



A graph-theory approach to optimisation of an acoustic absorber targeting a specific noise spectrum that approaches the causal optimum minimum depth

Ian Davis^{a,*}, Andrew McKay^b, Gareth J. Bennett^b

^a Efficient Energy Transfer (η ET) Department, Nokia Bell Labs, Blanchardstown Business & Technology Park, Dublin 15, Ireland

^b Department of Mechanical & Manufacturing Engineering, Trinity College Dublin, Dublin 2, Ireland

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ABSTRACT

Equivalent circuit analysis is a powerful tool for analysing acoustic systems where a lumped element model is valid. These equivalent circuits allow an overall impedance of the structure to be estimated which facilitates predictions of the reflectivity, transmissibility and/or absorptivity of the system. Complex acoustic systems are represented by non-planar equivalent circuits which are challenging to simplify to a single overall impedance value using traditional Kirchoff's Law simplifications. A two-point impedance method using graph theory allows the impedance of a circuit to be estimated without simplification. The graph theory method is applied to a type of acoustic absorber structure named SeMSA (Segmented Membrane Sound Absorber) which had previously been investigated for a two-segment cell design. This method allows the SeMSA analysis to be expanded to multi-sector designs with a wider parameter space. A local optimisation routine is applied to the graph theory impedance estimation to maximise acoustic absorption of SeMSA under consideration of absorber depth, causal optimality and the targeted noise spectra. Analytical predictions are validated using numerical simulations. The optimised multi-sector absorber demonstrates 70.5% white noise absorption in the 20–4500 Hz frequency range with an absorber depth of 16 mm and is just 0.5 mm from the theoretical minimum depth to achieve this absorption response.

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1. Introduction

Acoustic materials for sound absorption have evolved significantly from traditional bulk porous/fibrous absorbers to modern materials such as membrane-based metamaterials [1], absorbers consisting of arrangements of axially-coupled channels [2], acoustic black holes that direct acoustic waves to an absorptive core [3] and coiled Helmholtz resonators [4]. These modern materials typically target sub-wavelength absorption i.e. absorption coefficients of close to unity are achieved with material depths of order $\lambda/100$, where λ is the acoustic wavelength. A summary of recent development may be found in a

* Corresponding author.

E-mail address: davisisound@gmail.com (I. Davis).

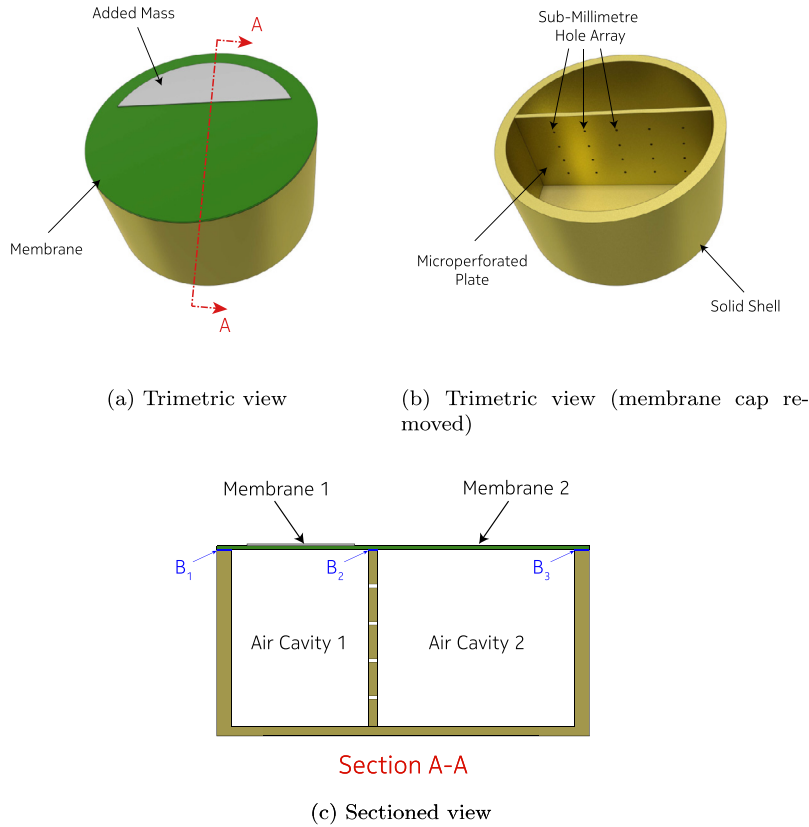


Fig. 1. A cylindrical Segmented Membrane Sound Absorber (SeMSA). The cylindrical volume is divided into two air cavity segments by a microperforated plate.

recent review paper by Yang et al. [5]. Traditional bulk absorbers such as melamine or polyurethane acoustic foams generally exhibit poor absorption performance below 1 kHz unless depths of order 100 mm are used [6].

While different metamaterials utilise different physical phenomena such as Fabry-Perot resonances [7] and elastic curvature profiles inside a membrane-platelet system that couple weakly with acoustic radiation [1], most metamaterials are based on resonant systems that absorb effectively in narrow frequency bands. Extending this frequency range to wider bandwidths is desirable in order to create broadband sound absorption in a small footprint. The key challenge for extending the absorption bandwidth of an acoustic absorber is at low frequencies where the absorber depth is smaller than the acoustic wavelength.

Another key consideration for the design of acoustic materials is the simplicity of their construction. In practical terms the efficacy of an acoustic absorbers (as typically characterised by its absorption per unit depth) must be considered alongside the cost and complexity of manufacturing the material. Furthermore, if an acoustic absorber is being manufactured at scale and across a variety of environments and targeted sound sources, it is desirable that the performance of the material may be predicted, tuned and optimised for the specific application under consideration of available volume, the sound source spectrum and other environmental factors.

Absorber technologies that are made up of linear combinations of resistive and reactive elements can be represented by lumped-element models. These lumped-element models allow the impedance of the acoustic material to be represented using equivalent mechanical or electric circuit models [8], and circuit simplifications using Kirchoff's and Ohm's Laws can allow the circuit to be simplified to a single overall impedance value. Once simplified, the circuit can be optimised to achieve a targeted absorption profile such as maximal white noise absorption in a prescribed frequency band in a given volume, as was demonstrated using the SeMSA (Segmented Membrane Sound Absorber) technology [6]. Donda et al. [9] also present an acoustic absorber that uses a network of lumped elements to target narrowband, high absorption with absorber depths of just $\lambda/527$.

In the SeMSA technology previously investigated in McKay et al. [6] and shown in Fig. 1, the incident sound causes resonance of the membrane-cavity systems which are coupled through a microperforated plate. The differential pressure between the two cavities drives air through the microperforates where there are large visco-thermal losses. Sound is also absorbed by damping inside the membranes. SeMSA had previously been investigated for a two-segment cylindrical design with an equivalent circuit model (see Fig. 2(a)) based on the impedances of the two membranes plus added masses (z_{m1}

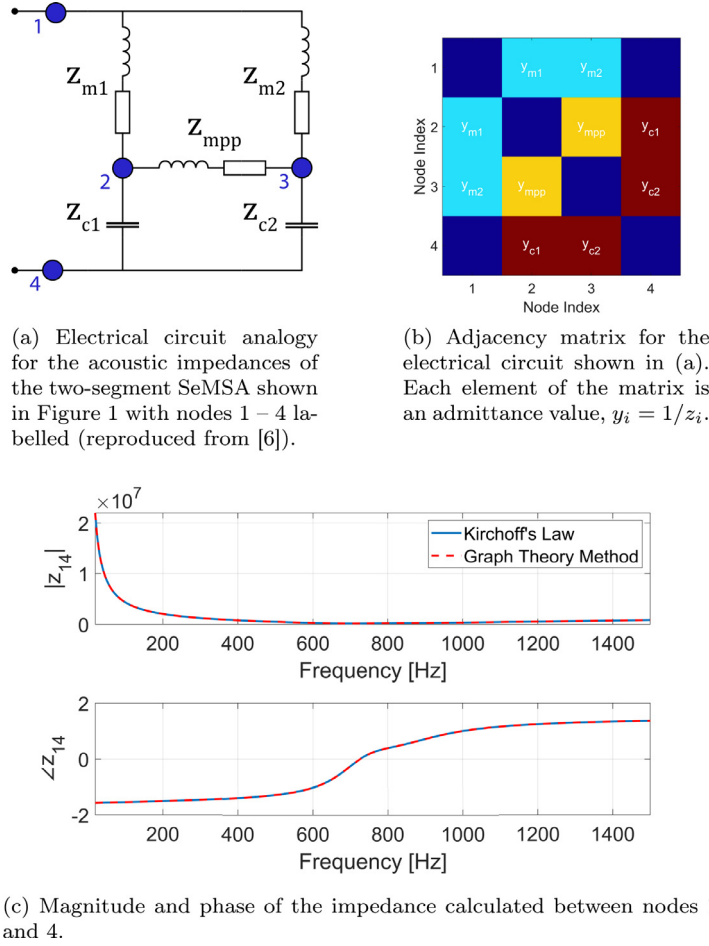


Fig. 2. Comparison between the impedance estimated by simplifying the equivalent circuit shown in (a) to the two-point impedance model based on Tzeng & Wu's graph theory approach [13] which analyses the adjacency matrix shown in (b).

and z_{m2}), the two air cavity volumes (z_{c1} and z_{c2}) and the microperforated plate dividing the two air cavities (z_{mpp}). A local absorption routine was used to tune the circuit impedances (and hence geometric parameters) to achieve significant white noise absorption in the range 20–1200 Hz at a range of overall depths.

In order to extend the analysis of SeMSA to more complex designs with more segments or sectors than the two-segment design shown in Fig. 1, a more complex equivalent circuit would be required than Fig. 2(a). As the number of sectors or segments increases there will be multiple microperforated plates dividing adjacent air cavities which results in a non-planar equivalent circuit. Adding additional parameters to the model allows for additional tuneability of the overall impedance which may extend the bandwidth of the absorber's performance or allow multiple narrowband frequency ranges to be targeted. However, as the complexity of the circuit increases it becomes more challenging to simplify the circuit using a system of $\Delta - Y$ and $Y - \Delta$ transforms. Example simplifications of three and four-sector SeMSA cells may be found in section S1 of the supplementary material. It is proposed that rather than attempting to simplify these complex, non-planar circuits that an alternative approach is taken by applying a graph theory method to calculate the impedance between any two nodes of any circuit regardless of its complexity (a two-point impedance model). This method does not necessitate simplification of the circuit.

In this paper the two-point impedance model based on graph theory is first validated against a Kirchhoff's Law simplification of a two-segment SeMSA impedance model. The graph theory method is then applied to multi-sector cylindrical SeMSA designs within a local optimisation routine in order to achieve maximal absorption in the frequency range 20–4500 Hz at a range of cell depths. This method may be applied to any acoustic system where a lumped element model is valid whether the goal is sound absorption or not and is particularly useful for analysing complex acoustic systems with multiple parallel branches per node. A cylindrical cell design has been selected to allow comparison with a previous study of a two-segment SeMSA design [6] and facilitate future experimental tests using a cylindrical impedance tube.

2. Two-point impedance model using graph theory

Graph theory is the study of structures which represent the relations between objects. The graphs that are studied are made up of multiple vertices or nodes and the connections between these nodes which are known as edges or links. Graph theory is applied in a wide range of fields from mathematical topology [10] to the analysis of social media interactions [11] and allows for very complex networks of vertices and edges to be studied.

A graph theory approach may be applied to solve the two-point resistances between arbitrary nodes in a real-valued resistance network regardless of the order or number of dimensions of the circuit [12]. This same approach may be extended to RLC circuits with complex-valued impedances [13]. Electrical circuits can be thought of as finite graphs where the edges are the circuit elements (RLC components) and the vertices are the points between these circuit elements.

Finite graphs are mathematically described by square matrices which represent the connectivity between nodes/edges in the network. The adjacency matrix is a symmetric matrix which describes the admittances (reciprocal of the impedances) between the $\mathcal{N}$ nodes in the network:

$$\mathbf{Y} = \begin{pmatrix} 0 & y_{12} & \dots & y_{1\mathcal{N}} \\ y_{21} & 0 & \dots & y_{2\mathcal{N}} \\ \vdots & \vdots & \ddots & \vdots \\ y_{\mathcal{N}1} & y_{\mathcal{N}2} & \dots & 0 \end{pmatrix} \quad (1)$$

The elements y_{ij} are the admittances between nodes i and j . The degree matrix is a diagonal matrix which describes the total admittance connected to each node in the network:

$$\mathbf{D} = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_{\mathcal{N}} \end{pmatrix} \quad (2)$$

Each element on the main diagonal of the degree matrix d_i is the sum of all admittances connected to node i :

$$d_i = \sum_{j=1}^{\mathcal{N}} y_{ij} \quad \text{where } i \neq j \quad (3)$$

A Laplace matrix $\mathbf{L}$ is formed from the adjacency and degree matrices of the network:

$$\mathbf{L} = \mathbf{D} - \mathbf{Y} \quad (4)$$

In order to calculate the impedance of the network represented by the Laplace matrix $\mathbf{L}$ the following eigenvalue equation is solved:

$$\mathbf{L}^\dagger \psi_\beta = \eta_\beta \psi_\beta, \quad \eta_\beta \geq 0, \quad \beta = 0, 1, \dots, \mathcal{N}. \quad (5)$$

where † denotes the hermitian conjugation, η_β are the eigenvalues and ψ_β are the eigenvectors of $\mathbf{L}^\dagger \mathbf{L}$. Details on the regularisation and potential for singularities when solving this equation will be omitted for brevity, but can be found in the original article [13]. Once this eigendecomposition has been performed the impedance between any two nodes i and j may be calculated as follows:

$$Z_{ij} = \sum_{\beta=2}^{\mathcal{N}} \frac{1}{\eta_\beta} (\psi_{\beta i} - \psi_{\beta j})^2 \quad (6)$$

The impedance of an acoustic system such as a sound absorber which may be described by a lumped-element model may be represented by an electrical circuit analogy, assuming that flow effects and non-linear effects may be assumed to be negligible. The impedance between any two nodes in this equivalent circuit may therefore be calculated using the graph theory method described above. In order to validate this approach, the impedance calculated for a two-segment cylindrical SeMSA absorber is found by simplifying the circuit (see Fig. 2(a)) to a single impedance value using Kirchoff's Law simplifications (see the original article [6] for detail) and compared with the impedance value calculated using Eq. 6. The two impedance values are compared for a range of frequencies in Fig. 2(c). The two impedance values are identical which demonstrates the efficacy of the graph theory approach. This graph theory method is applied to more complex acoustic systems in the following sections.

For an electroacoustic circuit analogy, the reflection coefficient of the material under normal-incidence can be calculated from the two-point impedance calculated from the graph theory method:

$$R_{1\mathcal{N}} = (\hat{z}_{1\mathcal{N}} - \hat{z}_0) / (\hat{z}_{1\mathcal{N}} + \hat{z}_0) \quad (7)$$

where $\hat{z}_0$ is the characteristic specific impedance of the fluid medium. Note the $(\hat{\cdot})$ symbol denotes that this is a specific impedance in units of Rayls or $\text{kg/m}^2\text{s}$. The other impedance elements discussed herein without a $(\hat{\cdot})$ symbol are in units

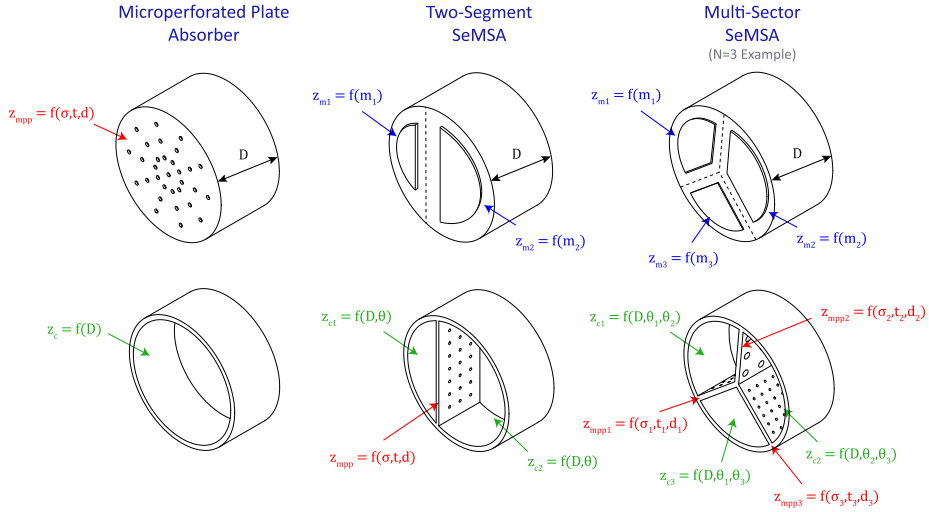


Fig. 3. Isometric view of the cylindrical sound absorbers under investigation: a microperforated plate backed by a sealed air cavity, a two-segment SeMSA design and a multi-sector SeMSA design. The acoustic impedances are labelled for each example.

of $\text{kg/m}^4\text{s}$ and must be converted to a specific impedance value before estimating the reflection coefficient. The normal-incidence acoustic absorption coefficient may be calculated from the reflection coefficient:

$$\alpha = 1 - |R_{1N}|^2 \quad (8)$$

Note that the acoustic impedances, reflection coefficients and absorption coefficients are dependent on the frequency f which has been omitted for brevity.

The key benefit of the graph theory method discussed herein is its simplicity. Alternative methods of analysing acoustic networks such as the generalised two-port mobility-matrix formalism introduced by Glav and Åbom [14] also account for flow effects, making it a very useful model for estimating impedances in flow ducts for example. The acoustic absorbers analysed herein are geometrically compact with respect to the acoustic wavelength and therefore a lumped element model is considered valid.

3. SeMSA with multiple sectors

3.1. Lumped element impedances

The proposed sound absorber design is an evolution of the two-segment SeMSA design to a more complex multi-sector design. The absorber geometry is still cylindrical, but instead of being subdivided into two segments the cylindrical volume is subdivided into N sectors. A cylindrical geometry is used so that the multi-sector SeMSA may be experimentally investigated using a cylindrical impedance tube (in the same manner as the two-segment design [6]), although the study herein is limited to analytical predictions with numerical validation. In order to analytically compare the more complex sectorised SeMSA designs with existing designs, the microperforated plate sound absorber backed by a sealed air cavity examined by Maa [15] as well as the two-segment SeMSA design are used as benchmarks. The three absorber types investigated are shown schematically in Fig. 3.

In a lumped element system the individual lumped impedances can be thought of as both the building blocks and tuneable elements of the overall system. For example, a woodwind instrument can be analysed using lumped elements to predict the tonality and timbre of its response with different fingerings [16]. For an absorber we seek to tune the elements like a musical instrument to achieve a desired absorption response. There are three building blocks we use in the microperforated plate absorber and SeMSA designs (each of these acoustic impedance terms have units $\text{kg/m}^4\text{s}$):

$$z_{mi} = \zeta + \frac{j\omega_0 m_i}{S_i^2} \quad (9)$$

$$z_{ci} = \frac{k_{ci}}{j\omega_0} \quad \text{where} \quad k_{ci} = \frac{c_0^2 \rho_0}{S_i D} \quad (10)$$

$$z_{mppi} = -\hat{z}_0 \left(\frac{jk_0}{\sigma_i} \frac{J_0\left(\frac{d_i}{2} \sqrt{\frac{-j\omega_0}{v_0}}\right)}{J_2\left(\frac{d_i}{2} \sqrt{\frac{-j\omega_0}{v_0}}\right)} [t + 0.85d_i \cdot \Psi(\sigma_i)] \right) / S_{mpp} \quad (11)$$

Table 1

Definitions of symbols representing the physical parameters listed in Table 2 for the acoustic absorbers shown in Fig. 3.

Symbol	Physical parameter	Units	Minimum value	Maximum value
N	Number of subdivisions in multi-sector cell		3	7
R	SeMSA cell radius	mm	20	40
D	SeMSA cell depth	mm	10	50
σ_i	Porosity of plate i		0	0.25
t_i	Thickness of plate i	mm	0.1	1
θ_i	Angle between plate i and plate $(i - 1)$	rads	$2\pi/N$ 0.25($2\pi/N$)	$2\pi/N + 0.25(2\pi/N)$
d_i	Diameter of holes in plate i	mm	0.2	1
m_i	Mass of membrane & added mass at sector i	g	0.02	1
ζ	Damping inside membrane	kg/m ⁴ s	2.78×10^5	2.78×10^5

Table 2

Parameter space for the sound absorbers under study (see Fig. 3) for a fixed cell radius R : a microperforated plate backed by a sealed air cavity, a two-segment SeMSA design and an N -sector SeMSA design where $N > 2$.

Absorber type	Z_m Parameters	Z_c Parameters	Z_{mpp} Parameters	No. of parameters
Microperforated Plate	–	D	$\sigma; t; d$	4
Two-Segment SeMSA	$m_1, m_2; \zeta$	$D; \theta$	$\sigma; t; d$	8
Three-Sector SeMSA	$m_1, m_2, m_3; \zeta$	$D; \theta_1, \theta_2$	$\sigma_1, \sigma_2, \sigma_3; t_1, t_2, t_3; d_1, d_2, d_3$	16
Four-Sector SeMSA	$m_1, m_2, m_3, m_4; \zeta$	$D; \theta_1, \theta_2, \theta_3$	$\sigma_1, \sigma_2, \sigma_3, \sigma_4; t_1, t_2, t_3, t_4; d_1, d_2, d_3, d_4$	21
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
N -Sector SeMSA	$m_1, m_2, \dots, m_N; \zeta$	$D; \theta_1, \dots, \theta_{N-1}$	$\sigma_1, \dots, \sigma_N; t_1, \dots, t_N; d_1, \dots, d_N$	$5N + 1$

where ω_0 is the angular frequency, $\hat{z}_0$ is the characteristic impedance of the fluid medium, ν_0 is the kinematic viscosity of the fluid medium, k_0 is the wavenumber, ρ_0 is the density of the fluid medium, $j = \sqrt{-1}$ and J_n are the n th order Bessel functions. The Fok function $\Psi(\sigma_i)$ accounts for hole-to-hole interactions [17]. Many alternative estimations of the acoustic impedance of a microperforated plate impedance are available; Eq. (11) was selected as it matches the implementation used by the COMSOL's interior perforated plate boundary condition [18] which eases comparisons between the analysis herein and a numerical model. This microperforated plate impedance model assumes that thermal effects are negligible compared to viscous effects, plate vibrations are negligible and non-linear effects are ignored. The equations describing impedances Z_m and Z_c may be found in Merhaut [8].

The cavity area term S_i varies depending on the type of absorber being investigated:

$$S_i = \begin{cases} S_{\text{cell}}, & \text{for a microperforated plate absorber, where } i = 1. \\ 0.5(\theta - \sin(\theta))R^2, & \text{for a two segment SeMSA design, where } i = 1. \\ S_{\text{cell}} - 0.5(\theta - \sin(\theta))R^2, & \text{for a two segment SeMSA design, where } i = 2. \\ S_{\text{cell}}(\theta_i/2\pi) & \text{for all multi-sector SeMSA designs.} \end{cases} \quad (12)$$

where $S_{\text{cell}} = \pi R^2$. Definitions for all other terms may be found in Table 1. All parameters are geometric and therefore freely tuneable with the exception of ζ , the damping factor inside the membrane. This parameter must be estimated empirically. In this study the damping parameter for 0.2 mm latex membrane as determined by McKay et al. [6] is applied. The parameter space increases as more sectors are added as per table 2. The microperforated plate thickness can be as low as 0.1 mm which would make the assumption of negligible plate vibrations inaccurate, however in a practical implementation a thicker plate may be used whose plate thickness is reduced locally to the microperforations by counter-boring larger diameter holes in series with the microperforated holes. For more details on this manufacturing process see McKay et al. [6].

3.2. Effect of cell size on the impedance of a SeMSA absorber

For the case of a circular cell, the area of the MPPs is proportional to the cell radius but the area of the membranes and projected area of the cavities are both proportional to the square of the radius. Eqs. (9)–(11) then show that $Z_{mi} \propto 1/r^4$ (or $Z_{mi} \propto 1/r^2$ if the areal density, m_i/S_i , is kept constant), $Z_{ci} \propto 1/r^2$ and $Z_{mppi} \propto 1/r$. This means that as the cell radius is changed, the impedances of the cell's components do not all change at the same rate which leads to a change in absorption response with cell size.

For most other SeMSA arrangements a similar argument holds: the size of the cell cannot be increased in a way that causes all the membrane areas and MPP areas to grow at the same rate. One counterexample is that of a rectangular two-segment SeMSA which is allowed to grow along one dimension to which the MPP is aligned and the areal density of the membrane is kept constant.

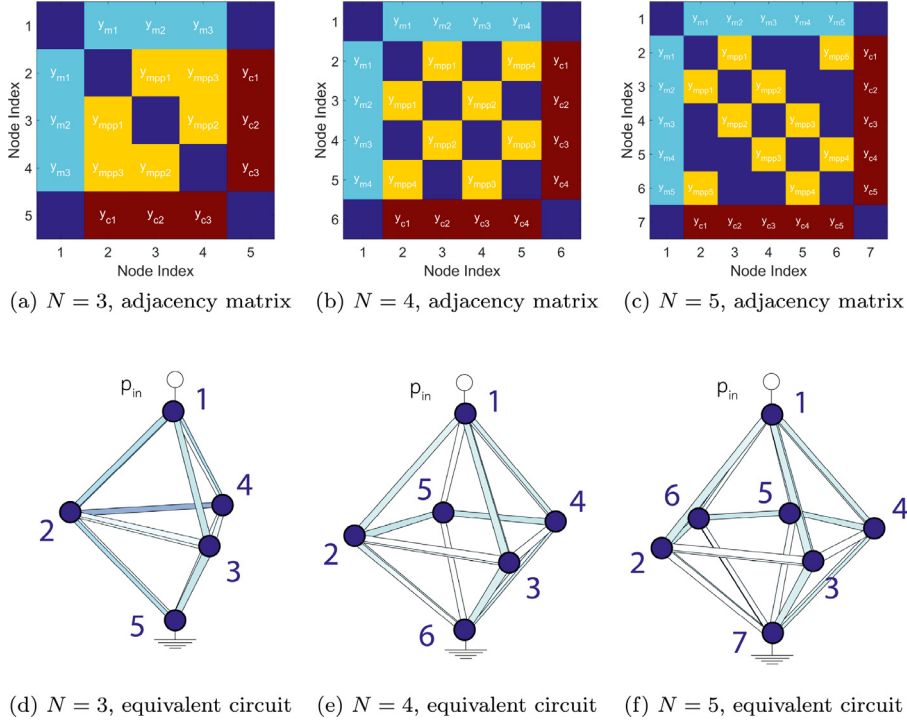


Fig. 4. Adjacency matrices and circuit layouts for multi-sector SeMSA designs. The vertical connections between node 1 and nodes $(2, \dots, N+1)$ have acoustic impedances z_{mi} . The vertical connections between node $N+2$ and nodes $(2, 3, \dots, N+1)$ have acoustic impedances z_{ci} . The horizontal connections joining nodes $(2, \dots, N+1)$ have acoustic impedances z_{mppi} .

3.3. Graph theory model

The Laplace matrix representation of the impedances of the equivalent circuits for the absorber designs under investigation will have an increasing number of nodes $\mathcal{N}$ as the number of designs parameters increases:

$$\mathcal{N} = \begin{cases} 2, & \text{for a microperforated plate absorber.} \\ N+2 & \text{for all SeMSA designs.} \end{cases} \quad (13)$$

The adjacency matrices for a range of multi-sector SeMSA designs are shown in Fig. 4. This demonstrates that the non-planar equivalent circuits for multi-sector SeMSA designs have a bipyrmaid structure. Note that the Laplace matrix approach to estimating the impedances is not limited to the SeMSA designs but could be applied to any acoustic system regardless of the shape and layout of its equivalent circuit, assuming a lumped-element model is valid.

4. Causal optimality

The acoustic causality constraint [7] tells us the theoretical minimum depth that can achieve a given absorption spectrum $\alpha(\lambda)$, where where α is the normal-incidence absorption coefficient and λ is the acoustic wavelength:

$$D \geq \frac{1}{4\pi^2} \frac{B_{\text{eff}}}{B_0} \left| \int_0^\infty \ln[1 - \alpha(\lambda)] d\lambda \right| = D_{\min} \quad (14)$$

where B_{eff} represents the bulk modulus of the sound absorber in the static limit. For the microperforated plate and SeMSA designs under investigation this static limit is equal to the bulk modulus of air B_0 because the membrane physics is accurately represented as a limp mass (under the assumption of no pre-tension and negligible bending stiffness). If the limp membrane was tensioned or stiff then B_{eff} would be equal to B_0 plus the additional stiffening effect. According to Eq. (14) it is not possible to achieve broadband perfect absorption in a finite thickness but it is possible to have peaks where the absorption is perfect at a single point. The equation also shows us that the minimum thickness of an absorber to achieve high absorption in a wide frequency band is dominated by the absorption response at low frequencies/large wavelengths.

In reference to the absorption coefficient spectra for the the acoustic absorbers under investigation, it is prudent to compare the actual depth of the absorber D used to achieve a given absorption spectrum $\alpha(f)$ with the theoretical minimum depth achievable as calculated by Eq. (14):

$$\Delta D = D_{\min}(\alpha) - D \quad (15)$$

As ΔD tends towards zero the absorption spectrum is closer to the causal limit that is achievable at the prescribed depth. For a single absorber design we can calculate the mean offset from the actual absorber depth and the causally optimal minimum depth:

$$\Delta D_m = \frac{1}{N_D} \left(\sum_{i=1}^{N_D} (D_{\min}(\alpha_i) - D_i) \right) \quad (16)$$

where i is the index of the depth of the absorber, α_i is the absorption spectrum generated by the absorber under investigation at depth i and N_D is the number of absorber depths investigated. In the results discussed below 21 absorber depths are tested from 10 mm to 50 mm at 2 mm increments.

5. Objective function for optimisation

The performance of an acoustic absorber may be quantified by linear averaging of the absorption coefficient spectrum in a defined frequency range to give an overall absorption coefficient, $\bar{\alpha}$. This approach was previously applied to the two-segment SeMSA design in the frequency range 20–1200 Hz. In order to extend this analysis, the overall absorption coefficient may be considered alongside the depth of the absorber (D) and how closely it reaches the causal limit (ΔD) to further quantify its effectiveness. Furthermore, simply averaging the absorption coefficient spectrum is only suitable for characterising the effectiveness of the absorber for spectrally-flat (white noise) absorption. If the targeted sound source for absorption has a non-white noise spectrum it is sensible to consider the power spectrum of the sound source when estimating an absorber's effectiveness.

Consider a source sound whose sound spectrum is described by the discrete auto-spectral density function $G_{xx}[k]$ where k is the frequency bin index. The sound power reduction that a sound absorber with a discrete absorption spectrum $\alpha[k]$ would achieve in Decibels is given by:

$$C_\alpha = 10 \log_{10}(1 - c_\alpha) \quad (17)$$

where c_α is the overall absorption coefficient weighted by the sound spectrum of the targeted noise source:

$$c_\alpha = \frac{\left(\sum_{k=k_1}^{k_2} \alpha[k] G_{xx}[k] \right)}{\left(\sum_{k=k_1}^{k_2} G_{xx}[k] \right)} \quad (18)$$

where k_1 is the frequency bin index of the lowest frequency and k_2 is the frequency bin index of the highest frequency in the bandwidth of interest.

The sound power reduction factor C_α provides a suitable cost function that can be minimised in an optimisation routine in order to generate the parameter set for a suitable N-SeMSA design. The steps for this optimisation are as follows:

1. The cell depth D , cell radius R and number of subdivisions N are prescribed/fixed.
2. An initial parameter set is selected as the mean values of the remaining free parameters in Table 2.
3. An optimisation routine (interior-point stochastic gradient descent algorithm) minimises the cost function C_α in Eq. (17) by modifying the free parameters within the bounds specified in Table 1. The absorption spectrum α is estimated from the acoustic impedance $z_{1,N}$ calculated from Eq. (6).
4. The optimal absorption spectrum is returned by the optimisation routine as well as the parameter set that achieved this result.

This procedure ensures that the optimisation routine generates the ideal SeMSA geometric parameters for the target sound spectrum.

6. Results

6.1. Analytical simulations

Fig. 5 shows the predicted absorption responses of the three absorber types tested: a microperforated plate, a two-segment SeMSA design and a range of multi-sector SeMSA designs with ($N = 3, \dots, 6$). The absorber size was fixed with $R = 40$ mm. The cost function in Eq. (17) has been minimised in order to maximise white noise absorption in the 20–4500 Hz frequency range ($G_{xx}[k] = 1$, for all k). This optimisation procedure was repeated for a range of absorber depths. The minimum depth (10 mm) case is shown in blue, the maximum depth (50 mm) case is shown in red and the intermediate depths are graded between blue and red. As the absorber depth is increased the absorption bandwidth extends to lower frequencies for each design, as is predicted by Eq. (14).

For the microperforated plate design, the bandwidth of the absorber does not extend as the absorber depth increases but the frequency response shifts to lower frequencies, bringing more absorption into the frequency range under study. For

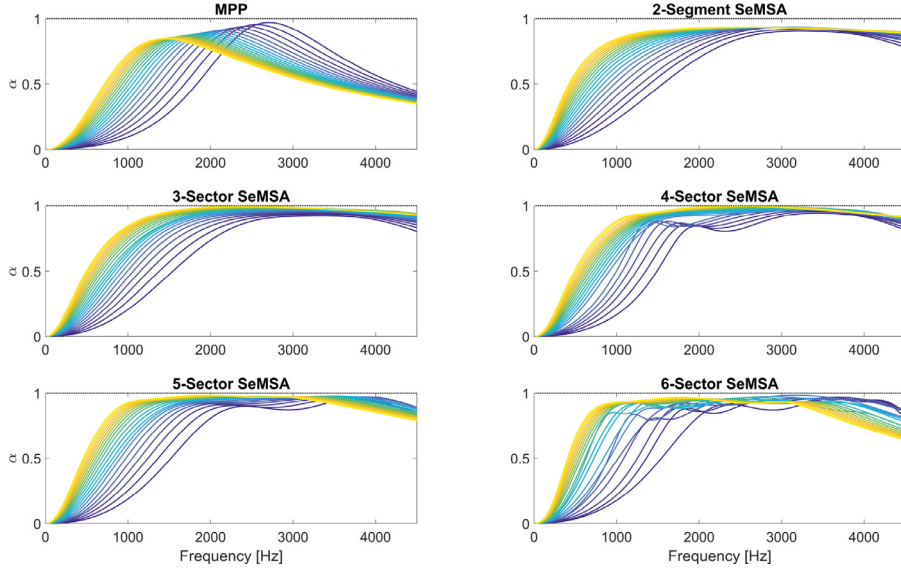


Fig. 5. Absorption spectra for a range of absorber depths from 10 mm (dark/blue) to 50 mm (light/yellow). Target spectrum = white noise. Frequency range = 20–4500 Hz. $R = 40$ mm. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

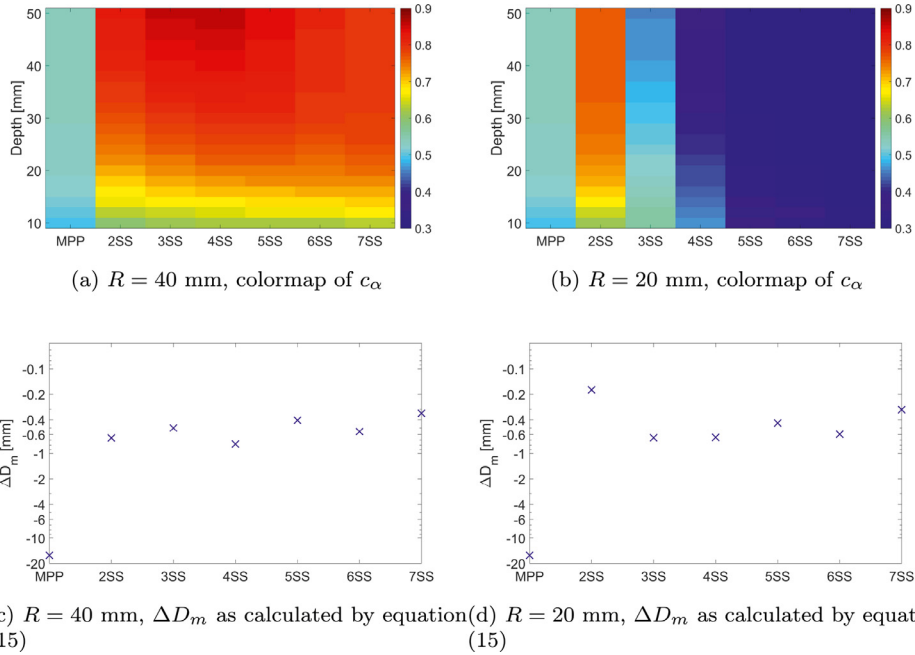


Fig. 6. Overall absorption coefficient and causal optimality of a microperforated plate (MPP), two-segment SeMSA (2SS) and multi-sector SeMSA (NSS, where $N > 2$) absorber. Target spectrum = white noise. Frequency range = 20–4500 Hz. Each curve is the absorption response at a different depth from 10 mm to 50 mm in 2 mm increments.

all the SeMSA designs the bandwidth extends considerably as the absorber depth increases. It is difficult to discern which SeMSA design performs best in terms of overall absorption by observing the absorption curves, so the overall absorption weighted by the input sound spectrum (c_α) is shown as a function of absorber depth in Fig. 6(a). For white noise c_α is equal to the linear average of the absorption curves shown in Fig. 5. The overall absorption increases markedly between the microperforated plate absorber (MPP) and all SeMSA designs. Between all SeMSA designs the overall absorption is quite similar, but peaks for a multi-sector design with $N = 4$ at the maximum depth. At smaller absorber depths closer to the minimum the absorption response is higher when more sectors are added. The overall absorption increases monotonically with absorber depth in each case.

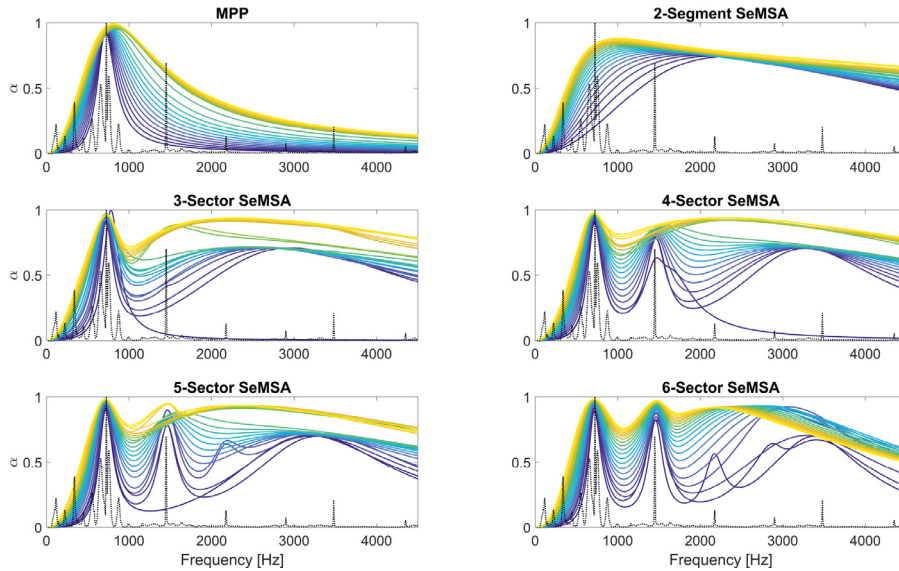


Fig. 7. Absorption spectra for a range of absorber depths from 10 mm (dark/blue) to 50 mm (light/yellow). Target spectrum = fan noise. Frequency range = 20–4500 Hz. $R = 40$ mm. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

The optimisation routine was repeated for a smaller absorber design with $R = 20$ mm which demonstrates very different behaviour to the $R = 40$ mm absorbers, see Fig. 6(b). In this example the two-segment SeMSA (2SS) clearly outperforms the other designs in terms of overall absorption at all depths. The SeMSA absorption performance decreases considerably as more sectors are added and actually performs worse than the MPP for $N > 3$. The individual sectors are constrained to smaller volumes as the number of sectors increases which limits the ability of the optimisation routine to extend the absorption response of the multi-sector absorber designs within this smaller overall absorber volume. Note that the microporated plate (MPP) absorber response is identical for the $R = 20$ mm and $R = 40$ mm cases since the impedance of an MPP absorber is unaffected by the cross-sectional dimensions of the absorber.

Fig. 6 (a) and (b) quantify the absorption performance of the absorber designs and demonstrate the benefits and potential limitations to adding more sectors to a SeMSA design. However, it is unclear by analysing the absorption coefficients alone whether we have achieved close to the theoretical limits of absorption performance at each absorber depth. In order to assess this, Eq. (16) was calculated using each of the absorption spectra in Fig. 5 as inputs. This quantifies the difference between the actual absorber depth and the minimum absorber depth that can theoretically generate the same absorption response, see Fig. 6(c) and (d). For the MPP absorber we see that the absorber is, on average, 18 mm from the causal optimum minimum depth to achieve the same absorption response. This shows a great deal of room for improvement, which explains how SeMSA is able to achieve significantly higher absorption responses within the same depth. For the SeMSA designs the ΔD_m value remains below 1 mm for all designs for both absorber sizes. Interestingly, this holds true even for multi-sector SeMSA designs which show lower overall absorption responses for $R = 20$ mm. This is due to the fact that the multi-sector SeMSA absorbers generate absorption peaks at relatively low frequencies which are close to what is theoretically possible at each depth, however the bandwidth of these absorption peaks is small and leads to lower c_α values. The absorption spectra for the $R = 20$ mm absorber designs may be found in section S2 of the supplementary material.

In real acoustic mitigation scenarios the target sound source(s) for absorption may not be spectrally flat as analysed above. The optimisation routine is repeated as above with the cost function (see Eq. (17)) modified by a change of input spectrum $G_{xx}[k]$. In this case the auto-spectral density function $G_{xx}[k]$ estimated for a compact axial fan (Sanyo Denki San Ace, 120 mm diameter) from microphone measurements taken inside an anechoic chamber is used to weight c_α in Eq. (18). The analysed frequency range is maintained at 20 Hz to 4500 Hz. The absorption spectra for a range of optimised $R = 40$ mm absorbers are shown in Fig. 7 that target the acoustic signature of this axial fan. The normalised auto-spectral density function of the fan is also shown in each sub-figure with black-dashed lines.

The absorption spectra for the fan noise case demonstrate very different behaviour when compared to the white noise example. Fan noise shows strong tonal peaks at the blade-pass frequency of the fan and its harmonics which is at 683 Hz for this fan design at this rotational speed. At low frequencies close to the blade-pass frequency (500–750 Hz) there is also considerable broadband noise generation. The optimisation routine produces absorption spectra which trade the wide bandwidth absorption observed for the white noise case for a narrowband absorption response centred close to the blade-pass frequency. This trade-off is illustrated in the causality constraint in Eq. (14). As more sectors are added to the multi-sector designs the absorption response may be further tuned to exhibit peaks of absorption at the harmonics of the fan blade-pass frequency. These additional absorption peaks allow the multi-sector design to achieve an overall absorption response that

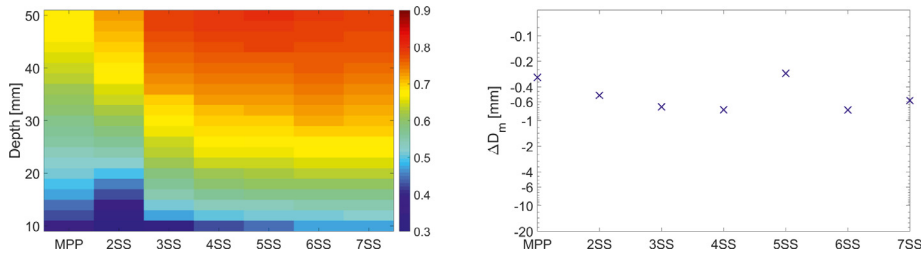


Fig. 8. Colormap of overall absorption coefficient c_α (left). Delta between the causally optimal minimum distance and the SeMSA cell depth for the absorption spectra shown in the previous section (right). Target spectrum = axial fan. Frequency range = 20–4500 Hz. $R = 40$ mm.

increases as more sectors are added, see Fig. 8. Interestingly the two-segment SeMSA design delivers the worst absorption response at low depths as the optimisation routine struggles to locate significant absorption close to the fan blade-pass frequency. As the absorber depth increases the superior bandwidth of the two-segment absorber leads to higher overall absorption than for the MPP design.

Given the wide bandwidth of sound being targeted for absorption in this study, the acoustic absorption performance of a bulk porous absorber is also considered for comparison. Absorption spectra for a melamine acoustic foam was extracted from a manufacturer's data sheet [19] and c_α was calculated for both the white noise and fan noise cases. For a 25 mm thick melamine foam sample c_α was estimated as 0.62 for white noise absorption and 0.48 for fan noise absorption which is considerable lower than the absorption values observed in Figs. 6 and 8.

6.2. Numerical simulations

The analytical predictions of the absorption responses of both the MPP and SeMSA absorbers described above are based on the graph theory approach described earlier in this article. In order to validate that these analytical predictions are in fact accurate predictions of the real absorption responses, the analytical predictions are compared with numerical simulations using COMSOL Multiphysics. However, the membrane model used within the simulation accounts for damping inside the membranes using a different physical model to an acoustic impedance value as listed in Table 1. To amend this, the effect of membrane damping value was not factored into the simulations i.e. any acoustic absorption occurs in the holes of the microperforated plates. In order to align the analytical predictions with this undamped model, the optimisation routine was run with $\zeta = 0$ and all other parameters in Table 1 were left free in the optimisation. The target spectrum to minimise the cost function in Eq. (17) is the axial fan spectrum as discussed above.

Fig. 9 (a) shows the geometry of the three sector SeMSA design that has been generated by the optimisation routine. The details of this model may be found in the Methods section below. Fig. 9(b) compares the absorption spectrum for this 10 mm deep absorber as predicted using the graph theory method (analytical) and as calculated by the COMSOL model (numerical). The two spectra show excellent agreement. Fig. 9(c) shows the integrated dissipation in the three microperforated plates (MPPs) inside the SeMSA cell, and the acoustic pressure amplitudes averaged over each sector are shown in Fig. 9(d). The annotations in Fig. 9(a) indicate the locations of MPPs and sectors 1–3. Each of the absorption peaks is associated with a spike in power dissipation in the microperforated plates which is driven by high acoustic velocities through the perforations driven by the motion of the membranes. Each sector's resonance also corresponds to a sharp increase in the acoustic pressure in all sectors, demonstrating the coupling of the acoustic system between sectors. The highest acoustic pressure amplitude is observed in a different sector at each resonance frequency. Fig. 9(e)–(g) show the acoustic pressure fields inside the SeMSA cell at the three resonance frequencies. These images illustrate the uniformity of the acoustic pressure field in the three sectors. This uniformity demonstrates the validity of the lumped-element model for analysing the physics of the SeMSA cell and explains the close agreement in the absorption spectra in Fig. 9(b).

7. Discussion

The presented graph theory method for calculating the two-point impedance of a circuit may be applied to any acoustic system e.g. waveguide, absorber or barrier where a lumped-element model is valid. The method has been applied to model the response of an optimised acoustic absorber under consideration of absorber size and the spectrum of the targeted sound source for absorption, generating the geometric parameters to maximise overall absorption. This method ensures that the most effective absorber possible per unit volume is generated for each absorber design investigated, as evident by the high overall absorption values and close proximity to causal optimal depths for the generated absorption spectra. Fundamentally the causal optimality constraint ensures that no absorber can exhibit perfect broadband response in a finite volume.

This study has demonstrated that adding sectors can improve the absorption response of SeMSA, however the ideal SeMSA design is generally dependent on the cross-sectional size of the absorber. For white noise absorption with a $R = 20$ mm absorber the highest absorption response is observed with a two-segment design. However, the absorption response of a larger ($R = 40$ mm) SeMSA design peaks for a four-sector design at higher absorber depth and a seven-sector design

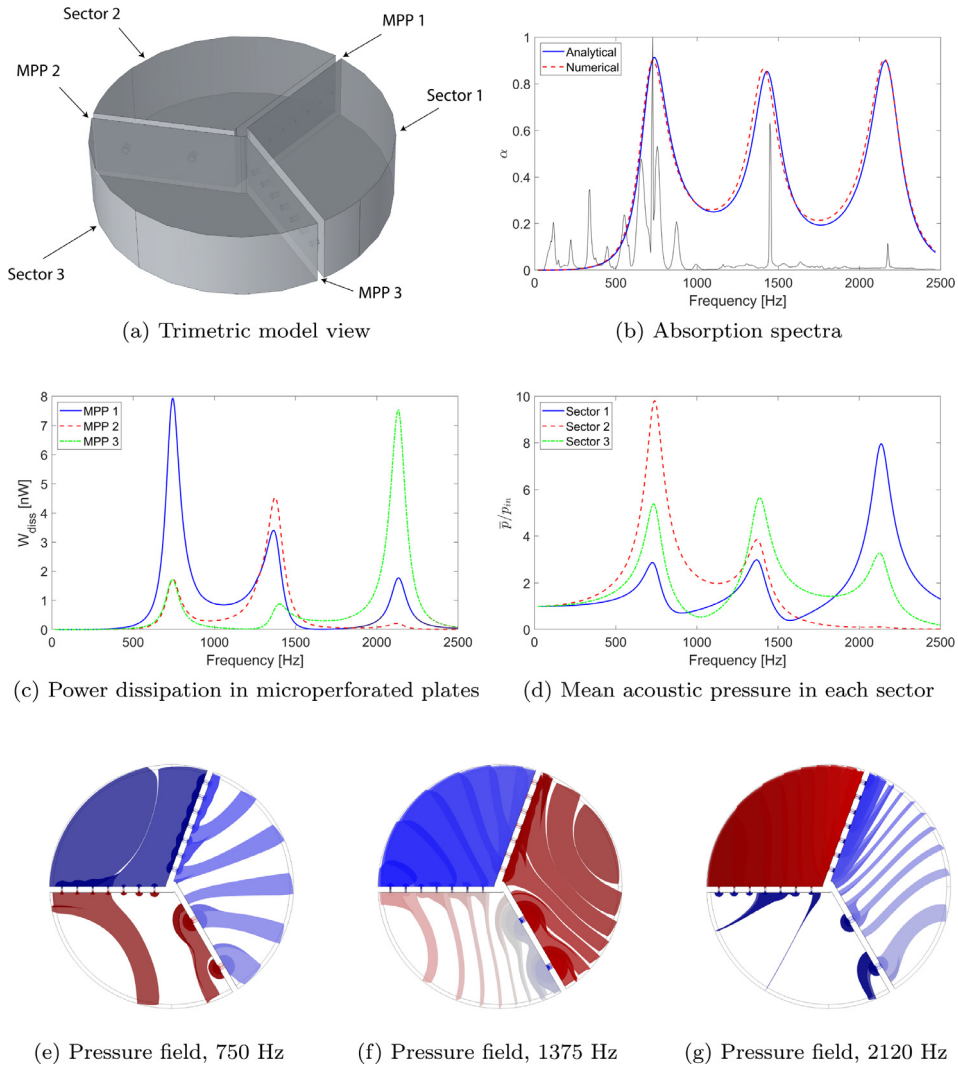


Fig. 9. Numerical simulation results for a three sector SeMSA cell, $R = 20$ mm, $D = 10$ mm. Target spectrum = axial fan. Frequency range = 20–2500 Hz. Analytical: $C = -2.91$ dB. Numerical: $C = -2.95$ dB.

at lower absorber depths. There is no single absorber which delivers the best absorption performance across all scenarios investigated; for practical implementation, the correct choice must be made under consideration of available space and the characteristics of the sound source. Furthermore, the manufacturing complexity of the absorber must be taken into consideration as well as the absorption performance. Spectral weighting functions such as A-weighting may also be factored into the cost function to account for psychoacoustic effects of sounds at different frequencies. Adding more sectors may increase the absorption bandwidth which is especially beneficial for the noise absorption of broadband sources, however manufacturing these absorbers at scale may be challenge without additive manufacturing.

The proximity of the absorber depth to the causal optimal minimum depth (ΔD) is not a sufficient predictor of a high-performing absorber in terms of overall absorption. However, this value does indicate that the absorber's low-frequency response is close to maximum for the prescribed absorber depth. In this analysis the back walls of the absorbers has not been factored into the absorber depths D . In accordance with the mass law, the ideal wall material should have a high density in order to keep the overall absorber volume as compact as possible while ensuring minimal sound leakage. It would be possible to extend the graph theory analysis to include the effects of this sound leakage with a given set of material properties and to optimise the absorber wall thicknesses for a finite volume.

The overall impedance of the absorbers has been the focus of this study, however it would also be possible to analyse the impedance between any pair of nodes within the equivalent circuit. This could be used for example to quantify losses through specific elements such as individual microperforated plates and membranes. This study has also focused on normal-

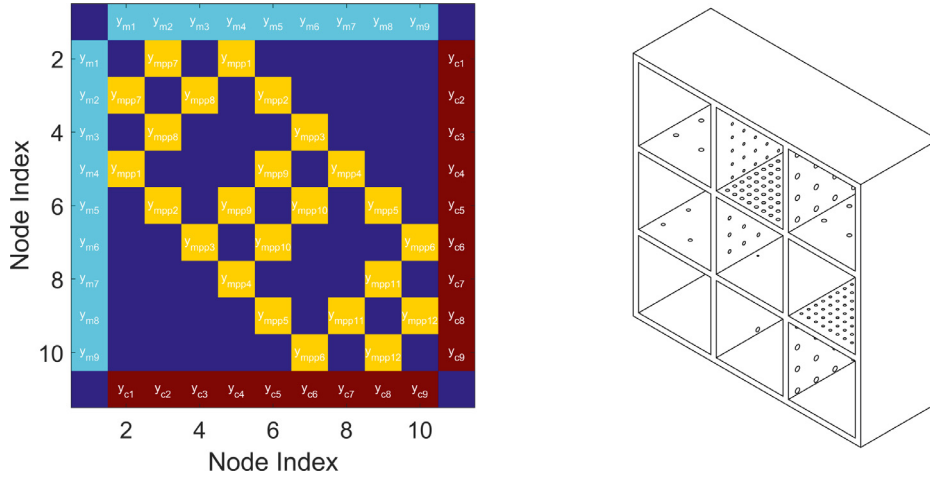


Fig. 10. Adjacency matrix (left) for a 3×3 rectangular grid form of SeMSA (right).

incidence absorption but could be extended to grazing incidence sound by applying a different potential at different points of the equivalent circuit. This analysis could be applied to acoustic liners.

The sound absorbers analysed in this study have all been cylindrical in order to aid future experimental investigations using a cylindrical impedance tube. An additional example of a rectangular grid SeMSA absorber is shown in Fig. 10. This geometry has the advantage that it can tessellate. The central rectangular sub-volume in the grid has four microperforated plates joining neighbouring sub-volumes which will further complicate the equivalent circuit. The graph theory method is therefore ideal for analysing such a complex design, which represents just one of a wide array of potential future possibilities for study using the methodology outlined in this article.

8. Methods

8.1. Analytical optimisation routine

The analytical optimisation routine was performed in MATLAB using the *fmincon* function for increasing depths, D . In order to prevent errors in the minimisation of the cost function where the solution converges on a local minima which is not globally optimal, a number of checks were in place. If, at the end of the optimisation routine at a given D the solution does not exceed the following thresholds:

- $c_\alpha > 0.1$, as calculated by Eq. (18),
- $\Delta D_m > -1$ mm, as calculated by Eq. (15).

the optimisation routine at the current depth is re-initialised with a random combination of the free parameters in the ranges defined by Table 1.

8.2. Numerical simulations

The numerical models investigated in this study were analysed in COMSOL Multiphysics in the frequency domain. The top faces of the model were modelled as a linear elastic membrane with the material properties of latex rubber. The membrane is fixed along the perimeter of each sector. Additional mass was added to the membranes covering each sector to match the prescribed m_i terms given by the analytical optimisation routine. A normally-incident, uniform acoustic pressure of 0.1 Pa was applied to each membrane for every frequency tested.

Each sector cavity in the cell was modelled using pressure acoustics domains (no thermoviscous losses are included). The microperforations were modelled using thermoacoustics domains in order to capture the thermoviscous losses inside the perforations with a boundary layer mesh. Twenty mesh nodes were placed inside the viscous penetration depth at the highest frequency tested in order to accurately capture the viscous boundary layer. The thermoacoustics domains were extended 0.5 mm either side of the ends of the perforations in order to capture end effects. The absorptive losses inside the SeMSA cell were quantified by integrating the total thermoviscous losses inside the thermoacoustic domains. All other boundaries in the simulation were modelled as hard walls i.e. have infinite acoustic impedance.

Additional information

The authors declare that the research was conducted in the absence of any relationships that could be construed as a potential conflict of interest.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRediT authorship contribution statement

Ian Davis: Conceptualization, Methodology, Software, Visualization, Writing - original draft. **Andrew McKay:** Investigation, Software, Data curation, Writing - review & editing. **Gareth J. Bennett:** Supervision, Writing - review & editing.

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Supplementary material

The supplementary material document includes the following:

- Examples of the Kirchoff's Law simplifications of the biyramid equivalent circuits for a three- and four-sector SeMSA design.
- Additional absorption spectra for a wide combination of parameters such as SeMSA cell radius, absorber depth, targeted frequency range and targeted sound spectrum.
- Colormaps of (c_α) and plots of ΔD_m for all the above.
- An additional numerical simulation for a four-sector SeMSA design, further validating the efficacy of the graph theory method.

Supplementary material associated with this article can be found, in the online version, at [10.1016/j.jsv.2021.116135](https://doi.org/10.1016/j.jsv.2021.116135)

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