LMU-TPW 97-21

Virasoro amplitude from the $S^N\mathbf{R}^{24}$ orbifold sigma model

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Abstract

Four tachyon scattering amplitude is derived from the $S^N\mathbf{R}^{24}$ orbifold sigma model in the large N limit. The closed string interaction is described by a vertex which is a bosonic analog of the supersymmetric one, recently proposed by Dijkgraaf, Verlinde and Verlinde.

1 Introduction

Compactification of M(atrix) theory [1] on a circle results in the $\mathcal{N} = 8$ two-dimensional supersymmetric $SU(N)$ Yang-Mills model [2]. It was recently suggested in [3, 4, 5] that in the large N limit the Yang-Mills theory describes non-perturbative dynamics of type IIA string theory, and the string coupling constant was argued to be inverse proportional to the Yang-Mills coupling. This suggestion looks very natural since in the IR limit the gauge theory is strongly coupled and the IR fixed point may be described by the $\mathcal{N} = 8$ supersymmetric conformal field theory on the orbifold target space $S^N\mathbf{R}^8$. The Hilbert space of the orbifold model is known [6] to coincide (to be precise, to contain) in the large N limit with the Fock space of the free second-quantized type IIA string theory. Using these facts, Dijkgraaf, Verlinde and Verlinde (DVV) [5] have suggested that perturbative string dynamics in the first order in the string coupling constant can be described by the $S^N\mathbf{R}^8$ supersymmetric orbifold conformal model

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perturbed by an irrelevant operator of conformal dimension $(3/2, 3/2)$. An explicit form of this operator V was determined in [5] and it nicely fits the conventional formalism of the light-cone string theory.

The described approach seems not to be limited only to the supersymmetric case. In particular, one can suggest [7] that the M(atrix) theory formulation for closed bosonic strings is provided by the large N limit of the two-dimensional Yang-Mills theory with 24 matter fields in the adjoint representation of the $U(N)$ gauge group. In this case, the IR limit of the gauge theory results in the $S^N\mathbf{R}^{24}$ orbifold conformal model. The closed bosonic string interactions are described via perturbation of the CFT action with a bosonic analog of the DVV vertex [7].

An important problem posed by the above-described stringy interpretation of the S^N orbifold sigma models is to obtain the usual string scattering amplitudes directly from the models. This problem seems to be nontrivial due to the nonabelian nature of the S^N orbifold models.

The aim of the present paper is to derive the four tachyon scattering amplitude from the $S^N\mathbf{R}^{24}$ orbifold conformal field theory perturbed by the bosonic analog of the DVV interaction vertex.

Obviously, the first step in constructing the scattering amplitudes consists in defining the incoming and outgoing asymptotic states $|i>$ and $|f>$. The free string limit $g_s \to 0$ implies that the asymptotic states should be identified with some states in the Hilbert space of the orbifold conformal field theory and, therefore, they should be created by some conformal fields. Then, by the conventional quantum field theory, the g_s^n-order scattering amplitude A can be extracted from the S-matrix element described as a correlator of n conformal fields $V(z_i)$ with the subsequent integration over the insertion points z_i:

$$< f|S|i > \sim \int \prod_i d^2 z_i < f|V(z_1)\dots V(z_n)|i > .$$

The construction of the asymptotic states $|i>$ and $|f>$ that can be identified with incoming and outgoing tachyons, and computation of the above-mentioned correlators in the $S^N\mathbf{R}^{24}$ orbifold CFT are the main questions we are dealing with in the paper to obtain the four tachyon scattering amplitude.

The paper is organized as follows. In the second section we remind the description of the Hilbert space of the orbifold model. In the third section the twist fields that create the states of the Hilbert space are introduced and their conformal dimensions are calculated. In the fourth section the scattering amplitude is calculated and is shown to coincide with the Virasoro one. In Conclusion we discuss unsolved problems.

2 $S^N\mathbf{R}^D$ orbifold sigma model

We consider two-dimensional field theory on a cylinder described by the action

$$S = \frac{1}{2\pi}\int d\tau d\sigma (\partial_\tau X_I^i \partial_\tau X_I^i - \partial_\sigma X_I^i \partial_\sigma X_I^i), \tag{2.1}$$

where $0 \le \sigma < 2\pi$, $i = 1, 2, \dots, D$, $I = 1, 2, \dots, N$ and the fields X take values in $S^N\mathbf{R}^D \equiv (\mathbf{R}^D)^N/S_N$.

As usual in orbifold models [8, 9], the field X^i can have twisted boundary conditions

$$X^i(\sigma + 2\pi) = gX^i(\sigma), \tag{2.2}$$

where g belongs to the symmetric group S_N.
Multiplying (2.2) by some element $h \in S_N$ and taking into account that X^i and hX^i describe the same configuration, one gets that all possible boundary conditions are in one-to-one correspondence with the conjugacy classes of the symmetric group. Therefore, the Hilbert space of the orbifold model is decomposed into the direct sum of Hilbert spaces of the twisted sectors corresponding to the conjugacy classes $[g]$ of S_N [6]

$$\mathcal{H}(S^N\mathbf{R}^D) = \bigoplus_{[g]} \mathcal{H}_{[g]}.$$

It is well-known that the conjugacy classes of S_N are described by partitions $\{N_n\}$ of N

$$N = \sum_{n=1}^{s} nN_n$$

and can be represented as

$$[g] = (1)^{N_1}(2)^{N_2}\cdots(s)^{N_s}. \tag{2.3}$$

Here N_n is the multiplicity of the cyclic permutation (n) of n elements.

In any conjugacy class $[g]$ there is the only element g_c that has the canonical block-diagonal form

$$g_c = diag(\underbrace{\omega_1, ..., \omega_1}_{N_1\ times}, \underbrace{\omega_2, ..., \omega_2}_{N_2\ times}, ..., \underbrace{\omega_s, ..., \omega_s}_{N_s\ times}), \tag{2.4}$$

where ω_n is an $n \times n$ matrix that generates the cyclic permutation (n) of n elements

$$\omega_n = \sum_{i=1}^{n-1} E_{i,i+1} + E_{n1}$$

and E_{ij} are matrix unities.
It is not difficult to show that ω_n generates the $\mathbf{Z}_n$ group, since $\omega_n^n = 1$, and that only the matrices ω_n^k from $\mathbf{Z}_n$ commute with ω_n. Since the centralizer subgroup C_g of any element $g \in [g]$ is isomorphic to C_{g_c} one concludes that

$$C_g = \prod_{n=1}^{s} S_{N_n} \times \mathbf{Z}_n^{N_n},$$

where the symmetric group S_{N_n} permutes the N_n cycles (n). It is obvious that the stabilizer C_g contains $\prod_{n=1}^{s} N_n! n^{N_n}$ elements.

Due to the factorization (2.3) of $[g]$, the Hilbert space $\mathcal{H}_{[g]} \equiv \mathcal{H}_{\{N_n\}}$ of each twisted sector can be decomposed into the N_n-fold symmetric tensor products of the Hilbert spaces $\mathcal{H}_{(n)}$ which correspond to the cycles of length n

$$\mathcal{H}_{\{N_n\}} = \bigotimes_{n=1}^{s} S^{N_n}\mathcal{H}_{(n)} = \bigotimes_{n=1}^{s} \left(\underbrace{\mathcal{H}_{(n)} \otimes \cdots \otimes \mathcal{H}_{(n)}}_{N_n\ times} \right)^{S_{N_n}}.$$

The space $\mathcal{H}_{(n)}$ is $\mathbf{Z}_n$ invariant subspace of the Hilbert space of a sigma model of Dn fields X_I^i with the cyclic boundary condition

$$X_I^i(\sigma+2\pi) = X_{I+1}^i(\sigma), \quad I = 1,2,...,n. \tag{2.5}$$

The fields $X_I(\sigma)$ can be glued together into one field $X(\sigma)$ that is identified with a long string of the length n. The states of the space $\mathcal{H}_{(n)}$ are obtained by acting by the creation operators of the string on eigenvectors of the momentum operator. These eigenvectors have the standard normalization

$$\langle \mathbf{q}|\mathbf{k}\rangle = \delta^D(\mathbf{q}+\mathbf{k})$$

and can be regarded as states obtained by acting by the operator $\mathrm{e}^{\,i\mathbf{k}x}$ on the vacuum state (that is not normalizable): $|\mathbf{k}\rangle = \mathrm{e}^{\,i\mathbf{k}x}|0\rangle$, $\langle \mathbf{q}| = \langle 0|\,\mathrm{e}^{\,i\mathbf{q}x}$.

The $\mathbf{Z}_n$ invariant subspace is singled out by imposing the condition

$$(L_0 - \bar{L}_0)|\Psi\rangle = nm|\Psi\rangle,$$

where m is an integer and L_0 is the canonically normalized L_0 operator of the single string.

If $D = 24$ then the Fock space of the second-quantized closed bosonic string is recovered in the limit $N \to \infty$, $\frac{n_i}{N} \to p_i^+$ [6], where the finite ratio $\frac{n_i}{N}$ is identified with the p_i^+ momentum of a long string. The $\mathbf{Z}_n$ projection reduces in this limit to the usual level-matching condition $L_0^{(i)} - \bar{L}_0^{(i)} = 0$. The individual p_i^- light-cone momentum is defined by means of the standard mass-shell condition $p_i^+ p_i^- = L_0^{(i)}$.

3 Twist fields

Let us consider conformal field theory of DN free scalar fields described by the action (2.1). It is convinient to perform the Wick rotation $\tau \to -i\tau$ and to map the cylinder onto the sphere: $z = \mathrm{e}^{\,\tau+i\sigma}$, $\bar{z} = \mathrm{e}^{\,\tau-i\sigma}$.

The vacuum state $|0\rangle$ of the CFT is annihilated by the momentum operators and by annihilation operators, and has to be normalizable. To be able to identify this vacuum state with the vacuum state of the untwisted sector of the orbifold sigma model we choose the following normalization of $|0\rangle$

$$\langle 0|0\rangle = R^{DN}.$$

Here R should be regarded as a regularization parameter of the sigma model. We regularize the sigma model by compactifying the coordinates x_I^i on circles of radius R. Then the norm of the eigenvectors of the momentum operators in the untwisted sector is given by

$$\langle \mathbf{q}|\mathbf{k}\rangle = (2\pi)^{-DN} \int_0^{2\pi R} d^{DN}x\,\mathrm{e}^{\,i(\mathbf{q}+\mathbf{k})x} = \prod_{I=1}^{N} \delta_R^D(\mathbf{q}_I + \mathbf{k}_I),$$

where $k_I^i = \frac{m_I^i}{R}$ and $q_I^i = \frac{n_I^i}{R}$ are momenta of the states, m_I^i and n_I^i are integers since we compactified the coordinates, and $\delta_R^D(\mathbf{k}) = R^D \prod_{i=1}^D \delta_{m^i 0}$ is the regularized δ-function. In the limit $R \to \infty$ one recovers the usual normalization of the eigenvectors.

As usual, the field $X(z,\bar{z})$ can be decomposed into the left- and right-moving components

$$2X(z,\bar{z}) = X(z) + \bar{X}(\bar{z}). \tag{3.6}$$

In what follows we shall mainly concentrate our attention on the left-moving sector.
Let $\sigma_g(z,\bar{z})$ be a primary field [10] that creates a vacuum of a twisted sector at the point z, i.e. the fields $X^i(z,\bar{z})$ satisfy the following monodromy conditions

$$X^i(z\,\mathrm{e}^{2\pi i}, \bar{z}\,\mathrm{e}^{-2\pi i})\sigma_g(0,0) = gX^i(z,\bar{z})\sigma_g(0,0).$$

It is clear that the twist field $\sigma_g(z,\bar{z})$ can be represented as the tensor product of the twist fields $\sigma_g(z)$ and $\bar{\sigma}_g(\bar{z})$ that create the vacuum states of the left- and right-moving sectors respectively: $\sigma_g(z,\bar{z}) = \sigma_g(z)\otimes\bar{\sigma}_g(\bar{z})$.

It is obvious that the conformal dimension Δ_g depends only on $[g]$. To calculate Δ_g let us suppose that g has the factorization (2.3). Then Δ_g is given by the equation

$$\Delta_g = \sum_{n=1}^{s} N_n \Delta_{(n)}, \tag{3.7}$$

where $\Delta_{(n)}$ denotes the conformal dimension of the twist field $\sigma_{(n)}$ that creates the vacuum state of the space $\mathcal{H}_{(n)}$ of the sigma model of Dn fields with cyclic boundary condition (2.5). Let the twist field $\sigma_{(n)}$ be located at $z=0$ and let us denote the vacuum state [1] as $|(n)\rangle = \sigma_{(n)}(0)|0\rangle$. Since the twist field $\sigma_{(n)}$ creates one long string we normalize the vacuum state $|(n)\rangle$ as

$$\langle(n)|(n)\rangle = R^D. \tag{3.8}$$

The fields $X(z)$ have the following decomposition in the vicinity of $z=0$

$$\partial X^i_I(z) = -i\frac{1}{n}\sum_m \alpha^i_m\,\mathrm{e}^{-\frac{2\pi i}{n}Im} z^{-\frac{m}{n}-1}, \tag{3.9}$$

where α^i_m $(m \neq 0)$ are the usual creation and annihilation operators with the commutation relations

$$[\alpha^i_m, \alpha^j_n] = m\delta^{ij}\delta_{m+n,0}, \tag{3.10}$$

and α^i_0 is proportional to the momentum operator [2].
The vacuum state $|(n)\rangle$ is annihilated by the operators α^i_m for $m \geq 0$.

Since $\sigma_{(n)}$ is a primary field, the conformal dimension $\Delta_{(n)}$ can be found from the equation

$$\langle(n)|T(z)|(n)\rangle = \frac{\Delta_{(n)}}{z^2}\langle(n)|(n)\rangle,$$

where $T(z)$ is the stress-energy tensor.
By using eqs. (3.9) and (3.10), one calculates the correlator

$$\langle(n)|\partial X^i_I(z)\partial X^j_I(w)|(n)\rangle = -\delta^{ij}\frac{(zw)^{\frac{1}{n}-1}}{n^2(z^{\frac{1}{n}} - w^{\frac{1}{n}})^2}\langle(n)|(n)\rangle.$$

[1] This vacuum state is a primary state of the CFT.
[2] $\alpha^i_0 = \frac{1}{2}p^i$ in string units $\alpha' = \frac{1}{2}$.

Taking into account that the stress-energy tensor is defined as

$$T(z) = -\frac{1}{2}\lim_{w\to z}\sum_{i=1}^{D}\sum_{I=1}^{n}\left(\partial X_I^i(z)\partial X_I^i(w) + \frac{1}{(z-w)^2}\right),$$

one gets

$$\Delta_{(n)} = \frac{D}{24}(n - \frac{1}{n}). \tag{3.11}$$

The excited states of this sigma model are obtained by acting on $|(n)\rangle$ by some vertex operators. In particular the state corresponding to a scalar particle with momentum $\mathbf{k}$ is given by

$$\sigma_{(n)}[\mathbf{k}](0,0)|0\rangle =: \mathrm{e}^{ik_I^i X_I^i(0,0)} : |(n)\rangle, \tag{3.12}$$

where the summation over i and I is assumed, $k_I^i = \frac{m_I^i}{R}$ is a momentum carried by the field $X_I^i(z,\bar{z})$ and $k^i = \sum_{I=1}^{n} k_I^i$ is a total momentum of the long string.
By using the definition of the vacuum state $|(n)\rangle$, one can rewrite eq.(3.12) in the form

$$\sigma_{(n)}[\mathbf{k}](0,0)|0\rangle =: \mathrm{e}^{i\frac{k^i}{\sqrt{n}}Y^i(0,0)} : |(n)\rangle, \tag{3.13}$$

where

$$Y^i(z,\bar{z}) = \frac{1}{\sqrt{n}}\sum_{I=1}^{n} X_I^i(z,\bar{z}). \tag{3.14}$$

The field $Y(z)$ is canonically normalized, i.e. the part of the stress-energy tensor depending on Y is $-\frac{1}{2} : \partial Y(z)\partial Y(z) :$, and has the trivial monodromy around $z = 0$.

It is obvious from eq.(3.13) that the conformal dimension of the primary field

$$\sigma_{(n)}[\mathbf{k}](z,\bar{z}) =: \mathrm{e}^{i\frac{k^i}{\sqrt{n}}Y^i(z,\bar{z})} : \sigma_{(n)}(z,\bar{z})$$

is equal to

$$\Delta_{(n)}[\mathbf{k}] = \Delta_{(n)} + \frac{\mathbf{k}^2}{8n} = \frac{D}{24}(n - \frac{1}{n}) + \frac{\mathbf{k}^2}{8n},$$

where the decomposition (3.6) was taken into account.

Due to eqs. (3.7) and (3.11), the conformal dimension of σ_g is given by

$$\Delta_g = \sum_{n=1}^{s} N_n\frac{D}{24}(n - \frac{1}{n}) = \frac{D}{24}(N - \sum_{n=1}^{s}\frac{N_n}{n}).$$

One can also introduce a primary field that creates scalar particles with momenta k_α^i, $\alpha = 1, 2, ..., N_1 + N_2 + \cdots + N_s \equiv N_{str}$

$$\sigma_g[\{\mathbf{k}_\alpha\}](z,\bar{z}) =: \mathrm{e}^{i\frac{k_\alpha^i}{\sqrt{n_\alpha}}Y_\alpha^i(z,\bar{z})} : \sigma_g(z,\bar{z}),$$

where $n_1 = n_2 = \cdots = n_{N_1} = 1$, $n_{N_1+1} = n_{N_1+2} = \cdots = n_{N_1+N_2} = 2$ and so on, Y_α^i corresponds to the cycle (n_α) and is defined by eq.(3.14), and the summation over i and α is assumed.

The conformal dimension of the field $\sigma_g[\{\mathbf{k}_\alpha\}]$ is equal to

$$\Delta_g[\{\mathbf{k}_\alpha\}] = \frac{D}{24}(N - \sum_{n=1}^{s} \frac{N_n}{n}) + \sum_\alpha \frac{\mathbf{k}_\alpha^2}{8n_\alpha}. \tag{3.15}$$

It is obvious that the two-point correlation function of the twist fields σ_{g_1} and σ_{g_2} is not equal to zero if and only if $g_1 g_2 = 1$. Taking into account the normalization (3.8), we find [3]

$$\langle \sigma_{g^{-1}}(\infty)\sigma_g(0)\rangle = R^{DN_{str}}.$$

It means that the fields $\sigma_{g^{-1}}$ and σ_g have the following OPE

$$\sigma_{g^{-1}}(z,\bar{z})\sigma_g(0,0) = \frac{R^{D(N_{str}-N)}}{|z|^{4\Delta_g}} + \cdots.$$

The two-point correlation function of $\sigma_{g^{-1}}[\{\mathbf{q}_\alpha\}]$ and $\sigma_g[\{\mathbf{k}_\alpha\}]$ is respectively equal to

$$\langle \sigma_{g^{-1}}[\{\mathbf{q}_\alpha\}](\infty)\sigma_g[\{\mathbf{k}_\alpha\}](0)\rangle = \prod_\alpha \delta_R^D(\mathbf{q}_\alpha + \mathbf{k}_\alpha). \tag{3.16}$$

The twist fields σ_g do not create twisted sectors of the orbifold CFT since they are not invariant with respect to the action of the symmetric group. An invariant twist field can be defined by summing up all the twist fields from one conjugacy class

$$\sigma_{[g]}(z,\bar{z}) = \frac{1}{N!}\sum_{h\in S_N} \sigma_{h^{-1}gh}(z,\bar{z}).$$

By using this definition, one can easily calculate the two-point correlation function

$$\langle \sigma_{[g]}(\infty)\sigma_{[g]}(0)\rangle = \frac{R^{DN_{str}}}{N!}\prod_{n=1}^{s} N_n! n^{N_n},$$

where $\prod_{n=1}^{s} N_n! n^{N_n}$ is the number of elements of the centralizer subgroup C_g.

The definition of the twist field $\sigma_{[g]}[\{\mathbf{k}_\alpha\}]$ is not so straightforward. Let us consider the element $g_c \in [g]$ that has the canonical block-diagonal form (2.4). There are $N_1 + \cdots + N_s = N_{str}$ fields $Y_\alpha(z,\bar{z})$ that have the trivial monodromy in the vicinity of the location of the twist field σ_{g_c}. According to eq.(3.14) they are defined as

$$Y_\alpha(z,\bar{z}) = \frac{1}{\sqrt{n_\alpha}}\sum_{I\in(n_\alpha)} X_I(z,\bar{z}).$$

Let us now consider the fields X which have the monodromy

$$X(z\,\mathrm{e}^{2\pi i}, \bar{z}\,\mathrm{e}^{-2\pi i}) = h^{-1}g_c h X(z,\bar{z}). \tag{3.17}$$

One can see from eq.(3.17) that the fields $Y_\alpha[h]$

$$Y_\alpha[h](z,\bar{z}) = \frac{1}{\sqrt{n_\alpha}}\sum_{I\in(n_\alpha)} (hX)_I(z,\bar{z})$$

[3]it is clear that $[g^{-1}] = [g]$ and therefore $\Delta_{g^{-1}} = \Delta_g$

have the trivial monodromy.
Then an invariant twist field $\sigma_{[g]}[\{\mathbf{k}_\alpha\}]$ is defined as follows

$$\sigma_{[g]}[\{\mathbf{k}_\alpha\}](z,\bar{z}) = \frac{1}{N!}\sum_{h\in S_N} :\mathrm{e}^{i\frac{k^i_\alpha}{\sqrt{n_\alpha}}Y^i_\alpha[h](z,\bar{z})}:\sigma_{h^{-1}gh}(z,\bar{z}). \tag{3.18}$$

One can easily check that the twist field $\sigma_{[g]}[\{\mathbf{k}_\alpha\}]$ is invariant with respect to the permutation of momenta $\mathbf{k}_\alpha$ which correspond to cycles (n_α) of the same length.

The interaction vertex proposed by DVV [5] is defined with the help of the twist field σ_{IJ} that corresponds to the group element $g_{IJ} = 1 - E_{II} - E_{JJ} + E_{IJ} + E_{JI}$ transposing the fields X_I and X_J.

The twist fields σ_g have the following OPE [4]

$$\sigma_{g_1}(z,\bar{z})\sigma_{g_2}(0) = \frac{1}{|z|^{2\Delta_{g_1}+2\Delta_{g_2}-2\Delta_{g_1g_2}}}\left(C^{g_1g_2}_{g_1,g_2}\sigma_{g_1g_2}(0) + C^{g_2g_1}_{g_1,g_2}\sigma_{g_2g_1}(0)\right) + \cdots. \tag{3.19}$$

Here the two leading terms appear because there are two different ways to go around the points z and 0. It is not difficult to see that g_1g_2 and g_2g_1 belong to the same conjugacy class and, hence, $\Delta_{g_1g_2} = \Delta_{g_2g_1}$.

Therefore, the twist field σ_{IJ} acting on the state $\sigma_g(0)|0\rangle$ creates the states $\sigma_{g_{IJ}g}(0)|0\rangle$ and $\sigma_{gg_{IJ}}(0)|0\rangle$. An arbitrary element g has a decomposition $(n_1)(n_2)\cdots(n_k)$ and describes a configuration with k strings. If the indices I and J belong, say, to the cycle (n_1) in the decomposition then the element $g_{IJ}g$ has the decomposition $(n_1^{(1)})(n_1^{(2)})(n_2)\cdots(n_k)$ with $n_1^{(1)} + n_1^{(2)} = n_1$ and, hence, describes a configuration with $k+1$ strings. If the index I belongs to the cycle (n_1) and the index J belongs to (n_2) then the element $g_{IJ}g$ has the decomposition $(n_1+n_2)(n_3)\cdots(n_k)$ and describes a configuration with $k-1$ strings. Thus, the twist field σ_{IJ} generates the elementary joining and splitting of strings.

To write down the DVV interaction vertex it is useful to come back to the Minkowskian space-time. Then the interaction is described by the translationally-invariant vertex

$$V_{int} = \frac{\lambda N}{2\pi}\sum_{I<J}\int d\tau d\sigma \sigma_{IJ}(\sigma_+,\sigma_-),$$

where λ is a coupling constant proportional to the string coupling, and $\sigma_\pm$ are light-cone coordinates: $\sigma_\pm = \tau \pm \sigma$.

If $D = 24$, then the twist field $\sigma_{IJ}(\sigma_+,\sigma_-)$ is a weight $(\frac{3}{2},\frac{3}{2})$ conformal field and the coupling constant λ has dimension -1.

Performing again the Wick rotation and the conformal map onto the sphere, one gets the following expression for V_{int} (and for $D = 24$)

$$V_{int} = -\frac{\lambda N}{2\pi}\sum_{I<J}\int d^2z|z|\sigma_{IJ}(z,\bar{z}),$$

where the minus sign appears because σ_{IJ} has conformal dimension $(\frac{3}{2},\frac{3}{2})$.

[4] Let us stress that there are other primary fields on the r.h.s. of the OPE, in particular, the field $\sigma_{g_1g_2}(0)\otimes\bar{\sigma}_{g_2g_1}(0)$. However, these fields will be nonessential in our consideration.

Thus, the action of the interacting $S^N\mathbf{R}^{24}$ orbifold sigma model is given by the sum

$$S_{int} = S_0 + V_{int}$$

In the next section we calculate the S-matrix element corresponding to the scattering of four tachyons and show that the scattering amplitude coincides with the Virasoro one.

4 Scattering amplitude

The S-matrix element at the second order in the coupling constant λ is given by the standard formula of quantum field theory

$$\langle f|S|i\rangle = -\frac{1}{2}\left(\frac{\lambda N}{2\pi}\right)^2 \langle f|\int d^2z_1 d^2z_2 |z_1||z_2| T\left(\mathcal{L}_{int}(z_1,\bar{z}_1)\mathcal{L}_{int}(z_2,\bar{z}_2)\right)|i\rangle, \tag{4.20}$$

where the symbol T means the time-ordering: $|z_1| > |z_2|$, and

$$\mathcal{L}_{int}(z,\bar{z}) = \sum_{I<J}\sigma_{IJ}(z,\bar{z}).$$

The initial state $|i\rangle$ describes two tachyons with momenta $\mathbf{k}_1$ and $\mathbf{k}_2$ and is created by the twist field $\sigma_{[g_0]}[\mathbf{k}_1,\mathbf{k}_2]$

$$|i\rangle = C_0\sigma_{[g_0]}[\mathbf{k}_1,\mathbf{k}_2](0,0)|0\rangle.$$

The element g_0 is taken in the canonical block-diagonal form

$$g_0 = (n_0)(N-n_0),$$

where $n_0 < N - n_0$.

The final state $\langle f|$ describes two tachyons with momenta $\mathbf{k}_3$ and $\mathbf{k}_4$ and is given by the formula (see [10])

$$\langle f| = C_\infty \lim_{z_\infty\to\infty}|z_\infty|^{4\Delta_\infty}\langle 0|\sigma_{[g_\infty]}[\mathbf{k}_3,\mathbf{k}_4](z_\infty,\bar{z}_\infty).$$

The element g_∞ has the canonical decomposition

$$g_\infty = (n_\infty)(N-n_\infty), \quad n_\infty < N - n_\infty.$$

The constants C_0 and C_∞ are chosen to be equal to

$$C_0 = \sqrt{\frac{N!}{n_0(N-n_0)}}, \quad C_\infty = \sqrt{\frac{N!}{n_\infty(N-n_\infty)}}$$

that guarantees the standard normalization of the initial and final states.

After the conformal transformation $z \to \frac{z}{z_1}$ eq.(4.20) acquires the form

$$\begin{aligned}\langle f|S|i\rangle &= -\frac{1}{2}\left(\frac{\lambda N}{2\pi}\right)^2\int d^2z_1 d^2z_2|z_1||z_2||z_1|^{2\Delta_\infty-2\Delta_0-6} \\ &\times \langle f|T\left(\mathcal{L}_{int}(1,1)\mathcal{L}_{int}(\frac{z_2}{z_1},\frac{\bar{z}_2}{\bar{z}_1})\right)|i\rangle,\end{aligned} \tag{4.21}$$

where Δ_0 and Δ_∞ are conformal dimensions of the twist fields $\sigma_{[g_0]}[\mathbf{k}_1,\mathbf{k}_2]$ and $\sigma_{[g_\infty]}[\mathbf{k}_3,\mathbf{k}_4]$

$$
\begin{aligned}
\Delta_0 &= N - \frac{1}{n_0} - \frac{1}{N-n_0} + \frac{\mathbf{k}_1^2}{8n_0} + \frac{\mathbf{k}_2^2}{8(N-n_0)}, \\
\Delta_\infty &= N - \frac{1}{n_\infty} - \frac{1}{N-n_\infty} + \frac{\mathbf{k}_3^2}{8n_\infty} + \frac{\mathbf{k}_4^2}{8(N-n_\infty)}.
\end{aligned}
\tag{4.22}
$$

Let us introduce the light-cone momenta of the tachyons [5] taking into account the mass-shell condition for the tachyonic states

$$
\begin{aligned}
k_1^+ &= \frac{n_0}{N}, \quad k_1^- k_1^+ - \mathbf{k}_1^2 \equiv -k_1^2 = -8, \\
k_2^+ &= \frac{N-n_0}{N}, \quad k_2^- k_2^+ - \mathbf{k}_2^2 \equiv -k_2^2 = -8, \\
k_3^+ &= -\frac{n_\infty}{N}, \quad k_3^- k_3^+ - \mathbf{k}_3^2 \equiv -k_3^2 = -8, \\
k_4^+ &= -\frac{N-n_\infty}{N}, \quad k_4^- k_4^+ - \mathbf{k}_4^2 \equiv -k_4^2 = -8.
\end{aligned}
$$

By using the light-cone momenta and the mass-shell condition, one can rewrite (4.22) in the form

$$
\begin{aligned}
\Delta_0 &= N + \frac{k_1^- + k_2^-}{8N}, \\
\Delta_\infty &= N - \frac{k_3^- + k_4^-}{8N}.
\end{aligned}
$$

Performing the change of variables $\frac{z_2}{z_1} = u$, one obtains

$$
\begin{aligned}
\langle f|S|i\rangle &= -\frac{1}{2}\left(\frac{\lambda N}{2\pi}\right)^2 \int d^2 z_1 |z_1|^{2\Delta_\infty - 2\Delta_0 - 2} \\
&\times \int d^2 u |u| \langle f| T\left(\mathcal{L}_{int}(1,1)\mathcal{L}_{int}(u,\bar{u})\right)|i\rangle.
\end{aligned}
$$

The integral over z_1 is obviously divergent. To understand the meaning of this divergency one should remember that we made the Wick rotation. Coming back to the σ,τ-coordinates on the cylinder, we get for the integral over z_1

$$
\int d^2 z_1 |z_1|^{2\Delta_\infty - 2\Delta_0 - 2} \to i \int d\tau d\sigma \, \mathrm{e}^{\,2i\tau(\Delta_\infty - \Delta_0)}.
$$

Integration over σ and τ gives us the conservation law for the light-cone momenta k_i^-

$$
\int d\tau d\sigma \, \mathrm{e}^{\,2i\tau(\Delta_\infty - \Delta_0)} = 4N(2\pi)^2 \delta(k_1^- + k_2^- + k_3^- + k_4^-).
$$

Thus, the S-matrix element is equal to

$$
\langle f|S|i\rangle = -i2\lambda^2 N^3 \delta(k_1^- + k_2^- + k_3^- + k_4^-) \int d^2 u |u| \langle f| T\left(\mathcal{L}_{int}(1,1)\mathcal{L}_{int}(u,\bar{u})\right)|i\rangle. \tag{4.23}
$$

So, to find the S-matrix element one has to calculate the correlator

$$\begin{aligned} F(u,\bar{u}) &= \langle f|T\left(\mathcal{L}_{int}(1,1)\mathcal{L}_{int}(u,\bar{u})\right)|i\rangle \\ &= C_0 C_\infty \sum_{I<J;K<L} \langle \sigma_{[g_\infty]}[\mathbf{k}_3,\mathbf{k}_4](\infty) T\left(\sigma_{IJ}(1,1)\sigma_{KL}(u,\bar{u})\right)\sigma_{[g_0]}[\mathbf{k}_1,\mathbf{k}_2](0,0)\rangle. \end{aligned} \tag{4.24}$$

In what follows we assume for definiteness that $n_0 < n_\infty$ and $|u| < 1$.

By using the definition (3.18) of $\sigma_{[g]}[\{\mathbf{k}_\alpha\}]$, and taking into account that the interaction vertex is S_N-invariant, and that any correlator of twist fields is invariant with respect to the global action of the symmetric group

$$\langle \sigma_{g_1}\sigma_{g_2}\cdots\sigma_{g_n}\rangle = \langle \sigma_{h^{-1}g_1h}\sigma_{h^{-1}g_2h}\cdots\sigma_{h^{-1}g_nh}\rangle, \tag{4.25}$$

we rewrite the correlator in the form

$$F(u,\bar{u}) = \frac{C_0C_\infty}{N!}\sum_{h_\infty\in S_N}\sum_{I<J;K<L}\langle \sigma_{h_\infty^{-1}g_\infty h_\infty}[\mathbf{k}_3,\mathbf{k}_4](\infty)\sigma_{IJ}(1,1)\sigma_{KL}(u,\bar{u})\sigma_{g_0}[\mathbf{k}_1,\mathbf{k}_2](0,0)\rangle.$$

Let us note that the correlator

$$\langle \sigma_{g_1}(\infty)\sigma_{g_2}(1,1)\sigma_{g_3}(u,\bar{u})\sigma_{g_4}(0,0)\rangle \tag{4.26}$$

does not vanish only if

$$g_1g_2g_3g_4 = 1 \quad or \quad g_1g_4g_3g_2 = 1. \tag{4.27}$$

It can be seen as follows. Due to the OPE (3.19) of σ_g, in the limit $u \to 0$ the correlator (4.26) reduces to the sum of three-point correlators $\langle \sigma_{g_1}\sigma_{g_2}\sigma_{g_3g_4}\rangle$ and $\langle \sigma_{g_1}\sigma_{g_2}\sigma_{g_4g_3}\rangle$. This sum does not vanish if one of the following equations is fulfilled:

$$g_1g_2g_3g_4 = 1, \quad g_1g_3g_4g_2 = 1, \quad g_1g_2g_4g_3 = 1, \quad g_1g_4g_3g_2 = 1. \tag{4.28}$$

From the other side in the limit $u \to 1$ one gets the sum of the correlators $\langle \sigma_{g_1}\sigma_{g_2g_3}\sigma_{g_4}\rangle$ and $\langle \sigma_{g_1}\sigma_{g_3g_2}\sigma_{g_4}\rangle$. This sum does not vanish if

$$g_1g_2g_3g_4 = 1, \quad g_1g_4g_2g_3 = 1, \quad g_1g_3g_2g_4 = 1, \quad g_1g_4g_3g_2 = 1. \tag{4.29}$$

Comparing eqs.(4.28) and (4.29), one obtains eq.(4.27).

However, the contribution of the terms satisfying the equation $h_\infty^{-1}g_\infty h_\infty g_0 g_{KL} g_{IJ} = 1$, coincides with the contribution of the terms which satisfy $h_\infty^{-1}g_\infty h_\infty g_{IJ}g_{KL}g_0 = 1$. To prove the statement let us note that the invariance of the action (2.1) with respect to the world-sheet parity symmetry $z \to \bar{z}$ (or $\sigma \to -\sigma$ in the Minkowskian space-time) leads to the following equality

$$\langle \sigma_{g_1}\sigma_{g_2}\cdots\sigma_{g_n}\rangle = \langle \sigma_{g_1^{-1}}\sigma_{g_2^{-1}}\cdots\sigma_{g_n^{-1}}\rangle, \tag{4.30}$$

since twist fields σ_g transform into $\sigma_{g^{-1}}$. Now taking into account eqs.(4.25) and (4.30), and that the elements g and g^{-1} belong to the same conjugacy class, one obtains the desired equality

$$\begin{aligned} &\sum_{I<J;K<L}\langle \sigma_{g_{IJ}g_{KL}g_0^{-1}}\sigma_{IJ}\sigma_{KL}\sigma_{g_0}\rangle = \sum_{I<J;K<L}\langle \sigma_{g_0g_{KL}g_{IJ}}\sigma_{IJ}\sigma_{KL}\sigma_{g_0^{-1}}\rangle \\ &= \sum_{I<J;K<L}\langle \sigma_{g_0^{-1}g_{KL}g_{IJ}}\sigma_{IJ}\sigma_{KL}\sigma_{g_0}\rangle. \end{aligned}$$

Thus, the function $F(u,\bar{u})$ is given by a sum of correlators of twist fields which can be schematically represented as

$$\mathcal{S} = \sum_{h_\infty \in S_N} \sum_{I<J;K<L} \langle \sigma_{h_\infty^{-1} g_\infty h_\infty} \sigma_{IJ} \sigma_{KL} \sigma_{g_0} \rangle,$$

where the elements h_∞, g_{IJ}, g_{KL} solve the equation $h_\infty^{-1} g_\infty h_\infty g_{IJ} g_{KL} g_0 = 1$.
We can fix the values of the indices K and L by using the action of the stabilizer of g_0 and the invariance (4.25) of the correlators

$$\begin{aligned}\mathcal{S} = \sum_{h_\infty \in S_N} \sum_{I<J} \quad & \Big(n_0(N-n_0) \langle \sigma_{h_\infty^{-1} g_\infty h_\infty} \sigma_{IJ} \sigma_{n_0 N} \sigma_{g_0} \rangle \\ & + (N-n_0) \langle \sigma_{h_\infty^{-1} g_\infty h_\infty} \sigma_{IJ} \sigma_{n_\infty N} \sigma_{g_0} \rangle \\ & + (N-n_0) \langle \sigma_{h_\infty^{-1} g_\infty h_\infty} \sigma_{IJ} \sigma_{n_0+n_\infty, N} \sigma_{g_0} \rangle \Big) . \end{aligned} \tag{4.31}$$

The first term in (4.31) corresponds to the joining of two incoming strings and the factor $n_0(N-n_0)$ appears since in this case the index K can take n_0 values, $K=1,...,n_0$, and the index L takes $N-n_0$ values, $L=n_0+1,...,N$. To fix $K=n_0$ and $L=N$ we have to use all elements of C_{g_0}. The second and the third terms correspond to the splitting of the string of length $N-n_0$ into two strings of lengths $n_\infty - n_0$ and $N-n_\infty$, and $N-n_0-n_\infty$ and n_∞ respectively. In these cases to fix the values of K and L one should use $N-n_0$ elements of the subgroup $\mathbf{Z}_{N-n_0}$ of C_{g_0} that does not act on the cycle (n_0).

Eq.(4.31) can be further rewritten in the form

$$\begin{aligned} \mathcal{S} \;\; = \;\; & n_0(N-n_0) n_\infty (N-n_\infty) \left(\sum_{I=1}^{n_\infty} \langle \sigma_{g_\infty(I)} \sigma_{I,I+N-n_\infty} \sigma_{n_0 N} \sigma_{g_0} \rangle \right. \\ + \;\; & \sum_{I=1}^{N-n_\infty} \langle \sigma_{g_\infty(I)} \sigma_{I,I+n_\infty} \sigma_{n_0 N} \sigma_{g_0} \rangle + \sum_{J=n_0+1}^{n_\infty} \langle \sigma_{g_\infty(J)} \sigma_{n_0 J} \sigma_{n_\infty N} \sigma_{g_0} \rangle \\ + \;\; & \left. \sum_{J=n_0+n_\infty+1}^{N} \langle \sigma_{g_\infty(J)} \sigma_{n_0 J} \sigma_{n_0+n_\infty, N} \sigma_{g_0} \rangle \right), \end{aligned} \tag{4.32}$$

where the elements g_∞ have to be found from the equation $g_\infty g_{IJ} g_{KL} g_0 = 1$.

Some comments are in order. The factor $n_\infty(N-n_\infty)$ is the volume of the stabilizer $\mathbf{Z}_{n_\infty} \times \mathbf{Z}_{N-n_\infty}$ of g_∞. The first two terms correspond to the splitting of the long string of length N into strings of lengths n_∞ and $N-n_\infty$. It can be achieved only if $J-I=N-n_\infty$ or $J-I=n_\infty$. In the third and fourth terms we fixed the value of I equal to n_0 by using the action of the subgroup $\mathbf{Z}_{n_0}$ of C_{g_0}. It gave the additional factor n_0. The third (fourth) term describes the joining of the strings of lengths n_0 and $n_\infty - n_0$ $(N-n_0-n_\infty)$ into one string of length n_∞ $(N-n_\infty)$. The total number of different correlators is, therefore, equal to $2(N-n_0)$. The diagramms corresponding to these four terms are depicted in Fig.1.

So, we need to compute the correlators (and the same correlators with the interchange $u \leftrightarrow 1$)

$$G(u,\bar{u}) = \langle \sigma_{g_\infty}[\mathbf{k}_3,\mathbf{k}_4](\infty) \sigma_{IJ}(1,1) \sigma_{KL}(u,\bar{u}) \sigma_{g_0}[\mathbf{k}_1,\mathbf{k}_2](0,0) \rangle, \tag{4.33}$$

where all possible elements $g_\infty, g_{IJ}, g_{KL}, g_0$ are listed in eq.(4.32).

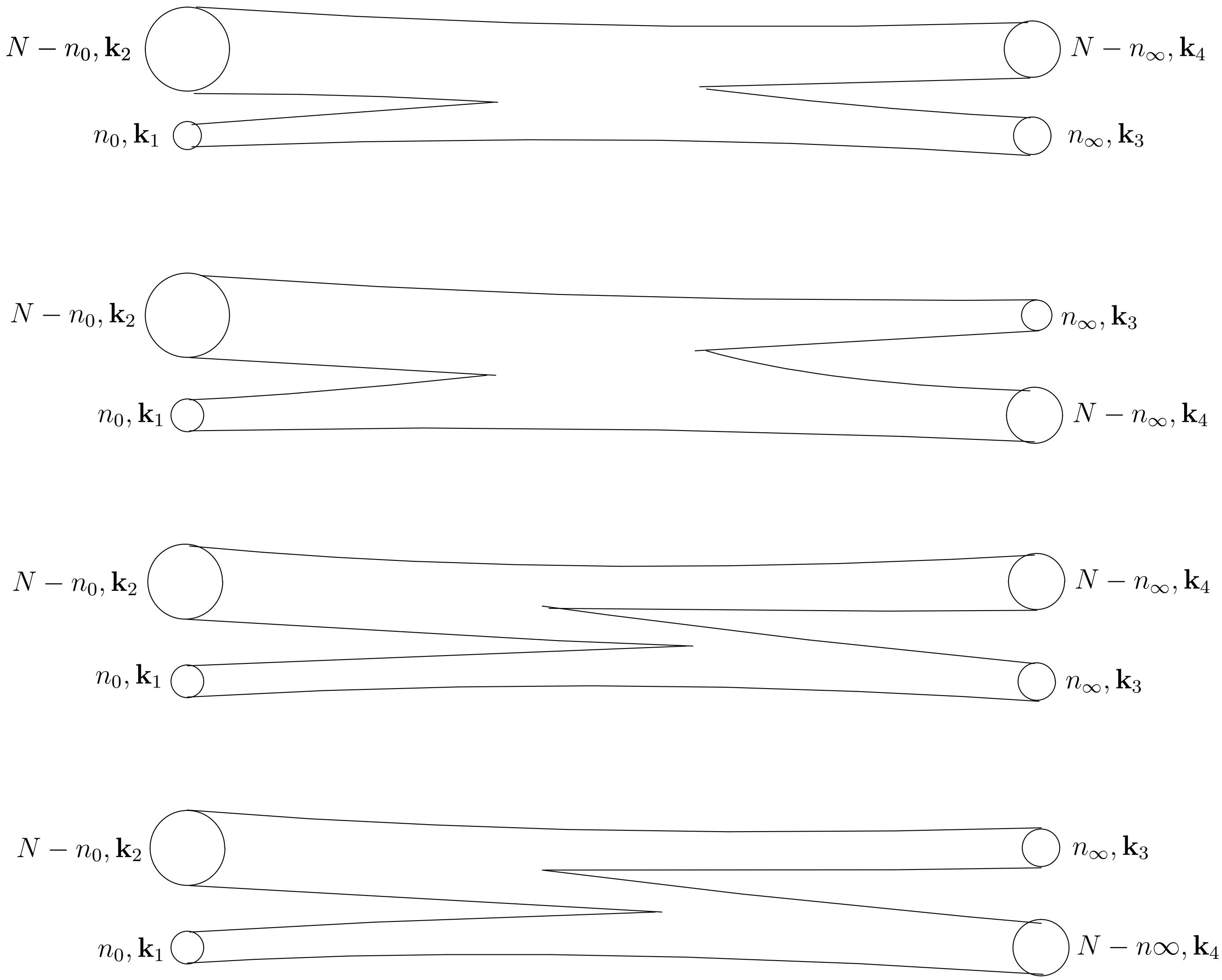

Figure 1: The diagramm representation of different correlators in eq.(4.32)

To calculate the correlator (4.33) we employ the stress-energy tensor method [11]. The idea of the method is as follows. Let us suppose that we know the following ratio

$$f(z,u) = \frac{\langle T(z)\phi_\infty(\infty)\phi_1(1)\phi_2(u)\phi_0(0)\rangle}{\langle \phi_\infty(\infty)\phi_1(1)\phi_2(u)\phi_0(0)\rangle}, \tag{4.34}$$

where $T(z)$ is the stress-energy tensor and ϕ are primary fields. Taking into account that the OPE of $T(z)$ with any primary field has the form

$$T(z)\phi(0) = \frac{\Delta}{z^2}\phi(0) + \frac{1}{z}\partial\phi(0) + \cdots,$$

one gets a differential equation on the correlator $G(u,\bar{u}) = \langle \phi_\infty(\infty)\phi_1(1)\phi_2(u)\phi_0(0)\rangle$

$$\partial_u \log G(u,\bar{u}) = H(u,\bar{u}),$$

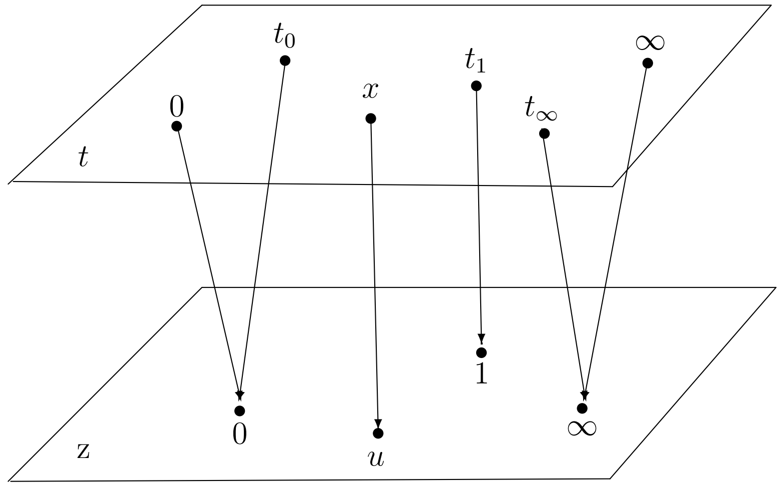

Figure 2: The N-fold covering of the z-sphere by the t-sphere.

where $H(u,\bar{u})$ is the second term in the decomposition of the function $f(z,u)$ in the vicinity of u

$$f(z,u) = \frac{\Delta_2}{(z-u)^2} + \frac{1}{z-u} H(u,\bar{u}) + \cdots .$$

In the same way one gets the second equation on $G(u,\bar{u})$ by using the stress-energy tensor $\bar{T}(\bar{z})$

$$\partial_{\bar{u}} \log G(u,\bar{u}) = \bar{H}(u,\bar{u}).$$

A solution of these two equations determines the correlator $G(u,\bar{u})$ up to a constant.

To calculate the ratio (4.34) we firstly find the following Green functions [5]

$$\begin{aligned} G^{ij}_{MS}(z,w) &= \frac{\langle \partial X^i_M(z) \partial X^j_S(w) \sigma_{g_\infty}[\mathbf{k}_3,\mathbf{k}_4](\infty)\sigma_{IJ}(1,1)\sigma_{KL}(u,\bar{u})\sigma_{g_0}[\mathbf{k}_1,\mathbf{k}_2](0,0)\rangle}{\langle \sigma_{g_\infty}[\mathbf{k}_3,\mathbf{k}_4](\infty)\sigma_{IJ}(1,1)\sigma_{KL}(u,\bar{u})\sigma_{g_0}[\mathbf{k}_1,\mathbf{k}_2](0,0)\rangle} \\ &\equiv \langle\langle \partial X^i_M(z) \partial X^j_S(w) \rangle\rangle . \end{aligned}$$

These Green functions have non-trivial monodromies around points $\infty, 1, u$ and 0, and, in fact, are different branches of one multi-valued function. However, this function is single-valued on the sphere that is obtained by gluing the fields X^i_I at $z=0$ and $z=\infty$. Thus to construct $G^{ij}_{MS}(z,w)$ we introduce the following map from this sphere onto the original one:

$$z = \frac{t^{n_0}(t-t_0)^{N-n_0}}{(t-t_\infty)^{N-n_\infty}} \frac{(t_1-t_\infty)^{N-n_\infty}}{t_1^{n_0}(t_1-t_0)^{N-n_0}} \equiv u(t). \tag{4.35}$$

Here the points $t=0$ and $t=t_0$ are mapped to the point $z=0$; $t=\infty$, $t=t_\infty \to z=\infty$, $t=t_1 \to z=1$ and $t=x \to z=u$ (see Fig.2). The map (4.35) may be viewed as the N-fold covering of the z-plain by the t-sphere on which the Green function is single-valued. The more detailed discussion of eq.(4.35) is presented in the Appendix.

[5] We consider the correlators for general values of D keeping in mind the application to the superstring case.

Due to the projective transformations, the positions of the points t_0, t_∞, t_1 depend on x and it is convinient to choose this dependence as follows

$$
\begin{aligned}
t_0 &= x-1, \\
t_\infty &= x - \frac{(N-n_\infty)x}{(N-n_0)x+n_0}, \\
t_1 &= \frac{N-n_0-n_\infty}{n_\infty} + \frac{n_0 x}{n_\infty} - \frac{N(N-n_\infty)x}{n_\infty((N-n_0)x+n_0)}.
\end{aligned}
$$

This choice leads to the following expression for the rational function $u(x)$

$$
\begin{aligned}
u = u(x) &= (n_0-n_\infty)^{n_0-n_\infty} \frac{n_\infty^{n_\infty}}{n_0^{n_0}} \left(\frac{N-n_0}{N-n_\infty}\right)^{N-n_\infty} \left(\frac{x+\frac{n_0}{N-n_0}}{x-1}\right)^{N} \\
&\times \left(\frac{x-\frac{N-n_0-n_\infty}{N-n_0}}{x}\right)^{N-n_0-n_\infty} \left(x - \frac{n_0}{n_0-n_\infty}\right)^{n_0-n_\infty}. \qquad (4.36)
\end{aligned}
$$

Since $n_0 < n_\infty$, the map $u(x)$ can be treated as the $2(N-n_0)$-fold covering of the u-sphere by the x-sphere, that means that an equation $u(x) = u$ has $2(N-n_0)$ different solutions. It is worthwhile to note that this number coincides with the number of nontrivial correlators in eq.(4.32) and, therefore different roots of eq.(4.36) correspond to different correlators (4.32). We see that the t-sphere can be represented as the union of $2(N-n_0)$ domains, and each domain V_{IJKL} contains the points x corresponding to the correlator (4.33). If we take on the u-plain the appropriate system of cuts, then every root of eq.(4.36) realizes a one-to-one conformal mapping of the cut u-plain onto the corresponding domain V_{IJKL}.

Let us now choose some root of eq.(4.36). One can always cut the z-sphere and numerate the roots $t_R(z)$ of eq.(4.35) in such a way that they have the same monodromies as the fields X do. Then the Green functions are obviously not equal to zero only if $\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4 = 0$ and are given by

$$
\begin{aligned}
G^{ij}_{MS}(z,w) &= -\delta^{ij}\frac{t'_M(z)t'_S(w)}{(t_M(z)-t_S(w))^2} - \frac{k_1^i k_1^j t'_M(z)t'_S(w)}{4t_M(z)t_S(w)} \\
&- \frac{k_1^i k_2^j t'_M(z)t'_S(w)}{4t_M(z)(t_S(w)-t_0)} - \frac{k_2^i k_1^j t'_M(z)t'_S(w)}{4(t_M(z)-t_0)t_S(w)} \\
&- \frac{k_2^i k_2^j t'_M(z)t'_S(w)}{4(t_M(z)-t_0)(t_S(w)-t_0)} - \frac{k_1^i k_4^j t'_M(z)t'_S(w)}{4t_M(z)(t_S(w)-t_\infty)} \\
&- \frac{k_2^i k_4^j t'_M(z)t'_S(w)}{4(t_M(z)-t_0)(t_S(w)-t_\infty)} - \frac{k_4^i k_1^j t'_M(z)t'_S(w)}{4(t_M(z)-t_\infty)t_S(w)} \\
&- \frac{k_4^i k_2^j t'_M(z)t'_S(w)}{4(t_M(z)-t_\infty)(t_S(w)-t_0)} - \frac{k_4^i k_4^j t'_M(z)t'_S(w)}{4(t_M(z)-t_\infty)(t_S(w)-t_\infty)}. \qquad (4.37)
\end{aligned}
$$

One can easily check that these functions have the singularity $-\frac{\delta^{ij}\delta_{MS}}{(z-w)^2}$ in the vicinity $z-w=0$ and proper monodromies at points $z = \infty, 1, u, 0$.

Recall that the stress-energy tensor is defined as

$$
T(z) = -\frac{1}{2}\lim_{w\to z}\sum_{i=1}^{D}\sum_{I=1}^{N}\left(\partial X_I^i(z)\partial X_I^i(w) + \frac{1}{(z-w)^2}\right).
$$

By using this definition and eq.(4.37), one gets [6]

$$
\begin{aligned}
\langle\langle T(z)\rangle\rangle &= \sum_M \left(\frac{D}{12}\left(\left(\frac{t''_M(z)}{t'_M(z)}\right)' - \frac{1}{2}\left(\frac{t''_M(z)}{t'_M(z)}\right)^2\right) + \frac{\mathbf{k}_1^2 (t'_M(z))^2}{8(t_M(z))^2}\right.\\
&+ \frac{\mathbf{k}_1\mathbf{k}_2 (t'_M(z))^2}{4t_M(z)(t_M(z)-t_0)} + \frac{\mathbf{k}_2^2 (t'_M(z))^2}{8(t_M(z)-t_0)^2}\\
&+ \frac{\mathbf{k}_1\mathbf{k}_4 (t'_M(z))^2}{4t_M(z)(t_M(z)-t_\infty)} + \frac{\mathbf{k}_2\mathbf{k}_4 (t'_M(z))^2}{4(t_M(z)-t_0)(t_M(z)-t_\infty)}\\
&+ \left.\frac{\mathbf{k}_4^2 (t'_M(z))^2}{8(t_M(z)-t_\infty)^2}\right).
\end{aligned}
$$

The term

$$
\left(\frac{t''}{t'}\right)' - \frac{1}{2}\left(\frac{t''}{t'}\right)^2 = \frac{t'''}{t'} - \frac{3}{2}\left(\frac{t''}{t'}\right)^2
$$

is the Schwartz derivative as one could expect from the very beginning. To get the differential equation on the correlator (4.33) one should expand $\langle\langle T(z)\rangle\rangle$ in the vicinity of $z = u$. This expansion is given by

$$
\begin{aligned}
\langle\langle T(z)\rangle\rangle &= \frac{D}{16(z-u)^2} - \frac{D}{16(z-u)u}\left(1 + \frac{2a_2}{a_0^2} - \frac{3a_1^2}{2a_0^3}\right)\\
&+ \frac{1}{4a_0(z-u)u}\left(\frac{\mathbf{k}_1^2}{x^2} + \frac{\mathbf{k}_2^2}{(x-t_0)^2} + \frac{2\mathbf{k}_1\mathbf{k}_2}{x(x-t_0)} + \frac{\mathbf{k}_4^2}{(x-t_\infty)^2}\right.\\
&+ \left.\frac{2\mathbf{k}_1\mathbf{k}_4}{x(x-t_\infty)} + \frac{2\mathbf{k}_2\mathbf{k}_4}{(x-t_0)(x-t_\infty)}\right) + \cdots .
\end{aligned}
\tag{4.38}
$$

Here the coefficients a_k are defined as follows

$$
a_k = \frac{(-1)^{k-1}}{k+2}\left(\frac{n_0}{x^{k+2}} + \frac{N-n_0}{(x-t_0)^{k+2}} - \frac{N-n_\infty}{(x-t_\infty)^{k+2}}\right).
$$

The first term shows that the conformal dimension of the twist field σ_{KL} is equal to $\frac{D}{16}$, as it should be, and the other terms lead to the following differential equation on $G(u,\bar{u})$

$$
\begin{aligned}
u\partial_u \log G(u,\bar{u}) &= -\frac{D}{16}\left(1 + \frac{2a_2}{a_0^2} - \frac{3a_1^2}{2a_0^3}\right)\\
&+ \frac{1}{4a_0}\left(\frac{\mathbf{k}_1^2}{x^2} + \frac{\mathbf{k}_2^2}{(x-t_0)^2} + \frac{2\mathbf{k}_1\mathbf{k}_2}{x(x-t_0)} + \frac{\mathbf{k}_4^2}{(x-t_\infty)^2}\right.\\
&+ \left.\frac{2\mathbf{k}_1\mathbf{k}_4}{x(x-t_\infty)} + \frac{2\mathbf{k}_2\mathbf{k}_4}{(x-t_0)(x-t_\infty)}\right).
\end{aligned}
\tag{4.39}
$$

[6] If all $\mathbf{k}_i = 0$, the expectation value of $T(z)$ in the presence of twist fields can be equivalently found by mapping with $t_M(z)$ the stress-energy tensor on the t-sphere onto the z-sphere with the subsequent summation over M (see e.g.[11])

It is useful to make the change of variables $u \to u(x)$. Then, performing simple but tedious calculations which are outlined in Appendix, one obtains the following differential equation on $G(u,\bar{u})$

$$\begin{aligned}
\partial_x \log G(u(x),\bar{u}(\bar{x})) &= -\frac{D}{16}\frac{d}{dx}\log u + \frac{d_0}{x} + \frac{d_1}{x-1} + \frac{d_2}{x+\frac{n_0}{N-n_0}} \\
&+ \frac{d_3}{x-\frac{N-n_0-n_\infty}{N-n_0}} + \frac{d_4}{x-\frac{n_0}{n_0-n_\infty}} - \frac{D}{24}(\frac{1}{x-\alpha_1}+\frac{1}{x-\alpha_2}). \qquad (4.40)
\end{aligned}$$

Here

$$\alpha_i = \frac{n_0}{n_0-n_\infty} + (-1)^i\sqrt{\frac{n_0 n_\infty (N-n_\infty)}{(n_0-n_\infty)^2(N-n_0)}}$$

are roots of the equation $x^2 a_0 = 0$ and the coefficients d_i are given by the following formulas

$$\begin{aligned}
d_0 &= \frac{D}{24} + \frac{n_0}{8(N-n_\infty)}(\mathbf{k}_4^2 - \frac{D}{3}) + \frac{N-n_\infty}{8n_0}(\mathbf{k}_1^2 - \frac{D}{3}) + \frac{1}{4}\mathbf{k}_1\mathbf{k}_4, \\
d_1 &= \frac{D}{24} - \frac{n_\infty}{8(N-n_\infty)}(\mathbf{k}_4^2 - \frac{D}{3}) - \frac{N-n_\infty}{8n_\infty}(\mathbf{k}_3^2 - \frac{D}{3}) + \frac{1}{4}\mathbf{k}_3\mathbf{k}_4, \\
d_2 &= \frac{D}{24} - \frac{n_0}{8(N-n_0)}(\mathbf{k}_2^2 - \frac{D}{3}) - \frac{N-n_0}{8n_0}(\mathbf{k}_1^2 - \frac{D}{3}) + \frac{1}{4}\mathbf{k}_1\mathbf{k}_2, \\
d_3 &= \frac{D}{24} + \frac{n_\infty}{8(N-n_0)}(\mathbf{k}_2^2 - \frac{D}{3}) + \frac{N-n_0}{8n_\infty}(\mathbf{k}_3^2 - \frac{D}{3}) + \frac{1}{4}\mathbf{k}_2\mathbf{k}_3, \\
d_4 &= \frac{D}{24} + \frac{n_0}{8n_\infty}(\mathbf{k}_3^2 - \frac{D}{3}) + \frac{n_\infty}{8n_0}(\mathbf{k}_1^2 - \frac{D}{3}) + \frac{1}{4}\mathbf{k}_1\mathbf{k}_3. \qquad (4.41)
\end{aligned}$$

Taking into account that the second equation on $G(u,\bar{u})$ has the same form with the obvious substitution $u \to \bar{u}, x \to \bar{x}$, one gets the solution of eq.(4.40)

$$\begin{aligned}
G(u,\bar{u}) &= C(g_0,g_\infty)\delta_R^D(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)|u|^{-\frac{D}{8}}|x-\alpha_1|^{-\frac{D}{12}}|x-\alpha_2|^{-\frac{D}{12}} \\
&\times |x|^{2d_0}|x-1|^{2d_1}|x+\frac{n_0}{N-n_0}|^{2d_2}|x-\frac{N-n_0-n_\infty}{N-n_0}|^{2d_3}|x-\frac{n_0}{n_0-n_\infty}|^{2d_4}. \qquad (4.42)
\end{aligned}$$

Here $x = x(u)$ is the root of equation $u = u(x)$ that corresponds to given values of the indices I, J, K, L, and $C(g_0, g_\infty)$ is a normalization constant which does not depend on $u, \bar{u}$.

To determine this constant let us consider an auxiliary correlator

$$G_0(u,\bar{u}) = \langle \sigma_{g_0^{-1}}[-\mathbf{k}_1,-\mathbf{k}_2](\infty)\sigma_{IJ}(1,1)\sigma_{IJ}(u,\bar{u})\sigma_{g_0}[\mathbf{k}_1,\mathbf{k}_2](0,0)\rangle, \qquad (4.43)$$

where $I = 1,\ldots,n_0,\ J = n_0+1,\ldots,N$.

Let us note that by using the action of C_{g_0} one can fix $I = n_0,\ J = N$. This correlator corresponds to the case $n_\infty = n_0$ and the rational function $u(x)$ is equal to

$$u(x) = \left(1 + \frac{2n_0 - N}{N-n_0}\frac{1}{x}\right)^{N-2n_0}\left(\frac{1+\frac{n_0}{N-n_0}\frac{1}{x}}{1-\frac{1}{x}}\right)^N. \qquad (4.44)$$

The root of eq.(4.44) that corresponds to the correlator (4.43) behaves as

$$\frac{1}{x} = \frac{1}{4n_0}(u-1) + o(u-1), \qquad \text{when} \quad u \to 1. \tag{4.45}$$

The following expression for the correlator $G_0(u,\bar{u})$ can be derived from eq.(4.42) in the limit $n_\infty \to n_0$

$$\begin{aligned} G_0(u,\bar{u}) &= C(g_0)R^D|u|^{-\frac{D}{8}}|x - \frac{N-2n_0}{2(N-n_0)}|^{-\frac{D}{12}} \\ &\times \; |x|^{2d_0}|x-1|^{2d_1}|x + \frac{n_0}{N-n_0}|^{2d_2}|x - \frac{N-2n_0}{N-n_0}|^{2d_3}, \end{aligned} \tag{4.46}$$

where the coefficients d_i are given by eq.(4.41) with the obvious substitution $n_\infty \to n_0$, $\mathbf{k}_3 = -\mathbf{k}_1$ and $\mathbf{k}_4 = -\mathbf{k}_2$.

Taking into account the OPE

$$\sigma_{IJ}(1,1)\sigma_{IJ}(u,\bar{u}) = \frac{R^{-D}}{|u-1|^{\frac{D}{4}}} + \cdots,$$

and the normalization (3.16) of two-point correlation functions, one gets

$$G_0(u,\bar{u}) \to \frac{R^D}{|u-1|^{\frac{D}{4}}}. \tag{4.47}$$

From the other side by using eqs.(4.45) and (4.46), one derives in the limit $u \to 1$

$$G_0(u,\bar{u}) \to C(g_0)R^D\left(\frac{1}{4n_0}|u-1|\right)^{-2(d_0+d_1+d_2+d_3-\frac{D}{24})} = \frac{R^D}{|u-1|^{\frac{D}{4}}}C(g_0)(4n_0)^{\frac{D}{4}}. \tag{4.48}$$

Comparing eqs.(4.47) and (4.48), one finds the normalization constant

$$C(g_0) = (4n_0)^{-\frac{D}{4}}. \tag{4.49}$$

Let us now consider the limit $u \to 0$. Taking into account the OPE

$$\begin{aligned} \sigma_{n_0N}(u,\bar{u})\sigma_{g_0}[\mathbf{k}_1,\mathbf{k}_2](0) &= \frac{C_{n_0N,g_0}^{g_{n_0N}g_0}(\mathbf{k}_1,\mathbf{k}_2)}{|u|^{\frac{D}{8}+2\Delta_{g_0}[\mathbf{k}_1,\mathbf{k}_2]-2\Delta_{g_{n_0N}g_0}[\mathbf{k}_1+\mathbf{k}_2]}}\sigma_{g_{n_0N}g_0}[\mathbf{k}_1+\mathbf{k}_2](0) \\ &+ \frac{C_{n_0N,g_0}^{g_0g_{n_0N}}(\mathbf{k}_1,\mathbf{k}_2)}{|u|^{\frac{D}{8}+2\Delta_{g_0}[\mathbf{k}_1,\mathbf{k}_2]-2\Delta_{g_{n_0N}g_0}[\mathbf{k}_1+\mathbf{k}_2]}}\sigma_{g_0g_{n_0N}}[\mathbf{k}_1+\mathbf{k}_2](0) + \cdots, \end{aligned} \tag{4.50}$$

one obtains

$$\begin{aligned} &G_0(u,\bar{u}) \to \\ &\frac{C_{n_0N,g_0}^{g_{n_0N}g_0}(\mathbf{k}_1,\mathbf{k}_2)}{|u|^{\frac{D}{8}+2\Delta_{g_0}[\mathbf{k}_1,\mathbf{k}_2]-2\Delta_{g_{n_0N}g_0}[\mathbf{k}_1+\mathbf{k}_2]}}\langle\sigma_{g_0^{-1}}[-\mathbf{k}_1,-\mathbf{k}_2](\infty)\sigma_{n_0N}(1)\sigma_{g_{n_0N}g_0}[\mathbf{k}_1+\mathbf{k}_2](0)\rangle \\ &+\frac{C_{n_0N,g_0}^{g_0g_{n_0N}}(\mathbf{k}_1,\mathbf{k}_2)}{|u|^{\frac{D}{8}+2\Delta_{g_0}[\mathbf{k}_1,\mathbf{k}_2]-2\Delta_{g_{n_0N}g_0}[\mathbf{k}_1+\mathbf{k}_2]}}\langle\sigma_{g_0^{-1}}[-\mathbf{k}_1,-\mathbf{k}_2](\infty)\sigma_{n_0N}(1)\sigma_{g_0g_{n_0N}}[\mathbf{k}_1+\mathbf{k}_2](0)\rangle. \end{aligned} \tag{4.51}$$

It is not difficult to show that the correlators $\langle\sigma_{g_0^{-1}}\sigma_{n_0N}\sigma_{g_{n_0N}g_0}\rangle$ and $\langle\sigma_{g_0^{-1}}\sigma_{n_0N}\sigma_{g_0g_{n_0N}}\rangle$ are equal to $C^{g_0g_{n_0N}}_{n_0N,g_0}$ and $C^{g_{n_0N}g_0}_{n_0N,g_0}$ respectively, and, moreover, are equal to each other. It follows from eqs.(4.25) and (4.30), and from the obvious symmetry property of the structure constant $C^{g_{n_0N}g_0}_{n_0N,g_0}(-\mathbf{k}_1,-\mathbf{k}_2)=C^{g_{n_0N}g_0}_{n_0N,g_0}(\mathbf{k}_1,\mathbf{k}_2)$:

$$\langle\sigma_{g_0^{-1}}\sigma_{n_0N}\sigma_{g_{n_0N}g_0}\rangle=\langle\sigma_{g_{n_0N}g_0}\sigma_{n_0N}\sigma_{g_0^{-1}}\rangle=\left\{\begin{array}{l}\langle\sigma_{g_{n_0N}g_0^{-1}}\sigma_{n_0N}\sigma_{g_0}\rangle=R^DC^{g_0g_{n_0N}}_{n_0N,g_0}\\ \langle\sigma_{g_0^{-1}g_{n_0N}}\sigma_{n_0N}\sigma_{g_0}\rangle=R^DC^{g_{n_0N}g_0}_{n_0N,g_0}\end{array}\right. . \tag{4.52}$$

Thus, the correlator $G_0(u,\bar u)$ in the limit $u\to0$ is expressed through the structure constant

$$C(n_0,\mathbf{k}_1;N-n_0,\mathbf{k}_2)\equiv C^{g_{n_0N}g_0}_{n_0N,g_0}(\mathbf{k}_1,\mathbf{k}_2)$$

as follows

$$G_0(u,\bar u)\to\frac{2R^DC^2(n_0,\mathbf{k}_1;N-n_0,\mathbf{k}_2)}{|u|^{\frac{D}{8}+2\Delta_{g_0}[\mathbf{k}_1,\mathbf{k}_2]-2\Delta_{g_{n_0N}g_0}[\mathbf{k}_1+\mathbf{k}_2]}}. \tag{4.53}$$

On the other hand, taking into account that in the limit $u\to0$ the root $x(u)$ behaves as

$$|x+\frac{n_0}{N-n_0}|\to Nn_0^{\frac{N-2n_0}{N}}(N-n_0)^{\frac{2n_0-2N}{N}}|u|^{\frac{1}{N}},$$

one gets from eq.(4.46)

$$G_0(u,\bar u)\to\frac{2^{\frac{D}{12}}R^DC(g_0)}{|u|^{\frac{D}{8}-\frac{2}{N}d_2}}N^{-\frac{D}{12}+4d_2}(N-n_0)^{-\frac{D}{12}+4\frac{n_0-N}{N}d_2}n_0^{\frac{D}{6}-4\frac{n_0}{N}d_2}. \tag{4.54}$$

Comparing eqs.(4.53) and (4.54), one obtains the following expression for the structure constant

$$C(n_0,\mathbf{k}_1;N-n_0,\mathbf{k}_2)=2^{-\frac{5D+12}{24}}N^{-\frac{D}{24}+2d_2}(N-n_0)^{-\frac{D}{24}-2\frac{N-n_0}{N}d_2}n_0^{-\frac{D}{24}-2\frac{n_0}{N}d_2}, \tag{4.55}$$

where

$$d_2\equiv d_2(n_0,\mathbf{k}_1;N-n_0,\mathbf{k}_2)=\frac{D}{24}-\frac{n_0}{8(N-n_0)}(\mathbf{k}_2^2-\frac{D}{3})-\frac{N-n_0}{8n_0}(\mathbf{k}_1^2-\frac{D}{3})+\frac{1}{4}\mathbf{k}_1\mathbf{k}_2. \tag{4.56}$$

It is now not difficult to express any three-point correlator of the form $\langle\sigma_{g^{-1}g_{IJ}}\sigma_{IJ}\sigma_g\rangle$ through the structure constant $C(n,\mathbf{k};m,\mathbf{q})$. First of all let us note that any twist field $\sigma_g[\{\mathbf{k}_\alpha\}]$ has the following decomposition into the tensor product of the twist fields $\sigma_{(n)}[\mathbf{k}]$

$$\sigma_g[\{\mathbf{k}_\alpha\}]=\bigotimes_{\alpha=1}^{N_{str}}\sigma_{(n_\alpha)}[\mathbf{k}_\alpha], \tag{4.57}$$

where the element g has the decomposition $(n_1)(n_2)\cdots(n_{N_{str}})$. [7]
Then, due to eq.(4.52), the structure constant $C(n,\mathbf{k};m,\mathbf{q})$ with arbitrary n and m is equal to

$$C(n,\mathbf{k};m,\mathbf{q})=R^{-D}\langle\sigma_{(-n-m)}[-\mathbf{k}-\mathbf{q}](\infty)\sigma_{IJ}(1)\sigma_{(n)}[\mathbf{k}]\otimes\sigma_{(m)}[\mathbf{q}](0)\rangle, \tag{4.58}$$

[7] we will use the notation $(-n_1)(-n_2)\cdots(-n_{N_{str}})$ for the decomposition of the element g^{-1}.

where $I \in (n)$ and $J \in (m)$.
By using eqs.(4.57) and (4.58), one can easily get the following expression for the three-point correlator

$$\begin{aligned}
&\langle \sigma_{g^{-1}g_{IJ}}[\{\mathbf{q}_\alpha\}](\infty)\sigma_{IJ}(1)\sigma_g[\{\mathbf{k}_\alpha\}](0)\rangle = \langle \sigma_g[\{\mathbf{k}_\alpha\}](\infty)\sigma_{IJ}(1)\sigma_{g^{-1}g_{IJ}}[\{\mathbf{q}_\alpha\}](0)\rangle \\
&= \langle \sigma_{(-n_1-n_2)}[\mathbf{q}] \bigotimes_{\alpha=3}^{N_{str}} \sigma_{(-n_\alpha)}[\mathbf{q}_\alpha](\infty)\sigma_{IJ}(1)\sigma_{(n_1)}[\mathbf{k}_1] \otimes \sigma_{(n_2)}[\mathbf{k}_2] \bigotimes_{\alpha=3}^{N_{str}} \sigma_{(n_\alpha)}[\mathbf{k}_\alpha](0)\rangle \\
&= \prod_{\alpha=3}^{N_{str}} \delta_R^D(\mathbf{q}_\alpha + k_\alpha)\langle \sigma_{(-n_1-n_2)}[\mathbf{q}](\infty)\sigma_{IJ}(1)\sigma_{(n_1)}[\mathbf{k}_1] \otimes \sigma_{(n_2)}[\mathbf{k}_2](0)\rangle \\
&= C(n_1, \mathbf{k}_1; n_2, \mathbf{k}_2)\delta_R^D(\mathbf{q} + k_1 + \mathbf{k}_2) \prod_{\alpha=3}^{N_{str}} \delta_R^D(\mathbf{q}_\alpha + k_\alpha), \qquad (4.59)
\end{aligned}$$

where $I \in (n_1)$ and $J \in (n_2)$.
It is now clear that the structure constant $C_{IJ,g}^{g_{IJ}g}$ in the OPE of σ_{IJ} and σ_g is just equal to $C(n_1, \mathbf{k}_1; n_2, \mathbf{k}_2)$, and that the structure constant $C_{IJ,g^{-1}g_{IJ}}^{g^{-1}}$ (which coincides with $C_{IJ,g^{-1}g_{IJ}}^{g_{IJ}g^{-1}g_{IJ}}$ due to eq.(4.52)) in the OPE

$$\begin{aligned}
&\sigma_{IJ}(u, \bar{u})\sigma_{g^{-1}g_{IJ}}[\{\mathbf{q}_\alpha\}](0) = \sum_{\mathbf{q}_1, \mathbf{q}_2} \frac{\delta_{\mathbf{q}_1+\mathbf{q}_2-\mathbf{q},0}}{|u|^{\frac{D}{8}+2\Delta_{g^{-1}g_{IJ}}[\{\mathbf{q}_\alpha\}]-2\Delta_g[\{\mathbf{q}_\alpha\}]}} \\
&\times \left(C_{IJ,g^{-1}g_{IJ}}^{g^{-1}}(\mathbf{q}_1, \mathbf{q}_2)\sigma_{g^{-1}}[\{\mathbf{q}_\alpha\}](0) + C_{IJ,g^{-1}g_{IJ}}^{g^{-1}}(\mathbf{q}_1, \mathbf{q}_2)\sigma_{g_{IJ}g^{-1}g_{IJ}}[\{\mathbf{q}_\alpha\}](0) \right) + \cdots \quad (4.60)
\end{aligned}$$

is equal to

$$C_{IJ,g^{-1}g_{IJ}}^{g^{-1}}(\mathbf{q}_1, \mathbf{q}_2) = R^{-D}C(n_1, \mathbf{q}_1; n_2, \mathbf{q}_2).$$

In particular, the structure constants $C_{n_\infty N,g_0}^{g_{n_\infty N}g_0}$ and $C_{n_0+n_\infty,N;g_0}^{g_{n_0+n_\infty,N}g_0}$, which will be used to find the normalization constant $C(g_0, g_\infty)$ are given by

$$\begin{aligned}
&C_{n_\infty N,g_0}^{g_{n_\infty N}g_0}(\mathbf{k}_1, \mathbf{k}_2) = R^{-D}C(n_\infty - n_0, \mathbf{k}_1; N - n_\infty, \mathbf{k}_2), \qquad (4.61) \\
&C_{n_0+n_\infty,N;g_0}^{g_{n_0+n_\infty,N}g_0}(\mathbf{k}_1, \mathbf{k}_2) = R^{-D}C(N - n_\infty - n_0, \mathbf{k}_1; n_\infty, \mathbf{k}_2).
\end{aligned}$$

Now we are ready to determine the normalization constant $C(g_0, g_\infty)$ by factorizing $G(u, \bar{u})$ in the limit $u \to 0$ on tree-point functions. According to eq.(4.36), $u \to 0$ in the following three cases

$$I) \;\; x \to -\frac{n_0}{N - n_0}; \qquad II) \;\; x \to \infty; \qquad III) \;\; x \to \frac{N - n_0 - n_\infty}{N - n_0}$$

and, conversely, any root $x_M = x_M(u)$ of eq.(4.36) tends to one of these values when $u \to 0$. Evidently, these three possible asymptotics correspond to three different choices of the indices K and L in eq.(4.32).

Let us begin with the case $K = n_0$, $L = N$. By using the OPE (4.50) and the normalization (3.16) of two-point correlators, one gets in the limit $u \to 0$

$$G(u, \bar{u}) \to \delta_R^D(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4)\frac{C(n_0, \mathbf{k}_1; N - n_0, \mathbf{k}_2)C(n_\infty, \mathbf{k}_3; N - n_\infty, \mathbf{k}_4)}{|u|^{\frac{D}{8}+2\Delta_{g_0}[\mathbf{k}_1, \mathbf{k}_2]-2\Delta_{g_{n_0N}g_0}[\mathbf{k}_1+\mathbf{k}_2]}}. \qquad (4.62)$$

In this case the root $x(u)$ has the following behaviour in the vicinity of $u=0$

$$|x+\frac{n_0}{N-n_0}| \to N n_0^{\frac{N-n_0}{N}} n_\infty^{-\frac{n_\infty}{N}} (N-n_0)^{\frac{n_0-2N}{N}} (N-n_\infty)^{\frac{n_\infty}{N}} |u|^{\frac{1}{N}}. \tag{4.63}$$

By using eqs.(4.42) and (4.63), one can easily find

$$\begin{aligned} G(u,\bar{u}) &\to \delta_R^D(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\frac{C(g_0,g_\infty)}{|u|^{\frac{D}{8}-\frac{2}{N}d_2}}\left(\frac{n_0 N(N-n_\infty)}{(N-n_0)^2(n_0-n_\infty)}\right)^{-\frac{D}{12}} \times \\ &\times \left(\frac{n_0}{N-n_0}\right)^{2d_0}\left(\frac{N}{N-n_0}\right)^{2d_1}\left(\frac{n_0(N-n_\infty)}{(N-n_0)(n_0-n_\infty)}\right)^{2d_4} \\ &\times \left(\frac{N-n_\infty}{N-n_0}\right)^{2d_3}\left(N n_0^{\frac{N-n_0}{N}} n_\infty^{-\frac{n_\infty}{N}}(N-n_0)^{\frac{n_0-2N}{N}}(N-n_\infty)^{\frac{n_\infty}{N}}\right)^{2d_2}. \end{aligned} \tag{4.64}$$

It is not difficult to verify that

$$\Delta_{g_0}[\mathbf{k}_1,\mathbf{k}_2] - \Delta_{g_{n_0 N} g_0}[\mathbf{k}_1+\mathbf{k}_2] = -\frac{1}{N}d_2,$$

as one should expect.

Comparing eqs.(4.62) and (4.64), one can obtain the normalization constant. However, for general values of D the corresponding expression looks rather complicated and will not be written down. For $D=24$ one should take into account that the coefficients d_i are given by the following simple formulas

$$\begin{aligned} d_0 &= 1+\frac{1}{4}k_1k_4, \quad d_1 = 1+\frac{1}{4}k_3k_4, \quad d_2 = 1+\frac{1}{4}k_1k_2, \\ d_3 &= 1+\frac{1}{4}k_2k_3, \quad d_4 = 1+\frac{1}{4}k_1k_3, \end{aligned} \tag{4.65}$$

where $k_ik_j \equiv \mathbf{k}_i\mathbf{k}_j - \frac{1}{2}k_i^+k_j^- - \frac{1}{2}k_i^-k_j^+$.

By using eq.(4.65), one easily obtains

$$C(g_0,g_\infty) = \frac{2^{-11}}{n_0(N-n_0)n_\infty(N-n_\infty)(n_\infty-n_0)^2}\left(\frac{N-n_0}{n_\infty-n_0}\right)^{2+\frac{1}{2}(k_1+k_3)k_4}. \tag{4.66}$$

Thus, we have found the normalization constant for N correlators which are presented in the first and second terms of eq.(4.32).

Let us now determine the normalization constant for $n_\infty - n_0$ correlators of the form $\langle\sigma_{g_\infty(J)}\sigma_{n_0 J}\sigma_{n_\infty N}\sigma_{g_0}\rangle$. By using the OPE (4.60) and eq.(4.61), one finds in the limit $u\to 0$

$$G(u,\bar{u}) \to \delta_R^D(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\frac{C(n_\infty-n_0,\mathbf{k}_2+\mathbf{k}_4;N-n_\infty,-\mathbf{k}_4)C(n_\infty-n_0,\mathbf{k}_2+\mathbf{k}_4;n_0,\mathbf{k}_1)}{|u|^{\frac{D}{8}+2\Delta_{g_0}[\mathbf{k}_1,\mathbf{k}_2]-2\Delta_{g_{n_\infty N}g_0}[\mathbf{k}_1,\mathbf{k}_2+\mathbf{k}_4,\mathbf{k}_4]}}. \tag{4.67}$$

Taking into account the behaviour of the root $x(u)$ in the vicinity of $u=0$

$$|x| \to \left((n_\infty-n_0)^{n_0-n_\infty}\frac{n_\infty^{n_\infty}}{n_0^{n_0}}\left(\frac{N-n_0}{N-n_\infty}\right)^{N-n_\infty}\right)^{\frac{1}{n_\infty-n_0}}|u|^{\frac{1}{n_0-n_\infty}},$$

one obtains from eq.(4.42)

$$G(u,\bar{u}) \to \frac{\delta^D_R(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)C(g_0,g_\infty)}{|u|^{\frac{D}{8}+\frac{2(d_0+d_1+d_2+d_3+d_4-\frac{D}{12})}{n_\infty-n_0}}}$$
$$\times\left((n_\infty-n_0)^{n_0-n_\infty}\frac{n_\infty^{n_\infty}}{n_0^{n_0}}\left(\frac{N-n_0}{N-n_\infty}\right)^{N-n_\infty}\right)^{\frac{2(d_0+d_1+d_2+d_3+d_4-\frac{D}{12})}{n_\infty-n_0}}. \quad (4.68)$$

A simple calculation shows that

$$\Delta_{g_0}[\mathbf{k}_1,\mathbf{k}_2]-\Delta_{g_{n_\infty N}g_0}[\mathbf{k}_1,\mathbf{k}_2+\mathbf{k}_4,\mathbf{k}_4]=\frac{d_0+d_1+d_2+d_3+d_4-\frac{D}{12}}{n_\infty-n_0}.$$

The normalization constant $C(g_0,g_\infty)$ can be now found from eqs.(4.67) and (4.68). For $D=24$ the computation drastically simplifies if one notes that

$$\begin{aligned}
&d_0+d_1+d_2+d_3+d_4-2=-1-\frac{1}{4}k_1k_3,\\
&d_2(n_\infty-n_0,\mathbf{k}_2+\mathbf{k}_4;N-n_\infty,-\mathbf{k}_4)=-\frac{N-n_0}{n_\infty-n_0}(1+\frac{1}{4}k_1k_3),\\
&d_2(n_\infty-n_0,\mathbf{k}_2+\mathbf{k}_4;n_0,\mathbf{k}_1)=-\frac{n_\infty}{n_\infty-n_0}(1+\frac{1}{4}k_1k_3).
\end{aligned}$$

Then, one can easily show that $C(g_0,g_\infty)$ is again given by eq.(4.66).

The normalization constant for the remaining $N-n_0-n_\infty$ correlators of the form

$$\langle\sigma_{g_\infty(J)}\sigma_{n_0J}\sigma_{n_0+n_\infty,N}\sigma_{g_0}\rangle$$

can be found in the same manner and is again defined by eq.(4.66).

Up to now we considered the correlators $G_{IJKL}(u,\bar{u})=\langle\sigma_{g_\infty}(\infty)\sigma_{IJ}(1)\sigma_{KL}(u,\bar{u})\sigma_{g_0}(0)\rangle$ with $|u|<1$. The correlators $G_{IJKL}(u,\bar{u})$ with $|u|>1$ can be calculated in the same way, and their dependence on u is given by eq.(4.42) as well. The normalization constant in this case is derived by studying the limit $u\to\infty$ and coincides with the previously found constant (4.66). The time-ordering, therefore, can be omitted, and to complete the computation of the S-matrix element we have to integrate the correlator $F(u,\bar{u})$ (4.24) over the complex plane. With the help of the momentum conservation law, the mass-shell condition and eq.(4.65), one can rewrite eq.(4.42) in the following form

$$\begin{aligned}
G_{IJKL}(u,\bar{u}) &= \delta^D_R(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)C(g_0,g_\infty)(n_\infty-n_0)^2|\frac{du}{dx}|^{-2}|u|^{-1}\frac{|x-\alpha_1|^2|x-\alpha_2|^2}{|x-\frac{n_0}{n_0-n_\infty}|^4}\\
&\times\left|\frac{x(x-\frac{N-n_0-n_\infty}{N-n_0})}{x-\frac{n_0}{n_0-n_\infty}}\right|^{\frac{1}{2}k_1k_4}\left|\frac{(x-1)(x+\frac{n_0}{N-n_0})}{x-\frac{n_0}{n_0-n_\infty}}\right|^{\frac{1}{2}k_3k_4},
\end{aligned}$$

where the equality

$$\frac{1}{u}\frac{du}{dx}=\frac{(n_0-n_\infty)(x-\alpha_1)^2(x-\alpha_2)^2}{x(x-1)(x-\frac{N-n_0-n_\infty}{N-n_0})(x-\frac{n_0}{n_0-n_\infty})(x+\frac{n_0}{N-n_0})}$$

was used.
Now the integral $\int d^2u|u|G_{IJKL}(u,\bar{u})$ can be easily calculated by changing the variables $u \to x$

$$\int d^2u|u|G_{IJKL}(u,\bar{u}) = \delta_R^D(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)C(g_0,g_\infty)(n_\infty-n_0)^2$$
$$\int_{V_{IJKL}} d^2x \frac{|x-\alpha_1|^2|x-\alpha_2|^2}{|x-\frac{n_0}{n_0-n_\infty}|^4}\left|\frac{x(x-\frac{N-n_0-n_\infty}{N-n_0})}{x-\frac{n_0}{n_0-n_\infty}}\right|^{\frac{1}{2}k_1k_4}\left|\frac{(x-1)(x+\frac{n_0}{N-n_0})}{x-\frac{n_0}{n_0-n_\infty}}\right|^{\frac{1}{2}k_3k_4}, \quad (4.69)$$

where we have taken into account that under this change of variables the u-sphere is mapped onto the domain V_{IJKL}.
Since the correlator $F(u,\bar{u})$ is equal to the sum

$$F(u,\bar{u}) = \frac{C_0C_\infty}{N!}2n_0(N-n_0)n_\infty(N-n_\infty)\sum_{IJKL}G_{IJKL}(u,\bar{u}),$$

where the summation goes over the set of indices listed in eq.(4.32), the integral $\int d^2u|u|F(u,\bar{u})$ is equal to

$$\int d^2u|u|F(u,\bar{u}) = \frac{2^{-10}\delta_R^D(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)}{\sqrt{n_0(N-n_0)n_\infty(N-n_\infty)}}\left(\frac{N-n_0}{n_\infty-n_0}\right)^{2+\frac{1}{2}(k_1+k_3)k_4}$$
$$\int d^2x \frac{|x-\alpha_1|^2|x-\alpha_2|^2}{|x-\frac{n_0}{n_0-n_\infty}|^4}\left|\frac{x(x-\frac{N-n_0-n_\infty}{N-n_0})}{x-\frac{n_0}{n_0-n_\infty}}\right|^{\frac{1}{2}k_1k_4}\left|\frac{(x-1)(x+\frac{n_0}{N-n_0})}{x-\frac{n_0}{n_0-n_\infty}}\right|^{\frac{1}{2}k_3k_4}, \quad (4.70)$$

Finally, performing the change of variables

$$\frac{n_\infty-n_0}{N-n_0}z = \frac{x(x-\frac{N-n_0-n_\infty}{N-n_0})}{x-\frac{n_0}{n_0-n_\infty}},$$

one gets the following expression for the integral

$$\int d^2u|u|F(u,\bar{u}) = \frac{2^{-9}\delta_R^D(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)}{\sqrt{n_0(N-n_0)n_\infty(N-n_\infty)}}\int d^2z|z|^{\frac{1}{2}k_1k_4}|1-z|^{\frac{1}{2}k_3k_4}. \quad (4.71)$$

The S-matrix element can be now found by using eq.(4.23) and by taking the limit $R\to\infty$:

$$\begin{aligned}\langle f|S|i\rangle &= -i\frac{\lambda^2 2^{-8}N^3\delta(k_1^-+k_2^-+k_3^-+k_4^-)\delta^D(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)}{\sqrt{n_0(N-n_0)n_\infty(N-n_\infty)}}\int d^2z|z|^{\frac{1}{2}k_1k_4}|1-z|^{\frac{1}{2}k_3k_4}\\ &= -i\frac{\lambda^2 2^{-8}N\delta(\sum_i k_i^-)\delta^D(\sum_i\mathbf{k}_i)}{\sqrt{k_1^+k_2^+k_3^+k_4^+}}\int d^2z|z|^{\frac{1}{2}k_1k_4}|1-z|^{\frac{1}{2}k_3k_4}. \quad (4.72)\end{aligned}$$

Let us now represent the light-cone momenta k_i^+ as $k_i^+ = \frac{m_i}{N}$ and rewrite eq.(4.72) in the following form

$$\langle f|S|i\rangle = -i\frac{\lambda^2 2^{-8}N\delta_{m_1+m_2+m_3+m_4,0}\delta(\sum_i k_i^-)\delta^D(\sum_i\mathbf{k}_i)}{\sqrt{k_1^+k_2^+k_3^+k_4^+}}\int d^2z|z|^{\frac{1}{2}k_1k_4}|1-z|^{\frac{1}{2}k_3k_4}. \quad (4.73)$$

In the limit $N \to \infty$ the combination $N\delta_{m_1+m_2+m_3+m_4,0}$ goes to $\delta(\sum_i k_i^+)$ and eq.(4.73) acquires the form

$$\langle f|S|i\rangle \;=\; -i\frac{\lambda^2 2^{-9}\delta^{D+2}(\sum_i k_i^\mu)}{\sqrt{k_1^+k_2^+k_3^+k_4^+}}\int d^2z|z|^{\frac{1}{2}k_1k_4}|1-z|^{\frac{1}{2}k_3k_4}. \tag{4.74}$$

Taking into account that the scattering amplitude A is related to the S-matrix as follows (see e.g. [12])

$$\langle f|S|i\rangle = -i\frac{\delta^{D+2}(\sum_i k_i^\mu)}{\sqrt{k_1^+k_2^+k_3^+k_4^+}}A(1,2,3,4),$$

one finally gets

$$A(1,2,3,4) = \lambda^2 2^{-9}\int d^2z|z|^{\frac{1}{2}k_1k_4}|1-z|^{\frac{1}{2}k_3k_4}$$

that is the well-known Virasoro amplitude.

5 Conclusion

In this paper we developed the technique for calculating scattering amplitudes of bosonic string states by using the interacting $S^N\mathbf{R}^{24}$ orbifold sigma model. The scattering amplitude turned out to be automatically Lorentz-invariant. It gives a strong evidence that the corresponding two-dimensional Yang-Mills model should possess the same invariance.

It would be of interest to trace the appearance of the loop amplitudes in the framework of the $S^N\mathbf{R}^{24}$ orbifold sigma model. Obviously, the one-loop amplitude requires the computation of the correlator of four Z_2-twist fields sandwiched between the asymptotic states that technically results in constructing the non-commutative Green functions in the presence of six twist fields. We note that cancellation of possible divergences in the amplitude may require the further perturbation of the CFT action by higher-order contact terms.

The next important problem to be solved is to consider the $S^N\mathbf{R}^8$ supersymmetric orbifold sigma model and to prove the DVV conjecture. It is not difficult to introduce twist fields for fermionic variables and to calculate their conformal dimensions. However, the calculation of four-point correlators of the twist fields is more complicated problem and is now under consideration. It is not excluded that the simplest way to solve the problem is to bosonize the fermion fields.

ACKNOWLEDGMENT The authors thank I.Y.Aref'eva, L.O.Chekhov, P.B.Medvedev and N.A.Slavnov for valuable discussions. One of the authors (S.F.) is grateful to Professor J.Wess for kind hospitality and the Alexander von Humboldt Foundation for the support. This work has been supported in part by the RFBI grants N96-01-00608, N96-01-00551 and by the ISF grant a96-1516.

Appendix A

In this Appendix we consider some properties of the map (4.35) and outline the derivation of the differential equation (4.40) for the four-point correlators (4.33).

Let us consider the map (4.35)

$$z = \frac{t^{n_0}(t-t_0)^{N-n_0}}{(t-t_\infty)^{N-n_\infty}} \frac{(t_1-t_\infty)^{N-n_\infty}}{t_1^{n_0}(t_1-t_0)^{N-n_0}} \equiv u(t). \tag{A.1}$$

This map is the N-fold covering of the z-sphere by the t-sphere. Obviously, it branches at the points $t = 0, t_0, t_\infty$ and ∞. To find other branch points we have to solve the following equation:

$$\begin{aligned} \frac{d\log z}{dt} &= \frac{n_0}{t} + \frac{N-n_0}{t-t_0} - \frac{N-n_\infty}{t-t_\infty} \\ &= \frac{n_\infty t^2 + ((N-n_0-n_\infty)t_0 - Nt_\infty)\, t + n_0 t_0 t_\infty}{t(t-t_0)(t-t_\infty)}. \end{aligned} \tag{A.2}$$

In general there are two different solutions t_1 and t_2 of this equation, and the map (A.1) has the following form in the vicinity of these points

$$z - z_i \sim (t-t_i)^2, \quad z_1 = 1 = u(t_1), \quad z_2 = u = u(t_2).$$

Due to the projective transformations, we can impose three relations on positions of branch points. However, we have already chosen the points 0 and ∞ as two branch points, therefore, only one relation remains to be imposed. Since the differential equation on the four-point correlator is written with respect to the point u, it is convinient not to fix the position of the point $t_2 \equiv x$. Then, the remaining relation that leads to the rational dependence of points t_0, t_∞ and t_1 on x looks as follows

$$t_0 = x - 1. \tag{A.3}$$

The point x is supposed to be a solution of eq.(A.2). Therefore, one can immediately derive from eqs.(A.2) and (A.3) that t_∞ is expressed through the point x as

$$t_\infty = x - \frac{(N-n_\infty)x}{(N-n_0)x + n_0}. \tag{A.4}$$

The second solution of eq.(A.2) can be now easily found and is given by

$$\begin{aligned} t_1 &= \frac{N-n_0-n_\infty}{n_\infty} + \frac{n_0 x}{n_\infty} - \frac{N(N-n_\infty)x}{n_\infty((N-n_0)x+n_0)} \\ &= \frac{n_0(x-1)\,((N-n_0)x + n_0 + n_\infty - N)}{n_\infty\,((N-n_0)x+n_0)}. \end{aligned} \tag{A.5}$$

The rational function $u(x)$ is defined by the following equation

$$u(x) = \frac{x^{n_0}(x-t_0)^{N-n_0}(t_1-t_\infty)^{N-n_\infty}}{(x-t_\infty)^{N-n_\infty} t_1^{n_0}(t_1-t_0)^{N-n_0}}. \tag{A.6}$$

By using eqs.(A.3),(A.4) and (A.5), one can derive the following relations

$$\begin{aligned} t_1 - t_0 &= \frac{(N-n_0)(x-1)\,((n_0-n_\infty)x - n_0)}{n_\infty\,((N-n_0)x+n_0)}, \\ t_1 - t_\infty &= \frac{((n_0-n_\infty)x-n_0)\,((N-n_0)x+n_0+n_\infty-N)}{n_\infty\,((N-n_0)x+n_0)}. \end{aligned}$$

Then the rational function $u(x)$ is found to be equal to

$$\begin{aligned} u = u(x) &= (n_0 - n_\infty)^{n_0 - n_\infty} \frac{n_\infty^{n_\infty}}{n_0^{n_0}} \left(\frac{N-n_0}{N-n_\infty}\right)^{N-n_\infty} \left(\frac{x + \frac{n_0}{N-n_0}}{x-1}\right)^N \\ &\times \left(\frac{x - \frac{N-n_0-n_\infty}{N-n_0}}{x}\right)^{N-n_0-n_\infty} \left(x - \frac{n_0}{n_0 - n_\infty}\right)^{n_0-n_\infty}. \end{aligned} \tag{A.7}$$

To obtain the differential equation (4.40) we need to know the decomposition of the roots $t_K(z)$ and $t_L(z)$ in the vicinity of $z = u$. Let us take the logarithm of the both sides of eq.(A.1):

$$\log\frac{z}{u} = n_0 \log\frac{t}{x} + (N-n_0)\log\frac{t-t_0}{x-t_0} - (N-n_\infty)\log\frac{t-t_\infty}{x-t_\infty}. \tag{A.8}$$

Decomposition of the l.h.s. of eq.(A.8) around $z = u$ and the r.h.s. of eq.(A.8) around $t = x$ gives:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\left(\frac{z-u}{u}\right)^k = (t-x)^2 \sum_{k=0}^{\infty} a_k (t-x)^k, \tag{A.9}$$

where the coefficients a_k are equal to

$$a_k = \frac{(-1)^{k-1}}{k+2}\left(\frac{n_0}{x^{k+2}} + \frac{N-n_0}{(x-t_0)^{k+2}} - \frac{N-n_\infty}{(x-t_\infty)^{k+2}}\right). \tag{A.10}$$

It is clear from eq.(A.9) that $t(z)$ has the following decomposition

$$t - x = \sum_{k=1}^{\infty} c_k (z-u)^{\frac{k}{2}}. \tag{A.11}$$

Substituting eq.(A.11) into eq.(A.9), one finds

$$\begin{aligned} &c_1^2 = \frac{1}{ua_0}, \quad c_2 = -\frac{a_1}{2ua_0}, \\ &2a_0c_1c_3 = -\frac{1}{2u^2} + \frac{5a_1^2}{4u^2a_0^3} - \frac{a_2}{u^2a_0^2}. \end{aligned} \tag{A.12}$$

Next coefficients are not important for us.
Then, by using the decomposition (A.11) and eq.(A.12), one gets

$$\begin{aligned} \left(\frac{t''}{t'}\right)' &= \frac{1}{2(z-u)^2} + O(1), \\ \left(\frac{t''}{t'}\right)^2 &= \frac{1}{4(z-u)^2} + \frac{3}{z-u}\left(\frac{c_2^2}{c_1^2} - \frac{c_3}{c_1}\right) + O(1), \\ \frac{c_2^2}{c_1^2} - \frac{c_3}{c_1} &= \frac{1}{4u}\left(1 + \frac{2a_2}{a_0^2} - \frac{3a_1^2}{2a_0^3}\right). \end{aligned}$$

Finally, taking into account that in the set of N roots $t_M(z)$ only two roots $t_K(z)$ and $t_L(z)$ have the decomposition (A.11), we obtain eqs.(4.38) and (4.39).

The coefficients a_k can be rewritten as the following functions of x:

$$
\begin{aligned}
a_0 &= \frac{n_0(n_0+n_\infty-N)}{2(N-n_\infty)x^2}+\frac{n_0(N-n_0)}{(N-n_\infty)x}+\frac{(N-n_0)(n_\infty-n_0)}{2(N-n_\infty)} \\
&= \frac{(N-n_0)(n_\infty-n_0)}{2(N-n_\infty)x^2}(x-\alpha_1)(x-\alpha_2), \\
a_1 &= \frac{n_0((N-n_\infty)^2-n_0^2)}{3(N-n_\infty)^2x^3}-\frac{n_0^2(N-n_0)}{(N-n_\infty)^2x^2} \\
&- \frac{n_0(N-n_0)^2}{(N-n_\infty)^2x}+\frac{(N-n_0)((N-n_\infty)^2-(N-n_0)^2)}{3(N-n_\infty)^2}, \\
a_2 &= -\frac{n_0((N-n_\infty)^3-n_0^3)}{4(N-n_\infty)^3x^4}+\frac{n_0^3(N-n_0)}{(N-n_\infty)^3x^3}+\frac{3n_0^2(N-n_0)^2}{2(N-n_\infty)^3x^2} \\
&+ \frac{n_0(N-n_0)^3}{(N-n_\infty)^3x}-\frac{(N-n_0)((N-n_\infty)^3-(N-n_0)^3)}{4(N-n_\infty)^3}.
\end{aligned}
\tag{A.13}
$$

To obtain the differential equation (4.40) we have to use the following important equalities on $\frac{1}{u}\frac{du}{dx}$, that can be derived by using eqs.(A.7) and (A.13)

$$
\begin{aligned}
\frac{1}{u}\frac{du}{dx} &= \frac{n_0+n_\infty-N}{x}-\frac{N}{x-1}+\frac{N}{x+\frac{n_0}{N-n_0}} \\
&+ \frac{N-n_0-n_\infty}{x-\frac{N-n_0-n_\infty}{N-n_0}}+\frac{n_0-n_\infty}{x-\frac{n_0}{n_0-n_\infty}}, \\
\frac{1}{u}\frac{du}{dx} &= \frac{4(N-n_\infty)^2x^4a_0^2}{(N-n_0)^2(n_0-n_\infty)x(x-1)(x-\frac{N-n_0-n_\infty}{N-n_0})(x-\frac{n_0}{n_0-n_\infty})(x+\frac{n_0}{N-n_0})} \\
&= \frac{(n_0-n_\infty)(x-\alpha_1)^2(x-\alpha_2)^2}{x(x-1)(x-\frac{N-n_0-n_\infty}{N-n_0})(x-\frac{n_0}{n_0-n_\infty})(x+\frac{n_0}{N-n_0})}.
\end{aligned}
$$

Finally, to get eq.(4.40) one should use the Lagrange interpolation formula for the ratio of two polynomials

$$
\frac{P(x)}{Q(x)}=\sum_i \frac{P(x_i)}{Q'(x_i)}\frac{1}{x-x_i},
$$

where x_i are the simple roots of $Q(x)$ and $degP < degQ$.

These equalities drastically simplify the derivation of eq.(4.40).

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