

The Method for Structural Reliability Analysis Considering the First Three Moments as Interval Variables

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ABSTRACT: In the traditional structural reliability method, the first three statistical moments of random variable are used to describe its uncertainty. But in practical engineering, due to the lack of statistical data, it is difficult to obtain the certain statistical moments of some random variables, and the failure probability is no longer the determined value. To solve this problem, this paper treats the uncertain first three moments as interval variables, and proposes the method to compute the interval of failure probability. This method uses the simplified third moment normal transformation model to map the limit state surface from the original space to the standard normal space. By distinguishing the statistical moment interval values corresponding to the upper and lower boundaries of the limit state strip in the normal space, limit state functions corresponding to the upper and lower boundaries of the limit state strip are established with the system reliability, and method of moments is used to calculate the interval of failure probability. In this paper, an assessment of structural reliability analysis considering the first three moments as interval variables is introduced. The example is provided to demonstrate the accuracy and efficiency of this approach in structural reliability analysis.

INTRODUCTION

Generally, the probabilistic reliability model is the most prevalent method to solve structural reliability problems. In the past research, many analysis approaches have been used in structural reliability analysis, which include the first order reliability method (FORM) (Hasofer and Lind 1974, Rackwitz and Fiessler 1978), second order reliability method (SORM) (Kiureghian et al. 1987, Kiureghian and Stefano 1991), Monte-Carlo simulation (Fu 1994, Melchers 1990), method of moments (Grigoriu 1983, Zhao and

Ono 2001), etc. When using these methods, the cumulative distribution functions (CDF) or probability density functions (PDF) of some random variables are usually assumed to be known, so the statistical moments in the PDF are also certain. The imprecise statistical moments can only be estimated due to the lack of distribution information and the limitation of the experimental samples. The statistical moments of random variables will change due to the additional sample information, resulting in statistical moment uncertainty. The uncertainty of statistical moments can substantially affect the failure

probability and reliability index (Ben-Haim and Elishakoff 1990, Kiureghian 1989).

To consider the uncertainty of statistical moments in reliability analysis, two approaches have been developed in past research. In the first approach, the statistical moments are treated as random variables, thereby the failure probability becomes a random variable which calls conditional failure probability (Kiureghian 1989), and the overall failure probability is defined as the expectation of the conditional failure probability. And a point-estimate method based on dimension-reduction integration and Bayes theory has been proposed to estimate the probability distribution and quantile of the conditional reliability index and conditional failure probability (Zhao et al. 2018).

The second approach involves treating statistical moments as interval variables. A reliability analysis method combining the probabilistic reliability method with a convex model was proposed when information is insufficient (Elishakoff and Colombi 1993, Zhu and Elishakoff 1996). For the failure probability of basic linear function $G(X)=R-S$, when R and S are normal random variables and independent of each other, the analytical expression of the reliability index interval is obtained (Qiu et al. 2008, Qiu et al. 2008). However, in practical engineering, R and S are not always normal distributions and function is mostly nonlinear. In more common situations, an optimization method to acquire the upper and lower bounds of the reliability index was proposed based on the FORM (Du 2008). A more efficient computing method was proposed, and two reliability analysis models were given based on the proof of monotonicity for the CDF of the random variable with respect to its statistical moments (Jiang et al. 2011, Jiang et al. 2012).

The goal of the present paper is to develop a methodology for solving the reliability analysis considering the uncertainty of the first three statistical moments. It is based on method of moments and the monotonicity analysis of third-moment normal transformation. Firstly, the

monotonicity proof and the interval analysis of skewness parameters are proposed in this paper. Next, an accurate and efficient procedure for calculating the interval of failure probability and reliability index is developed. Furthermore, the calculation of structural reliability with unknown probability distributions of random variables is also one of the characteristics of this paper. The paper includes examples to demonstrate the feasibility of the proposed method. Results indicate that the method is considerably more efficient than the Monte-Carlo simulation (MCS) method while maintaining sufficient accuracy.

1. PROPOSED METHOD

1.1. Monotonicity analysis

When considering the uncertainty of the statistical moment of a random variable X , it is usually treated as interval parameter Y , the limit state function is transformed from the original space to the standard normal space, and it is expressed as follows (Jiang et al. 2012)

$$G(X) = g(T(U, Y)) = G(U, Y) \quad (1)$$

where Y is an n -dimensional vector containing all interval statistical moments, which can be expressed as follow

$$Y \in [Y^L, Y^R], Y_i \in [Y_i^L, Y_i^R], i = 1, 2, \dots, n \quad (2)$$

where the superscripts L and R represent the lower boundary and upper boundary of the interval, respectively.

In traditional reliability analysis, the limit state defined by $G(X)=0$ is a single hyper-surface in u space. However, due to the existence of Y , the limit state of the hybrid reliability defined by $G(U, Y)=0$ changes to a strip enclosed by two bounding hyper surfaces S_L and S_R as shown in Fig. 1.

The above analysis shows the existence of the bounding hypersurfaces, which can further explain that the reliability index will become an interval when the statistical moments of random variables are interval variables. Therefore, it is critical to find the relationship between the bounding hypersurfaces and the values of interval

variables, which will be obtained by monotonicity analysis for the third-moment normal transformation below.

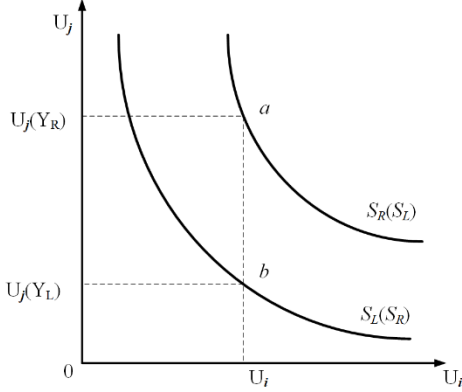


Figure 1: The transformation process of the limit state surface

For a group of independent random variables $X = \{x_1, x_2, \dots, x_n\}$ in $G(X)$, whether the probability distribution is known or not, a random variable x_i can be transformed into u space by the third-order moment normal transformation model as (Zhao and Ono 2000)

$$\frac{x_i - \mu_{xi}}{\sigma_{xi}} = x_{si} = S_u(u_i) = a_{1i} + a_{2i}u_i + a_{3i}u_i^2 \quad (3)$$

where x_{si} represents the standardized variable of x_i , $S_u(\cdot)$ represents that the independent variable is the function of a standard normal random variable u_i which can be written as

$$u_i = \begin{cases} \frac{-a_{2i} + \sqrt{a_{2i}^2 - 4a_{3i}(a_{1i} - x_{si})}}{2a_{3i}}, & a_{3i} \neq 0 \\ x_{si}, & a_{3i} = 0 \end{cases} \quad (4)$$

where a_{1i} , a_{2i} and a_{3i} are calculated as follows

$$a_{3i} = -a_{1i} = \text{sign}(\alpha_{3si})\sqrt{2} \cos\left[\frac{\pi + |\theta_i|}{3}\right] \quad (5)$$

$$a_{2i} = \sqrt{1 - 2a_{3i}^2} \quad (6)$$

$$\theta_i = \arctan\left(\frac{\sqrt{8 - \alpha_{3i}^2}}{\alpha_{3i}}\right) \quad (7)$$

To hold the equality, the value of α_{3i} is limited to $[-2\sqrt{2}, 2\sqrt{2}]$. Then the monotonicity of u_i with respect to u_{xi} , σ_{xi} and α_{3i} will be proved, respectively. Here the monotonicity of statistical moments when $a_{3i} \neq 0$ is considered.

Firstly, the first-order derivative of u_i with respect to u_{xi} can be obtained as

$$\frac{\partial u_i}{\partial \mu_{xi}} = \begin{cases} -\frac{1/\sigma_{xi}}{\sqrt{a_{1i}^2 + 4a_{2i}^2 + 4a_{2i}x_{si}}}, & \alpha_{3xi} \neq 0 \\ -\frac{1}{\sigma_{xi}}, & \alpha_{3xi} = 0 \end{cases} \quad (8)$$

In Eq. (8), it can be obtained easily that the u_i is always monotonically decreasing with respect to u_{xi} .

Secondly, the first-order derivative of u_i with respect to σ_{xi} can be obtained as

$$\frac{\partial u_i}{\partial \sigma_{xi}} = \begin{cases} \frac{\mu_{xi} - x_i}{\sigma_{xi}^2 \sqrt{a_{1i}^2 + 4a_{2i}^2 + 4a_{2i}x_{si}}}, & \alpha_{3xi} \neq 0 \\ \frac{\mu_{xi} - x_i}{\sigma_{xi}^2}, & \alpha_{3xi} = 0 \end{cases} \quad (9)$$

In Eq. (9), it can be found that the sign of $\frac{\partial u_i}{\partial \sigma_{xi}}$ is related to the value of x_i .

When $x_i > \mu_{xi}$, u_i is monotonically decreasing with respect to σ_{xi} , and when $x_i \leq \mu_{xi}$ the monotonicity is opposite.

Thirdly, the first-order derivative of u_i with respect to α_{3i} can be obtained as

$$\frac{\partial u_i}{\partial \alpha_{3i}} = \frac{(1 - u_i^2) \cdot \sqrt{1 - 2a_{1i}^2} + 2a_{1i}u_i}{\sqrt{1 - 2a_{1i}^2} \cdot (2a_{1i}u_i + a_{2i})} \cdot \frac{\partial a_{1i}}{\partial \alpha_{3i}} \quad (10)$$

In Eq. (10), it can be divided into three parts for facilitating the analysis.

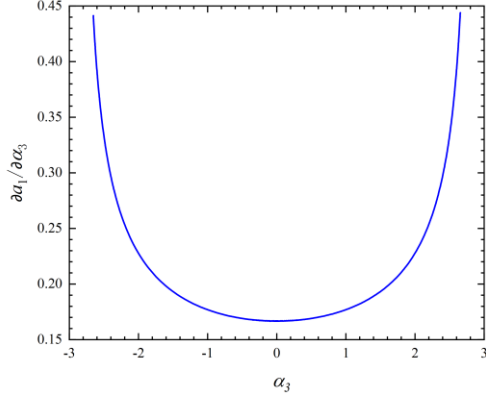


Figure 2: Judgment of monotonicity of a_1 with respect to skewness

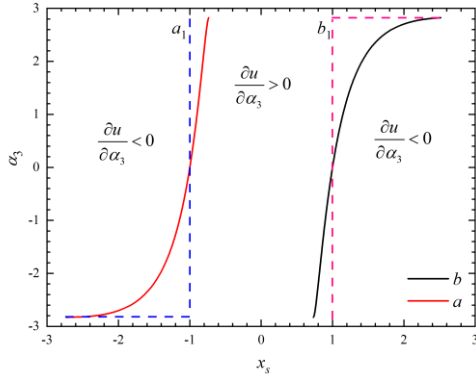


Figure 3. Judgment of monotonicity of u with respect to skewness

According to the Fig. 2 whose horizontal axis represents the skewness of random variables α_{3i} and the vertical axis represents the first-order derivative of a_{1i} with respect to α_{3i} , it can be evaluated that $G(U, Y)$ is positive. Similarly, $(2a_{1i}u_i + a_{2i})$ is also positive according to Eq. (4).

However, the sign of $(1 - u_i^2) + \frac{2a_{1i}u_i}{\sqrt{1 - 2a_{1i}^2}}$ cannot

be evaluated directly, and because the value of

$(1 - u_i^2) + \frac{2a_{1i}u_i}{\sqrt{1 - 2a_{1i}^2}}$ is related to x_{si} and α_{3i} , this

paper uses Fig. 3 to solve it.

In Fig. 3, the horizontal axis represents standardized random variables x_{si} and the vertical axis represents the skewness of random

variables α_{3i} . This figure shows that when $\frac{\partial u_i}{\partial \alpha_{3i}}$,

each given value of α_{3i} has two different values of x_{si} , two curves a and b are obtained. It is concluded that the value of $\frac{\partial u_i}{\partial \alpha_{3i}}$ is positive in the internal region surrounded by curves a and b , while the value of $\frac{\partial u_i}{\partial \alpha_{3i}}$ is negative in the region

corresponding to the left part of the curve a and the right part of the curve b . But for any given value of x_{si} , u_i is not monotonous with respect to α_{3i} .

So, in this paper, curve a is approximated as polygonal line a_1 , and line b is approximated as polygonal line b_1 . When $\alpha_{3i} = 0$, two values of x_{si} are -1 and 1 , respectively. After analysis, the results can be obtained as

$$\begin{cases} \frac{\partial u_i}{\partial \alpha_{3i}} \geq 0, -1 \leq x_{si} \leq 1 \\ \frac{\partial u_i}{\partial \alpha_{3i}} < 0, x_{si} < -1, x_{si} > 1 \end{cases} \quad (11)$$

In Eq. (11), the monotonicity of u_i with respect to α_{3i} depends on the standardized random variables x_{si} , but for any given value of x_{si} , u_i is monotonous with respect to α_{3i} . When $-1 \leq x_{si} \leq 1$, u_i is monotonically decreasing with respect to α_{3i} , and when $x_{si} < -1$ and $x_{si} > 1$ the monotonicity is opposite.

Therefore, through the analysis above, the monotonicity analysis of the statistical moments can be divided into two categories. The first category is independent of the value of random variables, that is, u_i is always monotonically decreasing function with respect to μ_{xi} . The second category is related to the value of random variables, for the monotonicity of u_i with respect to σ_{xi} , there is an inflection point so that the

monotonicity of σ_{xi} is opposite on both sides of the inflection point. Similarly, there are two inflection points in the monotonicity of u_i with respect to α_{3i} , and on both sides of each inflection point, the monotonicity of α_{3i} is opposite.

1.2. Structure Reliability calculation models

To obtain the Interval parameters values corresponding to the bounds of the limit state strips, it is necessary to determine the position relationship between the limit state strip and the origin point.

Since $G(X_{mean}) > 0$ holds in general and the origin point in the u space should be not inside the limit-state strip, the relative position can be obtained by calculating the gradient $\left. \frac{\partial G}{\partial U_i} \right|_{origin}$. The value of Y corresponding to the limit strip can be derived by combining monotonicity and gradient, and the results are summarized in Table. 1.

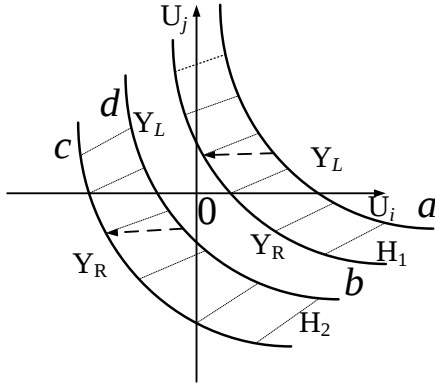


Figure 4: The relative position of the origin point and the limit state strips in u space

Due to the difference in the monotonicity of statistical moments, the shape of the limit state strip and their corresponding relationship with the boundary value of statistical moment interval parameters are different. Here three different limit state strips and two structural reliability calculation models will be built according to the different monotonicity.

When only mean uncertainty is considered, after determining the values of the mean interval parameters corresponding to the lower and upper boundaries of the limit state strip, the method of moments is used to calculate the reliability index of the upper and lower boundaries, and the intervals of reliability index and failure probability can be obtained.

Table 1: Interval parameters values corresponding to the bounds of the limit state strips

$\left. \frac{\partial G}{\partial U_i} \right _{origin}$	Monotonicity	Y on the lower bound	Y on the upper bound
< 0	Monotonically decreasing	Y^R	Y^L
< 0	Monotonically increasing	Y^L	Y^R
> 0	Monotonically decreasing	Y^L	Y^R
> 0	Monotonically increasing	Y^R	Y^L

Note: Monotonicity is the monotonicity of u with respect to statistical moments.

When the uncertainty of standard deviation is considered, according to the monotonicity of u with respect to σ in 2.2, there is an inflection point in u space, and the monotonicity of u with respect to σ at both ends of the inflection point is opposite. Therefore, the limit state strip will rotate, and the upper and lower limit state hyper-surfaces will become enveloping surfaces. Like the second strip, when considering the skewness uncertainty, the upper and lower limit state hyper-surfaces also rotate.

In the latter two cases, the boundary of the limit state strip changes to the envelope surface, and the values of standard deviation or skewness interval parameters corresponding to each small part of the envelope surface are different, and the corresponding failure modes are also different. Therefore, the system reliability problem is formed, and the system reliability based on method of moments is used to solve it in this paper (Zhao and Ang 2003), which avoids the solution

of the correlation coefficient between failure modes.

1.3. Calculating procedure

From the above analysis, it can be seen that for the structural reliability problem considering the uncertainty of statistical moments, two different calculation models will eventually be formed. Next, the calculation procedure considering the uncertainty of statistical moments will be summarized as follows

1. Divide the interval parameters of statistical moments into two classes, mean interval parameters Y_{1i} , $i=1,2,\dots, n$, and standard deviation and skewness interval parameters Y_{2j} , $j=1,2,\dots, m$.
2. If there are mean interval parameters, calculate the gradient $\left. \frac{\partial G}{\partial U_i} \right|_{origin}$, and then determine the corresponding value of the upper and lower bounds of the mean interval parameter Y_i in the limit state strip according to Table. 1.
3. Repeat 2 for $i=1,\dots, n$ to obtain the Y_{1L} and Y_{1R} .
4. According to whether Y_{2j} exists or not, to choose to calculate the interval of P_f and β or consider the 2^j combinations of Y_{2j}
5. Obtain the limit state function of the lower boundary series system and upper boundary parallel system.
6. Calculate the interval of reliability index and the failure probability are obtained using the system reliability based on method of moments.

2. ILLUSTRATIVE EXAMPLES

A four-bar truss as shown in Fig. 5 is investigated, which is modified from the numerical examples in Ref. (Du 2008). Considering that the vertical displacement δ at the tip joint of the truss is required to be less than an allowable value the LSF can be constructed as

$$G(X) = \delta_a - \delta = \delta_a - \frac{6Pl}{E} \left(\frac{3}{A_1} + \frac{\sqrt{3}}{A_2} \right) \quad (12)$$

where $\delta_a = 1.7\text{mm}$ is the allowable value and vertical force P is 160 kN; A_1 is the cross-sectional area of members 1-3 and A_2 is the area of member 4; the Young's Modulus E of the truss is 200 GPa and the length l is 500 mm. In this example, the mean of A_1 and the standard deviation of A_2 are treated as interval variable. The statistical information is given in Table. 2.

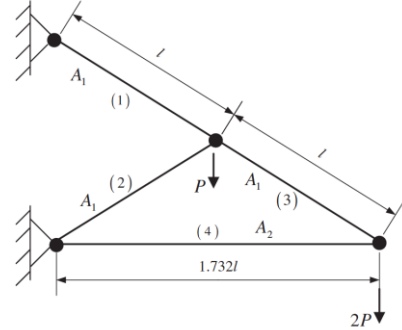


Figure 5: Four-bar truss

Table 2: Statistical information for Example 1

Random Variable	Mean	Standard Deviation	Distribution
$A_1(\text{mm}^2)$	[635.85, 777.15]	70.65	Normal
$A_2(\text{mm}^2)$	1256	[169.56, 207.24]	Normal

Table 3: Reliability analysis results for Example 1

Method	Reliability index β			
	Lower bound		Upper bound	
	Value	R.E. (%)	Value	R.E. (%)
MCS	1.6164	---	3.0618	---
SLMA	1.62	0.22	3.13	2.23
Proposed method	1.6198	0.21	3.0811	0.63

In comparison, the single-loop method based on the monotonicity analysis, also known as SLMA, is used (Jiang et al. 2012). Furthermore, Monte Carlo simulation (MCS) is used as an accuracy benchmark. Table 3 shows the bounds of the reliability index obtained using these approaches, as well as the relative error (denoted by R.E.) when compared to MCS results. Table 3

shows that the relative errors of both proposed method and the SLMA methods are less than 5%, indicating that both methods are accurate for this example.

3. CONCLUSIONS

This paper proposes a reliability analysis method that considers the uncertainty of the first three statistical moments. The following findings were derived from previous research:

1. The monotonicity of the standard normal random variable u with respect to the first three moments can be proved respectively based on the third-order moment normal transformation model.
2. Based on the monotonicity analysis, the study proposes a calculation procedure that requires no optimization and only two reliability analyses.
3. The results of the example are very close to MCS indicating the accuracy of the proposed method.

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