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Deformed Conformal Field Theory and the Symmetric Group

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Abstract

By a proposal of Dijkgraaf, the generating function of simple Hurwitz numbers admits a striking reformulation: it is the partition function of a $W_{1+\infty}$-deformed conformal field theory. We make this claim explicit via a series of Fock space isomorphisms. In so doing, we discover a Clifford and Heisenberg action on the class functions of the symmetric group, easily computable by a Young-diagrammatic calculus. Finally, to present the advantages of the physical perspective, we elaborate on Zagier's recent proof of the quasimodularity of the Bloch-Okounkov q-bracket to show that, in a CFT context, quasimodularity is a consequence of spin structures and spectral flow.

Declaration

I declare that this thesis has not been submitted as an exercise for a degree at this or any other university and it is entirely my own work. I confirm that the work has been completed in full compliance with relevant university policies, including the Academic Integrity policy, the Trinity Policy on Good Research Practice, and the guidance on AI and GenAI in Teaching, Learning, Assessment and Research.

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Luke Hamilton

24th April 2026

I would like to dedicate this thesis to my mother.

Summary

This thesis elaborates on an observation of Dijkgraaf in [Dij95], namely, that deforming a free Conformal Field Theory (CFT) by higher-spin currents lends its partition function a topological significance: the CFT correlators are none other than Hurwitz Numbers, which count covers of Riemann surfaces. To justify this coincidence without recourse the original string-theoretical arguments [GT93], [Dou95], we start with the Hurwitz generating function as calculated by the representation theory of the Symmetric Group, and express it as a trace over the class algebra of the symmetric group. By a series of isomorphisms of Vertex Operator Algebras (VOAs) we will make physical sense of this trace as the partition function of a CFT deformed by $W_{1+\infty}$ currents.

Throughout the thesis, we use the graphical calculus of Young Diagrams, whose original purpose was to index characters of the symmetric group. One new result we obtain is a 'quantisation' of characters: namely, the presence of a Heisenberg algebra, which adds or removes hooks in their corresponding Young diagrams. We formulate a Boson-Fermion Correspondence in the symmetric group to obtain a Clifford algebra acting diagonally on characters, which adds or removes arms or legs in their Young diagrams. This gives the required isomorphism of algebras; to write an isomorphism of Fock spaces, we map from the Young diagram associated to a character to the partition associated to a fermionic/bosonic state; we map the offset of an off-diagonal 'cut' of the Young diagram to the charge of a fermionic/bosonic state.

With the VOA isomorphism in hand, we follow Dijkgraaf's original argument to write the Hurwitz generating function as a sum of eigenvalues of certain operators. It turns out these eigenvalues, too, are evaluated in terms of Young diagrams: they are the difference of power-sums of the arm-lengths and leg-lengths, relative to the 'cut' of the Young diagram. We then proceed to write equivalent operators which live in the $W_{1+\infty}$ VOA. Hence we

equate the Hurwitz generating function and a character of the $W_{1+\infty}$ VOA.

The final two chapters of this thesis expand upon the physical significance of the preceding mathematics. First, we give a physical proof of the Jacobi triple product, by setting equal the partition functions of the free boson/fermion CFT, whose VOAs are isomorphic by the Boson-Fermion correspondence. We confirm the modularity one expects in a CFT partition function [Zhu96]. However, the $W_{1+\infty}$-deformed CFT does not enjoy this behaviour, which we demonstrate explicitly using results from [Dij97] and [GLT25]. Finally, we exhibit the phenomenon of QUASIMODULARITY. Usually this is proven using classical analytic techniques, e.g. [KZ95], [BO00]; instead we cast Zagier's new proof [Zag16] in the CFT context. We find that quasimodularity is a consequence of spectral flow and spin structures on the elliptic curve.

Contents

Introduction

This thesis answers the question: how does the partition function of a free, $c = 1$, genus 1 CONFORMAL FIELD THEORY (CFT) change, once its action is deformed by a higher spin current? For a clue, consider the well-known JACOBI TRIPLE PRODUCT:

$$\underbrace{q^{-\frac{1}{24}} \prod_{n \in \mathbb{Z}^{\geq 0}+\frac{1}{2}} (1 + zq^n)(1 + z^{-1}q^n)}_{\text{Fermionic}} = \underbrace{\frac{1}{\eta(q)} \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}n^2} z^n}_{\text{Bosonic}}.$$

This formula is an incarnation of the famous BOSON-FERMION CORRESPONDENCE—the CFT partition function $\mathrm{tr}(q^{L_0 - 1/24} z^{\alpha_0})$ can be evaluated as a trace over the either fermionic Fock space *or* bosonic Fock space. In [Dij95], a formula very similar to Jacobi's product is presented:

$$q^{-\frac{1}{24}} \prod_{n \in \mathbb{Z}^{\geq 0}+\frac{1}{2}} \left(1 + zq^n y^{\frac{1}{2}n^2}\right) \left(1 + z^{-1}q^n y^{-\frac{1}{2}n^2}\right). \tag{$*$}$$

Dijkgraaf interprets this product as a higher-spin partition function $\mathrm{tr}(y^{W_0} q^{L_0 - 1/24} z^{\alpha_0})$ over fermionic Fock space. If one evaluates instead over the charge 0 Fock bosonic space, this trace is the generating function of the simple HURWITZ NUMBERS!

This coincidence was originally explained with recourse to string theory [GT93], [Dou95]. We choose to work mathematically, by phrasing Dijkgraaf's formula as a trace over a different Fock space: the class algebra of the SYMMETRIC GROUP. The symmetric group is the natural setting to calculate Hurwitz numbers, and Dijkgraaf's observation gives us the evidence that the symmetric group admits a physical interpretation. For example, we will find the 'cut-and-join' operator of the Hurwitz literature [Gou94] is exactly the higher-spin operator W_0.

In this thesis we justify the above mixed metaphors, by a series of isomorphisms of VERTEX OPERATOR ALGEBRAS (VOAs). These are similar to the famous Vertex Algebras of [Bor86], but enjoy additional Virasoro structure, which allows them to axiomatically describe aspects of 2d CFT. It is through this technology that we construct the infinite dimensional Lie algebra $W_{1+\infty}$, in which the higher-spin operators live. Let us briefly describe the route we take.

Chapter 1: The Symmetric Group and Dijkgraaf's Formula

In Chapter 1, the story begins with the representation theory of the Symmetric Group S_n. We exhibit the inductive decomposition of representations, by indexing them with partitions, or equivalently, with Young Diagrams of n boxes, an essential piece of machinery in what follows. Ordering Young subdiagrams by inclusion gives the desired decomposition of representations, as paths through the lattice of Young subdiagrams.

Now the quantities of interest are the Hurwitz numbers, which count ramified coverings of a target Riemann surface by other (possibly disconnected) Riemann surfaces. Alternatively, by describing a ramified covering by its monodromy, as explained in Appendix A, we find the Hurwitz numbers count Factorisations of the Identity of S_n.

The classical solution to this problem, when the target has genus 0, is Frobenius's Formula. We give a generalisation of Frobenius's formula to genus g, originally appearing in [Bur11]. Interestingly, this formula also appears in a topological QFT context [DW90]—in this vein, we find that Hurwitz numbers are nicely described by a Fusion Algebra [Bea94], since the Riemann surfaces they count can be recursively degenerated into sums over thrice-punctured Riemann spheres. This result makes contact with the Verlinde formula [Ver88]. In particular, the genus 1 case simplifies to a trace of certain operators Π_k on the Class Algebra $\mathfrak{Z}$. This is our first Fock space.

The culmination of this chapter is Dijkgraaf's proof [Dij95] that summing these traces over $\mathfrak{Z}$ with simple ramification data produces the above infinite product formula $(*)$. The key step is to choose a basis indexed by Young diagrams, which diagonalises the operators Π_k on $\mathfrak{Z}$. The trace then turns into a sum over the eigenvalues of Π_k, which we call the Partition Invariants p_k.

If we want to translate Dijkgraaf's formula as a trace over different Fock spaces, our task, therefore, is to define analogous operators whose eigenvalues are the partition invariants p_k.

Chapter 2: Bosons and Fermions

With our task in mind, in Chapter 2, we formulate the Fermionic Fock space $\mathfrak{F}$ in terms of the exterior algebra, and the Bosonic Fock Space $\mathfrak{H}$ in terms of the symmetric algebra. We define the Clifford and Heisenberg algebras to act on these Fock spaces, before axiomatically constructing their Vertex Operator Algebras (VOAs). Following the exposition of [Kac98], we show that they are isomorphic by the Boson-Fermion correspondence, which goes back to [Sky71]. On the level of Fock spaces, fermionic states are Semi-infinite Wedges [BO00], which we index by partition and charge. The Boson-Fermion correspondence of the $\mathfrak{F}$ and $\mathfrak{H}$ VOAs sends wedges to Schur Functions, which are also indexed by partition and charge. Finally, we realise the VOA of the infinite Lie algebra $W_{1+\infty}$ over the Fock spaces $\mathfrak{F}$ and $\mathfrak{H}$. It is in this VOA where we find the special Π_k operators we wanted, having partition invariants p_k for eigenvalues.

Chapter 3: Boson and Fermions in the Symmetric Group

Now to close the circle of Chapters 1 and 2, in Chapter 3 we discuss a Fock space isomorphic to the class algebra $\mathfrak{Z}$: namely, the Class Functions $\mathfrak{X}$. This chapter presents an original result that 'quantises' the characters of the symmetric group. In particular, we discover a Heisenberg action of multiplication by the conjugacy class indicator functions δ_C. We then employ the Boson-Fermion correspondence to obtain Clifford operators on the class functions $\mathfrak{X}$.

We propose that characters form a 'fermionic' basis of class functions $\mathfrak{X}$: they can be written as a determinant of induced trivial characters, and this determinant is diagonalised as a product of Clifford operators. It is easy to explain the Clifford action through the Young-diagrammatic decomposition of characters mentioned above. The Clifford operators $\psi_k, \psi^\dagger_{-k}$ respectively insert arms or legs of length k into the Young diagram, moving the character around the Young lattice; Heisenberg operators $J_{\pm k}$ are bilinears in the Clifford operators, and so add or remove hooks of size k. This is proved using the Murnaghan-Nakayama theorem [Mur63], [Nak41]. We compose the boson-fermion correspondence map with the Frobenius Characteristic Map between characters and Schur polynomials, following the classic exposition of [Mac95]. Through the bijection between class functions $\mathfrak{X}$ and the class algebra $\mathfrak{Z}$, we can elaborate on an intriguing comment by Dijkgraaf [Dij95, §5]:

"The representation basis and the conjugacy class basis [...] correspond respectively to fermions and bosons."

The moral of Chapters 1–3 is that the Fock space isomorphisms:

$$\mathfrak{Z} \quad \cong \quad \mathfrak{X} \overset{\text{Frobenius Characteristic}}{\cong} \mathfrak{H} \overset{\text{Boson-Fermion}}{\cong} \mathfrak{F}$$

make physical sense of Dijkgraaf's formula $(*)$, as a trace over fermionic Fock space.

Chapter 4: Jacobi Triple Product

As a prototype for the physical arguments of Chapter 5, in Chapter 4 we prove the Jacobi Triple Product formula by equating $U(1)$ characters of the isomorphic $\mathfrak{H}$ and $\mathfrak{F}$ VOAs [Gin89]. Out of physical interest, we then show how to sum these charged characters over the four spin structures supported on the elliptic curve E, in order to obtain the expected modular CFT partition function on E [Zhu96].

Chapter 5: Deformed CFT

Having established Dijkgraaf's formula $(*)$ as a trace of $W_{1+\infty}$ operators over a Fock space, we can synthesise Chapters 1–4 as a description of the Partition Function of a $W_{1+\infty}$-deformed CFT.

One advantage of the physical formalism we present is having access to OPEs, which reveal analytic structure in the partition function, obscured in the combinatorial Fock space picture. For example, it is natural to expect the partition function of a CFT to enjoy modularity. We use the results of [Dij97], which were verified in a recent announcement [GLT25], to obtain some modularity properties. Unfortunately our hopes of interpreting formula $(*)$ as a generalised Jacobi form are dashed, since we find modular transforms of mixed weight. However, the deformation produces Quasimodular properties in the coefficients of the partition function. For the deformed partition function given by Dijkgraaf's formula $(*)$, quasimodularity was proven in [KZ95]; and in fact holds for a general $W_{1+\infty}$-deformed partition function, as shown in the seminal result of [BO00]. Both of these results are in the spirit of classical analysis. We cast Zagier's new proof [Zag16] of this quasimodularity in a CFT context, finding that quasimodularity is a consequence of spectral flow and spin structures on the elliptic curve.

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Chapter 1

The Symmetric Group and Dijkgraaf's Formula

1.1 Representation Theory of the Symmetric Group

To begin, we dive to the heart of the representation theory of finite groups. For a detailed exposition of the subject, consult e.g. [FH91, Part I].

Much of our work relies on the omniscience of the following functions:

Definition 1.1.1. *The* CHARACTER *of a representation* $\pi : G \to \mathrm{GL}(V)$ *of a group* G *over a field* $\mathbb{F}$ *is the function*

$$\chi_\pi : G \to \mathbb{F} : g \mapsto \mathrm{tr}(\pi(g)).$$

Definition 1.1.2. *The* CLASS FUNCTIONS $\mathfrak{X}(G)$ *of a finite group* G *are the functions* $f \in L^2(G)$ *that are invariant under conjugation, that is,* $f(h^{-1}gh) = f(g)$ *for all* $g, h \in G$.

The character $\chi_\pi(g)$ is an example of a class function, which depends only on the conjugacy class C_g of g and the isomorphism class of π. Define the shorthand $\chi_\pi(C_g) := \chi_\pi(g)$.

It is a striking fact that characters of the irreducible representations of G are orthonormal under the following inner product on $\alpha, \beta \in \mathfrak{X}(G)$:

$$(\alpha, \beta)_G := \frac{1}{|G|} \sum_{g \in G} \alpha(g)\beta(g^{-1}) \tag{1.1}$$

and indeed the characters of G form an orthonormal basis for *all* class functions on G [FH91, I.2.12].

Write $|z_{C_g}| := |z_g| = \#\{h \in G : gh = hg\} = \frac{|G|}{|C|}$ for the size of the centraliser of a representative element $g \in C_g$ in a conjugacy class. It is well-defined by the orbit-stabiliser theorem.

Proposition 1.1.1. *The* FIRST ORTHOGONALITY RELATION *of characters* $\chi_\pi, \chi_{\pi'} \in \mathfrak{X}(G)$ *is*

$$(\chi_\pi, \chi_{\pi'}) = \sum_{C \in \mathcal{C}} \frac{1}{|z_C|} \chi_\pi(C) \chi_{\pi'}(C^{-1}) = \delta_{\pi,\pi'}. \tag{1.2}$$

where $\mathcal{C}$ is the finite set of conjugacy classes of G.

Proposition 1.1.2. *The* SECOND ORTHOGONALITY RELATION *of conjugacy classes* $C, C' \in \mathcal{C}$ *is*

$$\sum_{\pi \in \mathcal{R}} \chi_\pi(C) \chi_\pi(C'^{-1}) = |z_C| \delta_{C,C'}. \tag{1.3}$$

where $\pi, \pi' \in \mathcal{R}$, the finite set of isomorphism classes of irreducible representations of G [FH91, I.2.21].

1.1.1 Class Algebra

> We decompose the group algebra by group Fourier transform, and find that the centre has two natural bases.

Let a group G act on a finite set X. This gives a permutation representation $\mathbb{F}[X]$ over $\mathbb{F}$, which is the vector space of linear combinations of formal symbols $\sum_{x \in X} \alpha_x [x]$, where $\alpha_x \in \mathbb{F}$. The character of this representation is given by the number of fixed points of the action:

$$\chi(g) = \#\mathrm{Fix}(g) = \#\{x \in X | g.x = x\}. \tag{1.4}$$

To see this, it is enough to note that $\chi(g) = \sum_{x \in X}$ coeff. of $[x]$ in $[g.x]$.

Now, if G is finite, we can replace X by G itself, and let G act on itself via $g.g' = gg'$ for $g, g' \in G$. This multiplicative structure lets us upgrade to:

Definition 1.1.3. *The* GROUP ALGEBRA $\mathbb{F}[G]$ *of a finite group G is the set of linear combinations $\sum_{g \in G} \alpha_g [g]$ of formal symbols $[g]$, for α_g in a field $\mathbb{F}$, and $g \in G$, with addition and multiplication inherited from $\mathbb{F}$ and G.*

Definition 1.1.4. *The* Regular Representation $\mathbb{F}[G]$ *of* G *is the linear, faithful representation afforded by the group action of* G *on itself.*

The module point of view is enlightening: given an $\mathbb{F}$-vector space V and a representation $\pi : G \to \mathrm{GL}(V)$ of G over $\mathbb{F}$, we place an $\mathbb{F}[G]$-module structure on V by

$$\left(\sum_{g\in G} \alpha_g[g]\right) v = \sum_{g\in G} \alpha_g \pi(g)(v), \quad \alpha_g \in \mathbb{F}, v \in V.$$

Conversely, if V is an $\mathbb{F}[G]$-module, V is also a representation of G, by

$$\pi : G \to \mathrm{GL}(V) : g \mapsto (v \mapsto [g]v)$$

and an $\mathbb{F}$-vector space, by $\alpha v = (\alpha[1])v$, where $1 \in G$. Now this language lets us write the regular representation of G as the $\mathbb{F}[G]$-module $\mathbb{F}[G]$.

Now we attempt to decompose the regular representation.

Theorem 1.1.1. (Maschke) *Every representation of a finite group* G *over* $\mathbb{F}$ *with characteristic not dividing* $|G|$ *is a direct sum of irreducible representations.*

Proof. (Sketch of [FH91, I.1.5]) Let $V^{||}$ be a subrepresentation of a representation V of G. There is a procedure for finding its supplement, by using a projection operator P on V. If $V = V^{||} \oplus V^{\perp}$ is a decomposition of *vector spaces* induced by a projection map p, define

$$P : V \to V : v \mapsto \frac{1}{|G|} \sum_{g\in G} gp(g^{-1}v).$$

This is a morphism of representations. Then $V = \operatorname{Im} P \oplus \ker P = V^{||} \oplus \ker P$, and $\ker P$ is a subrepresentation of G. As G has finitely many irreducible representations, finitely many iterations of this decomposition will terminate with $V \cong \bigoplus_{\pi\in\mathcal{R}} \pi^{\oplus m_\pi}$, where $m_\pi \in \mathbb{N}$ are multiplicities. □

In particular, for the regular representation over a field of characteristic 0 like $\mathbb{C}$, we find:

Corollary 1.1.1.

$$\mathbb{C}[G] \cong \bigoplus_{\pi\in\mathcal{R}} \pi^{\oplus \dim \pi}.$$

Proof. From Maschke's theorem, we know a representation is irreducible if and only if $(\chi_\pi, \chi_\pi) = 1$. In the regular representation, we have $\chi_{\mathrm{reg}}(g) = |G|\delta_{g1}$, so it is clearly reducible for non-trivial G. The result now follows directly from the orthogonality of characters (1.2), where the multiplicities of $\pi \in \mathcal{R}$ in the regular representation are:

$$(\chi_{\text{reg}}, \chi_\pi) = \frac{1}{|G|} \sum_{g \in G} \chi_{\text{reg}}(g) \chi_\pi(g^{-1}) = \dim \pi.$$

□

The central object in what follows is:

Definition 1.1.5. *The* Class Algebra $\mathfrak{Z}(G) := Z(\mathbb{C}[G])$ *is the centre of the group algebra.*

In order to analyse it, it is easier to first display the Maschke decomposition of Corollary 1.1.1 from the module perspective:

Proposition 1.1.3. *[FH91, I.3.29].*

$$\mathbb{C}[G] \cong \prod_{\pi \in \mathcal{R}} \operatorname{End}(V_\pi) \tag{1.5}$$

where V_π is the vector space associated to the representation π.

This isomorphism is a powerful tool in the representation theory of finite groups. It has a nice interpretation as:

Definition 1.1.6. *The* Group Fourier Transform *of $\varphi : G \to \mathbb{C}$ is*

$$\hat{\varphi}(\rho) := \sum_{g \in G} \varphi(g) \rho(g) \in \operatorname{End}(V_\pi),$$

where ρ is any representation of G.

The Fourier perspective of the isomorphism (1.5) views $\alpha = \sum_{g \in G} \alpha_g [g] \in \mathbb{C}[G]$ instead as the $L^2(G)$ function $\alpha : G \to \mathbb{C} : g \mapsto \alpha_g$. Extending a representation ρ from G to $\mathbb{C}[G]$ is then accomplished via a Fourier transform,

$$\rho(\alpha) = \sum_{g \in G} \alpha_g \rho(g) = \hat{\alpha}(\rho).$$

This gives the most natural description of the isomorphism (1.5):

$$\mathbb{C}[G] \cong \prod_{\pi \in \mathcal{R}} \operatorname{End}(V_\pi) : \alpha \mapsto (\hat{\alpha}(\pi))_{\pi \in \mathcal{R}}.$$

Furthermore the Fourier description also supplies the backwards direction of the isomorphism, by the Fourier inversion formula [FH91, I.3.32]:

$$\alpha(g) = \frac{1}{|G|} \sum_{\pi \in \mathcal{R}} \dim(\pi) \mathrm{tr}\left(\pi(g^{-1}) \cdot \hat{\alpha}(\pi)\right).$$

We now have the tools to analyse the class algebra: the central elements of $\prod_{\pi \in \mathcal{R}} \mathrm{End}(V_\pi)$ as a matrix algebra are linear combinations of the various identity elements $1_{\mathrm{End}(V_\pi)}$. Their image under Fourier inversion,

$$(0, \ldots, 1_{\mathrm{End}(V_\pi)}, \ldots 0) \mapsto \frac{\dim \pi}{|G|} \sum_{g \in G} \chi_\pi(g^{-1})[g], \tag{1.6}$$

will remain central in $\mathbb{C}[G]$. In fact, recalling that characters form a basis of class functions, a pleasant condition for centrality in $\mathbb{C}[G]$ is:

Proposition 1.1.4. $\alpha = \sum_{g \in G} \alpha_g[g] \in \mathfrak{Z}(G)$ *if and only if* $\alpha : g \mapsto \alpha_g$ *is a class function.*

Corollary 1.1.2. *There are as many irreducible representations of* G *as there are conjugacy classes in* G*, i.e.* $|\mathcal{R}| = |\mathcal{C}|$*.*

Proof. One clear choice of basis for the class functions $\mathfrak{X}(G)$ are the class indicator functions $\{\delta_C\}_{C \in \mathcal{C}}$, which return 1 if their argument is in C, and 0 otherwise. Their corresponding elements in $\mathfrak{Z}(G)$ are

$$e_C = \sum_{g \in C} [g], \quad C \in \mathcal{C}, \tag{1.7}$$

and this forms a basis by Proposition 1.1.4. Hence $\dim_{\mathbb{C}} \mathfrak{Z}(G) = |\mathcal{C}|$.

Finally, to arrive at the desired equality, recall the isomorphism (1.5). By Schur's Lemma, $\mathrm{End}(V_\pi) \cong \mathbb{C}$ is 1-dimensional, thus

$$|\mathcal{C}| = \dim_{\mathbb{C}} \prod_{\pi \in \mathcal{R}} Z(\mathrm{End}(V_\pi)) = |\mathcal{R}|.$$

□

This corollary puts the *irreducible representation basis* of the class algebra,

$$e_\pi = \frac{\dim \pi}{|G|} \sum_{C \in \mathcal{C}} \chi_\pi(C^{-1}) e_C,$$

which is the right-hand-side of equation (1.6), on equal footing with the *conjugacy class basis* from equation (1.7). We favour the representation basis since it diagonalises $\mathfrak{Z}(G)$, with

$$e_\pi e_{\pi'} = \delta_{\pi,\pi'} e_\pi, \tag{1.8}$$

as easily seen from the left-hand-side of (1.6). Passage between bases is via

$$e_C = \sum_{\pi \in \mathcal{R}} \frac{|C|\chi_\pi(C)}{\dim \pi} e_\pi, \tag{1.9}$$

as can easily be checked by using the second orthogonality relation (1.3). We will denote by $C = 1$ the conjugacy class of the identity, which is simply the identity. Then the distinguished element $e_1 \in \mathfrak{Z}(G)$ is the element $[1] \in \mathbb{C}[G]$.

1.1.2 Jucys-Murphy Decomposition

For the remainder of this thesis, G will be the SYMMETRIC GROUP S_n. We decompose S_n by partitions, via two distinct approaches.

First, we will index the conjugacy classes of S_n by the set $\mathcal{P}_n$ of integer partitions of n. We denote a partition $\lambda \in \mathcal{P}_n$ by $\lambda = (\lambda_1, \dots \lambda_{\ell(\lambda)})$, where $\lambda_1 + \dots + \lambda_{\ell(\lambda)} = n$, and $\ell(\lambda)$ is the length of the partition.

Definition 1.1.7. *The* CYCLE TYPE *of $\sigma \in S_n$ is the ordered tuple* $\operatorname{cyc}(\sigma) := (\sigma_1, \dots, \sigma_n)$, *where σ_i is the number of cycles of length i in the factorisation of σ into disjoint cycles.*

This is sometimes written $\operatorname{cyc}(\sigma) = (1^{\sigma_1} 2^{\sigma_2} \dots n^{\sigma_n})$. Notice that specifying a cycle type $\operatorname{cyc}(\sigma)$ of $\sigma \in S_n$ is equivalent to specifying a partition $\lambda = (1\sigma_1, 2\sigma_2, \dots, n\sigma_n)$ of n.

Proposition 1.1.5. *The conjugacy classes of S_n are indexed by partitions of n.*

Proof. Any conjugacy class of S_n consists of elements of equal cycle type. This is because the entries inside the cycle factorisation do not affect the conjugacy class, because conjugation in the symmetric group is simply a relabelling of elements:

$$\tau(a_1 a_2 \dots)\tau^{-1} = (\tau(a_1)\tau(a_2)\dots). \tag{1.10}$$

Therefore to specify a conjugacy class it is necessary and sufficient to specify cycle type, which by the above comment is equivalent to specifying a partition of n. □

Secondly, we will index the irreducible representation classes $\mathcal{R}_n$ of S_n by partitions, exploiting the rich inductive structure of S_n. That is, we will consider S_{n-1} as a subgroup of S_n, specified by the set of elements in S_n fixing n. This induces a subalgebra $\mathbb{C}[S_{n-1}]$ in $\mathbb{C}[S_n]$.

The essential piece of machinery is:

Definition 1.1.8. *The* Jucys-Murphy *element is*

$$\mathsf{JM}_n := (1,n) + (2,n) + \cdots + (n-1,n) \in \mathbb{C}[S_n].$$

For cosmetic reasons we let $\mathsf{JM}_1 := 0$. These elements were discovered independently by Jucys [Juc74] and Murphy [Mur81]. Recent literature, e.g. [VO04], understands them as generators of the maximal commutative subalgebra of the group algebra.

It is illuminating to write the Jucys-Murphy elements as the difference of central elements in $\mathfrak{Z}(S_n)$ and $\mathfrak{Z}(S_{n-1})$:

$$\mathsf{JM}_n = e_{\mathcal{T}_{(n)}} - e_{\mathcal{T}_{(n-1)}}, \tag{1.11}$$

where $\mathcal{T}_{(k)}$ is the conjugacy class of a transposition in S_k.

Theorem 1.1.2. *An irreducible representation $\pi \in \mathcal{R}_n$, when restricted to the subgroup $S_{n-1} \subset S_n$, splits into a sum of irreducible representations $\pi' \in \mathcal{R}_{n-1}$, with simple multiplicities.*

The strategy of proof we employ is classical, and relies on:

Definition 1.1.9. *The* Young Diagram *Y is a subset of $\mathbb{N}^2$ such that $(x,y) \in Y$ implies $(x',y') \in Y$ for $1 \leq x' \leq x$ and $1 \leq y' \leq y$, orienting the x-axis south and the y-axis east. The set of Young diagrams of n entries is denoted $\mathcal{Y}_n$.*

Write $\pi' < \pi$ if π' occurs in $\pi|_{S_{n-1}}$. We will set up a bijective correspondence between $\mathcal{R}_n$ and $\mathcal{Y}_n$, such that $\pi' < \pi$ if and only if the corresponding Young diagrams satisfy $Y' \subset Y$. The graphical avatar of a decomposition of π is a path descending from Y in Young's lattice:

Definition 1.1.10. Young's Lattice *is the set $\mathcal{Y} = \cup_{n=0}^{\infty} \mathcal{Y}_n$ partially ordered by subsets, as pictured in Figure 1.1.*

Proof of Theorem 1.1.2 following ([LZ04], A.1.B). By restricting once,

$$\pi|_{S_{n-1}} = \bigoplus_{\substack{\pi' < \pi \\ \pi' \in \mathcal{R}_{n-1}}} \pi'. \tag{1.12}$$

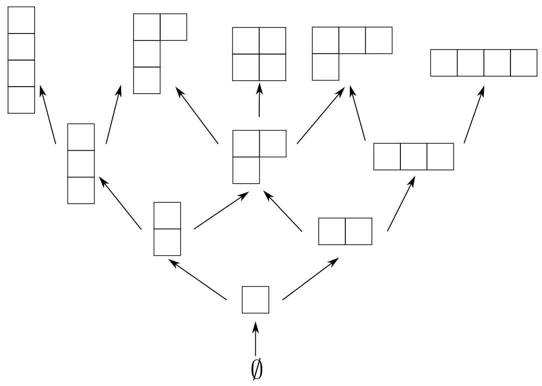

Figure 1.1: The frist few layers of Young's lattice.

Continuing this restriction, we can split $\pi|_{S_{n-2}}$ into a direct sum in $\mathcal{R}_{n-3}$, and so on. When we arrive at S_1, it is clear that π splits into a direct sum indexed by the chains $\Pi : \pi_1 < \cdots < \pi_n = \pi$, with $\pi_k \in \mathcal{R}_k$.

By equations (1.8) and (1.9), we find that multiplication of basis elements of the class algebra e_C and e_π is simply multiplication by a scalar, the so-called Class Multiplier $\nu_\pi(C)$:

$$e_C e_\pi = \frac{|C|\chi_\pi(C)}{\dim \pi} e_\pi =: \nu_\pi(C) e_\pi, \tag{1.13}$$

Define a left action of S_n on the set $\mathcal{R}_n$ as scalar multiplication by the class multiplier, $g\pi := \nu_\pi(C_g)\pi$.

In particular, the transposition element $(1^{n-2}2^1) \in \mathcal{T}_{(n)}$ acts on representations $\pi \in \mathcal{R}_n$ and every subrepresentation $\pi' < \pi$ as multiplication by a scalar $\nu_\pi(\mathcal{T}_{(n)}) \in \mathbb{Z}$. Using the expression (1.11), we find JM_n acts on π', for $\pi' \in \mathcal{R}_{n-1}$ as multiplication by

$$\lambda_n = \nu_\pi(\mathcal{T}_{(n)}) - \nu_{\pi'}(\mathcal{T}_{(n)})$$

So by applying the Jucys-Murphy elements all the way down the decomposition chain Π : $\pi_1 < \cdots < \pi_n = \pi$, we prescribe a series of n weights $\lambda(\Pi) = (\lambda_1(\Pi), \ldots, \lambda_n(\Pi)) \in \mathbb{Z}^n$ to the decomposition Π of π.

Conversely, these weights uniquely determine the decomposition, and thus π as well. The prescription is: a weight vector $\lambda \in \mathbb{Z}^n$ must satisfy (i) $\lambda_1 = 0$; (ii) $\forall j > 1$, $\exists i < j$ such that $|\lambda_j - \lambda_i| = 1$; (iii) if $\lambda_i = \lambda_j$ for some $i < j$ then $\lambda_i - 1$ and $\lambda_i + 1$ occur amongst $\{\lambda_k\}_{i<k<j}$. Furthermore two weight vectors $\lambda(\Pi), \lambda(\Pi')$ correspond to the same representation if and only if one is a permutation of the other. This means π is uniquely characterised by a

multiplicity-counting function $f(r) = \#\{i : \lambda_i = r\}$ of all the occurrences of r in the series.

To establish the desired bijection between the Young diagrams $\mathcal{Y}_n$ of n cells and the irreducible representation classes $\mathcal{R}_n$, we first map $Y \in \mathcal{Y}_n$ to $f : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ satisfying $\sum_r f(r) = n$, and having finite support, where inside its support

$$f(r+1) - f(r) \in \{0, |r| - |r+1|\}.$$

This is accomplished by inscribing $x - y$ at each $(x, y) \in Y$:

$$\begin{array}{|c|c|c|c|l}
\hline
0 & 1 & 2 & 3 & \cdots \\
\hline
-1 & 0 & 1 & \ddots & \\
\cline{1-3}
-2 & -1 & \ddots & \ddots & \\
\cline{1-2}
-3 & \ddots & \ddots & & \\
\cline{1-1}
\multicolumn{1}{c}{\vdots} & & & &
\end{array} \tag{1.14}$$

The required function is $f(r) = \#\{(x, y) \in Y : x - y = r\}$. Pictorially one constructs $f(r)$ from Y by counting the number of boxes in the r^{th} off-diagonal $x = r + y$ in $Y \subset \mathbb{N}^2$. So f must sum to the number of boxes of Y. The support is the finite interval between $\min\{y : (x, y) \in Y\}$ and $\max\{x : (x, y) \in Y\}$. Finally, if the r^{th} off-diagonal of Y is non-empty, then the $(r+1)^{\text{th}}$ off-diagonal is either the same length, or possibly one box shorter for $r \geq 0$, or possibly one box longer for $r < 0$.

Conversely, given f with the aforementioned conditions, the corresponding Young diagram is $Y = \{(x, y) \in \mathbb{N}^2 : \min(x, y) \leq f(x - y)\}$. The pictorial construction, in which $f(r)$ is the length of the r^{th} off-diagonal, is made obvious by writing $Y = \cup_{r \in \mathbb{Z}} \cup_{x \in \mathbb{N}} \{(x, x - r) : \min(x, x - r) \leq f(r)\}$. This is the union of diagonals of length $f(r)$, finite since f has finite support, and since the length of the diagonals sums to n, we have $Y \in \mathcal{Y}_n$ as desired. $\square$

Remark 1.1.1. *A Young diagram $Y \in \mathcal{Y}_n$, if drawn, is quite literally a partition $\lambda \in \mathcal{P}_n$ of n into the number of boxes in each row. So there is a four-fold bijection $\mathcal{C}_n \cong \mathcal{P}_n \cong \mathcal{Y}_n \cong \mathcal{R}_n$. In what follows, where the meaning is obvious, we freely write an integer partition λ in place of its corresponding representation π_λ or conjugacy class C_λ.*

1.1.3 Class Multipliers

Our last task is to compute the class multipliers $\nu_\pi(\mathcal{T})$, as they determine the weights of the decomposition.

It turns out that the inscription procedure (1.14) can be used to calculate arbitrary class multipliers $\nu_\pi(C)$, and so deserves a name:

Definition 1.1.11. *The* CONTENTS *of a Young diagram $Y \in \mathcal{Y}_n$ is the tuple*

$$c(Y) = \big(x - y \,|\, (x, y) \in Y\big) \subset \mathbb{Z}^n.$$

It is well-known [FH59, 1.1] that the following elements $\{Z_i\}_{i=0,\dots,n}$ form an alternative basis of the class algebra $\mathfrak{Z}(S_n)$:

$$Z_k = \sum_{\substack{\sum_{i=1}^{\ell(\lambda)} i\lambda_i = n \\ \sum_{i=1}^{\ell(\lambda)} \lambda_i = k}} e_{C_\lambda}, \tag{1.15}$$

We interpret Z_k as a sum over classes in $\mathcal{C}_n$ having k cycles, i.e. a sum over all elements $\sigma \in S_n$ such that $\sum_{i=1}^n \mathrm{cyc}(\sigma)_i = k$, where $\mathrm{cyc}(\sigma)$ is the cycle type of Definition 1.1.7. Equivalently this is a sum over Young diagrams of n boxes with k rows.

Proposition 1.1.6. *[Juc74, §3] The elementary symmetric polynomials e_m in the Jucys-Murphy elements form a basis of the class algebra $\mathfrak{Z}(S_n)$ via*

$$Z_{n-m} = e_m(\mathsf{JM}_1, \dots, \mathsf{JM}_n) \quad \textit{for } m = 0, \dots, n.$$

See Section 3.1 for a primer on the symmetric polynomials $\mathfrak{S}_n$. Now $\{e_m\}_{m=0,\dots,n}$ freely generate $\mathfrak{S}_n$, so there is a homomorphism between the $\mathfrak{S}_n$ and $\mathfrak{Z}(S_n)$ given by $f \mapsto f(\mathsf{JM}_1, \cdots, \mathsf{JM}_n)$.

Proposition 1.1.7.

$$f(\mathsf{JM}_1, \dots, \mathsf{JM}_n) e_{\pi_\lambda} = f(c(\lambda)) e_{\pi_\lambda},$$

where $f \in \mathfrak{S}_n$ is any symmetric function [Juc74, §4].

This result allows us to calculate class multipliers in terms of contents.

Example 1.1.1. *The transposition basis element in $\mathfrak{Z}(S_n)$ can be expressed in terms of the first power-sum polynomial p_1,*

$$e_{\mathcal{T}_{(n)}} = \sum_{i=1}^n \mathsf{JM}_i = p_1(\mathsf{JM}_1, \dots, \mathsf{JM}_n),$$

by definition of the Jucys-Murphy elements. So to calculate the class multiplier $\nu_\pi(\mathcal{T}_{(n)})$, by Proposition 1.1.7 we need only evaluate the sum of the contents of the partition λ associated to π:

$$\begin{aligned}\nu_\lambda(\mathcal{T}_{(n)}) = p_1(c(\lambda)) &= \sum_{i=1}^{\ell(\lambda)} \left(\frac{(\lambda_i - i)(\lambda_i - i + 1)}{2} - \frac{(\lambda_i^\dagger - i)(\lambda_i^\dagger - i + 1)}{2} \right) \\ &= \frac{1}{2} \sum_{i=1}^{\ell(\lambda)} \left((\lambda_i - i)^2 - (\lambda_i^\dagger - i)^2 \right)\end{aligned} \tag{1.16}$$

In the above calculation, $\lambda^\dagger$ is the Conjugate Partition *to λ, defined in terms of their corresponding Young diagrams by*

$$Y_{\lambda^\dagger} = Y_\lambda^\top \tag{1.17}$$

i.e. transposing x and y.

1.2 Frobenius's Formula

How many ways can we factorise the identity of a finite group? We give a modern topological interpretation of Frobenius's classical solution, and show that it satisfies the axioms of a fusion algebra.

Our goal is to calculate the number of factorisations of the identity $1 \in G$:

$$Z_0(G; C_1, \ldots, C_k) := \frac{1}{|G|} \#\{(c_1, \ldots, c_k) \in C_1 \times \cdots \times C_k : \prod_i^k c_i = 1\}, \tag{1.18}$$

for G a finite group, with fixed data of conjugacy classes $C_1, \ldots, C_k \in \mathcal{C}$. An elegant computation of this number by representation-theoretic means is:

Theorem 1.2.1. (Frobenius's Formula)

$$Z_0(G; C_1, \ldots, C_k) := \frac{|C_1| \cdots |C_k|}{|G|^2} \sum_{\pi \in \mathcal{R}} \frac{\chi_\pi(C_1) \cdots \chi_\pi(C_k)}{(\dim \pi)^{k-2}}. \tag{1.19}$$

Before we prove this formula, note the special case:

Example 1.2.1. *For $k = 2$, equation (1.19) reduces to the second orthogonality relation (1.3), to*

$$Z(C, C'^{-1}) = \frac{1}{|z_C|} \delta_{CC'},$$

which indeed matches equation (1.18).

Proof of Theorem 1.2.1. This formula was first produced [Fro00] while the representation theory of S_n was nascent. We take the elegant approach of [LZ04, A.1.9]: let the product $e_{C_1} \cdots e_{C_k}$ act by left multiplication on both sides of the isomorphism (1.5), then equate the traces of this action.

On the left-hand-side, the multiplication is by a formal sum of $[c_1 \cdots c_k]$ for $c_i \in C_i$. Now the trace of left multiplication by $[g]$ on $\mathbb{C}[G]$ is either $|G|$ if $g = 1$, and otherwise 0. This trace is essentially the character (1.4) of the regular representation $\mathbb{C}[G]$. Since the non-vanishing trace occurs for $c_1 \cdots c_k = 1$, the trace of $e_{C_1} \cdots e_{C_k}$-multiplication on $\mathbb{C}[G]$ is $|G|^2 Z(G; C_1, \ldots, C_k)$.

Now the isomorphism (1.5) was a Fourier transform, so the Fourier-transformed $\prod_{i=1}^k e_{C_i}$-multiplication acts on π as multiplication by $\prod_{i=1}^k \nu_\pi(C_i)$, and so the trace of this action is $(\dim \pi)^2 \prod_{i=1}^k \nu_\pi(C_i)$, noting that $\mathrm{End}(V_\pi)$ is a $(\dim \pi)^2$-dimensional space.

Equating these two traces gives

$$Z_0(G; C_1, \ldots, C_k) = \sum_\pi \left(\frac{\chi_\pi(1)}{|G|} \right)^2 \prod_{i=1}^k \nu_\pi(C_i) \tag{1.20}$$

which, after substituting the class multiplier definition (1.13), is exactly Frobenius's formula. $\square$

The *topological* interpretation of Frobenius's formula is very elegant. On the k-punctured genus 0 Riemann surface $\Sigma_{g=0} - \{P_1, \ldots, P_k\}$, with each puncture decorated by a label $(1, \ldots, n)$, the fundamental group is simply

$$\pi_1(\Sigma - \{P_1, \ldots, P_k\}) = \langle \gamma_1, \ldots, \gamma_k \,|\, \gamma_1 \cdots \gamma_k = 1 \rangle$$

and $Z_0(G; C_1, \ldots, C_k)$ counts the number of homomorphisms $\rho : \pi_1(\Sigma_0) \to G$ with prescribed data $\rho(\gamma_i) = C_i$ around each puncture. So long as G acts faithfully on some finite set, e.g. $G = S_n$ acting on $\{1, \ldots, n\}$, our interpretation is: each ρ corresponds to a (not necessarily connected) covering of Σ_0 with ramification points $\{P_1, \ldots, P_k\}$, such that the permutation of this finite set induced by the local monodromy about P_i belongs to the conjugacy class C_i. Then Z_0 counts exactly the number of such coverings.

Naturally, we can play the same game with the coverings of genus $g > 0$ Riemann surfaces. The fundamental group of the genus g surface Σ_g is

$$\pi_1(\Sigma_g) = \langle \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g \,|\, [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] = 1 \rangle,$$

where $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$. We can apply the Seifert-Van Kampen theorem to derive the appropriate expression for the genus g punctured Riemann surface $\Sigma_g - \{P_1, \dots, P_k\}$:

$$Z_g(G; C_1, \dots, C_k) := \frac{1}{|G|} \#\mathrm{Hom}\left(\pi_1(\Sigma_g - \{P_1, \dots, P_k\}), G\right)$$
$$= \frac{1}{|G|} \#\left\{(a_1, \dots, a_g, b_1, \dots, b_g, c_1, \dots, c_k) \in G^{2g} \times C_1 \times \dots \times C_k \mid [a_1, b_1] \cdots [a_g, b_g] c_1 \cdots c_k = 1\right\}. \tag{1.21}$$

Remark 1.2.1. *The upshot is that $Z_g(C_1, \dots, C_k)$ is a count of monodromy representations of cycle type $\{\lambda_1, \dots, \lambda_k\}$, where $\lambda_i = \mathrm{cyc}(C_i)$ and $C_i \in \mathcal{C}_d$. Crucially, this gives the identification with the connected Hurwitz number of Definition A.1,*

$$Z_g(C_1, \dots, C_k) = H_{h \xrightarrow{d} g}(\lambda_1, \dots, \lambda_n),$$

due to the bijection between monodromy representations and covers of Riemann surfaces, given in Theorem A.2. See Proposition A.1 for the proof.

The following formula, an arbitrary genus analogue of Frobenius's [Fro00], was discovered in physics in the context of topological QFT with finite gauge group [DW90]. It goes back in a sense to an exercise by Burnside [Bur11, §238].

Theorem 1.2.2. *For $g \geq 0$,*

$$Z_g(G; C_1, \dots, C_k) = \sum_{\pi} \left(\frac{\dim \pi}{|G|}\right)^{2-2g} \prod_{i=1}^{k} \nu_\pi(C_i). \tag{1.22}$$

There is a simple generalisation of the $g = 0$ proof above, which hinges on the first orthogonality relation (1.2). Instead, to make contact with the upcoming section on fusion algebras, we base our proof around the so-called *'Kommutator'*, coined in [CM16, §9.2]:

Definition 1.2.1. *The* Kommutator *is*

$$\mathfrak{K} := \sum_{C \in \mathcal{C}} |z_C| e_C^2 \in \mathfrak{Z}(G).$$

One should think of $\mathfrak{K}$ as a way to express the sum of all commutators in G (with multiplicity) as an element in the class algebra $\mathfrak{Z}(G)$. More precisely, denote by M the set of ordered monomials in the formal expansion of the quadratic $\sum_{C \in \mathcal{C}} e_C^2$, without multiplying any elements of the group. Consider the function

$$\kappa : G \times G \to M : (a, b) \mapsto [a, b]$$

This is surjective; furthermore, for every $a \in G$ which appears in a monomial of the form $a\hat{a}$ in M, the number of pairs whose commutator produces $a\hat{a}$ is $|\kappa^{-1}(a\hat{a})| = |z_a|$.

We can now express Z as the solution to a multiplication problem in the class algebra.

Lemma 1.2.1.
$$Z_g(G; C_1, \ldots, C_k) = \frac{1}{|G|}[e_1]\mathfrak{K}^g e_{C_1} \cdots e_{C_k}. \tag{1.23}$$
where $[e_1]$ *is an instruction to select the coefficient of* $e_1 = 1 \in \mathfrak{Z}(G)$.

Proof. We are concerned with counting all the tuples

$$(a_1, \ldots, a_g, b_1, \ldots, b_g, c_1, \ldots, c_k) \in G^{2g} \times C_1 \times \cdots \times C_k$$

such that
$$[a_1, b_1] \cdots [a_g, b_g] c_1 \cdots c_k = 1 \tag{$*$}$$
Expanding $\left(\sum_{C \in \mathcal{C}} e_C^2\right)^g e_{C_1} \cdots e_{C_k}$ as a sum of monomials, we can find terms of the form

$$a_1\hat{a}_1 \cdots a_g\hat{a}_g c_1 \cdots c_k$$

which satisfies $(*)$ when $\hat{a}_i = (b_i a_i b_i^{-1})^{-1}$. Such a term arises $\prod_{i=1}^g |z_{a_i}|$ times. In order to introduce this multiplicity into our count, we weight our sum appropriately, by inserting a factor of $|z_C|$ in front of e_C^2 in the sum $\left(\sum_{C \in \mathcal{C}} e_C^2\right)^g$. But this is exactly $\mathfrak{K}^g$. Therefore, by selecting the identity coefficient of $\mathfrak{K}^g C_1 \cdots C_k$, we obtain the desired count of tuples. □

Proof of Theorem 1.2.2. We rewrite $\mathfrak{K}$ using equation (1.9) and the orthonormality $e_\pi e_{\pi'} = \delta_{\pi\pi'} e_\pi$ of the representation basis (1.8):

$$\mathfrak{K} = \sum_C |z_C| e_C^2 = \sum_C |z_C| \sum_\pi \left(\frac{|C|\chi_\pi(C)}{\dim \pi}\right)^2 e_\pi.$$
Rearranging and using the identities $\sum_{g \in G} \chi_\pi(g)^2 = |G|$ and $|z_C| = |G|/|C|$, we find

$$\mathfrak{K} = \sum_\pi \frac{|G|}{(\dim \pi)^2} \sum_C |C| \chi_\pi(C)^2 e_\pi = \sum_\pi \frac{|G|}{(\dim \pi)^2} \left(\sum_{g \in G} \chi_\pi(g)^2\right) e_\pi = \sum_\pi \left(\frac{|G|}{\dim \pi}\right)^2 e_\pi.$$

We can also simplify the product by using the class multipliers (1.13) and orthonormality (1.8):

$$\begin{aligned} e_{C_1} \cdots e_{C_k} &= \left(\sum_{\pi_1} \nu_{\pi_1}(C_1) e_{\pi_1}\right) \cdots \left(\sum_{\pi_k} \nu_{\pi_k}(C_k) e_{\pi_k}\right) \\ &= \sum_\pi \left(\prod_{i=1}^k \nu_\pi(C_i)\right) e_\pi. \end{aligned}$$

Incorporating both of these expressions into equation (1.23) gives

$$Z_g(G; C_1, \ldots, C_k) = \frac{1}{|G|}[e_1] \sum_{\pi} \left(\frac{|G|}{\dim \pi} \right)^{2g} \left(\prod_{i=1}^{k} \nu_\pi(C_i) \right) e_\pi.$$

Simply return to the conjugacy class basis (1.9) with

$$e_\pi = \frac{\dim \pi}{|G|} \sum_{C \in \mathcal{C}} \chi_\pi(C) e_C = \frac{(\dim \pi)^2}{|G|} e_1 + \ldots,$$

to extract the coefficient of the identity e_1.

□

1.2.1 Fusion algebra

Definition 1.2.2. *[Sch08, §11.4] The* FUSION ALGEBRA $(V, 1, F)$, *where* V *is finite-dimensional* $\mathbb{C}$*-vector space with a distinguished element* $1 \in V$*; where* $F = (F_g)_{g \in \mathbb{Z}_{\geq 0}}$ *is a family of functions on the symmetric algebra* $F_g : \mathsf{S}(V) \to \mathbb{C}$ *which for every* $g \in \mathbb{Z}^{\geq 0}$*, and every* $f_1, \ldots, f_n \in V$*, satisfy*

1. (Constant Vacuum) $F_0(1, \ldots, 1) := F_0(1)$.

2. (Linearity) $f \mapsto F_0(f_1, \ldots, f_j, f, f_{j+1}, \ldots, f_n)$ *is* $\mathbb{C}$*-linear in* $f \in V$.

3. (Non-degeneracy) $(f_1, f_2) \mapsto F_0(f_1, f_2)$ *is non-degenerate.*

Define a pair of bases of V*:*

$$(b_j), (b^j) \quad \textit{such that } F_0(b_i, b^j) = \delta_i^j. \tag{1.24}$$

Then in addition the FUSION RULES *must hold:*

F1. (Genus reduction) *For* $g > 0$,

$$F_g(f_1, \ldots, f_n) = \sum_j F_{g-1}(b_j, b^j, f_1, \ldots, f_n).$$

F2. (Puncture reduction)

$$F_{g+g'}(f_1, \ldots, f_n, f'_1, \ldots, f'_m) = \sum_j F_g(f_1, \ldots, f_n, b_j) F_{g'}(b^j, f'_1, \ldots, f'_m).$$

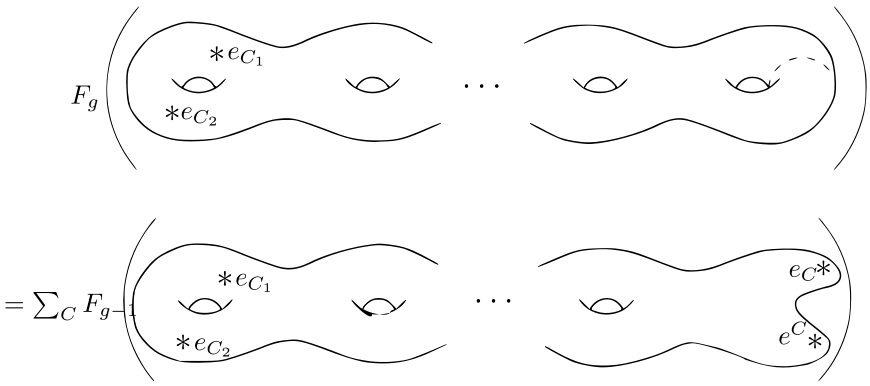

Figure 1.2: Fusion by genus reduction.

This notion arose in the study of the Verlinde formula when calculating dimensions of conformal blocks, e.g. [Bea94, II.5]. We have met a fusion algebra already.

Proposition 1.2.1. $\big(\mathfrak{Z}(S_d), e_1, (Z_g)_{g\in\mathbb{Z}_{\geq 0}}\big)$ *is a fusion algebra.*

Proof. By the class algebra version of the Frobenius formula in Lemma 1.2.1, Z_g is a function on $\mathfrak{Z}(S_d)$ viewed as a vector space, so by abuse of notation, we will write

$$Z_g(e_{C_1}, \dots, e_{C_n}) = Z_g(C_1, \dots, C_n)$$

interchangeably. From the results of the previous section, the class algebra $\mathfrak{Z}(S_d)$ equipped with Z_g satisfies conditions (1) to (4) of a fusion algebra, as follows.

Firstly, Z_g is defined on the symmetric algebra of $\mathfrak{Z}(S_d)$ because by definition (1.21) of Z_g, we are counting the set

$$\begin{aligned} M := \{(a_1, \dots, a_g, b_1, \dots, b_g, c_1, \dots, c_n) \in S_d^{2g} \times C_1 \times \cdots \times C_n \mid \\ [a_1, b_1] \cdots [a_g, b_g] c_1 \cdots c_n = 1\}. \end{aligned}$$

Since $c_i c_{i+1} = c_{i+1}(c_{i+1}^{-1} c_i c_{i+1})$, we can swap arbitrary C_i, C_{i+1}.

The constancy of vacuum condition (1) holds by definition. The linearity condition (2) can be checked using the class algebra expression for Z_0 in Lemma 1.2.1. Finally, condition (3) becomes exactly the second orthogonality relation (1.3), which is non-degenerate by definition.

The fusion rules (F1) and (F2) are so-called 'degeneration' formulas, of which there are many variants in the Hurwitz literature, e.g. [CM16, 7.5.1]. We give a combinatorial proof of (F1). Define

$$N_C := \{(a_1, \ldots, a_{g-1}, b_1, \ldots, b_{g-1}, \hat{c}, c, c_1, \ldots, c_n) \in S_d^{2g-2} \times C \times C \times C_1 \times \cdots \times C_n \mid \\ [a_1, b_1] \cdots [a_{g-1}, b_{g-1}]\hat{c}cc_1 \cdots c_n = 1\}.$$

There is a map $M \to \bigcup_{C \in \mathcal{C}} N_C$ given by

$$(a_1, \ldots, a_g, b_1, \ldots, b_g, c_1, \ldots, c_n) \\ \mapsto (a_1, \ldots, a_{g-1}, b_1, \ldots, b_{g-1}, (a_g b_g a_g^{-1}), b_g^{-1}, c_1, \ldots, c_n)$$

Our topological interpretation of M tells us that we have exchanged two 'genus generators' for two 'puncture generators.' Before and after the exchange, the multiplication of all factors remains equal to the identity. Also note that c and $\hat{c}$ are in the same conjugacy class, since in S_d every element is conjugate to its inverse, as can be seen through equation (1.10).

The map is surjective. To see this, take a tuple in N_C for some conjugacy class C. Then its inverse image in M consists of tuple where $b_g = c^{-1}$, and a_g satisfies the equation

$$a_g c^{-1} a_g^{-1} = \hat{c}.$$

But since c, c are in the same conjugacy class C, there are precisely $|z_C|$ solutions.

We define the pair of bases (1.24) of $\mathfrak{Z}(S_d)$ to be e_C and

$$e^C := |z_C| e_{C^{-1}} = |z_C| e_C,$$

which satisfy $Z_0(e_C, e^{C'}) = \delta_{C,C'}$, by Example 1.2.1.

Finally we can compute the size of M by adding the sizes of the inverse images:

$$|M| = \sum_{C \in \mathcal{C}} |z_C||N_C|.$$

Hence

$$Z_g(e_{C_1}, \ldots, e_{C_n}) = \sum_{C \in \mathcal{C}} Z_{g-1}(e_C, e^C, e_{C_1}, \ldots, e_{C_n}).$$

An alternative one-line proof of (F1) uses the Kommutator equation (1.23) for Z_g:

$$Z_g(e_{C_1}, \ldots, e_{C_n}) = \frac{1}{d!}[e_1]\mathfrak{K}\mathfrak{K}^{g-1} e_{C_1} \cdots e_{C_n} = \sum_C Z_{g-1}(e_C, e^C, e_{C_1}, \ldots, e_{C_n}).$$

where the last equality follows from substituting Definition 1.2.1 of the first factor of $\mathfrak{K}$.

As for fusion rule (F2), we factorise the Kommutator equation (1.23) for $Z_{g+g'}$ into two parts, then factorise the coefficient selection $[e_1]fh = \sum_C \frac{1}{|C|}[e_{C^{-1}}]f[e_C]h$ for $f, h \in \mathfrak{Z}(S_d)$:

$$\begin{aligned}
Z_{g+g'}(e_{C_1}, \ldots, e_{C_n}, e_{C'_1}, \ldots, e_{C'_m}) &= \frac{1}{d!}[e_1]\mathfrak{K}^{g+g'} e_{C_1} \cdots e_{C_n} e_{C'_1} \cdots e_{C'_m} \\
&= \frac{1}{d!} \sum_C \frac{1}{|C|}[e_{C^{-1}}]\mathfrak{K}^g e_{C_1} \cdots e_{C_n} [e_C]\mathfrak{K}^{g'} e_{C'_1} \cdots e_{C'_m} \\
&= \sum_C \frac{1}{d!}[e_1]\mathfrak{K}^g e_C e_{C_1} \cdots e_{C_n} \frac{|z_C|}{d!}[e_1]\mathfrak{K}^{g'} e_{C^{-1}} e_{C'_1} \cdots e_{C'_m} \\
&= \sum_C Z_g(e_C, e_{C_1}, \ldots, e_{C_n}) Z_{g'}(e^C, e_{C'_1}, \ldots, e_{C'_m}).
\end{aligned}$$

□

Many other parallels emerge as easy consequences of our definition of fusion. First we present the analogue of the genus g Frobenius formula (1.23), which is a type of Verlinde formula.

Lemma 1.2.2. *In the notation $N^{b^k}_{f_i f_j} := F_0(f_i, f_j, b^k)$, the product*

$$f_i \cdot f_j := \sum_k N^{b^k}_{f_i f_j} b_k$$

induces on V the structure of a commutative and associative algebra with identity 1.

Proposition 1.2.2. (Verlinde Formula, c.f. [Ver88, 3.14])

$$F_g(f_1, \ldots, f_n) = F_0(\mathfrak{K}^g \cdot f_1 \cdot \ldots \cdot f_n),$$

for $\mathfrak{K} = \sum_j b_j \cdot b^j$.

Proof. Firstly, we proceed by induction on $n \in \mathbb{Z}_{\geq 0}$ to prove:

$$F_g(f_1, \ldots, f_n) = F_g(f_1 \cdot \ldots \cdot f_n).$$

The case $n = 1$ is trivial. Assume this holds for $n > 1$. We may use fusion rules and the inductive hypothesis to deduce

$$
\begin{aligned}
F_g(f_1,\dots,f_n) &\overset{F2}{=} \sum_j F_0(f_1,f_2,b^j)F_g(b_j,f_3,\dots,f_n)\\
&= \sum_j N_{f_1 f_2}^{b^j} F_g(b_j,f_3,\dots,f_n)\\
&\overset{(2)}{=} F_g\left(\sum_j N_{f_1 f_2}^{b^j} b_j, f_3,\dots,f_n\right)\\
&= F_g(f_1\cdot f_2, f_3,\dots,f_n)\\
&= F_g(f_1\cdot\ldots\cdot f_n).
\end{aligned}
$$

With this in hand, we can rewrite:

$$
F_g(f) \overset{F1}{=} \sum_j F_{g-1}(b_j,b^j,f) \overset{(2)}{=} F_{g-1}\left(\sum_j b_j\cdot b^j\cdot f\right) = F_{g-1}(\mathfrak{K}\cdot f),
$$

and therefore $F_g(f) = F_0(\mathfrak{K}^g\cdot f)$ by iteration. Set $f = f_1\cdot\ldots\cdot f_n$ and we are done. □

Remark 1.2.2. *We briefly draw an analogy to rational CFTs on the torus with N primary operators* $\{\Phi_i\}_{i=0,\dots,N-1}$*, each corresponding to an irreducible representation* $\mathcal{H}_i$ *of some chiral algebra. The characters of this theory are* $Z_i = \mathrm{tr}_{\mathcal{H}_i}(q^{L_0-c/24})$*, and the fusion rules are* $\phi_i\times\phi_j = \sum_k N_{ij}^k\phi_k$ *[Ver88, 2.2]. A modular S-transformation* $\tau\mapsto -1/\tau$ *acts as unitary matrix multiplication* $Z_i\mapsto\sum_j S_i^j Z_j$*. Surprisingly, this matrix diagonalises the fusion rules [Ver88, 3.11],*

$$
N_{ij}^k = \sum_n \frac{S_j^n S_i^n S_n^{\dagger k}}{S_0^n}.
$$

Likewise, multiplication in the conjugacy class basis of the class algebra is encoded the structure constants

$$
e_{C_\alpha}e_{C_\beta} = \sum_{C\in\mathcal{C}} N_{C_\alpha C_\beta}^{C} e_C \tag{1.25}
$$

determined by $Z_0(G;C_\alpha,C_\beta,C_\gamma^{-1}) = \frac{1}{|S_d|}N_{C_\alpha C_\beta}^{C_\gamma}$*. Just as with fusion rules, the matrix* $N_\alpha = (N_{C_\alpha C_\beta}^{C_\gamma})_{\beta,\gamma\in\mathcal{C}_d}$ *can be diagonalised. The base change is given by the class multiplier matrix* $\nu = (\nu_\pi(C))_{\pi\in\mathcal{R}_d, C\in\mathcal{C}_d}$*.*

1.3 Dijkgraaf's Formula

We now turn to Dijkgraaf's celebrated quasimodular formula [Dij95], originally written down by Douglas [Dou95]. We will follow Dijkgraaf's topological proof closely. See Appendix A for a primer on Hurwitz theory.

The primary object of interest is the generating function of simple disconnected Hurwitz numbers:

$$H^{\bullet}(q,x) := \sum_{(d,b,g)\in \mathsf{RH}_h} H^{\bullet}_{g\xrightarrow{d}h}(\mathcal{T}^b_{(d)})q^d \frac{x^b}{b!}$$

These are the weighted number of degree d covers of the genus h Riemann surface by Riemann surfaces of arbitrary genus g, possibly disconnected, with the ramification data of a transposition $\mathcal{T}_{(d)}$ in S_d at b many branch points. The sum is over the set RH_h of solutions of the Riemann-Hurwitz formula (6.1) for simple ramification $k_i = 2$ at b many branch points, and genus h target curve. Since the ramification indices sum to $\sum_{i=1}^b k_i = b$, the solution set is

$$\mathsf{RH}_h := \{(d,b,g)\,|\, 2g-2 = d(2h-2)+b \text{ and } d,b,g\in\mathbb{Z}_{\geq 0}\}.$$

Henceforth we count covers of the elliptic curve by fixing $h := 1$. This choice weakens the Riemann-Hurwitz constraints to $b = 2g-2$ and $d \in \mathbb{Z}_{\geq 0}$.

Define the infinite product

$$Z(w,q,x) := \prod_{a\in\frac{1}{2}+\mathbb{Z}^{\geq 0}} \left(1+wq^a e^{\frac{x}{2}a^2}\right)\left(1+w^{-1}q^a e^{-\frac{x}{2}a^2}\right). \tag{1.26}$$

Theorem 1.3.1.

$$H^{\bullet}(q,x) = w^0 \textit{ coeff. of } Z(w,q,x). \tag{1.27}$$

Proof following [Dij95, §4]. By Proposition 1.2.2, the $h=1$ case of our Verlinde formula for Z is a trace:

$$\begin{aligned} Z_1(e_{C_1},\dots,e_{C_n}) &= Z_0(\mathfrak{K}e_{C_1}\cdots e_{C_n}) \\ &= \sum_C Z_0(e^C, e_{C_1}\cdots e_{C_n}e_C) \\ &= \mathrm{tr}_{\mathfrak{Z}(S_d)}(e_{C_1}\cdots e_{C_n}) \end{aligned} \tag{1.28}$$

for the inner product $Z_0(\cdot,\cdot)$ on $\mathfrak{Z}(S_d)$. Following Remark 1.2.1, we know the Hurwitz numbers are exactly calculated by Z, and in particular $H^{\bullet}_{g\xrightarrow{d}1}(\mathcal{T}^b) = Z_1(e^b_{\mathcal{T}_{(d)}})$.

Putting these formulas together, the Hurwitz potential is:

$$
\begin{aligned}
H^\bullet(q,x) &= \sum_{\mathsf{RH}_1} \mathrm{tr}_{\mathfrak{Z}(S_d)}(e^b_{\mathcal{T}(d)}) q^d \frac{x^b}{b!} \\
&= \sum_{d\in\mathbb{Z}_{\geq 0}} q^d \mathrm{tr}_{\mathfrak{Z}(S_d)}\left(e^{xe_{\mathcal{T}}}\right) \\
&= \sum_{\lambda\in\mathcal{P}} q^{|\lambda|} e^{x\nu_\lambda(\mathcal{T}_{|\lambda|})}
\end{aligned}
\tag{1.29}
$$

where ν_λ are the class multipliers defined in (1.13).

Dijkgraaf's key insight was to recognise $|\lambda|$ and ν_λ as Partition Invariants:

$$
p_k(A|B) := \frac{1}{k!}\left(\sum_{a\in A}\left(a+\frac{1}{2}\right)^k - \sum_{b\in B}\left(-\frac{1}{2}-b\right)^k\right), \quad A, B \subset \mathbb{Z}_{\geq 0}. \tag{1.30}
$$

Definition 1.3.1. *The* Frobenius Coordinates *of a partition $\lambda \in \mathcal{P}$ are*

$$
Y_\lambda = (A_\lambda \,|\, B_\lambda) = (a_1, \ldots, a_{d(\lambda)} \,|\, b_1, \ldots, b_{d(\lambda)})
$$

where $d(\lambda) \in \mathbb{N}$ is the length of the longest diagonal ($x = -y$) in the Young diagram Y_λ of λ. The positive integers $a_0 > \cdots > a_{d(\lambda)} \geq 0$, the arms of Y, and $b_0 > \cdots > b_{d(\lambda)} \geq 0$, the legs of Y, are the number of cells to the right and below, respectively, the cells of the $x = -y$ diagonal.

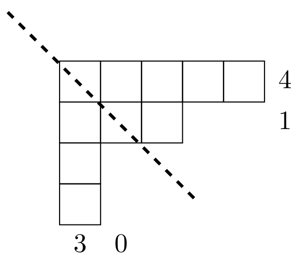

Figure 1.3: A Young diagram of the partition $\lambda = (5,3,1,1)$ with Frobenius coordinates $(4,1|3,0)$.

When evaluated on Frobenius coordinates, these are exactly the eigenvalues we need:

$$
p_1(Y_\lambda) = \sum_{i=1}^{d(\lambda)} (a_i + b_i + 1) = |\lambda|, \quad p_2(Y_\lambda) = \frac{1}{2}\sum_{i=1}^{d(\lambda)}\left((a_i+\frac{1}{2})^2 - (-\frac{1}{2}-b_i)^2\right) = \nu_\pi(\mathcal{T}),
$$

where the latter class multiplier calculation is given in Example 1.1.1.

We could also consider $(A \,|\, B)$, made up of two arbitrary finite subsets $A, B \subset \mathbb{Z}_{\geq 0}$. This will not in general be coordinates of a partition, unless $p_0(A \,|\, B) = \#A - \#B = 0$. This forces $(A \,|\, B) = Y_\lambda$ for some partition λ.

Now expanding the infinite product $Z(w, q, x)$ as a power series, each term is associated to a choice of two finite subsets $A, B \subset \mathbb{Z}_{\geq 0}$:

$$Z(w, q, x) = \sum_{A,B} w^{p_0(A \,|\, B)} q^{p_1(A \,|\, B)} e^{x p_2(A \,|\, B)} \tag{1.31}$$

Selecting just the w^0 term in this expansion enforces $p_0(A \,|\, B) = 0$, so that $(A \,|\, B) = Y_\lambda$ for some partition λ. Therefore the sum is

$$w^0 \text{ coefficient of } Z(w, q, x) = \sum_{\lambda \in \mathcal{P}} q^{p_1(Y_\lambda)} e^{x p_2(Y_\lambda)} = \sum_{\lambda \in \mathcal{P}} q^{|\lambda|} e^{x \nu_\lambda(\mathcal{T}_{|\lambda|})}$$

which exactly equals the trace expression (1.29) for the Hurwitz potential.

□

Remark 1.3.1. *In the string theory of an elliptic curve target space, one would include the so-called free energy of* $1/24$. *In the sequel we include this physically motivated term, understanding it as the contribution of a degree zero 'constant map' anomaly. Mathematically, this is not present in the Hurwitz theory, but is necessary for quasimodularity of the generating function.*

First split the generating function into

$$H^\bullet(q, \lambda) = \sum_{d,b \geq 0} H^\bullet_{g \xrightarrow{d} 1}(\mathcal{T}^b) q^d \frac{\lambda^b}{b!} = \sum_{d \geq 0} H^\bullet_{1 \xrightarrow{d} 1}(\emptyset) q^d + \sum_{d,b \geq 1} H^\bullet_{g \xrightarrow{d} 1}(\mathcal{T}^b) q^d \frac{\lambda^b}{b!},$$

where we have distinguished the covers for which both target and source have genus 1, *and are therefore unramified, with* $b = 0$ *by the Riemann-Hurwitz formula.*

The disconnected Hurwitz number $H^\bullet_{1 \xrightarrow{d} 1}(\emptyset)$ *with degree* $d \geq 1$ *is exactly the number of conjugacy classes in* S_d, *which equals the number of partitions* $p(d)$ *of* d. *So by a standard generating function argument,*

$$1 + H^\bullet_{1 \to 1}(q) = 1 + \sum_{d \geq 1} p(d) q^d = \frac{1}{\phi(q)},$$

the inverse of the Euler function $\phi(q) = \prod_{d=1}^\infty (1 - q^d)$. *By the disconnected-connected relation* $e^{H^\circ} = 1 + H^\bullet$, *[CM16, 10.2.1], we see* $H^\circ_{1 \to 1}(\emptyset) = -\ln \phi(q)$.

The degree $d=0$ free energy prescription is to set $\frac{d}{dt}|_{q=0} H^\circ_{1\to 1}(q) := -1/24$, where $q = e^t$ [Dij95]. After integration, one finds

$$H^\circ_{1\to 1}(q) = -\frac{1}{24}\ln q - \ln \phi(q) = -\ln \eta(q)$$

where $\eta(q)$ is the Dedekind eta function, the free energy of an elliptic curve string theory. In order to implement this in the trace formalism of the above proof, we express equation (1.29) as

$$H^\bullet(q,x) = \mathrm{tr}_{\oplus_{d=0}^{\infty} \mathfrak{z}(S_d)} \left(q^L e^{xW} \right),$$

in terms of the operators

$$L|_{\mathfrak{z}(S_d)} := d, \quad W|_{\mathfrak{z}(S_d)} := e_{\mathcal{T}_{(d)}}.$$

The free-energy prescription suggested by string theory simply changes L to $L - 1/24$.

Chapter 2

Bosons and Fermions

Dijkgraaf's formula, proved at the end of Chapter 1, seems physical by nature: it is a trace over a 'Fock space' of the class algebra. This chapter is dedicated to exploring various Fock spaces, and algebras on them.

2.1 Free Fermions

2.1.1 Infinite Wedge

We will first construct the Fock space for free fermions. We will construct a vector space from the infinite subalgebra of the exterior algebra, in order to impose Fermi-Dirac statistics.

Definition 2.1.1. *The* Exterior Algebra *of a vector space V is*

$$\bigwedge V := T(V)/I,$$

where

$$T(V) = \bigoplus_{k=0}^{\infty} V^{\otimes k}$$

is the tensor algebra of V, and I is the (two-sided) ideal generated by elements of the form $v \otimes v$ for $v \in V$, with wedge product

$$u \wedge w := u \otimes w \ (mod\ I).$$

Given a ring $\mathbb{B}$ of characteristic 0, define a $\mathbb{C}$-vector space $V(\mathbb{B})$ whose basis elements are $\underline{i}$, where the index runs over all $i \in \mathbb{B}$.

Definition 2.1.2. *The* FERMIONIC FOCK SPACE *[Kac83, 14.9]*

$$\mathfrak{F}(\mathbb{B}) := \operatorname{span}_{\mathbb{C}} \left\{ \bigwedge_{k\in\mathbb{N}} \underline{i_k} \;\middle|\; \substack{i_k \in \mathbb{B},\, \forall k \in \mathbb{N} \\ i_k > i_{k+1} \\ i_{k+1} = i_k - 1,\, \forall k \gg 0} \right\}$$

is the $\mathbb{C}$-vector space whose basis elements are SEMI-INFINITE WEDGES *taken from the subalgebra $V(\mathbb{B})^{\otimes\infty}/I$ of the exterior algebra $\bigwedge V(\mathbb{B})$, and we index the entries of said wedges by the natural numbers $\mathbb{N} := \mathbb{Z}_{\geq 1}$.*

For our purposes, the entries in the wedge will take values in $\mathbb{Z}$ or $\mathbb{F} := \mathbb{Z} + \frac{1}{2}$, which we call the RAMOND (R) and NEVEU-SCHWARZ (NS) sectors, respectively.

Definition 2.1.3. *The* VACUUM VECTOR *of charge $m \in \mathbb{B} + \frac{1}{2}$ is*

$$|m\rangle_{\mathbb{B}} = \underline{m - 1/2} \wedge \underline{m - 3/2} \wedge \cdots \in \mathfrak{F}(\mathbb{B}),$$

where $\mathbb{B}$ is either $\mathbb{Z}$ or $\mathbb{F}$.

For example, $|1/2\rangle_R = \underline{0} \wedge \underline{-1} \wedge \underline{-2} \wedge \cdots$ and $|0\rangle_{\mathrm{NS}} = \underline{-1/2} \wedge \underline{-3/2} \wedge \underline{-5/2} \wedge \cdots$.

Definition 2.1.4. *A vector $v \in \mathfrak{F}(\mathbb{B})$ has* CHARGE *m if v differs from $|m\rangle_{\mathbb{B}}$ in finitely many entries.*

Definition 2.1.5. *The* WEDGE COORDINATES *of a partition $\lambda \in \mathcal{P}$ at charge m are*

$$X_{\lambda;m} := \left\{ \lambda_j - j + \frac{1}{2} + m \right\}_{j\in\mathbb{N}} \subset \mathbb{F} + m.$$

One way to think of the $X_{\lambda;m}$ coordinates is as the entry-wise difference of a charge m vector $\bigwedge_{k\geq 1} \underline{i_k}$ and the charge m vacuum $|m\rangle$, given by $\underline{i_k} = \underline{\lambda_k - j + \frac{1}{2} + m}$, creating a finite integer partition λ with $\lambda_k = 0$ for $k > N \gg 0$. Therefore we can compactly label any charge m vector by

$$|\lambda; m\rangle := \bigwedge_{i \in X_{\lambda;m}} \underline{i}.$$

Notably, the charge 0 wedge coordinates $X_{\lambda;0}$ are in bijection with the Frobenius coordinates $(a_i|b_i)_{i=1,\ldots,d(\lambda)}$ of the Young diagram Y_λ from Definition 1.3.1 [Zag16, §3],

$$\begin{aligned} \mathbb{F}_{\geq 0} \cap X_{\lambda;0} &= \left\{ a_1 + \frac{1}{2}, \ldots, a_{d(\lambda)} + \frac{1}{2} \right\} \\ \mathbb{F}_{<0} - X_{\lambda;0} &= \left\{ -b_1 - \frac{1}{2}, \ldots, -b_{d(\lambda)} - \frac{1}{2} \right\}. \end{aligned} \tag{2.1}$$

In this notation, the vacuum vectors are given by trivial partition $|m\rangle = |\emptyset; m\rangle$. In fact *any* subset $X \subset \mathbb{Z} + m$ that is bounded above and whose complement is bounded below defines a state $\bigwedge_{i \in X} \underline{i} \in \mathfrak{F}(\mathbb{Z} + m)$. This gives a DIRAC SEA interpretation to the fermionic Fock space: the holes are legs; the occupations are arms.

Clearly, we can grade the Fock space into sectors of charge m, $\mathfrak{F}(\mathbb{B}) = \bigoplus_{m \in \mathbb{B}+1/2} \mathfrak{F}^{(m)}(\mathbb{B})$, where $\mathfrak{F}^{(m)} = \{|\lambda; m\rangle\}_{|\lambda \in \mathcal{P}}$. We require $\{|\lambda; m\rangle\}_{\lambda \in \mathcal{P}, m \in \mathbb{B}+1/2}$ to be an orthonormal basis of $\mathfrak{F}(\mathbb{B})$, with inner product

$$\langle \mu; m | \lambda; n \rangle = \delta_{\mu,\lambda} \delta_{m,n}, \tag{2.2}$$

à la Dirac bra-ket notation.

Example 2.1.1. *A quick way to calculate the wedge coordinates $X_{\lambda;m}$ from a Young diagram Y_λ is to cut the Young diagram in two. In the Neveu-Schwarz sector with $m \in \mathbb{Z}$, the cut is along the diagonal $x = -y - m$; in the Ramond sector with $m \in \mathbb{F}$, we cut the zig-zag $x = -\lfloor y + 1/2 + m \rfloor$. Then in the k^{th} row of the diagram, count the number of boxes $i_k \in \mathbb{F} + m$ to the right of the diagonal (negative if the boxes are to the left). See Figure 2.1.*

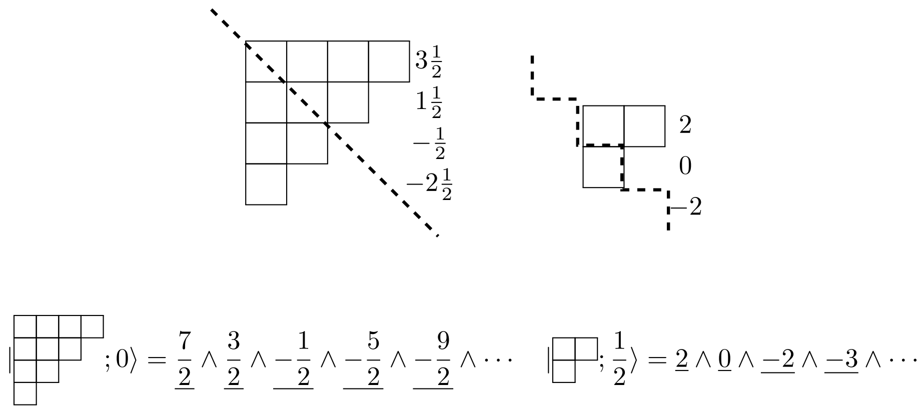

Figure 2.1: Cutting Y_λ along the m^{th} diagonal determines the charge m wedge coordinates $X_{\lambda;m}$. On the left is a Neveu-Schwarz cut, and on the right is a Ramond cut.

Remark 2.1.1. *We can rewrite Frobenius's partition invariants (1.30) in terms of the wedge coordinates $X_{\lambda;0}$ of the Neveu-Schwarz sector $\mathfrak{F}^{(0)}(\mathbb{F})$:*

$$p_k(Y_\lambda) = \frac{1}{k!} \left(\sum_{\mathbb{F}_{\geq 0} \cap X_{\lambda;0}} x^k - \sum_{\mathbb{F}_{<0} - X_{\lambda;0}} x^k \right) = \frac{1}{k!} \sum_{j \in \mathbb{N}} \left((\lambda_j - j + 1/2)^k - (-j + 1/2)^k \right) \tag{2.3}$$

using equation (2.1). The fact that these invariants take the form of differences *is an artifact of 'normal ordering' the Frobenius coordinates relative to charge* 0*, that is, relative to the* $x = -y$ *diagonal of* Y_λ.

This suggests a generalisation of partition invariants. Cut along the m^{th} *diagonal of* Y_λ*, by the procedure of Example 2.1.1, dividing it into an unequal number of arms and legs. Define the* Charged Partition Invariants*:*

$$p_k(X_{\lambda;m}) := \frac{1}{k!}\left(\sum_{\mathbb{F}_{\geq 0}\cap X_{\lambda;m}} x^k - \sum_{\mathbb{F}_{<0}-X_{\lambda;m}} x^k\right). \tag{2.4}$$

As an application of these invariants, consider the charge $m \in \mathbb{Z}$ *Neveu-Schwarz sector* $\mathfrak{F}^{(m)}(\mathbb{F})$*. The difference in number of arms and legs is* $p_0(X_{\lambda;m}) = m$*. By substituting the invariants into (1.31), we immediately see that the Dijkgraaf's partition function (1.26) is*

$$Z(w,q,x) = \sum_{m\in\mathbb{Z},\lambda\in\mathcal{P}} w^{p_0(X_{\lambda;m})} q^{p_1(X_{\lambda;m})} e^{xp_2(X_{\lambda;m})} \tag{2.5}$$

In particular, we can view the w^m *coefficient as a trace over charge* m *Neveu-Schwarz Fock space.*

Introduce the wedging and contracting operators ψ_j and $\psi_j^\dagger$ for $j \in \mathbb{B}$, whose action on $v \in \mathfrak{F}(\mathbb{B})$ is

$$\psi_j v = \underline{j} \wedge v,$$
$$\psi_j^\dagger v = \begin{cases} u & \text{if } v \text{ can be expressed as } \underline{j} \wedge u \text{ for some wedge } u \in \mathfrak{F}(\mathbb{B}), \\ 0 & \text{otherwise.} \end{cases}$$

Wedging and contracting are adjoint under the inner product (2.2) on $\mathfrak{F}(\mathbb{B})$.

Definition 2.1.6. *The* Clifford Algebra Cl *of* V*, where* V *is a vector space equipped with a symmetric bilinear form* $(.|.)$*, is the tensor algebra quotient*

$$\mathsf{Cl}(V) := T(V)/I$$

where I *is the (two-sided) ideal generated by elements of the form* $u\otimes v - (u|v)\mathbb{1}_V$ *for* $u, v \in V$.

It is natural to define a Clifford module over $\mathfrak{F}(\mathbb{F})$. Indeed, the wedge and contraction operators, together with a central element $\mathbb{1}$, generate $\mathsf{Cl} := \mathsf{Cl}(V(\mathbb{F}) \oplus V(\mathbb{F})^\dagger)$ with the bilinear form $(\underline{i}|\underline{j}^\dagger) = \delta_{ij}$, $(\underline{i}|\underline{j}) = 0 = (\underline{i}^\dagger|\underline{j}^\dagger)$, where we decorate the elements of the second vector space with a dagger (which is purely cosmetic, and does not denote conjugation). After quotienting, the vector space elements $\underline{i} \in V(\mathbb{B})$ and $\underline{i}^\dagger \in V(\mathbb{B})^\dagger$ correspond to $\psi_i \in \mathsf{Cl}$ and $\psi_i^\dagger \in \mathsf{Cl}$, as they obey standard Clifford anticommutation relations:

$$\{\psi_i, \psi_j^\dagger\} = \delta_{ij}\mathbb{1}, \quad \{\psi_i, \psi_j\} = 0 = \{\psi_i^\dagger, \psi_j^\dagger\}, \quad \text{for all } i, j \in \mathbb{F}. \tag{2.6}$$

Proposition 2.1.1. *The* Cl*-module* $\mathfrak{F}$ *is irreducible.*

Proof. Any finite linear combination of wedges can be projected onto a single wedge by applying a finite series of $\psi_i\psi_i^\dagger$ and $\psi_i^\dagger\psi_i$ operations; any single wedge is obtainable from any other by a finite number of wedge and contraction operations. Hence any non-zero submodule must be the full Cl-module. □

Consider the Clifford action on vacua. The vacuum state of charge $m = 0$ is a 'Dirac sea' in which all negative modes are 'empty' and all positive modes are 'occupied':

$$\psi_j|0\rangle = 0 \text{ for } j \leq 0, \quad \psi_j^\dagger|0\rangle = 0 \text{ for } j > 0.$$

Definition 2.1.7. *The* Creation *operators with respect to a vacuum* $|m\rangle$ *are* $\psi_{j\leq m}^\dagger$ *and* $\psi_{j>m}$*. The* Annihilation *operators with respect to a vacuum* $|m\rangle$ *are* $\psi_{j\leq m}$ *and* $\psi_{j>m}^\dagger$*.*

Excited states, i.e. those differing from the vacuum, are obtained by filling holes in the wedge with creation operators or making holes in the wedge with annihilation operators:

$$|\lambda; m\rangle = \prod_{i=1}^{d(\lambda)} \left(\psi^\dagger_{-b_i-m-1/2}\psi_{a_i+m+1/2}\right)|m\rangle. \tag{2.7}$$

in terms of the Frobenius coordinates $(a_i|b_i)_{i=1,\ldots,d(\lambda)}$. The arms a_i and legs b_i of a Young diagram respectively represent positive and negative excitations of the vacuum.

Finally, we define a useful notion to tame divergences when dealing with the infinite wedge:

Definition 2.1.8. *The* Normal Ordering *of a product* $\Psi \in \mathsf{Cl}$ *of fermionic operators, with respect to a vacuum* $|m\rangle$*, is the rearrangement* $:\!\Psi\!:_m$ *that moves, with respect to* $|m\rangle$*, all annihilation operators to the right and creation operators to the left, introducing an appropriate sign for each exchange of neighbouring operators. When we normal order relative to the Dirac sea* $|0\rangle$*, we drop the subscript,* $:\!\Psi\!: = :\!\Psi\!:_0$*.*

Normal ordering can be used to give an alternative interpretation of charge: namely, as the eigenvalue of the CHARGE operator $\alpha_0 := \sum_{j\in\mathbb{F}} {:}\psi_j\psi_j^\dagger{:}$

$$\alpha_0|\lambda;m\rangle = m|\lambda;m\rangle.$$

Another interpretation of the charge eigenvalue, obtained from the anti-commutation relations (2.6), is that it counts of the difference of wedges and contractions:

$$\alpha_0 \prod_{a\in A}\psi_a \prod_{b\in B}\psi_b^\dagger|0\rangle = (\#A - \#B)\prod_{a\in A}\psi_a \prod_{b\in B}\psi_b^\dagger|0\rangle$$

where $A, B \subset \mathbb{F}$ both have no repeated entries.

Since $[\alpha_0, \psi_j] = +\psi_j$ and $[\alpha_0, \psi_j^\dagger] = -\psi_j^\dagger$, we say that the operators $\psi_j, \psi_j^\dagger$ have charges ± 1 respectively, leading to the following definition:

Definition 2.1.9. *An operator $\Phi \in \mathsf{Cl}$ has charge m if $[\alpha_0, \Phi] = m\Phi$.*

Finally we note an important combinatorial result:

Theorem 2.1.1. (Wick)

$$\langle m|v_1\cdots v_n w_n \cdots \bar{w}_1|m\rangle = \det_{1\le i,j\le n}\langle m|v_i w_j|m\rangle,$$

for arbitrary linear combinations $v_i = \sum_j v_{ij}\psi_j, w_i = \sum_j w_{ij}\psi_j^\dagger$ [AZ13, §2.7].

2.1.2 $\mathsf{gl}(\infty)$ Representation

We begin with the observation that the commutation relation between quadratic Clifford elements

$$[\psi_m\psi_n^\dagger, \psi_{m'}\psi_{n'}^\dagger] = \delta_{nm'}\psi_m\psi_{n'}^\dagger - \delta_{mn'}\psi_{m'}\psi_n^\dagger, \tag{2.8}$$

is the commutation relation for the elementary matrices $(e_{mn})_{ij} := \delta_{im}\delta_{jn}$. To align with literature, in this section we consider the Clifford elements as acting in the Neveu-Schwarz sector $\mathfrak{F} = \mathfrak{F}(\mathbb{F})$.

Definition 2.1.10. $\mathsf{gl}(\infty)$ *is the Lie algebra of the matrices*

$$\{(a_{ij})_{i,j\in\mathbb{F}}\,|\, a_{ij}\in\mathbb{C},\ \textit{all but a finite number of } a_{ij} \textit{ are } 0\},$$

whose Lie bracket is the standard commutator.

Consider the map

$$r : \mathsf{gl}(\infty) \to \mathsf{Cl} : e_{ij} \mapsto \psi_i \psi_j^\dagger.$$

which is an embedding of $\mathsf{gl}(\infty)$ into the subalgebra of charge 0 operators in Cl. It preserves the respective bracket $r([\Phi, \Psi]) = r(\Phi)r(\Psi) - r(\Psi)r(\Psi)$. Since $\mathfrak{F}$ is a Cl-module by Proposition 2.1.1, it is also $\mathsf{gl}(\infty)$-module, where the action on vectors is a derivation:

$$\begin{aligned} r(a)(\underline{i_1} \wedge \underline{i_2} \wedge \cdots) = & \, a \cdot \underline{i_1} \wedge \underline{i_2} \wedge \cdots \\ & + \underline{i_1} \wedge a \cdot \underline{i_2} \wedge \cdots + \ldots \end{aligned}$$

while the action on the entries of the wedge is linear, by $a \cdot \underline{i} = \sum_{j \in \mathbb{F}} a_{ij} \underline{j}$.

We now try to extend our representation r to

$$\overline{\mathsf{gl}}(\infty) := \{(a_{ij})_{i,j \in \mathbb{F}} \mid a_{ij} = 0 \text{ for } |i - j| \gg 0\},$$

whose importance stems from that fact that many algebras can be embedded in $\overline{\mathsf{gl}}(\infty)$ but not $\mathsf{gl}(\infty)$, for example those containing elements with infinitely many diagonal entries. However, such an extension is impossible, for if we naïvely use r again, we encounter an anomalous representation of diagonal elements, e.g.

$$r(\mathrm{diag}(a_i)_{i \in \mathbb{F}})|0\rangle = \sum_{i \in \mathbb{F}} a_i \psi_i \psi_i^\dagger |0\rangle = \sum_{i \in \mathbb{F}^{\leq 0}} a_i |0\rangle$$

generally diverges. To correct this, we must normal order the representation relative to the vacuum $|0\rangle$,

$$\hat{r}(e_{ij}) := {:}r(e_{ij}){:} \tag{2.9}$$

Note that normal ordering only hits the diagonal elements, $\hat{r}(e_{ii}) = r(e_{ii}) - \mathbb{1}$ for $i \leq 0$. From this it can be checked [Kac98, §5.3] that for $a, b \in \overline{\mathsf{gl}}(\infty)$, the bracket is given by

$$[\hat{r}(a), \hat{r}(b)] = \hat{r}([a, b]) + \Psi(a, b)\mathbb{1}$$

in terms of a 2-cocycle

$$\Psi(a, b) := \mathrm{tr}\left([\sum_{i \in \mathbb{F}^{\leq 0}} e_{ii}, a]b \right). \tag{2.10}$$

So by normal ordering, we have centrally extended $\overline{\mathsf{gl}}(\infty)$ to

$$\widehat{\mathsf{gl}}(\infty) := \overline{\mathsf{gl}}(\infty) + \mathbb{C}K$$

with the bracket $[a, b] = ab - ba + \Psi(a, b)K$. By letting $\hat{r}(K) = \mathbb{1}$, we obtain a projective representation $\hat{r}$ of the Lie algebra $\widehat{\mathsf{gl}}(\infty)$ in the Fock space $\mathfrak{F}$.

2.2 Fermion-Boson Correspondence

We give an algebraic Fermion-Boson correspondence. The other direction of this correspondence, which is VOA-theoretic, will appear in Theorem (2.4.1).

First we define a Fock space which respects Bose-Einstein statistics, using the symmetric algebra.

Definition 2.2.1. *The* SYMMETRIC ALGEBRA *over a vector space V is*

$$\mathsf{S}(V) := T(V)/I$$

where $T(V)$ is the tensor algebra of V and I is the (two-sided) ideal generated by elements $u \otimes v - v \otimes u$, for $u, v \in V$.

It is clear that $\mathsf{S}(V)$ is isomorphic to the polynomial ring over basis elements of V. We consider the infinite-dimensional vector space $\mathbb{C}^\infty$, and label its coordinates by $\{t_i\}_{i\in\mathbb{N}}$, so that $S(\mathbb{C}^\infty) = \mathbb{C}[t_1, t_2, \ldots]$.

In view of the upcoming boson-fermion correspondence (2.4.1), we also would like to impose a grading by charge, so we tensor by the group algebra $\mathbb{C}[w, w^{-1}]$ of a charge lattice $\mathbb{Z}$. The full Fock space is:

Definition 2.2.2. *The* CHARGED BOSONIC FOCK SPACE *is*

$$\mathfrak{H} := \mathbb{C}[t_1, t_2 \ldots] \otimes \mathbb{C}[w, w^{-1}] = \bigoplus_{n\in\mathbb{Z}} \mathfrak{H}^{(n)},$$

where $\mathfrak{H}^{(n)} := w^n\mathbb{C}[t_1, t_2, \ldots]$.

Definition 2.2.3. *The* HEISENBERG-WEYL ALGEBRA *is*

$$\mathsf{He} = \mathbb{C}[\ldots, t_{-1}, t_0, t_1, \ldots] \oplus \mathbb{C}\mathbb{1}$$

where $\mathbb{1}$ is central, subject to commutation relations

$$[t_m, t_n] = m\delta_{m+n,0}\mathbb{1}. \tag{2.11}$$

We will represent He by an irreducible $\mathfrak{H}$-module on which the central elements t_0 and $\mathbb{1}$ act as multiplication by a fixed integer charge and 1 respectively. By slightly abusing notation and letting the letter t denote both vector space and algebra elements, we easily define an action of $t_m \in \mathsf{He}$ on each $\mathfrak{H}^{(n)}$:

$$t_m = \begin{cases} \partial_{t_m} & m > 0 \\ w\partial_w = n & m = 0 \\ -mt_{-m} & m < 0 \end{cases} \tag{2.12}$$

Theorem 2.2.1. (Stone-von Neumann) *Once one fixes a non-zero scalar by which the central element* $\mathbb{1} \in \mathsf{He}$ *acts, there is a unique irreducible representation of* He *[Kac83, §9.13].*

It is easily checked that the operators (2.12) satisfy the commutation relations (2.11). Hence $\mathfrak{H}^{(n)}$ is a He-module for all $n \in \mathbb{Z}$, all of which are irreducible and isomorphic by the Stone-von Neumann theorem. What is more surprising is:

Proposition 2.2.1. $\mathfrak{F}^{(n)}$ *is an irreducible* He*-module for all* $n \in \mathbb{Z}$.

Proof following [Kac83, 14.9]. If we can verify the commutation relations (2.11), then the Stone-von Neumann theorem (2.2.1) will guarantee irreducibility. The construction is to realise He as a subalgebra of $\widehat{\mathsf{gl}}(\infty)$, and represent the result in the Fock space $\mathfrak{F}$.

Consider $\mathsf{sl}(\infty) = \{a \in \mathsf{gl}(\infty) \,|\, \mathrm{tr}(a) = 0\}$, whose central extension $\widehat{\mathsf{sl}}(\infty)$ is given by the following 2-cocycle, induced from the $\widehat{\mathsf{gl}}(\infty)$ cocycle (2.10):

$$\psi(e_{ij}, e_{ji}) = -\psi(e_{ji}, e_{ij}) = 1 \quad \text{for } i \leq 0 \text{ and } j \geq 1,$$
$$\psi(e_{ij}, e_{mn}) = 0 \quad \text{otherwise.}$$

Then an embedding $\rho : \mathsf{He} \to \widehat{\mathsf{sl}}(\infty)$ is given by

$$s_m := \rho(t_m) := \sum_{j \in \mathbb{Z}} e_{j,j+m}$$

since $\psi(s_m, s_n) = m\delta_{m+n,0}$. These matrices s_m have entries 1 on the m^{th} off-diagonal, and 0 elsewhere. The final step is to represent $\widehat{\mathsf{sl}}(\infty)$ in $\mathfrak{F}$. We map s_m to Cl by the projective representation $\hat{r}$, defined in equation (2.9):

$$\alpha_m := \hat{r}(s_m) = :\sum_{j \in \mathbb{Z}} \psi_j \psi^\dagger_{j+m}: \tag{2.13}$$

which satisfy the Heisenberg commutation relations (2.12)

$$[\alpha_m, \alpha_n] = m\delta_{m+n,0}. \tag{2.14}$$

$\square$

2.3 Vertex Operator Algebras

The culmination of this chapter is the Boson-Fermion correspondence, which has a natural formulation in the language of vertex operator algebras, which we now formulate axiomatically.

2.3.1 Axioms

Take $V = \oplus_{n\in\mathbb{Z}_{\geq 0}} V_n$ a graded vector space of STATES, satisfying $\dim V_n < \infty$ for all n. The basic object is a FIELD on V, a formal power series

$$\phi(z) = \sum_{n\in\mathbb{Z}} \phi_{(j)} z^{-j-h} \in \mathrm{End}(V)[\![z, z^{-1}]\!],$$

whose MODES are $\phi_{(j)} \in \mathrm{End}(V)$ satisfying $\phi_{(j)} V_n \subset V_{n-j}$ for all n, and $\phi_{(j)} V = 0$ for $j \gg 0$. The WEIGHT associated to this field $\phi(z)$ is h.

Most of the subtlety in a VA is a consequence of locality:

Definition 2.3.1. *Two fields $\phi(z), \psi(w)$ are* MUTUALLY LOCAL *if*

$$(z-w)^N [\phi(z), \psi(w)] = 0, \quad N \gg 0.$$

Definition 2.3.2. *A (graded)* VERTEX ALGEBRA (VA) $(V, |0\rangle, Y)$ *is specified by the data:*

i (State Space) *A (graded) vector space V;*

ii (State-Field Correspondence) *A linear map of V_n to the space of fields* $\mathrm{End}(V)[\![z, z^{-1}]\!]$, *written $\phi \mapsto Y(\phi, z) := \sum_{n\in\mathbb{Z}} \phi_{(j)} z^{-j-n}$;*

iii (Vacuum) $|0\rangle \in V_0$*;*

satisfying axioms:

I (Vacuum)

(a) $Y(|0\rangle, z) = 1_V$*;*

(b) $\lim_{z\to 0} Y(\phi, z)|0\rangle = \phi$ *for $\phi \in V$;*

II (Translation) $[T, Y(\phi, z)] = \partial_z Y(\phi, z)$, *where $T \in \mathrm{End}(V)$ is defined by*

$$T(\phi) = \phi_{(-2)}|0\rangle. \tag{2.15}$$

III (Locality) *All fields $Y(\phi, z), Y(\psi, w)$ are mutually local with respect to each other, for all $\phi, \psi \in V$.*

We will generally work with:

Definition 2.3.3. *A* (Conformal) Vertex Operator Algebra (VOA) $(V, |0\rangle, Y, \omega, c)$ *is a VA $(V, |0\rangle, Y)$ with additional data:*

iv (Virasoro Element) $\omega \in V_2$*;*

v (Central Charge) $c \in \mathbb{C}$*;*

such that $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ obeys an additional axiom:

IV   *i* $L_{-1} = T$*;*

ii $L_0|_{V_n} = n1_{V_n}$*;*

iii $L_2\omega = \frac{1}{2}c|0\rangle$.

Let us derive a basic implication of the VA axioms, which builds a field from a state and translation.

Proposition 2.3.1. *In a VA $(V, |0\rangle, Y)$,*

$$Y(\phi, z)|0\rangle = e^{zT}(\phi), \quad \phi \in V. \tag{2.16}$$

Proof. According to the translation axiom (II),

$$[T, \phi_{(n)}] = -n\phi_{(n-1)}. \tag{2.17}$$

By (I.a), $|0\rangle_{(n)} = \delta_{n,-1}$, so that

$$T|0\rangle = 0. \tag{2.18}$$

According to the vacuum axiom (I.b),

$$\phi_{(n)}|0\rangle = 0 \quad \text{for } n \geq 0, \quad \phi_{(-1)}|0\rangle = \phi.$$

To put all these implications together, apply T to both sides of equation (2.15) $n-1$ times, and using equations (2.18) and (2.17), we find $T^n(\phi) = n!\phi_{-n-1}|0\rangle$. The result follows. □

An alternative derivation is to consider the differential equation for $f(z) \in V[\![z, z^{-1}]\!]$,

$$\partial_z f(z) = T f(z).$$

for which both sides of equation (2.16) are solutions, by the translation axiom (II). The coincidence of initial conditions follows from letting $z = 0$ and invoking the vacuum axiom (I), so equality follows.

Corollary 2.3.1. *(VA Skew-Symmetry) [Kac98, §4.2]*

$$Y(\Phi, z)\Psi = e^{zT} Y(\Psi, -z)\Phi.$$

Proof. From axiom (III), $Y(\phi, z)Y(\psi, w)|0\rangle = Y(\psi, w)Y(\phi, z)|0\rangle$. Substituting equation (2.16) gives

$$Y(\phi, z)e^{wT}(\psi) = Y(\psi, w)e^{zT}(\phi).$$

Simply apply the 'shift' identity, derived from axiom (II),

$$Y(\phi, z)e^{wT} = e^{wT} Y(\phi, z - w) \tag{2.19}$$

on both sides, and set $w = 0$, to obtain the result. □

In the study of VOA representations, the following object will be useful:

Definition 2.3.4. *Let V be a VOA. The* Character *of a V-module W with highest weight $h \in \mathbb{C}$ is*

$$\chi(q) := q^{-c/24} \sum_{n \in \mathbb{Z}_{\geq 0}} \dim W_{h+n} q^{h+n},$$

where $W = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} W_{h+n}$ is the weight space decomposition into $(h+n)$-eigenspaces of L_0.

2.3.2 Existence and Uniqueness

Split a VA field $\phi(z)$ of weight h into

$$\phi_+(z) := \sum_{j > -h} \phi_{(j)} z^{-j-h}, \quad \phi_-(z) := \sum_{j \leq -h} \phi_{(j)} z^{-j-h}$$

Definition 2.3.5. *The* Normally Ordered *product of two fields $\phi(z), \psi(w)$ is*

$$:\!\phi(z)\psi(z)\!: = \phi_-(z)\psi(z) + \psi(z)\phi_+(z).$$

Proposition 2.3.2. *[Kac98, 2.3(iii)] The fields $\phi(z)$, $\psi(w)$ are mutually local if and only if*

$$\phi(z)\psi(w) = \sum_{j=0}^{N-1} \frac{1}{(z-w)^{j+1}}\bigg|_{|z|>|w|} \eta^j(w) + :\phi(z)\psi(w):$$

for some fields $\eta^j(w)$ and $N \in \mathbb{N}$.

An equivalent statement is that the fields $\phi(z)$, $\psi(w)$ are mutually local if and only if

$$[\phi_{(m)}, \psi_{(n)}] = \sum_{j=0}^{N-1} \begin{pmatrix} m \\ j \end{pmatrix} \eta^j_{(m+n-j)}, \quad m, n \in \mathbb{Z}. \tag{2.20}$$

for $\eta^j \in \mathrm{End}(V)$, [Kac98, 2.3(iv)].

Abbreviating to just the singular part of a mutually local product gives:

Definition 2.3.6. Operator Product Expansion (OPE) *of fields is*

$$\phi(z)\psi(w) \sim \sum_{j=0}^{N-1} \frac{1}{(z-w)^{j+1}} \eta^j(w).$$

We have arrived at the working physicist's OPE, whose importance stems from the fact that the singular part of the OPE encodes *all* the brackets (2.20), and so provides a complete description of the product of mutually local fields.

Definition 2.3.7. *For $n > 0$, the n^{th}* Product *$\phi(w) \underset{(n)}{\times} \psi(w)$ is the $\eta_n(w)$ coefficient of the $\phi(z)\psi(w)$ OPE:*

$$\phi(w) \underset{(n)}{\times} \psi(w) := \mathrm{Res}_{z=w}\left([\phi(z), \psi(w)](z-w)^n\right)$$

The Leibnitz rule for the normally ordered product $\partial{:}\phi(z)\psi(z){:} = {:}\partial\phi(z)\psi(z){:} + {:}\phi(z)\partial\psi(z){:}$ allows one to extend the n^{th} product to encompass negative integers:

$$\phi(z) \underset{(-n-h)}{\times} \psi(z) = :\partial^{(n)}\phi(z)\psi(z): \tag{2.21}$$

We shall need the following uniqueness theorem, originally due to Goddard.

Theorem 2.3.1. (Uniqueness) *[Kac98, 4.4] Let V be a VA and $\phi(z)$ an* $\mathrm{End}(V)$*-valued field which is mutually local with respect to all fields $Y(\eta, z)$, $\eta \in V$. If $\phi(z)|0\rangle = e^{zT}(\phi)$ for some $\phi \in V$, then $\phi(z) = Y(\phi, z)$.*

Corollary 2.3.2. *Let V be a VA. For all $\phi, \psi \in V$ and $n \in \mathbb{Z}$,*

$$Y(\phi_{(n)}\psi, z) = Y(\phi, z) \underset{(n)}{\times} Y(\psi, z).$$

Proof. First we show both sides of

$$\left(Y(\phi,z) \underset{(n)}{\times} Y(\psi,z)\right)|0\rangle = Y(\phi_{(n)}\psi,z)|0\rangle$$

satisfy the differential equation

$$\partial f(z) = [T, f(z)]$$

for $f(z) \in V[\![z,z^{-1}]\!]$. The LHS is a solution since both ∂ and $\mathrm{ad}(T)$ act as derivations on n^{th} products. By the translation axiom (II), the RHS is a solution. The coincidence of initial conditions follows from applying the vacuum axiom (I.b), which tells us both sides have constant term $\phi_{(n)}\psi_{(-1)}$.

Now using equation (2.16), we find

$$Y(\phi,z) \underset{(n)}{\times} Y(\psi,z)|0\rangle = Y(\phi_{(n)}\psi,z)|0\rangle = e^{zT}(\phi_{(n)}\psi)$$

By Dong's Lemma [Kac98, §3.2], $Y(\phi,z) \underset{(n)}{\times} Y(\psi,z)$ is mutually local with respect to all fields $Y(\eta,z)$, $\eta \in V$, so the result follows from the Uniqueness Theorem 2.3.1. □

One use-case is: given a VA $(V,|0\rangle,Y)$, with some vectors $\phi^\alpha \in V_{h_\alpha}$ and fields $\phi^\alpha(z) = \sum_{j\in\mathbb{Z}} \phi^\alpha_{(j)} z^{-j-h_\alpha}$ of weight h_α,

$$Y\left(\phi^{\alpha_1}_{(-j_1-h_{\alpha_1})}\cdots\phi^{\alpha_1}_{(-j_n-h_{\alpha_n})}|0\rangle, z\right) = \frac{1}{j_1!\cdots j_n!}{:}\partial^{j_1}\phi^{\alpha_1}(z)\cdots\partial^{j_n}\phi^{\alpha_n}(z){:}. \tag{2.22}$$

This is a combination of equation (2.21) and Corollary 2.3.2.

Alternatively, starting with algebraic data, it is possible to build a VA. The construction is due to the following existence theorem, sometimes called the 'reconstruction' theorem, together with VA uniqueness, due to Theorem 2.3.1.

Theorem 2.3.2. (Existence) *[Kac98, 4.5], [Fre+95, 3.1] Take a collection of vectors* $\{\phi^\alpha \in V_{h_\alpha} \mid \alpha \in A\}$*, for* A *some indexing set, including* $\phi^0 = |0\rangle$*, and associated fields* $\phi^\alpha(z) = \sum_{j\in\mathbb{Z}} \phi^\alpha_{(j)} z^{-j-h_\alpha}$ *of weight* h_α*, including* $\phi^0(z) = 1_V$*. If the fields* $\phi^\alpha(z)$ *enjoy the following properties for all* $\alpha \in A$*:*

1. $\phi^\alpha(z), \phi^\beta(z)$ *are mutually local for all* $\beta \in A$*;*

2. $\lim_{z\to 0}\phi^\alpha(z)|0\rangle = \phi^\alpha$*;*

3. *the state space* V *is spanned by*

$$\phi^{\alpha_s}_{(j_1-h_{\alpha_1})}\cdots\phi^{\alpha_n}_{(j_n-h_{\alpha_n})}|0\rangle, \quad j_1,\ldots,j_n \in \mathbb{Z};$$

4. there exists $T \in \mathrm{End}V$ such that $T|0\rangle = 0$ and $[T, \phi^\alpha(z)] = \partial\phi^\alpha(z)$.

Then the formula

$$Y\left(\phi^{\alpha_1}_{(j_1-h_{\alpha_1})}\cdots\phi^{\alpha_1}_{(j_n-h_{\alpha_n})}|0\rangle, z\right) := \phi^{\alpha_1}(z) \underset{(j_1-h_{\alpha_1})}{\times} \left(\phi^{\alpha_2}(z) \underset{(j_2-h_{\alpha_2})}{\times} \left(\cdots\left(\phi^{\alpha_n}(z) \underset{(j_n-h_{\alpha_n})}{\times} 1_V\right)\right)\right)$$

defines a unique VA structure on V given by $(V, |0\rangle, Y)$ with translation T and state-field correspondence $Y(\phi^\alpha, z) = \phi^\alpha(z)$.

The final tool we need before proceeding with examples is the famous Borcherds commutator formula, which we present in OPE form:

Proposition 2.3.3. (Borcherds OPE) *[Kac98, 4.6] In the domain $|z| > |w|$, for all $\phi, \psi \in V$,*

$$Y(\phi, z)Y(\psi, w) = \sum_{n=0}^{\infty} \frac{Y(\phi_{(n)}\psi, w)}{(z-w)^{n+1}} + {:}Y(\phi, z)Y(\psi, w){:}.$$

This OPE is a direct consequence of Corollary 2.3.2.

2.3.3 Heisenberg and Clifford VOA

As an example of the preceding axioms, we now reconstruct VOAs from the Heisenberg and Clifford algebras, and their respective Fock spaces.

The bosonic Fock space $\mathfrak{H}^{(0)}$ of charge 0, given in Definition 2.2.2, admits a VOA structure as follows.

Example 2.3.1. (Charge 0 Heisenberg VOA) *The endomorphisms of the state space will be elements of the Heisenberg algebra He, defined in equation (2.12). We need a notion of bosonic normal ordering on He. Since $t_m t_n = t_n t_m$ if $n \neq m$, we can easily place positive (annihilation) modes before negative (creation) modes:*

$$ {:}t_m t_n{:} = \begin{cases} t_n t_m & \text{if } n = -m,\ n < 0, \\ t_m t_n & \text{otherwise.} \end{cases}$$

Define the fields

$$t_k(z) := \frac{1}{k!}\partial^{(k-1)}t(z) \in \mathsf{He}[\![z, z^{-1}]\!]$$

with a base case, sometimes called the bosonic current, a field of weight 1*,*

$$t(z) = t_1(z) := \sum_{n\in\mathbb{Z}} t_n z^{-n-1}.$$

Clearly the vectors

$$t_{k_1} t_{k_2} \cdots t_{k_n}, \quad k_i \in \mathbb{Z}, \tag{2.23}$$

span the space $\mathfrak{H}^{(0)}$*. Let* $1 \in \mathbb{C} \subset \mathfrak{H}^{(0)}$ *be the vacuum element, and let the translation* T *be defined by* $T(1) = 0$ *and* $[T, t_m] = mt_{m+1}$*. By the Reconstruction Theorem 2.3.2 there exists a Heisenberg VOA over* $\mathfrak{H}^{(0)}$*, with state-field correspondence* Y *given by* $Y(t_k, z) = t_k(z)$*, which sends the state* $t_{k_1}t_{k_2}\cdots t_{k_n} \in \mathfrak{H}^{(0)}$ *to the normally ordered product*

$$Y(t_{k_1}t_{k_2}\cdots t_{k_n}, z) = {:}Y(t_{k_1}, z)\cdots Y(t_{k_n}, z){:}.$$

The $t(z)$ *OPE is*

$$t(z)t(w) \sim \frac{\mathbb{1}}{(z-w)^2}.$$

Finally, to upgrade this VA to a VOA, we are free to choose the Virasoro elements from the one-parameter family $\omega_\lambda := \frac{1}{2}t_1^2 + \lambda t_2$*, for* $\lambda \in \mathbb{C}$*, with central charge* $c_\lambda = 1 - 12\lambda^2$*. We will take* $\lambda = 0$ *in the future, so that the central charge is* $c = 1$*, and the Virasoro field is*

$$L(z) := Y(\omega_0, z) = \frac{1}{2}{:}t(z)^2{:}. \tag{2.24}$$

In particular, the mode $L_0 = \sum_{n\in\mathbb{Z}} {:}t_k t_{-k}{:}$ *can be used to calculate the character of the* $\mathfrak{H}^{(0)}$ *VOA at* $c = 1$*, given in Definition 2.3.4. The vacuum vector has weight* $h = 0$*, and any other vector (2.23) has weight equal to the sum of its modes* $h = \sum_{i=1}^n k_i$*. Thus the weight grading is*

$$\mathfrak{H}^{(0)} = \bigoplus_{h\in\mathbb{Z}_{\geq 0}} \mathfrak{H}_h^{(0)}, \quad \dim \mathfrak{H}_h^{(0)} = p(h),$$

where $p(h)$ *is the number of partitions of* h *into natural numbers, giving the character*

$$\chi(q) = \sum_{n\in\mathbb{Z}_{\geq 0}} p(n)q^n = \prod_{n\in\mathbb{N}} \frac{1}{1-q^n} = q^{1/24}\frac{1}{\eta(q)} \tag{2.25}$$

where $\eta(q) := q^{1/24}\prod_{n\in\mathbb{N}}(1-q^n)$ *is the* Dedekind Eta Function*.*

The fermionic Fock space $\mathfrak{F}$, given in Definition 2.1.2, also admits a VOA structure. Here we describe the Neveu-Schwarz sector $\mathfrak{F}(\mathbb{F})$.

Example 2.3.2. (Clifford VOA) *The fields will be* Cl*-valued, as in Definition 2.1.6. Define the fields*

$$\psi_{k+1/2}(z) := \frac{1}{k!}\partial^{(k)}\psi(z) \in \mathsf{Cl}[\![z, z^{-1}]\!], \quad \psi^{\dagger}_{-k-1/2}(z) := \frac{1}{k!}\partial^{(k)}\psi^{\dagger}(z) \in \mathsf{Cl}[\![z, z^{-1}]\!],$$

for $k \in \mathbb{N}$*, with base cases*

$$\psi_{1/2}(z) = \psi(z) := \sum_{j\in\mathbb{F}} \psi_j z^{-j-1/2}, \quad \psi^{\dagger}_{-1/2}(z) = \psi^{\dagger}(z) := \sum_{j\in\mathbb{F}} \psi^{\dagger}_{-j} z^{-j-1/2}, \tag{2.26}$$

called the charge ± 1 *fermionic fields, each of weight* $1/2$*.*

Let $|0\rangle \in \mathfrak{F}$ *be the vacuum element, and let the translation* T *be defined by* $T|0\rangle = 0$ *and* $[T, \psi_j] = (-j+1/2)\psi_{j-1}$ *and* $[T, \psi^{\dagger}_{-j}] = (-j+1/2)\psi^{\dagger}_{-j+1}$*. Lastly observe that we can span* $\mathfrak{F}$ *by* $|\lambda; m\rangle$*, which by equation (2.7) is a product of operators* $\psi_{k+1/2}$ *and* $\psi^{\dagger}_{-k-1/2}$ *applied to* $|m\rangle$*, where* $k \in \mathbb{Z}_{\geq 0}$*. Therefore, by the Reconstruction Theorem 2.3.2 there exists a Clifford VOA over* $\mathfrak{F}$*, with state-field correspondence* Y *given by*

$$\begin{aligned} Y(|\lambda; m\rangle, z) &= Y\left(\prod_{i=1}^{d(\lambda)} \left(\psi_{m+a_i+1/2}\psi^{\dagger}_{-m-b_i-1/2}\right)|m\rangle, z\right) \\ &= :\prod_{i=1}^{d(\lambda)} \left(Y(\psi_{m+a_i+1/2}, z)Y(\psi^{\dagger}_{-m-b_i-1/2}, z)\right) Y(|m\rangle, z): \end{aligned}$$

where $Y(\psi_{k+1/2}|0\rangle, z) = \psi_{k+1/2}(z)$ *and* $Y(\psi^{\dagger}_{-k-1/2}|0\rangle, z) = \psi^{\dagger}_{-k-1/2}(z)$*, and*

$$Y(|m\rangle, z) = :\partial^{(m-1)}\psi(z)\cdots\partial\psi(z)\psi(z): \quad Y(|{-m}\rangle, z) = :\partial^{(m-1)}\psi^{\dagger}(z)\cdots\partial\psi^{\dagger}(z)\psi^{\dagger}(z): \tag{2.27}$$

for $m > 0$*, and* $Y(|0\rangle, z) = 1$*. The vacuum field has the above expression since* $|\pm m\rangle$*,* $m > 0$*, can be written in terms of* $|0\rangle$ *as*

$$|m\rangle = \psi_{m+1/2}\cdots\psi_{1/2}|0\rangle, \quad |{-m}\rangle = \psi^{\dagger}_{-m-1/2}\cdots\psi^{\dagger}_{-1/2}|0\rangle.$$

The various OPEs are

$$\psi(z)\psi^{\dagger}(w) \sim \frac{\mathbb{1}}{z-w}, \quad \psi(z)\psi(w) \sim 0, \quad \psi^{\dagger}(z)\psi^{\dagger}(w) \sim 0.$$

To promote this VA to a VOA, we note the one-parameter family of Virasoro elements [Kac98, 5.1]

$$\omega_{\lambda} = (1-\lambda)\psi_{3/2}\psi^{\dagger}_{-1/2}|0\rangle + \lambda\psi^{\dagger}_{-3/2}\psi_{1/2}|0\rangle \tag{2.28}$$

The central charge, in terms of this parameter, is $c_\lambda = -12\lambda^2 + 12\lambda - 2$. Henceforth we set $\lambda = 1/2$, so that $c = 1$. The Virasoro field is therefore

$$L(z) := Y(\omega_{1/2}, z) = \frac{1}{2}(:\partial\psi(z)\psi^\dagger(z): + :\psi(z)\partial\psi^\dagger(z):) \tag{2.29}$$

Remark 2.3.1. *In terms of the field*

$$\alpha(z) := :\psi(z)\psi^\dagger(z):$$

the Virasoro field of the $c = 1$ Clifford VOA is

$$L(z) = \frac{1}{2}:\alpha(z)^2:$$

c.f. the Virasoro field (2.24) of the $c = 1$ Heisenberg VOA. This is a VOA version of the algebraic Fermion-Boson correspondence of Equation (2.13).

2.3.4 $W_{1+\infty}$ VOA

We use the technology of Section 2.1.2 to establish a free fermion realisation of the infinite Lie algebra $W_{1+\infty}$, by embedding the generators in $\widehat{\mathfrak{gl}}(\infty)$. A VOA is constructed in the manner of the previous section.

Definition 2.3.8. *[Sch08, §5.1] The* Witt Algebra Wi *is the Lie algebra of the diffeomorphism group of the circle S^1, equivalent to the algebra of vector fields on $\mathbb{C}^*$ with coordinate z, whose generators are*

$$l_n = -z^{n+1}\partial_z, \quad n \in \mathbb{Z},$$

with commutation relations

$$[l_n, l_m] = (n - m)l_{n+m}.$$

Definition 2.3.9. *[Sch08, §5.3] The* Virasoro Algebra $\mathsf{Vir}_c = \mathsf{Wi} \oplus \mathbb{C}\mathbb{1}$ *is the unique non-trivial central extension of the Witt algebra* Wi, *with commutation relations*

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}\mathbb{1} \tag{2.30}$$

at Central Charge c.

We consider a higher-order generalisation of the Witt algebra: the Lie algebra D of differential operators on the punctured complex plane $x \in \mathbb{C}^*$, with generators [Fre+95, §1]:

$$J_n^k := x^{n+k}(-\partial_x)^k, \quad n \in \mathbb{Z}, k \in \mathbb{Z}_{\geq 0}. \tag{2.31}$$

The generators act on $E_j := x^{-j-1/2}, j \in \mathbb{F}$ by

$$J_n^k E_j = (-1)^k \left[-j - \frac{1}{2}\right]_k E_{j-n}$$

where we have used the falling factorial $[s]_l := s(s-1)\dots(s-l+1)$. This allows us to construct the following embedding

$$\rho : \mathsf{D} \to \overline{\mathsf{gl}}(\infty) : J_n^k \mapsto (-1)^k \sum_{j \in \mathbb{F}} \left[-j - \frac{1}{2}\right]_k e_{j-n,j}, \tag{2.32}$$

recalling the notation e_{ij} for the elementary matrices of $\mathsf{gl}(\infty)$.

Now we use the cocycle Ψ given in equation (2.10), which defined the central extension $\widehat{\mathsf{gl}}(\infty)$ of $\bar{\mathsf{gl}}(\infty)$. Lift the $\bar{\mathsf{gl}}(\infty)$ cocycle Ψ to a D cocycle $\psi := \Psi \circ \rho^{-1}$.

Definition 2.3.10. *[Kac98, §5.3] The $W_{1+\infty}$* Algebra $\widehat{\mathsf{D}} = \mathsf{D} + \mathbb{C}C$ *is the central extension of* D *by the cocycle* ψ*.*

To realise the $W_{1+\infty}$ algebra in $\widehat{\mathsf{gl}}(\infty)$, we extend the embedding ρ to

$$\hat{\rho} : \widehat{\mathsf{D}} \to \widehat{\mathsf{gl}}(\infty)$$

by letting $\hat{\rho}(C) := K$, where K is the central element of $\widehat{\mathsf{gl}}(\infty)$.

We can define a translation by the derivation $T := \partial_x$ of the Lie algebra D, which lifts to the central extension $\widehat{\mathsf{D}}$ by letting $T(C) = 0$. To check this, it suffices to notice $\psi([\partial_x, a], b) = -\psi(a, [\partial_x, b])$ for $a, b \in \mathsf{D}$. It is immediate that $[T, J^k(z)] = \partial_z J^k(z)$.

To construct a VA from this data, we need a graded vector space for $\widehat{\mathsf{D}}$ to act on. A common construction [Awa+95] is to realise $\widehat{\mathsf{D}}$ by the Clifford VOA:

$$W : \widehat{\mathsf{D}} \xrightarrow{\hat{\rho}} \widehat{\mathsf{gl}}(\infty) \xrightarrow{\hat{r}} \mathsf{Cl},$$

where we use the $\widehat{\mathsf{gl}}(\infty)$ embedding $\hat{r}$ defined in (2.9). This lets us define the weight $k+1$ Clifford field,

$$\Phi^k(z) := \sum_{n \in \mathbb{Z}} W(J_n^k) z^{-n-(k+1)} \in \mathsf{Cl}[\![z, z^{-1}]\!], \quad k \in \mathbb{Z}_{\geq 0}.$$

The fields $\Phi^k(z)$ are all mutually local [Kac98, §2.10]. We can write an attractive expression using the fermionic fields $\psi(z), \psi^\dagger(z)$ defined in (2.26):

Proposition 2.3.4.

$$\Phi^k(z) = {:}\psi(z)(-\partial)^k\psi^\dagger(z){:}. \tag{2.33}$$

Proof. By definition of the embedding (2.32) , we have

$$\Phi^k(z) = (-1)^k \sum_{i,j\in\mathbb{F}} \left[-j-\frac{1}{2}\right]_k {:}\psi_i\psi^\dagger_{-j}{:}z^{-i-j-(k+1)}. \tag{2.34}$$

To make the identification, differentiate k many times the normal ordered product ${:}\psi(z)\psi^\dagger(w){:}$ and set $z=w$, to obtain the above sum. $\square$

The modes $\Phi^k_n = W(J^k_n)$ of this field annihilate the vacuum, $\Phi^k_n|0\rangle = 0$, for $n+k\geq 1$. To see this one must note that falling factorials $[s]_l$ vanish for $l \geq s+1$. Now since Φ^k_n are charge 0 operators, linear combinations of the vectors

$$\Phi^{k_m}_{n_m}\dots\Phi^{k_1}_{n_1}|0\rangle, \quad n_i + k_i \geq 1 \text{ for } 1\leq i\leq m$$

produce any $|\lambda;0\rangle$, $\lambda\in\mathcal{P}$, and so span $\mathfrak{F}^{(0)}$. Define a state-field correspondence Y by:

$$Y(\Phi^k_n|0\rangle, z) := \frac{1}{(-n-k-1)!}\partial^{(-n-k-1)}\Phi^k(z). \tag{2.35}$$

These data determine a $W_{1+\infty}$ VA.

To upgrade to a $W_{1+\infty}$ VOA we must determine a Virasoro element. To do this, we use a popular basis for computation in D, namely:

$$L^k_n := x^n(-D)^k, \quad n\in\mathbb{Z}, k\in\mathbb{Z}_{\geq 0}, \tag{2.36}$$

where $D = x\partial_x$ is the Euler derivative. The passage between these bases is via

$$J^k_n = (-1)^k x^n [D]_k = L^k_n + \dots \tag{2.37}$$

In this language it is easier to calculate products, e.g. $x^r f(D) x^s g(D) = x^{r+s} f(D+s) g(D)$ for polynomials f, g. Furthermore, the cocycle takes the form [KR93, 1.5.5]:

$$\psi(x^r f(D), x^s g(D)) = \begin{cases} \sum_{-r\leq j\leq -1} f(j)g(j+r) & \text{if } r=-s>0, \\ 0 & \text{if } r+s\neq 0 \text{ or } r=s=0. \end{cases}$$

Example 2.3.3.

$$\psi(L^0_n, L^0_m) = n\delta_{n+m,0}, \quad \psi(L^1_n, L^1_m) = -\frac{n^3-n}{6}\delta_{n+m,0}, \quad \psi(L^0_n, L^1_m) = \frac{n^2-n}{2}\delta_{n+m,0}, \quad n\geq 0.$$

Introduce a grading $\widehat{\mathsf{D}} = \bigoplus_{k\in\mathbb{Z}_{\geq 0}} \widehat{\mathsf{D}}^{(k)}$ by the differential order k of the operator L_n^k. By the calculations in the above example, $\widehat{\mathsf{D}}^{(0)}$ is isomorphic to the Heisenberg algebra He with central element C. Moreover $\widehat{\mathsf{D}}^{(1)}$ contains a 1-parameter family of Virasoro algebra Vir_c, with generators

$$L_k(\lambda) := L_k^1 + \lambda(k+1)L_k^0,$$

satisfying the Virasoro relations (2.30), with central element C and central charge

$$c_\lambda = -12\lambda^2 + 12\lambda - 2.$$

We can therefore define a Virasoro element [Fre+95, 3.1]:

$$\omega_\lambda := \Phi^1_{-2}|0\rangle - \lambda\Phi^0_{-2}|0\rangle, \tag{2.38}$$

for $\lambda \in \mathbb{C}$, with corresponding Virasoro field

$$L(z) = Y(\omega_\lambda, z) = \Phi^1(z) - \lambda\partial\Phi^0(z) = \sum_{k\in\mathbb{Z}} L_k(\lambda)z^{-k-2}.$$

This provides all data needed to reconstruct, by Theorem 2.3.2, a $W_{1+\infty,c}$ VOA at central charge c, isomorphic to the charge 0 Clifford VOA $\mathfrak{F}^{(0)}$. The isomorphism sends respective Virasoro elements (2.28) to (2.38).

Remark 2.3.2. *The history of the $W_{1+\infty}$ algebra begins as an extension of the Virasoro algebra, in the context of extending the symmetry algebra of $2d$ CFT to higher conformal spin: the currents of spin $h = 2, \ldots, N$ and their non-linear commutators can be arranged into the so-called W_N algebra [Zam85]. Depending on how the background charge scales with N, there are many $N \to \infty$ limits of W_N. One such limit is w_∞, the Lie algebra of area-preserving diffeomorphisms of $2d$ phase space. Now w_∞ only admits a central extension in its spin $h = 2$ (Witt) sector [Awa+95, §1]. However there exists a deformation [PRS90] of w_∞ that admits a central extension in all spin $h = 2, \ldots, \infty$ sectors, called the W_∞ Lie algebra. Including the spin 1 current after the fact produces the very same $W_{1+\infty}$ algebra we have discussed above.*

2.4 Boson-Fermion Correspondence

The following construction marks the beautiful culmination of this chapter's ideas. We will roughly follow [Kac98, 5.2].

We begin recalling Example 2.3.2, the field-theoretic interpretation of the algebraic Fermion-Boson correspondence of Proposition 2.2.1. There, we found a Heisenberg representation in $\mathfrak{F}$ via the modes $\alpha_k \in \mathsf{Cl}$ defined in equation (2.13).

Can we attempt the opposite direction, expressing the fermionic fields $\psi(z), \psi^\dagger(z)$ in terms of the bosonic field $\alpha(z)$? This is clearly impossible, since $\alpha(z)$ is a charge 0 operator, while $\psi(z), \psi^\dagger(z)$ have charges ± 1 respectively.

The way around this is to consider the conjugate pairing of α_n/n with α_{-n}, for $n > 0$, induced by the Heisenberg relations $[\alpha_m, \alpha_n] = m\delta_{m+n,0}$. To do the same with the charge operator α_0, we must introduce a new operator $\beta \in \mathsf{Cl}$, conjugate to α_0 in the sense that $[\alpha_k, \beta] = \delta_{k0}, [\beta, \mathbb{1}] = 0$. It is defined such that $w = e^\beta$ is the charge shift operator:

$$w\psi_n w^{-1} = \psi_{n+1}, \quad w\psi_n^\dagger w^{-1} = \psi_{n-1}^\dagger,$$

which therefore acts on vacua as

$$w^{\pm 1}|m\rangle = |m \pm 1\rangle.$$

Now we make a leap-of-faith definition. Define $\alpha_\pm(z) := \sum_{k>0} \alpha_{\pm k} \frac{z^{\mp k}}{k}$.

Definition 2.4.1. *The* Veneziano Field *is*

$$\begin{aligned}\phi(z) &= \beta + \alpha_0 \ln z + \sum_{k>0} \frac{\alpha_{-k}}{k} z^k - \sum_{k>0} \frac{\alpha_k}{k} z^{-k} \\ &= \beta + \alpha_0 \ln z + \alpha_-(z) - \alpha_+(z).\end{aligned}$$

Clearly $\alpha(z) = \partial\phi(z)$.

Proposition 2.4.1. *The exponential of the Veneziano field corresponds to the charge m vacuum field:*

$$Y(|m\rangle, z) = {:}e^{m\phi(z)}{:}.$$

Proof following [Kac98, §5.2]. Define an auxiliary field

$$\eta_m(z) := w^{-m}\Gamma_-(z)^{-m} Y(|m\rangle, z)\Gamma_+(z)^m \in \mathsf{Cl}[\![z, z^{-1}]\!]$$

where

$$\Gamma_{\pm}(z) := e^{\alpha_{\pm}(z)}. \tag{2.39}$$

We claim:

$$\eta_m(z) = z^{m\alpha_0}.$$

Consider the OPE of the charge m field $Y(|m\rangle, z)$ with the bosonic field. From the Borcherds OPE, Proposition 2.3.3, we find

$$\alpha(w)Y(|m\rangle, z) \sim \sum_{k\geq 0} \frac{Y(\alpha_k|m\rangle, z)}{(w-z)^{k+1}} = \frac{mY(|m\rangle, z)}{w-z},$$

since $\alpha_k|m\rangle = \delta_{0k}m|m\rangle$. Equivalently,

$$[\alpha_k, Y(|m\rangle, z)] = mz^k Y(|m\rangle, z).$$

We find that α_k commutes with $\eta_m(z)$ for all $k \in \mathbb{Z}$.

From Equation (2.27), we have an expression for the field $Y(|m\rangle, z)$ in terms of ψ or $\psi^\dagger$ for $m > 0$ or $m < 0$ respectively. In particular, apply the field

$$Y(|m\rangle, z) = {:}\partial^{(m-1)}\psi(z)\cdots\partial\psi(z)\psi(z){:}, \quad m > 0,$$

to the charge n vacuum:

$$\begin{aligned} Y(|m\rangle, z) &: \mathfrak{F}^{(n)} \to \mathfrak{F}^{(n+m)}[\![z, z^{-1}]\!] \\ &: |n\rangle \mapsto z^{mn}|n+m\rangle + \ldots, \end{aligned}$$

where we have omitted the higher energy non-vacuum states. Due to the charge shift operator w^{-m}, the auxiliary field has the following effect on the vacuum:

$$\begin{aligned} \eta_m(z) &: \mathfrak{F}^{(n)} \to \mathfrak{F}^{(n)}[\![z, z^{-1}]\!] \\ &: |n\rangle \mapsto z^{mn}|n\rangle + \ldots \end{aligned}$$

We know that the representation of the Heisenberg algebra He given by α_k acting on $\mathfrak{F}^{(n)}$ is irreducible, by Proposition 2.2.1. So if η_n commutes with all α_k, it must be the central element $\mathbb{1} \in \mathsf{He}$. Therefore

$$\eta_m(z)|_{\mathfrak{F}^{(n)}} = z^{mn}\mathbb{1}_{\mathfrak{F}^{(n)}}$$

and so $\eta_m(z) = z^{m\alpha_0}$ as claimed. Exponentiate the Veneziano field $\phi(z)$ and rearrange to obtain:

$${:}e^{m\phi(z)}{:} = w^m\Gamma_-(z)^m z^{m\alpha_0}\Gamma_+(z)^{-m} = Y(|m\rangle, z).$$

$\square$

To realise a fermionic field acting on bosonic Fock space $\mathfrak{H}$, we must write down a different version of the field $:e^{m\phi(z)}:$ with Heisenberg-valued coefficients—namely, define

$$\Psi_m(z) := w^m z^{mt_0} \Gamma_-(z)^m \Gamma_+(z)^{-m} \in \mathsf{He}[\![z, z^{-1}]\!] \tag{2.40}$$

where we have $w \in \mathsf{He}$, and the fields $\Gamma_\pm(z) \in \mathsf{He}[\![z, z^{-1}]\!]$ are

$$\Gamma_-(z) = e^{-\sum_{j>0} t_j z^j}, \quad \Gamma_+(z) = e^{\sum_{j>0} \partial_{t_j} \frac{z^{-j}}{j}}$$

substituting the Heisenberg operators $t_j \in \mathsf{He}$, defined in equation (2.12), in place of the Heisenberg operators $\alpha_j \in \mathsf{Cl}$. One important use of this field is to implement charge in the following VOA:

Example 2.4.1. (Charged Heisenberg VOA) *To upgrade the uncharged Heisenberg VOA over $\mathfrak{H}^{(0)}$ from Example 2.3.1 to a Heisenberg VOA over full Fock space $\mathfrak{H}$, the only difference in VOA data is a new state-field correspondence:*

$$Y(t_{k_1} t_{k_2} \cdots t_{k_N} w^m, z) = :Y(t_{k_1}, z) \cdots Y(t_{k_N}, z) \Psi_m(z):.$$

One important implication is the shifting of vacuum energy by charge. Writing $L_0 = \frac{1}{2}:t(z)^2: = \frac{t_0^2}{2} + \sum_{n \in \mathbb{N}} t_{-n} t_n$, we find that the weight of a state in $\mathfrak{H}^{(n)}$ is

$$L_0(t_{k_1} t_{k_2} \cdots t_{k_N} w^n) = \left(\frac{n^2}{2} + \sum_{i=1}^{N} k_i \right) (t_{k_1} t_{k_2} \cdots t_{k_N} w^n). \tag{2.41}$$

Combining Proposition 2.4.1 with the Fermion-Boson correspondence noted of Remark 2.3.1, the final VOA isomorphism is:

Theorem 2.4.1. (Boson-Fermion Correspondence) *There exists an isomorphism $\sigma : \mathfrak{F} \to \mathfrak{H}$ of VOAs, such that*

$$\sigma|n\rangle = w^n$$

and

$$\sigma:\psi(z)\psi^\dagger(z):\sigma^{-1} = t(z)$$

and in general, for charge $m > 0$,

$$\sigma:\partial^{m-1}\psi(z) \cdots \partial\psi(z)\psi(z):\sigma^{-1} = \Psi_m(z),$$
$$\sigma:\partial^{m-1}\psi^\dagger(z) \cdots \partial\psi^\dagger(z)\psi^\dagger(z):\sigma^{-1} = \Psi_{-m}(z).$$

2.4.1 Schur Functions

We now make the Boson-Fermion map σ more explicit: the famous Schur functions $s_\lambda(t) \in \mathfrak{H}$ become the analogues of $|\lambda; 0\rangle \in \mathfrak{F}$.

We begin by defining a generating function:

$$\Gamma_\pm(\mathbf{x}) := e^{\alpha_\pm(\mathbf{x})} \tag{2.42}$$

where $\alpha_\pm(\mathbf{x}) = \sum_{k>0} x_{\pm k}\alpha_{\pm k}$, and we have introduced the generating function variables $\mathbf{x} = \{\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots\}$. In the previous section, we took $x_{\pm k} = z^{\pm k}/k$.

By definition of the vacua, we know $\langle n|\alpha_-(\mathbf{x}) = 0 = \alpha_+(\mathbf{x})|n\rangle$, and so

$$\langle n|\Gamma_-(\mathbf{x}) = \langle n|, \quad \Gamma_+(\mathbf{x})|n\rangle = |n\rangle. \tag{2.43}$$

We will now expand more the complicated states $\Gamma_-(\mathbf{x})|\lambda; n\rangle$ and $\langle\lambda; n|\Gamma_+(\mathbf{x})$ as linear combinations of basis states, whose coefficients are the famous Schur functions.

Definition 2.4.2. *The* ELEMENTARY SCHUR POLYNOMIAL $s_k(\mathbf{x})$ *is defined implicitly by*

$$e^{\sum_{k\geq 1} x_k z^k} = \sum_{k\geq 0} s_k(\mathbf{x}) z^k,$$

or explicitly by

$$s_k(\mathbf{x}) = \sum_{|\mu|=k} \prod_{i=1}^{\ell(\mu)} \frac{x_i^{\mu_i}}{\mu_i!}.$$

The fermionic modes transform diagonally under the adjoint action of $\Gamma_\pm(\mathbf{x})$, with elementary Schur functions for coefficients:

Lemma 2.4.1. *[AZ13, §2.9]*

$$\Gamma_\pm(\mathbf{x})\psi_i\Gamma_\pm(\mathbf{x})^{-1} = \sum_{k\in\mathbb{Z}^{\geq 0}} \psi_{i\mp k} s_k(\mathbf{x}), \quad \Gamma_\pm(\mathbf{x})\psi_i^\dagger\Gamma_\pm(\mathbf{x})^{-1} = \sum_{k\in\mathbb{Z}^{\geq 0}} \psi_{i\pm k}^\dagger s_k(-\mathbf{x}).$$

Definition 2.4.3. *The* SCHUR FUNCTION *associated to a partition* $\lambda \in \mathcal{P}$ *is defined by the Jacobi-Trudi formulas [Mac95, I §3.4]*

$$s_\lambda(\mathbf{x}) = \det_{i,j=1,\ldots,\ell(\lambda)} s_{\lambda_i+j-i}(\mathbf{x}). \tag{2.44}$$

Definition 2.4.4. *A* HOOK *of a Young diagram* Y *is a Young diagram* $h \subset Y$ *of at least one box, with one arm of length* $a(h) \geq 0$ *and one leg of length* $b(h) \geq 0$.

The Schur function s_λ can be calculated as a determinant of Schur functions of the hooks $(1+a_i, \underbrace{1,\ldots,1}_{b_i})$ composed of the i^{th} arm-leg pair in the of the Young diagram Y_λ. Here we are using $(a_i|b_i)_{i=1,\ldots,d(\lambda)}$, the Frobenius coordinates of Y_λ from Definition 1.3.1. The formula is:

Theorem 2.4.2. *(Giambelli) [Mac95, I §3 Ex. 9]*

$$s_\lambda(\mathbf{x}) = \det_{i,j=1,\ldots,d(\lambda)} s_{(a_i|b_j)}(\mathbf{x}),$$

Proposition 2.4.2. *[AZ13, §2.10.2]*

$$\Gamma_-(\mathbf{x})|n\rangle = \sum_{\lambda\in\mathcal{P}} (-1)^{b(\lambda)} s_\lambda(\mathbf{x})|\lambda; n\rangle, \quad \langle n|\Gamma_+(\mathbf{x}) = \sum_{\lambda\in\mathcal{P}} (-1)^{b(\lambda)} s_\lambda(\mathbf{x})\langle\lambda; n|, \tag{2.45}$$

where $b(\lambda) = \sum_{i=1}^{d(\lambda)}(b_i+1)$.

Proof. We treat just the first formula, since the second follows from a similar argument.

We will need the expression for the Schur polynomial of a hook, which is an easy case of the Jacobi-Trudy definition (2.44) of Schur functions:

$$s_{(a|b)}(\mathbf{x}) = (-1)^b \sum_{m=0}^{b} s_{a+1-m}(\mathbf{x}) s_{b+m}(-\mathbf{x}). \tag{2.46}$$

Using Wick's Theorem 2.1.1, Lemma 2.4.1, and Giambelli's Theorem, 2.4.2, we find

$$\begin{aligned}
\langle\lambda; n|\Gamma_-(\mathbf{x})|n\rangle &= \langle n|\psi^\dagger_{n+a_1}\cdots\psi^\dagger_{n+a_{d(\lambda)}}\psi_{n-b_{d(\lambda)}-1}\cdots\psi_{n-b_1-1}\Gamma_-|n\rangle \\
&\underset{(2.1.1)}{=} \det_{1\le i,j\le d(\lambda)} \langle n|\psi^\dagger_{n+a_i}\psi_{n-b_j-1}\Gamma_-|n\rangle \\
&\underset{(2.43)}{=} \det_{1\le i,j\le d(\lambda)} \langle n|\Gamma_-^{-1}\psi^\dagger_{n+a_i}\Gamma_-\Gamma_-^{-1}\psi_{n-b_j-1}\Gamma_-|n\rangle \\
&\underset{(2.4.1)}{=} \det_{1\le i,j\le d(\lambda)} \sum_{k=0}^{a(\lambda)} s_{a_i-k}(\mathbf{x}) s_{b_j+k+1}(-\mathbf{x}) \\
&\underset{(2.46)}{=} \det_{1\le i,j\le d(\lambda)} (-1)^{b_j+1} s_{(a_i|b_j)}(\mathbf{x}) \\
&\underset{(2.4.2)}{=} (-1)^{b(\lambda)} s_\lambda(\mathbf{x}).
\end{aligned}$$

Since the $|\lambda; n\rangle$ basis of $\mathfrak{F}$ is orthonormal, we have proved the original statement. □

One can generalise this statement to non-vacuum states, using the following coefficients:

Definition 2.4.5. *The* SKEW SCHUR FUNCTIONS $s_{\lambda\backslash\mu}$ *are given by the modified Jacobi-Trudi formula,*

$$s_{\lambda\backslash\mu}(\mathbf{x}) = \det_{1\leq i,j\leq \ell(\lambda)} s_{\lambda_i - i + \mu_j - j}(\mathbf{x}).$$

Proposition 2.4.3. *[AZ13, §2.10.2]*

$$\Gamma_-(\mathbf{x})|\lambda; n\rangle = \sum_{\mu} (-1)^{b(\mu)-b(\lambda)} s_{\mu\backslash\lambda}(\mathbf{x})|\mu; n\rangle$$

$$\Gamma_+(\mathbf{x})|\lambda; n\rangle = \sum_{\mu} (-1)^{b(\lambda)-b(\mu)} s_{\lambda\backslash\mu}(\mathbf{x})|\mu; n\rangle$$

With the machinery of this section, we make the Boson-Fermion correspondence map explicit. Set the generating function variables to be the Heisenberg Fock space elements, $x_k := t_{-k}$.

Corollary 2.4.1. *The Fock space map σ of Theorem 2.4.1 is*

$$\sigma|\lambda; n\rangle = \sum_{m\in\mathbb{Z}} w^m \langle m|\Gamma_+(t)|\lambda; n\rangle = (-1)^{b(\lambda)} w^n s_\lambda(t)$$

The upshot is that Schur functions are the counterparts to fermionic states inside bosonic Fock space, a theme which will be expanded upon in Chapter 3.

2.4.2 Partition Invariant Operators

As an application of the correspondence between Heisenberg and Clifford VOAs, we show how to realise the $W_{1+\infty}$ fields in the Heisenberg VOA. We define $W_{1+\infty}$ fields whose eigenvalues are the partition invariants p_k.

Recall the $W_{1+\infty}$ fermionic field $\Phi^{k-1}(z) = {:}\psi(z)(-\partial)^{k-1}\psi^\dagger(z){:}$ of weight k, defined in equation (2.33). A bosonic expression of this field is available as follows: write down a Taylor expansion

$${:}e^{\phi(z)-\phi(w)}{:} - 1 = \sum_{k\in\mathbb{N}} \frac{(z-w)^k}{k!} W_k(z),$$

for weight $k \in \mathbb{N}$ bosonic fields $W_k(z)$, in terms of the Veneziano field $\phi(z)$ of Definition 2.4.1. We can determine this expansion through the well-known identity [AZ13, 2.92],

$${:}\psi(z)\psi^\dagger(w){:} = \frac{{:}e^{\phi(z)-\phi(w)}{:} - 1}{z-w}.$$

The result is:

Proposition 2.4.4. *[LZ21, 2.11] [FKN92, §4]*

$$\frac{1}{k!}W_k(z) = \frac{1}{(k-1)!}{:}\psi(z)\partial^{k-1}\psi^\dagger(z){:} = \sum_{\sum_{i\geq 1} ik_i = k} \frac{1}{\prod_{i\geq 1} k_i!}{:}\prod_{i\geq 1}\left(\frac{1}{i!}\partial^i\phi(z)\right)^{k_i}{:}. \tag{2.47}$$

We can write a more attractive $W_{1+\infty}$ field by using the $L_n^k = z^n(-D)^k$ generators (2.36) of $W_{1+\infty}$ instead of the $J_n^k = z^n(-\partial)^k$ generators. The prescription is to sandwich the L_0^k between the normally ordered product of fermions [Awa+95, §3.1], similar to how we sandwiched J_0^k between a normally ordered product in equation (2.33).

Definition 2.4.6. *The* Partition Invariant (PI) Field *of weight $k+1$ is*

$$\Pi^k(z) := \frac{1}{k!}{:}\psi(z)(-D)^k\psi^\dagger(z){:} = \sum_{i,j\in\mathbb{F}}(j+1/2)^k{:}\psi_i\psi^\dagger_{-j}{:}z^{-i-j-1}, \quad k\in\mathbb{Z}_{\geq 0}.$$

There is a simple bosonic expression for the PI field, derived by a similar manner to the $W_k(z)$ field above. It is proportional to the top term of equation (2.47):

$$\Pi_k(z) = \frac{1}{(k+1)!}{:}(\partial\phi(z))^{k+1}{:} \tag{2.48}$$

The zero modes of $\Pi_k(z)$, which we call PI Operators, are

$$\Pi_k := \frac{1}{k!}\sum_{i\in\mathbb{F}}(-i+1/2)^k{:}\psi_i\psi_i^\dagger{:}. \tag{2.49}$$

Proposition 2.4.5. *The eigenvalues of the PI operator are given by the partition invariants p_k of equation (2.4):*

$$\Pi_k|\lambda;m\rangle = p_k(X_{\lambda;m})|\lambda;m\rangle.$$

Proof. To see this, notice that ${:}\psi_i\psi_i^\dagger{:}$ can detect if an entry is present in wedge coordinates:

$${:}\psi_i\psi_i^\dagger{:}|\lambda;m\rangle = \left(\begin{cases}1 & i\in\mathbb{F}_{\geq 0}\cap X_{\lambda;m}\\ -1 & i\in\mathbb{F}_{<0}-X_{\lambda;m}\\ 0 & \text{otherwise.}\end{cases}\right)|\lambda;m\rangle.$$

Combine this observation with the definition of partition invariants p_k in equation (2.4), and the definition of wedge coordinates, $X_{\lambda;m} = \{\lambda_j - j + 1/2 + m\}_{j\in\mathbb{N}}$, to obtain the eigenvalues of (2.49). □

At this point, we have the necessary tools to express Dijkgraaf's partition function (1.26) as a trace over NS fermionic Fock space: we substitute PI operators for their eigenvalues in equation (2.5, giving

$$Z(w,q,x) = \mathrm{tr}_{\mathfrak{F}(\mathbb{F})}(w^{\Pi_0}q^{\Pi_1}e^{x\Pi_2}). \tag{2.50}$$

Therefore by Theorem 1.3.1, the simple Hurwitz generating function is a trace over charge 0 Fock space:

$$H^\bullet(q,x) = \mathrm{tr}_{\mathfrak{F}^{(0)}}\left(q^{\Pi_1} e^{x\Pi_2}\right).$$

Chapter 3

Bosons and Fermions in the Symmetric Group

In Chapter 2, we defined bosonic and fermionic operators acting on Fock spaces. This chapter constructs analogous operators acting on the class functions of the symmetric group.

3.1 Class Functions

3.1.1 Symmetric Functions

Any discussion of symmetric functions must defer to the meticulous exposition of [Mac95, I §2]. We satisfy ourselves with some definitions.

Definition 3.1.1. *The ring of* Symmetric Polynomials *in n variables is the S_n-invariant subring*

$$\mathfrak{S}_n := \mathbb{Z}[t_1, \ldots, t_n]^{S_n}.$$

We can grade this ring as $\mathfrak{S}_n = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} \mathfrak{S}_n^k$, where $\mathfrak{S}_n^k$ consists of the homogeneous symmetric functions of degree k. One obvious $\mathbb{Z}$-basis of $\mathfrak{S}_n$ is composed of the symmetric monomials:

$$m_\lambda(t) = \sum_\mu \prod_{i=1}^n t_i^{\mu_i}, \quad \lambda \in \mathcal{P}, \mu \in \{\text{permutations of } \lambda\}.$$

This makes the grading clear: m_λ constitute a basis for $\mathfrak{S}_n^k$, where $\ell(\lambda) \leq n$ and $|\lambda| = k$. For our purposes, we will need symmetric *functions* in infinitely many variables, formulated as a projective limit,

$$\mathfrak{S} := \lim_{\substack{\leftarrow \\ n}} \mathfrak{S}_n,$$

where the projections are

$$\mathfrak{S} \to \mathfrak{S}_n : m_\lambda \mapsto \begin{cases} m_\lambda(t_1, \ldots, t_n) & \ell(\lambda) \leq n, \\ 0 \end{cases} \quad \text{for } n \in \mathbb{Z}_{\geq 0}.$$

Here $\{m_\lambda\}_{\lambda \in \mathcal{P}}$ constitutes a basis for $\mathfrak{S}$. But in practice, one generally uses the following polynomial generators of $\mathfrak{S}_n$, with $0 \leq k \leq n$:

$$\begin{aligned} \textit{(Elementary)} \quad e_k &:= \sum_{i_1 < \ldots < i_k} t_{i_1} \cdots t_{i_k} = m_{(1^k)}, \\ \textit{(Complete)} \quad h_k &:= \sum_{|\lambda| = k} m_\lambda, \\ \textit{(Power-sum)} \quad p_k &:= \sum t_i^k = m_{(k)}. \end{aligned}$$

Let

$$p_\lambda := \prod_{i=1}^{\ell(\lambda)} p_{\lambda_i}$$

where p_k is the k^{th} power-sum symmetric function. These are linearly independent, and in fact $\{p_\lambda\}_{\lambda \in \mathcal{P}_n}$ forms a $\mathbb{Q}$-basis of $\mathfrak{S}_n$. The basis change to elementary and complete functions is given by [Mac95, I §2]:

$$e_k = \sum_{|\lambda|=n} (-1)^{|\lambda| - \ell(\lambda)} \frac{1}{|z_\lambda|} p_\lambda, \quad h_k = \sum_{|\lambda|=n} \frac{1}{|z_\lambda|} p_\lambda, \tag{3.1}$$

The last tool we need to define is a $\mathbb{Z}$-valued bilinear form on basis elements of $\mathfrak{S}$, defined by [Mac95, I §4]

$$\langle h_\lambda, m_\mu \rangle := \delta_{\lambda\mu}, \quad \lambda, \mu \in \mathcal{P}, \tag{3.2}$$

in terms of which the power-sum polynomials form an orthogonal basis $\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} |z_\lambda|$, while the Schur functions satisfy $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$, i.e. $\{s_\lambda\}_{\lambda \in \mathcal{P}_n}$ form an orthonormal basis of $\mathfrak{S}_n$.

3.1.2 Frobenius Characteristic Map

Let $\mathfrak{X}_n := \mathfrak{X}(S_n)$ be the space of class functions of S_n from Definition 1.1.2. Our principal concern will be the infinite dimensional, graded algebra

$$\mathfrak{X} := \bigoplus_{n=0}^{\infty} \mathfrak{X}_n,$$

with multiplication

$$\alpha \star \beta := \mathrm{Ind}_{S_n \times S_m}^{S_{n+m}} (\alpha \otimes \beta) \in \mathfrak{X}_{n+m}, \quad \alpha \in \mathfrak{X}_n, \beta \in \mathfrak{X}_m \tag{3.3}$$

defined by induction:

Definition 3.1.2. *For H a subgroup of a finite group G, the* Induction *of the class function $\alpha \in \mathfrak{X}(H)$ to G is:*

$$\mathrm{Ind}_H^G(\alpha)(s) = \frac{1}{|H|} \sum_{\substack{t \in G \\ t^{-1}st \in H}} \alpha(t^{-1}st) \in \mathfrak{X}(G).$$

The counterpart to induction is the standard Restriction $\mathrm{Res}_H^G(\alpha) \in \mathfrak{X}(H)$ of the class function $\alpha \in \mathfrak{X}(G)$ to H. They are adjoint in the following sense:

Proposition 3.1.1. (Frobenius Reciprocity) *[Ser71, II.7.2] For H a subgroup of a finite group G,*

$$(\alpha, \mathrm{Res}_H^G(\beta))_H = (\mathrm{Ind}_H^G(\alpha), \beta)_G,$$

under the standard inner product $(\cdot,\cdot)_G$ on class functions defined in (1.1).

We can now describe the well-known route [Mac95, I §7] between the symmetric functions and class functions of the symmetric group. Recall from Definition 1.1.7 that the cycle type $\mathrm{cyc}(\sigma) := (1^{\sigma_1} \dots n^{\sigma_n})$ of $\sigma \in S_n$ also defines a partition $(1\sigma_1, 2\sigma_2, \dots, n\sigma_n)$ of n.

Definition 3.1.3. *The* Cycle Map *is*

$$\psi : S_n \to \mathfrak{S}_n : \sigma \mapsto \prod_{k=1}^{n} p_{\sigma_k},$$

Definition 3.1.4. *The* Frobenius Characteristic Map *is contraction with the cycle map:*

$$\mathrm{ch}_n : \mathfrak{X}_n \to \mathfrak{S}_n : f \mapsto (f, \psi_n)_{S_n} = \sum_{|\lambda|=n} \frac{1}{|z_{C_\lambda}|} f(C_\lambda) p_\lambda.$$

This clearly admits a linear extension to $\mathrm{ch} : \mathfrak{X} \to \mathfrak{S}$. The basic fact is that:

Theorem 3.1.1. *The Frobenius characteristic* ch *is an isometry.*

To prove this we need to distinguish the TRIVIAL CHARACTER $\eta_n \in \mathfrak{X}_n$ of the trivial representation of S_n, which maps every element of S_n to 1. This has Frobenius characteristic $\mathrm{ch}(\eta_n) = \sum_{|\lambda|=n} \frac{1}{|z_\lambda|} p_\lambda = h_n$, the complete symmetric polynomial, by equation (3.1).

Now define the class function

$$\chi_\lambda := \det(\eta_{\lambda_i - i + j})_{1 \leq i,j \leq n} \in \mathfrak{X}_n, \tag{3.4}$$

in terms of the character η_λ of S_n induced by the character of the trivial representation of $S_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \cdots$, [Mac95, I §3]. Explicitly the character induced by the trivial representation of S_λ is given by

$$\eta_\lambda := \eta_{\lambda_1} \star \eta_{\lambda_2} \star \cdots, \quad \lambda \in \mathcal{P}_n.$$

From this, one immediately clear property of χ_λ is its Frobenius characteristic: namely, the Schur function s_λ, as defined by the Jacobi-Trudy formula (2.44):

$$\mathrm{ch}(\chi_\lambda) = \det(h_{\lambda_i - i + j})_{1 \leq i,j \leq n} =: s_\lambda.$$

Proof of Theorem 3.1.1. (Sketch of [Mac95, I §7] First of all, isometry follows from

$$\langle \mathrm{ch}(\alpha), \mathrm{ch}(\beta) \rangle_{\mathfrak{S}_n} = \sum_{|\lambda|=n} \frac{1}{|z_{C_\lambda}|} \alpha(C_\lambda)\beta(C_\lambda) = (\alpha, \beta)_{S_n}$$

where $\alpha, \beta \in \mathfrak{X}_n$.

Next we verify that ch is a ring homomorphism as follows. Notice that it is possible to embed $S_n \times S_m$ in S_{n+m} by making S_m and S_n act on complementary subsets of $\{1, \ldots, m+n\}$. There are many such actions, but they are all related by a conjugation. Thus the product $\sigma \times \tau \in S_{n+m}$ of $\sigma \in S_n$ and $\tau \in S_m$ is well defined up to conjugacy, with cycle type $\mathrm{cyc}(\sigma \times \tau) = \mathrm{cyc}(\sigma) \cup \mathrm{cyc}(\tau)$, which implies

$$\psi_{n+m}(\sigma \times \tau) = \psi_n(\sigma)\psi_m(\tau). \tag{3.5}$$

Then by Frobenius reciprocity,

$$\begin{aligned}\mathrm{ch}(\alpha \star \beta) = \left(\mathrm{Ind}_{S_n \times S_m}^{S_{n+m}}(\alpha \star \beta), \psi\right)_{S_{n+m}} = \left(\alpha \star \beta, \mathrm{Res}_{S_{n+m}}^{S_n \times S_m}(\psi)\right)_{S_n \times S_m} &= (\alpha, \psi)_{S_n}(\beta, \psi)_{S_m} \\ &= \mathrm{ch}(\alpha)\mathrm{ch}(\beta).\end{aligned}$$

Secondly we show that ch is bijective. Since ch is an isometry, we have by equation (3.1.2) that

$$(\chi_\lambda, \chi_\mu) = \langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}.$$

Since the number of irreducible representation classes in S_n is exactly the number of partitions of n, which equals the dimension of $\mathfrak{X}_n$, we conclude that $\{\chi_\lambda\}_{|\lambda|=n}$ forms a basis of $\mathfrak{X}_n$.

Therefore, ch is a isomorphism from $\mathfrak{X}_n \cong \mathfrak{S}_n$, and so $\mathfrak{X} \cong \mathfrak{S}$ also. $\square$

Proposition 3.1.2. *[Mac95, I §7] If $\lambda \in \mathcal{P}_n$ is the partition corresponding to an irreducible representation $\pi \in \mathcal{R}_n$, then*

$$\chi_\lambda = \chi_{\pi_\lambda}$$

3.2 Heisenberg Algebra

In this section, we uncover a Heisenberg algebra acting on class functions in $\mathfrak{X}$, which plays the rôle of a Fock space. In the character basis of $\mathfrak{X}$, the Heisenberg operators applied to χ_λ will add or remove hooks from λ.

We will define Heisenberg operators to act on class functions by restriction or induction, i.e. removing cycles from or inserting cycles into the argument. First, some notation.

Definition 3.2.1. *The* Contraction $\mathrm{tr}_\phi : \mathfrak{X}_{n+m} \to \mathfrak{X}_n$ *of a class function $\alpha \in \mathfrak{X}_{n+m}$ with respect to $\phi \in \mathfrak{X}_m$ is*

$$\mathrm{tr}_\phi : \alpha \mapsto \left(\sigma \mapsto \frac{1}{|S_m|} \sum_{\tau \in S_m} \phi(\tau)\alpha(\sigma \times \tau) \right).$$

Recall δ_{C_λ} was the indicator function of an λ conjugacy class. We normalise it by the centraliser:

$$\delta_\lambda = |z_\lambda| \delta_{C_\lambda}.$$

We will use the indicator function $\delta_{(m)}$ of the m-cycle conjugacy class extensively. This takes value $|z_{(m)}| = m$ if its argument is an m-cycle, and 0 otherwise. In particular, we define the negative Heisenberg operators as multiplication by this indicator function.

Definition 3.2.2. *The* HEISENBERG OPERATORS $J_{\pm m} : \mathfrak{X}_n \to \mathfrak{X}_{n\mp m}$ *are*

$$J_{-m}(\alpha) := \alpha \star \delta_{(m)} = \mathrm{Ind}_{S_n\times S_m}^{S_{n+m}}(\alpha \otimes \delta_{(m)}),$$
$$J_m(\alpha) := \mathrm{tr}_{\delta_{(m)}} \mathrm{Res}_{S_m\times S_{n-m}}^{S_n}(\alpha),$$

where $m \in \mathbb{N}$, *and set* $J_0 := 0$.

Proposition 3.2.1. *The operators* J_m *and* J_{-m} *are adjoint in the following sense:*

$$(J_{-m}\alpha, \beta)_{S_{n+m}} = (\alpha, J_m\beta)_{S_n}, \quad \alpha \in \mathfrak{X}_n, \beta \in \mathfrak{X}_{n+m}.$$

Proof. Expand the left-hand-side using Frobenius reciprocity, the definition of the inner product, and the definition of the indicator function:

$$\begin{aligned}
(\alpha \star \delta_{(m)}, \beta)_{S_{n+m}} &= (\alpha \otimes \delta_{(m)}, \mathrm{Res}_{S_n\times S_m}^{S_{n+m}}(\beta))_{S_n\times S_m} \\
&= \frac{1}{n!m!} \sum_{(\sigma,\tau)\in S_n\times S_m} \alpha(\sigma)\delta_{(m)}(\tau)\mathrm{Res}_{S_n\times S_m}^{S_{n+m}}(\sigma,\tau) \\
&= \frac{1}{n!m!} \sum_{\sigma\in S_n} \alpha(\sigma) \sum_{\tau\in C_{(m)}} |z_{(m)}|\mathrm{Res}_{S_n\times S_m}^{S_{n+m}}(\sigma,\tau) \\
&= \frac{1}{n!} \sum_{\sigma\in S_n} \alpha(\sigma)\beta(\sigma\times(m)),
\end{aligned}$$

where last line follows from the fact that $\mathrm{Res}_{S_n\times S_m}^{S_{n+m}}(\sigma,\cdot)$ is constant over the conjugacy class $C_{(m)}$, which has size $|C_{(m)}| = (m-1)!$, and $|z_{(m)}| = m$. Since we also have

$$(\mathrm{tr}_{\delta_{(m)}}\beta)(\sigma) = \frac{1}{m!} \sum_{\tau\in C_{(m)}} \delta_{(m)}\beta(\sigma\times\tau) = \frac{1}{m!}|C_{(m)}||z_{(m)}|\beta(\sigma\times(m)) = \beta(\sigma\times(m)), \quad (3.6)$$

we can equate

$$(\alpha \star \delta_{(m)}, \beta)_{S_{n+m}} = \frac{1}{n!} \sum_{\sigma\in S_n} \alpha(\sigma)(\mathrm{tr}_{\delta_{(m)}}\beta)(\sigma) = (\alpha, \mathrm{tr}_{\delta_{(m)}}\beta)_{S_n}.$$

□

Remark 3.2.1. *A simple generalisation of this argument shows that in* $\mathfrak{X}$, *multiplication is adjoint to contraction: schematically,* $(\cdot \star \phi, \cdot) = (\cdot, \mathrm{tr}_\phi \mathrm{Res}\cdot)$.

To explain why we named $J_{\pm m}$ as Heisenberg operators, we show they form a representation of He, acting on a Fock space $\mathfrak{X}$.

Theorem 3.2.1.

$$[J_{-m}, J_l] = m\delta_{ml}.$$

To prove Theorem 3.2.1, the fundamental tool is:

Theorem 3.2.2. (Mackey) *If H, K are subgroups of a finite group G, and $\chi \in \mathfrak{X}(H)$ is a character of H, then*

$$\mathrm{Res}^G_K \mathrm{Ind}^G_H(\chi) = \bigoplus_{\bar{s} \in K\backslash G/H} \mathrm{Ind}^K_{H_s}\left(\mathrm{Res}^{{}^sH}_{H_s}({}^s\chi)\right) \in \mathfrak{X}(K),$$

where s is a representative of the double coset $\bar{s}$, ${}^sH = sHs^{-1}$, $H_s = {}^sH \cap K$, and ${}^s\chi(g) = \chi(s^{-1}gs)$ [Ser71, II.7.3].

Some machinery is necessary in order to apply Mackey's theorem to the symmetric group. We follow the exposition of [Zel81, A].

Definition 3.2.3. *The* Young Subgroup

$$S_\lambda \cong S_{\lambda_1} \times \cdots \times S_{\lambda_n}$$

where $|\lambda| = k$, is the subgroup of S_k consisting of permutations that leave invariant the blocks L_i of length λ_i partitioning $\{1, \dots, k\}$,

$$L_1 = \{1, \dots, \lambda_1\}, L_2 = \{\lambda_1{+}1, \dots, \lambda_1{+}\lambda_2\}, \dots, L_n = \{\lambda_1{+}\cdots{+}\lambda_{n-1}{+}1, \dots, \lambda_1{+}\cdots{+}\lambda_n\}.$$

That is, permutations $\sigma \in S_\lambda$ do not mix entries between blocks, $\sigma(L_i) = L_i$.

Proposition 3.2.2. *Double cosets $S_\mu \backslash S_k / S_\lambda$, where $|\lambda| = |\mu| = k$, are parametrised by matrices*

$$K = (k_{ij})_{1 \le i \le \ell(\lambda), 1 \le j \le \ell(\mu)}$$

such that $k_{ij} \in Z_{\ge 0}$, and

$$\sum_j k_{ij} = \lambda_i, \quad 1 \le i \le \ell(\lambda) \quad \textit{and} \quad \sum_i k_{ij} = \mu_j, \quad 1 \le j \le \ell(\mu). \tag{3.7}$$

Proof following [Zel81, A.3]. Let L_i be the blocks of λ and M_j be the blocks of μ. Given $\sigma \in S_k$, the construction is to count the elements mixed by σ between L_i and M_j:

$$k_{ij} = \#\{\sigma(L_i) \cap M_j\},$$

which automatically satisfies the conditions (3.7). For all $s \in S_\mu$, $s' \in S_\lambda$, and $\sigma \in S_k$, we have

$$\begin{aligned} s'\sigma s(L_i) \cap M_j &= s'\sigma(L_i) \cap M_j \\ &= s'\left(\sigma(L_i) \cap s'^{-1}(M_j)\right) \\ &= s'\left(\sigma(L_i) \cap M_j\right). \end{aligned}$$

Hence $\#\{s'\sigma s(L_i) \cap M_j\} = \#\{\sigma(L_i) \cap M_j\}$, i.e. k_{ij} does not change when σ varies in its double coset $S_\mu \backslash S_k / S_\lambda$. If

$$\#\{\sigma(L_i) \cap M_j\} = \#\{\sigma'(L_i) \cap M_j\}$$

then σ and σ' lie in the same double coset.

Conversely, given $K = (k_{ij})_{1 \le i \le \ell(\lambda), 1 \le j \le \ell(\mu)}$ satisfying the conditions (3.7), we construct σ in the double coset as follows.

Since the length of L_i is $\lambda_i = k_{i1} + \cdots + k_{i\ell(\mu)}$, we can split each L_i into subsets $L_{i1} \cup \ldots \cup L_{i\ell(\mu)}$, where

$$L_{i1} = \{1, \ldots, k_{i1}\}, L_{i2} = \{k_{i1}+1, \ldots, k_{i1}+k_{i2}\}, \text{ etc.}$$

Similarly split $M_j = M_{1j} \cup \ldots \cup M_{\ell(\lambda)j}$. Then let $\sigma \in S_k$ be the permutation sending L_{ij} to M_{ij}. By construction, this permutation mixes k_{ij} many elements, i.e. $\{\sigma(L_i) \cap M_j\} = k_{ij}$. □

Proof of Theorem 3.2.1. Our strategy is to calculate $J_m J_{-l}$ using Mackey's formula, which switches restriction and induction, recovering $J_{-l} J_m$. First, specialise Proposition 3.2.2 to $S_l \times S_r \backslash S_k / S_m \times S_n$. This double coset is parametrised by

$$K = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a+b=m, c+d=n, a+c=l, b+d=r \right\}. \tag{3.8}$$

For all double coset representatives $s \in K$,

$$\begin{aligned} (S_m \times S_n)_s &= (S_l \times S_r) \cap s(S_m \times S_n)s^{-1} = S_a \times S_c \times S_b \times S_d \\ (S_l \times S_r)_s &= (S_m \times S_n) \cap s^{-1}(S_l \times S_r)s = S_a \times S_b \times S_c \times S_d \end{aligned}$$

so we see that $s \in S_k$ fixes $\{1, \ldots, a\}$ and $\{a+b+c+1, \ldots, d\}$, and switches the two parts $\{a+1, \ldots, a+b\}$ and $\{a+b+1, \ldots, a+b+c\}$. Finally we apply Mackey's theorem 3.2.2:

$$\begin{aligned} J_m J_{-l}(\alpha) &= \mathrm{tr}_{\delta_{(m)}} \mathrm{Res}^{S_k}_{S_m \times S_n} \mathrm{Ind}^{S_k}_{S_l \times S_r} (\delta_{(l)} \otimes \alpha) \\ &= \mathrm{tr}_{\delta_{(m)}} \sum_{\bar{s} \in S_l \times S_r \backslash S_k / S_m \times S_n} \mathrm{Ind}^{S_m \times S_n}_{(S_l \times S_r)_s} \left(\mathrm{Res}^{{}^s(S_l \times S_r)}_{(S_l \times S_r)_s} \left({}^s(\delta_{(l)} \otimes \alpha) \right) \right) \\ &= \mathrm{tr}_{\delta_{(m)}} \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K} \mathrm{Ind}^{S_m \times S_n}_{S_a \times S_b \times S_c \times S_d} \left(\mathrm{Res}^{S_l \times S_r}_{S_a \times S_c \times S_b \times S_d} \left(\delta_{(l)} \otimes \alpha \right) \right). \end{aligned}$$

Since $\mathrm{Res}^{S_l}_{S_a\times S_c}(\delta_{(l)})$ vanishes unless $a=l$ or $c=l$, we reduce the sum to two double cosets:

$$\left(\begin{smallmatrix} l & m-l \\ 0 & n \end{smallmatrix}\right), \quad \left(\begin{smallmatrix} 0 & m \\ l & n-l \end{smallmatrix}\right),$$

each still satisfying the parametrisation (3.8). For these two remaining summands, we can push the restriction through to obtain

$$\mathrm{tr}_{\delta_{(m)}}\left(\underbrace{\mathrm{Ind}^{S_m\times S_n}_{S_l\times S_{m-l}\times S_n}\left(\delta_{(l)}\otimes \mathrm{Res}^{S_r}_{S_{m-l}\times S_n}(\alpha)\right)}_{(I)}+\underbrace{\mathrm{Ind}^{S_m\times S_n}_{S_m\times S_l\times S_{n-l}}\left(\delta_{(l)}\otimes \mathrm{Res}^{S_r}_{S_m\times S_{n-l}}(\alpha)\right)}_{(II)}\right).$$

The contraction of (I) collapses down to

$$\mathrm{tr}_{\delta_{(m)}}\left(\mathrm{Ind}^{S_m}_{S_l\times S_{m-l}}\left(\delta_{(l)}\otimes\beta\right)\right)=\frac{1}{|S_m|}\sum_{\substack{\sigma\in S_m\\ \mathrm{cyc}(\sigma)=(m)}}\ \sum_{\substack{t\in S_m\\ t^{-1}\sigma t\in S_l\times S_{m-l}}}(\delta_{(l)}\otimes\beta)(t^{-1}\sigma t)$$

where $\beta=\mathrm{Res}^{S_r}_{S_{m-l}}(\alpha)$ is one block of the above restriction of α. This expression is a sum over m-cycles, but must also be a sum over elements in $S_l\times S_{m-l}$. So it vanishes unless $m=l$, in which case $n=r$, and so $\mathrm{tr}_{\delta_{(m)}}(\mathrm{I})$ is exactly

$$\mathrm{tr}_{\delta_{(m)}}\mathrm{Ind}^{S_m\times S_n}_{S_m\times S_n}\left(\delta_{(m)}\otimes\mathrm{Res}^{S_n}_{S_n}(\alpha)\right)=m\alpha,$$

using the fact that $\mathrm{tr}_{\delta_{(m)}}\delta_{(m)}=m$ from equation (3.6).

As for the contraction $\mathrm{tr}_{\delta_{(m)}}(\mathrm{II})$, moving tensor products around produces the desired result:

$$\begin{aligned}\mathrm{tr}_{\delta_{(m)}}\mathrm{Ind}^{S_n\times S_m}_{S_l\times S_{n-l}\times S_m}\left(\delta_{(l)}\otimes\mathrm{Res}^{S_r}_{S_{n-l}\times S_m}(\alpha)\right)&=\mathrm{Ind}^{S_{l+r-m}}_{S_l\times S_{r-m}}\left(\delta_{(l)}\otimes\left(\mathrm{tr}_{\delta_{(m)}}\mathrm{Res}^{S_r}_{S_m\times S_{r-m}}(\alpha)\right)\right)\\&=J_{-l}J_m(\alpha).\end{aligned}$$

□

3.2.1 Character Basis

We find that Heisenberg operators have a graphical interpretation in terms of Young diagrams corresponding to characters, which matches the well-understood representation of Heisenberg operators on the symmetric functions.

First let us explain the Heisenberg representation on symmetric functions, as it has been noted before, e.g. [Mac95, I §5 Ex. 3.d)], and is easy to describe.

Proposition 3.2.3. *Under the standard inner product on $\mathfrak{S}$, the adjoint of multiplication by $p_n \in \mathfrak{S}$ is*

$$p_n^* = n\frac{\partial}{\partial p_n}$$

Proof. Recalling orthogonality of power-sums under the inner product (3.2) on $\mathfrak{S}$,

$$\langle p_n^* p_\lambda, p_\nu \rangle = \langle p_\lambda, p_n p_\nu \rangle = \begin{cases} |z_\lambda| & \text{if } \lambda = \nu \times (n), \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $p_n^* p_\lambda = \frac{z_\lambda}{z_\mu} p_\mu = n m_n(\lambda) p_\mu$, so long as n is a part of λ, and μ is obtained from removing a single part n from λ. Here $m_n(\lambda)$ is the multiplicity of the part n in λ, which is exactly the effect of applying the derivative $\partial/\partial p_n$. □

Define the operators P_n on $\mathfrak{S}$ for $n \in \mathbb{Z}$ as:

$$P_n \cdot = \begin{cases} \frac{\partial}{\partial p_n} \cdot & n > 0 \\ 1 \cdot & n = 0 \\ -n p_{-n} \cdot & n < 0. \end{cases} \tag{3.9}$$

Since power-sum symmetric functions form a $\mathbb{Q}$-basis of $\mathfrak{S}$, we can always write $f \in \mathfrak{S}$ in power-sums, so the operators (3.9) are well-defined. It is straightforward to verify these operators form a representation of the Heisenberg algebra He (2.12) over $\mathfrak{S}$,

$$[P_m, P_n] = m\delta_{m+n,0}.$$

Recall from Section 2.4.1 that states in bosonic Fock space $\mathfrak{H}^{(0)}$ were expressed in a symmetric basis of Schur functions s_λ. We now propose that characters χ_λ form an attractive and physically relevant basis for $\mathfrak{X}$, as they correspond to the Schur basis s_λ of $\mathfrak{S}$ through the Frobenius characteristic map, by equation (3.1.2). Furthermore there is a graphical calculus one can use to quickly evaluate the Heisenberg action on a character.

Definition 3.2.4. *The* Skew Character $\chi_{\lambda\backslash\mu}$ *is defined by* $\mathrm{ch}(\chi_{\lambda\backslash\mu}) = s_{\lambda\backslash\mu}$, *where* $s_{\lambda\backslash\mu}$ *is the skew Schur function of Definition 2.4.5.*

Recalling Definition 2.4.4 of the hooks of a Young diagram Y, collect them into a set $\mathrm{hk}(Y)$.

Theorem 3.2.3. (Murnaghan-Nakayama) *If $\mu = (\mu_1, \ldots, \mu_m)$ is a partition of n, then*

$$\chi_\lambda(\mu) = \sum_{\substack{h \in \mathrm{hk}(Y_\lambda) \\ |h| = \mu_k}} (-1)^{b(h)} \chi_{\lambda \setminus h}\left((\mu_1, \ldots, \widehat{\mu_k}, \ldots, \mu_m)\right),$$

for $k = 1, \ldots, \ell(\lambda)$ [Mur63, 5.2], [Nak41].

A particular case of the Murnaghan-Nakayama theorem can be used to define operators $j_{\pm m} : \mathfrak{X}_n \to \mathfrak{X}_{n \mp m}$ on characters χ_λ in $\mathfrak{X}_n$, whose action is to remove hooks of size m:

$$(j_m \chi_\lambda)(\mu) := \chi_\lambda\left((m) \times \mu\right) \overset{\substack{\text{Thm.} \\ 3.2.3}}{=} \sum_{\substack{\eta \in \mathrm{hk}(Y_\lambda) \\ |\eta| = m}} (-1)^{b(\eta)} \chi_{\lambda \setminus \eta}(\mu), \tag{3.10}$$

or add hooks of size m:

$$(j_{-m} \chi_\lambda)(\mu) = \sum_{\substack{\nu \\ |\nu| = |\lambda| + m}} \sum_{\substack{\eta \in \mathrm{hk}(Y_\nu) \\ \nu \setminus \eta = \lambda}} (-1)^{b(\eta)} \chi_\nu(\mu), \tag{3.11}$$

where $m > 0$. Set $j_0 := 0$. These operators acting on character Fock space $\chi_\lambda \in \mathfrak{X}$ are designed to mimic the Heisenberg operators α_m acting on the fermionic Fock space $|\lambda; 0\rangle \in \mathfrak{F}$, as defined in Definition 2.1.2.

Example 3.2.1.

$$\begin{aligned} \alpha_{-3}|\ydiagram{2,1}; 0\rangle &= \sum_{i \in \mathbb{F}} {:}\psi_i \psi^\dagger_{i-3}{:} (3/2 \wedge -1/2 \wedge -5/2 \wedge -7/2 \wedge \cdots) \\ &= |\ydiagram{5,1}; 0\rangle + |\ydiagram{2,1,1,1,1}; 0\rangle - |\ydiagram{2,2,2}; 0\rangle - |\ydiagram{3,3}; 0\rangle. \end{aligned}$$

$$j_{-3}\chi_{\ydiagram{2,1}} = \chi_{\ydiagram{5,1}} + \chi_{\ydiagram{2,1,1,1,1}} - \chi_{\ydiagram{2,2,2}} - \chi_{\ydiagram{3,3}}.$$

This Young-diagrammatic interpretation of the bases χ_λ for $\mathfrak{X}$ and $|\lambda; 0\rangle$ for $\mathfrak{F}^{(0)}$ was how we originally discovered the relation between hooks and Heisenberg operators. This fact also foreshadows the correspondence between $|\lambda; 0\rangle$ and χ_λ, developed in Section 3.3.

Remark 3.2.2. *One can view our $j_{\pm m}$ operators as generalising the formula [OP06, 3.1.2]*

$$\prod_{i=1}^{\ell(\mu)} \alpha_{-\mu_i} |0\rangle = \sum_{|\lambda| = |\mu|} \chi_\lambda(\mu) |\lambda; 0\rangle$$

in the wedge basis $\mathfrak{F}$, which itself is an incarnation of the Murnaghan–Nakayama theorem 3.2.3.

Finally, to close the circle, we claim

Proposition 3.2.4. *In the character basis of $\mathfrak{X}$, Definition 3.2.2 specialises to:*

$$J_m \chi_\lambda = j_m \chi_\lambda, \quad m \in \mathbb{Z}.$$

Proof. Applying equation (3.6) to χ_λ, we reproduce exactly the definition of j_{+m}, $m > 0$,

$$(J_{+m}\chi_\lambda)(\mu) := (\mathrm{tr}_{\delta_{(m)}}\chi_\lambda)(\mu) = \chi(\mu \times (m)) =: (j_{+m}\chi_\lambda)(\mu).$$

So $J_m = j_m$. To prove the $J_{-m} = j_{-m}$ case, we first demonstrate that $j_{\pm m}$ are adjoint:

$$(j_m\chi_\lambda, \chi_\mu) = (\chi_\lambda, j_{-m}\chi_\mu), \quad m > 0.$$

Substituting the definition (3.11) of j_{-m}, the right-hand-side is

$$\sum_{\substack{\nu \\ |\nu|=|\mu|+m}} \sum_{\substack{\eta\in\mathrm{hk}(\mu) \\ \nu\backslash\eta=\mu}} (-1)^{b(\eta)}\delta_{\lambda\nu} = \sum_{\substack{\eta\in\mathrm{hk}(\lambda) \\ \lambda\backslash\eta=\mu \\ |\eta|=m}} (-1)^{b(\eta)}$$

where we have used the first orthogonality relation (1.2) between characters, $(\chi_\pi, \chi_{\pi'}) = \delta_{\pi,\pi'}$. Substituting the definition (3.10) of j_m, the left-hand-side is

$$\sum_{\substack{\eta\in\mathrm{hk}(\lambda) \\ |\eta|=m}} (-1)^{b(\eta)}\delta_{\lambda\backslash\eta,\mu} = \sum_{\substack{\eta\in\mathrm{hk}(\lambda) \\ \lambda\backslash\eta=\mu \\ |\eta|=m}} (-1)^{b(\eta)},$$

which verifies adjointness of $j_{\pm m}$.

Now by Proposition 3.2.1 the adjoint of J_m is J_{-m}. But we have already shown that $J_m = j_m$, and that j_{-m} is the adjoint of j_m. Therefore $J_{-m} = j_{-m}$. □

Proposition 3.2.5. $\mathfrak{S}$ *and* $\mathfrak{X}$ *are isomorphic as* He*-modules.*

Proof. Under the Fock space isomorphism ch,

$$\mathrm{ch}(\delta_\sigma) = p_\sigma,$$

hence

$$p_m . s_\lambda = \mathrm{ch}(\delta_{(m)} \star \chi_\lambda) = \mathrm{ch}(J_m\chi_\lambda). \tag{3.12}$$

By the isometry of ch, the adjoint operator $m\partial/\partial p_m$ (Proposition 3.2.3) is sent to the adjoint operator J_m (Proposition 3.2.1). The result follows the Stone-von Neumann theorem 2.2.1. □

3.3 Vertex Operators

> The Frobenius characteristic map sends $\mathfrak{X}$ to bosonic Fock space $\mathfrak{H}$, and so lays a path, via the Boson-Fermion correspondence, to realise fermionic operators on $\mathfrak{X}$. We propose that fermionic operators act diagonally in the character basis of $\mathfrak{X}$.

By Theorem 3.1.1, we can now substitute the He-module $\mathfrak{X}$ for the He-module $\mathfrak{H}^{(0)}$,

$$\mathfrak{X} \overset{\text{ch}}{\cong} \mathfrak{H}^{(0)} : \chi_\lambda \mapsto s_\lambda.$$

and by Proposition 3.2.5 we obtain a VOA isomorphism sending the Heisenberg VOA over $\mathfrak{H}^{(0)}$ of Example 2.3.1, to the Heisenberg VOA over $\mathfrak{X}$. Our task for the rest of this section is to apply the Boson-Fermion correspondence of Theorem 2.4.1 in this new Fock space $\mathfrak{X}$.

Recall the notation $\eta_n \in \mathfrak{X}_n$ for the character of the trivial representation of S_n. Denote the character of the sign representation of S_n by $\varepsilon_n \in \mathfrak{X}_n$.

Remark 3.3.1. *The elements in the class algebra $Z(\mathbb{C}[S_n])$ associated to the trivial character η_n and the sign character ε_n are*

$$e_{\text{triv},n} := \sum_{\sigma \in S_n} [\sigma], \quad e_{\text{sign},n} := \sum_{\sigma \in S_d} \text{sgn}(\sigma)[\sigma]$$

Product expressions of the above are available:

$$\begin{aligned} e_{\text{triv}} &= \sum_{i=0}^{n} Z_{n-m} = \sum_{i=0}^{n} e_m(\mathsf{JM}_1, \ldots, \mathsf{JM}_n) = \prod^{n}(1_{S_k} + \mathsf{JM}_k) \\ e_{\text{sign}} &= \prod^{n}(1_{S_k} - \mathsf{JM}_k) \end{aligned}$$

where JM_k are the Jucys-Murphy elements of Definition 1.1.8, and Z_k, defined in equation (1.15), is an elementary polynomial in the Jucys-Murphy elements by Proposition 1.1.6. Since our party of distinguished elements has grown large, we collect them in a dictionary in Table 3.1.

We collect the relations (3.1) between elementary, complete, and power-sum symmetric functions into the generating functions:

$$\sum_{n \geq 0} h_n z^n = e^{\sum_{k \geq 1} \frac{1}{k} p_k z^n}, \quad \sum_{n \geq 0} e_n z^n = e^{-\sum_{k \geq 1} \frac{(-1)^k}{k} p_k z^k}. \tag{3.13}$$

$Z(\mathbb{C}[S_d])$	$\mathfrak{X}_d$	$\mathfrak{S}_d$
$\lvert z_\lambda\rvert e_{C_\lambda}$	δ_λ	p_λ
$\frac{d!}{\dim \pi_\lambda} e_{\pi_\lambda}$	χ_{π_λ}	s_λ
e_{triv}	η_d	h_d
e_{sign}	ε_d	e_d

Table 3.1: Translations of distinguished elements between $Z(\mathbb{C}[S_d]) \cong \mathfrak{X}_d \cong \mathfrak{S}_d$.

Take the image of these relations under the Frobenius characteristic map to obtain

$$\sum_{n\geq 0} \eta_n z^n = e^{\sum_{k\geq 1} \frac{1}{k}\delta_{(k)} z^k}, \quad \sum_{n\geq 0} \varepsilon_n z^n = e^{-\sum_{k\geq 1} \frac{(-1)^k}{k}\delta_{(k)} z^k}. \tag{3.14}$$

Introduce the following operators $H_{\pm m}, E_{\pm m} : \mathfrak{X}_n \to \mathfrak{X}_{n\mp m}$ for $m > 0$, defined as multiplication by trivial and sign characters:

$$H_{-m}(\alpha) := \alpha \star \eta_m = \mathrm{Ind}_{S_n\times S_m}^{S_n}(\alpha \otimes \eta_m), \quad E_{-m}(\alpha) := \alpha \star \varepsilon_m = \mathrm{Ind}_{S_n\times S_m}^{S_{n+m}}(\alpha \otimes \varepsilon_m),$$

and their adjoints:

$$H_m(\alpha) = \mathrm{tr}_{\eta_m} \mathrm{Res}_{S_m\times S_{n-m}}^{S_n}(\alpha), \quad E_m(\alpha) = \mathrm{tr}_{\varepsilon_m} \mathrm{Res}_{S_m\times S_{n-m}}^{S_n}(\alpha).$$

Now if we multiply a class function by (3.14), we have obtained the vertex operators $H_-(z), E_-(z)$ and the adjoint vertex operators $H_+(z), E_+(z)$,

$$H_\pm(z) = \sum_{n\geq 1} H_{\pm n} z^{\mp n} = e^{\sum \frac{1}{n} J_{\pm n} z^{\mp n}} \quad E_\pm(z) = \sum_{n\geq 1} E_{\pm n}(-z)^{\mp n} = e^{-\sum \frac{1}{n} J_{\pm n} z^{\mp n}} \tag{3.15}$$

At this point, we have almost reproduced the exponential of the Veneziano field, as in Proposition 2.4.1—but charge is still missing from the story. We use the Fock space $\mathbb{C}[\mathbb{Z}] \otimes \mathfrak{X}$, where the rôle of the group algebra of the integer charge lattice $\mathbb{Z}$ is to keep track of charge. Represent said charge by e^β, where β is a formal parameter. The multiplication $e^{m\beta} \star e^{n\beta} = e^{(m+n)\beta} \in \mathbb{C}[\mathbb{Z}] \otimes \mathfrak{X}$ is simple charge shift. Define a charge operator J_0 on $\mathbb{C}[\mathbb{Z}]$,

$$J_0 e^{m\beta} := \partial_\beta e^{m\beta} = m e^{m\beta}.$$

Some notation is needed to let a character carry charge: define the Charged Character

$$\chi_{\lambda;m} := e^{m\beta} \star \chi_\lambda \in \mathbb{C}[\mathbb{Z}] \otimes \mathfrak{X}_{|\lambda|}.$$

Finally we can construct Neveu-Schwarz fermionic operators on the Fock space $\mathbb{C}[\mathbb{Z}]\otimes\mathfrak{X}$, following Theorem 2.4.1:

$$\psi(z) = H_-(z)e^{\beta}z^{\partial_\beta+1/2}E_+(z), \quad \psi^\dagger(z) = E_-(z)e^{-\beta}z^{-\partial_\beta+1/2}H_+(z).$$

Given the fermionic fields, we can deduce their modes via the contour integral approach to VOAs, as explained in the appendix of [FLM88]. We write the NS mode expansion $\psi(z) = \sum_{i\in\mathbb{F}}\psi_i z^{-i-1/2}, \psi^\dagger(z) = \sum_{i\in\mathbb{F}}\psi_i^\dagger z^{-i-1/2}$ in terms of [FLM88, A.2.18]

$$\psi_j := \oint \frac{dz}{2\pi i z} z^j \psi(z), \quad \psi_j^\dagger := \oint \frac{dz}{2\pi i z} z^j \psi^\dagger(z).$$

Proposition 3.3.1.

$$\psi_{i_1}\cdots\psi_{i_m}\chi_{\emptyset;0} = \det(\eta_{\lambda_k-k+j})_{1\le k,j\le m}\chi_{\emptyset;m}$$

where $i_k = \lambda_k - k + m + 1/2$, ordered as $i_1 \ge \cdots \ge i_m$, are the wedge coordinates of Definition 2.1.5.

Proof. We have vertex operator commutation relations,

$$E_+(z_1)H_-(z_2) = (1+\frac{z_2}{z_1})H_-(z_2)E_+(z_1), \quad e^{\ln(z)\partial_\beta}e^{\beta} = ze^{\beta}e^{\ln(z)\partial_\beta}$$

valid in the region $|z_1| > |z_2|$, which can be seen as a special case of the Baker-Campbell-Hausdorff formula,

$$e^Ae^B = e^Be^Ae^{[A,B]}, \quad \text{for } [A,[A,B]] = 0 = [B,[A,B]],$$

using the commutation relation $[J_m, J_{-l}] = m\delta_{ml}$ of Theorem 3.2.1, and $[\partial_\beta, \beta] = 1$. Hence [Kac83, §14.8]

$$\psi(z_1)\psi(z_2) = (z_1 - z_2)H_-(z_1)H_-(z_2)E_+(z_1)E_+(z_2)e^{2\beta}z_1^{\partial_\beta+1/2}z_2^{\partial_\beta+1/2}. \tag{3.16}$$

We define the normally ordered product

$$:\psi(z_1)\psi(z_2)\cdots\psi(z_m): := \prod_{k=1}^m H_-(z_k) \times \prod_{k=1}^m z_k^{\partial_\beta+1/2} \times e^{m\beta} \times \prod_{k=1}^m E_+(z_k), \tag{3.17}$$

so that the equation (3.16) is $\psi(z_1)\psi(z_2) = (z_1 - z_2):\psi(z_1)\psi(z_2):$ [FLM88, §8.4.25].

By a similar commutation argument, one can prove the following

$$\prod_{k=1}^m \psi(z_k) = \prod_{1\le i<j\le m}(z_i - z_j):\prod_{k=1}^m \psi(z_k):$$

valid in the region $\gamma \subset \mathbb{C}^m$ cut out by $|z_i| > |z_j|, 1 \le i < j \le m$ [FLM88, A.2.9]. We recognise $\prod_{1\le i<j\le m}(z_i - z_j) = \det(z_j^{m-i})_{1\le i,j\le m}$ as the Vandermonde determinant. This lets us write the product of modes as a multiple contour integral over γ,

$$\begin{aligned}\left(\prod_{k=1}^{m} \psi_{i_k}\right) \chi_{\emptyset;0} &= \frac{1}{(2\pi i)^m} \oint \cdots \oint \prod_{k=1}^{m} \left(\psi(z_k) z_k^{i_k} \frac{dz_k}{z_k}\right) \chi_{\emptyset;0} \\ &= \frac{1}{(2\pi i)^m} \oint \cdots \oint \det(z_j^{m-i}) \left(:\prod_{k=1}^{m} \psi(z_k):\right) \chi_{\emptyset;0} \prod_{k=1}^{m} \left(z_k^{i_k} \frac{dz_k}{z_k}\right)\end{aligned}$$

Applying the definition of the normally ordered product (3.17) to $\chi_{\emptyset;0}$ gives

$$:\prod_{k=1}^{m} \psi(z_k):\chi_{\emptyset;0} = e^{m\beta} \prod_{k=1}^{m} \left(\sum_{n\ge 0} \eta_n z_k^n\right) (z_1 \cdots z_m)^{1/2}.$$

where by definition $H_-(z)\chi_{\emptyset;0} = \sum_{n\ge0} H_{-n}\chi_{\emptyset;0} z^n = \sum_{n\ge0} \eta_n z^n$. Expanding the determinant as a sum over permutations, then taking the residue, we have

$$\begin{aligned}\left(\prod_{k=1}^{m} \psi_{i_k}\right) \chi_{\emptyset;0} &= \frac{e^{m\beta}}{(2\pi i)^m} \oint \cdots \oint \prod_{k=1}^{m} \left(\sum_{n\ge 0} \eta_n z_k^n\right) \left(\prod_{i=1}^{m} z_k^{i_k+1/2}\right) \\ &\qquad \times \left(\sum_{\sigma\in S_m} \operatorname{sgn}(\sigma) z_1^{m-\sigma(1)} \cdots z_m^{m-\sigma(m)}\right) \left(\frac{dz_k}{z_k}\right) \\ &= e^{m\beta} \sum_{\sigma\in S_m} \eta_{-(m+i_1+1/2)+\sigma(1)} \cdots \eta_{-(m+i_k+1/2)+\sigma(m)} \\ &= e^{m\beta} \det(\eta_{-(m+i_k+1/2)+j})_{1\le k,j\le m} \\ &= \chi_{\emptyset;m} \det(\eta_{\lambda_k-k+j})_{1\le k,j\le m},\end{aligned}$$

where in the last line we moved from wedge coordinates $i_k \in X_{\lambda;m}$ to partition coordinates $\lambda_k := -i_k + k - m - 1/2$. □

As an application, we substitute the determinant formula (3.4) for characters:

$$\chi_\lambda = \det(\eta_{\lambda_k-k+j})_{1\le k,j\le n},$$

into Proposition 3.3.1, to immediately obtain

$$\psi_{i_1} \cdots \psi_{i_m} \chi_{\emptyset;0} = \chi_{\lambda;0} \star \chi_{\emptyset;m} = \chi_{\lambda;m}.$$

Essentially we have diagonalised the determinant as a product of operators—in this sense the fermionic operators act diagonally on the character basis of $\mathbb{C}[\mathbb{Z}] \otimes \mathfrak{X}$.

Remark 3.3.2. *We conjecture that* $\chi_{\lambda;m} \star \chi_{\mu;n} = \psi_{i_1} \cdots \psi_{i_m} \psi_{j_1} \cdots \psi_{j_n} \chi_{\emptyset;0}$ *is a sum over characters* χ_π *running over unions of wedge coordinates such that* $X_{\pi;m+n} = X_{\lambda;m} \cup X_{\lambda;n}$*, and the union is in descending order. Heuristically the sum is over mergers of arms and legs from one diagram into the other (with the charge telling us how to decompose the diagram into arms and legs). This could give an alternative method of computing Littlewood-Richardson coefficients* $c^\nu_{\lambda\mu} := (\chi_\nu, \chi_\lambda \star \chi_\mu)$.

We now have a method to edit characters in terms of the Young-diagrammatic calculus of Section 3.2.1. Given a character $\chi_{\lambda;0}$ with corresponding diagram Y_λ, applying $\psi_{\pm k}$ to $\chi_{\lambda;0}$ shifts the charge by $+1$, then inserts/removes an arm of length $k \in \mathbb{F}_{\geq 0}$ into Y_λ. Likewise, $\psi^\dagger_{\mp k}$ shifts the charge by -1, then inserts/removes a leg of length k. The lengths are relative to the diagonal cut in the Young diagram, which is determined by the charge, as explained in Example 2.1.1. The location of the insertion/removal is that which preserves the descending arm and leg length property that defines a Young diagram. See the example in Figure 3.1.

Remark 3.3.3. *Charge* 0 *operators insert hooks. For example, the Heisenberg operators* $j_{\pm k}$ *defined in equations (3.10, 3.11) insert/remove every possible arm+leg hook of total size k which still results in a Young diagram. Intuitively this is because under the Boson-Fermion correspondence they are sums of bilinears* $\psi_j \psi^\dagger_{-j+k}$.

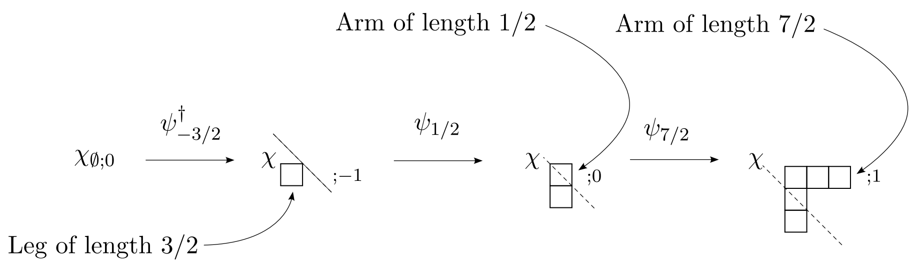

Figure 3.1: An example of the character $\chi_{\lambda;1} = \psi_{7/2}\psi_{1/2}\psi^\dagger_{-3/2}\chi_{\emptyset;0}$ constructed from fermionic operators. Lengths are measured relative to the diagonal, as explained in Example 2.1.1.

This discussion can be summarised as follows: the Frobenius Characteristic map ch composed with the Boson-Fermion correspondence σ provides an isomorphism:

$$\mathbb{C}[\mathbb{Z}] \otimes \mathfrak{X} \overset{\text{ch}}{\cong} \mathfrak{H} \overset{\sigma}{\cong} \mathfrak{F},$$

where characters $\chi_{\lambda;m}$ correspond to fermionic states $w^m s_\lambda \in \mathfrak{H}$ or $|\lambda; m\rangle \in \mathfrak{F}$. Meanwhile, conjugacy class indicator functions δ_λ correspond to bosonic states $p_\lambda \in \mathfrak{H}$ or $\prod_{i=1}^{\ell(\lambda)} \alpha_{-\lambda_i}|0\rangle \in \mathfrak{F}$.

In the class algebra, this translates to (c.f. Table 3.1) the representation basis $\{e_\pi\}_{\pi\in\mathcal{R}}$ being fermionic, and the conjugacy class basis $\{e_C\}_{C\in\mathcal{C}}$ being bosonic, thus explaining a comment by Dijkgraaf [Dij95, §5]:

> "The representation basis and the conjugacy class basis [...] correspond respectively to fermions and bosons."

3.3.1 Cut-and-Join

The partition invariant (PI) operators of the $W_{1+\infty}$ algebra, defined in 2.49, admit a bosonic expression by equation (2.48), which we write as an operator on $\mathfrak{X}$:

$$\Pi_k = \frac{1}{(k+1)!} \sum_{i_1+\cdots+i_{k+1}=0} :J_{i_1}\cdots J_{i_{k+1}}:.$$

where the normal ordering of Heisenberg operators is the usual one, see Example 2.3.1.

Definition 3.3.1. *The* Cut-and-Join *operator on $\mathfrak{S}$ is*

$$W = \frac{1}{2}\sum_{i,j\in\mathbb{Z}} \left((i+j)p_i p_j \frac{\partial}{\partial p_{i+j}} + ij p_{i+j} \frac{\partial^2}{\partial p_i \partial p_j} \right)$$

where p_i is the i^{th} power-sum symmetric polynomial.

By definition (3.9) of the He-module $\mathfrak{S}$, we can rearrange the cut-and-join operator more attractively as:

$$W = \frac{1}{6} \sum_{\substack{i,j,k\in\mathbb{Z} \\ i+j+k=0}} :P_i P_j P_k:.$$

By Proposition 3.2.5, the He-modules $\mathfrak{S}$ and $\mathfrak{X}$ are isomorphic under the Frobenius characteristic map, so that $p_i s_\lambda = p_i \mathrm{ch}(s_\lambda) = \mathrm{ch}(J_i \chi_\lambda)$. Hence we can interpret W as a PI operator on symmetric functions $\mathfrak{S}$, i.e.

$$W s_\lambda = \mathrm{ch}(\Pi_2 \chi_\lambda). \tag{3.18}$$

It can be proved [Gou94, §3.1] that under the cycle map ψ of Definition 3.1.3, the cut-and-join operator acts as multiplication by the transposition basis element $e_{\mathcal{T}}$ of the class algebra:

$$W\psi(\sigma) = \psi(e_{\mathcal{T}}\sigma).$$

Therefore

$$W\psi(e_\pi) = \psi(e_{\mathcal{T}} e_\pi) = \nu_\pi(\mathcal{T})\psi(e_\pi),$$

where e_π is a representation basis element of the class algebra. This implies that the eigenstates of W are Schur functions, with transposition class multipliers $\nu_\pi(\mathcal{T})$ for eigenvalues. An alternative approach, using our knowledge of PI operators, is:

$$Ws_\lambda = \mathrm{ch}(\Pi_2\chi_\lambda) = \nu_\lambda(\mathcal{T})\mathrm{ch}(\chi_\lambda) = \nu_\lambda(\mathcal{T})s_\lambda,$$

where we have used equation (3.18) followed by Proposition 2.4.5.

Remark 3.3.4. *At this stage, it is natural to ask if there are elements of the class algebra, other than $e_{\mathcal{T}}$, which have partition invariants $\frac{1}{k!}\left(\sum_{i=1}^{\ell(\lambda)}(\lambda_i - i)^k - (\lambda_i^\dagger - i)^k\right)$ for eigenvalues. But this is exactly the power-sum of contents $p_{k-1}(c(\lambda))$, as can be seen by a similar argument to Example 1.1.1. In that example, the PI operator was $e_{\mathcal{T}_{(n)}} = p_1(\mathsf{JM}_1, \ldots, \mathsf{JM}_n) \in Z(\mathbb{C}[S_n])$, where JM are the Jucys-Murphy elements of Definition 1.1.8. A straightforward generalisation tells us that the power-sum $p_{k-1}(\mathsf{JM}_1, \ldots, \mathsf{JM}_n) \in Z(\mathbb{C}[S_n])$ has partition invariants for eigenvalues. It is known that $p_{k-1}(\mathsf{JM}_1, \ldots, \mathsf{JM}_n)$ is a linear combination of the m-cycle basis elements $e_{C_{(m)}}$, $m \leq k$ [Las13, 6.1], [LT01, §3].*

Remark 3.3.5. *The original cut-and-join formulation [Gou94] arose in the combinatorics of multiplication of $\sigma \in S_n$ by a transposition $\tau := (i\, j) \in S_n$. Suppose $\mathrm{cyc}(\sigma) = \lambda = (\lambda_1, \ldots, \lambda_m)$, so there is a factorisation $\sigma = c_1 \cdots c_m \in S_n$. The possible cycle types of $\sigma\tau$ are:*

(Cut) *If i, j are in the same cycle c_k, then this cycle is 'cut':*

$$\mathrm{cyc}(\sigma\tau) = (\lambda_1, \ldots, \widehat{\lambda_k}, \ldots, \lambda_m, r, s)$$

where $r + s = \lambda_k$.

(Join) *If i, j are in different cycles $c_k \neq c_l$, then these cycles are 'joined':*

$$\mathrm{cyc}(\sigma\tau) = (\lambda_1, \ldots, \widehat{\lambda_k}, \ldots, \widehat{\lambda_l}, \ldots, \lambda_k + \lambda_l, \ldots, \lambda_m).$$

Chapter 4

Jacobi Triple Product

In this brief chapter, we evaluate the characters of a genus 1 free CFT over both fermionic and bosonic Fock space to give a physical proof of the famous Jacobi triple product. We then construct the modular invariant partition function.

4.1 Genus 1 CFT

We briefly set up the free fermion CFT. For more authoritative material on CFT see [Sch08, §2, §5, §9] and [Gin89, §1-3]. For detail on the free fermion, see [Gin89, §6–8].

4.1.1 Conformal Group

The group of conformal coordinate transformations of $\mathbb{C}$ can be described as the holomorphic coordinate transformations with non-vanishing derivative, given by $z \mapsto f(z), \bar{z} \mapsto \bar{f}(\bar{z})$. We write an infinitesimal holomorphic transform as the addition of a convergent Laurent series $\epsilon(z)$ [Sch08, 5.2],

$$z \mapsto z + \epsilon(z) := z - \sum_{n \in \mathbb{Z}} \epsilon_n z^{n+1}, \quad \bar{z} \mapsto \bar{z} + \bar{\epsilon}(\bar{z}) := \bar{z} - \sum_{n \in \mathbb{Z}} \bar{\epsilon}_n \bar{z}^{n+1} \tag{4.1}$$

whose corresponding vector fields have basis $l_n := -z^{n+1}\partial_z, \bar{l}_n := -\bar{z}^{n+1}\partial_{\bar{z}}$. These form an algebra, closed under commutation relations

$$[l_n, l_m] = (n-m)l_{n+m}, \quad [\bar{l}_n, \bar{l}_m] = (n-m)\bar{l}_{n+m}, \quad [l_n, \bar{l}_m] = 0. \tag{4.2}$$

which we identify as nothing other than $\mathsf{Wi} \otimes \bar{\mathsf{Wi}}$, two copies of the Witt algebra of Definition 2.3.8. But it is impossible to promote this Lie algebra to a group of conformal transformations [Sch08, §5.2]. The best we can do is:

Definition 4.1.1. *The* Global Conformal Group *of the Riemann sphere* $\mathbb{CP}^1$ *consists of invertible holomorphic functions* $f : \mathbb{CP}^1 \mapsto \mathbb{CP}^1$.

Proposition 4.1.1. *The global conformal group of* $\mathbb{CP}^1$ *has infinitesimal generators* $\{l_{-1}, l_0, l_1, \bar{l}_{-1}, \bar{l}_0, \bar{l}_1\}$.

This can be seen by noting $l_n, \bar{l}_n$ are non-singular at $z = 0$ if and only if $n \geq -1$. Furthermore, $l_n, \bar{l}_n$ are non-singular at $z = \infty$ if and only if $n \leq 1$. It is easily seen from the relations (4.2) that $\{l_n\}_{-1\leq n\leq 1}$ is a closed subalgebra of Wi.

In particular, the generators of the conformal transforms are

$$\begin{aligned} \text{translations: } & l_{-1}, \bar{l}_{-1}; \qquad \text{rotations: } i(l_0 - \bar{l}_0); \\ \text{dilatations: } & l_0 + \bar{l}_0; \qquad \text{special conformal: } l_1, \bar{l}_1. \end{aligned} \tag{4.3}$$

This can be phrased instead in terms of the Möbius Group $\mathrm{PSL}(2,\mathbb{C}) := \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$.

Corollary 4.1.1. *The global conformal group of the Riemann sphere* $\mathbb{CP}^1$ *is isomorphic to the Möbius group* $\mathrm{PSL}(2,\mathbb{C})$ *[Sch08, §2.3].*

The conformal transformations in this language are

$$\text{translations: } \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}; \qquad \text{rotations: } \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}; \tag{4.4}$$

$$\text{dilatations: } \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}; \qquad \text{special conformal: } \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}. \tag{4.5}$$

The question is now: how can we conformally transform the fields of a VA? We follow the exposition of [Kac98, §4.9–4.10]. Given a graded VA $(V, |0\rangle, Y)$ from Definition 2.3.2, the 'shift' identity of equation (2.19) gives the **translation** $z \mapsto z + b$,

$$e^{bT} Y(\phi, z) e^{-bT} = Y(\phi, z + b), \quad \phi \in V.$$

By the grading of V, we assume the existence of a diagonal operator H on V such that

$$[H, Y(\phi, z)] = z\partial Y(\phi, z) + Y(H\phi, z),$$

or equivalently $[H, \phi_{(n)}] = -n\phi_{(n)}$, then we can integrate this up to $a^H \phi_{(n)} a^{-H} = a^{-H}\phi_{(n)}$, which gives the **dilatation** and **rotation** $z \mapsto az$,

$$a^H Y(\phi, z) a^{-H} = Y(a^H \phi, az).$$

Finally we must assume the existence of an operator T^* on V such that

$$[T^*, Y(\phi, z)] = z^2\partial Y(\phi, z) + 2zY(H\phi, z) + Y(T^*\phi, z).$$

This integrates up to the **special conformal transformation** $z \mapsto \frac{z}{cz+1}$,

$$e^{-cT^*} Y(\phi, z) e^{cT^*} = Y\left(e^{-c(1+cz)T^*}(1+cz)^{-2H}\phi, \frac{z}{1+cz}\right)$$

It can be checked that $\mathrm{span}_{\mathbb{C}}\{T, H, T^*\}$ generate a $\mathsf{sl}(2, \mathbb{C})$ Lie algebra with commutation relations

$$[H, T^*] = -T^*, \quad [T^*, T] = 2H,$$

which we realise by matrices

$$T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad 2H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T^* = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

By exponentiating this Lie algebra to the Lie group $\mathrm{SL}(2, \mathbb{C})$, and substituting, we find the Möbius transforms of equation (4.4):

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} Y(\phi, z) \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} = Y(\phi, z+a), \qquad \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} Y(\phi, z) \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} = Y(a^H\phi, az),$$

$$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} Y(\phi, z) \begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix} = Y\left(e^{-c(1+cz)T^*}(1+cz)^{-2H}\phi, \frac{z}{1+cz}\right).$$

This is the origin of the additional Virasoro axiom of VOA Definition 2.3.3: it supplies exactly the operators we need: $T = L_{-1}, H = L_0, T^* = L_1$. We now explore some consequences of this axiom.

Proposition 4.1.2. *In a VOA $(V, |0\rangle, Y, \omega, c)$, the Virasoro OPE is*

$$Y(\omega, z)Y(\omega, w) \sim \frac{\frac{c}{2}\mathbb{1}}{(z-w)^4} + \frac{2Y(\omega, w)}{(z-w)^2} + \frac{\partial Y(\omega, w)}{z-w}.$$

Proof. The weight 2 Virasoro field $Y(\omega, z)$, due to the Borchards OPE of Proposition 2.3.3, has OPE

$$Y(\omega, z)Y(\omega, w) \sim \frac{\frac{c}{2}\mathbb{1}}{(z-w)^4} + \frac{Y(L_1\omega, w)}{(z-w)^3} + \frac{Y(L_0\omega, w)}{(z-w)^2} + \frac{\partial Y(\omega, w)}{z-w}$$

By mutual locality, exchanging z and w should not change this equation, which implies $L_1\omega = 0$, eliminating the cubic term, and by definition, $L_0\omega = 2\omega$. □

By using equation (2.20), we can extract the commutation relations of $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ from this OPE:

Corollary 4.1.2.

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}\mathbb{1}, \quad [L_n, \mathbb{1}] = 0, \quad n \in \mathbb{Z},$$

These are exactly the Virasoro relations at central charge c from Definition 2.3.9.

Definition 4.1.2. *Given a VOA V and $\phi \in V$, the field $Y(\phi, w)$ is a* PRIMARY FIELD *of weight h if*

$$Y(\omega, z)Y(\phi, w) \sim \frac{h}{(z-w)^2}Y(\phi, w) + \frac{1}{z-w}\partial Y(\phi, w).$$

Remark 4.1.1. *To return to the local v.s. global discussion of holomorphic transforms which began this section, we note the definition of primary fields for a chiral CFT, which can contain an anti-holomorphic dependence: a CFT primary field $\Phi(z, \bar{z})$ of weight $(h, \bar{h}) \in \mathbb{R}^2$, under the holomorphic coordinate transformations with non-vanishing derivatives $z \mapsto f(z), \bar{z} \mapsto \bar{f}(\bar{z})$, transforms as [Gin89, §2.1]*

$$\Phi(z, \bar{z}) \mapsto (\partial f)^h (\bar{\partial}\bar{f})^{\bar{h}} \Phi(f(z), \bar{f}(\bar{z})).$$

Explicitly, under the holomorphic transform (4.1), a primary field in a CFT transforms as

$$\Phi(z, \bar{z}) \mapsto \Phi(z, \bar{z}) + (h\partial\epsilon(z) + \epsilon(z)\partial)\Phi(z, \bar{z}) + (\bar{h}\bar{\partial}\bar{\epsilon}(\bar{z}) + \bar{\epsilon}(\bar{z})\bar{\partial})\Phi(z, \bar{z}). \tag{4.6}$$

Were $\Phi(z)$ purely holomorphic, this would reproduce the OPE of Definition 4.1.2.

4.1.2 Dirac Fermion

We describe the free complex Dirac fermion CFT on an elliptic curve, paying particular attention to the difference between the Ramond and Neveu-Schwarz sectors.

To write down a CFT, we work with Euclidean $\mathbb{R}^2$ space and time coordinates σ_1 and σ_0, or alternatively $z := \sigma_0 + i\sigma_1 \in \mathbb{C}$. To eliminate any infrared divergences [Gin89, §2.2], we compactify the space coordinate, $\sigma_1 \sim \sigma_1 + 2\pi$, thus defining a cylinder. The conformal map $z \mapsto \zeta = e^z$ sends this cylinder to the plane $\mathbb{C}$, c.f. Figure 4.1.

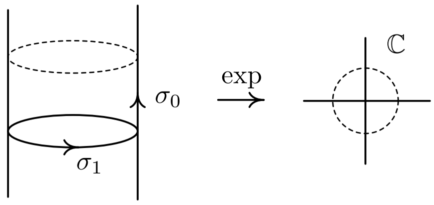

Figure 4.1: Conformal map of the compactified space-time cylinder to $\mathbb{C}$.

To convert the cylinder to the torus, we perform the usual discrete identification of two periodic cycles. For convenience, rotate $z \mapsto iz$ first, so that one period is $z \sim z + 2\pi$, and the other is $z \sim z + 2\pi\tau$. This defines an elliptic curve, parametrised by $\tau \in \mathbb{H}$ in the upper complex half-plane.

We can now describe the $c = 1$ Dirac fermion living on the elliptic curve E, parametrised by the elliptic modulus $\tau \in \mathbb{H}$ in the upper complex half-plane. The Dirac operator is calculated by Pauli matrices σ_x, σ_y:

$$\not{\partial} = \sigma_x \partial_{\sigma_0} + \sigma_y \partial_{\sigma_1} = \begin{pmatrix} 0 & \partial_{\sigma_0} - i\partial_{\sigma_1} \\ \partial_{\sigma_0} + i\partial_{\sigma_1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \partial \\ \bar{\partial} & 0 \end{pmatrix},$$

where $z = \sigma_0 + \tau\sigma_1$, and the Euclidean measure is $2i dz \wedge d\bar{z} = 4 \operatorname{Im} \tau d\sigma_0 \wedge d\sigma_1$. The action is thus

$$S := \frac{1}{2\pi \operatorname{Im} \tau} \int_E d^2 z\, \psi(z, \bar{z})(\bar{\partial} - i\bar{A})\psi^\dagger(z, \bar{z}) + \bar{\psi}(z, \bar{z})(\partial + iA)\bar{\psi}^\dagger(z, \bar{z}),$$

where $A, \bar{A}$ are components of a flat $U(1)$ connection on E. We have split the Dirac fermion into two independent chiral (Weyl) components $\psi, \bar{\psi}$. For conciseness' sake, we will treat only the first chiral field ψ; identical considerations apply to $\bar{\psi}$.

Now the connection A on E is fully specified by the holonomy of ψ around the spatial and temporal cycles of E, which are sometimes known as the a-cycle and b-cycle. Fix

$$\oint_{a\text{-cycle}} \frac{dz}{2\pi i} A = a, \quad \oint_{b\text{-cycle}} \frac{dz}{2\pi i} A = b,$$

i.e. the holonomy around the spatial/a-cycle $z \sim z + 2\pi$ is $e^{2\pi i a}$, and the holonomy around the temporal/b-cycle $z \sim z + 2\pi\tau$ is $e^{2\pi i b}$. We will concentrate on the cases $a, b \in \{0, 1/2\}$, which cause periodicity or anti-periodicity around their respective cycles.

We can import the results of Example 2.3.2 describing the Clifford VOA of a complex holomorphic fermion $\psi(z), \psi^\dagger(z)$ on the plane $\mathbb{C}$. Under the exponential map $z \mapsto \zeta = e^z$, the primary fields of weight $h = 1/2$ transform as $\psi(z) \mapsto z^{1/2}\psi(z)$ and $\psi^\dagger(z) \mapsto z^{1/2}\psi^\dagger(z)$. Examining the new mode expansions of the fermionic fields (2.26),

$$\psi(z) = \sum_{j\in\mathbb{Z}+a} \psi_j z^{-j-1/2}, \quad \psi^\dagger(z) = \sum_{j\in\mathbb{Z}+a} \psi^\dagger_{-j} z^{-j-1/2},$$

we identify $a = 0$ (periodic boundary conditions) with the Ramond fermion and $a = 1/2$ (anti-periodic boundary conditions) with the Neveu-Schwarz fermion. The Virasoro field $L(z) := Y(\omega, z)$ is not primary, and under the exponential map picks up an anomalous term from the Schwartzian derivative, [Gin89, §7.1]

$$L(z) \mapsto \zeta^2 L(\zeta) - \frac{c}{24} = \sum_{n\in\mathbb{Z}} \left(L_n - \frac{1}{24}\delta_{n0}\right) e^{-nz}.$$

In particular, the L_0 mode on the cylinder has been raised by $1/24$, known as the Vacuum Energy Constant of the cylinder.

From Proposition 2.4.1, we know the $c = 1$ Clifford VOA is generated by the Veneziano field $\phi(z)$:

$$\alpha(z) = -i\partial\phi(z), \quad \psi(z) = {:}e^{+i\phi(z)}{:}, \quad \psi^\dagger(z) = {:}e^{-i\phi(z)}{:},$$

and by Remark 2.3.1, we can write the Virasoro field as

$$L(z) = -\frac{1}{2}{:}\alpha(z)^2{:}.$$

This allows us to examine the weights of various fields. Take the OPE of ${:}e^{\pm im\phi(z)}{:}$ with L, using the Borcherds formula of Proposition 2.3.3,

$$\begin{aligned} -\frac{1}{2}{:}\alpha(z)^2{:}\,{:}e^{\pm im\phi(w)}{:} &\sim \frac{Y(L_0|\pm m\rangle, w)}{(z-w)^2} + \frac{Y(L_{-1}|\pm m\rangle, w)}{z-w} \\ &\sim \frac{m^2/2}{(z-w)^2}{:}e^{\pm im\phi(z)}{:} + \frac{1}{z-w}\partial{:}e^{\pm im\phi(z)}{:}, \end{aligned} \tag{4.7}$$

where we have used the state-field correspondence $:e^{\pm im\phi(w)}: = Y(|\pm m\rangle, w)$ of Theorem 2.4.1, and the fact that the eigenvalue $L_0|\pm m\rangle = \frac{m^2}{2}|\pm m\rangle$ is simply the sum of the modes from $\pm 1/2$ to $\pm m \pm 1/2$. Therefore by Definition 4.1.2, $:e^{\pm im\phi(z)}:$ is a primary field of weight $\frac{m^2}{2}$. So we can be decompose the Fock space by charge $m \in \mathbb{Z}$ into infinitely many $\mathsf{Vir}_{c=1}$-modules of highest weight $m^2/2$, since L_{-n} is a charge 0 operator, so the descendent states of a charge m state will also be charge m. See Table 4.1 and 4.2 for some sample low-energy states.

Finally, we must modify the state-field correspondence, since we have introduced boundary conditions on the fields. The standard trick [Gin89, §6] to implement corresponding boundary conditions on the level of *states* is to first define the TWIST VERTEX OPERATOR:

$$\mu_a(z) := :e^{i(-a+1/2)\phi(z)}:$$

with OPEs

$$\psi(z)\mu_a(0) \sim O(z^{-a+1/2}), \quad \psi^\dagger(z)\mu_a(0) \sim O(z^{a-1/2}),$$

possessing square-root branch cuts. Intuitively, when a field e.g. $\psi(z)$ is transported around $\mu_a(w)$, it is twisted by a holonomy factor $e^{2\pi ia}$ from passing the branch cut. The twist vertex operator $\mu_a(z)$ has weight $h = \frac{1}{2}(a - 1/2)^2$ by equation (4.7). In the Neveu-Schwarz sector $a = 1/2$, the state-field correspondence produces the familiar vacuum state $Y(|0\rangle_{\mathrm{NS}}, z) = \mu_{1/2}(z) = \mathbb{1}$, of weight $h = 0$. The Ramond sector $a = 0$ is more interesting: the state-field correspondence defines *two* states, corresponding to $\mu_0(z), \mu_0^\dagger(z)$:

$$Y\left(|\pm 1/2\rangle_R, z\right) = :e^{\pm \frac{i}{2}\phi(z)}:$$

i.e. the Ramond sector has two vacua of conjugate charge $\pm 1/2$, so surprisingly, the vacua have weight $h = 1/8$. Referring back to the Definition 2.1.3 of the Fock space vacua, we identify

$$|1/2\rangle_R = 0 \wedge -1 \wedge -2 \wedge \cdots, \quad |-1/2\rangle_R = -1 \wedge -2 \wedge -3 \wedge \cdots$$

Note that this raises the vacuum energy constant to $-1/24 + 1/8 = 1/12$. Indeed, the general value of this constant for $0 \le a \le 1$ is

$$N(a) := \frac{1}{12} - \frac{a(1-a)}{2}, \tag{4.8}$$

c.f. [Gin89, §7.1] for a zeta-function regularisation argument.

Remark 4.1.2. *The CFT partition function is intimately related to the VOA character. The equal-time surfaces* $\sigma_0 =$ *const. on the cylinder* $z = \sigma_0 + i\sigma_1$*, under the exponential map, become circles of constant radius on the plane* $\zeta \in \mathbb{C}$*. So dilatations of* $\mathbb{C}$ *are just the time translations on the cylinder, while rotations of* $\mathbb{C}$ *are space translations. After a rotation* $z \mapsto iz$ *and the discrete identification of* $z \sim z + 2\pi$ *and* $z \sim z + 2\pi\tau$*, we find that shifts in* $\operatorname{Re} z$ *are space translations, and shifts in* $\operatorname{Im} z$ *are time translations.*

Now in the Hamiltonian formalism, the energy operator H *is the zero mode of the field that generates time translations on the cylinder; the momentum operator* P *is the zero mode of the field that generates space translations on the cylinder. We know by (4.3) that after the rotation of* z*, dilatations and rotations are respectively generated by* $L_0 \pm \bar{L}_0$*, so we can identify*

$$H = L_0 + \bar{L}_0 - \frac{c + \bar{c}}{24}, \quad P = L_0 - \bar{L}_0 - \frac{c - \bar{c}}{24}$$

The partition function in the Hamiltonian operator formalism is:

$$\begin{aligned} Z(q) &:= \operatorname{tr}\left(e^{2\pi i \operatorname{Re}\tau P} e^{-2\pi \operatorname{Im}\tau H}\right) \\ &= \operatorname{tr}\left(q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}\right), \end{aligned} \tag{4.9}$$

where $q = e^{2\pi i \tau}$ *is the nome of* τ*. We can recognise the holomorphic part of this trace as Definition 2.3.4 of the VOA Virasoro character, after a zero mode shift by* $-c/24$*.*

L_0-eigenvalue	Neveu-Schwarz State	Diagram
0	$\|0\rangle$	$\|0\rangle$
${}^1\!/\!_2$	$\psi_{1/2}\|0\rangle, \psi^\dagger_{-1/2}\|0\rangle$	$\|1\rangle, \|-1\rangle$
1	$\psi_{1/2}\psi^\dagger_{-1/2}\|0\rangle$	$\|\yng(1);0\rangle$
${}^3\!/\!_2$	$\psi_{3/2}\|0\rangle, \psi^\dagger_{-3/2}\|0\rangle$	$\|\yng(1);1\rangle, \|\yng(1);-1\rangle$
2	$\psi_{3/2}\psi_{1/2}\|0\rangle, \psi_{3/2}\psi^\dagger_{-1/2}\|0\rangle, \psi^\dagger_{-3/2}\psi_{1/2}\|0\rangle, \psi^\dagger_{-3/2}\psi^\dagger_{-1/2}\|0\rangle$	$\|2\rangle, \|\yng(2);0\rangle, \|\yng(1,1);0\rangle, \|-2\rangle$
${}^5\!/\!_2$	$\psi_{5/2}\|0\rangle, \psi_{1/2}\psi^\dagger_{-3/2}\psi^\dagger_{-1/2}\|0\rangle, \psi^\dagger_{-5/2}\|0\rangle, \psi_{3/2}\psi_{1/2}\psi^\dagger_{-1/2}\|0\rangle$	$\|\yng(2);1\rangle, \|\yng(2);-1\rangle, \|\yng(1,1);-1\rangle, \|\yng(1,1);1\rangle$
3	$\psi_{5/2}\psi_{1/2}\|0\rangle, \psi_{3/2}\psi^\dagger_{-3/2}\|0\rangle, \psi_{5/2}\psi^\dagger_{-1/2}\|0\rangle, \psi_{1/2}\psi^\dagger_{-5/2}\|0\rangle, \psi^\dagger_{-5/2}\psi^\dagger_{-1/2}\|0\rangle$	$\|\yng(1);2\rangle, \|\yng(2,1);0\rangle, \|\yng(3);0\rangle, \|\yng(1,1,1);0\rangle, \|\yng(1);-2\rangle$

Table 4.1: Low-energy states of the NS sector $\mathfrak{F}(\mathbb{F})$.

$\frac{1}{8}+L_0$-eigenvalue	Ramond State	Diagram
$\frac{1}{8}+0$	$\|{}^1\!/\!_2\rangle, \|-{}^1\!/\!_2\rangle$	$\|{}^1\!/\!_2\rangle, \|-{}^1\!/\!_2\rangle$
$\frac{1}{8}+1$	$\psi_1\|{}^1\!/\!_2\rangle, \psi^\dagger_{-1}\|{}^1\!/\!_2\rangle, \psi_1\|-{}^1\!/\!_2\rangle, \psi^\dagger_{-1}\|-{}^1\!/\!_2\rangle$	$\|{}^3\!/\!_2\rangle, \|\yng(1);-{}^1\!/\!_2\rangle, \|\yng(1);{}^1\!/\!_2\rangle, \|-{}^3\!/\!_2\rangle$
$\frac{1}{8}+2$	$\psi_2\|{}^1\!/\!_2\rangle, \psi_1\psi^\dagger_{-1}\|{}^1\!/\!_2\rangle, \psi^\dagger_{-2}\|{}^1\!/\!_2\rangle, \psi_2\|-{}^1\!/\!_2\rangle, \psi_1\psi^\dagger_{-1}\|-{}^1\!/\!_2\rangle, \psi^\dagger_{-2}\|-{}^1\!/\!_2\rangle$	$\|\yng(1);{}^3\!/\!_2\rangle, \|\yng(1,1);{}^1\!/\!_2\rangle, \|\yng(1,1);-{}^3\!/\!_2\rangle, \|\yng(2);{}^1\!/\!_2\rangle, \|\yng(2);-{}^1\!/\!_2\rangle, \|\yng(1);-{}^3\!/\!_2\rangle$

Table 4.2: Low-energy states of the R sector $\mathfrak{F}(\mathbb{Z})$.

4.2 $U(1)$ Characters

Definition 4.2.1. *The classical* Jacobi Theta Function *is [WW15, §21.1]*

$$\vartheta(w,\tau) := \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}n^2} w^n.$$

Proposition 4.2.1. (Jacobi Triple Product)

$$\frac{\vartheta(w,q)}{\eta(q)} = \underbrace{\frac{1}{\eta(q)}\sum_{n\in\mathbb{Z}} q^{\frac{1}{2}n^2} w^n}_{\chi_{\mathfrak{H}}} = \underbrace{q^{-1/24}\prod_{j\in\mathbb{F}_{\geq 0}} (1+wq^j)(1+w^{-1}q^j)}_{\chi_{\mathfrak{F}(\mathbb{F})}}.$$

To prove this famous identity [WW15, §21.3], we equate characters χ of the VOAs $\mathfrak{H}$ and $\mathfrak{F}$, which are isomorphic by the Boson-Fermion correspondence 2.4.1.

Proof. **Fermionic Side.** Given the $c = 1$ holomorphic fermion CFT of Section 4.1, the $U(1)$ character of the Neveu-Schwarz sector $\mathfrak{F}(\mathbb{F})$ is:

$$\chi_{\mathfrak{F}(\mathbb{F})}(w,q) := \mathrm{tr}_{\mathfrak{F}(\mathbb{F})}\left(w^{\alpha_0} q^{L_0 - 1/24}\right),$$

This trace is easily calculated by decomposing the Fock space. Fix $j \in \mathbb{F}_{>0}$ and consider the 2d Clifford algebra generated by $\psi_j, \psi^\dagger_{-j}$ with representation V_j spanned by

$$|0\rangle_{\mathrm{NS}}, \psi_j|0\rangle_{\mathrm{NS}}, \psi_j^\dagger|0\rangle_{\mathrm{NS}}, \psi_j\psi^\dagger_{-j}|0\rangle_{\mathrm{NS}}.$$

Projecting the trace onto this basis gives

$$\mathrm{tr}_{V_j}(w^{:\psi_j\psi_j^\dagger:} q^{j:\psi_j\psi_j^\dagger:}) = \mathrm{tr}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & wq^j & 0 & 0 \\ 0 & 0 & w^{-1}q^j & 0 \\ 0 & 0 & 0 & q^{2j} \end{pmatrix} = (1+wq^j)(1+w^{-1}q^j). \tag{4.10}$$

Therefore

$$\begin{aligned} \chi_{\mathfrak{F}(\mathbb{F})}(w,q) &= q^{-1/24}\mathrm{tr}_{\mathfrak{F}(\mathbb{F})}\left(w^{\sum_{j\in\mathbb{F}}:\psi_j\psi_j^\dagger:} q^{\sum_{j\in\mathbb{F}} j:\psi_j\psi_j^\dagger:}\right) \\ &= q^{-1/24}\prod_{j\in\mathbb{F}_{\geq 0}} \mathrm{tr}_{V_j}\left(w^{:\psi_j\psi_j^\dagger:} q^{j:\psi_j\psi_j^\dagger:}\right) \\ &= q^{-1/24}\prod_{j\in\mathbb{F}_{\geq 0}} (1+wq^j)(1+w^{-1}q^j). \end{aligned} \tag{4.11}$$

Bosonic Side. The partition function of the free boson is the $U(1)$ character

$$\chi_{\mathfrak{H}}(w,q) = \mathrm{tr}_{\mathfrak{H}}\left(w^{\alpha_0} q^{L_0 - 1/24}\right).$$

Recall the calculation (2.25) of the character of the $\mathfrak{H}^{(0)}$ VOA, $\chi_{\mathfrak{H}^{(0)}}(q) = q^{1/24}\frac{1}{\eta(q)}$. We promoted this VOA to the full $\mathfrak{H}$ VOA in Example 2.4.1. Since the charge m vacuum has L_0 weight $\frac{m^2}{2}$ by equation (2.41), the charge m character is $\chi_{\mathfrak{H}^{(m)}}(q) = q^{m^2/2}\chi_{\mathfrak{H}^{(0)}}(q)$. Putting this together, we find:

$$\chi_{\mathfrak{H}}(w,q) = q^{-1/24}\sum_{n\in\mathbb{Z}}\chi_{\mathfrak{H}^{(0)}}(q)q^{\frac{n^2}{2}}w^n = \frac{1}{\eta(q)}\sum_{n\in\mathbb{Z}}q^{\frac{n^2}{2}}w^n = \frac{\vartheta(w,q)}{\eta(q)}. \tag{4.12}$$

Now, by the Boson-Fermion Correspondence 2.4.1, the trace $\chi(w,q)$ can be evaluated over either $\mathfrak{H}$ or $\mathfrak{F}$. Equating expressions (4.12) and (4.11), we obtain the desired identity. $\square$

4.2.1 Spin Structures

> We demand that conformally equivalent elliptic curves should have the same partition function. This forces us to consider the four spin structures supported on elliptic curves.

A unifying object for what follows is:

Definition 4.2.2. *The* Jacobi Theta Function of Characteristic α,β *is [Pol98, 7.2.36]:*

$$\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z,\tau) := q^{\alpha^2/2}w^{\alpha}e^{2\pi i\alpha\beta}\vartheta(z+\alpha\tau+\beta,\tau). \tag{4.13}$$

Example 4.2.1.

$$\vartheta\begin{bmatrix}0\\ 0\end{bmatrix}(w,\tau) = \sum_{n\in\mathbb{Z}}q^{\frac{n^2}{2}}w^n \qquad \vartheta\begin{bmatrix}0\\ 1/2\end{bmatrix}(w,\tau) = \sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{n^2}{2}}w^n$$
$$\vartheta\begin{bmatrix}1/2\\ 0\end{bmatrix}(w,\tau) = \sum_{n\in\mathbb{F}}q^{\frac{n^2}{2}}w^n \qquad \vartheta\begin{bmatrix}1/2\\ 1/2\end{bmatrix}(w,\tau) = \sum_{n\in\mathbb{F}}(-1)^n q^{\frac{n^2}{2}}w^n.$$

Proposition 4.2.2. (Ramond Jacobi Identity)

$$\vartheta\begin{bmatrix}1/2\\ 0\end{bmatrix}(w,q) = q^{1/8}(w^{1/2}+w^{-1/2})\prod_{j\in\mathbb{Z}_{>0}}(1-q^j)(1+wq^j)(1+w^{-1}q^j).$$

Proof. The argument of Theorem 4.2.1 must be modified, as the Ramond sector $\mathfrak{F}(\mathbb{Z})$ presents new features: the presence of zero modes $\psi_0, \psi_0^\dagger$ shifts the α_0 charge by ± 1, but do not affect the L_0 energy eigenvalue. For example, as was discussed in Section 4.1, the Ramond vacua $|\pm 1/2\rangle_R$ both have weight $1/8$.

These Ramond vacua must provide a representation of the 2d Clifford algebra generated by $\psi_0, \psi_0^\dagger$ and $\{\psi_0, \psi_0^\dagger\} = 1$. We can write this Clifford representation through Pauli matrices:

$$\psi_0 = \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \psi_0^\dagger = \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

acting on $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = |1/2\rangle_R$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix} = |-1/2\rangle_R$.

The calculation of the Ramond $U(1)$ character plays out similarly to the Neveu-Schwarz character of equation (4.11): on the 2d vacuum representation with basis $|1/2\rangle_R, |-1/2\rangle_R$, the projected trace is

$$\mathrm{tr}(w^{:\psi_0\psi_0^\dagger:}q^0) = \mathrm{tr}\begin{pmatrix} w^{1/2} & 0 \\ 0 & w^{-1/2} \end{pmatrix} = w^{1/2} + w^{-1/2}.$$

On the representation of the Clifford algebra generated by $\psi_j, \psi_{-j}^\dagger$, $j \in \mathbb{Z}_{>0}$, the projected trace is $(1+wq^j)(1+w^{-1}q^j)$, following equation (4.10). Therefore

$$\chi_{\mathfrak{F}(\mathbb{Z})}(z,q) = q^{-1/24}\mathrm{tr}_{\mathfrak{F}(\mathbb{Z})}\left(w^{\alpha_0}q^{L_0+1/8}\right) = q^{1/12}(w^{1/2}+w^{-1/2})\prod_{j\in\mathbb{Z}_{>0}}(1+wq^j)(1+w^{-1}q^j).$$

Bosonically, we reproduce the character of equation (4.12), modified to be a sum over $m \in \mathbb{F}$. By consulting the theta functions of Example 4.2.1, this can be written as

$$\chi_{\mathfrak{H}}(z,q) = \frac{1}{\eta(q)}\vartheta\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}(w,q).$$

Equating characters of the isomorphic VOAs produces the desired result. $\square$

The general $U(1)$ character, twisted by the holonomy $we^{2\pi i b}, e^{2\pi i a}$ around the time and space cycles respectively, is

$$b\underset{a}{\square}(w,q) := q^{N(a)}\mathrm{tr}_{\mathfrak{F}(\mathbb{Z}+a)}\left((-1)^{\alpha_0}e^{2\pi i b\alpha_0}w^{\alpha_0}q^{L_0}\right). \tag{4.14}$$

where $N(a)$ is the vacuum energy constant (4.8). The next proposition can be proved in exactly the same manner as Propositions 4.2.2 and 4.2.1.

Proposition 4.2.3.

$$b\underset{a}{\square}(w,q) = \frac{1}{\eta(q)}\vartheta\begin{bmatrix} 1/2-b \\ 1/2-a \end{bmatrix}(w,q).$$

Reading off the theta functions of Example 4.2.1, we find the $U(1)$ characters for each of the four spin structures on the elliptic curve:

$$
\begin{aligned}
1/2\,\underset{1/2}{\square}(w,q) &= \frac{1}{\eta(q)}\vartheta\begin{bmatrix}0\\0\end{bmatrix}(w,q) \quad & 1/2\,\underset{0}{\square}(w,q) &= \frac{1}{\eta(q)}\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}(w,q)\\
0\,\underset{1/2}{\square}(w,q) &= \frac{1}{\eta(q)}\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}(w,q) \quad & 0\,\underset{0}{\square}(w,q) &= \frac{1}{\eta(q)}\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}(w,q).
\end{aligned}
\tag{4.15}
$$

Definition 4.2.3. *The* THETANULLWERT *is the restriction of the Jacobi theta function* $\vartheta\begin{bmatrix}\alpha\\\beta\end{bmatrix}(z,\tau)$ *to* $z=0$.

Remark 4.2.1. *Note the vanishing of the Thetanullwert*

$$
\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}(0,\tau) = i\sum_{n\in\mathbb{F}}(-1)^{n-1/2}q^{\frac{n^2}{2}} = 0, \tag{4.16}
$$

since there is an involution $n\mapsto -n$ *of this summation over* $\mathbb{F}$*, that sends terms to their negative. We will return to this idea of an involution of a theta function summation, when we prove quasimodularity in Chapter 5.*

Now there is a natural relationship between the various Thetanullwerte, revealed by the action of:

Definition 4.2.4. *The* MODULAR GROUP *is* $\mathrm{PSL}(2,\mathbb{Z})/\mathbb{Z}_2$.

This group has a well-defined linear fractional action on $\tau\in\mathbb{H}$:

$$
\tau\mapsto\frac{a\tau+b}{c\tau+d},\qquad \begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\mathrm{PSL}(2,\mathbb{Z}). \tag{4.17}
$$

Remark 4.2.2. *A geometric viewpoint sees* $\mathrm{PSL}(2,\mathbb{Z})$ *as the group of disconnected diffeomorphisms of the torus, where the action (4.17) cuts along either of the non-trivial cycles, then re-glues after a twist by* 2π*. We follow [Gin89, §7.3] in defining generators of* $\mathrm{PSL}(2,\mathbb{Z})$*:*

$$
U:\tau\mapsto\frac{\tau}{1+\tau},\qquad T:\tau\mapsto\tau+1,
$$

which cut and glue along the space and time cycles, respectively, c.f. Figure 4.2.2.

Now the traditional generators of $\mathrm{PSL}(2,\mathbb{Z})$ *are* T *and* $S=T^{-1}UT^{-1}$ *satisfying* $S^2=$

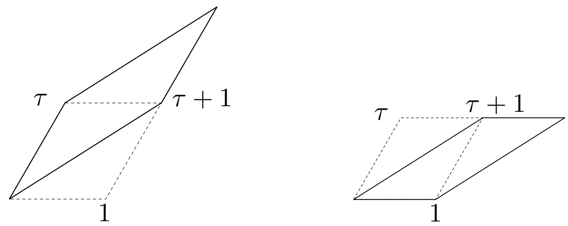

Figure 4.2: Modular maps U and T acting on the fundamental parallelogram of E, parametrised by τ.

$1 = (ST)^3$. *Ignoring prefactors, the restricted characters* $b\underset{a}{\square}(0,\tau)$ *transform as:*

$$
\begin{array}{cccc}
1/2\underset{1/2}{\square} \underset{T}{\longleftrightarrow} 0\underset{1/2}{\square} & 1/2\underset{0}{\square}\,\underset{T}{\circlearrowleft} & 0\underset{0}{\square}\,\underset{T}{\circlearrowleft} \\
0\underset{1/2}{\square} \underset{S}{\longleftrightarrow} 1/2\underset{0}{\square} & 1/2\underset{1/2}{\square}\,\underset{S}{\circlearrowleft} & 0\underset{0}{\square}\,\underset{S}{\circlearrowleft}
\end{array}
\tag{4.18}
$$

This is clear from Figure 4.2.2. After transforming the elliptic curve by T, the new "time" cycle, from lower left corner to the upper right corner, sees both old time and space boundary conditions $e^{2\pi ib}, e^{2\pi ia}$, to give an overall boundary condition of $e^{2\pi i(a+b)}$ on the new time cycle. Meanwhile,

$$S : \tau \mapsto -\frac{1}{\tau}$$

is an inversion, which swaps the space and time cycles, and therefore swaps their boundary conditions.

Let us fix the prefactors of the modular transforms (4.18) of the restricted character $b\underset{a}{\square}(0,\tau)$. Via Poisson resummation, one can show that under $T \in \mathrm{PSL}(2,\mathbb{C})$, the Thetanullwerte and the Dedekind eta function transform as [Gin89, §7.6]:

$$\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix} \leftrightarrow \vartheta\begin{bmatrix}0\\1/2\end{bmatrix}, \quad \vartheta\begin{bmatrix}1/2\\0\end{bmatrix} \mapsto \sqrt{i}\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}, \quad \eta \mapsto e^{i\pi/12}\eta$$

while under $S \in \mathrm{PSL}(2,\mathbb{C})$, they transform as

$$\vartheta\begin{bmatrix}\alpha\\\beta\end{bmatrix} \mapsto (-i\tau)^{1/2}\vartheta\begin{bmatrix}\alpha\\\beta\end{bmatrix}, \quad \eta \mapsto (-i\tau)^{1/2}\eta$$

This determines the prefactors of the T transform of the $z = 0$ characters:

$$1/2\underset{1/2}{\square} \mapsto e^{-i\pi/12}\,0\underset{1/2}{\square}, \qquad 0\underset{1/2}{\square} \mapsto e^{-i\pi/12}\,1/2\underset{1/2}{\square}, \qquad 1/2\underset{0}{\square} \mapsto e^{i\pi/6}\,1/2\underset{0}{\square},$$

while there are no prefactors for the S transform.

To construct a modular-invariant partition function, it is necessary to reintroduce the other chiral field $\bar{\psi}$ in the theory, as we ignored it from Section 4.1 onwards. We enforce identical boundary conditions on both ψ and $\bar{\psi}$. Then the character (4.9) in full chirality is:

$$bb\underset{aa}{\square}(w,\bar{w},q,\bar{q}) := |q|^{1/6-a(1-a)}\mathrm{tr}_{\mathfrak{F}(\mathbb{Z}+a)}\left((-1)^{\alpha_0-\bar{\alpha}_0}w^{\alpha_0}\bar{w}^{\bar{\alpha}_0}q^{L_0}\bar{q}^{\bar{L}_0}e^{2\pi i b(\alpha_0-\bar{\alpha}_0)}\right) = |b\underset{a}{\square}(w,q)|^2$$

The partition function of the free Dirac fermion is the sum of $U(1)$ characters over the four spin structures of the elliptic curve, [Gin89, §8.2]

$$Z(z,\tau) := \frac{1}{2}\sum_{a,b\in\{0,1/2\}}\left|b\underset{a}{\square}(w,q)\right|^2. \tag{4.19}$$

Proposition 4.2.4. *$Z(0,\tau)$ is modular invariant.*

Proof. After noting the vanishing of one of the $0\underset{0}{\square}(0,\tau)$ character by equation (4.16), we see that the above transformations of the Thetanullwerte simply swap terms in the sum over spin structures:

$$Z(0,\tau) = \frac{1}{2}\left|1/2\underset{0}{\square}(0,\tau)\right|^2 + \frac{1}{2}\left|0\underset{1/2}{\square}(0,\tau)\right|^2 + \frac{1}{2}\left|1/2\underset{1/2}{\square}(0,\tau)\right|^2$$

□

Remark 4.2.3. *We restricted our study of $\vartheta\begin{bmatrix}\alpha\\\beta\end{bmatrix}(z,\tau)$ to the Thetanullwert $z=0$ in order to demonstrate their modular transformation—a more honest treatment of both modular and elliptic transforms requires the framework of Jacobi Forms, as presented for the interested reader in Appendix B.*

Chapter 5

Deformed Conformal Field Theory

We deform the chiral fermionic CFT of the preceding chapter. This furnishes insights into Dijkgraaf's partition function Z which are not obvious from the Hurwitz-theoretic view—in particular, Z has a modular transform of mixed weight, and has quasimodular coefficients.

5.1 Chiral Deformations

We give a description of a $c = 1$ CFT deformed by $W_{1+\infty}$ fields. Using contact algebra results of [Dij97], recently proven by [GLT25], we study the effect of modular transformations on the partition function.

5.1.1 Dijkgraaf's Condition

Definition 5.1.1. *A* Chiral Field *is a holomorphic field* $\Phi(z)$ *of weight* $(h, 0)$*, for* $h \geq 0$.

By the Reconstruction Theorem 2.3.2, the space of chiral fields in a 2d CFT can be arranged into a Chiral VOA W [Dij97, §2.1], with state-field translation $Y(\Phi, z) = \Phi(z)$,

$\Phi \in W$. We will denote the Virasoro field by $L(z)$, and the translation by

$$T := L_{-1} = \partial_z$$

which raises the weight of an operator by one.

The first-order term in the OPE of chiral fields induces a Lie algebra structure on the quotient by the ideal TW,

$$V := W/TW,$$

given by

$$[\Phi, \Psi] := \Phi(z) \underset{(0)}{\times} \Psi(z),$$

for $\Psi, \Phi \in V$, recalling the notation for the n^{th} product of Definition 2.3.7.

Proposition 5.1.1. *V equipped with the bracket $[\cdot,\cdot]$ is a Lie algebra.*

Proof. First consider the skew-symmetry of the n^{th} product:

Lemma 5.1.1.

$$\Phi(z) \underset{(n)}{\times} \Psi(z) \equiv (-1)^{n+1} \Psi(z) \underset{(n)}{\times} \Phi(z)$$

where $\Phi(z), \Psi(z)$ are bosonic fields.

Proof. This identity is confirmed by comparing coefficients on both sides of the VA skew-symmetry of Corollary 2.3.1, which gives

$$\Phi(z) \underset{(n)}{\times} \Psi(z) = -\sum_{j \in \mathbb{Z}_{\geq 0}} (-1)^{n+j} \partial^j \left(\Psi(z) \underset{(n+j)}{\times} \Phi(z) \right), \tag{5.1}$$

for all $n \in \mathbb{Z}$. $\square$

This is the original skew-symmetry formula of Borcherds [Bor86]. Our bracket is the $n = 0$ case, and is therefore skew-symmetric modulo derivatives.

Note that the weight of the product $\Phi(z) \underset{(n)}{\times} \Psi(z)$ is given by $h_\Phi + h_\Psi - n$. Hence, the bracket closes, as $[\Phi, \Psi]$ has weight $h_\Phi + h_\Psi \geq 0$.

Lemma 5.1.2. *[Kac98, 4.8.3] In a VA V, for $\phi, \psi, \eta \in V$ and $m, n, k \geq 0$,*

$$\sum_{j \in \mathbb{Z}_{\geq 0}} \binom{m}{j} (\phi \underset{(n+j)}{\times} \psi) \underset{(m+k-j)}{\times} \eta = \sum_{j \in \mathbb{Z}_{\geq 0}} \binom{n}{j} (-1)^j \phi \underset{(m+n-j)}{\times} (\psi \underset{(k+j)}{\times} \eta)$$
$$- \sum_{j \in \mathbb{Z}_{\geq 0}} \binom{n}{j} (-1)^{j+n} \psi \underset{(k+n-j)}{\times} (\phi \underset{(m+j)}{\times} \eta).$$

This result is also due to Borcherds. The $m, n, k = 0$ case produces the Jacobi identity, up to total translation, for the bracket on V. Therefore V is a Lie algebra. $\square$

Remark 5.1.1. *One can identify this Lie algebra with the Lie algebra of Noether charges of W, where the isomorphism is via a contour integral around $0 \in \mathbb{C}$,*

$$Q : W \to V : \Psi \mapsto \oint \frac{dz}{2\pi i} \Psi(z)$$

which satisfies $[Q(\Psi), Q(\Phi)] = Q([\Psi, \Phi])$. In particular, $Q(\Phi) = \Phi_{(h-1)}$ for a weight h field.

Henceforth we require the Lie algebra structure on V to be abelian, i.e. the bracket $[\cdot, \cdot]$ vanishes. The appropriate condition is:

Condition 5.1.1. *A sub-VOA $A \subset W$ of a chiral VOA W satisfies* Dijkgraaf's Condition *if:*

I $A \cap W_1 = \emptyset$.

II $[\Phi, \Psi] \in TA$ *for all* $\Phi, \Psi \in A$,

Assume henceforth that there exists $A \subset W$ satisfying Dijkgraaf's Condition (5.1.1), so the bracket is a translation. Redefine

$$V := A/TA.$$

We will try to determine structure constants $c_{ij}^k \in \mathbb{C}$ such that:

$$[\Phi_j, \Phi_i] - T \sum_{k \in \mathbb{N}} c_{ij}^k \Phi_k \in T^2 A.$$

A formal solution of this equation in V is $T^{-1}[\Phi_j, \Phi_i] \equiv \sum_{k \in \mathbb{N}} c_{ij}^k \Phi_k$. This can be made precise:

Proposition 5.1.2. *[GLT25, 2.8] Let A satisfy Dijkgraaf's Condition (5.1.1). Then there exists a well-defined inverse $T^{-1} : A \underset{(0)}{\times} A \to A_{>0}$ of translation T.*

This allows us to strip off the translation from the bracket, to define the following product [Dij97, 2.9]:

Definition 5.1.2. *The* Chiral Product *in an abelian VOA $V = A/TA$, where the VOA A satisfies Dijkgraaf's Condition (5.1.1), is*

$$B(\Phi_i, \Psi_j) := T^{-1}[\Phi_i, \Psi_j] := \sum_{k \in \mathbb{N}} c_{ij}^k \Phi_k.$$

This product is well-defined modulo translations. It is commutative but non-associative. To see the necessity of Dijkgraaf's Condition (I) which excludes weight 1 fields, note that the weight of $B(\Phi, \Psi)$ is $h_\Phi + h_\Psi - 2$. So if $h_\Phi = 1 = h_\Psi$, then $B(\Phi, \Psi)$ is weight 0, and therefore a multiple of $\mathbb{1}$, the vacuum field. Proposition 5.1.2 does not apply, as $\mathbb{1} \notin A_{>0}$ is not the inverse translation of any field. To see this, suppose $\mathbb{1} = T^{-1}\Phi$ for some weight 1 operator Φ; but then $\Phi = T\mathbb{1} = 0$, a contradiction.

Remark 5.1.2. *The antisymmetric part of the chiral product yields a second Lie algebra structure on V:*

$$[\![\Psi, \Phi]\!] := B(\Psi, \Phi) - B(\Phi, \Psi). \tag{5.2}$$

The Jacobi identity for this bracket follows from the identity

$$[\![B(\Psi, \Upsilon), B(\Phi, \Upsilon)]\!] = B([\![\Psi, \Phi]\!], \Upsilon). \tag{5.3}$$

for $\Psi, \Phi, \Upsilon \in V$. Curiously, equations (5.3) and (5.2) together give B the interpretation of a flat, torsion-free linear connection on V viewed as a vector space.

We shall also need:

Definition 5.1.3. *The* Contact Product *on a VOA A satisfying Dijkgraaf's Condition (5.1.1) is*

$$C(\Phi, \Psi) := \Phi \underset{(1)}{\times} \Psi - B(\Phi, \Psi), \quad \Phi, \Psi \in A.$$

From equation (5.1) it follows that, in V, we have [Dij97, 2.10]

$$C(\Phi, \Psi) \equiv B(\Psi, \Phi) \quad \Phi, \Psi \in A.$$

Example 5.1.1. (Free Field) *Our main example is the $W_{1+\infty}$ VOA at $c = 1$, constructed in Section 2.3.4, which we determined was isomorphic to the uncharged Clifford VOA $\mathfrak{F}^{(0)}$. Recall that we constructed the state-field correspondence (2.35) of this VOA from the weight $k+1$ fields $\Phi^k(z) = {:}\psi(z)\partial^k\psi^\dagger(z){:}$, and the translation $T = \partial$. Define a subalgebra A of $W_{1+\infty}$, consisting of the fields ${:}\partial^a\psi\partial^b\psi^\dagger{:}$. Passing to the abelian subalgebra $V := A/\partial A$, we find a unique field Φ_n of weight $h = 1 + a + b = 1 + n$ in the class*

$${:}\partial^a\psi\partial^b\psi^\dagger{:} \equiv {:}\psi\partial^n\psi^\dagger{:} \, (\mathrm{mod}\, \partial).$$

We consider only Φ_n for $n \geq 1$, i.e. restrict to the W_∞ algebra, which discards the $U(1)$ field $\Phi_0 = {:}\psi\psi^\dagger{:}$ of weight 1 from $W_{1+\infty}$. The OPEs are can be determined by Wick's theorem *to be:*

$$\Phi_m \underset{(0)}{\times} \Phi_n \equiv 0 \, (\mathrm{mod}\, \partial), \quad \Phi_m \underset{(1)}{\times} \Phi_n = (m+n)\Phi_{m+n-1}, \quad m, n \in \mathbb{N}.$$

From the 0^{th} *product, we learn* V *satisfies Dijkgraaf's Condition (5.1.1). The chiral product is*

$$B(\Phi_m, \Phi_n) = n\Phi_{m+n-1}, \tag{5.4}$$

i.e. the structure constants are $c^l_{jk} = k\delta_{l,j+k-1}$. *The second Lie bracket on* V, *mentioned in Remark 5.1.2, is*

$$[\![\Phi_n, \Phi_m]\!] = (n-m)\Phi_{n+m-1}. \quad m, n \in \mathbb{N}.$$

Clearly this Lie algebra structure on V *is isomorphic to the positive half of the Witt algebra* Wi *of Definition 2.3.8,*

$$\Phi_n \mapsto l_{n-1} = -z^n \frac{\partial}{\partial z}, \quad n \in \mathbb{N}.$$

5.1.2 Dijkgraaf's Master Equation

We write the Lagrangian of a chirally deformed CFT. A brief discussion follows, concerning a technical result of [GLT25]: a differential equation relating the partition function in Lagrangian and Hamiltonian formalisms.

The approach of this section is to take the free $c = 1$ CFT with associated chiral VOA W, subalgebra $A \subset W$ satisfying Dijkgraaf's Condition 5.1.1, and abelian subalgebra $V = A/TA$. We then deform this CFT by adding to the free Lagrangian the fields $\Phi_i \in V$ of weight h_i, with couplings t_i for $i \in \mathbb{N}$.

However, we would like to preserve conformality of the theory. Under a modular transform, $\tau \mapsto \frac{a\tau+b}{c\tau+d}$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{Z})$, the linear coordinate $z \in E$ transforms as $z \mapsto \frac{z}{c\tau+d}$. Since the fields transform as

$$\Phi_i(z) \mapsto (c\tau+d)^{h_i}\Phi_i(z),$$

we can preserve conformality by requiring the couplings to have τ-dependence, obeying an inverse modular transformation:

$$t_i \mapsto (c\tau+d)^{-h_i} t_i. \tag{5.5}$$

Definition 5.1.4. *The* CHIRALLY DEFORMED CFT *of a free CFT with action S_0 has action*

$$S(t) := S_0 + \frac{1}{2\pi}\int_E \frac{d^2 z}{\operatorname{Im}\tau}\sum_{j\in\mathbb{N}} t_j \Phi_j.$$

where $\Phi_j \in V = A/TA$ and $t_j(\tau)$ satisfy transformation (5.5).

We have normalised the action by $\operatorname{Im}\tau$ to make the measure invariant under a modular transform, since $\operatorname{Im}\tau \mapsto |c\tau+d|^{-2}\operatorname{Im}\tau$.

Next we write the partition function corresponding to $S(t)$ in the Hamiltonian operator formalism:

$$Z(s;\tau) := \operatorname{tr}\left(\prod_{j\in\mathbb{N}} \exp q^{L_0 - c/24}\left(s_j \oint_a dz \Phi_j(z)\right)\right). \tag{5.6}$$

where $q = e^{2\pi i\tau}$, and Hamiltonian couplings s_j are generally not equal to the Lagrangian couplings. We must ensure that function is well-defined perturbatively.

The fundamental result, due to Zhu, gives a well-defined perturbative expansion in terms of the CONFORMAL BLOCK on the elliptic curve, which is a family of maps $S := (S_n)_{n\in\mathbb{N}}$, where $S_n : A^{\otimes n} \to F_n$ maps to F_n, the space of meromorphic functions $f(z_1,\dots,z_n)$ on $\mathbb{C}^n$ which are doubly periodic in each variable with period $m + n\tau$, for $m, n \in \mathbb{Z}$. We will abuse notation to write the function itself as

$$S(\Phi_{i_1}\otimes\cdots\otimes\Phi_{i_n}) := S((\Phi_{i_1}, z_{i_1}),\dots,(\Phi_{i_n}, z_{i_n})).$$

This function must satisfy a long list of properties which is given in authoritative detail in [Zhu96, 4.1.1]. A Feynman diagrammatic approach to calculating the conformal block of the elliptic curve is given in [GLT25, §3.1].

The partition function (5.6) admits a perturbative expansion:

$$Z(s;\tau) = \sum_{n\in\mathbb{Z}_{\geq 0}} \frac{1}{n!}\sum_{I\subset\mathbb{N}^n}\left(\prod_{i\in I} s_i\right)\langle\Phi_I\rangle, \tag{5.7}$$

where $\Phi_I := \otimes_{i\in I}\Phi_i \in A^n$ for $I \in \mathbb{N}^n$, and the correlators $\langle\cdot\rangle : F_n \to \mathbb{C}$ are multiple a-cycle integrals over conformal blocks, defined by

$$\langle\Phi_I\rangle := \prod_{i\in I}\oint_a dz_i S((\Phi_{i_1}, z_{i_1}),\dots,(\Phi_{i_n}, z_{i_n})).$$

which are convergent and modular [Zhu96, 4.4.3, 5.3.3].

Now a multiple a-cycle contour integral generally depends on the order of integration. However, Dijkgraaf's condition guarantees the following result:

Proposition 5.1.3. *In an VOA A satisfying Condition 5.1.1,*

$$\left[\oint_a dz_i, \oint_a dz_j\right] S(\Phi_I) = 0, \quad i, j \in I, \tag{5.8}$$

where $\Phi_I = \otimes_{i\in I}\Phi_i \in A^{\otimes n}$ *for* $I \in \mathbb{N}^n$.

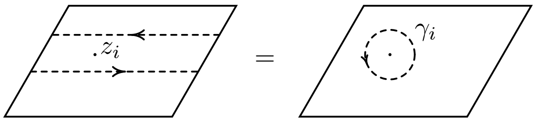

Figure 5.1: Commutator of a-cycle contour integrals, depicted in the fundamental parallelogram of E.

Proof. We have the following a-cycle contour integral identity:

$$\left[\oint_a dz_i, \oint_a dz_j\right] = \oint_a dz_i \oint_{\gamma_i} dz_j$$

by the contour deformation of Figure 5.1.2, where γ_i is the contour about z_i. This is well-known as the radial ordering contour [Gin89, §2.1].

We use three of the defining axioms [Zhu96, 4.1.1] of the conformal block:

1. *(Symmetry)* $S(\Phi_{i_1} \otimes \cdots \otimes \Phi_{i_n}) = S(\Phi_{\sigma(i_1)} \otimes \cdots \otimes \Phi_{\sigma(i_n)})$ for any permutation σ of n;

2. *(Vacuum)* $S(|0\rangle \otimes \Phi_{i_1} \otimes \cdots \otimes \Phi_{i_n}) = S(\Phi_{i_1} \otimes \cdots \otimes \Phi_{i_n})$;

3. *(Residue)*

$$\oint_{\gamma_i} dw(w-z_i)^k S((\Psi, w), (\Phi_{i_1}, z_{i_1}), \ldots, (\Phi_{i_n}, z_{i_n})) = S((\Phi_{i_1}, z_{i_1}), \ldots, (\Psi_{(k)}\Phi_i, z_i), \ldots, (\Phi_{i_n}, z_{i_n})). \tag{5.9}$$

We set $k = 0$ in equation (5.9) to obtain

$$\left[\oint_a dz_i, \oint_a dz_j\right] S(\Phi_I) = \oint_a dz_i \partial_{z_i} S(B_{j\to i}\Phi_i) = 0,$$

where we have extended the chiral product from Definition 5.1.2 to $A^{\otimes n}$ as follows:

$$B_{i\to j}(\Phi_1 \otimes \cdots \otimes \Phi_n) := \Phi_1 \otimes \cdots \otimes \underbrace{|0\rangle}_{i^{\text{th}}} \otimes \cdots \otimes \underbrace{B(\Phi_i, \Phi_j)}_{j^{\text{th}}} \otimes \cdots \otimes \Phi_n$$

for $1 \le i \ne j \le n$. $\square$

Now that we have a well-defined expansion (5.7), our next task is to solve for the Hamiltonian coupling constants $s_j(t)$ in terms of Lagrangian coupling constants t_j, $j \in \mathbb{N}$. If we accomplish this, we can use the modular transform of the Lagrangian couplings (5.5) to determine the modular transform of the partition function (5.6).

Write the 'mixed' partition function of the deformation,

$$Z(t, s; \tau) := \operatorname{tr}\left(\exp \sum_{j\in\mathbb{N}} \left(t_j \int_E \frac{d^2 z}{2\pi \operatorname{Im}\tau} \Phi_j(z) + s_j \oint_a dz \Phi_j(z)\right)\right). \tag{5.10}$$

Solving for the Hamiltonian couplings $s_j(t)$ in terms of the Lagrangian couplings t_j, $j \in \mathbb{N}$, amounts to solving $Z(t, 0; \tau) = Z(0, s(t); \tau)$. We need the following technical result:

Theorem 5.1.1. *[Dij97, 3.34] The mixed partition function Z of Equation (5.10) satisfies*

$$\left(\frac{\partial}{\partial t_j} - \frac{1}{2\operatorname{Im}\tau} \sum_{k,l\in\mathbb{N}} c^l_{jk} t_k \frac{\partial}{\partial t_l} - \frac{\partial}{\partial s_j} - \frac{1}{2\operatorname{Im}\tau} \sum_{k,l\in\mathbb{N}} c^l_{jk} s_k \frac{\partial}{\partial s_l}\right) Z(t, s; \tau) = 0, \quad j \in \mathbb{N},$$

where c^l_{jk} are the structure constants of the chiral product $B(\Phi_j, \Phi_k) = \sum_{l\in\mathbb{N}} c^l_{jk} \Phi_l$.

Remark 5.1.3. *The basic philosophy of the recent proof of this theorem [GLT25, 3.29] is to trade each integration in the mixed partition function (5.10) for a contour integration* plus a contact interaction, *as follows [Dij97, 3.10], [GLT25, 3.16].*

$$\int_E \frac{d^2 z_i}{\operatorname{Im}\tau} \Phi_I(z) = \underbrace{\oint_a dz_i \Phi_I(z) + \sum_{j\neq i} \partial_{z_i}\left(\frac{\operatorname{Im} z_j}{\operatorname{Im}\tau} (B_{i\to j}\Phi_I)(z)\right)}_{\widehat{\oint}_a} + \frac{1}{2\operatorname{Im}\tau} (C_{i\to j}\Phi_I)(z).$$

However, if we naïvely contour integrate over multiple a-cycles $a_1, \ldots, a_n$, the multiple integral $\widehat{\oint}_{a_1} \cdots \widehat{\oint}_{a_n}$ will also *generate a contact interaction, i.e. the property (5.8) is ruined, $[\widehat{\oint}_a dz_i, \widehat{\oint}_a dz_j] \neq 0$. To remedy this, a so-called 'modified' a-cycle contour integral $\widetilde{\oint}_{a^n}$ is introduced in [GLT25, 3.22], whose relevance is the fact that it does not generate a contact term, and it preserves double periodicity.*

Write the mixed partition function as a perturbative expansion in the coupling constants:

$$Z(s, t; \tau) = \sum_{m,n\in\mathbb{Z}_{\geq 0}} \frac{1}{m!n!} \left\langle (\sum_{j\in\mathbb{N}} t_j \Phi_j)^{\otimes m} \otimes (\sum_{k\in\mathbb{N}} s_k \Phi_k)^{\otimes n} \right\rangle, \tag{5.11}$$

where the mixed correlators $\langle\cdot\rangle_{m,n} : A^{\otimes(m+n)} \to \mathbb{C}$ are defined by

$$\langle \Phi_I \rangle_{m,n} := \int_{E^m} \prod_{j=1}^{m} \frac{d^2 z_j}{\operatorname{Im}\tau} \widetilde{\oint}_{a^n} \prod_{k=m+1}^{m+n} dz_k S(\Phi_I),$$

By induction, we can indeed trade integrals inside the mixed correlator for a modified contour integral plus contact terms [Dij97, 3.35], [GLT25, 3.27]:

$$\langle\Phi_I\rangle_{m,n} = \langle\Phi_I\rangle_{m-1,n+1} + \frac{1}{2\operatorname{Im}\tau}\sum_{\substack{1\le k\le m+n\\ k\neq m}}\sum_{l\in\mathbb{N}} c^l_{i_m i_k}\langle\Phi_{i_1}\otimes\cdots\otimes\underbrace{|0\rangle}_{m^{th}}\otimes\cdots\otimes\underbrace{\Phi_l}_{k^{th}}\otimes\cdots\otimes\Phi_{i_{m+n}}\rangle_{m,n}. \tag{5.12}$$

Assuming these results, we simply apply $\prod_{j=1}^m\prod_{k=1}^n \frac{\partial}{\partial t_{i_j}}\frac{\partial}{\partial s_{i_k}}$ to both sides of equation (5.11), and substitute equation (5.12) to arrive at the statement of Theorem 5.1.1.

Example 5.1.2. (Stress-Energy Deformation) *The simplest deformation of a free CFT is a Lagrangian perturbation $-tT(z)$, where $T(z)$ is the holomorphic component of the weight 2 quasi-primary stress-energy $T(z,\bar z)$. The stress-energy field deforms the action by $\delta S = -\int_E \frac{d^2z}{2\operatorname{Im}\tau}tT$, and the Hamiltonian by $\delta H = \oint_a dzs(t)T = -2\pi s(t)L_0$.*

Theorem 5.1.1 gives

$$(\partial_t - \frac{1}{2\operatorname{Im}\tau}t\partial_t - \partial_s - \frac{1}{2\operatorname{Im}\tau}s\partial_s)Z(s,t;\tau) = 0.$$

To see this, note the chiral product identity $B(T,\Phi)=\Phi$, which arises from the leading term of the OPE of quasi-primary fields $T(z)\Phi(w)\sim\frac{1}{z-w}\partial_w\Phi(w)+\ldots$. So if we only deform by T, there is only one structure constant $c^1_{11}=1$.

Introduce new variables $a = 1+\frac{s}{2\operatorname{Im}\tau}, b = 1-\frac{t}{2\operatorname{Im}\tau}$ to simplify the above:

$$(a\partial_a + b\partial_b)Z(a,b;\tau) = 0.$$

With the initial condition $Z(a,0;\tau) = Z(0,b;\tau)$, we find that the couplings are related by $a = b^{-1}$, or

$$1+\frac{s}{2\operatorname{Im}\tau} = \frac{1}{1-\frac{t}{2\operatorname{Im}\tau}}. \tag{5.13}$$

Notice that under a modular transformation, the left-hand-side is invariant, thus $s\mapsto(c\tau+d)^{-2}s$ must have modular weight -2.

As a sanity check, we have the following:

Example 5.1.3. (Stress-Energy Deformation as Complex Structure Deformation) *We alternatively view Example 5.1.2 as a deformation of $\bar\partial$ into a Beltrami differential with $\mu := -\frac{t}{2\operatorname{Im}\tau}$,*

$$\bar\partial\mapsto\bar\partial+\mu\partial.$$

In terms of the real coordinates $z = x + \tau y, \bar{z} = x + \bar{\tau} y$, and $\partial = \frac{\bar{\tau}\partial_x - \partial_y}{\bar{\tau}-\tau}, \bar{\partial} = \frac{-\tau\partial_x + \partial_y}{\bar{\tau}-\tau}$, this is

$$\bar{\partial} \mapsto \frac{\frac{\mu\bar{\tau}-\tau}{1-\mu}\partial_x + \partial_y}{\frac{\bar{\tau}-\tau}{1-\mu}}$$

which corresponds to a deformation of the modulus, or complex structure, of E:

$$\tau \mapsto \tau(t) = \tau + \frac{2i \operatorname{Im} \tau \mu}{1-\mu}, \quad \bar{\tau} \mapsto \bar{\tau}.$$

The Hamiltonian coupling is $s(t) = -i(\tau(t) - \tau) = \frac{2 \operatorname{Im} \tau \mu}{1-\mu}$, which replicates equation (5.13) as desired.

5.1.3 Modularity

Following [Dij97, §4.3], we find a general solution to the PDE in Theorem 5.1.1 when the chiral algebra is W_∞. We examine the modular behaviour of these solutions.

We consider the VOA W_∞, whose abelian subalgebra V we studied in Example 5.1.1, consisting of fields $\Phi_k(z) \equiv :\psi(z)\partial^k\psi^\dagger(z):$ of weight $k+1$, where $k \in \mathbb{N}$. By equation (5.4), the structure constants of the chiral product are $c^l_{jk} = k\delta_{l,j+k-1}$. We will always include Φ_1, i.e. we treat the deformation as being around the free action with $t_i = \delta_{i1}$. For this reason, we will use normalised couplings

$$a_i = \delta_{i1} + \frac{1}{2 \operatorname{Im} \tau} s_i, \quad b_i = \delta_{i1} - \frac{1}{2 \operatorname{Im} \tau} t_i,$$

and rewrite the PDE as

$$(A_j + B_j) Z(a, b; \tau) = 0, \quad j \in \mathbb{Z}_{\geq 0}, \tag{5.14}$$

where $B_j = \sum_{k\in\mathbb{N}} k b_k \frac{\partial}{\partial b_{k+j}}$ and $A_j = \sum_{k\in\mathbb{N}} k a_k \frac{\partial}{\partial a_{k+j}}$, $j \geq 0$.

Remark 5.1.4. *The differential operators A_i and B_i each form a representation of the strictly positive part of the Witt algebra $\mathsf{Wi}_{>0}$ of Definition 2.3.8 under the standard commutator, since for example $[A_i, A_j] = (i-j)A_{i+j}$, $i, j \in \mathbb{N}$.*

By the Boson-Fermion correspondence 2.4.1, instead of realising $W_{1+\infty}$ in the Clifford VOA $\mathfrak{F}^{(0)}$, as detailed in Section 2.3.4, we can realise it in the Heisenberg VOA $\mathfrak{H}^{(0)}$. We calculated the bosonic counterpart to Φ_k in equation (2.47). Under the quotient $V = A/\partial A$, the total derivatives in this equation vanish and we find that the abelian subalgebra V consists of

$$\Phi_k := :\psi \partial^k \psi^\dagger: \equiv \frac{1}{k+1} :(\partial\phi)^{k+1}:.$$

Because the fields are simply powers of $\partial\phi(z)$, we can collect the couplings into holomorphic functions on $\mathbb{C}^*$:

$$a(z) = \sum_{n\in\mathbb{N}} a_n z^n, \quad b(z) = \sum_{n\in\mathbb{N}} b_n z^n,$$

and express the bosonic deformed action of Definition 5.1.4 as

$$S = \int d^2 z \partial\phi\bar{\partial}\phi + :a(\partial\phi(z)):.$$

The mixed partition function of the deformation is

$$Z(a,b) = \mathrm{tr}\left(\exp(\int_E :a(\partial\phi): + \oint_a :b(\partial\phi):)\right) =: Z[a(z), b(z)],$$

This bosonic partition functional makes manifest the action of A_j, B_j on the coupling constants: A_j, B_j act as $\mathsf{Wi}_{>0}$ vector fields acting on holomorphic functions $a(z), b(z)$. The differential equations (5.14) express the invariance of the functional $Z[a(z), b(z)]$ under the $\mathsf{Wi}_{>0}$.

From the $\mathsf{Wi}_{>0}$ vector fields we can construct an arbitrary diffeomorphism $f : \mathbb{C} \to \mathbb{C}$ that fixes the origin $f(0) = 0$, such that $Z[a \circ f, b \circ f] = Z[a, b]$. In particular, setting $a(z) = z$ and $f(z) = b^{-1}(z)$ gives

$$Z[z, b(z)] = Z[b^{-1}(z), z]. \tag{5.15}$$

Recall we are attempting to solve this equation subject to the condition $Z(a, 0; \tau) = Z(0, b; \tau)$. This implies $a(b(z)) = z$, an equation which can be solved order-by-order to yield:

$$\begin{aligned}
a_1 &= \frac{1}{b_1} \\
a_2 &= -\frac{b_2}{b_1^3} \\
a_3 &= -\frac{b_3}{b_1^4} + \frac{2b_2^2}{b_1^5} \\
a_4 &= -\frac{b_4}{b_1^5} + \frac{5b_2 b_3}{b_1^6} - \frac{5b_2^3}{b_1^7}, \quad \text{etc.}
\end{aligned}$$

Under a modular transformation $\tau \mapsto \frac{a\tau+b}{c\tau+d}$, the Lagrangian coupling transformation (5.5) induces the following Hamiltonian coupling transformation:

$$
\begin{aligned}
s_1 &\mapsto (c\tau+d)^{-2}s_1 \\
s_2 &\mapsto (c\tau+d)^{-3}s_2 \\
s_3 &\mapsto (c\tau+d)^{-4}s_3 - 2ics_2^2(c\tau+d)^{-5}, \quad \text{etc.}
\end{aligned}
\tag{5.16}
$$

where as early as s_3 we see modular transformations of mixed weight.

5.2 Quasimodularity and Spectral Flow

> We discuss the spectral flow of partition invariants, which can be used to prove a quasimodularity theorem of [Zag16]. For a primer on quasimodular forms, see Appendix C.

In equation (2.50), we expressed Dijkgraaf's partition function (1.26) as a trace, over fermionic Fock space, of partition invariant (PI) operators Π_k, $0 \le k \le 3$.

Now we study the full $W_{1+\infty}$ character [Awa+95, §3.1]. We understand this to be the partition function of a free CFT deformed by the abelian subalgebra V of $W_{1+\infty}$, consisting of weight $k+1$ fields $\Pi_k(z) = \frac{1}{(k+1)!}{:}\alpha(z)^{k+1}{:} \equiv \frac{1}{k!}{:}\psi(z)\partial^k\psi^\dagger(z){:}$, with Hamiltonian coupling constants s_k. We express this character as a trace over Neveu-Schwarz Fock space:

$$
\begin{aligned}
Z(q_0,q_1,q_2,\dots) &:= \prod_{k\in\mathbb{Z}_{\ge 0}} q_k^{N_k} \prod_{j\in\mathbb{F}_{\ge 0}} \left(1+q_0\prod_{k\in\mathbb{N}} q_k^{\frac{1}{k!}j^k}\right)\left(1+q_0^{-1}\prod_{k\in\mathbb{N}} q_k^{-\frac{1}{k!}(-j)^k}\right) \\
&= \sum_{\substack{m\in\mathbb{Z}\\ \lambda\in\mathcal{P}}} \prod_{k\in\mathbb{Z}_{\ge 0}} q_k^{p_k(X_{\lambda;m})+N_k} \\
&= \operatorname{tr}_{\mathfrak{F}(\mathbb{F})}\left(\prod_{k\in\mathbb{Z}_{\ge 0}} q_k^{\Pi_k+N_k}\right),
\end{aligned}
\tag{5.17}
$$

for nomes $q_k = e^{s_k}$, $q_1 = q = e^{2\pi i\tau}$ and vacuum normalisation constants N_k yet to be determined.

As in the proof of Dijkgraaf's formula, we take the constant coefficient with respect to q_0:

$$
H(q_1,q_2,\dots) := q_0^0 \text{ coefficient of } Z(q_0,q_1,q_2,\dots) = \operatorname{tr}_{\mathfrak{F}^{(0)}}\left(\prod_{k\in\mathbb{N}} q_k^{\Pi_k+N_k}\right),
$$

to obtain a trace over the charge 0 Fock space. Develop this into a Taylor series in the couplings:

$$H(\tau, s_2, s_3, \ldots) = \sum_{\alpha=(\alpha_2,\ldots)} H_\alpha(\tau)\frac{s^\alpha}{\alpha!}$$

where the sum is over the multi-index α of $\mathbb{Z}_{\geq 0}$ with almost all entries $\alpha_j = 0$. Naturally enough the coefficients are correlators:

$$H_\alpha(\tau) = \mathrm{tr}_{\mathfrak{F}^{(0)}}\left(q^{\Pi_1 - 1/24}\prod_{k\geq 2}\Pi_k^{\alpha_k}\right).$$

where the multi-index α defines a monomial in the polynomial ring of PI operators $\mathbb{Q}[\Pi_2, \Pi_3, \ldots]$. Define the larger rings

$$R_\infty := \mathbb{Q}[\mathbb{1}, \Pi_1, \Pi_2, \ldots], \quad R_{1+\infty} := \mathbb{Q}[\mathbb{1}, \Pi_0, \Pi_1, \ldots].$$

We have a natural derivation on polynomials $R_{1+\infty}$ given by the chiral product with Π_0, which we abbreviate to

$$\nabla\cdot := B(\Pi_0, \cdot). \tag{5.18}$$

By equation (5.4), we have $\nabla\Pi_k = \Pi_{k-1}$ for $k > 0$ and $\nabla\Pi_0 = \mathbb{1}$. We also have a grading of these rings by weight, $\mathrm{wt}(\Pi_k) := k+1$, $\mathrm{wt}(\mathbb{1}) = 0$. Denote the graded sectors by $R^{(k)}$. The weight $\nabla^n P$ of a polynomial $P \in R^{(k)}_{1+\infty}$ is $\mathrm{wt}(P) = k - n$.

Now the monomial inside the correlator $H_\alpha(\tau)$ has weight $\mathrm{wt}\left(\prod_{k\geq 2}\Pi_k^{\alpha_k}\right) = \sum_{k\geq 2}(k+1)\alpha_k$.

Theorem 5.2.1. *[BO00, 0.2] $\eta(\tau)H_\alpha(\tau)$ is quasimodular of weight $\sum_{k\geq 2}(k+1)\alpha_k$.*

This theorem was motivated by the special case of Dijkgraaf's formula (2.50) for $H(\tau, s_2) = \sum_{n\in\mathbb{Z}_{\geq 0}} H_n(\tau)s_2^{2n}$, where $\eta(\tau)H_n(\tau)$ was shown to be quasimodular of weight $6n$ in [KZ95]. We give a proof along the lines of [Zag16, §4], presented in the language of Fock spaces.

Lemma 5.2.1. (Spectral Flow)

$$\langle\lambda; m+n|\Pi_k|\lambda; m+n\rangle = \langle\lambda; m|e^{n\nabla}\Pi_k|\lambda; m\rangle, \quad n \in \mathbb{F}\cup\mathbb{Z}.$$

Proof. Recall the charged wedge coordinates $X_{\lambda;m}$ of Definition 2.1.5 were coordinates for the fermionic Fock space, where $m \in \mathbb{Z}$ covered the Neveu-Schwarz sector $\mathfrak{F}(\mathbb{F})$ and $m \in \mathbb{F}$ covered the Ramond sector $\mathfrak{F}(\mathbb{Z})$. The charge shift by n is

$$\mathfrak{F}(\mathbb{B}) \to \mathfrak{F}(\mathbb{B}+n) : X_{m;\lambda} \mapsto X_{\lambda;m} + n = X_{\lambda;m+n}. \tag{5.19}$$

For $n \in \mathbb{F}$ this defines a bijection between the sectors $\mathfrak{F}(\mathbb{F})$ and $\mathfrak{F}(\mathbb{Z})$, and vice-versa; for $n \in \mathbb{Z}$, the charge shift leaves the wedge coordinates in the same sector.

Define the formal generating series for the wedge $X_{\lambda;m}$ in either sector:

$$f_{X_{\lambda;m}}(T) := \sum_{x \in X_{\lambda;m}} T^x,$$

which is convergent for $T > 1$. Therefore we can define a meromorphic function $f_{X_{\lambda;m}}(e^z)$, which evaluated on the charge 0 vacuum gives

$$f_{X_{\emptyset;0}}(e^z) = \frac{e^{z/2}}{e^z - 1} = \frac{1}{2\sinh(z/2)} = z^{-1} - \frac{1}{24}z + \frac{7}{5760}z^3 - \frac{31}{967680}z^5 + \ldots =: \sum_{k \in \mathbb{Z}_{\geq 0}} N_{k-1} z^{k-1}.$$

These $N_{k-1} \in \mathbb{Q}$ can be seen as vacuum normalisation constants for the PI operators Π_{k-1} of weight $k \in \mathbb{N}$, at central charge $c = 1$. For general wedge coordinates $X_{\lambda;m}$ in either Neveu-Schwarz or Ramond sectors $m \in \mathbb{Z} \cup \mathbb{F}$, we obtain

$$f_{X_{\lambda;m}}(e^z) = \sum_{k \in \mathbb{Z}_{\geq 0}} p_{k-1}(X_{\lambda;m}) z^{k-1} + f_{X_{\emptyset;0}}(e^z) = \sum_{k \in \mathbb{Z}_{\geq 0}} \left(p_{k-1}(X_{\lambda;m}) + N_{k-1} \right) z^{k-1}$$

whose coefficients are the partition invariants $p_k(X_{\lambda;m}) = \frac{1}{k!} \sum_{x \in X_{\lambda;m}} x^k$ defined in (2.4), offset by the normalisation N_k. We have defined by hand a trivial partition invariant $p_{-1} := 0$.

Notice that the charge shift (5.19) by n simply scales the generating function

$$f_{X_{\lambda;m+n}}(T) = f_{X_{\lambda;m}+n}(T) = T^n f_{X_{\lambda;m}}(T).$$

Hence,

$$\sum_{n \in \mathbb{Z}_{\geq 0}} p_{k-1}(X_{\lambda;m+n}) z^{k-1} = f_{X_{\lambda;m+n}}(e^z) = e^{nz} f_{X_{\lambda;m}}(e^z) = \sum_{k \in \mathbb{Z}_{\geq 0}} \sum_{j=0}^{k} p_{k-1-j}(X_{\lambda;m}) \frac{n^j}{j!} z^{k-1},$$

which implies

$$p_k(X_{\lambda;m+n}) = (e^{nd} p_k)(X_{\lambda;m}), \tag{5.20}$$

where we employ the derivation $dp_k := p_{k-1}$ for $k \geq 0$.

Recall from Proposition 2.4.5 that the eigenvalues of Π_k are exactly the partition invariants p_k. This derivation d is naturally implemented at the level of PI operators by the derivation ∇ of $R_{1+\infty}$, defined in equation (5.18). Therefore equation (5.20) implies the result:

$$\langle \lambda; m+n | \Pi_k | \lambda; m+n \rangle = \langle \lambda; m | e^{n\nabla} \Pi_k | \lambda; m \rangle.$$

□

Remark 5.2.1. *This matches the expected spectral flow in the $c = 1$ $W_{1+\infty}$ algebra, c.f. [Awa+95, 2.9]*

Example 5.2.1. *We can use this Lemma to calculate spectral flows of the partition invariants. Since $\nabla\Pi_0 = \mathbb{1}$ and $\nabla\mathbb{1} = 0$, we can expand:*

$$e^{n\nabla}\Pi_k = \sum_{l=0}^{k} \frac{n^l}{l!}[k]_l \Pi_{k-l} + \frac{n^{k+1}}{k+1}\mathbb{1},$$

where $[k]_l := (k)(k-1)\cdots(k-l+1)$ is the falling factorial, with $[k]_0 := 1$. The first few partition invariants are shifted by:

$$\begin{aligned}
p_0(X_{\lambda;m+n}) &= m+n, \\
p_1(X_{\lambda;m+n}) &= p_1(X_{\lambda;m}) + nm + \frac{n^2}{2}, \\
p_2(X_{\lambda;m+n}) &= p_2(X_{\lambda;m}) + 2np_1(X_{\lambda;m}) + n^2 m + \frac{n^3}{3}.
\end{aligned}$$

Notably, the energy eigenvalue of $|\lambda; 0\rangle$ is shifted by $n^2/2$, which is consistent with the energy shift of equation (2.41). This is also consistent with the vacuum energy shift of $1/8$ which we found in equation (4.8), when moving from the Neveu-Schwarz vacuum $|0\rangle$ to the Ramond vacua $|\pm 1/2\rangle$.

We can write the $W_{1+\infty}$ character (5.17) in various spin structures, $a, b \in \{0, 1/2\}$:

$$b\underset{a}{\square}(q_0, q_1, q_2, \ldots) := \mathrm{tr}_{\mathfrak{F}(\mathbb{Z}+a)}\left((-1)^{\Pi_0} e^{2\pi i b \Pi_0} \prod_{k\in\mathbb{Z}_{\geq 0}} q_k^{\Pi_k + N_k} \right).$$

In the Ramond-Ramond spin structure, $a = 0 = b$, the $W_{1+\infty}$ correlators all vanish:

Lemma 5.2.2.

$$\mathrm{tr}_{\mathfrak{F}(\mathbb{Z})}\left((-1)^{\Pi_0} q^{\Pi_1 - 1/24} P\right) = 0, \quad \textit{for } P \in R_\infty.$$

Proof. The following involution [Zag16, §4] on the wedge coordinates of the Ramond sector,

$$\mathfrak{F}(\mathbb{Z}) \to \mathfrak{F}(\mathbb{Z}) : X \mapsto \begin{cases} X - \{0\} & \text{if } 0 \in X, \\ X \cup \{0\} & \text{if } 0 \notin X, \end{cases}$$

changes the charge eigenvalue p_0 by ± 1, but leaves the higher eigenvalues p_k unchanged for $k \in \mathbb{N}$. So by expressing the trace as a sum over eigenvalues, we find the terms of the sum cancel in pairs, due to the presence of $(-1)^{p_0(X)}$. $\square$

Remark 5.2.2. *This involution is reminiscent of Section 4.2.1, where we calculated the $U(1)$ character of equation (4.14) in the Ramond-Ramond spin structure by using the Jacobi theta function of Definition 4.2.2:*

$$0\underset{0}{\square}(z,\tau) = \mathrm{tr}_{\mathfrak{F}(\mathbb{Z})}\left((-1)^{\Pi_0} q^{\Pi_1 - 1/24} w^{\Pi_0}\right) = \frac{1}{\eta(q)}\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}(w,q) = \frac{1}{\eta(q)}\sum_{n\in\mathbb{F}}(-1)^n q^{\frac{n^2}{2}} w^n,$$

In Remark 4.2.1 we showed $0\underset{0}{\square}(0,\tau) = 0$, since there exists an involution $n \mapsto -n$ of the summation over $n \in \mathbb{F}$, that pairs sectors of opposite charge $\pm n$; due to the presence of $(-1)^{\Pi_0}$, the traces over sectors of charge $\pm n$ cancel with one another.

Abbreviate the Ramond-Ramond theta function as:

$$\Theta(z) := \vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}(z,\tau) = \sum_{n\in\mathbb{F}}(-1)^n q^{\frac{n^2}{2}} e^{nz}. \tag{5.21}$$

Lemma 5.2.3.

$$\mathrm{tr}_{\mathfrak{F}^{(0)}}\left(q^{\Pi_1-1/24}\Theta(\nabla)P\right) = 0, \quad \textit{for } P \in R_\infty.$$

Proof. Insertion of the theta operator $\Theta(\nabla)$ into the charge 0 Neveu-Schwarz trace has the effect of summing over charge shifted traces:

$$\begin{aligned}
\mathrm{tr}_{\mathfrak{F}^{(0)}}\left(q^{\Pi_1-1/24}\Theta(q)P\right) &= \sum_{\substack{n\in\mathbb{F}\\ \lambda\in\mathcal{P}}}\langle\lambda;0|(-1)^n q^{\Pi_1-1/24+\frac{n^2}{2}}\left(e^{n\nabla}P\right)|\lambda;0\rangle \\
&= \sum_{\substack{n\in\mathbb{F}\\ \lambda\in\mathcal{P}}}\langle\lambda;n|(-1)^{\Pi_0} q^{\Pi_1-1/24}P|\lambda;n\rangle \\
&= \mathrm{tr}_{\mathfrak{F}(\mathbb{Z})}\left((-1)^{\Pi_0}q^{\Pi_1-1/24}P\right) \\
&= 0.
\end{aligned}$$

To arrive at the second equality and fourth equalities, we used Lemma 5.2.1 and Lemma 5.2.2 respectively. □

Lemma 5.2.4.

$$R^{(k)}_{1+\infty} = \nabla R^{(k+1)}_{\infty} + \Pi_0 R^{(k-1)}_{1+\infty}.$$

Proof. This can be seen by lexicographically ordering the monomial basis of $R^{(k)}_{1+\infty}$. □

We are now ready to prove quasimodularity. Define a series

$$\frac{\Theta(z)}{\Theta'(0)} =: \sum_{n\in\mathbb{Z}_{\geq 0}} H_n(\tau) z^{n+1},$$

in terms of $\Theta(z)$ given in equation (5.21), whose first few non-zero coefficients are:

$$H_0 = 1, \quad H_2 = \frac{E_2}{24}, \quad H_4 = \frac{4E_2^2 - 2E_4}{5760}, \quad H_6 = \frac{35E_2^3 - 42E_2E_4 + 16E_6}{2903040}, \quad \text{etc.}$$

where E_k are the standard Eisenstein series given in Appendix C. In general, H_n is quasi-modular of weight n, [Zag16, §2].

Proof of Theorem 5.2.1 following [Zag16, §4]. We claim the more general statement is true: that $\eta(q)\mathrm{tr}_{\mathfrak{F}^{(0)}}\left(q^{\Pi_1-1/24}P\right)$ is quasimodular of $\mathrm{wt}(P)$, for $P \in R_\infty$. We proceed by induction on the weight $k = \mathrm{wt}(P) \in \mathbb{N}$ of polynomials $P \in R_\infty$.

The base case of $k = 0$ is $P \in \mathbb{Q}\mathbb{1}$. Recalling the calculation $\mathrm{tr}_{\mathfrak{F}^{(0)}}(q^{\Pi_1}) = q^{1/24}\eta(q)^{-1}$ from equation (2.25), we find the trace $\eta(q)\mathrm{tr}_{\mathfrak{F}^{(0)}}(q^{\Pi_1-1/24}P) = P$ is constant and so trivially quasimodular. The presence of the $\eta(q)$ factor here is essential.

Notice that $P \in \Pi_0 R_{1+\infty}$ gives a vanishing trace, since $\Pi_0|_{\mathfrak{F}^{(0)}} = 0$. Hence, by Lemma 5.2.4, we can assume $P \in R_{1+\infty}^{(k)}$ can be expressed as $P = \nabla Q$ for $Q \in R_\infty^{(k+1)}$.

Now assume the inductive hypothesis for all polynomials of weight less than k. By applying Lemma 5.2.3 to Q we find

$$\begin{aligned}
0 &= \sum_{n\in\mathbb{Z}_{\geq 0}} H_n(q)\eta(q)\mathrm{tr}_{\mathfrak{F}^{(0)}}\left(q^{\Pi_1-1/24}\nabla^{n+1}Q\right) \\
&= \sum_{n\in\mathbb{Z}_{\geq 0}} H_n(q)\eta(q)\mathrm{tr}_{\mathfrak{F}^{(0)}}\left(q^{\Pi_1-1/24}\nabla^{n}P\right) \\
&= \eta(q)\mathrm{tr}_{\mathfrak{F}^{(0)}}\left(q^{\Pi_1-1/24}P\right) + \eta(q)H_2(q)\mathrm{tr}_{\mathfrak{F}^{(0)}}\left(q^{\Pi_1-1/24}\nabla^2P\right) + \eta(q)H_4(q)\mathrm{tr}_{\mathfrak{F}^{(0)}}\left(q^{\Pi_1-1/24}\nabla^4P\right) + \cdots
\end{aligned}$$

Since $\mathrm{wt}(\nabla^n P) = \mathrm{wt}(P) - n = k - n$, by the inductive hypothesis, we see that all the traces $\eta(q)\mathrm{tr}_{\mathfrak{F}^{(0)}}\left(q^{\Pi_1-1/24}\nabla^n P\right)$ are quasimodular of weight $k-n$ for $n > 0$. Since H_n is quasimodular of weight n, it follows that the leading term $\eta(q)\mathrm{tr}_{\mathfrak{F}^{(0)}}\left(q^{\Pi_1-1/24}P\right)$ must be quasimodular of weight k. It follows by induction that this is true for all $P \in R_\infty$. $\square$

Chapter 6

Appendices

A Hurwitz Theory

We briefly detail the topological interpretation of the Frobenius formula of Chapter 1, in the case $G = S_n$. Namely, the formula calculates Hurwitz numbers.

Hurwitz numbers, simply put, are a count of covers of Riemann surfaces, weighted by the size of their automorphism group. An automorphism of a map of Riemann surfaces $f : X \to Y$ is a holomorphic map $\phi : X \to X$ such that $f = f \circ \phi$. Such automorphisms form a group $\mathrm{Aut}(f)$ under the binary operation of composition.

For a degree d map of Riemann surfaces $f : X \to Y$, the fibre over a point is $f^{-1}(P) = \{R_1, \ldots, R_m\}$ for some $1 \leq m \leq d$. At each point R_i, the map f admits a local expression $f : z \mapsto (z - R_i)^{k_{R_i}}$, where $k_{R_i} \in \mathbb{N}$ is the Ramification Index at $R_i \in X$. The Ramification Profile of f at $P \in Y$ is the tuple consisting of ramification indices of points R_i above P:

$$\lambda = (k_{R_1}, \ldots, k_{R_m}).$$

Note that λ will always be a partition of d. Some partitions have customary names:

- Unramified: $\lambda = (1, \ldots, 1)$,
- Simply ramified: $\lambda = (2, 1, \ldots, 1)$,
- Fully ramified: $\lambda = (d)$.

Definition A.1. *Let Y be a connected, compact Riemann surface of genus h. Fix points $P_1, \ldots, P_n \in Y$ and let $\lambda_{P_1}, \ldots, \lambda_{P_n}$ each be partitions of d. The* CONNECTED HURWITZ NUMBER

$$H^{\circ}_{g \xrightarrow{d} h}(\lambda_1, \ldots, \lambda_n) = \sum_{[f]} \frac{1}{|\mathrm{Aut}(f)|},$$

is the weighted sum over the isomorphism classes of all covers $f : X \to Y$ satisfying

1. *f is holomorphic, non-zero, of degree d.*

2. *X is compact of genus h.*

3. *The ramification profile of f at P_i is λ_{P_i}.*

4. *X is connected.*

The genus h of a valid Hurwitz cover is determined by the ramification data and the target genus h. It can be found from the following formula:

Theorem A.1. (Riemann-Hurwitz) *Let $f : X \to Y$ be a non-constant, degree d, holomorphic map of compact Riemann Surfaces. Denote by g the genus of X, and h the genus of Y. Then*

$$2g - 2 = d(2h - 2) + \sum_{z \in X} (k_z - 1), \tag{6.1}$$

Example A.1. *The genus zero Hurwitz number, fully ramified over 0 and ∞, and unramified everywhere else, is*

$$H^{\circ}_{0 \xrightarrow{d} 0}((d), (d)) = \frac{1}{d}$$

To see this, apply the Riemann-Hurwitz formula. The cover must satisfy $f : \mathbb{CP}^1 \to \mathbb{CP}^1 : z \mapsto z^d$, which has automorphism group

$$\mathrm{Aut}(f) = \{\phi(z) = cz : c^d = 1\},$$

which is isomorphic to the d^{th} roots of unity. Hence $|\mathrm{Aut}(f)| = d$.

A.1 Monodromy Representations

To make contact with our representation theory of the symmetric group S_d, developed in Chapter 1, we shift from counting covers to counting monodromy representations of covers.

Given a cover $f : X \to Y$ satisfying the conditions of Definition A.1, let us study the effect of lifting the fundamental group of the target Y to the source X. A loop $\gamma : [0,1] \to Y - \{P_1, \ldots, P_n\}$, based at $Q \in Y$, can be lifted to possibly many paths starting and finishing at possibly different points in the inverse image $f^{-1}(Q)$. For a given loop $\tilde{\gamma}$, there is clearly a bijection sending the starting point to the end point:

$$\sigma_\gamma : f^{-1}(Q) \to f^{-1}(Q) : \tilde{\gamma}(0) \mapsto \tilde{\gamma}(1)$$

Hence a loop γ based at Q yields a permutation σ_γ of the preimages of Q. It is natural to decorate these preimages with labels, in order to keep track of the action of the permutation σ_γ. Define the LABELLING (L, Q) of the preimage of Q by the bijection $L : f^{-1}(Q) \to \{1, \ldots, d\}$. Since it is a bijection, Q is necessarily unramified.

Definition A.2. *The* MONODROMY REPRESENTATIONS *of a cover $f : X \to Y$ with labelling (L, P) are the group homomorphisms*

$$\Phi : \pi_1(Y - \{P_1, \ldots, P_n\}, Q) \to S_d : \gamma \mapsto \sigma_\gamma.$$

If, in addition, the subgroup $\mathrm{Im}(\Phi)$ *of S_d acts transitively on the set $\{1, \ldots, d\}$, then we can say that Φ is a* connected *monodromy representation.*

Since the permutation produced by concatenating two loops is equivalent to the product of their permutations, that is, $\sigma_{\gamma * \eta} = \sigma_\eta \sigma_\gamma$, the monodromy representation $\Phi(\gamma)$ depends only on the homotopy class of the loop γ, and is therefore well-defined.

The monodromy representation is a purely topological construction, in that it is ignorant of the position of the punctures $\{P_1, \ldots, P_n\}$. The reason for the 'connected' terminology is that a monodromy representation Φ is connected if and only if X, the source of the cover, is a connected Riemann surface. Now we explain how to encode the ramification profiles of a covering in a monodromy representation, as follows.

Recall from Definition 1.1.7 that the CYCLE TYPE of $\sigma \in S_d$ is $\mathrm{cyc}(\sigma) := (1^{\sigma_1} \ldots d^{\sigma_d})$, where σ_i is number of cycles of length i in the cycle-factorisation of σ. Note that a cycle type $\mathrm{cyc}(\sigma)$ is equivalent to a partition of d, since $1\sigma_1 + \ldots + d\sigma_d = d$.

Let a cover f satisfying the conditions of Definition A.1 have a branch point P, with preimages $f^{-1}(P) = \{R_1, \ldots, R_m\}$, and ramification profile $\lambda_P = (k_{R_1}, \ldots, k_{R_m})$ at P. Let γ be a simple loop based at Q, winding once around P. Then the monodromy representation $\sigma_\gamma \in S_d$ of the loop γ has cycle type given by the partition $\lambda = (k_{R_1}, \ldots, k_{R_m})$ of d. To see why, take $R_i \in f^{-1}(P)$, and choose a chart around R_i such that the local expression of f around R_i is $f : z \mapsto z^{k_{R_i}}$. Choose a chart around P such that locally the loop is a concatenation $\gamma = \alpha * \beta * \alpha^{-1}$, where in local coordinates, β is the loop $|f(z)| = 1$, and α joins the base point Q to the point $f(z) = 1$. Then σ_γ permutes the ${k_{R_i}}^{\text{th}}$ roots of unity $z^{R_i} = 1$. That is, as the lifted loop $\tilde{\gamma}$ around R_i progresses through k_{R_i} distinct points of $f^{-1}(Q)$, the base loop γ passes k_{R_i} many times through Q. This occurs at each of the points $R_1, \ldots, R_m$ in $f^{-1}(P)$. Hence the monodromy representation σ_γ consists of m disjoint cycles of lengths $\{k_{R_1}, \ldots, k_{R_m}\}$, which is exactly the ramification profile λ_P of f at P.

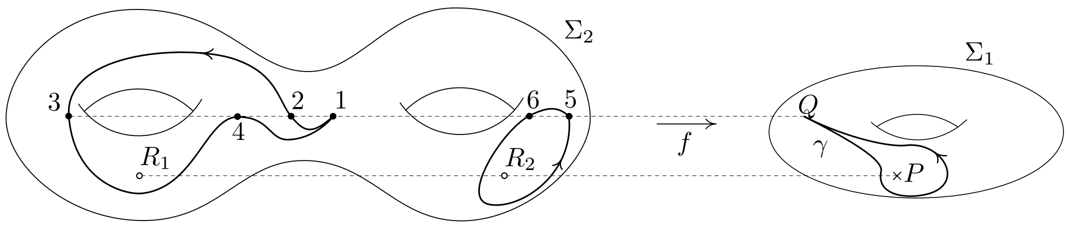

Figure 6.1: A Hurwitz cover of Σ_1 by Σ_2, with degree $d = 6$ and ramification profile $\lambda_P = (4, 2)$ at a single branch point p. The monodromy representation σ_γ of a loop $\gamma \in \pi_1(\Sigma_1 - \{P\}, Q)$ has cycle type $\text{cyc}(\sigma_\gamma) = (2^1 4^1)$.

This gives a map from Hurwitz covers to monodromy representations. In fact, more is true:

Theorem A.2. *[CM16, 7.5] There is a bijection between*

$$\left\{ \begin{array}{l} \textit{Isomorphism classes of maps } f : X \xrightarrow{d} Y \textit{ of connected Riemann surfaces,} \\ \textit{with labelling } L \textit{ over a base point } Q\textit{, and ramification profiles } \{\lambda_1, \ldots, \lambda_n\} \\ \textit{over branch points } \{P_1, \ldots, P_n\}. \end{array} \right\}$$

and

$$\left\{ \begin{array}{l} \textit{Connected monodromy representations } \Phi : \pi_1(Y - \{P_1, \ldots, P_n\}, Q) \to S_d \\ \textit{of cycle type } \{\lambda_{P_1}, \ldots, \lambda_{P_n}\}\textit{, where } \lambda_i \textit{ are partitions of } d. \end{array} \right\}$$

This gives us yet another rephrasing of our counting problem:

Proposition A.1. *[CM16, 7.3.1] Let M° be the set of connected monodromy representations from Theorem A.2. Then*

$$H^\circ_{g \xrightarrow{d} h}(\lambda_1, \ldots, \lambda_n) = \frac{|M^\circ|}{|S_d|},$$

where g is determined by the Riemann-Hurwitz formula (6.1).

Proof. Given a base point P, there are always $d!$ many ways to label the preimages $f^{-1}(P)$. An automorphism of f produces an isomorphism of labelled maps with base point P. This is a *relabelling*: that is, if $\phi \in \mathrm{Aut}(f)$ and (L, P) is a labelling, then $(\phi \bullet L, P)$ is a relabelling, where $\phi \bullet L := L \circ \phi^{-1}$. This defines the free left group action of $\mathrm{Aut}(f)$ on the set of labels of $f^{-1}(P)$.

For a given map f, the number n_f of isomorphism classes of labels of $f^{-1}(P)$ counts the number of monodromy representations of f, each of which arises from different choices of labelling of $f^{-1}(P)$. The orbit-stabiliser theorem implies that $d! = n_f|\mathrm{Aut}(f)|$, so that

$$H^{\circ}_{g \xrightarrow{d} h}(\lambda_1, \ldots, \lambda_n) = \sum_{[f]} \frac{1}{|\mathrm{Aut}(f)|} = \sum_{[f]} \frac{n_f}{d!} = \frac{|M|}{d!}.$$

□

Now we relax the connected condition in Definition A.2, i.e. drop the requirement that the monodromy representations act transitively on $\{1, \ldots, d\}$. We call these the *disconnected* monodromy representations $M^{\bullet}$. The corresponding Hurwitz number $H^{\bullet}_{g \xrightarrow{d} h}(\lambda_1, \ldots, \lambda_n) = |M^{\bullet}|/|S_d|$ is the Disconnected Hurwitz Number. This is defined identically to Definition A.1, except we remove condition (4). So topologically speaking, disconnected Hurwitz numbers count (possibly) disconnected covers.

B Jacobi Forms

To examine the modular and elliptic properties of the Jacobi theta functions of Section 4.2.1, it is natural to place them into the framework of Jacobi Forms, as follows.

Fix the nomes $q = e^{2\pi i\tau}$ and $w = e^{2\pi iz}$. For a holomorphic function $\phi : \mathbb{C} \times \mathbb{H} \to \mathbb{C}$, define a modular transform, parametrised by $m, k \in \mathbb{N}$:

$$\left(\phi|_{k,m} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(z, \tau) := (c\tau + d)^{-k} e^{-2\pi i m \frac{cz^2}{c\tau+d}} \phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right),$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$. Define also an elliptic transform:

$$\left(\phi|_m \begin{bmatrix} \alpha \\ \beta \end{bmatrix}\right)(z, \tau) := q^{m\alpha^2} w^{2m\alpha} \phi(z + \alpha\tau + \beta, \tau), \quad \alpha, \beta \in \mathbb{Z}. \tag{6.2}$$

Definition B.1. *[EZ85, §1] An* SL$(2,\mathbb{Z})$ JACOBI FORM *of* INDEX $k \in \mathbb{N}$ *and* WEIGHT $m \in \mathbb{N}$ *is a holomorphic function* $\phi : \mathbb{C} \times \mathbb{H} \to \mathbb{C}$*, invariant under modular and elliptic transforms:*

$$\phi|_{k,m} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \phi, \quad \phi|_m \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \phi,$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$*, which has a Fourier development of the form*

$$\phi(z,\tau) = \sum_{n=0}^{\infty} \sum_{\substack{r \in \mathbb{Z} \\ r^2 \leq 4mn}} c(n,r) q^n w^r.$$

It is a routine matter to verify

$$(\phi|_{k,m}A)\,|_{k,m}A' = \phi|_{k,m}(AA'), \quad (\phi|_{k,m}A)\,|_{k,m}(AX) = (\phi|_m X)\,|_{k,m}A,$$

for $A, A' \in \mathrm{PSL}(2,\mathbb{Z})$ and $X = (\alpha,\beta), X' = (\alpha',\beta') \in \mathbb{Z}^2$. The case of repeated elliptic transforms is

$$(\phi|_m X)\,|_m X' = \phi|_m(X + X'), \tag{6.3}$$

Hence Jacobi forms admit a a group action by $\mathrm{PSL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$. However, we would prefer if the translation $z \mapsto z + \alpha\tau + \beta$ in equation (6.2) was an honest elliptic transform, where $(\alpha,\beta) \in \mathbb{R}^2$. But this creates an obstruction which ruins the group action in (6.3):

$$\begin{aligned}(\phi|_m X)\,|_m X' &= q^{m(\alpha'^2+\alpha^2+2\alpha'\alpha)} w^{2m(\alpha'+\alpha)} e^{2\pi i m\alpha\beta'} \phi(z + \alpha'\tau + \beta' + \alpha z + \beta, \tau) \\ &= e^{2\pi i m\alpha\beta'} \left(\phi|_m(X+X')\right).\end{aligned}$$

Since $e^{2\pi i m\alpha\beta'}$ will not in general be equal to 1, the group action does not close.

The remedy [EZ85, §1(7)] is to redefine the elliptic transform of equation (6.2) by:

$$\left(\phi|_m \begin{bmatrix} \alpha \\ \beta \end{bmatrix}\right)(z,\tau) := q^{m\alpha^2} w^{2m\alpha} e^{2\pi i m\alpha\beta} \phi(z + \alpha\tau + \beta, \tau), \quad \alpha,\beta \in \mathbb{R}.$$

The effect of our redefinition on the obstruction is

$$(\phi|_m X)\,|_m X' = e^{2\pi i m(\alpha\beta' - \alpha'\beta)} \phi|_m(X + X').$$

So the last step needed, to absorb this extra factor, is to introduce a real scalar action on Jacobi forms:

$$(\phi|_m[\kappa])\,(z,\tau) := e^{2\pi i m\kappa} \phi(z,\tau), \quad \kappa \in \mathbb{R}.$$

and then make the following central extension of the group $\mathbb{R}^2$ by the group $\mathbb{R}$:

Definition B.2. *The* HEISENBERG GROUP *is*

$$H_{\mathbb{R}} := \left\{ \left(\begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \kappa \right) \mid \alpha, \beta, \kappa \in \mathbb{R} \right\},$$

with multiplication $\left(\begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \kappa \right) \cdot \left(\begin{bmatrix} \alpha' \\ \beta' \end{bmatrix}, \kappa' \right) = \left(\begin{bmatrix} \alpha + \alpha' \\ \beta + \beta' \end{bmatrix}, \kappa + \kappa' + \alpha'\beta - \alpha\beta' \right).$

Collecting the various factors together, the final definition is:

Definition B.3. *The* ELLIPTIC TRANSFORM *of Jacobi forms is*

$$\left(\phi|_m \left(\begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \kappa \right) \right) (z, \tau) := q^{m\alpha^2} w^{2m\alpha} e^{2\pi i m \alpha\beta + \kappa} \phi(z + \alpha\tau + \beta, \tau), \quad ([\alpha, \beta], \kappa) \in H_{\mathbb{R}}.$$

We now have a group action of $\mathrm{PSL}(2, \mathbb{Z}) \ltimes H(\mathbb{R})$ on Jacobi forms.

Example B.1. *The Jacobi theta functions (4.13),* $\vartheta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z, \tau)$ *are* weak *Jacobi forms, which are a generalisation of Jacobi forms to half-integer weight* $m \in \frac{1}{2}\mathbb{Z}$ *and index* $k \in \frac{1}{2}\mathbb{N}$.

The transformation properties of the classical Jacobi theta function are as follows: the elliptic transform is:

$$\vartheta(z + 1, \tau) = \vartheta(z, \tau), \quad \vartheta(z + \tau) = w^{-1} q^{-1/2} \vartheta(w, \tau).$$

The generators T *and* S *of* $\mathrm{PSL}(2, \mathbb{Z})$ *act as*

$$\vartheta(z, \tau + 1) = \vartheta(z + 1/2, \tau), \quad \vartheta(z/\tau, -1/\tau) = (-i\tau)^{1/2} e^{\pi i z^2/\tau} \vartheta(z, \tau).$$

see e.g. [Pol98, 7.2.32] or [WW15, §21.4] for a full treatment. The Jacobi theta functions are weak Jacobi forms of weight $m = 1/2$ *and index* $k = 1/2$*, with a multiplier system of order* 8*. The Heisenberg action on theta functions is a classic result, see e.g. [Car66].*

Proposition B.1. (Thetanullwert) *[EZ85, §7] Given a Jacobi form* $\phi(z, \tau)$*, the function* $\tau \mapsto \phi(0, \tau)$ *is a modular form.*

C Modular and Quasimodular Forms

We record the basics facts of the theory of quasimodular forms, in particular defining the Eisenstein series generators G_2, G_4, and G_6 of quasimodular forms.

Fix a subgroup $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$ of the modular group, with finite index. Define the so-called slash operator on functions $f : \mathbb{H} \to \mathbb{C}$ by

$$\left(f|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(\tau) := (c\tau + d)^{-k} f(\frac{a\tau + b}{c\tau + d})$$

Definition C.1. *A* Modular Form *of* weight k *is a holomorphic function* $f : \mathbb{H} \to \mathbb{C}$ *satisfying*

$$f|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} = f, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma. \tag{6.4}$$

The modular forms constitute a ring $M(\Gamma) := \bigoplus_{k \geq 0} M^{(k)}(\Gamma)$ which we have graded by weight [KZ95, §1].

Let us give an example. Provided it converges, a candidate modular form of weight k is the series $\sum_{\gamma \in \Gamma}(1|_k \gamma)$, where $1(\tau) := 1$ is the constant function. Consider an extension of the linear fractional action of Γ to $\bar{\mathbb{H}} := \mathbb{H} \cup \{\infty\}$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}.\infty = \frac{a}{c}$ when $c \neq 0$, and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}.\infty := \infty$ when $c = 0$, and let $\Gamma_\infty := \{\pm \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in \Gamma\}$ be the stabiliser of the cusp at infinity.

Definition C.2. *The* Eisenstein Series *for* Γ *of weight* k *is* $E_k := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma}(1|_k \gamma)$.

Henceforth we fix $\Gamma := \mathrm{SL}(2,\mathbb{Z})$, which is generated by $S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and in particular, Γ_∞ is generated by T. Now if we take two arbitrary matrices $\gamma, \gamma' \in \Gamma$ with the same bottom row entries c, d, we can show $\gamma' = T^n \gamma$, for some $n \in \mathbb{Z}$. Due to the unit determinant, the entries c, d must be coprime. So the sum over $\Gamma_\infty \backslash \Gamma$ reduces to the elegant expression:

Definition C.3. *The* Eisenstein Series *for* $SL(2,\mathbb{Z})$ *of weight* $k > 2$ *is*

$$E_k(\tau) := \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} \frac{1}{(c\tau + d)^k} \tag{6.5}$$

Proposition C.1. *[Bru+08, §2.1] For $k > 2$, the Eisenstein series E_k converges absolutely to a modular form of weight k.*

It is standard to normalise the Eisenstein series E_k by the Riemann zeta function $\zeta(k)$, as it admits a nice Fourier expansion [Bru+08, §2.2]:

$$\mathbb{G}_k(\tau) := \frac{(k-1)!}{(2\pi i)^k}\zeta(k)E_k(\tau) = -\frac{B_k}{2k} + \sum_{n\in\mathbb{N}} \sigma_{k-1}(n)q^n,$$

where B_k denotes the k^{th} Bernoulli number, and $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$.

In the case $k = 2$, we have no guarantee of absolute convergence of the Eisenstein series (6.5). However we can, by analogy, define the function:

$$\mathbb{G}_2(\tau) := -\frac{1}{24} + \sum_{n\in\mathbb{N}} \sigma_1(n)q^n,$$

and undoing the normalisation defines $E_2(\tau) := 1 - 24\sum_{n\in\mathbb{N}} \sigma_1(n)q^n$. We now proceed to the well-known anomalous transformation in the case $k = 2$:

Proposition C.2. *[BO00, §3]*

$$\left(\mathbb{G}_2|_2\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right)(\tau) = \mathbb{G}_2(\tau) - \frac{c}{4\pi i(c\tau + d)}.$$

So G_2 is not a modular form! Let us develop some theory which classifies objects with these anomalous modular transformations.

If we assume a modular form f on $\mathrm{SL}(2,\mathbb{Z})$ is holomorphic as $\operatorname{Im}\tau \to \infty$, we can develop a Fourier expansion, convergent for $|q| < 1$,

$$f(\tau) = \sum_{n\geq 0} a_n q^n, \tag{6.6}$$

where $q = e^{2\pi i\tau}$. It is fruitful to consider functions $F(\tau)$ which transform as (6.4), but have expansion:

$$F(\tau) = \sum_{m=0}^{M} f_m(\tau)\frac{1}{(\operatorname{Im}\tau)^m}, \tag{6.7}$$

for some $f_m(\tau)$ having a Fourier expansion of the form (6.6), $0 \leq m \leq M$, and $M \geq 0$. These functions are the Almost-holomorphic Modular Forms of weight k, which form a ring $\widehat{M}$.

Example C.1. *We note the modular transform of* $\operatorname{Im}\tau$ *is simply*

$$\frac{1}{\operatorname{Im}\frac{a\tau+b}{c\tau+d}} = \frac{|c\tau+d|^2}{\operatorname{Im}\tau} = \frac{(c\tau+d)^2}{y} - 2ic(c\tau+d).$$

Therefore the function

$$\mathbb{G}_2(\tau) + \frac{1}{8\pi\operatorname{Im}\tau}$$

is an almost-holomorphic modular form of weight 2.

Definition C.4. *A holomorphic function* $f : \mathbb{H} \to \mathbb{C}$ *is a* Quasimodular Form *of weight* k *if there exists an almost-holomorphic modular form* $F \in \widehat{M}$ *of weight* k*, whose leading term is* f*, i.e.* $f = f_0$ *in the expansion (6.7).*

The primary example of a quasimodular form is G_2, by Example C.1.

The quasimodular forms constitute a ring $\widetilde{M}$. We can tie all of our objects together by the following propositions:

Proposition C.3. *[BO00, 3.4] The leading term map*

$$\widehat{M} \to \widetilde{M} : \sum_{m=0}^{M} f_m(\tau)\frac{1}{(\operatorname{Im}\tau)^m} \mapsto f_0(\tau)$$

is an isomorphism of rings.

Proposition C.4. *[KZ95, §1] Every quasimodular form can be written as a polynomial in the Eisenstein series with coefficients that are modular forms:*

$$\widetilde{M} = M \otimes \mathbb{C}[\mathbb{G}_2].$$

This Proposition combines nicely with the well-known fact that:

Proposition C.5. *[Bru+08] The modular forms are generated by the Eisenstein series* $\mathbb{G}_4$ *and* $\mathbb{G}_6$*, i.e.* $M = \mathbb{C}[\mathbb{G}_4, \mathbb{G}_6]$.

So we conclude that the ring of quasimodular forms is given by $\widetilde{M} = \mathbb{C}[\mathbb{G}_2, \mathbb{G}_4, \mathbb{G}_6]$.

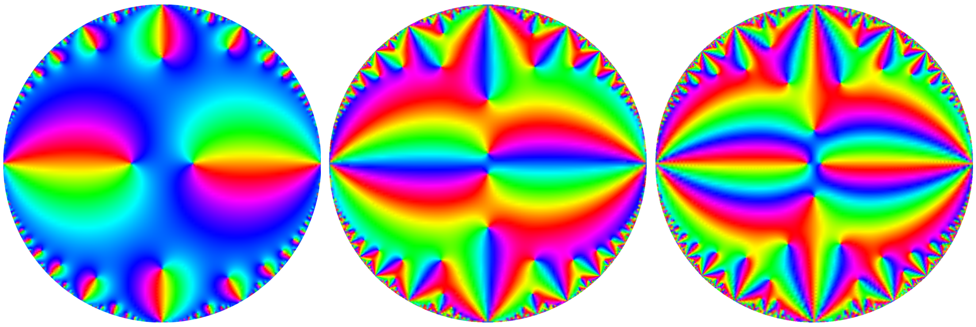

Figure 6.2: Mathematica plots of Eisenstein series G_2, G_4, G_6 in the unit circle $|q| < 1$, coloured by complex argument.

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