^4} \quad ; \quad 0 < \epsilon(n) < 1$$

where $\gamma = 0 \cdot 5772156644...$ is Euler's constant.

Experiment with this formula for $n = 5$.

A Socratic Dialogue

Prologue

You may think of the dialogue you are about to read, as I often did while writing it, as being between a "master" and a "pupil" – the master in his middle years, well-versed in mathematics and as devoted and passionate about his craft as any artist is about his art; the pupil on the threshold of adulthood, articulate in speech, adventuresome of mind and enthusiastically receptive to any knowledge the more learned teacher may care to impart.

Their conversation - the exact circumstances of which are never described - is initiated by the master, one of whose tasks is to persuade his disciple that the concept of number is more subtle than might first be imagined. Their mathematical journey starts with the teacher guiding the student, by way of questions and answers, through a beautifully simple geometrical demonstration (believed to have originated in ancient India), which establishes the existence of a certain number, the understanding of whose nature is destined to form a major part of the subsequent discussion between the enquiring duo.

Strong as the master's motivation is to have the young person glimpse a

little of the wonder of mathematics, stronger still is his desire to see that his protégé gradually becomes more and more adept at mathematical reasoning so that he may experience the pure pleasure to be had from simply "finding things out" for himself. This joy of discovery is soon felt by the young learner, who having embarked upon an exploration, is richly rewarded when, after some effort, he chances upon a sequence of numbers that he surmises is inextricably linked to the mysterious number lately revealed by the master. Enthralled by this fortunate occurrence, he immediately finds himself in the grip of a burning curiosity to know more about this number and its connection with the sequence that has already captivated him. Thus begins this tale told over five chapters.

I made every effort to have the first four chapters as self-contained as possible. The use of mathematical notation is avoided whenever words can achieve the same purpose, albeit in a more lengthy manner. When mathematical notation is used, nothing beyond high school algebra of the simplest kind is called on, but in ways that show clearly the need for this branch of mathematics. While the algebra used is simple, it is often clever, revealing that a few tools handled with skill can achieve a great deal. If readers were to appreciate nothing more than this aspect of algebra – its power to prove things in general – then this work will not have been in vain.

Unfortunately, to have the fifth chapter completely self-contained would have meant sacrificing exciting material, something I didn't wish to do, preferring to reward the reader for the effort taken to reach this point, when it

is hoped he will understand enough to appreciate the substance of what is being related.

Throughout the dialogue, so as to distinguish between the two speakers, the following typographical conventions are used:
The master's voice – assured, but gently persuasive – is fixed firmly at the left edge of the column. The pupil's voice – deferential, but eager and enquiring – is moved slightly inward from the margin by means of the typographical indentation —.

The best conversations between teacher and students are both serious and playful, and my hope is that the readers of this book will sense that something of that spirit – real learning coupled with real pleasure – coexist in this dialogue.

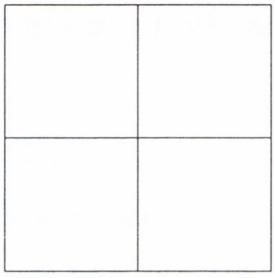
Asking the Right Questions

I'd like you to draw a square made from four unit squares.

— A unit square is one where each of the sides is one unit long?

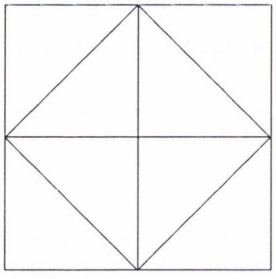
Yes.

— Well, that shouldn't be too hard.



— Will this do?

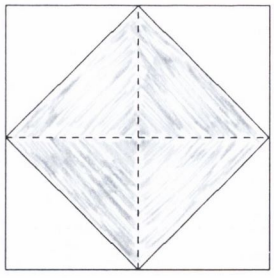
Perfect. Now let me add the following diagonals to your drawing.



You see that by doing this a new square is formed.

— I do. One that uses a diagonal of each of the unit squares for its four sides.

Let's shade this square and call it the "internal" square.



Now, I want you to tell me the area of this internal square.

— Let me think. The internal square contains exactly half of each unit square and so must have half the area of the large square. So it has an area of 2 square units.

Exactly. Now, what is the length of any one of those diagonals that forms a side of the internal square?

— Off-hand I don't think I can say. I know that to get the area of a rectangular region you multiply its length by its breadth.

"Length by breadth", as you say, meaning multiply the length of one side by the length of a side at right angles to it.

— So, for a square, this means that you multiply the length of one side by itself, since length and breadth are equal.

Yes.

— But where does this get me? As I said, I don't know the length of the side.

As you say. But if we let s stand for the length of one of the sides, then what could you say about s?

— I suppose there is no way that we could have this little chat without bring-

ing letters into it?

There is, but at the cost of the discussion being more longwinded than it need be. Incidentally, why did I chose the letter s?

— Because it is the initial of the word side?

Precisely. It is very common to use the initial of the word describing the quantity which you're looking for.

— So s stands for the length of the side of the internal square. I hope you are not going make me do algebra.

Just a very small amount – for the moment. So can you tell me something about the number s?

— When you multiply s by itself you get 2.

Exactly, because the area of the internal square is 2 (squared units). Do you recall that $s \times s$ is often written as s^2?

— I do. My algebra isn't *that* rusty.

So you are saying that the number s "satisfies the equation":

$$s^2 = 2$$

In words, "s squared equals two".

— Okay, so the number s when multiplied by itself gives 2. Doesn't this mean that s is called the square root of 2?

Well, it would be more accurate to say that s is *a* square root of 2. A number is said to be a square root of another if, when multiplied by itself, it gives the other number.

— So 3 is a square root of 9 because $3 \times 3 = 9$.

As is -3 because $-3 \times -3 = 9$ also.

— But most people would say that the square root of 9 is 3.

True. It is customary to call the positive square root of a number its square root. And since s is the length of the side of a square, it is obviously a positive quantity, so we may say ...

— ... that s is the square root of 2.

Sometimes, we simply say "root two", it being understood that it's a square root that is involved.

— And not some other root like a cube root?

Yes. Now the fact that 3 is the square root of 9 is often expressed mathematically by writing $\sqrt{9} = 3$.

— I've always liked this symbol for the square root.

It was first used by a certain Christoff Rudolff in 1525, in the book *Die Coss*, but I won't go into the reasons why he chose it.

[See chapter note 1]

— Can we say goodbye to s and write $\sqrt{2}$ in its place from now on?

If we want to, but we'll still use s if it serves our purposes.

— So we have shown that the diagonal of a unit square is $\sqrt{2}$ in length.

Indeed we have. This wonderful way of establishing the existence of the square root of 2 originated in India thousands of years ago.

[See chapter note 2]

– You'd have to say that it is quite simple.

Which makes it all the more impressive.

— So what number is $\sqrt{2}$?

As the equation $s^2 = 2$ says, it is that number that, when multiplied by itself, gives 2 exactly. This means no more or no less than what the equation

$$\sqrt{2} \times \sqrt{2} = 2$$

says it means: $\sqrt{2}$ is that number that when multiplied by itself gives 2.

— I know, but what number does $\sqrt{2}$ actually stand for? I mean $\sqrt{16} = 4$ and 4 is what I would call a tangible number.

I understand. You have given me a concrete value for $\sqrt{16}$, namely the number 4. You want me to do the same for $\sqrt{2}$, that is, to show you some number of a type with which you are familiar, and that when squared, gives 2.

— Exactly. I'm simply asking what the concrete value of s is, that makes $s^2 = 2$.

I can convince you quite easily that $\sqrt{2}$ is not a natural number.

— The natural numbers are the ordinary counting numbers, $1, 2, 3$, and so on.

Precisely.

– Even though 2 itself is a natural number? The natural numbers 9 and 16 have square roots that are also natural numbers.

That's true, they do.

— But you are saying that 2 doesn't.

I am. One way of seeing this is to write the first few natural numbers in order of increasing magnitude in a line, and beneath them on a second line

write their corresponding squares:

1	2	3	4	5	6	7	...
1	4	9	16	25	36	49	...

The three dots, or ellipsis, at the end of a line means that the pattern continues without stopping.

— Well, I can see straight away that the number 2 is missing from the second row.

As are

$$3,\quad 5,\quad 6,\quad 7,\quad 8,\quad 10,\quad 11,\quad 12,\quad 13,\quad 14,\quad 15,\quad 17\ldots$$

— I would say that there are a lot more numbers missing than are present.

Yes, in a sense "most" of the natural numbers are absent from this second line. The numbers $1, 4, 9, 16, \ldots$ that appear on it are known as the *perfect squares*.

— And those numbers that are missing from this line are not perfect squares.

Correct: 49 is a perfect square but 48 is not.

— I think I see now why there is no natural number squaring to 2. The first natural number squares to 1 while the second natural number squared is 4, so 2 gets skipped over.

That's about it.

— Alright, it is fairly obvious, now at any rate, that there is no natural number that squares to 2, but surely there is some fraction whose square is 2?

By fraction, you mean a common fraction where one whole number is divided

by another whole number?

— That's what I mean, $\frac{7}{5}$, for example. Are there other types of fractions?

There are, but when we say "fraction" we mean one whole number divided by another one. The number being divided is the numerator and the one doing the dividing is called the denominator.

— The number on the top is the numerator and the number on the bottom is the denominator.

That's it exactly. In your example, the whole number 7 is the numerator while the whole number 5 is the denominator.

— Now mustn't there be some fraction close to this one that squares to give 2 exactly?

Why did you say close to this one?

— Because my calculator tells me that 7/5 is $1 \cdot 4$ in decimal form; and when I multiply this by itself I get $1 \cdot 96$, which is fairly close to 2.

Agreed. Let me show you how we can see this for ourselves without a calculator but using a little ingenuity instead. Since

$$\left(\frac{7}{5}\right)^2 = \frac{49}{25}$$

$$\stackrel{!}{=} \frac{50-1}{25}$$

Desert Island Math

$$= \frac{50}{25} - \frac{1}{25}$$

$$= \quad 2 - \frac{1}{25}$$

we can say that the fraction 7/5 when squared underestimates 2 by the amount 1/25.

— And according to my calculator $1/25 = 0 \cdot 04$, which is just $2 - 1 \cdot 96$. By the way, why did you put the exclamation point over the second = sign?

To indicate that the step being taken is quite a clever one.

— It certainly wouldn't have occurred to me, which I know is not saying much.

Well, I don't lay any claim to originality for taking this step. I have seen many similar such tricks used by others in the past and, after all, I knew what it was I wanted to show.

— At least I can see why it's clever.

Good. Why?

— By writing the numerator 49 as $50 - 1$, you were able to divide the 50 by 25 to get 2 exactly and the 1 by 25 to get 1/25 as the measure of the underestimate.

A useful trick if you're stranded on a desert island without any calculating devices other than your own poor head.

— Pure do-it-yourself mathematics! I suppose using a calculator to get the value of something you wouldn't be able to calculate for yourself is a form of cheating?

Do you mean like asking for the decimal expansion of $\sqrt{2}$, for example?

— Well, something like that. I wouldn't have a clue how to get the decimal expansion of $\sqrt{2}$ using my own very limited powers.

```
      1 · 4
  5 | 7 · 00
      5
      20
      20
      00
```

I'm sure you do your mental abilities an injustice. If we know and understand how to get a decimal expansion of a number "by hand", then we don't contravene the DIY philosophy if we use a calculator to save labour.

— Are you saying that because I know how to get the decimal expansion of 7/5 or 3/11 by long division, even though I wouldn't like to be pressed on why the procedure works, I may use a calculator to avoid the "donkey work" involved with such a task?

I think we'll let this be a policy. We'll assume that if we were put to it we could explain to ourselves and others the "ins and outs" of the long-division algorithm.

algorithm: step by step procedure

— Of course, completely!

Decimal expansions, or "decimals" as we often say for short, have certain advantages, one being that they convey the magnitude of a number more readily than their equivalent fractions do. When a number is expressed in decimal form, it is easy to say geometrically where it is located on the number line:

```
      0 · 272 · · ·
 11 | 3 · 000
      22
      80
      77
      30
      22
      80
      ⋮
```



No matter how long the decimal expansion of a number may be, we still know between which two whole numbers it lies on this number line.

— So we can see quite easily from $1 \cdot 4$ that it is a number between 1 and 2,

whereas it is not as easy to see this from the fraction 7/5?

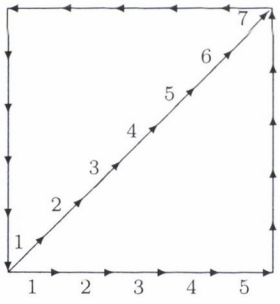
The fraction 7/5 is perhaps too simple. It is not too difficult to mentally determine the two whole numbers between which it is located on the number line, but who can say without resorting to a calculation where the fraction 103993/33102 is positioned on the same line.

— I see the point, or should I say I do not see the (decimal) point!

Hmm! Speaking of the fraction 7/5, you might like to get a box of matches and construct a square with five matches on each side.

— Does this mean that the five matches between them make up the unit-length?

You can certainly think of it this way, if you like. Now you'll find that seven matches will fit along the diagonal:



These seven matches do not stretch the full length of the diagonal since 7/5 underestimates $\sqrt{2}$.

— That they don't is barely visible.

True, but the gap is there.

— This is a rather neat way of visualising 7/5 as an approximation to $\sqrt{2}$.

Yes it is, Isn't it? Looked at another way it says that the ratio $7 : 5$ is close to the ratio $\sqrt{2} : 1$. Now, where were we?

— Looking for a fraction that squares to 2.

Indeed, so Let's continue the quest. Any further thoughts?

— There must be some fraction a little bit bigger than $7/5$ that squares to give 2 exactly.

Well, there are an awful lot of fractions just a little bit bigger than $7/5$.

— I know. Isn't there an infinity of fractions between $1 \cdot 4$ and $1 \cdot 5$ alone?

Yes, but that this is so we can leave for another time. Why do you mention $1 \cdot 5$?

— Simply because $(1 \cdot 4)^2 = 1 \cdot 96$ is less than 2 while $(1 \cdot 5)^2 = 2 \cdot 25$ is greater than 2.

So?

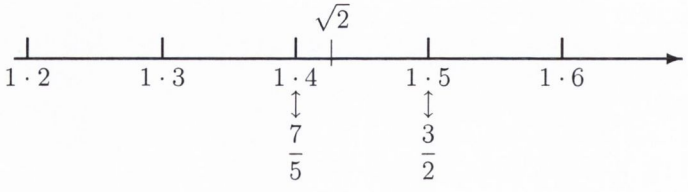
— Doesn't this mean that the square root of 2 lies between these two values?

It does. In fact since $1.5 = 3/2$ we may write that

The symbol $<$ means less than.

$$\frac{7}{5} < \sqrt{2} < \frac{3}{2}$$

Let me display this arithmetic "inequality" on the number line:



Notice that I have placed $\sqrt{2}$ to the right of $1 \cdot 4$ and closer to $1 \cdot 4$ than to $1 \cdot 5$ because $3/2$ squared overestimates 2 by $1/4$, which is much more than

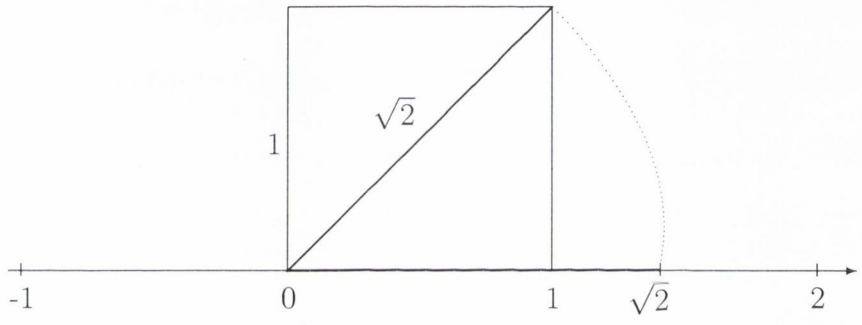
the 1/25 by which 7/5 squared underestimates $\sqrt{2}$.

— But how do you locate $\sqrt{2}$ on the number line if you don't know what fraction it is?

A good question. The answer is that you do so geometrically.

— I'd like to see how.

It's easy to construct a unit square geometrically on the interval that stretches between 0 and 1:



Now imagine the diagonal with one end at 0 and of length $\sqrt{2}$ being rotated clockwise about the point 0 until its other end lies on the number line.

— At a point $\sqrt{2}$ from 0. Very smart.

Of course, this is an ideal construction where everything can be done to perfection.

— I understand. It is the method that counts.

Yes.

An Exploration

— But to return to the point I was making: surely among the infinity of fractions lying between $1 \cdot 4$ and $1 \cdot 5$ there is one that squares to give 2 exactly.

Well if there is, how do you propose finding it?

— That's what's bothering me.

I'm sure you'll agree that it's not wise to begin checking fraction after fraction in this infinity of fractions without having some kind of plan.

— Absolutely, it could take forever. What would you suggest?

Thinking about the problem a little to see if we can find some systematic way of attacking it.

— Sounds as if we are about to go into battle.

A mental battle. Let us begin our campaign by examining the implications of expressing the number $\sqrt{2}$ as a fraction.

— This could get interesting. What are you going to call this fraction?

Well, since we don't know it, at least not yet, we must keep our options open. One way of doing this is to use distinct letters, one to stand for its numerator and the other for its denominator.

— Here comes some more algebra.

Only a little, used as scaffolding as it were, just to get us started.

— Well, I'll stop you if I think I'm losing the drift of the discussion.

Let's call the numerator of the fraction m and the denominator n.

— So if the fraction were 7/5, which I know it is not, then m would be 7 and

n would equal 5.

Or put slightly differently, if $m = 7$ and $n = 5$ then

$$\frac{m}{n} = \frac{7}{5}$$

— I'm with you.

Now if

$$\sqrt{2} = \frac{m}{n}$$

then

$$\sqrt{2} \times \sqrt{2} = \frac{m}{n} \times \frac{m}{n}$$

Agreed?

— I think so. You are simply squaring both sides of the original equation.

I am, and I do so in this elaborate manner to highlight the presence of $\sqrt{2} \times \sqrt{2}$.

— Which by definition is 2.

Yes, a simple but vital use of the defining property of $\sqrt{2}$, which allows us to write that

$$2 = \left(\frac{m}{n}\right)^2$$

We can turn this equation around and write

$$\left(\frac{m}{n}\right)^2 = 2$$

to put the emphasis on the fraction m/n. What is the equation saying about m/n?

— That its square is 2.

Exactly. And since

$$\left(\frac{m}{n}\right)^2 = \frac{m^2}{n^2}$$

we can say that

$$\frac{m^2}{n^2} = 2$$

or that

$$m^2 = 2n^2$$

— So this equation is a consequence of writing $\sqrt{2}$ as m/n.

It is indeed. Now let us see what we can learn from it.

— I'll leave this to you.

I'm sure it won't be long before you join in. For one thing, $m^2 = 2n^2$ tells us is that if we are to find a fraction that is equal to $\sqrt{2}$, then we must find two perfect squares, one of which is twice the other.

— What are perfect squares again? Oh, I remember, $1, 4, 9, 16, \ldots$.

That's right, a perfect number is one that is the square of a natural number.

— Well, this is a task that I can definitely undertake.

Be my guest.

— Why don't I make out a list of the first twenty squares along with their doubles and see if I can find a match between some square and the double of some other square.

An excellent plan. Nothing like a bit of"number crunching", as it's called, to really get one thinking.

— Of course, I'm going to use a calculator just to speed things up.

Naturally. Nobody doubts that you can multiply one number by itself.

— Here's the table I get:

Natural Number	Number Squared	Twice Number Squared
1	1	2
2	4	8
3	9	18
4	16	32
5	25	50
6	36	72
7	49	98
8	64	128
9	81	162
10	100	200
11	121	242
12	144	288
13	169	338
14	196	392
15	225	450
16	256	512
17	289	578
18	324	648
19	361	722
20	400	800

— The three columns show, in turn, the first twenty natural numbers, their squares, and twice these squares.

Great. We can think of the second column as corresponding to m^2 numbers and the third column as corresponding to numbers of the form $2n^2$.

— I'm not sure I understand what you are saying here.

I'll explain by example. We may think of the number 196 in the second column as being an m^2 number, where $m = 14$, while we may consider the number 450 in the third column as being a $2n^2$ number where $n = 15$.

— Let me test myself to see if I have got the idea. I can think of 16 in the second column as an m^2 number with $m = 4$, while I can think of the 648 in the third column as corresponding to $2n^2$, with $n = 18$, because $2(18)^2 = 648$. Do I pass?

With honours. Now if you can find an entry in the second column that matches an entry in the third column, you will have found values for m and an n which make $m^2 = 2n^2$ and so you'll have a fraction m/n equal to $\sqrt{2}$.

— As easy as that? So fingers crossed as I look at each entry of the second column of this table and then look upwards from its location along the third column for a possible match.

Of course! A time-saving observation. As you say, you need only look upwards because the corresponding entries in the third column are bigger than those in the second.

— Unfortunately, I can't find a single entry in the second column that is equal to any entry in the third column.

So the second and third columns have no element in common.

— Not that I can see. I'm going to experiment a little more by calculating the next ten perfect squares along with their doubles.

Good for you.

— This time I get:

Natural Number	Number Squared	Twice Number Squared
21	441	882
22	484	968
23	529	1058
24	576	1152
25	625	1250
26	676	1352
27	729	1458
28	784	1568
29	841	1682
30	900	1800

— I realise that this is not much of an extension to the previous table.

Maybe, but perhaps you'll get a match this time.

— I'm scanning the second column to see if any entry matches anything in the previous third column or the new third column.

Any luck?

— I'm afraid not. However, I notice that there are some near misses in the first table.

What do you mean by "near misses"? A discovery?

— There are entries in the second column that are either just one less or one more than an entry in the third column.

I'm more than curious; please elaborate.

— Well, take the number 9 in the second column. It is 1 more than the 8 in the third column.

True. Any others?

— There is a 49 in the second column that is 1 less than the 50 appearing

in the third column.

Again, true. Any more?

— Yes. There is a 289 in the second column and a 288 in the third column.

Again, as you observed, with a difference of 1 between them. Did you find any more examples?

— Not that I can see in these two tables, except, of course, at the very beginning. There is a 1 in the second column and a 2 in the third column.

Indeed there is.

— But I don't know what to make of these near misses.

However, you seem to have hit upon something interesting, exciting even, so let's take a little time out to mull over your observations.

— Fine by me, but you'll have to do the thinking.

Why don't we look at the case of the 9 in the second column and the 8 in the third column. What is the m number corresponding to this 9 in the second column, and what is the n number corresponding to the 8 in the third column?

— Let me think. I would say that $m = 3$ and that $n = 2$.

And you'd be right. Your observation tells us that

$$m^2 = 2n^2 + 1$$

where $m = 3$ and $n = 2$.

— Because $3^2 = 2(2)^2 + 1$?

Exactly. Now let us move on to the case of the number 49 in the second

column and the 50 in the third column.

— Here the $m = 7$ and $n = 5$ since $2(5)^2 = 2(25) = 50$.

This time

$$m^2 = 2n^2 - 1$$

where $m = 7$ and $n = 5$.

— Could I try the next case?

By all means.

— The number 289 corresponds to $m = 17$ since in this case $m^2 = 289$. On the other hand, the number 288 correspond to $n = 12$ since $2(12)^2 = 2(144) = 288$.

No argument there.

— This time

$$m^2 = 2n^2 + 1$$

where $m = 17$ and $n = 12$.

So we're back to 1 over.

— There seems to be an alternating pattern with these pairs of near misses.

There does indeed. For the sake of completeness, you should look at the first case.

— You mean the case with 1 in the second column and 2 in the third column?

None other; the smallest case, so to speak.

— Okay. Here $m = 1$ and $n = 1$.

And what is the value of $m^2 - 2n^2$ on this occasion?

— This time

$$m^2 = 2n^2 - 1$$

Does this fit the alternating pattern?

— It does.

Which is great.

— But returning to the original reason for constructing the tables, I haven't found a single square among the first thirty perfect squares that is equal to twice another square.

True, and that means that, as of yet, you haven't found a fraction m/n that squares to 2. But, on the other hand, you have found a number of very interesting fractions.

— I have? I would have thought that I've only found pairs of natural numbers that are within 1 of each other.

In a sense, you could say that. But you actually have discovered fractions with the property that the square of their numerator is within 1 of double the square of their denominator.

— I'm afraid you'll have to elaborate.

Of course. You remember we said, when you observed that 9 in the second column was 1 greater than the 8 in the third column, that the 9 corresponded to m^2 where $m = 3$, while the 8 corresponded to $2n^2$ where $n = 2$?

— I do.

Furthermore, $m^2 - 2n^2 = 1$, in this case.

— That's correct.

Suppose now that we form the fraction

$$\frac{m}{n} = \frac{3}{2}$$

Then can't we say that the equation $m^2 - 2n^2 = 1$ tells us that this fraction is such that the square of its numerator is 1 more than twice the square of its denominator?

— It seems to. I'll have to think a little more about this. Yes: $3^2 = 9$ and $2(2)^2 = 8$.

Try another one. Ask yourself, "What fraction is associated with the observation that the 49 in the second column is 1 less than the 50 in the third column?"

— Okay. Here $m = 7$ and $n = 5$, so the fraction is $7/5$, right?

Absolutely. Now what can you say about the numerator and denominator of this fraction?

— That the square of the numerator is 1 less than twice the square of its denominator.

Exactly.

— I think I understand now. You are saying that every time we observe the near-miss phenomenon we actually find a special fraction.

Yes. You looked for a fraction whose numerator squared would match twice its denominator squared; you didn't find one, but instead you found fractions whose numerators squared are within 1 of twice their denominators squared.

— That's a nice way of looking at it.

Often when you look for something specific you chance upon something else.

— So I suppose you could say that I found the next best thing.

I think we can say this, and not a bad reward for your labours.

— Actually, I'm really curious to know if there are any more than just these four misses and to see if the plus or minus pattern continues to hold.

Let's hope so. Why don't we do a little more exploring?

— I'd be happy to but should we not stick to our original mission of finding a difference of exactly 0?

Very nicely put. Finding an m and n such that $m^2 = 2n^2$ means that the difference $m^2 - 2n^2$ would be 0.

— Thanks.

However, I think we'll indulge ourselves and investigate your observation about near misses a little further, particularly as it looks so promising.

— Okay. I'll extend my tables and then go searching.

You could do that, but it might be an idea to look more carefully at what you have already found.

— Like good scientists would.

As you say. Begin by cataloguing the specimens found to date and examine them carefully for any clues.

— Will do.

Time to Reflect

— Beginning with the smallest, and listing them in increasing order, the fractions are

$$\frac{1}{1}, \quad \frac{3}{2}, \quad \frac{7}{5}, \quad \frac{17}{12}$$

Not many as of yet, but tantalising.

— What secrets do they hold, if any?

Indeed. Can you spot some connection between them?

— Just like one of those sequence puzzles, "What is the next number in the sequence?", except here it looks harder because there are fractions and not ordinary numbers.

A puzzle certainly, but one we have encountered quite naturally.

— And not just made up for the sake of it.

Yes, something like that.

— I hope this is an easy puzzle.

It is always best to be optimistic so I advise that you say to yourself, "This is sure to be easy", and look for simple connections.

— Optimism it is then, but where to start?

It is often a good idea to begin by examining a pair of terms some way out along a sequence rather than at the very beginning of it.

— Right. On that advice I'll see if I can spot a connection between

$$\frac{7}{5} \quad \text{and} \quad \frac{17}{12}$$

and if I think I have found one, I'll check it on the earlier fractions.

Very sensible. Happy hunting!

— I think I'll begin by focusing on the denominator 12 of the fraction 17/12.

Following a very definite line of enquiry, as they say.

— I think I have spotted something already.

Which is?

— That $12 = 7 + 5$, the next denominator looks as if it might be the sum of the numerator and denominator of the previous fraction.

If it's true, it will be a big breakthrough. I must say that was pretty quick.

— I must check the earlier terms of the sequence

$$\frac{1}{1}, \quad \frac{3}{2}, \quad \frac{7}{5}, \quad \frac{17}{12}$$

to see if this rule holds also for their denominators.

Fingers crossed, then.

— I obviously cannot check the first fraction, 1/1.

Why not?

— Because there is no fraction before it.

A good point.

— But the second fraction, 3/2, has denominator 2, which is just $1 + 1$, the sum of the numerator 1 and denominator 1 of the first fraction 1/1. This is getting exciting.

That's great. How about the third fraction 7/5?

— Right, Mr. 7/5, let's see if you fit the theory. Your denominator is 5, is it not? Indeed it is, and the sum of the numerator and denominator of the previous fraction, 3/2, is $3 + 2$, which I'm happy to say is none other than 5.

This is fantastic! Who would have thought?

Great again! Now is there an equally simple rule for the numerators?

— I hope so, because discovering that rule for the denominators gave me a great thrill.

We couldn't ask for more than that.

— Right, back to the drawing board. So is there a connection between the numerator 17 of the fraction 17/12 and the numbers 7 and 5 from the previous fraction 7/5?

It would be marvellous if there were.

— If I'm not mistaken, there is. It's simply that $17 = 7 + (2 \times 5)$.

Well spotted, though not quite as simple as the rule for the denominators.

— No, but still easy enough.

Once you see it. How do you interpret this rule?

— Doesn't it say that the next numerator is obtained by adding the numerator of the previous fraction to twice the denominator of the previous fraction?

Indisputable. You had better check this rule on the other fractions.

— It works for the fraction 3/2 since $3 = 1 + 2 \times 1$, and it also works for 7/5 since $7 = 3 + 2 \times 2$.

This is wonderful. So how would you summarise the overall rule, which allows one to go from one fraction to the next?

— Well, the general rule obtained by combining the denominator rule and the numerator rule seems to be:

> To get the denominator of the next fraction, add
> the numerator and denominator of the previous fraction;
> to get the numerator of the next fraction, add
>
> the numerator of the previous fraction to twice its denominator.

Well done! And a fairly straightforward rule, at that.

— Isn't it amazing?

Indeed it is. After all, there was no reason to believe that there had to be any rule whatever connecting the fractions, but to find that there is one and that it's simple is remarkable.

— I must now apply this general rule to 17/12 to see what fraction comes out and to see if it has the property that its top squared minus twice the bottom squared is either 1 or -1.

Let's hope that the property holds.

— According to the rule, the next fraction has a denominator of $17+12 = 29$ and a numerator of $17 + 2 \times 12 = 41$, and so is 41/29.

Good. And now what are we hoping for?

— Based on the pattern displayed by the previous four fractions, that $(41)^2 - 2(29)^2$ will work out to be -1.

Perform the acid test.

— Here goes:

$$41^2 - 2(29)^2 = 1681 - 2(841) = 1681 - 1682 = -1$$

This is fantastic !

So now you have found another example of a perfect square that is within 1 of twice another perfect square – the whole point of this investigative detour – *without* having to go to the bother of extending your original tables.

— That's true. Our more thorough examination of the four cases we found seems to have paid off.

A little thought can save a lot of computing.

— I know that I couldn't have spotted this example with my tables because they give only the first thirty perfect squares along with their doubles; but can we be sure that there is not an m value between 17 and 41 that gives a square that is within 1 of twice another perfect square?

An excellent question. At the moment we can't be sure without checking. However, if there is such an m value, then it corresponds to a fraction m/n that doesn't fit in with the above rule. Of course, this doesn't exclude the possibility of there being such a value. However, if you check, you won't find any such value.

— I must calculate the next fraction generated by the rule to see if it also satisfies the plus or minus 1 property, to give it a name. Applying the rule to $41/29$ gives $41 + 29 = 77$ as the next denominator and $47 + 2 \times 29 = 99$ as the numerator.

So 99/70 is the next fraction to be tested.

— I predict that $m^2 - 2n^2 = 1$ in this case. The calculation

$$99^2 - 2(70)^2 = 9801 - 2(4900) = 9801 - 9800 = 1$$

verifies this. Great!

Bravo! What now?

— Obviously, we can apply the rule over and over again and so generate an infinite sequence beginning with

$$\frac{1}{1}, \quad \frac{3}{2}, \quad \frac{7}{5}, \quad \frac{17}{12}, \quad \frac{41}{29}, \quad \frac{99}{70}, \ldots$$

True, you can generate an infinite number of fractions using the rule but...

— How can we be sure that all the fractions of this sequence have the property that $m^2 - 2n^2$ is either plus or minus 1 without checking each, which is clearly out of the question.

Yes, this is a bit of a problem. It might be that answering such a question may prove difficult or even impossible.

— And can we say that these are the only fractions having this plus or minus 1 property?

My, my, what truly mathematical questions! You need have no fear that you and mathematics are strangers if you can think up questions like this.

— I don't know about that. Normally, I know I wouldn't even dream of asking questions such as these, but at the moment my mind seems to be totally engrossed by these particular fractions.

Ah, well, I've read somewhere that you really only see a person's true intelligence when his or her affections are fully engaged.

— Perhaps tomorrow I won't care, but right now I really want to know if all the fractions generated by the rule actually obey the plus or minus 1 property; and I also want to know if these are the only fractions that do so.

Good for you. In mathematics it often seems that asking questions is the easy part, whereas it is the answering of them that is hard. But asking the right questions is a very important part of any investigation, whether it be mathematical or otherwise.

— The good detectives always ask the right questions.

Well, by the end at any rate.

— But can you tell me if my questions have answers; and if they do, what are their answers?

They do, but I am not going to tell you what they are. I don't want to spoil the fun you'll have in trying to answer them for yourself later.

— Later could be an eternity away if it is left up to me on my own.

That remains to be seen. Anyway, you have opened up a rich vein for further exploration with your observation that there are squares whose doubles are within plus or minus 1 of other squares, and with your recent rule, both of which we'll come back to soon.

— So, are we going to return to our original investigation?

Not just yet.

Squeezing $\sqrt{2}$

Before leaving the fractions

$$\frac{1}{1}, \quad \frac{3}{2}, \quad \frac{7}{5}, \quad \frac{17}{12}, \quad \frac{41}{29}, \quad \frac{99}{70}, \ldots$$

let me show you how they connect with the number $\sqrt{2}$ itself.

— Although none of them is $\sqrt{2}$?

Correct. But each of them can be thought of as an *approximation* to $\sqrt{2}$. In fact, each successive fraction provides a better approximation to $\sqrt{2}$ than its predecessor.

— I hope you don't mind my saying so, but I would be much more interested in finding the exact fraction instead of approximations, however good they might be.

I appreciate that you are impatient to get on with your searching, but follow me for just a little longer so that I can show you how simply but cleverly we can use these fractions to close in on the location of $\sqrt{2}$ on the number line.

— Alright. Maybe I'll learn something that will help with my search.

Perhaps we should look for clues anywhere we can. Now we know that

$$\begin{aligned} 1^2 &= 2(1)^2 - 1 \\ 3^2 &= 2(2)^2 + 1 \\ 7^2 &= 2(5)^2 - 1 \\ 17^2 &= 2(12)^2 + 1 \\ 41^2 &= 2(29)^2 - 1 \end{aligned}$$

$$99^2 = 2(70)^2 + 1$$

— Yes.

These equations are either of the form

$$m^2 = 2n^2 - 1$$

or

$$m^2 = 2n^2 + 1$$

alternating between one and the other.

— Agreed. And I would bet that this jumping between -1 and 1 continues forever, though I have no idea how to prove it.

Now let us divide each of the equations

$$\begin{aligned} 1^2 &= 2(1)^2 - 1 \\ 3^2 &= 2(2)^2 + 1 \\ 7^2 &= 2(5)^2 - 1 \\ 17^2 &= 2(12)^2 + 1 \\ 41^2 &= 2(29)^2 - 1 \\ 99^2 &= 2(70)^2 + 1 \end{aligned}$$

by their corresponding n^2 values to get

$$\left(\frac{1}{1}\right)^2 = 2 - \frac{1}{1^2}$$

$$\begin{aligned}
\left(\frac{3}{2}\right)^2 &= 2+\frac{1}{2^2} \\
\left(\frac{7}{5}\right)^2 &= 2-\frac{1}{5^2} \\
\left(\frac{17}{12}\right)^2 &= 2+\frac{1}{12^2} \\
\left(\frac{41}{29}\right)^2 &= 2-\frac{1}{29^2} \\
\left(\frac{99}{70}\right)^2 &= 2+\frac{1}{70^2}
\end{aligned}$$

Are you with me?

— Just about. I'm still mentally dividing across the second equation by 2^2, putting it beneath the 3^2 and placing the combination 3/2 under one umbrella with the power of 2 outside.

Takes practice, but it's all legal.

— I'll accept this, since you did it, but I'm a little rusty when it comes to fractions and powers, so I can be slow. Anyway, I'm happy with this last set of equations now.

This simple but clever idea gives us equations that are very informative. They tell, in turn, how close the square of each fraction is to the number 2. Can you see why?

— I'll have to take time on this. What is the equation

$$\left(\frac{17}{12}\right)^2 = 2+\frac{1}{12^2}$$

saying? That when we square 17/12 we get 2 plus the fraction 1/144?

Yes. And?

— Since 1/144 is small, the fraction 17/12 Isn't a bad approximation of $\sqrt{2}$.

Not bad at all.

— I think I see now why the approximations are getting better and better. As we move down through the set of equations, the fractions on the very right-hand side get smaller and smaller.

True. So?

— So the corresponding fractions squared on the right-hand side are getting closer and closer to 2, which I take it means that the fractions themselves are better and better approximations of $\sqrt{2}$.

Excellent. We can say more.

— We can?

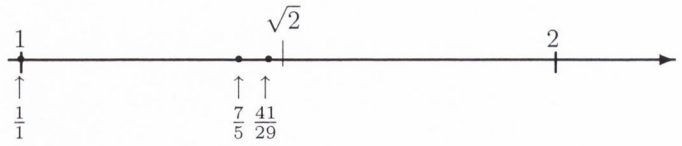
We can say that the alternate fractions

$$\frac{1}{1}, \quad \frac{7}{5}, \quad \frac{41}{29}$$

are three underestimates of $\sqrt{2}$, each being a better approximation of $\sqrt{2}$ than its predecessor.

— Because the minus sign before the last fraction in each equation tells us that these fractions squared are less than 2 by some amount.

That's it. The fraction 1/1 is the smallest of these fractions, and 41/29 is the largest:



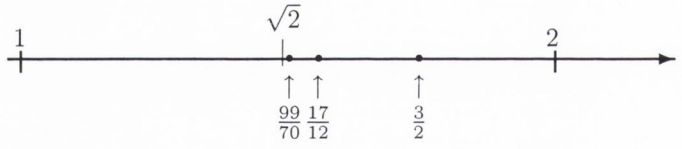
You'll understand why I make this point in a moment.

— But the fractions we skipped

$$\frac{3}{2}, \quad \frac{17}{12}, \quad \frac{99}{70},$$

on the other hand, are three overestimates of $\sqrt{2}$, which become progressively closer to $\sqrt{2}$.

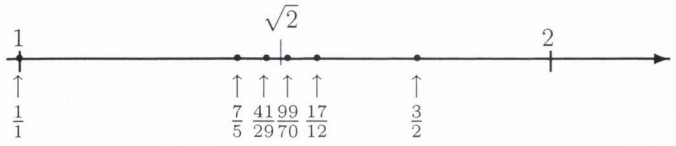
Right again. When these fractions are squared, they give 2 plus something positive. Note that this time the first fraction is the largest and the last one the smallest.



This is the opposite of the previous case.

— I think I see what you're driving at. The underestimates are creeping up on $\sqrt{2}$ from the left while the overestimates are creeping back toward $\sqrt{2}$ from the right.

That's right as we can see when we show all six fractions together on the number line:



Here is one way of summarising this information:

$$\frac{1}{1} < \frac{7}{5} < \frac{41}{29} < \sqrt{2} < \frac{99}{70} < \frac{17}{12} < \frac{3}{2}$$

— I know we haven't proven anything yet about the fractions in the sequence that follow the first six:

$$\frac{1}{1}, \quad \frac{3}{2}, \quad \frac{7}{5}, \quad \frac{17}{12}, \quad \frac{41}{29}, \quad \frac{99}{70}$$

but it looks, then, as if the very first fraction is the smallest of all the fractions in the sequence, while the second of the fractions is the largest of them all. Another interesting observation.

— If this the case, all the fractions, except for $1/1$, are greater than or equal to $1.4 = 7/5$ and less than or equal to $1.5 = 3/2$.

It would appear that way.

— So the fractions alternate between being underestimates and overestimates of $\sqrt{2}$ simply because of the plus and minus property.

Yes. Actually, it is very handy to have the fractions alternate between being underestimates and overestimates of $\sqrt{2}$ because we can use them to place $\sqrt{2}$ into narrower and narrower intervals of the number line.

— As if you were squeezing $\sqrt{2}$.

You could say that. For example, taking only the fraction 7/5 on the left of $\sqrt{2}$ and the fraction 3/2 to the right of $\sqrt{2}$ we get the inequality

$$\frac{7}{5} < \sqrt{2} < \frac{3}{2}$$

which you may recognise.

— Something I mentioned earlier?

Yes, you said that $1 \cdot 4$ squared is less than 2 but that $1 \cdot 5$ squared is greater than 2.

– A pure accident.

Maybe, or a sign of deep mathematical intuition.

— Without doubt. So now we can improve on this and say that

$$\frac{41}{29} < \sqrt{2} < \frac{99}{70}$$

Correct. We cannot say, at least not yet, exactly how close 99/70 is to $\sqrt{2}$ in terms of fractions or in decimal terms because we don't know how to calculate

$$\frac{99}{70} - \sqrt{2}$$

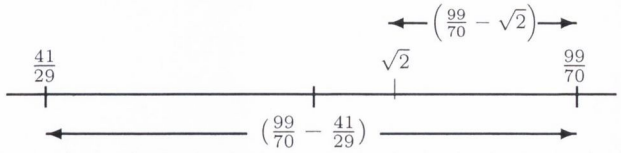
— But we could, if we could only find the fraction that is the same as $\sqrt{2}$.

Certainly, but this fraction is eluding us at the moment. Still we can *estimate* how close the fraction 99/70 is to $\sqrt{2}$.

— How?

Let us look at the interval between 41/29 and 99/70 under the microscope,

as it were.



It may not strike you as a remarkable observation, but we can now at least say that the distance between $\sqrt{2}$ and 99/70 is less than the length of the interval from 41/29 to 99/70, in which $\sqrt{2}$ resides.

— This seems obvious from the picture you have just drawn.

In fact, since we know that 99/70 is greater than $\sqrt{2}$ but closer to it than the fraction 41/29, we may say that 99/70 is within *half* the length of the interval between 41/29 and 99/70.

— Of course; simple but clever.

The length of the interval between 99/70 and 41/29 is

$$\frac{99}{70} - \frac{41}{29} = \frac{(99 \times 29) - (70 \times 41)}{70 \times 29} = \frac{1}{2030}$$

— Pretty narrow.

Since 2030 is bigger than 2000, we can say that the fraction 1/2030 is smaller than 1/2000. So the length of the interval is less than

$$\frac{1}{2000} = 0 \cdot 0005$$

— Less than 5 ten-thousandths of a unit.

Yes. This means that

$$\left(\frac{99}{70} - \sqrt{2}\right) < \frac{0 \cdot 0005}{2} = 0 \cdot 00025$$

– an estimate that shows with minimum computation that 99/70 is within $0 \cdot 00025$ of $\sqrt{2}$.

— Very smart.

Why don't you use your rule to show that the next two terms in the sequence

$$\frac{1}{1}, \quad \frac{3}{2}, \quad \frac{7}{5}, \quad \frac{17}{12}, \quad \frac{41}{29}, \quad \frac{99}{70} \quad \cdots$$

are the fractions

$$\frac{239}{169} \quad \text{and} \quad \frac{577}{408}$$

respectively, and verify that $239^2 - 2(169)^2 = -1$, with $577^2 - 2(408)^2 = 1$?

— So the next two fractions also follow the plus or minus 1 pattern.

Yes, but these two facts prove nothing about the remaining fractions.

— I realise this.

However, you might like to use these two new arrivals to show that

$$\frac{1}{1} < \frac{7}{5} < \frac{41}{29} < \frac{239}{169} < \sqrt{2} < \frac{577}{408} < \frac{99}{70} < \frac{17}{12} < \frac{3}{2}$$

— A further homing in on where $\sqrt{2}$ lives. It's very impressive how much can be said with just simple mathematics.

True, but it does help to have good observations to work on.

— A lesson I've learned from all of this in relation to the search for a fraction

exactly matching $\sqrt{2}$ is that it could be an awfully long search.

Why?

— Well, we have just shown that the leading six fractions of the sequence

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \frac{239}{169}, \frac{577}{408} \cdots$$

provide successively improving approximations of $\sqrt{2}$, and I suspect that the fractions further out this sequence do even better.

For the sake of argument we'll grant for the moment that they do.

— Judging from the numerators and denominators of the first eight terms, I'm guessing that the numerators and denominators grow longer and longer as we move further out the sequence.

Another interesting observation that we might discuss in more detail later. But for the moment, where is this line of reasoning taking you?

— Well, it suggests that the actual fraction exactly matching $\sqrt{2}$ may also have a very large numerator and denominator and, if so, searching could take a very long time to find it.

You have a point.

— For example, even if $\sqrt{2}$ were the fraction 99/70 with its very modest numerator and denominator, I would have to search as far as the ninety-ninth perfect square before hearing a click.

And if $\sqrt{2}$ were the fraction

$$\frac{3515043237929985687829131076921717644468626388 41}{2485510909704211897294694733728148710290930026 29}$$

which it isn't by the way, although it is very very close, you could be...

— ...searching for the rest of my life.

What the Ancients Knew

So are you going to give up on the search idea?

— Maybe, but I'd still like to test just a few more squares in the hope of getting lucky, even though I now realise that it is a most impractical method. And one that would not produce a positive result no matter how far you, or countless millions of others armed with all the computing power in the world, were prepared to search.

— What did you say?

That you would never succeed. Your search would be in vain.

— Are you telling me that, of the infinity of fractions lying between $1 \cdot 4$ and $1 \cdot 5$ there is not one that squares to give 2 exactly?

That's correct. There *isn't* a fraction between these two numbers that squares to give 2 exactly.

— But if there isn't such a fraction – and how on earth could you be convinced that there isn't – what kind of number is it that, when squared, gives 2? Or are you going to say that there is no such number?

Ah! A moment of truth has arrived. These crucial questions, which our opening geometrical demonstration has forced upon us, are ones we must attempt to answer.

— Am I to understand that $\sqrt{2}$ is definitely not a fraction?

Yes, there is no *rational* number that, when squared, gives 2. Integers and fractions are known collectively as rational numbers. Put another way, there is no rational number that measures the length of the diagonal of a unit square.

— Incredible! Of the infinity of fractions between $1 \cdot 4 = 7/5$ and $1 \cdot 5 = 3/2$ you are absolutely certain that there isn't a single one of them that squares to give 2 exactly?

Absolutely.

— But how do you know for certain that such a fraction doesn't exist?

I know because the Ancient Greeks proved that it is impossible. I will show you one beautiful numerical proof.

— It must be a very deep proof that shows that there isn't a number that squares to 2 exactly.

No, that would be going too far! I'm definitely not saying that there isn't a "number" whose square is exactly 2. All I am saying is that there isn't an integer or a fraction which when squared gives 2 exactly. There is a difference.

— But what other numbers are there besides the rational numbers, as you have just called them?

This is the mystifying point about the length of the diagonal of a unit square. It presents us with a paradox – an apparent contradiction – about the nature of numbers.

— So all along you have known that my search was futile.

Futile in its ambition but not otherwise. I didn't want to give the game away You are not the first to believe with complete conviction that there must be a fraction, however hard it might be to find, that squares to give 2 exactly Besides, I wanted you to enjoy exploring and discovering, to experience the pleasure of finding things out for yourself.

— I must gather my thoughts. I would not deny that the diagonal of the unit square has a length. In fact, this length is obviously greater than 1 unit and as we know, less than $1 \cdot 5$ units. Yet you tell me that the length of this diagonal cannot be expressed as a unit plus a certain fraction of a unit.

That's right. While the rational numbers are perfectly adequate for the world of commerce, they are not up to the task of measuring the *exact* length of a diagonal of a unit square. No matter how close a rational number may come to measuring the length, there will always be an error, microscopically small perhaps, but nevertheless an error. Always. The ancient way of putting this was to say that the diagonal and side of a square are *incommensurable.*

— So if we insisted on thinking that *all* numbers are the ones with which we are familiar, namely the rationals, then we'd be forced to say that there is no number of units that measure this diagonal, or that there is no number whose square is 2.

Yes, but why restrict ourselves to such a viewpoint?

— It seems natural.

Maybe, but perhaps it seems this way simply because most people's experi-

ence is limited to dealing with rational numbers. However, as you have said, if we were to insist on maintaining that rational numbers are the only type of number, then we'd have to be prepared to live in a world where there are lengths which are not measurable and where certain numbers have no square roots.

— So we must accept that there are other types of numbers.

For mathematicians the proof that no unit plus a fraction of a unit can hope to exactly measure the diagonal forces us to broaden our notions of what constitutes a number. When we do this, the paradox surrounding $\sqrt{2}$ simply dissolves.

— So what "number" measures the diagonal of a unit square?

The one whose square is 2 and that we denote by $\sqrt{2}$. We admit the existence of this number because it makes its presence necessary by being the length of a legitimate quantity – the diagonal of a unit-square.

— So the length of any side of the internal square we talked about at the beginning is simply $\sqrt{2}$, with no need for further elaboration.

Yes. $\sqrt{2}$ is a number between $1 \cdot 4$ and $1 \cdot 5$ that is not a rational number but that, when squared, gives 2. As we have already said, $\sqrt{2}$ is defined by the equation $\sqrt{2} \times \sqrt{2} = 2$, which is the mathematical way of saying that $\sqrt{2}$ is the positive number that squares to give 2.

— So $\sqrt{2}$ is a new type of number.

Yes, new or different, but we have not proved this yet. Because it is not a rational number, it is called an *irrational* number. Not that there is anything

unreasonable about it. It is so named because it cannot be expressed as the *ratio* of two integers in the way that the fractions are.

— So the word rational in "rational number" is used because of the word ratio, while the term "irrational" in connection with $\sqrt{2}$ is used because it cannot be so expressed.

Quite so. The number $\sqrt{2}$ is as real as any fraction. In fact, $\sqrt{2}$ is just one of an infinite number of irrational real numbers that exist "outside" the realm of the rational numbers.

— Can you show me some other irrational numbers?

Yes, the positive square roots of each of the other numbers missing from the list of perfect squares we made out some time ago can also be shown to be irrational numbers.

— This means that

$$\sqrt{2}, \quad \sqrt{3}, \quad \sqrt{5}, \quad \sqrt{7}, \quad \sqrt{8}, \quad \sqrt{10} \ldots$$

are all irrational numbers.

Yes.

— This is why there is an infinity of these numbers.

Certainly, but the collection of irrational numbers contains not just all these *surds*, as they are sometimes called, but a whole galaxy of other weird and wonderful numbers, the most famous being π.

— Ah, π! The ratio of the length of the circumference of any circle to the length of its diagonal. I thought that π was the fraction 22/7.

This is only an approximation of its true value, just like 7/5 is an approximation of $\sqrt{2}$.

— Reality is a lot more complicated than I naïvely thought.

Perhaps we should say that the world of mathematics is a lot more complicated than one might think at first. However, speaking of reality, the collection of rational numbers and the collection of irrational numbers between them constitute the set of *real* numbers.

— The idea that there are new specimens of numbers other than the "usual ones" used in arithmetic takes a little getting used to.

You're not the first person who was more than a little perplexed by these new numbers. The minds of the ancient Greeks were bewildered when these irrational numbers thrust their existence upon the Greeks' consciousness. Legend has it that they were positively perturbed by the intrusion of these new quantities into their reality. They experienced an intellectual and philosophical crisis.

— They did? Why?

Well, there was a brotherhood of Pythagoreans, followers of the famous philosopher and mathematician Pythagoras, which was devoted to the pursuit of higher learning, in particular mathematics. They were very well respected and considered to know all that there was to know. They believed that everything could be quantified by the familiar rational numbers.

— A reasonable enough belief, or should I say a rational belief?

Yes, a very justifiable one. After all, these numbers are the only ones needed

for commercial transactions, and they are equally adequate at describing various other physical phenomena. They also suffice for most measuring purposes that occur in practice.

— Even though they cannot be used to give the measure of the diagonal of a unit square.

Yes, the issue about the new nature of $\sqrt{2}$ and its cousins, $\sqrt{3}$, $\sqrt{5}$, ... was a theoretical one rather than a concern with "practical" measurement. The Greeks were fully aware that even if fractions could not measure the diagonal of a unit square exactly, they could measure it to any desired degree of accuracy. For example, a length of

$$577/408 = 1 + \frac{169}{408}$$

units measures the "true" length of the diagonal to well within a hundred-thousandth of a unit.

— Which is less than one-hundredth of a millimetre if the unit is a metre.

[See chapter note 3]

There is evidence that this approximation to $\sqrt{2}$ was known to the Babylonians around 1600 B.C. This is many centuries before the ancient Greeks whom I mentioned, because a Babylonian tablet from that time gives $1; 24, 51, 10$ as an approximation to $\sqrt{2}$.

Tablet 7289
Yale Collection

— What does $1; 24, 51, 10$ stand for?

It's shorthand for

$$1 + \frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3}$$

The Babylonians used a sexagesimal system.

— What is this when it is simplified?

The fraction

$$\frac{30547}{21600}$$

which, as you can see, is not 577/408 exactly.

— Why, then, is it thought that they knew of "our" 577/408?

Because in base 60,

$$\frac{577}{408} = 1; 24, 51, 10, 35 \ \ldots$$

it is suspected that they just truncated (shortened) the sexagesimal expansion of this fraction.

— To three places, as we'd say.

Yes.

— How did the Babylonians find such approximations?

It is not exactly known, but there is speculation that they knew of a method of approximation.

— Was it a different method from the one using the sequence of fractions we have discovered?

It is related to this but faster.

— Faster sounds interesting.

Accelerated, we might even say. This method also gave them $1; 25$ as an approximation to $\sqrt{2}$. Convert $1; 25$ to base 10 to see what it is.

— I'll try. Since they used a sexagesimal system

$$1; 25 = 1 + \frac{25}{60} = \frac{85}{60} = \frac{17}{12}$$

What is 1;24 as a fraction?

this fraction, just like 577/408, is in our sequence.

It is indeed, the fourth in the sequence. It is not as good an approximation to $\sqrt{2}$ as 577/408, which is the eighth entry in the same sequence. As we said before, it doesn't do a bad job of approximating $\sqrt{2}$.

— So these Mesopotamians must have known their mathematics.

And quite a deal, by all accounts.

— I should be able to verify for myself that 577/408 approximates $\sqrt{2}$ as closely as you say.

You should and, what is significant, without knowing anything about the decimal expansion of $\sqrt{2}$.

— Hmm. I didn't appreciate this point before.

I didn't emphasise it prior to this.

— Please remind me of how exactly I would begin to go about this verification.

Recall that the fraction 239/169 is the one before 577/408 in our short sequence and that it underestimates $\sqrt{2}$, whereas 577/408 overestimates it.

— I think I remember now. Since 577/408 is closer to $\sqrt{2}$ than 239/169 is, its distance from $\sqrt{2}$ is less than half the distance between these two fractions.

Yes. This distance is 1/68952, as you can check.

— And what do we say now?

Since $50000 < 68592$, we say that 1/68592 is less than 1/50000. Hence the length of the interval is less than one fifty-thousandth of a unit, and so 577/408 is within a hundred-thousandth of a unit of $\sqrt{2}$.

— So we're done.

Yes. Maybe now is a good time to use what we know to get some idea of the leading digits in the decimal expansion of $\sqrt{2}$.

— How are you going to do this?

Convert the fractions in the inequality

$$\frac{239}{169} < \sqrt{2} < \frac{577}{408}$$

to decimal form.

— Using a calculator I hope.

Yes, because in theory this is something we can do ourselves by hand.

— And so we are free to use a calculator to save time.

We get

$$1 \cdot \underline{4142}011834319526627\ldots < \sqrt{2} < 1 \cdot \underline{4142}156862745098039\ldots$$

working to twenty decimal places.

— Some calculator! I see that both expansions agree to four decimal places. Is it safe so to say that

$$\sqrt{2} = 1 \cdot 4142\ldots$$

which, I think, is pretty good.

It is. And because we did everything ourselves I think we can take a bow. We'll have fun improving on all of this at a later stage.

— But to get back to the ancient Greeks. You were saying that it is the

nature of numbers that was of primary interest to these learned men.

Indeed. Such was their conviction that the rational numbers described all of nature exactly that their motto was "All is number".

— By which they meant the rational numbers.

Yes.

— I'm glad to see I'm in good company.

You could certainly say that.

— So they had their colours well and truly nailed to the mast.

This proclamation took on the status of an incontrovertible truth. It became a creed.

— Oh! The discovery of the existence of $\sqrt{2}$ must have come as a shock.

A most unwelcome one we are told, because it challenged their cherished belief about the nature of numbers.

— They took this whole business about numbers really seriously then?

I don't know how true much of the early lore surrounding the discovery of $\sqrt{2}$ is, but one story has it one of the brotherhood leaked the news that all was not well with the accepted dogma. For this breach of faith he was taken on a sea trip and cast overboard.

[See chapter note 4]

— You're kidding me!

Well, if it is true, it goes toward answering your question as to how seriously they took their mathematics.

— So his number was up!

As fate would have it, the number $\sqrt{2}$ is referred to in some quarters as

Pythagoras's constant.

— I wonder what the Pythagorean brotherhood would have to say about that. But how did they come to know for certain that $\sqrt{2}$ is not a rational number? Surely they must have thought that $\sqrt{2}$ is actually a rational number but that they simply lacked the means to find it?

Perhaps they came upon numerical evidence similar to what you found in your search but I don't know. I do know that their main mathematical focus was geometry.

— Of course, the famous theorem of Pythagoras.

Actually, it may have been this very theorem that first brought $\sqrt{2}$ to the attention of the ancient Greeks.

— So they were the cause of the downfall of their own philosophy that "all is number".

You could say that. Coming back to what we were saying about searching, these clever Greeks would have known that the search method is one that, no matter how many perfect squares may have been checked, still leaves an infinite number of possibilities to be tested.

— I hope they figured that out faster than I did.

I'm sure they were fully aware that any finite quantity, no matter what its size, is as nothing against the backdrop of infinity.

— Still, they must have suspected right from the very moment the $\sqrt{2}$ problem reared its head that their doctrine of number was in trouble.

I'd be inclined to agree: they must have known that the doctrine wasn't as

all-embracing as they originally proclaimed. It may be that some were realy intrigued that $\sqrt{2}$ does not dwell in the infinite realm of rational numbers but is something that is "outside of it", as it were. Certainly minds over the centuries have been charmed by this aspect of numbers.

— Did the ancient Greeks find the proof you mentioned fairly soon after observing that there was more to $\sqrt{2}$ than meets the eye?

As far as I know, quite a stretch of time elapsed, about 300 years or so, before someone found an argument that turned suspicion into fact and established the irrationality of $\sqrt{2}$ once and for all. However, I don't know if the argument described by Euclid, which I am about to show you was the first because there are many ingenious proofs of the irrationality of $\sqrt{2}$.

Elements X, 115a

— But they must all be very difficult. It cannot be easy to be convinced that of the infinity of rational numbers, not one squares to give 2.

Not at all. The proof we are about to discuss is a magnificent *reductio ad absurdum* argument.

— Which means?

In this case, you assume that there is a rational number that, when squared, gives 2, and then you show that this assumption leads to a contradiction or, put another way, reduces to something absurd. This form of logic – bequeathed to us by these Greek scholars – has been used ever since, throughout mathematics.

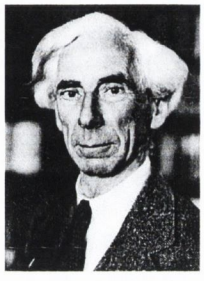
— If you arrive at a contradiction, you say that the assumption you made at the start is the cause of the trouble.

Yes, and you conclude that it must be false and must be abandoned.

— So if the assumption is false then its opposite is true?

Precisely.

— After thinking that $\sqrt{2}$ *must* be a fraction and having been frustrated in a vain search, I cannot wait to see this proof of irrationality. It must be a wonderful mathematical work of art.



(1872-1970)

A work of art indeed. Bertrand Russell once said, "Mathematics, rightly viewed, possesses not only truth but supreme beauty... sublimely pure, capable of a stern perfection as only the greatest art can show." Judge for yourself whether the proof merits this accolade.[12]

[12]The complete dialogue can be found on the Core CD.

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