

Investigations on the effect of non-linear models on the reliability of partial safety factor designs

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ABSTRACT: Most modern structural design codes, such as the Eurocode, are based on the partial safety factor concept. The partial safety factors are calibrated on linear limit states. They provide simplified rules on the application of partial safety factors to non-linear limit states. Teichgräber et al. (2023) investigate those rules and their effect on the structural reliability for non-linear structural response functions with a single action based on a novel measure of non-linearity. In this paper we extend our investigations and the measure of non-linearity to the case of non-linear structural response functions with two actions applied. We perform a parameter study and a sensitivity analysis to investigate which characteristics of non-linearities have which effect on the structural reliability given different design options.

1. INTRODUCTION

Most modern structural design codes are based on the semi-probabilistic partial safety factor (PSF) concept (Ellingwood et al. 1980; Galambos et al. 1982; Ellingwood et al. 1982; Deutsches Institut für Normung 1981). The PSFs ensure sufficient structural reliability of the resulting design. They are calibrated in such a way that on average a desired target reliability is achieved for the case of linear models (Ditlevsen and Madsen 1996; Baravalle 2017; Köhler et al. 2019). In practice, PSFs are also applied to non-linear models. Various options exist to apply the PSFs to non-linear models. The question is, if and which of these design options lead to structural reliabilities sufficiently close to the target reliability. We investigate this in Teichgräber et al. (2023) for the case of a single action. Those

investigations are based on a novel measure of non-linearity. This paper extends the previous work to limit states with two actions.

2. NON-LINEARITIES IN THE PARTIAL SAFETY FACTOR CONCEPT

Following the PSF concept (e.g., CEN 2002) a design is verified by ensuring that the following inequality holds:

$$e_d \leq r_d \quad (1)$$

where e_d is the design action effect and r_d is the design resistance. e_d and r_d are the outputs of the structural response function t_S and the resistance function t_R .

We assume t_R to be a one-dimensional linear function through the origin with respect to a ma-

terial strength. Therefore, r_d can be derived as

$$r_d = \frac{m_k}{\gamma_M} \quad (2)$$

where m_k is the characteristic value of a material strength typically defined as a lower quantile value of the random variable M describing the material strength and γ_M is the corresponding PSF.

t_S can be non-linear. The non-linearity of t_S leads to various options how to apply the PSFs and, therefore, how to derive e_d . We first summarize the case of a single action previously investigated in Teichgräber et al. (2023). We then investigate the case of two actions.

2.1. ONE-DIMENSIONAL ACTION CASE

In the one-dimensional action case two basic design options regarding e_d exist (CEN 2002):

$$\text{Design option (1):} \quad e_d = t_S(\gamma_F \cdot l_k) \quad (3)$$

$$\text{Design option (2):} \quad e_d = \gamma_F \cdot t_S(l_k) \quad (4)$$

To quantify how the two basic design options effect the structural reliability a measure of the non-linearity of t_S is needed. In Teichgräber et al. (2023) we define the following two measures of non-linearity:

$$y_0 = \frac{t_S(0)}{t_S(l_k)} \quad (5)$$

$$\kappa = \frac{(t_S(l_d) - t_S(l_k)) \cdot l_k}{(t_S(l_k) - t_S(0)) \cdot (l_d - l_k)} \quad (6)$$

y_0 is a measure of the amount of initial actions (e.g., due to prestress). If $y_0 = 0$ no initial action is present. If $y_0 = 1$ the action effect of the initial action is equal to the action effect of the characteristic action. κ is a measure of the curvature at the characteristic action. If $\kappa > 1$ then t_S is classified to be convex, if $\kappa = 1$ then t_S is classified to be without curvature and if $\kappa < 1$ then t_S is classified to be concave.

This measure has proven to be well suited within multiple application examples and parameter studies (see Fußeder et al. 2021, 2023; Teichgräber et al. 2023). These investigations have shown that that not only the degree of non-linearity of the structural

response function but also the interaction of non-linear structural response functions with probabilistic properties (distribution types and parameters) and semi-probabilistic properties (choices of target reliabilities, PSFs and characteristic values) have a strong effect on the structural reliability. For this reason, it is impossible to homogenize the safety level perfectly with respect to non-linear models without leaving the scope of the PSF concept. However, there is some potential to partly homogenize the safety level. In the one-dimensional action case, this seems to be especially necessary if the structural response function has strong concave behavior or if large initial force are present. Both can lead to heavy over-design. Cases of under-design are possible if the structural response function is convex; however, the under-design is limited and appears to be an acceptable trade-off for the simplicity of the PSF format.

2.2. TWO-DIMENSIONAL ACTION CASE

In the two-dimensional action case we further extend the measure of non-linearity and perform a parameter study and a sensitivity analysis on the structural reliability resulting from the two basic design options (Eqs. 3 and 4) in case of various degrees of non-linearity of the structural response function t_S .

Remark: Some two-dimensional non-linear t_S may be linear in the superposition of the two action effects, meaning that t_S can be linearly separable in the following sense:

$$t_S(x, y) = t_{S1}(x) + t_{S2}(y) \quad (7)$$

If this is the case, the investigations of the one-dimensional action case are applicable to t_{S1} and t_{S2} separably. If this is not the case, the following investigations are relevant.

2.2.1. DESIGN CHOICES

Design option (1) directly follows from the one-dimensional case (Eq. 3) and is defined as

$$\text{Design option (1):} \quad e_d = t_S(\gamma_{F1} \cdot l_{k1}, \gamma_{F2} \cdot l_{k2}) \quad (8)$$

where l_{k1} and l_{k2} are the characteristic values of the actions L_1 and L_2 and γ_{F1} and γ_{F2} are the corresponding PSFs.

Design option (2) is not as straightforward to be transferred to the two-dimensional case. Hereby, we follow the German national Annex of the EN1990:2002 (CEN 2010) which applies the PSF of the dominating action to the action effect and applies the PSF the remaining PSF directly to the action; however, scaled by the PSF of the dominating action. Hence, design option (2) in the two-dimensional case is defined as

Design option (2): (9)

$$e_d = \max \left\{ \gamma_{F1} \cdot t_S \left(l_{k1}, \frac{\gamma_{F2}}{\gamma_{F1}} \cdot l_{k2} \right), \gamma_{F2} \cdot t_S \left(\frac{\gamma_{F1}}{\gamma_{F2}} \cdot l_{k1}, l_{k2} \right) \right\}$$

EN1990:2002 gives the instruction to use design option (1) if “the action effect increases more than the action” and option (2) if “the action effect increases less than the action”. In the two-dimensional action case this categorization is not well-defined. We interpret the EN1990:2002 and the German national Annex such that the maximum e_d following from design option (1) and (2) should be chosen.

2.2.2. MEASURE OF NON-LINEARITY

We extend the two measures of Sec. 2.1 to six measures in the 2-dimensional action case as follows:

$$y_0 = \frac{t_S(0,0)}{t_S(l_{1k}, l_{2k})} \quad (10)$$

$$\kappa_1 = \frac{(t_S(l_{1d}, 0) - t_S(l_{1k}, 0)) \cdot l_{1k}}{(t_S(l_{1k}, 0) - t_S(0, 0)) \cdot (l_{1d} - l_{1k})} \quad (11)$$

$$\kappa_2 = \frac{(t_S(0, l_{2d}) - t_S(0, l_{2k})) \cdot l_{2k}}{(t_S(0, l_{2k}) - t_S(0, 0)) \cdot (l_{2d} - l_{2k})} \quad (12)$$

$$\kappa_{12} = \frac{(t_S(l_{1d}, l_{2d}) - t_S(l_{1k}, l_{2k}))}{(t_S(l_{1k}, l_{2k}) - t_S(0, 0))} \quad (13)$$

$$\cdot \frac{\sqrt{l_{1k}^2 + l_{2k}^2}}{\sqrt{(l_{1d} - l_{1k})^2 + (l_{2d} - l_{2k})^2}}$$

$$r_1 = \frac{t_S(l_{1k}, 0) - t_S(0, 0)}{t_S(l_{1k}, l_{2k}) - t_S(0, 0)} \quad (14)$$

$$r_2 = \frac{t_S(0, l_{2k}) - t_S(0, 0)}{t_S(l_{1k}, l_{2k}) - t_S(0, 0)} \quad (15)$$

The interpretation of y_0 , κ_1 , κ_2 and κ_{12} is analogous to the one-dimensional case: y_0 measures the amount of initial action. κ_1 , κ_2 measures the curvature of t_S at $(l_{1k}, 0)$ and $(0, l_{2k})$ in the direction of the respective action. κ_{12} measures curvature of t_S at (l_{1k}, l_{2k}) in the directions of the origin and the design point (l_{1d}, l_{2d}) , respectively.

r_1 and r_2 have no analogy in the one-dimensional case. They measure the ratio of the action effect if only one characteristic action is applied relative to the case where both characteristic actions are applied. If $r_1 = r_2$ the action effect of only action 1 or action 2 applied is the same. If $r_1 > r_2$ or $r_1 < r_2$ the action effect of action 1 or action 2 applied solely is greater than the respective other action applied solely. If $r_1 + r_2 = 1$ then t_S is approximately linear regarding actions below the characteristic value. If $r_1 + r_2 > 1$ then t_S is approximately convex in the perpendicular direction of the combined action regarding actions below the characteristic action level (the superposition of both actions is below the linear case). If $r_1 + r_2 < 1$ t_S is approximately concave in the perpendicular direction of the combined action regarding actions below the characteristic action level (the superposition of both actions is above the linear case).

Similar to the measure of the one-dimensional action case, the determination of this measure is a by-product of a PSF design as it is only based on evaluations of t_S at characteristic action and design action. This makes it applicable in everyday practice. However, the interpretation of the six measures – instead of only two measures in the one-dimensional action case – and a potential adaptation of the PSF concept based on these six measures is significantly more complicated. One goal of the following parameter study is to identify which of the measures is needed in the sense that knowledge of this measure provides knowledge about the structural reliability.

3. PARAMETER STUDY

The parameter study systematically investigates the effect of non-linearity on the structural reliability.

3.1. SETUP

We investigate the case of log-normally distributed material strength M with $\text{COV}[M] = 0.1$ and a Gumbel distributed actions L_1 and L_2 with $\text{COV}[L_1] = \text{COV}[L_2] = 0.3$. M , L_1 and L_2 are assumed to be independent.

The characteristic actions l_{1k} and l_{2k} are defined as the 98% quantile values of L_1 and L_2 . The characteristic material strength is defined as the 5% quantile of M . The PSF associated with the actions are chosen as $\gamma_{L1} = 1.5$ and $\gamma_{L2} = 1.5$. The PSF on the material strength is chosen such that a target reliability of $\beta_{TRG} = 4.3$ is achieved in corresponding linear limit state function (LSF). $\beta_{TRG} = 4.3$ is in the common range of structural reliability index targets (Köhler et al. 2019; JCSS 2001). The corresponding linear LSF is defined by a t_S that is a plane spanned by two straight lines through the origin and the points $(l_{1k}, 0, t_S(l_{1k}, 0))$ and $(0, l_{2k}, t_S(0, l_{2k}))$. The six non-linear measures of this plane are $y_0 = 0$, $\kappa_1 = 1$, $\kappa_2 = 1$, $\kappa_{12} = 1$. The values of r_1 and r_2 depend on the angle of the plane. Hence, the values of r_1 and r_2 are not only varying for different non-linear t_S , but also for different linear t_S . We, therefore, choose γ_M differently for different values of r_1 and r_2 . The reason for this is that –even in the linear case– two actions applied in various ratios can lead to the same design but to different structural reliabilities. This effect occurs in practice, but is filtered out in this work through the calibration of γ_M in order to isolate the non-linear effect.

We investigate a portfolio of non-linear structural response functions t_S . The various t_S of the portfolio are generated through various combinations of the measures of non-linearity. We consider $y_0 \in \{0, 0.2, \dots, 0.8\}$ covering non-linear functions with initial actions (e.g., prestress) up to 80% of the characteristic action. Moreover, we consider $\kappa_1 \in \{0, 0.25, \dots, 2\}$, $\kappa_2 \in \{0, 0.25, \dots, 2\}$ and $\kappa_{12} \in \{0, 0.25, \dots, 2\}$ covering various combinations of convex and concave non-linear behavior in the individual and the combined direction of action. Finally, we consider $r_1 \in \{0.1, 0.3, \dots, 0.9\}$ and $r_2 \in \{0.1, 0.3, \dots, 0.9\}$ to cover various weightings of the two individual action effects. Since $r_1 < 1$ and $r_2 < 1$, only cases in which the com-

bined action effect is greater than the individual action effect at characteristic action level are considered; however, since κ_1 and κ_2 can be smaller than κ_{12} , the combined action effect might be lower than the individual action effect at higher action levels.

Given a combination of the measures of non-linearity, we define t_S as a bi-linear function along each axis of action and the combined direction; hence, t_S is defined such that it forms three straight lines from $(0, 0, t_S(0, 0))$ towards the characteristic points $(l_{1k}, 0, t_S(l_{1k}, 0))$, $(0, l_{2k}, t_S(0, l_{2k}))$ and $(l_{1k}, l_{2k}, t_S(l_{1k}, l_{2k}))$, which then change direction towards $(l_{1d}, 0, t_S(l_{1d}, 0))$, $(0, l_{2d}, t_S(0, l_{2d}))$ and $(l_{1d}, l_{2d}, t_S(l_{1d}, l_{2d}))$. In between those lines we linearly interpolate parallel to the non-dominating action. Fig. 1 shows possible variations of t_S .

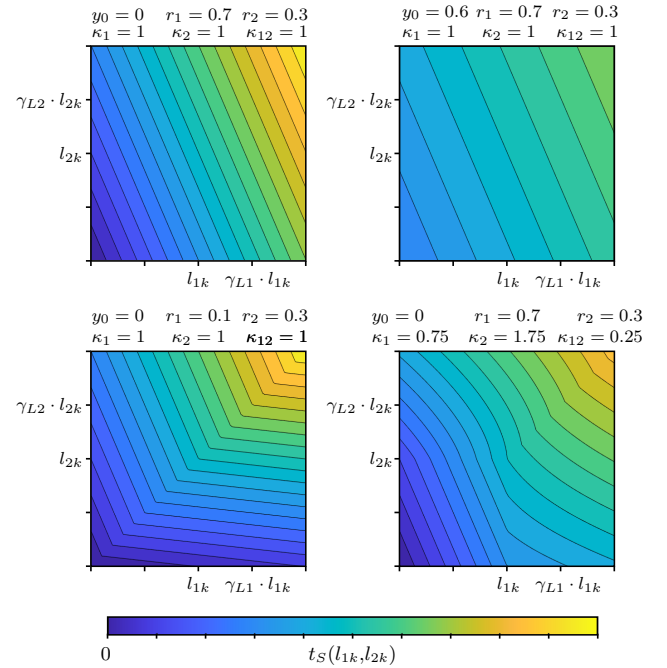


Figure 1: Contour plots of a linear t_S (top left) and 3 non-linear t_S (top right and bottom left and right).

3.2. RELIABILITY ANALYSIS

We calculate the probability of failure for each structural response functions within the considered portfolio via the following LSF

$$g(M, L_1, L_2) = \frac{\gamma_M \cdot e_d}{m_k} \cdot M - t_S(L_1, L_2) \quad (16)$$

where e_d is either calculated following Eq. 8 or Eq. 9. The probability of failure $\Pr(F)$ follows to

$$\Pr(F) = \int_{\{g(m, l_1, l_2) < 0\}} f_M(m) \cdot f_{L_1}(l_1) \cdot f_{L_2}(l_2) dm dl_1 dl_2 \quad (17)$$

where f_M , f_{L_1} and f_{L_2} are the probability density function of M , L_1 and L_2 . The corresponding reliability index β is calculated as

$$\beta = -\Phi^{-1}(\Pr(F)) \quad (18)$$

Moreover, we analyze the influence of y_0 , κ_1 , κ_2 , κ_{12} , r_1 and r_2 on the reliability index β by performing a variance-based global sensitivity analysis. We calculate the first order Sobol indices S_i and the total order Sobol indices S_i^T as

$$S_i = \frac{\text{Var}[E[\beta|X_i = x_i]]}{\text{Var}[\beta]} \quad (19)$$

$$S_i^T = \frac{\text{Var}[E[\beta|\mathbf{X}_{-i} = \mathbf{x}_{-i}]]}{\text{Var}[\beta]} \quad (20)$$

where X_i represents one measure of non-linearity and the vector $\mathbf{X}_{-i}$ represents all but one measure of non-linearity.

The first order Sobol indices measure how much percent of the variance of the structural reliability is directly caused by each variance of the measures of non-linearity. The total order Sobol additionally takes all variance with other measures of non-linearity into account.

3.3. RESULTS

Fig. 2 shows box-plots of the structural reliability indices resulting from design option (1) and design option (2). Moreover, it shows the structural reliability indices resulting from the maximum design value of option (1) and (2), which corresponds to the design following EN1990:2002. In the majority of cases EN1990:2002 follows design option (2).

The smaller the range of a box-plot (i.e., ranges between maximum and minimum value or between the 25 % and 75 % quantile), the better the resulting structural reliability can be predicted via the respective measure of non-linearity. Box-plots that

only marginally change with respect to varying values of a non-linearity measure indicate that the resulting structural reliability is not sensitive to this measure. This is the case for κ_1 and κ_2 .

To quantify the sensitivity of the structural reliability to the measures of non-linearity, we calculate the first order and the total order Sobol indices (see Table 1).

	i	S_i	S_i^T
Design option (1)	y_0	0.21	0.25
	κ_1	0.04	0.04
	κ_2	0.04	0.04
	κ_{12}	0.32	0.32
	r_1	0.17	0.27
	r_2	0.17	0.27
Design option (2)	y_0	0.22	0.24
	κ_1	0.06	0.10
	κ_2	0.06	0.10
	κ_{12}	0.20	0.26
	r_1	0.15	0.27
	r_2	0.15	0.27

Table 1: First order Sobol indices S_i and total order Sobol indices S_i^T of the six measures of non-linearity with respect to the structural reliability.

The first order Sobol indices sum up to $\sum(S_i) = 0.95$ and $\sum(S_i) = 0.84$ for design option (1) and (2) respectively. In general first Sobol indices of independent input variables sum up to a maximum of 1. The difference $1 - \sum(S_i) = 0.05$ and $1 - \sum(S_i) = 0.16$ (for design option (1) and (2) respectively) measures the effect to the structural reliability due to interactions of the measures of non-linearity. These numbers are low, indicating limited interaction. Which of the measures of non-linearity is responsible for the interaction can be understood by comparing the first order Sobol indices to the total order Sobol indices. This shows that r_1 and r_2 are the measures with the highest amount of interaction.

4. DISCUSSION

The uniform discretization of the measures of non-linearity to generate the portfolio mimics a uniform distribution of the measures of non-linearity.

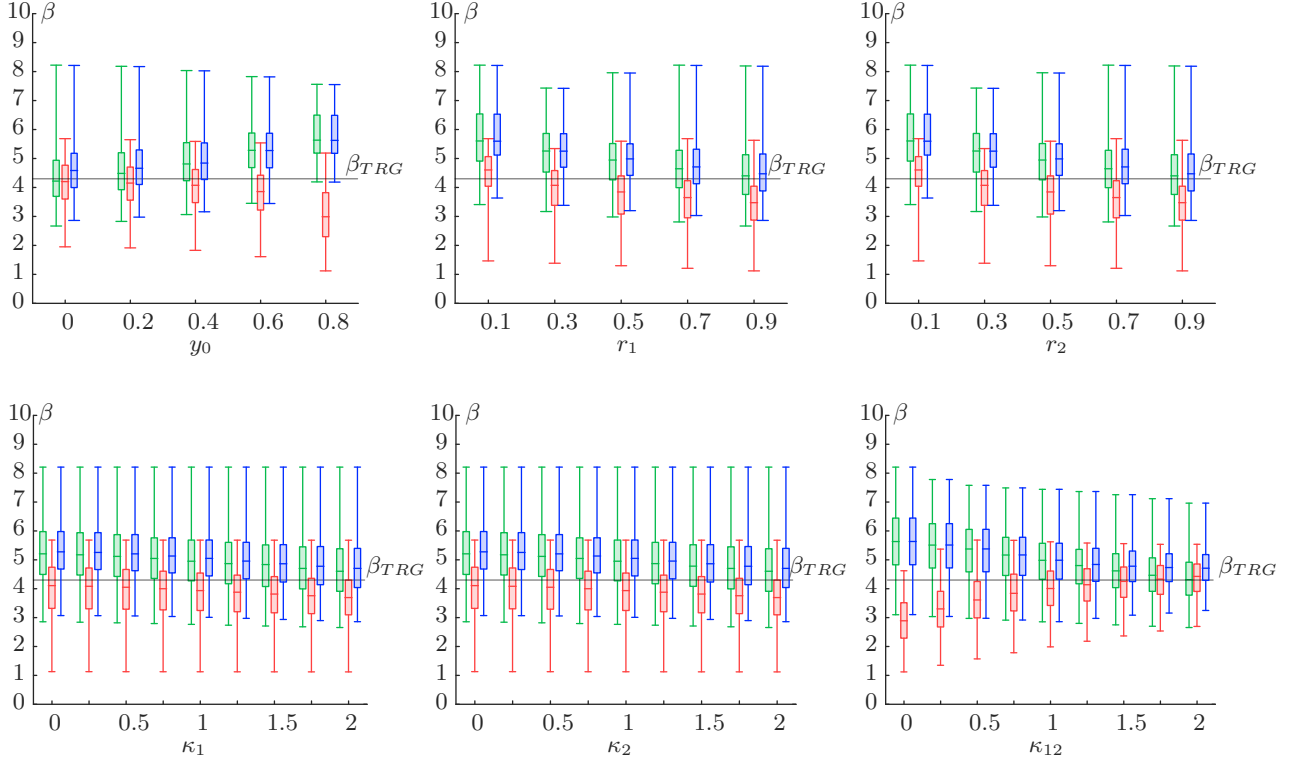


Figure 2: Box-plots of the structural reliability indices within the considered portfolio of non-linear structural response functions of design option (1) (red), design option (2) (green) and the design resulting from the maximum of design value of design option (1) or (2) (blue).

However, uniform distributions are certainly not a realistic assumption. Therefore, the considered portfolio should not be considered to be representative. Consequently, the box-plots shown in Fig. 2 are not representative. The parameter study should rather be understood as a general investigation of what can happen if non-linear structural models are used within the design.

The two actions applied are symmetric, in the sense that they have the same mean, the same variance and the same distribution type. If the two actions would be asymmetrically, we expect smaller Sobol indices for r_1 and r_2 . Moreover, we expect greater Sobol indices for κ_1 and smaller Sobol indices for κ_2 or vice versa. The reason is the following: Changes in the ratio of the two considered action effects change the resulting structural reliability. Thereby, the structural reliability is specifically sensitive to changes of this ratio, if the ratio is close to 1. The symmetrically chosen actions lead to a ratio of 1 between the actions and a ratio of 1 between the action effects considering the mean value of r_1

and r_2 . Therefore, the structural reliability is sensitive to the values of r_1 and r_2 . By contrast, if the ratio of the two action effects were far from 1, one of the two actions would be dominating. The action effect in the direction of the dominating action would be more sensitive to the respective κ -value.

Various properties have been investigated for the one-dimensional action case in Teichgräber et al. (2023). We expect that some results of those investigations can be transferred to the two-dimensional action case including the following three observations:

First, structural reliability is not very sensitive to the exact functional form of t_s for given measures of non-linearity. Hence, other functional forms (e.g. polynomial) will result in similar results.

Second, the probabilistic setup significantly interacts with non-linearities affecting the resulting structural reliabilities. Especially the ratio of the uncertainty of the resistance side to the uncertainty of the action side strongly influences the resulting structural reliability. In general, the structural reliability

bilities of both design options increases/decreases with increasing uncertainty of the resistance side or decreasing uncertainty of the action side if t_S is dominated by convex (κ values above 1) /concave (κ values below 1) non-linear behavior. This effect is strongest for design option (1) and non-linear behavior.

Third, the semi-probabilistic setup significantly interacts with non-linearities, affecting the resulting structural reliability. The approach how the PSFs are calibrated within the parameter study in order to achieve the target reliability for the linear case strongly influences the structural reliability of the non-linear cases. In this paper we chose $\gamma_{L1} = 1.5$ and $\gamma_{L2} = 1.5$ and calibrated the PSF of the resistance side. This can lead to design values of the PSF concept which are far from the FORM-design point (which would be more ideal); however, this is a realistic approach, as PSF are not ideally chosen in practice as well. If the PSF are calibrated such that the design values of the PSF concept and the FORM-design point coincide, we expect design option (1) to be close to the target reliability for all t_S that do not have a strong concave behavior (κ values much below 1). By contrast, the structural reliabilities following from design option (2) are unaffected by the way of calibrating the PSF in the one-dimensional action case. This mostly also holds for the two-dimensional action case.

5. CONCLUSION

Our investigations show that some characteristics of non-linearities of structural response functions have a greater influence on the structural reliability than other types of non-linearities. Hereby, the different types of non-linearities are represented via the different measures of non-linearity. The amount of initial action measured via y_0 and the curvature in the direction of the combined action measured via κ_{12} have the greatest influence on the structural reliability. Moreover, the ratio of the two action effects and the combined action effect due to non-linearities¹ measured via r_1 and r_2 has medium influence on the structural reliability. Finally, the

curvature in the direction of the individual actions measured via κ_1 and κ_2 has a small influence regarding the structural reliability. We expect similar for larger numbers of actions, meaning that the amount of initial action and the curvature in the direction of the combined action likely remain the most important measures. If this is confirmed, a potential future adaptation of the PSF concept to guide engineers which design option to choose could be mainly based on the measure of initial action and the measure of the curvature in the combined direction of action.

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