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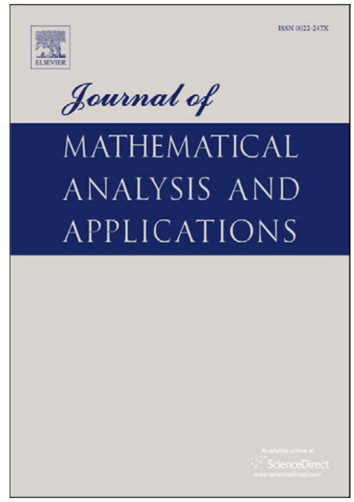

Weyl's theorem and tensor products: A counterexample

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WEYL'S THEOREM AND TENSOR PRODUCTS: A COUNTEREXAMPLE

DEREK KITSON, ROBIN HARTE, AND CARLOS HERNÁNDEZ

Abstract. Approximately fifty percent of Weyl's theorem fails to transfer from Hilbert space operators to their tensor product. As a biproduct we find that the product of circles in the complex plane is a limaçon.

0. Introduction

To say that "Weyl's theorem holds", for a bounded operator $T \in B(X)$ on a Banach space X, is to assert [2],[4],[5] that

0.1 $$\sigma(T) \setminus \omega_{ess}(T) = \pi_{00}^{left}(T) \equiv \text{iso}\ \sigma(T)\cap\pi_0^{left}(T)\ :$$

the complement in the spectrum of the Weyl spectrum coincides with the isolated eigenvalues of finite multiplicity. One half of this is the assertion that "Browder's theorem holds" for $T \in B(X)$, which is to say [2],[4],[5] that

0.2 $$\omega_{ess}(T) = \beta_{ess}(T)$$

the Weyl and the Browder spectrum coincide. We would like to apologize for this somewhat anachronistic terminology, which over the years has been allowed to solidify. In this note we offer an example of two operators $A, B \in B(X)$, each of which satisfies (0.2), and indeed (0.1), whose tensor product $A \otimes B \in B(X) \otimes B(X) \subseteq B(X \otimes X)$ does not. We remark that these operators are constructed from rather simple building blocks, as are the resulting spectra. For the record we recall that the *Fredholm, Weyl* and *Browder* essential spectrum $\sigma_{ess}(T)$, $\omega_{ess}(T)$ and $\beta_{ess}(T)$ collect complex numbers λ for which $T - \lambda I$ fails to be, respectively, Fredholm, Weyl or Browder; an operator is Fredholm if it has finite dimensional null space and closed range of finite codimension, is Weyl if in addition these two finite dimensions are equal, and is Browder if in addition it has finite ascent and descent [2],[3],[4]. We are writing $\pi^{left}(T)$ for the *eigenvalues* of $T \in B(X)$, $\pi_0^{left}(T)$ for the eigenvalues of finite (geometric) multiplicity and $\pi_{00}^{left}(T)$ for those which are also isolated in the spetrum $\sigma(T)$.

1. Shifts

Consider, in the Banach algebra $B(X)$ with $X = \ell_2$, the forward and backward shifts and the standard weight u, v, w: for arbitrary $x = (x_1, x_2, x_3, \ldots) \in X$ and $n \in \mathbb{N}$

1.1 $$(ux)_1 = 0\ ,\ (ux)_{n+1} = x_n\ ;\ (vx)_n = x_{n+1}\ ;\ (wx)_n = \tfrac{1}{n}x_n\ .$$

Evidently

1.2 $$vu = 1 \neq uv \in 1 + B_{00}(X)$$

where $B_{00}(X) \subseteq B(X)$ is the ideal of finite rank operators, and

1.3 $$\sigma(u) = \sigma(v) = \sigma^{right}(u) = \sigma^{left}(v) = \mathbb{D}\ ;\ \sigma^{left}(u) = \sigma^{right}(v) = \mathbb{S}\ ,$$

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where $\mathbb{D}$ is the closed unit disc and $\mathbb{S} = \partial\mathbb{D}$ is its boundary the unit circle; we remark that also the disc $\mathbb{D} = \eta\mathbb{S}$ is the *connected hull* ([3] Definition 7.11.1) of the circle. The Fredholm and Weyl essential spectrum [2],[3],[4],[5] are given by

1.4 $$\sigma_{ess}(u) = \sigma_{ess}(v) = \mathbb{S}\ ;\ \omega_{ess}(u) = \omega_{ess}(v) = \mathbb{D}\ .$$

The discrepancy between the Weyl and the Fredholm spectrum of the shifts u and v is reflected in their *spectral pictures* [1]: each has one hole, of index -1 and +1 respectively. The interior of the unit disc gives also the set of all eigenvalues of v, each of which is of geometric multiplicity 1 , while u has no eigenvalues:

1.5 $$\pi^{left}(v) = \pi_0^{left}(v) = \mathbb{D}\setminus\mathbb{S}\ ;\ \pi^{left}(u) = \emptyset\ :$$

specifically

1.6 $$v^{-1}(0) = (1-uv)(X) = \mathbb{C}\delta_1 = \{(\lambda, 0, 0, \dots) : \lambda \in \mathbb{C}\}$$

and

1.7 $$|\lambda| < 1 \Longrightarrow (v-\lambda)u = 1-\lambda u \in B(X)^{-1}\ ,\ (v-\lambda)^{-1}(0) = (1-\lambda u)^{-1}v^{-1}(0)\ .$$

For the standard weight

1.8 $$\sigma(w) = \mathbb{O}\cup\mathbb{N}^{-1}\ ;\ \sigma_{ess}(w) = \beta_{ess}(w) = \mathbb{O}\ ;\ \pi^{left}(w) = \pi_0^{left}(w) = \mathbb{N}^{-1}\ ,$$

where $\mathbb{O} = \{0\}$ is the origin and $\mathbb{N}$ the natural numbers. For the projection uv

1.9 $$\sigma(uv) = \{0,1\} = \mathbb{O}\cup(\mathbb{O}+1)\ ;\ \sigma_{ess}(uv) = \omega_{ess}(uv) = \beta_{ess}(uv) = \{1\} = \mathbb{O}+1$$

and

1.10 $$\pi^{left}(uv) = \mathbb{O}\cup(\mathbb{O}+1)\ ;\ \pi_0^{left}(uv) = \mathbb{O}\ .$$

Finally the product vw is quasinilpotent:

1.11 $$\sigma(vw) = \sigma_{ess}(vw) = \pi_0^{left}(vw) = \mathbb{O}\ .$$

2. Sums

Our counterexample will be the tensor product of two direct sums: with

2.1 $$A = (1-uv)\oplus(\tfrac{1}{2}u-1)\ ,\ B = -(1-uv)\oplus(\tfrac{1}{2}v+1)$$

we have

2.2 $$\sigma(A) = (\tfrac{1}{2}\mathbb{D}-1)\cup\mathbb{O}\cup(\mathbb{O}+1)\ ,\ \sigma(B) = (\mathbb{O}-1)\cup\mathbb{O}\cup(\tfrac{1}{2}\mathbb{D}+1)\ ,$$

2.3 $$\sigma_{ess}(A) = (\tfrac{1}{2}\mathbb{S}-1)\cup\mathbb{O}\ ,\ \sigma_{ess}(B) = \mathbb{O}\cup(\tfrac{1}{2}\mathbb{S}+1)\ ,$$

and ([4] Theorem 5), noting that $\omega_{ess}(1-uv) = \sigma_{ess}(1-uv)$,

2.4 $$\omega_{ess}(A) = \beta_{ess}(A) = (\tfrac{1}{2}\mathbb{D}-1)\cup\mathbb{O}\ ,\ \omega_{ess}(B) = \beta_{ess}(B) = \mathbb{O}\cup(\tfrac{1}{2}\mathbb{D}+1)\ .$$

The reader may notice that we are trying to lay out the spectra as they would appear along the real line: she is invited to draw the pictures.

Weyl's theorem (0.1) does not always [4] transfer to direct sums, and also does not always transfer back; for example Weyl's theorem holds for v and for the direct sum $v\oplus vw$ but fails for the product vw: recall (1.11) together with

2.5 $$\sigma(v\oplus vw) = \omega_{ess}(v\oplus vw) = \mathbb{D}\ ;\ \pi^{left}(v\oplus vw) = \mathbb{D}\setminus\mathbb{S}\ ;\ \pi_{00}^{left}(vw) = \mathbb{O}\ .$$

3. Products

We claim that Weyl's theorem holds for A and for B, while Browder's theorem fails for $A \otimes B$. Specifically ([3] Theorem 11.7.5)

3.1 $$\sigma(A \otimes B) = \sigma(A)\sigma(B) = (\tfrac{1}{2}\mathbb{D} - 1)(\tfrac{1}{2}\mathbb{D} + 1) \cup \mathbb{O} \cup (\mathbb{O} + 1) \cup (\tfrac{1}{2}\mathbb{D} + 1)\ ,$$

and we observe that the product of the discs

3.2 $$(\tfrac{1}{2}\mathbb{D} - 1)(\tfrac{1}{2}\mathbb{D} + 1) = \mathbf{L} \subseteq -1 + \tfrac{1}{2}\mathbb{D} + \tfrac{1}{2}\mathbb{D} + \tfrac{1}{4}\mathbb{D} = -1 + \tfrac{5}{4}\mathbb{D}$$

is a (solid) *limaçon* $\mathbf{L}$ whose intersection with the real line is $\left[-\frac{9}{4}, -\frac{1}{4}\right]$, and hence on aggregate

3.3 $$\sigma(A \otimes B) = \mathbf{L} \cup \mathbb{O} \cup (\tfrac{1}{2}\mathbb{D} + 1)\ .$$

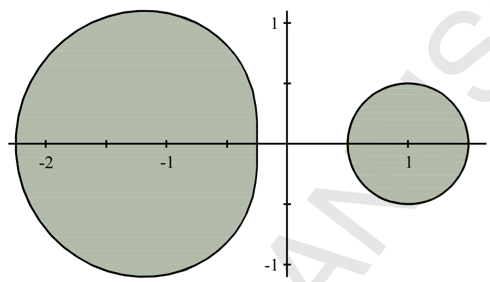

Figure 1. $\sigma(A \otimes B)$

For the essential spectrum (cf [3] Theorem 11.6.4)

3.4 $$\sigma_{ess}(A \otimes B) = \sigma_{ess}(A)\sigma(B) \cup \sigma(A)\sigma_{ess}(B)\ ,$$

and we have

$$\sigma_{ess}(A)\sigma(B) = \big((\tfrac{1}{2}\mathbb{S} - 1) \cup \mathbb{O}\big)\big((\mathbb{O} - 1) \cup \mathbb{O} \cup (\tfrac{1}{2}\mathbb{D} + 1)\big) = \mathbf{L} \cup \mathbb{O} \cup (\tfrac{1}{2}\mathbb{S} + 1)\ ,$$

$$\sigma(A)\sigma_{ess}(B) = \big((\tfrac{1}{2}\mathbb{D} - 1) \cup \mathbb{O} \cup (\mathbb{O} + 1)\big)\big(\mathbb{O} \cup (\tfrac{1}{2}\mathbb{S} + 1)\big) = \mathbf{L} \cup \mathbb{O} \cup (\tfrac{1}{2}\mathbb{S} + 1)\ ,$$

and hence also

3.5 $$\sigma_{ess}(A \otimes B) = \mathbf{L} \cup \mathbb{O} \cup (\tfrac{1}{2}\mathbb{S} + 1)\ .$$

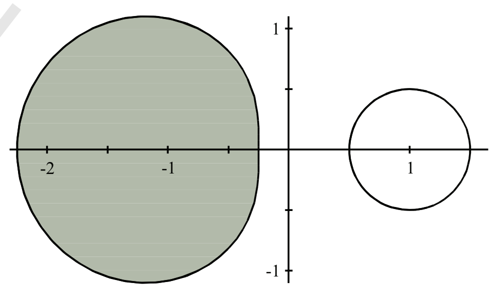

Figure 2. $\sigma_{ess}(A \otimes B)$

For the Browder spectrum [2],[3],[4] we have ([2] Theorem 5)

3.6 $$\beta_{ess}(A\otimes B)=\beta_{ess}(A)\sigma(B)_{\cup}\sigma(A)\beta_{ess}(B)\ ,$$

and we get

$$\sigma(A)\beta_{ess}(B)=\big((\tfrac{1}{2}\mathbb{D}-1)\cup\mathbb{O}\cup(\mathbb{O}+1)\big)\big(\mathbb{O}\cup(\tfrac{1}{2}\mathbb{D}+1)\big)=\mathbf{L}\cup\mathbb{O}\cup(\tfrac{1}{2}\mathbb{D}+1))=\beta_{ess}(A)\sigma(B)\ ,$$

giving also

3.7 $$\beta_{ess}(A\otimes B)=\mathbf{L}\cup\mathbb{O}\cup(\tfrac{1}{2}\mathbb{D}+1))=\sigma(A\otimes B)\ .$$

So far therefore

3.8 $$\mathbf{L}\cup\mathbb{O}\cup(\tfrac{1}{2}\mathbb{S}+1)=\sigma_{ess}(A\otimes B)\subseteq\omega_{ess}(A\otimes B)\subseteq\beta_{ess}(A\otimes B)=\mathbf{L}\cup\mathbb{O}\cup(\tfrac{1}{2}\mathbb{D}+1))\ .$$

4. Index

The essential spectrum $\sigma_{ess}(A\otimes B)$ also has one hole, and the determination of $\omega_{ess}(A\otimes B)$ depends on its spectral picture, which by (3.8) rests entirely on the index of the operators $A\otimes B-\lambda I\otimes I$ for $|\lambda-1|<\frac{1}{2}$, and by index continuity specifically for $\lambda=1$:

Theorem 1. *With the notation above,* index$(A\otimes B-I\otimes I)=0$, *and hence*

4.1 $$\sigma_{ess}(A\otimes B)=\omega_{ess}(A\otimes B)\neq\beta_{ess}(A\otimes B)\ .$$

Proof. There are just two points $(\lambda,\mu)\in\sigma(A)\times\sigma(B)$ with $\lambda\mu=1$, namely $(1,1)$ and $(-1,-1)$: now ([1] Teorema 4.5; cf [5] Theorem 3)

4.2 $$\text{Index}(A\otimes B-I\otimes I)=\text{index}(A+I)\dim(B+I)^{-1}(0)+\dim(A-I)^{-1}(0)\text{index}(B-I)=-1+1=0\ .$$

Hence $1\notin\omega_{ess}\,(A\otimes B)$, and by index continuity the whole of the interior of $\frac{1}{2}\mathbb{D}+1$ is excluded from the Weyl spectrum. □

For a more hands-on computation of its index, the operator $A\otimes B$, which is a tensor product of direct sums, can be thought of as block diagonal operator matrix: we interpret

4.3 $$(a\oplus c)\otimes(b\oplus d)=(a,c)\otimes(b,d)=(a\otimes b,a\otimes d,c\otimes b,c\otimes d)\ .$$

Specifically, with $c=\frac{1}{2}u-1$, $d=\frac{1}{2}v+1$ and $p=1-uv$,

4.4 $$A\otimes B=(-p\otimes p,p\otimes d,-c\otimes p,c\otimes d)\ .$$

The first component $-p\otimes p$ has essential spectrum $\{0\}=\mathbb{O}$, with no holes, and the essential spectrum of the fourth component $\mathbf{L}$ also has no holes; neither includes the point $1\in\mathbb{C}$. The essential spectrum of the second component $p\otimes d$ is the union $\mathbb{O}\cup(\frac{1}{2}\mathbb{S}+1)$ and has one hole, containing the point $1\in\mathbb{C}$, with index $+1$, and finally the essential spectrum of the third component $c\otimes -p$ is the same, except that the hole has index -1. The direct sum of the second and third components therefore has the same essential spectrum, now with index zero. This therefore gives back Theorem 1.

5. Limaçon

Every singleton set $\{\alpha\}=\alpha+\mathbb{O}$ is obtained by translation of the origin $\mathbb{O}=\{0\}$, and every disc $\alpha+\beta\mathbb{D}$ by translation and scaling of the unit disc $\mathbb{D}=\{|z|\leq 1\}$; the same is true of circles $\alpha+\beta\mathbb{S}=\partial(\alpha+\beta\mathbb{D})$. We can add points to get points and add discs to get discs: since $\mathbb{D}$ is absolutely convex we have

5.1 $$\beta\mathbb{D}+\gamma\mathbb{D}=(\beta+\gamma)\mathbb{D}\ .$$

Similarly we can multiply points:

5.2 $$(\alpha+\mathbb{O})(\beta+\mathbb{O})=\alpha\beta+\mathbb{O}\ .$$

The product of a point and a disc is again a disc (degenerate if the point is the origin!); when we come to the product of two discs however the results are more complicated than we might at first suppose. The product of discs centred at the origin is easy:

5.3 $$(\beta\mathbb{D})(\gamma\mathbb{D}) = \beta\gamma\mathbb{D}\ .$$

When the centres move away from the origin and from one another the results [6] are *limaçons*. For a simple example look at

5.4 $$\sigma((v-1)\otimes(u+1)) = \sigma(v-1)\sigma(u+1) = (\mathbb{D}-1)(\mathbb{D}+1) \subseteq 1+3\mathbb{D}\ ;$$

if we start by multiplying the corresponding circles, and starting at their intersection the origin move round each in positive direction, we reach the following parametrised curve Γ_0:

5.5 $$(1-e^{it})(-1+e^{it}) = -(1-\cos t - i\sin t)^2 = -2(1-\cos t)(\cos t + i\sin t)\ .$$

Evidently in polar coordinates this curve has equation $r = 2(1-\cos\theta)$. If we now vary the points on each circle independently we find that we stay within, and fill out, the connected hull $\eta\Gamma_0$.

Turning to the set (3.2), the same strategy traces out the curve

5.6 $$(1-\tfrac{1}{2}e^{it})(-1+\tfrac{1}{2}e^{it}) = -\tfrac{3}{4} + (1-\tfrac{1}{2}\cos t)e^{it}\ ,$$

so that $\partial\mathbf{L} = -\frac{3}{4}+\Gamma$, and $\mathbf{L} = -\frac{3}{4}+\eta\Gamma = \eta(-\frac{3}{4}+\Gamma)$, where Γ has polar equation $r = 1-\frac{1}{2}\cos\theta$.

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