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Thermodynamics of Nanoelectromechanics

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Abstract

Taking inspiration from nanoelectromechanical experiments, we introduce a microscopic toy model consisting of a resonant-level quantum dot coupled to two electrodes and a single-mode oscillator. We derive a Fokker-Planck equation that describes the evolution of the oscillator in the quasi-adiabatic limit. The oscillator can be treated as a quantum flywheel, converting electrical power into stored mechanical energy. Calculating the non-equilibrium steady state of the oscillator, we find that energy dependence in the dot-reservoir coupling is required to produce self-sustained oscillations, corroborating previous findings. We investigate the self-sustained oscillations through the lens of work extraction. We characterise the threshold for self-sustained oscillations using two approaches to quantifying work deposition in non-equilibrium quantum thermodynamics: ergotropy and non-equilibrium free energy. We find that ergotropy acts as an order parameter for self-sustained oscillations, becoming non-zero only when a population inversion is achieved.

Using this toy model, we investigate the thermodynamics of autonomous quantum clocks. The periodic mechanical motion of the system behaves as a clockwork, similar to the swinging of a pendulum, while induced oscillations in the electrical current can be used to read out the ticks. We derive formulae calculating the current and current noise in the nanoelectromechanical model, which allows us to infer statistical properties of the clock's ticks from the current auto-correlation function. Similar to previous studies, we find that the electromechanical clock exhibits a tradeoff between accuracy, resolution, and dissipation. However, our work goes beyond the distribution of individual ticks, investigating how clock accuracy varies over different integration times by computing the Allan variance. We observe non-monotonous features of the Allan variance, which can be attributed to temporal correlations between ticks; these correlations yield a precision advantage for timekeeping over the timescales that the correlations persist.

Finally, we investigate the quantum Mpemba effect from the perspective of non-equilibrium quantum thermodynamics by studying the relaxation dynamics of quantum systems coupled to a Markovian heat bath, as described by Davies maps. In discrete systems starting from a state with coherences in the energy eigenbasis, we demonstrate that an exponential speedup to equilibrium will always occur if the state is transformed to a diagonal state in the energy eigenbasis, provided that a complex eigenvalue defines the spectral gap of the generator. When the transformed state has a higher non-equilibrium free energy, we argue using thermodynamic reasoning that this is a genuine quantum Mpemba effect. Furthermore, we show how a unitary transformation on an initial state can always be constructed to yield the effect and demonstrate our findings by studying the dynamics of both the non-equilibrium free energy and the irreversible entropy production in single and multi-qubit examples. In continuous systems, we explore the effect in classical and quantum systems performing an eigenmode decomposition on the generator. We investigate an ergotropic Mpemba effect in quantum Gaussian states; displaced states are found to be a better resource for storing ergotropy than squeezed states due to an exponential slowdown in the relaxation of the ergotropy in displaced states.

List of publications

The results exposed in this thesis are based on the following publications:

1. **O. Culhane**, M. T. Mitchison, and J. Goold,
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3. M. Moroder, **O. Culhane**, K. Zawadzki, and J. Goold,
The thermodynamics of the quantum Mpemba effect,
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4. I. Medina, **O. Culhane**, F. Binder, G. T. Landi and J. Goold,
The ergotropic Mpemba effect in systems with continuous degrees of freedom,
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Introduction

Recent advances in experimental mesoscopic physics have opened the door to exploring a plethora of physical phenomena. Some of the most active areas of quantum thermodynamic research are currently taking place in the fields of quantum batteries and work extraction [1–7], timekeeping [8–13] and anomalous thermalisation [14–18]. Nanoelectromechanical systems that incorporate mechanical degrees of freedom into electronic circuits are an exciting platform to investigate these fields. The incorporation of mechanical degrees of freedom in these systems allows a connection to thermodynamic work; these systems can be coupled to thermal reservoirs that allow a large amount of experimental control, and the state of the resonator can be reconstructed [19]. Platforms such as carbon nanotubes have been used to generate interesting phenomena such as self-sustained oscillations [20], explore anomalous relaxation [21], while membranes have been experimentally used to probe the fundamental limits of timekeeping [9]. This thesis sets out to generate a theoretical framework to understand these thermodynamic fields through the lens of nanoelectromechanics.

Drawing inspiration from recent experiments [20], we construct a toy model comprising a single-level quantum dot coupled to two electrodes. The dot is coupled to a single-mode oscillator that allows an interplay between the electronic and mechanical degrees of freedom. From this simple model, self-sustained oscillations are observed above a threshold bias. We investigate these self-sustained oscillations through the lens of extractable work, using concepts in thermodynamics — the ergotropy [22] and non-equilibrium free energy [23]. The oscillator acts like a flywheel or quantum battery [1], extracting work from the electronic degrees of freedom and storing them in the mechanical degrees of freedom.

Electromechanical systems can also play a role in understanding the fundamental limitations of our ability to measure time. While current atomic optical clocks are incredibly accurate, achieving an error of about one second in the lifetime of the universe [24], they also require a lot of resources to operate. Instead, One can ask how thermodynamics limits our ability to measure time - how efficient can we make a clock? The quest for answers has generated significant interest, spawning many models and papers trying to understand the fundamental constraints. At the most fundamental level, a clock is an out-of-equilibrium system that produces a continuous time reference. For accurate accounting, we require that the resources used to generate this non-equilibrium state be time-independent; such systems require some method of temporal probability concentration to generate accurate ticks [25].

We will employ an electromechanical model to further understand the fundamental resources necessary for timekeeping. The electromechanical setup resembles a classical pendulum clock, where a mechanical resonator generates time dependence. At the same time, the ticks are registered by monitoring the induced oscillations in the electronic current. This platform is sufficient to observe trade-offs between entropy production and precision of quantum clocks while probing underlying temporal correlations that have largely been neglected in the literature on quantum clocks.

This boom in microscopic physics also led to excitement in other fields of physics. The Mpemba effect is one such example of a phenomenon that has recently been rejuvenated by research in microscopic physics. A longstanding piece of folklore in many cultures, the idea that hot water freezes faster than cold water has generated has spawned arguments among physicists about whether the effect is merely an old wives tale. However, in 2017, Lu and Raz described how a Mpemba effect can occur in classical microscopic system by analysing overlaps between initial states and the eigenmodes of the dynamics [14]; this was soon after experimentally verified in colloidal systems [18] and has contributed to an explosion of interest in the Mpemba effect.

The interest in the topic can be demonstrated by a simple arXiv search. While throughout 2020, there were four submissions to the arXiv repository with the keyword Mpemba, in 2024, there were fifty-four submissions. The Mpemba effect has been demonstrated in a diverse range of physical systems such as clathrate hydrates [26], polymers [27], mag-

netic alloys [28], carbon nanotube resonators [21], granular gases [29] and dilute atomic gases [30]. The Mpemba effect has spread into various subfields, with Mpemba-like effects being recorded in random quantum circuits [31], symmetry restoration [32–36] and systems coupled to squeezed thermal reservoirs [37]. While these aspects are fascinating, little work has been done to set the quantum Mpemba effect in a thermodynamic framework. In this thesis, we discuss the Mpemba effect in quantum Davies maps, using the non-equilibrium free energy as a thermodynamic distance measure for the Mpemba effect in quantum systems, connecting this measure to the classical framework.

We introduce the ergotropic Mpemba effect to help preserve work stored (or, more accurately, slow down discharging) in continuous variable systems. By taking the simple model of an oscillator weakly coupled to a thermal bath, we find that storing work through displacing the initial state results in an exponential slowdown of the ergotropic dissipation relative to squeezing the initial state.

We start this thesis by reviewing some background theory of quantum systems, focusing on open quantum systems, thermodynamics and phase space representations of quantum systems. We will also briefly overview stochastic calculus and the Fokker-Planck equation. Chapter 2 discusses previous experimental and theoretical work in electromechanical systems and introduces our toy model. In Chapter 3, we investigate the toy model through the lens of work extraction. We find that self-sustained oscillations occur when the coupling between the system and the leads has energy dependence; we also find that the ergotropy acts as an order parameter for the self-sustained oscillation transition. We then probe the electric currents through the oscillator and propose to use the system as a quantum clock in Chapter 4. This model goes beyond previous analyses of quantum clocks, as the clockwork is based on a continuous degree of freedom. While we see a trade-off between the precision and dissipation in this model, the ticks are temporally correlated, requiring more robust tools to analyse the clock’s behaviour.

In Chapter 5, we investigate the thermodynamics of the Mpemba effect. We find that a strong Mpemba effect can arise during the thermalization of a quantum state governed by a Davies map, by exploiting the block structure of the generator. If a complex eigenvalue pair describes the spectral gap, a unitary can always be defined, which removes the over-

lap between the initial state and the slowest decaying eigenmode. This rotated initial state has a higher non-equilibrium free energy, and we argue through thermodynamic arguments that this constitutes a genuine Mpemba effect. Finally, Chapter 6 merges electromechanics and the Mpemba effect. We perform an eigenfunction decomposition of both a classical oscillator undergoing dissipation when coupled to a heat bath and a quantum oscillator thermalising via a Davies map. We then restrict our analysis to Gaussian states, finding an ergotropic Mpemba effect where states “charged” using displacement operators dissipate stored ergotropy slower than states “charged” via squeezing operations.

Throughout this thesis, we will work in units $\hbar = k_b = 1$.

Chapter 1

Theory of Quantum Systems

This section will outline the background theory necessary for understanding the results in Chapters 2-6. We begin by discussing open quantum systems - outlining the derivation of the Lindblad master equation, and touching on other techniques used to study these systems. We will briefly introduce quantum thermodynamics, focusing primarily on work extraction via the ergotropy and non-equilibrium free energy. We will then outline the fundamentals of stochastic calculus and Fokker-Planck equations — methods used to describe stochastic processes. Finally, we will briefly discuss the thermodynamics of timekeeping, using an example of a classical pendulum clock.

1.1 Open Quantum Systems

The evolution of a closed quantum system is described in density matrix formalism by the von Neumann equation

$$\frac{d\hat{\rho}(t)}{dt} = -i[\hat{H}(t), \hat{\rho}(t)], \quad (1.1)$$

where $\hat{\rho}(t)$ denotes the density matrix describing the state of the system, $\hat{H}(t)$ represents the Hamiltonian of the system, which may have time dependence and $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ denotes the commutator. Eq. 1.1 describes the unitary evolution of a closed quantum system. However, in physical manifestations, systems of interest are generally, to some extent, coupled to an external environment (commonly referred to as a bath, lead or reservoir). This environment will affect the system's evolution, often leading to effects such as decoherence. These systems are known as open quantum systems. They are significant as they allow the construction of a framework of system-environment interactions and how quantum systems

are measured. While Eq. 1.1 describes the evolution of the extended system (system plus environment), due to the size of the environment, numeric calculations using Eq. 1.1 are simply intractable. Approximations are necessary to encapsulate the system evolution — one of the most commonly used are GKSL master equations.

GKSL Master Equation

GKSL master equations, named after Gorini, Kossakowski, Sudarshan and Lindblad, often shortened to Lindblad master equations for brevity, are a class of master equations originally used in quantum optics but have since been seen widespread use throughout quantum research to describe open quantum system evolution. The mathematical foundations were originally described by Gorini, Kossakowski and Sudarshan [38] and Lindblad [39] independently. Such master equations define a dynamical semigroup acting on the system — $\mathcal{N}_t \rho(0)$ — where $\mathcal{N}_t$ is a convex-linear, completely positive and trace-preserving quantum operation. These operations obey the semi group properties $\mathcal{N}_t \mathcal{N}_s = \mathcal{N}_{t+s}$ for $t, s > 0$ and $\text{Tr}(\mathcal{N}_t \hat{\rho}) \hat{A}$ is a continuous function of t for any state matrix $\hat{\rho}$ and Hermitian operator $\hat{A}$.

Multiple derivations of the Lindblad master equation exist [40–44]; here we will follow the microscopic derivation as outlined in Milburn and Wiseman [40]. Consider a quantum system evolving in the Schrödinger picture

$$\dot{\hat{\rho}}_{\text{tot}} = -i [\hat{H}_S + \hat{H}_E + \hat{V}, \hat{\rho}_{\text{tot}}(t)], \quad (1.2)$$

here $\hat{\rho}_{\text{tot}}$ is the total system and bath state, $\hat{H}_S$ is the system Hamiltonian, $\hat{H}_E$ is the environment Hamiltonian and $\hat{V}$ is the interaction between the system and bath. Moving to the interaction picture Eq. 1.2 becomes

$$\dot{\hat{\rho}}_{\text{tot,IP}}(t) = -i [\hat{V}_{\text{IP}}(t), \hat{\rho}_{\text{tot,IP}}(t)], \quad (1.3)$$

where $\hat{V}_{\text{IP}}(t) = \exp(i(\hat{H}_S + \hat{H}_E)t) \hat{V} \exp(-i(\hat{H}_S + \hat{H}_E)t)$ is the system bath coupling in the interaction picture and $\hat{\rho}_{\text{tot,IP}}(t) = \exp(-i(\hat{H}_S + \hat{H}_E)t) \hat{\rho}_{\text{tot}} \exp(i(\hat{H}_S + \hat{H}_E)t)$ is the density matrix in the interaction picture. Formally integrating Eq. 1.3:

$$\hat{\rho}_{\text{tot,IP}}(t) = \hat{\rho}_{\text{tot,IP}}(0) - i \int_0^t dt_1 [\hat{V}_{\text{IP}}(t_1), \hat{\rho}_{\text{tot,IP}}(t_1)]. \quad (1.4)$$

We will drop the IF subscripts on the density matrix and the interaction operator to simplify notation. Substituting this back into the equation of motion for the total system and tracing out the environmental degrees of freedom results in an equation of motion for the system

$$\dot{\hat{\rho}}(t) = -i\text{Tr}_E \left(\left[\hat{V}(t), \hat{\rho}_{\text{tot}}(0) \right] \right) - \int_0^{t_1} dt_1 \text{Tr}_E \left(\left[\hat{V}(t), \left[\hat{V}(t_1), \hat{\rho}_{\text{tot}}(t_1) \right] \right] \right). \quad (1.5)$$

This is a second-order expansion in the system-bath coupling. While this equation is exact, there is still implicit dependence on the total density matrix due to the presence of $\hat{\rho}_{\text{tot}}(t_1)$ in the integral. Eq. 1.5 is typically brought to second order as it is often the lowest-order term, giving a non-vanishing contribution to the final master equation. We can now introduce our first assumption of the derivation — that there are no initial correlations between the system and the environment, i.e. the initial state can be written as a product state between the system and the bath. As the system and the bath are interacting, one would expect a build-up of correlations between the bath and the system; however, if the timescale of relaxation of the bath (τ_B) and timescale of relaxation of correlations (τ_{SB}) are much smaller than the typical timescale of the system (τ_S) then one can consider this assumption satisfied. The interaction Hamiltonian can then split into two parts, $\hat{V}(t) = \hat{V}_S(t) + \hat{V}_{SE}(t)$, where $\hat{V}_S(t)$ acts non-trivially on the system Hilbert space and where $\text{Tr} \left[\hat{V}_{SE}(t) \hat{\rho}_{\text{tot}}(0) \right] = 0$.

We now introduce our second assumption, known as the Born approximation — as the system only weakly affects the bath, it is permissible to replace $\hat{\rho}_{\text{tot}}(t_1)$ by $\hat{\rho}(t_1) \otimes \hat{\rho}_E(0)$. We will use another approximation to generate a time-convolutionless master equation; we replace $\hat{\rho}(t_1)$ with $\hat{\rho}(t)$; this is justified if the integrand is small except in the region $t_1 \approx t$. This occurs if the system couples equally to many energy levels - i.e. a wideband approximation. Putting these assumptions together, one arrives at the time-convolutionless master equation

$$\dot{\hat{\rho}}(t) = -i \left[\hat{V}_S(t), \hat{\rho}(t) \right] - \int_0^t dt_1 \text{Tr}_E \left(\left[\hat{V}(t), \left[\hat{V}(t_1), \hat{\rho}(t) \otimes \rho_E(0) \right] \right] \right). \quad (1.6)$$

This equation is not necessarily a Markovian master equation, as it has time-dependent coefficients. To form a Markov master equation, one needs one additional approximation, replacing the lower bound in the integral of Eq. 1.6 with $-\infty$. This again uses the fact that the integrand is sharply peaked at $t_1 \approx t$. Together, these two approximations are known as the Markov approximation. Markovianity will be discussed in more detail in Section 1.3.

This generates the Redfield master equation [45]

$$\dot{\hat{\rho}}(t) = -i[\hat{V}_S(t), \hat{\rho}(t)] - \int_{-\infty}^t dt_1 \text{Tr}_E \left([\hat{V}(t), [\hat{V}(t_1), \hat{\rho}(t) \otimes \rho_E(0)]] \right). \quad (1.7)$$

This equation does not necessarily preserve the positivity of the system and warrants one more approximation before expressing it in GKSL form — the rotating wave or secular approximation. This approximation is performed to remove terms in $\hat{V}$ that rotate so quickly that they average out to zero on a timescale of interest in the system. Applying this approximation and performing algebraic manipulations [40], one acquires the master equation in Lindblad form

$$\dot{\hat{\rho}}(t) = \mathcal{L}\hat{\rho}(t) = -i[\hat{H}, \hat{\rho}(t)] + \sum_{k=1}^K \mathcal{D}[\hat{L}_k] \hat{\rho}, \quad (1.8)$$

where we have defined the dissipation superoperator as $\mathcal{D}[\hat{A}] = \hat{A}\hat{\rho}\hat{A}^\dagger - 1/2\{\hat{A}^\dagger\hat{A}, \hat{\rho}\}$, where $\hat{A}_k$ are the Lindblad jump operators related to the interaction term $\hat{V}$.

In this thesis, Lindblad master equations are considered in Chapter 5, where we consider Davies maps, a form of GKSL master equation describing the thermalisation of quantum systems defined by Davies [46]. We will also briefly consider Lindblad master equation and unravellings of master equations in the Summary and Outlook section of Chapter 4 to move away from the quasi-adiabatic limit considered in Chapter 2.

Other Methods

The GKSL master equation is an extremely useful tool for describing the evolution of quantum systems. However, due to approximations such as the weak coupling limit, the Markov approximation and the rotating wave approximation, the GKSL master equation is limited. For example, in interacting Fermionic systems, the coupling between the system and the environment can be strong, violating the weak coupling approximation, requiring using Green's functions techniques [47] to analyse the system. In chapter 3, we consider the solutions to such a system. Other approximations, such as the rotating wave approximation, may be invalid for systems such as in quantum Brownian motion; in these systems, the dynamics of the reduced system are slow compared to the correlation times of the environment; see chapter 6 for a brief outline.

1.2 Quantum Thermodynamics

Thermodynamics traces its origins to the Industrial Revolution. While 18th-century Britain was the world's foremost industrial power, the foundations of thermodynamics were not laid in Britain but in France [48]. In Britain, coal was abundant, fueling the engines that drove the industrial boom, making even inefficient engines profitable. In contrast, coal was far scarcer south of the Channel, which spurred a pressing need to improve engine efficiency and understand the physical limits of their performance. This drive to understand the fundamental limits of efficiency led to the work of Sadi Carnot [49], whose insights laid the foundations of thermodynamics as the study of the relationships between heat and work and a rejection of the fictitious caloric theory [50].

These ideas spread and were expanded on by notable figures such as Joule [51] and Thomson [52]. Through the past two centuries, thermodynamics has flourished and developed through its restructuring based on statistical mechanics [53]. Thermodynamics has been modified through the twentieth century by its coupling with information theory and quantum physics, and it continues to be understood further today. Through this thesis, we will use thermodynamics through the lens of work extraction; because of this, we will focus primarily on the first and second laws of thermodynamics and specifically on the ergotropy and non-equilibrium free energy.

1.2.1 The First Law and Ergotropy

The first law states

- 1st law: Energy is a globally conserved quantity, and energy changes can be decomposed into two forms: work and heat.

This can be stated mathematically as

$$\Delta E = \delta Q + \delta W, \quad (1.9)$$

where ΔE is the change in energy, δQ is the heat generation during the process, and δW is the work done on the system. The energy of a quantum system is given as $E = \text{Tr} \{ \hat{\rho} \hat{H} \}$. The concepts of heat and work are more difficult to define in quantum systems. One can

decompose the energy change in the following way

$$\frac{dE}{dt} = \frac{d}{dt} \text{Tr} \{ \hat{\rho}(t) \hat{H}(t) \} = \text{Tr} \left\{ \frac{d}{dt} [\hat{\rho}(t)] \hat{H}(t) \right\} + \text{Tr} \left\{ \hat{\rho}(t) \frac{d}{dt} [\hat{H}(t)] \right\}. \quad (1.10)$$

The first term on the r.h.s. is related to changes in the system's configuration and can be associated with the heat produced in a process — δQ . The second term relates to the energy changes due to changes in external constraints and can be associated with the work done — δW . Considering a closed system evolving unitarily via Eq. 1.1, δQ is zero and no heat is generated. One can, however, input and extract work from the quantum system by modulating the system Hamiltonian.

Consider some Hamiltonian, $\hat{H}(t) = \hat{H} + \hat{V}(t)$, where $\hat{V}(t)$ is some externally controlled parameter that accounts for work transfer. We consider a process to be cyclic if the external control parameter vanishes at some initial time and some final time, i.e. $\hat{V}(0) = 0$ and $\hat{V}(\tau) = 0$ for some later time τ . A quantum state, $\hat{\rho}$ is passive if no work can be extracted from the system via a cyclic control protocol ie

$$W = \text{Tr} \{ \hat{H} \hat{\rho} \} - \text{Tr} \{ \hat{H} \hat{U} \hat{\rho} \hat{U}^\dagger \} \leq 0 \quad \forall \hat{U}. \quad (1.11)$$

A state that is not passive is known as a nonpassive. The total amount of work extractable from such a system via cyclic variation of the Hamilton parameters is known as the ergotropy — $\mathcal{W}_E(\hat{\rho})$ — and is given by the difference between the energy of the state and its unitarily connected passive state, equivalently

$$\mathcal{W}_E(\hat{\rho}) := \text{Tr} [\hat{H} \hat{\rho}] - \min_{U \in \mathcal{U}} \{ \text{Tr} [\hat{H} \hat{U} \hat{\rho} \hat{U}^\dagger] \}, \quad (1.12)$$

where $\mathcal{U}$ is the set of unitary operators acting on $\hat{\rho}$. The ergotropy was coined by Allahverdyan and coworkers [22] in 2004. The optimal unitary extracting the most work diagonalises the system's state in the Hamiltonian eigenbasis and orders the eigenvalues of $\hat{\rho}$ into non-increasing order. If the state of the system is given by $\hat{\rho} = \sum_k r_k |r_k\rangle \langle r_k|$, where $r_k \geq r_{k+1}$, and the Hamiltonian is expressed in its energy eigenbasis as $\hat{H} = \sum \epsilon_k |\epsilon_k\rangle \langle \epsilon_k|$ with $\epsilon_k \leq \epsilon_{k+1}$ then the ergotropy of the system is given by

$$\mathcal{W}_E = \sum_{j,k} r_j \epsilon_k (|\langle r_j | \epsilon_k \rangle|^2 - \delta_{jk}). \quad (1.13)$$

If $[\hat{\rho}, \hat{H}] = 0$ and $r_k \geq r_{k+1}$ then the state is passive, a thermal state is always passive, but not all passive states are thermal [54].

The ergotropy is primarily seen in the context of batteries — a physical system that stores energy [55, 1] — as a useful tool to separate useful energy that can be extracted as work from the total energy of the system. The ergotropy can be divided into coherent and incoherent parts [56] and extended to open quantum systems [57, 58]. In Chapter 2, we consider an oscillator coupled to an electronic system - the oscillator acts as a flywheel storing energy similar to the oscillator in [59]. Ergotropy acts as a useful parameter for quantifying the amount of useful energy in the battery and defining the threshold for population inversion. In Chapter 6, we consider the ergotropy in an oscillator coupled to a thermal bath, investigating methods to slow down the dissipation of the ergotropy.

Finally, a note on the definition of the heat and work as outlined in Eq. 1.10. Classical thermodynamics is built upon taking the thermodynamic limit where the system size is infinite or at least large enough for fluctuations to be negligible. Quantum thermodynamics and much of modern classical thermodynamics analyse systems that are small enough for fluctuations to be significant [60, 61]. Due to these fluctuations, the work injected or extracted from a system can vary significantly. Tools such as the two-point measurement scheme [62] have been developed to understand the role of fluctuations in work and heat.

1.2.2 The Second Law and the Non-Equilibrium Free Energy

The second law states

- 2nd law: No work can be extracted from an equilibrium closed system via the cyclic variation of an external parameter.

This is known as the Thomson formulation of the second law. As noted above, no work can be extracted via cyclic permutation of the Hamiltonian. However, the state generated by having N copies of the passive state is not necessarily passive with respect to the Hamiltonian $\hat{H}^N = \sum_{i=1}^N \hat{H}_i$ where $\hat{H}_i = \mathbb{1}_1 \otimes \cdots \otimes \mathbb{1}_{i-1} \otimes \hat{H}_i \otimes \mathbb{1}_{i+1} \otimes \cdots \otimes \mathbb{1}_N$. A passive state that remains passive given any number of copies of the state is known as a completely passive state, in fact the thermal state and ground states are the only states that are completely passive [55, 54]. The concept of complete passivity was used in the 1970s to define the

formula for thermal equilibrium states in quantum systems using the second law [63]. The thermal state of a quantum system is an extension of the Gibbs state

$$\hat{\rho}_\beta = \frac{1}{Z} e^{-\beta \hat{H}}, \quad Z = \text{Tr} \{ e^{-\beta \hat{H}} \}, \quad (1.14)$$

where Z is the partition function. The thermal state is the unique state that maximises the entropy of a quantum system, given a system's energy. The Gibbs state is parametrised by the inverse temperature β . The entropy of a quantum system arises from information theory's Shannon entropy, $S(\vec{p}) = -\sum p_i \ln(p_i)$, where $\vec{p}$ is some probability distribution with elements p_i . The Von Neumann entropy gives the quantum counterpart

$$H(\hat{\rho}) = -\text{Tr} \{ \hat{\rho} \ln(\hat{\rho}) \}. \quad (1.15)$$

We can extend our description of extractable work from the previous subsection to open quantum systems. The evolution of such systems is not unitary but rather specified via a CPTP generator — see Section 1.1. Entropy can be generated in such a setup in two ways: via energy entering and exiting the thermal reservoir or through changes in the entropy of the system of interest. The total entropy production of the extended open system is given as

$$\Sigma = \Delta S + \frac{Q}{T} \geq 0, \quad (1.16)$$

The system's entropy can either increase or decrease. However, the total entropy of the combined system and bath must increase. The time derivative of the entropy production is given by

$$\dot{\Sigma}(t) = \dot{S}(t) + \dot{\Phi}(t). \quad (1.17)$$

Where the entropy rate of change has been divided into two parts, $\dot{S}$ is the entropy production rate of the system, and $\dot{\Phi}$ is the entropy flux from the system to the environment [64], the time derivative of the second term on the RHS of Eq. 1.16. For an open quantum system weakly coupled to a thermal bath, the entropy production rate is given by Spohn [65]

$$\dot{\Sigma} = -\frac{d}{dt} S(\hat{\rho}(t) \| \hat{\tau}), \quad (1.18)$$

where $\hat{\tau}$ is the thermal fixed point of the evolution and $S(\hat{\rho}||\hat{\sigma})$ is the relative entropy between the state $\hat{\rho}$ and the state $\hat{\sigma}$:

$$S(\hat{\rho}||\hat{\sigma}) = \text{Tr} \{ \hat{\rho} (\log(\hat{\rho}) - \log(\hat{\sigma})) \}. \quad (1.19)$$

The relative entropy is always positive. The relative entropy is monotonical under any CPTP map due to the data processing inequality

$$S(\mathcal{E}(\hat{\rho})||\mathcal{E}(\hat{\sigma})) \leq S(\hat{\rho}||\hat{\sigma}), \quad (1.20)$$

this inequality ensures a positive entropy production rate in thermal relaxation. Now, consider extracting work from the extended system-bath framework. The number of allowed operations on the system is not restricted to unitaries but rather a much broader class of operations - thermal operations. The total amount of work extractable given such constraints is given by the difference between the non-equilibrium free energy of the state $\hat{\rho}$ and the equilibrium free energy of the corresponding thermal state $\hat{\tau}_\beta$ [66]

$$\mathcal{W}_F(\hat{\rho}) = F_{\text{neq}}(\hat{\rho}) - F_{\text{eq}}(\hat{\tau}_\beta), \quad (1.21)$$

where the equilibrium free energy is given by $F_{\text{eq}}(\hat{\tau}_\beta) = \beta^{-1} \ln(Z)$ where Z is the partition function defined above. The non-equilibrium free energy is given as

$$F_{\text{neq}} = E(\hat{\rho}) - \beta^{-1} S(\hat{\rho}). \quad (1.22)$$

As the non-equilibrium free energy encapsulates all unitary operations, the non-equilibrium free energy is an upper bound to the ergotropy of a quantum system

$$\mathcal{W}_F \geq \mathcal{W}_E. \quad (1.23)$$

The non-equilibrium free energy can also be related to the relative entropy of a system [23]

$$\begin{aligned}
 F_{\text{neq}} &= E(\hat{\rho}) - \beta^{-1} S(\hat{\rho}) \\
 &= -\beta^{-1} \left[\text{Tr} \left\{ \hat{\rho} \log \left(e^{-\beta \hat{H}} \right) \right\} - \text{Tr} \left\{ \hat{\rho} \log (\hat{\rho}) \right\} \right] \\
 &= -\beta^{-1} \left[\text{Tr} \left\{ \hat{\rho} \log (\tau) \right\} - \text{Tr} \left\{ \hat{\rho} \log (\hat{\rho}) \right\} \right] + \beta^{-1} \log \left(\frac{1}{\mathcal{Z}} \right) \\
 &= \beta^{-1} D(\hat{\rho} \| \hat{\tau}) - \mathcal{F}_{\text{eq}}.
 \end{aligned} \tag{1.24}$$

Where $E(\hat{\rho}) = \text{Tr} \{ \hat{H} \hat{\rho} \}$ is the internal energy of the system. Thus, the entropy production rate is given by

$$\dot{\Sigma} = -\beta \frac{d}{dt} F_{\text{neq}}. \tag{1.25}$$

In Chapter 3, we compare the extractable work using the non-equilibrium free energy to the work extractable using the ergotropy. In Chapter ?? the non-equilibrium free energy is used as a distance measure displaying the Mpemba effect in a thermalisation process. It is possible to divide the total entropy produced as [67, 68]

$$D(\hat{\rho}(t) \| \hat{\tau}_{\beta}) = \mathcal{P}(p(t) \| \hat{\tau}_{\beta}) + \mathcal{C}(\hat{\rho}(t)), \tag{1.26}$$

where $\mathcal{P}(p(t) \| t_{\beta}) = \sum_n p_n(t) \ln \frac{p_n(t)}{t_{\beta}^n}$ is the classical relative entropy of the time-evolving population vector and the thermal state and $\mathcal{C}(\hat{\rho}(t)) = S(\Delta(\hat{\rho}(t))) - S(\hat{\rho}(t))$ is known as the relative entropy of coherence where $\Delta(\hat{\rho}_s(t))$ is the state dephased in the systems' energy eigenbasis. In Chapter 5, we study how a rotation that exponentially speeds up the thermal relaxation affects the division of the entropy production.

1.3 Stochastic Calculus

In physics and everyday life, we often encounter random processes — whether predicting tomorrow's weather (in Ireland, rain is always a reliable guess) or modelling the flow of electrons in mesoscopic systems. Such randomness can arise from factors like incomplete information or quantum effects. These processes can be mathematically framed as stochastic processes, where time-dependent random variables influence the system's evolution. Stochastic calculus is the mathematical field that seeks to clarify otherwise uncertain systems, finding applications ranging from predicting stock prices in finance to modelling microscopic systems in physics. Understanding the erratic motion of colloidal particles

suspended in water was a key problem in developing this branch of mathematics. These particles experience random impacts from water molecules, resulting in erratic motion. The following simple equation describes this phenomenon [69]:

$$\frac{dv}{dt} + \gamma v = \Gamma(t). \quad (1.27)$$

Here v represents the particle's velocity, γ is a damping coefficient, and $\Gamma(t)$ is a rapidly varying and highly irregular term that introduces randomness into an otherwise deterministic system. This equation is known as a Langevin equation and has general form

$$\frac{dx}{dt} = A(x, t) + B(x, t)\zeta(t), \quad (1.28)$$

where $A(x, t)$ is a drift term, $B(x, t)$ is a diffusion term and $\zeta(t)$ is a noise term. $\zeta(t)$ is continuous in time and stationary, meaning its statistical properties are invariant to changes in time. The noise term here has zero mean, $\mathbb{E}(\zeta(t)) = 0$ — as any non-zero mean could be incorporated into the drift term. The noise is normalised such that its correlation function integrates to unity

$$\int_{-\infty}^{\infty} d\tau \mathbb{E}[\zeta(t)\zeta(t + \tau)] = 1. \quad (1.29)$$

To proceed, we now introduce the concept of Markovianity; the conditional probability of the system, x_n at some time t_n , only depends on the value of x_{n-1} at the previous time t_{n-1} . This assumption encodes memorylessness into a given process and was used in the derivation of the Lindblad master equation in Section 1.1, where the approximation was used to remove the dependence of the evolution of the state from its past. To ensure that Eq. 1.28 is Markovian we require

$$\mathbb{E}[\zeta(t)\zeta(t')] = \delta(t - t'). \quad (1.30)$$

This noise is known as white noise due to the fact its spectrum is flat. In the most general case, the noise would have some frequency dependence and be non-Markovian, known as coloured noise. Frequently, one assumes that the noise considered in Eq. 1.30 noise is Gaussian. This is often justified using the central limit theorem, where the microscopic fluctuations acting on the system of interest are uncorrelated and occur on a timescale much shorter than that of the relevant system dynamics, this assumption along with the requirement that $\zeta(t)$ is continuous warrant the use of Gaussian white noise. There are various examples

of processes that break these assumptions. Jump processes, which will be discussed later, make instant and discrete jumps; thus, they are not continuous. Other processes, such as Cauchy processes, have infinitesimal noise increments but are drawn from processes with a probability density with infinite variance; see Jacobs [70] for a more in-depth discussion.

Now, actually solving Eq. 1.28 is not trivial due to the noise term $\zeta(t)$. To tackle this problem, we introduce the Wiener process; we solve Eq. 1.28 using a finite difference method by taking x_0 and repeatedly adding Δx ,

$$x_n = x_0 + \sum_{i=0}^{n-1} A(x, t) \Delta t + \sum_{i=0}^{n-1} B(x, t) \Delta W(t), \quad (1.31)$$

where we have here defined the Wiener process $\Delta W(t)$. The Wiener process is the integral of the white noise process $\zeta(t)$ and has the paradoxical phenomenon that it is everywhere continuous yet nowhere differentiable. Each Wiener process is independent and has the same probability density

$$P(\Delta W(t)) = \frac{1}{\sqrt{2\pi\Delta t}} e^{-\frac{(\Delta W)^2}{2\Delta t}}, \quad (1.32)$$

i.e. has zero mean and a variance of Δt . We can take the continuum limit to form a stochastic differential equation, much like in deterministic difference equations. However, there is a subtlety in how this limit is defined and leads to two formalisms for stochastic differential equations.

Due to the Wiener increment not being a smooth function of time one can define separate interpretations of the Wiener process — Stratonovich and Ito integrals. Explicit details can be found in Gardiner [71] and Jacobs [70]. The Stratonovich integrals arise naturally when taking the limit of continuous coloured noise processes. The advantage of Stronovich integrals is that the normal chain rules of calculus apply; however, in the Stratonovich interpretation, the noise is not independent of the system, and the formalism loses the martingale properties of the Ito formalism.

Ito calculus, on the other hand, sacrifices the standard chain rule of calculus. This formalism will be the preferred formalism for this thesis. An Ito differential equation is given by

$$dx = A(x, t)dt + B(x, t)dW, \quad (1.33)$$

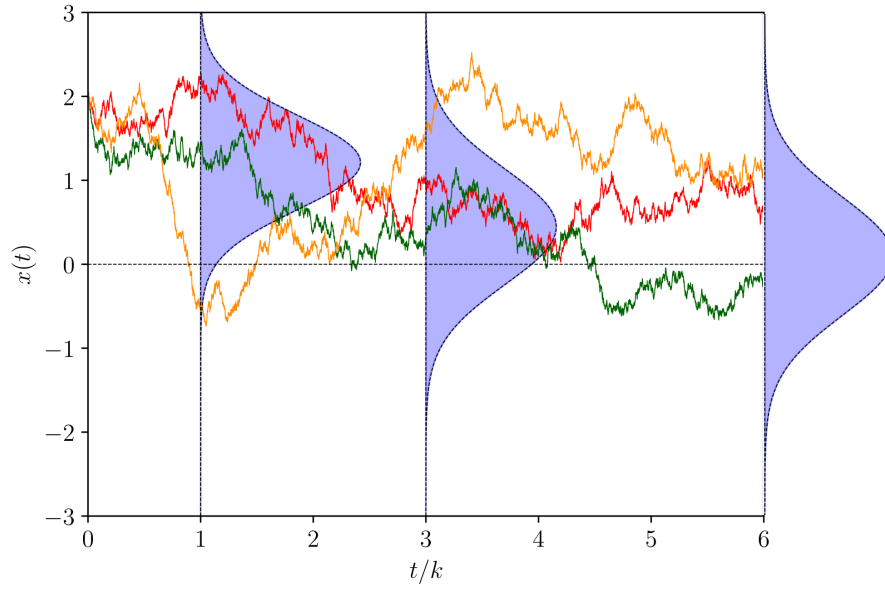


Figure 1.1: Plot highlighting sample trajectories of an Ornstein-Uhlenbeck process. The paths differ dramatically due to the Wiener noise acting on the process. The blue insets show the probability distributions of the process at different times.

where we have defined the Ito increment dW . The infinitesimal Wiener increment has mean zero and variance equal to dt . When using Ito's framework, one needs to consider two key formulae

$$(dW(t))^2 = dt, \quad (1.34)$$

$$(dW(t))^{N+2} = 0 \quad (N > 0). \quad (1.35)$$

Proofs for these formulae can be found in Gardiner [71]. Eq. 1.34 says that the integral of $(dW)^2$ is deterministic and naturally leads to the correction to the chain rule in Ito calculus. Consider a change of variable in an Ito stochastic differential equation from $x(t)$ defined in Eq. 1.33 to an arbitrary function $f[x(t)]$; the chain rule is given by Ito's lemma

$$df[x(t)] = \left\{ A(x, t)f'[x(t)] + \frac{1}{2}B(x, t)^2 f''[x(t)] \right\} dt + B(x, t)f'[x(t)]dW(t). \quad (1.36)$$

This differs from the normal chain rule due to the inclusion of the $f''[x(t)]$ term. This arises because $dW^2 = dt$, requiring an expansion of $df[x(t)]$ to second order.

To demonstrate the use of Ito calculus and Ito's lemma, we consider an Ornstein-Uhlenbeck

process

$$dx = -kxdt + \sqrt{D}dW(t), \quad (1.37)$$

this equation describes a particle under the influence of a linear drift term with a constant noise term. This equation can be solved using the following substitution

$$y = xe^{kt}, \quad (1.38)$$

and recalling Eq. 1.34

$$\begin{aligned} dy &= dx e^{kt} + dx e^{kt} + x de^{kt} \\ &= (-kxdt + \sqrt{D}dW(t)) e^{kt} dt + (-kxdt + \sqrt{D}dW(t)) e^{kt} + kxe^{kt} dt \\ &= (-kxdt + \sqrt{D}dW(t)) e^{kt} + kxe^{kt} dt \\ &= \sqrt{D}dW(t) e^{kt}, \end{aligned} \quad (1.39)$$

where any term higher order than dt vanishes. Integrating and resubstituting gives the result

$$x(t) = x(0)e^{-kt} + \sqrt{D} \int_0^t e^{-k(t-t')} dW(t'). \quad (1.40)$$

If the initial condition is Gaussian, then this evolution will preserve the Gaussianity of the system with a mean value

$$\mathbb{E}[x(t)] = \mathbb{E}[x(0)] e^{-kt}, \quad (1.41)$$

and variance

$$\text{Var}[x(t)] = \left(\text{Var}[x(0)] - \frac{D}{2k} \right) e^{-2kt} + \frac{D}{2k}. \quad (1.42)$$

In Fig. 1.1, we show sample trajectories of the OU process, demonstrating the fluctuating effect of the Wiener increment in calculations. In Section 1.4, we will solve this example using Fokker-Planck methods. We will expand on the Ornstein-Uhlenbeck example in Appendix A to construct a toy model to help understand the behaviour of electric currents in a nanomechanical setup. In Chapter 3, we construct a Langevin equation describing the equation of motion of an oscillator coupled to an electromechanical system and solve the equation using numerical methods.

Rather than looking at stochastic process trajectories, one can probe the probability dis-

tribution's evolution. Gardiner [71] outlines the derivation of the differential Chapman-Kolmogorov equation

$$\begin{aligned} \frac{\partial p(\mathbf{x}, t | \mathbf{x}_0, t_0)}{\partial t} = & - \sum_i \frac{\partial}{\partial x_i} [f_i(\mathbf{x}, t) p(\mathbf{x}, t | \mathbf{x}_0, t_0)] + \sum_{i,j} \frac{1}{2} \frac{\partial^2}{\partial x_i \partial x_j} [g_{ij}(\mathbf{x}, t) p(\mathbf{x}, t | \mathbf{x}_0, t_0)] \\ & + \int d\mathbf{z} [W(\mathbf{x} | \mathbf{z}, t) p(\mathbf{x}, t | \mathbf{x}_0, t_0) - W(\mathbf{z} | \mathbf{x}, t) p(\mathbf{x}, t | \mathbf{x}_0, t_0)]. \end{aligned} \quad (1.43)$$

Here $p(\mathbf{x}, t | \mathbf{x}_0, t_0)$ is the probability density at some time t conditioned on the initial density at time t_0 , the first term on the RHS of the first line is a continuous drift term, the second term is a diffusion term, the final terms on the second line are discontinuous jump terms. Setting $W = 0$ Eq. 1.43 becomes the Fokker Planck equation, which is the subject of Section 1.4, setting $f = g = 0$, Eq. 1.43 becomes a jump process which we will briefly discuss now.

1.3.1 Jump processes

Jump processes describe the evolution of a system characterised by discontinuous jumps described by the master equation

$$\frac{\partial p(\mathbf{x}, t | \mathbf{x}_0, t_0)}{\partial t} = \int d\mathbf{z} [W(\mathbf{x} | \mathbf{z}, t) p(\mathbf{x}, t | \mathbf{x}_0, t_0) - W(\mathbf{z} | \mathbf{x}, t) p(\mathbf{x}, t | \mathbf{x}_0, t_0)]. \quad (1.44)$$

Here $W(\mathbf{x} | \mathbf{z}, t)$ denotes the probability of a jump occurring between state $\mathbf{z}$ and $\mathbf{x}$ at a time t . The simplest case of a jump process is the telegraph process, where a signal fluctuates between being on and off. The master equation for a telegraph process is given as

$$\frac{d}{dt} \begin{pmatrix} p_{\text{on}} \\ p_{\text{off}} \end{pmatrix} = \begin{pmatrix} -\lambda_1 & \lambda_2 \\ \lambda_1 & -\lambda_2 \end{pmatrix} \begin{pmatrix} p_{\text{on}} \\ p_{\text{off}} \end{pmatrix}, \quad (1.45)$$

where $p_{\text{on/off}}$ is the probability of the signal being on/off and $\lambda_{1/2}$ are the jump rates from $p_{\text{on/off}}$ to $p_{\text{off/on}}$. One can also tackle jump processes using trajectory methods where instead of defining a Wiener increment, one instead defines a Poisson increment dN , which takes the value 1 with probability λdt where λ is the jump rate and 0 with probability $1 - \lambda dt$. In Appendix A, we construct a bistable process to better understand the behaviour of the clock in Chapter 4.

1.4 The Fokker-Planck Equation

Setting the jump terms to zero in Eq. 1.43 one generates the Fokker-Planck equation; in a one-dimensional system, the Fokker Planck equation is given as

$$\frac{\partial p(x, t|x_0, t_0)}{\partial t} = -\frac{\partial}{\partial x} (A(x, t)p(x, t|x_0, t_0)) + \frac{1}{2} \frac{\partial^2}{\partial x^2} (B(x, t)^2 p(x, t|x_0, t_0)) = L_{\text{FP}} p(x, t|x_0, t_0), \quad (1.46)$$

here $A(x, t)$ denotes the drift coefficient, $B(x, t)$ denotes the diffusion coefficient, L_{FP} is the Fokker Planck generator. This Fokker-Planck equation is equivalent to a one-dimensional Ito differential equation. Consider a one-dimensional Ito stochastic process

$$dx = A(x, t)dt + B(x, t)dW, \quad (1.47)$$

as before dW denotes the Wiener increment. We calculate the expectation value of some function $f(x)$ of the probability distribution given by $p(x, t|x_0, t_0)$. Using Ito's Lemma, the differential equation describing $f(x)$ is given as

$$df(x) = \left\{ A(x, t) \frac{\partial f(x)}{\partial x} + \frac{1}{2} B(x, t) \frac{\partial^2 f(x)}{\partial x^2} \right\} dt + B(x, t) \frac{\partial f(x)}{\partial x} dW. \quad (1.48)$$

We take the expectation value of both sides of the equation, noting that $\langle B(x, t) \frac{\partial f(x)}{\partial x} dW \rangle = 0$:

$$\frac{d\langle f(x) \rangle}{dt} = \left\langle \left\{ A(x, t) \frac{\partial f(x)}{\partial x} + \frac{1}{2} B(x, t) \frac{\partial^2 f(x)}{\partial x^2} \right\} \right\rangle. \quad (1.49)$$

One can find the evolution of the expectation value of $f(x)$ through the probability distribution conditioned on the initial distribution at t_0 ,

$$\langle f(x) \rangle = \int dx f(x) p(x, t|x_0, t_0), \quad (1.50)$$

taking the derivative with respect to time on both sides of the equation, one finds

$$\int dx f(x) \frac{\partial p(x, t|x_0, t_0)}{\partial t} = \int dx \left\{ A(x, t) \frac{\partial f(x)}{\partial x} + \frac{1}{2} B(x, t) \frac{\partial^2 f(x)}{\partial x^2} \right\} p(x, t|x_0, t_0), \quad (1.51)$$

One can integrate each term on the right-hand side of the equation by parts; the first term is given as

$$\int dx A(x, t) \partial_x f p(x, t|x_0, t_0) = f(x, t) A(x, t) p(x, t|x_0, t_0) - \int dx f(x, t) \partial_x [A(x, t) p(x, t|x_0, t_0)], \quad (1.52)$$

where the first term on the r.h.s. is a surface term that, for most purposes, goes to zero; for example, in a system with infinite boundary conditions, normalisation of the distribution ensures that this term is zero. A similar integration by parts can be performed on the diffusion term, excluding the negligible surface terms this is given by

$$\int dx B(x, t) \frac{\partial^2 f(x)}{\partial x^2} p(x, t|x_0, t_0) = \int dx f(x, t) \frac{\partial^2}{\partial x^2} [B(x, t)^2 p(x, t|x_0, t_0)], \quad (1.53)$$

putting everything together, one finds

$$\int dx f(x) \frac{\partial p(x, t|x_0, t_0)}{\partial t} = \int dx f(x) \left\{ -\frac{\partial}{\partial x} [A(x, t) p(x, t|x_0, t_0)] + \frac{1}{2} \frac{\partial^2}{\partial x^2} [B(x, t)^2 p(x, t|x_0, t_0)] \right\}, \quad (1.54)$$

recalling that $f(x)$ is some arbitrary function, we arrive at the one-dimensional Fokker-Planck equation

$$\frac{\partial p(x, t|x_0, t_0)}{\partial t} = -\frac{\partial}{\partial x} [A(x, t) p(x, t|x_0, t_0)] + \frac{1}{2} \frac{\partial^2}{\partial x^2} [B(x, t)^2 p(x, t|x_0, t_0)]. \quad (1.55)$$

One can consider various boundary conditions depending on the system of interest; we will only consider systems with infinite boundary conditions throughout this thesis.

The Fokker-Planck is often formally derived as a second-order Kramers-Moyal expansion of the transition probability of a Markovian stochastic variable, i.e.

$$p(x, t + \tau) = \int dx' p(x, t + \tau|x', t) p(x', t), \quad (1.56)$$

the derivation is not relevant to this work; we will just quote the result [72]

$$\frac{\partial p(x, t)}{\partial t} = \sum_{n=1}^{\infty} \left(-\frac{\partial}{\partial x} \right)^n D^{(n)}(x, t) p(x, t). \quad (1.57)$$

where $D(x, t)$ are related to the moments of the conditional probability distribution $P(x, t + \tau|x', t)$. If this expansion is taken to the second-order term, one retrieves the Fokker-Planck

equation [73]. The Fokker-Planck equation can be extended to many-dimensional systems

$$\frac{dp(\vec{x}, t | \vec{x}_0, t_0)}{dt} = - \sum_i \frac{\partial}{\partial x_i} [A_i(\vec{x}, t) p(\vec{x}, t)] + \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} [B_{ij}(\vec{x}, t) p(\vec{x}, t)]. \quad (1.58)$$

One can define a probability current for a corresponding Fokker-Planck equation as

$$J(\vec{x}, t) = \sum_i [A_i(\vec{x}, t) p(\vec{x}, t)] - \frac{1}{2} \sum_{i,j} \frac{\partial}{\partial x_j} [B_{ij}(\vec{x}, t) p(\vec{x}, t)]. \quad (1.59)$$

The system is said to obey detailed balance if the stationary solution has vanishing probability currents. Detailed balance is a microscopic property of stochastic systems where, for the stationary solution, each possible transition balances with the reverse transition. The criterion for detailed balance is

$$p(\vec{x}, \tau; \vec{x}', 0) p_{ss}(\vec{x}') = p(\epsilon(\vec{x}'), \tau; \epsilon(\vec{x}), 0) p_{ss}(\vec{x}) \quad (1.60)$$

here p_{ss} is the stationary solution and $\epsilon(\vec{x})$ transforms the variable to its time-reversed counterpart (i.e. $a_i \rightarrow a_i$ if the variable a_i is even under time reversal or $a_i \rightarrow -a_i$ if the variable a_i is odd under time reversal). An example of a system that obeys detailed balance is any classical system whose fluctuations are generated from an interaction with a thermal bath. Such systems obey a fluctuation-dissipation theorem that relates the fluctuation magnitude to the damping magnitude. For example, for an oscillator coupled to a thermal bath, the dissipation and fluctuation terms are related by

$$D = \frac{2\Gamma T}{m} \quad (1.61)$$

where D is the diffusion strength, Γ is the dissipation strength, T is the temperature of the thermal bath and m is the mass of the oscillator.

As done in the previous section, we will take the example of an Ornstein-Uhlenbeck process to show the Fokker-Planck equation. The Ornstein-Uhlenbeck Fokker Planck equation is given by

$$\frac{\partial p(x, t | x_0, t_0)}{\partial t} = k \frac{\partial}{\partial x} (x p(x, t | x_0, t_0)) + \frac{1}{2} D \frac{\partial^2}{\partial x^2} p(x, t | x_0, t_0). \quad (1.62)$$

This equation can be solved by converting the distribution to the Fourier domain and using the method of characteristics. Assuming a deterministic initial state, the probability of the

system evolves according to

$$p(x, t|x_0, t_0) = \sqrt{\frac{k}{2\pi D(1 - e^{-2kt})}} \exp\left[-\frac{k(x - e^{-kt}x_0)^2}{2D(1 - e^{-2kt})}\right]. \quad (1.63)$$

This is a Gaussian distribution with mean given as $\mathbb{E}[p(x, t|x_0, t_0)] = e^{-kt}x_0$ and variance $\text{Var}[p(x, t|x_0, t_0)] = (1 - e^{-2kt})\frac{D}{2k}$. In general, finding analytic solutions to the Fokker-Planck equation is extremely difficult — numerous numerical techniques for solving Fokker-Planck equations have been constructed. One can convert the Fokker Planck equation into an Ito stochastic differential equation and perform a Monte Carlo simulation [74]. Often, these trajectories are calculated using stochastic Euler or Runge-Kutta-like methods [70]. These techniques have the advantage of being highly adaptable to parallelisation and useful at tackling complex systems, however often need large trajectory sizes to produce the probability distribution function, especially in the tails [75]. In Chapter 3, we solve a Fokker-Planck equation using a trajectories approach.

One can also use finite element approaches, discretising the spatial dimensions into finite elements; the element-level equations are then solved using a finite timestep [76, 77]. Other methods include cell-mapping [78] and finite difference methods [79]. Fokker Planck equations have a wide range of uses in many different systems, including meteorological forecasting [80], plasma physics [74], options pricing [81] and motion planning in robotics [82]. In Chapter 2, we derive a Fokker-Planck equation to describe an oscillator in a nanoelectromechanical setup. In Chapter 6, we use a Fokker-Planck equation to describe the evolution of a harmonic oscillator coupled to a quantum bath.

1.5 Quasi Probability Distributions

Often, it is difficult to interpret quantum mechanics through the density matrix formalism alone. Through the past century, researchers in quantum optics have developed alternative methods to work with and visualise quantum systems; one such technique is phase space representations. These techniques allow quantum mechanical systems to be described in the language of classical phase space, mapping quantum states onto quasi-probability distributions. To motivate the use of quasi-probability distributions, we express the expectation

value of some operator as

$$\langle \hat{F} \rangle = \text{Tr} \{ \hat{\rho} \hat{F} \} = \int w(\alpha, \alpha^*) f(\alpha, \alpha^*) d\alpha d\alpha^* \quad (1.64)$$

where $w(\alpha, \alpha^*)$ plays the role of the quasi-probability distribution on some plane α, α^* , $\hat{F}$ is some operator we wish to calculate the expectation number of and $f(\alpha, \alpha^*)$ is a function that refers to the operator $\hat{F}$. Due to the fact that, in general, quantum operators do not commute, whereas the c-number analogues of quantum operators do, there are many functions, $w(\alpha, \alpha^*)$ and $f(\alpha, \alpha^*)$, that can be associated with $\langle \hat{F} \rangle$.

Consider, for example, the bosonic operator $\hat{F} = \hat{a}^\dagger \hat{a}$ where $\hat{a}$ is the annihilation operator. Writing the operator in the form $\hat{F} = \hat{a}^\dagger \hat{a}$ is known as normal operator ordering as all the annihilation operators are on the right-hand side of all the creation operators. Using the bosonic commutation relations $[\hat{a}, \hat{a}^\dagger] = 1$, the operator $\hat{F}$ can be rewritten as $\hat{F} = \hat{a} \hat{a}^\dagger - 1$, this is known as an anti normal operator ordering representation — all the annihilation operators are on the left-hand side of all creation operators. The third commonly used ordering is the symmetric ordering, where every possible ordering is equally represented. The symmetric ordering of the operator $\hat{F}$ is given as $\frac{1}{2} (\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger - 1)$. These orderings are equivalent and will generate the same expectation value on the left-hand side of Eq. 1.64. Thus, the quasi-probability function on the right-hand side of Eq. 1.64 needs to be modified depending on the choice of ordering.

Quasi-probability distributions are often defined by their relation to the coherent state:

$$|\alpha\rangle = \exp\left(-\frac{1}{2}|\alpha|^2\right) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \quad (1.65)$$

The set of coherent states forms an overcomplete basis and acts as eigenvectors of the annihilation operator, i.e. $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$. The Husimi Q-function [83] quasi-probability distribution is defined as

$$Q(\alpha, \alpha^*) = \frac{1}{\pi} \langle \alpha | \hat{\rho} | \alpha \rangle. \quad (1.66)$$

This can straightforwardly be shown to be the quasiprobability distribution corresponding

to anti-normally expressed operators as follows

$$\begin{aligned}
 \text{Tr} \left\{ \hat{a}^s (\hat{a}^\dagger)^r \hat{\rho} \right\} &= \text{Tr} \left\{ \frac{1}{\pi} \int d\alpha^2 \hat{a}^s |\alpha\rangle \langle \alpha| (\hat{a}^\dagger)^r \hat{\rho} \right\} \\
 &= \text{Tr} \left\{ \frac{1}{\pi} \int d\alpha^2 \alpha^s |\alpha\rangle \langle \alpha| (\alpha^*)^r \hat{\rho} \right\} \\
 &= \frac{1}{\pi} \int d\alpha^2 \alpha^s (\alpha^*)^r \langle \alpha | \hat{\rho} | \alpha \rangle \\
 &= \int d\alpha^2 \alpha^s (\alpha^*)^r Q(\alpha, \alpha^*),
 \end{aligned} \tag{1.67}$$

here we have introduced the identity $1 = \frac{1}{\pi} \int d\alpha^2 |\alpha\rangle \langle \alpha|$. The Q-function is a non-negative function. However, not all normalisable Q-functions correspond to positive definite normalisable density matrices.

The corresponding quasi-probability distribution for normally ordered operators is the Glauber-Sudarshan P-function [84, 85]. The P-function is most intuitively defined by expressing the density operator in a diagonal representation of coherent states

$$\hat{\rho} = \int d\alpha^2 P(\alpha, \alpha^*) |\alpha\rangle \langle \alpha|, \tag{1.68}$$

performing a similar proof as done in Eq. 1.67 it can be shown that the P-function is the corresponding probability distribution to normally ordered operator representations

$$\langle (\hat{a}^\dagger)^s \hat{a}^r \rangle = \int d\alpha^2 (\alpha^*)^s \alpha^r P(\alpha, \alpha^*). \tag{1.69}$$

Unlike the Q-function, the P-function can take negative values and is often extremely sharply peaked — for example, the P-function representation of a coherent state is a delta function. The final quasi-probability distribution we will look at, and the most important distribution for this thesis, is the Wigner function [86]. The Wigner function is the corresponding quasi-probability distribution for symmetrically ordered products and is most succinctly defined as

$$W(\alpha, \alpha^*) = \frac{1}{\pi^2} \int d\lambda^2 \text{Tr} \left\{ \hat{\rho} \exp \left(\lambda (\hat{a}^\dagger - \alpha^*) - \lambda^* (\hat{a} - \alpha) \right) \right\}. \tag{1.70}$$

$$\langle \{ (\hat{a}^\dagger)^s \hat{a}^r \}_{\text{sym}} \rangle = \int d\alpha^2 (\alpha^*)^s \alpha^r W(\alpha, \alpha^*). \tag{1.71}$$

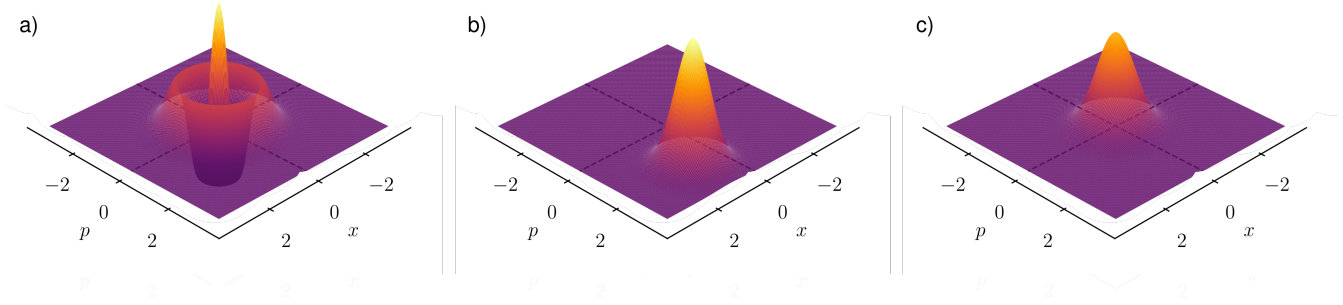


Figure 1.2: Sample Wigner distributions of a) the Fock state $|n = 2\rangle$, b) a coherent state where $\alpha = 2$, c) thermal state where $\beta = 2$.

The Wigner function's most interesting property and the rationale behind Wigner's original framework is the function's connection with position and momentum operators. The position and momentum operators are connected to the operation and annihilation operators by the following relation

$$\hat{x} = \sqrt{\frac{1}{2m\omega}} (\hat{a}^\dagger + \hat{a}), \quad \hat{p} = i\sqrt{\frac{m\omega}{2}} (\hat{a}^\dagger - \hat{a}), \quad (1.72)$$

where m is the mass of the oscillator and ω is the oscillator frequency. Wigner originally derived the Wigner distribution as a quasiprobability distribution for the momentum and position of a particle. In terms of position and momentum operators, the Wigner function is given as

$$W(x, p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-ip\zeta) \langle x + \frac{1}{2}\zeta | \hat{\rho} | x - \frac{1}{2}\zeta \rangle d\zeta \quad (1.73)$$

This acts as a quasi-probability distribution for symmetrically ordered operators in terms of momentum and position. One can relate the Wigner function expressed in terms of the quadratures x and p and the expressed in terms of α and α^* by

$$W(x, p) = \frac{1}{2} W(\alpha, \alpha'). \quad (1.74)$$

The Wigner function itself is not positive definite; for pure states, the only states that are positive semi-definite are Gaussian states [87]. Gaussian states are considered in more

detail in Chapter 6. Another interesting feature of Wigner functions is that their marginal distributions are true probability distributions;

$$\int dq W(q, p) = \langle p | \hat{\rho} | p \rangle, \quad (1.75)$$

$$\int dp W(q, p) = \langle q | \hat{\rho} | q \rangle. \quad (1.76)$$

This feature of the Wigner function is used in Chapter 4 to calculate current noise through an electromechanical system. The evolution of quantum systems can often be described in terms of the systems Wigner function, akin to a classical Fokker-Planck equation described in Section 1.4. In Fig. 3.3, we plot sample Wigner functions for a Fock state, coherent state and a thermal state. One can also derive useful operator correspondences between density matrix formalism and Wigner function formalism

$$\hat{x}\hat{p} \leftrightarrow \left(x + \frac{i}{2} \frac{\partial}{\partial p}\right) W(x, p), \quad (1.77a)$$

$$\hat{p}\hat{x} \leftrightarrow \left(x - \frac{i}{2} \frac{\partial}{\partial p}\right) W(x, p), \quad (1.77b)$$

$$\hat{p}\hat{p} \leftrightarrow \left(p - \frac{i}{2} \frac{\partial}{\partial x}\right) W(x, p), \quad (1.77c)$$

$$\hat{p}\hat{p} \leftrightarrow \left(p + \frac{i}{2} \frac{\partial}{\partial x}\right) W(x, p). \quad (1.77d)$$

A full review of the Wigner function and its properties can be found in references [43, 88, 89]. Finally, given the Wigner distribution, one may wish to reconstruct the density matrix, this is done in Chapter 3, to understand the thermodynamics of an oscillator system. The method to do this can be found in reference [90]; here, we briefly outline how to perform this calculation. Each element in the density matrix $\hat{\rho} = \gamma_{n,m} |n\rangle \langle m|$ is given by

$$\gamma_{n,m} = \frac{1}{\pi} \int d^2 \xi \chi(\xi) f_{nm}(\xi), \quad (1.78)$$

where χ is the symmetric characteristic function;

$$\chi(\xi) = \text{Tr} \{ \hat{\rho} \hat{D}(\xi) \}. \quad (1.79)$$

Here $\hat{D}(\chi) = \exp(x\hat{a}^\dagger - x^*\hat{a})$ is the Glauber displacement operator. The second term in the

integrand in Eq. 1.78 is $f_{nm}(\zeta) = \langle n | \hat{D}(-\zeta) | m \rangle$, when $n > m$ this is given by

$$f_{nm} = \sqrt{\frac{m!}{n!}} (-\zeta)^{n-m} e^{-|\zeta|^2/2} \mathcal{L}_m^{n-m}(|\zeta|^2), \quad (1.80)$$

where $\mathcal{L}_m^{n-m}(|\zeta|^2)$ is the generalised Laguerre polynomial of degree m . The systems we consider in Chapter 3 are rotationally symmetric, thus the off-diagonal terms of the density matrix vanish. For the diagonal elements Eq. 1.80 simplifies to

$$f_{nn} = e^{-|\zeta|^2/2} \mathcal{L}_n^0(|\zeta|^2). \quad (1.81)$$

To convert the Wigner function to the characteristic function, one needs to perform an inverse Fourier transform

$$\chi(\zeta) = \frac{1}{\pi^2} \int d\alpha W(\alpha) \exp(-\alpha \zeta^* + \alpha^* \zeta). \quad (1.82)$$

This expression is simplified significantly due to the symmetric nature of the Wigner function in Chapter 3. Converting to polar coordinates, the characteristic function is

$$\chi(r) = 2\pi \int du u \mathcal{J}_0(2ru) W(u), \quad (1.83)$$

where u is the radial value for the second integral, $\mathcal{J}_0$ is the Bessel function of the first kind, and r is the radial value of the Wigner function. The full equation describing the transform is therefore given by

$$\rho_{nn} = 8\pi \int \int du dr u \mathcal{J}_0(2ru) W(u) r e^{-|r|^2/2} \mathcal{L}_n^0(r^2). \quad (1.84)$$

1.6 Thermodynamics of Clocks

Accurate and precise timekeeping is a cornerstone of modern society. Clocks are vital in daily life, from coordinating meetings to navigating unfamiliar cities with GPS. A clock requires a system that undergoes some regular periodic event; counting these periodic events gives a time reference. The earliest clocks were based on celestial events such as the moon's phases, the Earth's rotation or the Earth's orbit around the sun. As technology developed, so did the ability to construct clocks based on shorter timescales. The development of the pendulum clock based on mechanical resonators in the seventeenth century [91] and quartz

crystal clocks in the early twentieth century [92] are two examples of leaps in the precision of timekeeping.

In the 1950s, the atomic clock was born [93]; these clocks are based on observing an atom's frequency absorption. These clocks were more stable and accurate than any clock before, using the hyperfine transition frequency of a caesium 133 atom. These clocks were so accurate they warranted a redefinition of the second [94]. Since the development of atomic clocks, work has continued to construct more precise clocks, with the development of laser-cooled caesium fountain clocks [95] and optical atomic clocks [24]. These clocks are so accurate they have achieved a precision corresponding to an error of about one second in the lifetime of the universe; these clocks have been used to measure gravitational redshift on the scale of millimetres [96].

However, despite this spectacular precision, fundamental gaps in our understanding of timekeeping still remain. While the fundamental cost of engines, heat pumps and refrigerators was explored during the Industrial Revolution, the energetic cost of timekeeping was largely ignored. With the growth of quantum control, the energetic costs of timekeeping have become more relevant. Poor timekeeping has been shown to affect the fidelity of time-dependent control protocols in quantum thermodynamics [97–99] and quantum computation [100, 101].

Within quantum theory, time holds its special place; whereas Hermitian operators can represent other physical quantities such as position and momentum, it is not possible to find a conjugate operator to the Hamiltonian that acts as a time observable [102]. Thus, in order to measure time, one needs to look at the evolution of some reference system. One fruitful and long-standing [103–105] strategy to address this problem is to formulate and study models of clocks as dynamical systems. Recent work in this direction has shown that basic physical considerations, stemming from information theory on the one hand [106–110] and thermodynamics on the other [8, 13, 11, 25], strongly constrain both the internal structure and the achievable performance of timekeeping devices. For example, it is now understood that accurate clocks must produce a lot of entropy [8, 11], quantum clocks can be more accurate than classical clocks with the same number of accessible states [110, 109, 12], and that there exists a tradeoff between a clock's accuracy and its timing resolution [10].

Similar constraints on temporal precision known as thermodynamic or kinetic uncertainty relations have been discovered in other, seemingly unrelated contexts, such as biomolecular systems [111–113] or mesoscopic transport setups [114–116].

In this thesis, we will focus on ticking clocks, which produce a continuous time reference to an external observer, as opposed to stopwatch-like systems (e.g. the Larmor clock [117]) that measure only a single time interval. Any quantum ticking clock comprises two distinct parts: the clockwork and the register [107]. The clockwork is a non-equilibrium system whose dynamics generate the ticks, which are then recorded by the register. Separation of clockwork and register ensures that the tick readout does not disrupt the clockwork [118]. Although almost any out-of-equilibrium system can act as a clockwork, accurate timekeeping requires temporal probability concentration [25], whereby the irreversible process that generates each tick is permitted to occur only at well-spaced time intervals. For example, while a simple hourglass exploits a freefalling mass to indicate the passage of time, the escapement of a pendulum clock allows a mass to fall incrementally — thereby generating the next tick — only after one full period of the swinging pendulum has elapsed [11, 119]. However, much of this new understanding of quantum ticking clocks has been garnered via theoretical analysis of simple toy models. The connection between the abstract theory of autonomous quantum clocks and concrete experimental implementations remains lacking.

1.6.1 Classical Pendulum Clock

The concepts of the cost of timekeeping can be demonstrated in a classical toy model of a pendulum clock as outlined by Milburn [11]. This continuous variable model acts as a prelude to the electromechanical quantum clock model discussed in Chapter 4.

In the absence of friction, a pendulum will oscillate with a constant amplitude and frequency for small oscillations. However, the oscillator’s motion will eventually stop due to collisions with air molecules and other environmental interactions. To combat this degradation, a pendulum clock is coupled to an escapement mechanism acting as an external drive — some weight or spring driving the system out of equilibrium. The escapement inputs energy into the pendulum’s motion to prevent the decay of the clock’s motion. The pendulum will eventually reach a limit cycle: a stable, steady state of sustained oscillatory motion due

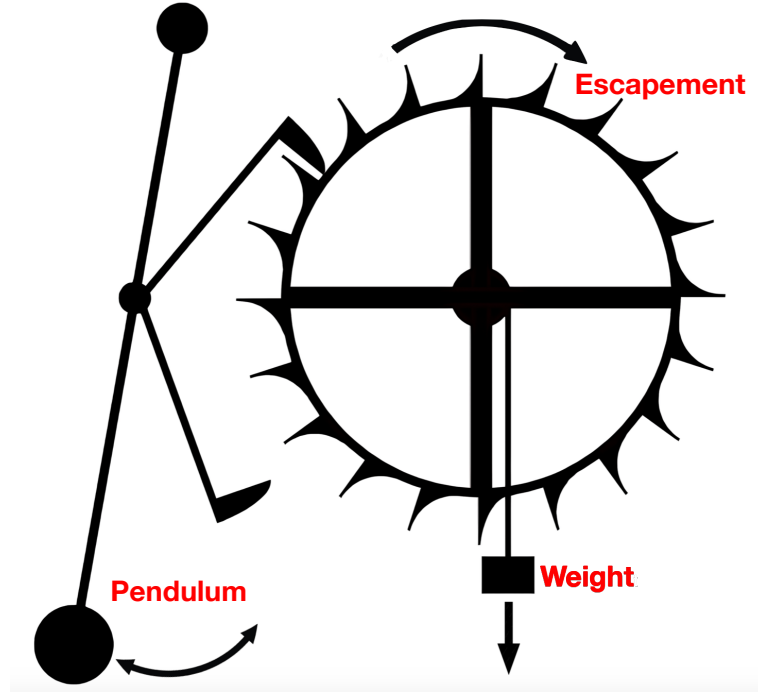


Figure 1.3: Cartoon depicting a classical pendulum clock. The pendulum is coupled to a mechanical escapement mechanism, the escapement mechanism provides a drive for the pendulum, preventing the oscillations dying. The escapement is powered by some external force, here a physical weight but could be a spring or battery. The escapement can be coupled to a clock face to provide a register.

to some non-linearity in the system. A schematic cartoon of a simple pendulum clock is shown in Fig. 1.3. The escapement considered is a Graham escapement mechanism [120]. The system described here acts as the clockwork, as discussed previously. The escapement mechanism can be coupled to a clockface to act as a register. We focus on the pendulum clockwork and write the unitless equations of motion of the pendulum as [11]

$$\dot{x} = y, \quad (1.85)$$

$$\dot{y} = -x - \Gamma y + K(x, y) + \sqrt{D}\eta(t), \quad (1.86)$$

here $K(x, y)$ is the driving force from the escapement mechanism, given by

$$K(x, y) = -\kappa \operatorname{sign}(\sin(\phi_0)x - \cos(\phi_0)y), \quad (1.87)$$

the drive acts as a constant force that changes direction every half period, κ denotes the strength of the drive. Γ represents the frictional damping rate. $\eta(t)$ represents a Langevin white noise term as discussed in Section 1.3. D is the diffusion constant related to the damping rate by $D = 2\Gamma$. We have set the mass and frequency of the oscillator to 1. These

equations can be rewritten in Ito form

$$dx = ydt \quad (1.88)$$

$$dy = (-x - \Gamma y + K(x, y))dt + \sqrt{D}dW(t), \quad (1.89)$$

where we have introduced the Wiener increment $dW(t)$. We define a clock tick as the oscillator's time to complete a full 2π rotation. To simplify these calculations, we convert these equations of motion into polar coordinates

$$r = \sqrt{x^2 + y^2}, \quad (1.90)$$

$$\phi = \arctan\left(\frac{y}{x}\right). \quad (1.91)$$

Using Ito's lemma, as discussed in Section 1.3 we can write the equations of motion of the oscillator in polar form:

$$dr = \frac{\partial r}{\partial x}dx + \frac{\partial r}{\partial y}dy + \frac{1}{2}\frac{\partial^2 r}{\partial y^2}dy^2, \quad (1.92)$$

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{1}{2}\frac{\partial^2 \phi}{\partial y^2}dy^2. \quad (1.93)$$

Substituting the derivatives into these equations, one arrives at the result

$$dr = \cos\phi dx + \sin\phi dy + \frac{\cos^2\phi}{2r}dy^2, \quad (1.94)$$

$$d\phi = -\frac{\sin\phi}{r}dx + \frac{\cos\phi}{r}dy - \frac{\cos\phi\sin\phi}{2r^2}dy^2. \quad (1.95)$$

Substituting the values of dx , dy and dy^2 and removing terms higher order than dt , one arrives at the stochastic equations describing the phase and the amplitude of the oscillations of the pendulum clock

$$dr = \left[-r \sin^2(\phi)\Gamma + \sin(\phi)K(\phi) + \frac{\cos^2(\phi)}{2r}D \right]dt + \sin(\phi)\sqrt{D}dW, \quad (1.96)$$

$$d\phi = \left[-1 - \Gamma \sin(\phi)\cos(\phi) + \frac{\cos(\phi)}{r}K(\phi) + \frac{\cos(\phi)\sin(\phi)}{2r^2}D \right]dt + \frac{\cos(\phi)}{r}\sqrt{D}dW. \quad (1.97)$$

The driving term has been rewritten as

$$K(t) = -\kappa \text{sign}[\cos(\phi(t) + \phi_0)]. \quad (1.98)$$

We simplify these equations using the phase averaging technique outlined in [121]. We wish to remove the rapidly oscillating trigonometric terms. First, considering the deterministic part of Eqs. 1.96 and 1.97, to a first approximation, we simply time average over a period, assuming that the amplitude and the phase do not change, resulting in truncated equations. For our purposes, it is not necessary to resort to higher-order truncations. Assuming that the correlation times of the random fluctuations are small relative to the relaxation time of the amplitude of the oscillator — we have already assumed delta correlated fluctuations — the fluctuations acting on the oscillator can be phase averaged by replacing $\cos(\phi)dW$ and $\sin(\phi)dW$ with $\frac{1}{\sqrt{2}}dW_1$ and $\frac{1}{\sqrt{2}}dW_2$ respectively. Here, dW_1 and dW_2 denote independent Wigner increments. This phase averaging does not affect the deterministic terms as the fluctuation terms act in opposite directions to partially compensate each other, diminishing the effect of fluctuations. Using phase averaging Eqs. 1.96 and 1.97 can be rewritten as

$$dr = \left[-\frac{\Gamma r}{2} + \frac{2\kappa}{\pi} \cos(\phi_0) + \frac{D}{4r} \right] dt + \sqrt{\frac{D}{2}} dW_1, \quad (1.99)$$

$$d\phi = \left[-1 + \frac{2\kappa}{r\pi} \sin(\phi_0) \right] dt + \frac{1}{r} \sqrt{\frac{D}{2}} dW_2. \quad (1.100)$$

From these equations, one can calculate the limit cycle by setting $dr = 0$ and solving for the deterministic part of Eq. 1.99. The steady-state amplitude is given by

$$r_* = \frac{2\kappa}{\pi\Gamma} \cos(\phi_0) + \sqrt{\frac{4\kappa^2}{\pi^2\Gamma^2} \cos^2(\phi_0) + \frac{D}{4}} \quad (1.101)$$

The final approximation to simplify the equation of motion for the oscillator is a linearisation, replacing the amplitude term with the limit cycle amplitude, r_* .

$$d\phi = \tilde{\omega} dt + \frac{\sigma}{r_*} dW_2, \quad (1.102)$$

where we have defined the renormalised frequency $\tilde{\omega} = -1 + \frac{2\kappa}{r\pi} \sin(\phi_0)$ and $\sigma = \sqrt{D/2}$. The noise acting on the oscillator is proportional to the inverse of the limit cycle. We can thus reduce the noise in the system by increasing the size of the limit cycle. One method of increasing the limit cycle is increasing the force the escapement provides — κ . To quantify how varying κ alters the performance of the clock, we introduce the concept of the tick distribution or waiting-time distribution (WTD). The WTD is defined as the probability, $W(\tau)$, of a time delay τ between two consecutive ticks. In simple models of autonomous

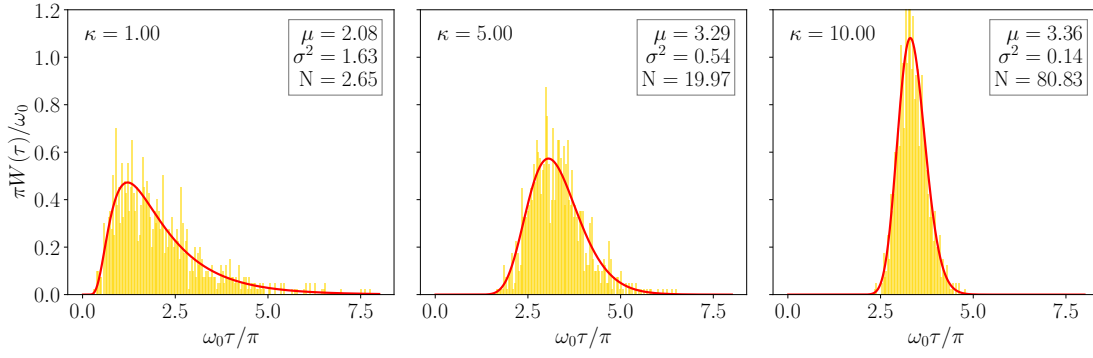


Figure 1.4: Waiting time distribution of the pendulum clock described by Eq. 1.102. Yellow bars are generated from numeric simulations of Eq. 1.102. Red solid curves are lines of best fit to match the inverse Gaussian distribution with fitting parameters μ and σ . As the driving force κ is increased the mean of the waiting time distribution increases and the variance decreases, indicating that increasing the work done by the displacement increases the precision of the clock. Parameters of the model are $T = 1$, $\phi_0 = \pi/4$ and $\Gamma = 1$.

clocks, each tick is assumed to be generated by an identical process resetting the system to some initial condition. This leads to the waiting times being independent and identically distributed (i.i.d.) random variables. If the ticks are i.i.d. the WTD fully characterises the statistical properties of the time signal. In this toy model, the ticks are i.i.d. due to the linearisation of the phase's evolution. We construct the WTD for the oscillator for various values of κ in Fig. 1.4. Increasing the force provided by the escapement both increases the mean of the WTD — reducing the frequency of the oscillator, and decreases the variance of the ticks — reducing the noise acting on the oscillator.

To further elucidate the effect of varying κ on the clock's performance, we introduce two precision measures when considering toy models — the accuracy and resolution defined in [8]. Given some process where each tick is assumed to be i.i.d, suppose that n ticks spaced by waiting times $\tau_1, \tau_2, \dots, \tau_n$ have been generated. The clock's time reading is $t_n = \sum_{i=1}^n \tau_i$, with average $\mathbb{E}[t_n] = n\mu$ and root-mean-square error $\Delta t_n = \sqrt{n}\sigma$ due to the assumption of i.i.d. ticks. We define the resolution of the clock to be the inverse of the mean ticking time

$$\nu_{\text{tick}} = 1/\mu. \quad (1.103)$$

The accuracy is the effective number of ticks N such that $\Delta t_N = \mu$, which leads directly to

$$N = \left(\frac{\mu}{\sigma}\right)^2. \quad (1.104)$$

A simple way to interpret the accuracy is the number of ticks that have elapsed before the reading is out by a single tick. From Fig. 1.4, it is clear that as the force generated by the escapement increases, the clock's resolution decreases while the clock's accuracy increases.

To complete this analysis of the simple pendulum clock, we want to consider the trade-off between the performance of the clock and the heat generated through operation. As discussed previously, there should be a trade-off between the amount of heat generated and the precision of the clock. The total work done by the escapement over a single cycle is given by

$$W = \kappa r_* \left[\int_{\tau_+} \sin(\phi(t)) dt + \int_{\tau_-} |\sin(\phi(t))| dt \right], \quad (1.105)$$

where $\tau_{+/-}$ denotes the time the escapement was producing a force κ or $-\kappa$ on the oscillator. Since the process is a stochastic process, the exact amount of work done by the pendulum per cycle will fluctuate. As we are assuming we are in a limit cycle of the oscillator, the work provided by the escapement is spent preventing the decay of the oscillator's cycle, i.e. all the work done by the escapement throughout a cycle is lost to the environment. Thus, the total heat generated throughout a cycle is determined by the total work the escapement does over the cycle. The entropy production per tick is then given by $\Delta S_{\text{tick}} = W\beta$ and we can define an entropy production rate as

$$\dot{\Sigma} = W\beta\nu = \beta\nu\kappa r_* \left[\int_{\tau_+} \sin(\phi(t)) dt + \int_{\tau_-} |\sin(\phi(t))| dt \right]. \quad (1.106)$$

In Fig. 1.5, we plot the accuracy and resolution of the clock as a function of the average entropy production rate $\langle \dot{\Sigma} \rangle$. There is a clear trade-off between the accuracy of the clock and the amount of entropy generated - to generate more precise ticks, one needs to generate more entropy.

In this analysis, we have only focused on how the work done by the escapement affects the clock's performance. Other factors, such as the temperature of the environment and the mass of the oscillator, will also affect the clock's performance. It is worth noting that when constructing macroscopic pendulum clocks, the noise source considered here is insignificant. As the limit cycle of such clocks is macroscopic, the effects of fluctuations on the pendulum are minimal. Internal friction between moving parts, temperature variations, and frequency shifts due to the clock's operation play a much more significant role and are

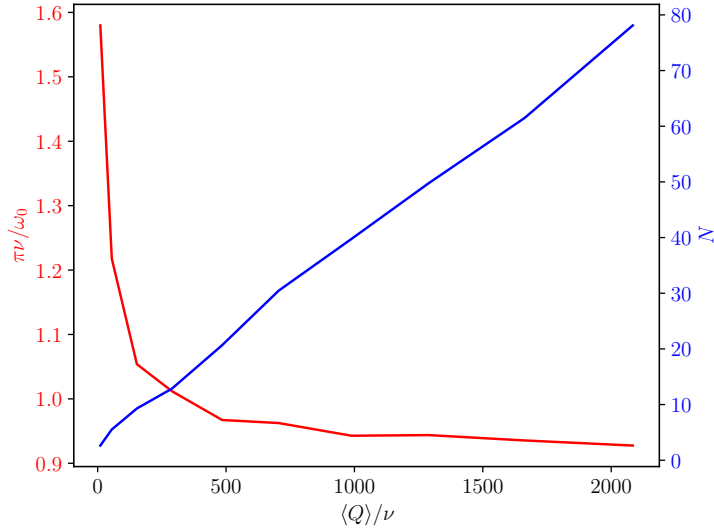


Figure 1.5: Accuracy (red) and resolution (blue) of the pendulum clock model given in Eq. 1.102 plotted against the average entropy produced per ticking cycle. The entropy production is varied by varying the driving force from the oscillator, κ . The accuracy increases as a function of the entropy produced by the clockwork due to the reduction of the impact of the noise term. The resolution decreases due to the larger limit cycle. The parameters used are the same as in Fig. 1.4.

beyond the scope of this thesis.

One final note on the assumption of i.i.d. ticks in this setup. As noted before, the linearisation procedure and phase averaging used in the derivation ensure that the system acts as a renewal process — where the system’s initial state is reset after each tick. In essence, as we assume the pendulum’s amplitude is fixed on the limit cycle, the clock’s state resets to $\phi = 0$, $r = r_*$ after each tick. Without this approximation, the effects would be twofold. After each tick, the amplitude of oscillation would not be reset, and because of this, the process would not be a renewal process. The second effect would be introducing an element of non-Markovianity into the process, and ticks would be temporarily correlated. We discuss this point in more detail and use methods to quantify the correlations between ticks and the merit of a clock without the i.i.d. assumption in Chapter 4.

Nano-Electromechanical Model

This chapter serves as a brief introduction to nano-electromechanical systems (NEMS). These systems amalgamate mechanical degrees of freedom with electronic systems. We outline some previous experiments exploring NEMS and their applications and delve into theoretical models to describe these systems. This chapter introduces the electromechanical model used in the following chapters, ultimately deriving a Fokker-Planck equation describing the motion of an oscillator degree of freedom in the quasi-adiabatic limit. Much of the content in this chapter is adapted from the papers “Extractable work in quantum electromechanics” [122] and “Powering an autonomous clock with quantum electromechanics” [123].

2.1 Electromechanics

Remarkable advances in engineering mesoscopic systems have led to vast control over such systems. Experiments such as the cooling of micro-mechanical oscillators to their ground state [124] and levitating mesoscopic particles with optical traps [125] demonstrate this progress. These systems have also allowed researchers to test the limits of thermodynamics, showcasing innovations like energy harvesting using quantum dots [126], tunable heat currents [127], and dissipation imaging in quantum systems [128]. Of particular interest for this thesis is the field of electromechanics. Electromechanics encapsulates a wide range of systems that integrate mechanical degrees of freedom into electronic devices [129–136]. These systems act as experimental platforms to explore quantum mechanics in systems ranging from a few atoms to nanometer scales, including nanotubes, cantilevers, and mem-

branes. This field bridges quantum mechanics and practical engineering, offering new insights into system dynamics. Various books and reviews have been written detailing NEMS and their applications [137–140].

NEMS are ideal candidates for quantum sensors. In force sensing experiments, the displacement of an oscillator is correlated to the external force causing the displacement. This displacement is coupled to a measured electric signal. These systems can then create highly accurate displacement sensors, for example, $0.5 \text{ pm}/\sqrt{\text{Hz}}$ in carbon nanotube oscillators [141] and extremely accurate force sensors, for example, $40 \text{ zN}/\sqrt{\text{Hz}}$ in silicon carbide nanowires [142]. These systems are also used in mass sensing, where the mass of an adsorbed particle can be related to the change in the eigenfrequency of an oscillator. For instance, the mass of systems as small as gold nanoparticles [143] and chromium atoms [144] can be measured. The charge of quantum dots can also be inferred by calculating variances in eigenfrequencies of nano-oscillators [145].

NEMS are also at the forefront of research in the measurement of time. These systems have been used to probe the thermodynamic limits of our ability to measure time. In macroscopic systems, it is extremely challenging to probe the fundamental limits of timekeeping due to the size and complexity of the systems involved; NEMS provide a platform to test how the second law limits our ability to construct arbitrarily accurate clocks. In Ref [9], using an optomechanical setup, a membrane excited by white noise, it was possible to probe the fundamental limitations of timekeeping, finding a trade-off between entropy production and precision. Nanomechanical oscillators are also used to generate some of the most accurate clocks in our everyday lives. While most clocks in mobile phones use oscillators on the microscale, research has been done to create ultrastable oscillators on the nanoscopic scale [146, 147].

2.2 Previous Theoretical Models of NEMS

Previous theoretical papers have created toy models to further understand NEMS. One popular toy model are shuttle models, where the central island oscillates perpendicularly to the electrodes. Electrons are loaded onto the central dot close to a turning point of the os-

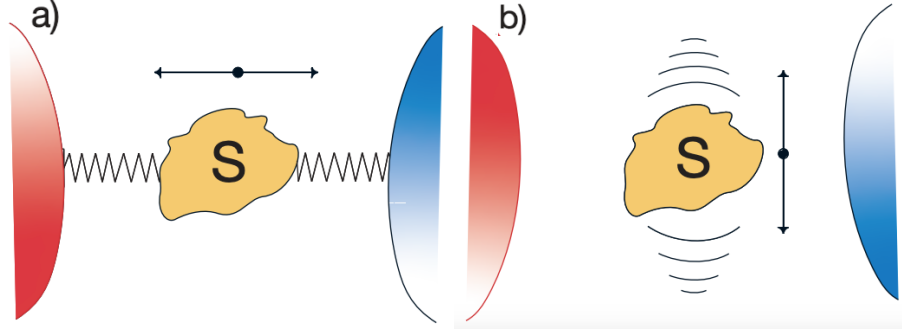


Figure 2.1: Cartoon outlining the difference between shuttle models and the model discussed in this thesis. In the shuttle models, the central island, S, moves to and from each electrode. In the model explored here the central island moves parallel to the electrodes, there is no motion to and from each electrode.

cillator and are unloaded at the opposite turning point [148–153]. An explicit positional dependence of the coupling rate between the central island and the electrodes describes this mechanical motion. The coupling term often has exponential dependence of the form $\exp[-x/l]$ [149, 150]. The electrostatic force acting on the oscillator is directed perpendicular to the electrodes and in the line of motion of the oscillator. These models exhibit a transition to a shuttling regime (self-sustained oscillations), where one electron is transported per oscillator cycle above a certain threshold bias. This is seen in the Wigner function, which describes the oscillator, taking an annulus shape. Such models have been proposed for rectification [150] as well as autonomous machines [152]. These models are inspired by experimental works in fullerene molecules [154] and gold grain systems [155].

This thesis considers models where the mechanical degree of freedom moves parallel to the electrodes. The oscillator experiences an electrostatic force proportional to the excess charge on the oscillator. This coupling can be achieved via a nearby gate electrode as in Refs. [129, 20, 156]. In Fig. 2.1, we highlight the difference between these models and shuttle models. In these models, the coupling between the central dot and electronic reservoirs has no explicit positional dependence - any positional dependence must be contained implicitly in the energy dependence of the spectral density.

Various mathematical techniques have been developed to analyse non-resonant transport in these systems. One can treat the oscillator classically, where the energy scales of the voltage is larger than the energy scales of the tunnelling and oscillating frequencies [157–160]. One assumes that the dot can only have $n = 0$ or $n = 1$ electrons and the force exerted when

the dot is occupied is classical. One can then define a sequential tunnelling description for the electron transport through the system using rate equations. With these approximations, the oscillator's position varies slowly between successive tunnelling events, and one can perform an adiabatic elimination to remove the electronic degrees of freedom. From this, a Wigner function describing the evolution of the oscillator can be generated.

Other models define a Feynman-Vernon path integral to describe the oscillator [161, 162]. Using an adiabatic approximation, assuming the mechanical motion is slow compared to electronic tunnelling rates, one can integrate out electronic degrees of freedom exactly. This technique is non-perturbative in the system-leads coupling due to the exact electronic solutions. The oscillator action is expanded in terms of powers of the oscillator frequency, with second-order powers describing fluctuations and higher-order powers describing non-Gaussianity.

Linear-response theory mixed with the adiabatic approximation can also be used to solve for the oscillator dynamics [163–165]. Assuming weak coupling between the oscillator and the electronic degrees of freedom, one can generate a Langevin equation describing the oscillator's motion. The back action and noise are determined by the quantum noise spectrum of the electronics, calculated via a Keldysh non-equilibrium Green's function.

One can also move to stronger interaction strengths using a polaron transformation [166]. This effectively eliminates the interaction term between the central island and the oscillator; from this, one can define a rate equation in the weak coupling limit. Other techniques use scattering theory techniques to solve for the motion of the oscillator [167–169]. One can use a scattering matrix describing single particle scattering by the central dot to describe the mean force and fluctuations acting on the oscillator. From this, a Langevin equation can be constructed to describe the oscillator dynamics.

2.3 Self-Sustained Oscillations

A fascinating phenomenon in NEMS is the occurrence of self-sustained oscillations. Self-sustained oscillations occur when a periodic output is produced by some input that lacks any corresponding periodicity. Self-sustained oscillations are seen throughout nature, from

the laser to the beating heart. An infamous example of self-sustained oscillations occurred in the Tacoma Narrows bridge disaster in 1940, where a strong wind caused oscillatory motion in the bridge, eventually leading to the complete collapse of the bridge [170].

Experimentally, self-sustained oscillations have been observed in NEMS in suspended carbon nanotube systems [20, 129, 156]. A carbon nanotube is doubly clamped between two electrodes above a set of electrode gates. These gates allow for the control of the tunnelling rates between the electrodes and the nanotube, couple the electronic system to the mechanical system and allow for the position measurement of the oscillator. The nanotube's motion is measured via a radio frequency circuit for time-resolved measurements. Through heterodyne detection, self-sustained oscillations are observed in the nanotube. These self-sustained oscillations are present in the Husimi-Q function plots as an annulus in phase space. Below a threshold bias, a thermal blob is seen; once the threshold bias is met, the Q-function takes on the characteristic annulus shape analogous to that of a laser setup.

Such self-sustained oscillations are present in many theoretical works. In the electron shuttle model, self-sustained oscillations occur due to the positional dependence in the coupling [149–153]. Self-sustained oscillations are also seen in other NEMS toy models [168, 169, 164]. Self-sustained oscillations in NEMS will be the subject of the next chapter of this thesis.

2.4 Model

The schematic of the model used in this and the following chapters is given in Fig. 6.3. Here, we have a central quantum dot described by a single resonant electronic level. The central dot is coupled to two leads and a single-mode oscillator. Electrons hopping on and off the central dot generate an electrostatic force acting on the oscillator. The resultant oscillator movement shifts the energy of the quantum dot, augmenting the electronic dynamics, thus generating a feedback loop between the electronic and mechanical degrees of freedom. The

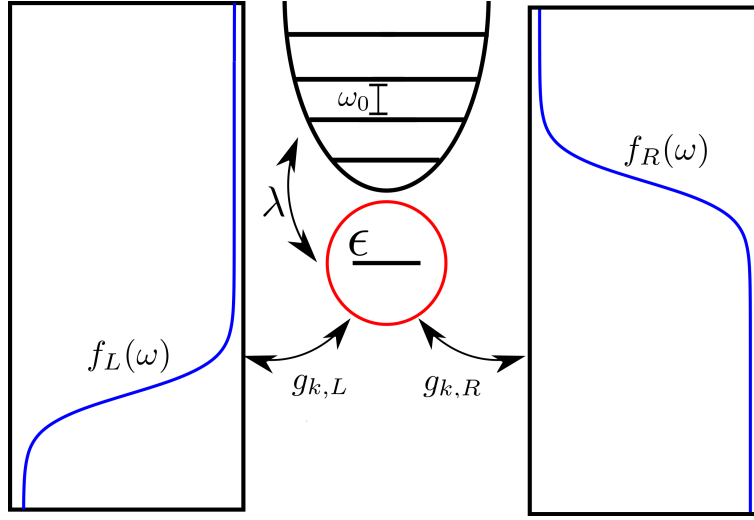


Figure 2.2: Schematic depiction of the electromechanical system. A central quantum dot is sandwiched between two electrodes. The dot is also coupled to a single-mode quantum harmonic oscillator representing the mechanical degree of freedom. The electrodes are described by Fermi-Dirac distributions $f_\alpha(\omega)$ at the same temperature but with differing chemical potentials.

Hamiltonian of the system is described by the following equations:

$$\hat{H}_S = \epsilon \hat{c}^\dagger \hat{c}, \quad (2.1)$$

$$\hat{H}_B = \sum_k \left(\Omega_{kL} \hat{d}_{kL}^\dagger \hat{d}_{kL} + \Omega_{kR} \hat{d}_{kR}^\dagger \hat{d}_{kR} \right), \quad (2.2)$$

$$\hat{H}_T = \sum_k \left(g_{kL} \left[\hat{c}^\dagger \hat{d}_{kL} + \hat{d}_{kL}^\dagger \hat{c} \right] + g_{kR} \left[\hat{c}^\dagger \hat{d}_{kR} + \hat{d}_{kR}^\dagger \hat{c} \right] \right), \quad (2.3)$$

$$\hat{H}_V = \frac{\hat{p}^2}{2m} + \frac{m\omega_0^2 \hat{x}^2}{2} - F\hat{n}\hat{x}, \quad (2.4)$$

$$\hat{H} = \hat{H}_S + \hat{H}_B + \hat{H}_T + \hat{H}_V. \quad (2.5)$$

Eq. 2.1 is the bare Hamiltonian of a resonant level quantum dot with free energy ϵ and annihilation [creation] operator $\hat{c}$ [$\hat{c}^\dagger$]. The dot is coupled to two leads which are described by an infinite collection of non-interacting free fermions as shown in eq. 2.2 with annihilation [creation] operators $\hat{d}_{k,\alpha}$ [$\hat{d}_{k,\alpha}^\dagger$] and dispersion relations $\Omega_{k,\alpha}$, where the index $\alpha = L, R$ denotes the left or the right bath with mode index k . Eq. 2.3 describes the interaction between the dot and the electrodes, with energy-dependent coupling constants g_{kL} . The effect of the electrodes on the quantum dot is fully characterised by their temperature T_α and their chemical potential μ_α . Their spectral densities determine the tunnelling rate between the lead and the dot

$$\kappa_\alpha(\omega) = 2\pi \sum_k g_{k\alpha}^2 \delta(\omega - \Omega_{k\alpha}). \quad (2.6)$$

Eqs. [2.1-2.3] describe the resonant level model, a model thoroughly explored in many textbooks [47, 41]. Eq. 2.4 sets this model apart from the resonant level model; this term describes a single mode oscillator with mass m , natural frequency ω_0 , described by the position and momentum operators $\hat{x}$ and $\hat{p}$ respectively. The oscillator experiences a static force where the mean force per unit charge is described by the coupling constant F , and $\hat{n}$ describes the excess occupation on the quantum dot, $\hat{n} = \hat{c}^\dagger \hat{c} - \langle \hat{c}^\dagger \hat{c} \rangle_{F=0}$. $\langle \hat{c}^\dagger \hat{c} \rangle_{F=0}$, in the absence of the electromechanical coupling, i.e. calculating the steady-state occupation with the coupling constant set to zero while keeping all other parameters fixed. The effect of the oscillator is to modify the energy of the quantum dot, altering the occupation of the dot. Using the excess occupation ensures that the average force acting on the oscillator from the electronic degrees of freedom while the oscillator is at the origin is zero.

This model is much simpler than the previous experiments, ignoring factors such as spin degrees of freedom, electronic interactions, non-linearities and intrinsic damping of the vibrational modes. However, as we will see, this model contains a wealth of interesting physics with a description of self-sustained oscillators [122] and a platform to explore clock models [123].

2.5 Quasi-Adiabatic Limit - Fokker Planck Equation Derivation

We explore this model through a quasi-adiabatic limit approximation, where the motion of the oscillator is slow in comparison to the characteristic rate of electron tunnelling Γ and the thermal correlation time β , i.e.

$$\min(\beta^{-1}, V), \Gamma \gg \omega_0, |\lambda|, \quad (2.7)$$

where $\lambda = Fx_0/2$ characterises the electromechanical interaction energy and $x_0 = (2/m\omega_0)$ is the mechanical zero point fluctuation. The quasi-adiabatic approximation is justified from the perspective of the experiments of Ref. [20]; these parameters are on the order of $\Gamma \sim 10$ GHz, $k_B T \sim 1$ GHz, $\omega_0 \sim 0.1$ GHz and $\lambda \gtrsim 0.1$ GHz, for which Ineq. (2.7) holds. From this, we can derive a Wigner function that describes the evolution of the oscillator.

Compressing all the electronic components of the Hamiltonian into a single term, $\hat{H}_{\text{el}} = \hat{H}_S + \hat{H}_B + \hat{H}_T$, the Hamiltonian of the system is given as

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega_0^2 \hat{x}^2}{2} - F\hat{x}\hat{n} + \hat{H}_{\text{el}}. \quad (2.8)$$

We use the Wigner transform of the density matrix,

$$\hat{\rho}(x, p) = \frac{1}{\pi} \int dy \langle x+y | \hat{\rho} | x-y \rangle e^{-i2yp}, \quad (2.9)$$

which is an operator-valued function of x and p that acts on the electronic Hilbert space. As discussed in chapter 1.5, the Wigner function of the oscillator is given by tracing both sides of eq. 2.9, $W(x, p) = \text{Tr}\{\hat{\rho}(x, p)\}$, while the electronic state is given by $\hat{\rho}_{\text{el}} = \int dx \int dp \hat{\rho}(x, p)$.

The time evolution of the system follows the Liouville-von Neumann equation $d\hat{\rho}/dt = -i[\hat{H}, \rho]$. Using the operator correspondences for the Wigner function [43], we obtain the corresponding equation of motion

$$\begin{aligned} \frac{\partial}{\partial t} \hat{\rho}(x, p) = & -i \left[\hat{H}_{\text{el}} - Fx\hat{n}, \hat{\rho}(x, p) \right] - \frac{p}{m} \frac{\partial}{\partial x} \hat{\rho}(x, p) + m\omega_0^2 x \frac{\partial}{\partial p} \hat{\rho}(x, p) \\ & - \frac{F}{2} \frac{\partial}{\partial p} \{ \hat{n}, \hat{\rho}(x, p) \}. \end{aligned} \quad (2.10)$$

Physically, the first two terms on the RHS of the first line represent the evolution of the electronic degrees of freedom conditioned on a particular oscillator position x . The final two terms on the first line represent the free mechanical evolution, and the second line represents the force on the oscillator due to the electronic charge.

The adiabatic assumption is that the electronic degrees of freedom, evolving under the Hamiltonian $\hat{H}_{\text{el}} - Fx\hat{n}$, relax to a stationary state much faster than the characteristic evolution timescale of the oscillator. The timescale separation of Inequality 2.7 ensures this, which justifies the adiabatic elimination of the electronic degrees of freedom from the mechanical equations of motion. Following the standard procedure [71], the elimination is

carried out by defining the following operators:

$$\mathcal{L}_x = -i[\hat{H}_{\text{el}} - Fx\hat{n}, \bullet], \quad (2.11)$$

$$\mathcal{H} = \left(m\omega_0^2 x - F\langle\hat{n}\rangle_x\right) \frac{\partial}{\partial p} - \frac{p}{m} \frac{\partial}{\partial x}, \quad (2.12)$$

$$\mathcal{V} = -\frac{F}{2} \frac{\partial}{\partial p} \{\hat{n} - \langle\hat{n}\rangle_x, \bullet\}, \quad (2.13)$$

such that $\partial_t \hat{\rho}(x, p) = (\mathcal{L}_x + \mathcal{H} + \mathcal{V})\hat{\rho}(x, p)$. Here, $\langle\bullet\rangle_x = \text{Tr}\{\bullet\hat{\rho}_x\}$ denotes an average with respect to the electronic stationary state, $\hat{\rho}_x$, defined by

$$\hat{\rho}_x = \lim_{t \rightarrow \infty} e^{\mathcal{L}_x t} \hat{\rho}_0 \hat{\rho}_B, \quad (2.14)$$

where $\hat{\rho}_B$ denotes the initial state of the leads and $\hat{\rho}_0$ is an arbitrary initial reference state for the electronic system S , i.e. the local charge degrees of freedom coupled to the oscillator. Here, we take the leads to be infinitely large and prepared in equilibrium at certain temperatures and chemical potentials, i.e. $\hat{\rho}_B \propto \prod_{\alpha} e^{-\beta_{\alpha}(\hat{H}_{\alpha} - \mu_{\alpha}\hat{N}_{\alpha})}$, where $\hat{H}_{\alpha}$ and $\hat{N}_{\alpha}$ are respectively the Hamiltonian and number operator of lead α , while β_{α} and μ_{α} are the corresponding inverse temperatures and chemical potentials. Strictly speaking, the steady-state density operator $\hat{\rho}_x$ describes the entire system-lead composite and exists only asymptotically in time. However, we will shortly see that the only relevant properties of $\hat{\rho}_x$ are those describing the vicinity of the localised charge (in particular, the noise spectrum associated with $\hat{n}$), which can be meaningfully assumed to relax quickly to the stationary state. Note that $\hat{\rho}_x$ depends on x but not on p , and satisfies $\mathcal{L}_x \hat{\rho}_x = 0$ and $\text{Tr}\{\hat{\rho}_x\} = 1$.

Let us now define a projector on the space of operator-valued functions of x and p , as

$$\mathcal{P}\hat{A}(x, p) = \text{Tr}\{\hat{A}(x, p)\}\hat{\rho}_x, \quad (2.15)$$

whose orthogonal complement is denoted by $\mathcal{Q} = 1 - \mathcal{P}$. The projector obeys the relations

$$\mathcal{P}^2 = \mathcal{P}, \quad (2.16a)$$

$$\mathcal{P}\mathcal{L}_x = 0 = \mathcal{L}_x\mathcal{P}, \quad (2.16b)$$

$$\mathcal{P}\mathcal{V}\mathcal{P} = 0, \quad (2.16c)$$

$$[\mathcal{P}, \mathcal{H}] = \frac{p}{m} \frac{\partial \hat{\rho}_x}{\partial x} \text{Tr}\{\bullet\}, \quad (2.16d)$$

$$\mathcal{P}[\mathcal{P}, \mathcal{H}] = 0. \quad (2.16e)$$

Eq. (2.16a) is the defining property of a projector, which follows here from the normalisation of $\hat{\rho}_x$. The first equality in Eq. (2.16b) follows from the definition (2.14) of $\hat{\rho}_x$, while the second equality holds since $\mathcal{L}_x$ generates a trace-preserving evolution. Eq. (2.16c) follows from the fact that the operator $\hat{n} - \langle \hat{n} \rangle_x$ has vanishing mean in the stationary state $\hat{\rho}_x$, which itself is independent of p . Finally, Eq. (2.16d) can be proved by a direct calculation, while Eq. (2.16e) is satisfied because $\partial \hat{\rho}_x / \partial x$ is traceless.

Let $\hat{r} = \mathcal{P}\hat{\rho}(x, p)$ denote the projected density matrix and $\hat{q} = \mathcal{Q}\hat{\rho}(x, p)$ its orthogonal complement. Applying these projectors to the left-hand side of Eq. (2.10), inserting appropriate factors of $1 = \mathcal{P} + \mathcal{Q}$, and using the properties (2.16), we obtain

$$\frac{\partial \hat{r}}{\partial t} = \mathcal{P}\mathcal{H}\hat{r} + \mathcal{P}(\mathcal{H} + \mathcal{V})\hat{q}, \quad (2.17)$$

$$\frac{\partial \hat{q}}{\partial t} = \mathcal{L}_x\hat{q} + \mathcal{Q}(\mathcal{H} + \mathcal{V})\hat{q} + \mathcal{Q}(\mathcal{H} + \mathcal{V})\hat{r}. \quad (2.18)$$

To proceed we define the unitless parameter $\epsilon = \frac{\omega_0}{\Gamma} \ll 1$, we can thus introduce the new operators

$$\bar{\mathcal{H}} = \Gamma \left[(m\omega_0 x - m\lambda \langle \hat{n} \rangle_x) \frac{\partial}{\partial p} - \frac{p}{m\omega_0} \frac{\partial}{\partial x} \right], \quad (2.19)$$

$$\bar{\mathcal{V}} = -\frac{m\Gamma\lambda}{2} \frac{\partial}{\partial p} \{ \hat{n} - \langle \hat{n} \rangle_x, \bullet \}. \quad (2.20)$$

We can thus rewrite the operators acting on the mechanical degrees of freedom as

$$\mathcal{H} = \epsilon \bar{\mathcal{H}} \quad \mathcal{V} = \epsilon \bar{\mathcal{V}} \quad (2.21)$$

Inserting these into Eq. 2.18 one sees a separation of timescales

$$\frac{\partial \hat{q}}{\partial t} = \mathcal{L}_x \hat{q} + \epsilon \left(\mathcal{Q} \left(\bar{\mathcal{H}} + \bar{\mathcal{V}} \right) \hat{q} + \mathcal{Q} \left(\bar{\mathcal{H}} + \bar{\mathcal{V}} \right) \hat{r} \right). \quad (2.22)$$

From the adiabatic condition, $\epsilon \ll 1$, we see that $\hat{q}$ relaxes quickly to its steady state on the timescale of the oscillator and can therefore be approximately replaced in Eq. (2.17) by its stationary value. To proceed, we formally integrate Eq. (2.18) to obtain

$$\hat{q}(t) = \mathcal{G}(t) \hat{q}(0) + \int_0^t dt' \mathcal{G}(t-t') \mathcal{Q}(\mathcal{H} + \mathcal{V}) \hat{r}(t'), \quad (2.23)$$

$$\mathcal{G}(t) = \exp \{ [\mathcal{L}_x + \mathcal{Q}(\mathcal{H} + \mathcal{V})] t \}. \quad (2.24)$$

We now make the crucial assumption that $\hat{q}(0) = 0$, i.e. the electronic degrees of freedom begin in the stationary state $\hat{\rho}_x$ (or reach it after a negligibly short time). Substituting the solution (2.23) into Eq. (2.17) and shifting the integration variable $t' \rightarrow t - t'$ yields

$$\begin{aligned} \frac{\partial \hat{r}}{\partial t} &= \mathcal{P} \mathcal{H} \hat{r}(t) + \int_0^t dt' \mathcal{P}(\mathcal{H} + \mathcal{V}) \mathcal{G}(t') \mathcal{Q}(\mathcal{H} + \mathcal{V}) \hat{r}(t-t') \\ &\approx \mathcal{P} \mathcal{H} \hat{r}(t) + \int_0^\infty dt' \mathcal{P}(\mathcal{H} + \mathcal{V}) e^{\mathcal{L}_x t'} \mathcal{Q}(\mathcal{H} + \mathcal{V}) \hat{r}(t). \end{aligned} \quad (2.25)$$

On the second line, we have made the following approximations: (i) we can rewrite the propagator $\mathcal{G}(t')$ as

$$\mathcal{G}(t') = \exp \left\{ \left[\mathcal{L}_x + \epsilon \mathcal{Q}(\bar{\mathcal{H}} + \bar{\mathcal{V}}) \right] t' \right\} \approx \exp \{ \mathcal{L}_x t' \}, \quad (2.26)$$

to lowest non-trivial order due to the quasi-adiabatic condition (Born approximation), (ii) assuming that the integrand decays rapidly as a function of t' , we approximate $\hat{r}(t-t') \approx \hat{r}(t)$ and extend the upper integration limit to infinity (Markov approximation).

To further simplify the integrand, we use Eqs. (2.16) and the obvious property $\mathcal{P} \mathcal{Q} = 0$ to write

$$\begin{aligned} &\mathcal{P}(\mathcal{H} + \mathcal{V}) e^{\mathcal{L}_x t'} \mathcal{Q}(\mathcal{H} + \mathcal{V}) \mathcal{P} \\ &= \mathcal{P}([\mathcal{P}, \mathcal{H}] + \mathcal{H} \mathcal{P} + \mathcal{V}) \mathcal{Q} e^{\mathcal{L}_x t'} \mathcal{Q}(\mathcal{P} \mathcal{H} - [\mathcal{P}, \mathcal{H}] + \mathcal{V}) \mathcal{P} \\ &= \mathcal{P} \mathcal{V} e^{\mathcal{L}_x t'} (\mathcal{V} - [\mathcal{P}, \mathcal{H}]) \mathcal{P}. \end{aligned} \quad (2.27)$$

Tracing over the electronic degrees of freedom finally yields a formal master equation for

the Wigner function, $W(x, p) = \text{Tr}[\hat{\rho}(x, p)]$, in the form

$$\frac{\partial}{\partial t} W = \mathcal{H} W - \int_0^\infty dt' \text{Tr} \left\{ \mathcal{V} e^{\mathcal{L}_x t'} (\mathcal{V} - [\mathcal{P}, \mathcal{H}]) \hat{\rho}_x \right\} W. \quad (2.28)$$

The first term describes the free evolution of the oscillator, while the second term describes diffusion and damping due to the fluctuating charge.

It is straightforward to evaluate

$$\int_0^\infty dt \text{Tr} \left\{ \mathcal{V} e^{\mathcal{L}_x t} \mathcal{V} \hat{\rho}_x \right\} = \frac{1}{2} D(x) \frac{\partial^2}{\partial p^2}, \quad (2.29)$$

where the diffusion coefficient is given by

$$D(x) = F^2 \int_{-\infty}^\infty dt \langle \{\delta \hat{n}(t), \delta \hat{n}(0)\} \rangle_x, \quad (2.30)$$

and we defined $\delta \hat{n}(t) = \hat{n}(t) - \langle \hat{n} \rangle_x$ as the charge fluctuation operator evaluated in the Heisenberg picture, with $\hat{n}(t) = e^{\mathcal{L}_x^\dagger t} \hat{n}$ and $\mathcal{L}_x^\dagger = i[\hat{H}_{\text{el}} - Fx\hat{n}, \bullet]$ the adjoint Liouvillian. The remaining contribution to Eq. (2.28) is evaluated as

$$\int_0^\infty dt \text{Tr} \left\{ \mathcal{V} e^{\mathcal{L}_x t} [\mathcal{P}, \mathcal{H}] \hat{\rho}_x \right\} = \gamma(x) \frac{\partial}{\partial p} p, \quad (2.31)$$

where we defined the damping rate

$$\gamma(x) = \frac{F}{m} \int_0^\infty dt \text{Tr} \left\{ \hat{n}(t) \frac{\partial \hat{\rho}_x}{\partial x} \right\}. \quad (2.32)$$

We express $\gamma(x)$ in terms of the charge noise spectrum as follows. First, note that consistency with the Markov approximation requires the integrand of Eq. (2.32) to decay to zero for large times. Any differentiable function $g(t)$ that vanishes at infinity, $\lim_{t \rightarrow \infty} g(t) = 0$, satisfies

$$-\int_0^\infty dt g(t) = \int_0^\infty dt \int_t^\infty dt' \frac{dg}{dt'} = \int_0^\infty dt t \frac{dg}{dt}, \quad (2.33)$$

where the first equality follows from the definition of integration and the second equality follows after switching the order of integration, $\int_0^\infty dt \int_t^\infty dt' = \int_0^\infty dt' \int_0^{t'} dt$, then carrying out the trivial integral over t and relabelling dummy variables. For the particular case of $g(t) = \text{Tr}\{\hat{n}(t) \partial \hat{\rho}_x / \partial x\}$, we have $dg/dt = \text{Tr}\{\hat{n}(t) \mathcal{L}_x \partial \hat{\rho}_x / \partial x\}$. By differentiating $\mathcal{L}_x \hat{\rho}_x$ with

respect to x , we obtain

$$\mathcal{L}_x \frac{\partial \hat{p}_x}{\partial x} = -iF[\hat{n}, \hat{p}_x]. \quad (2.34)$$

Putting everything together, we get

$$\gamma(x) = \frac{F^2}{m} \int_0^\infty dt \langle it [\hat{n}(t), \hat{n}(0)] \rangle_x. \quad (2.35)$$

Since the noise statistics are stationary (see Sect. 1.3) (2.30) and (2.35) can be related to the noise spectrum of the fluctuating charge-induced force as

$$D(x) = S_x(0) \quad (2.36)$$

$$m\gamma(x) = \left. \frac{dS_x(\omega)}{d\omega} \right|_{\omega=0} \quad (2.37)$$

$$S_x(\omega) = F^2 \int_{-\infty}^\infty dt e^{i\omega t} [\langle \hat{n}(t)\hat{n}(0)_x \rangle - \langle \hat{n} \rangle] \quad (2.38)$$

It is now straightforward to write down the equation of motion for the Wigner function

$$\frac{\partial W}{\partial t} = \frac{\partial}{\partial p} \left(m\omega_0^2 x - F \langle \hat{n} \rangle_x + \gamma(x)p \right) W - \frac{p}{m} \frac{\partial W}{\partial x} + \frac{D(x)}{2} \frac{\partial^2 W}{\partial p^2}. \quad (2.39)$$

Eq. (2.39) is a genuine Fokker-Planck equation for the Wigner function of the oscillator. Its solution may take negative values if the initial condition for $W(x, p)$ is non-classical. However, whenever $W(x, p) \geq 0$ it may be interpreted as a probability distribution associated with the Langevin equation:

$$m\ddot{x} + m\gamma(x)\dot{x} + m\omega_0^2 x = F\langle \hat{n} \rangle_x + \sqrt{D(x)}\xi(t), \quad (2.40)$$

The Wigner function is then reconstructed as

$$W(x', p, t) = \mathbb{E}[\delta(x' - x(t))\delta(p - m\dot{x}(t))], \quad (2.41)$$

where $\mathbb{E}[\bullet]$ denotes an average over Langevin trajectories $x(t)$ evolved up to a given time t . The Wigner function generates expectation values with symmetric $\hat{x} - \hat{p}$ operator order-

ing [43], such as

$$\text{Tr}\{\hat{x}\hat{\rho}(t)\} = \int dx \int dp x W(x, p, t), \quad (2.42)$$

$$\text{Tr}\{\hat{p}\hat{\rho}(t)\} = \int dx \int dp p W(x, p, t). \quad (2.43)$$

The Langevin equation can also be used to reconstruct symmetrised two-time correlation functions, such as $C(t, t + \tau) = \frac{1}{2} \langle \{\hat{x}(t + \tau), \hat{x}(t)\} \rangle = \text{Re} \langle \hat{x}(t + \tau) \hat{x}(t) \rangle$. Using the Born-Markov assumption and the quantum regression formula [43], this can be written as

$$\langle \hat{x}(t + \tau) \hat{x}(t) \rangle = \text{Tr} \{ \hat{x} \mathcal{E}(t + \tau, t) (\hat{x} \hat{\rho}(t)) \}, \quad (2.44)$$

where $\mathcal{E}(t + \tau, t)$ is the time evolution superoperator. In phase space, the expression $\mathcal{E}(t + \tau, t) (\hat{x} \hat{\rho}(t))$ corresponds to the solution of the Fokker-Planck equation (2.39) with initial condition

$$\hat{x} \hat{\rho}(t) \rightarrow \bar{W}(x, p, t) = x W(x, p, t) + \frac{i}{2} \frac{\partial W(x, p, t)}{\partial p}, \quad (2.45)$$

where $W(x, p, t)$ is the Wigner function associated with $\hat{\rho}(t)$. This evolves to

$$\int dx' dp' \Pi(x, p, t + \tau | x', p', t) \bar{W}(x', p', t), \quad (2.46)$$

with $\Pi(x, p, t + \tau | x', p', t)$ the Green function of the Fokker-Planck equation, i.e. its formal solution for a point initial condition $\delta(x - x') \delta(p - p')$. Taking the real part of Eq. (2.44) and using Eq. (2.42), we thus obtain

$$\begin{aligned} \text{Re} \langle \hat{x}(t + \tau) \hat{x}(t) \rangle &= \int dx \int dx' \int dp \int dp' x \Pi(x, p, t + \tau | x', p', t) x' W(x', p', t) \\ &= \int dx \int dx' x x' P(x, t + \tau; x', t) \\ &= \mathbb{E}[x(t + \tau)x(t)], \end{aligned} \quad (2.47)$$

where

$$P(x, t + \tau; x', t) = \int dp \int dp' \Pi(x, p, t + \tau | x', p', t) W(x', p', t) \quad (2.48)$$

is the joint probability distribution for the random variables $x(t + \tau)$ and $x(t)$ given that $x(t)$ is distributed according to $P(x, t) = \int dp W(x, p, t)$.

Finally, let us briefly recap the assumptions underlying Eq. (2.39). We assume that the

electronic degrees of freedom are initially in the stationary state, $\mathcal{Q}\hat{\rho}(0) = 0$, and relax quickly back to this state when perturbed, $\mathcal{L}_x \gg \mathcal{H}, \mathcal{V}$. These assumptions are well satisfied in the quasi-adiabatic regime defined by Eq. (5). Eq. (2.39) then follows from a Born-Markov approximation for the projected density matrix (2.15). Note, however, that we make no assumptions about the Hamiltonian $\hat{H}_{\text{el}}$ of the electronic system or the operator $\hat{n}$, which could, in principle, describe an arbitrary interacting system so long as the quasi-adiabatic assumption is satisfied. This is a significant advantage compared to previous derivations of the quasi-adiabatic Langevin equation, which have typically required either non-interacting electrons [161, 162] or transport in the sequential tunnelling regime [157–160, 151]. Our approach is most similar in its assumptions to the work of Clerk and Bennett [164, 165], but with the additional benefit that we provide a rigorous derivation that connects directly to the quantum state of the oscillator through its Wigner function, which may be highly non-Gaussian.

Extractable Work in Electromechanics

NEMS act as a powerful platform to explore and test the core principles and applications of thermodynamics. An area of intense interest in the thermodynamics community are quantum batteries - physical systems that store energy. These systems are gathering this interest due to the potential of exploiting collective quantum effects [2, 6, 3, 171, 172] to boost performance. Quantum batteries are typically far from equilibrium and strongly influenced by fluctuations and environmental noise [7, 173–175], leading to the development of advanced control strategies to mitigate these effects [176–179]. However, there remains a wide gap between the existing theoretical literature —largely focused on quantum optical settings [180–183]—and the physics of mesoscopic electronic devices. NEMS can bridge these two fields, utilising the mechanical degrees of freedom as a battery or flywheel to store useful energy.

This chapter explores this connection in the electromechanical model discussed in Section 2. We solve the noise spectrum of the electronic degrees of freedom in both the wide-band limit and with a Lorentzian spectral density. To achieve negative damping and generate ergotropy in the oscillator, one needs energy dependence in the spectral density. We show the lasing transition as a function of the bias across the leads and calculate the g_2 correlation function. Finally, we calculate the oscillator’s ergotropy and non-equilibrium free energy and show that the ergotropy acts as an order parameter for the lasing transition in the NEMS. This chapter is adapted from the paper “Extractable work in quantum electromechanics” [122].

3.1 Negative Damping

Consider the electromechanical setup described in Chapter 2. To achieve self-sustained oscillations, one needs to find a regime where the damping on the oscillator in Eq. 2.40 becomes negative for some x . This negative damping causes the oscillator to speed up rather than slow it down.

To solve Eq. 2.40 one needs to derive an expression for the excess occupation $\langle \hat{n} \rangle_x$ and the electronic noise spectrum $S_x(\omega)$ for each value of x . To achieve this, we use an equation of motion approach. The electronic Hamiltonian conditioned on the oscillator position x is defined by

$$\hat{H}_x = \hat{H}_S + \hat{H}_B + \hat{H}_T - Fx\hat{c}^\dagger\hat{c}, \quad (3.1)$$

where $\hat{H}_S + \hat{H}_B + \hat{H}_T$ is given in Eqs. (2.1)–(2.3). We ignore the constant term proportional to N_0 since this merely shifts the zero of energy. The Heisenberg equations of motion for this system are

$$\frac{d}{dt}\hat{c} = -i\left(\epsilon_x\hat{c} + \sum_{\alpha \in L,R} \sum_k g_{k\alpha}\hat{d}_{k\alpha}\right), \quad (3.2)$$

$$\frac{d}{dt}\hat{d}_{k\alpha} = -i\left(\Omega_{k\alpha}\hat{d}_{k\alpha} + g_{k\alpha}\hat{c}\right), \quad (3.3)$$

where $\epsilon_x = \epsilon - Fx$. Formally integrating Eq. (3.3) and inserting the result into Eq. (3.2), one finds

$$\frac{d\hat{c}}{dt} = -i\epsilon_x\hat{c}(t) + \sum_{\alpha} \left(\hat{\zeta}_{\alpha}(t) - \int_{t_0}^{\infty} dt' \chi_{\alpha}(t-t')\hat{c}(t') \right), \quad (3.4)$$

where we introduced the memory kernel $\chi_{\alpha}(t)$ and the noise operator $\hat{\zeta}_{\alpha}(t)$, defined by

$$\chi_{\alpha}(t-t') = \Theta(t-t') \sum_k g_{k\alpha}^2 e^{-i\Omega_{k\alpha}(t-t')}, \quad (3.5)$$

$$\hat{\zeta}_{\alpha}(t) = -i \sum_k g_{k\alpha} e^{-i\Omega_{k\alpha}(t-t_0)} \hat{d}_{k\alpha}(t_0), \quad (3.6)$$

with t_0 defining the initial time at which the leads are prepared in thermal equilibrium. Since we are interested in the steady state, we take the limit $t_0 \rightarrow -\infty$. Eq. (3.4) is then easily

solved in the Fourier domain to obtain

$$\tilde{c}(\omega) = G(\omega)\tilde{\zeta}(\omega), \quad (3.7)$$

where $G(\omega)$ is the retarded Green function

$$G(\omega) = [\omega - \epsilon_x - \tilde{\chi}_L(\omega) - \tilde{\chi}_R(\omega)]^{-1}, \quad (3.8)$$

with $\tilde{\chi}_\alpha$ the self energy of electrode $\alpha = L, R$,

$$\tilde{\chi}_\alpha(\omega) = \oint \frac{d\omega'}{2\pi} \frac{\kappa_\alpha(\omega')}{\omega - \omega'} - i \frac{\kappa_\alpha(\omega)}{2}, \quad (3.9)$$

where $\oint$ is a principal-value integral and κ_α is the spectral density of the quantum dot coupled with the left or right electrode. The self-energy term describes the influence of the electrodes on the central system; the real part describes an energy shift in the system, and the imaginary part describes the broadening of the energy level [47].

The non-equilibrium steady state depends on the statistical properties of the noise operator. We have assumed the initial state of the reservoirs are thermal and uncorrelated, i.e. $\langle \hat{d}_{\alpha,k}^\dagger(t_0) \hat{d}_{\alpha',k'}(t_0) \rangle = f_\alpha(\Omega_k) \delta_{\alpha\alpha'} \delta_{kk'}$, where $f_\alpha(\omega) = (e^{\beta_\alpha(\omega - \mu_\alpha)} + 1)^{-1}$ is the Fermi-Dirac distribution. The statistics of the noise operator can be found in the frequency domain, where

$$\tilde{\zeta}_\alpha(\omega) = -\sqrt{2\pi}i \sum_k g_{k\alpha} \delta(\omega - \Omega_{k\alpha}) \hat{d}_{k\alpha}(t_0). \quad (3.10)$$

One can calculate the two-point noise spectrum to be

$$\langle \tilde{\zeta}_\alpha^\dagger(\omega) \tilde{\zeta}_{\alpha'}(\omega') \rangle = \kappa_\alpha(\omega) \delta(\omega - \omega') f_\alpha(\omega) \delta_{\alpha\alpha'}. \quad (3.11)$$

Substituting this back into Eq. (3.7) of the dot

$$\langle \tilde{c}^\dagger(\omega) \tilde{c}(\omega') \rangle = \sum_{\alpha \in L, R} A(\omega) \kappa_\alpha(\omega) f_\alpha(\omega), \quad (3.12)$$

where $A(\omega) = |G(\omega)|^2$ is the spectral function. To find the steady state occupation, we evaluate the Fourier transform at the time $t = t'$ to obtain

$$\langle \hat{c}^\dagger \hat{c} \rangle_x = \frac{1}{2\pi} \sum_{\alpha \in L, R} \int A(\omega) \kappa_\alpha(\omega) f_\alpha(\omega). \quad (3.13)$$

For the parameters we consider, with $\frac{1}{2}(\mu_L + \mu_R) = \epsilon$ so that the dot is half-filled on average, this yields $N_0 = \langle \hat{c}^\dagger \hat{c} \rangle_0 = \frac{1}{2}$ and thus $\langle \hat{n} \rangle_x = \langle \hat{c}^\dagger \hat{c} \rangle_x - \frac{1}{2}$.

To find the dissipation rate and diffusion coefficient, one needs to calculate the charge noise spectrum

$$S_x(\omega) = F^2 \int_{-\infty}^{\infty} dt e^{i\omega t} \left[\langle \hat{n}(t) \hat{n}(0) \rangle_x - \langle \hat{n} \rangle_x^2 \right]. \quad (3.14)$$

One thus needs to calculate the two-point correlation function of the operator

$\hat{c}^\dagger \hat{c}$: $\langle \hat{c}^\dagger(t) \hat{c}(t) \hat{c}^\dagger(0) \hat{c}(0) \rangle$. We proceed using a similar method to the occupation function using a Green's function approach

$$\langle \hat{c}^\dagger(\omega) \hat{c}(\omega') \hat{c}^\dagger(\nu) \hat{c}(\nu') \rangle = G^*(\omega) G(\omega') G^*(\nu) G(\nu') \langle \hat{\zeta}^\dagger(\omega) \hat{\zeta}(\omega') \hat{\zeta}^\dagger(\nu) \hat{\zeta}(\nu') \rangle \quad (3.15)$$

To calculate $\langle \hat{\zeta}^\dagger(\omega) \hat{\zeta}(\omega') \hat{\zeta}^\dagger(\omega'') \hat{\zeta}(\omega''') \rangle$ we make use of Wick's theorem using the identity

$$\langle \hat{d}_{k\alpha}^\dagger \hat{d}_{k'\alpha'}^\dagger \hat{d}_{j\beta}^\dagger \hat{d}_{j'\beta'}^\dagger \rangle = \delta_{k'k} \delta_{\alpha'\alpha} f(\Omega_k) \delta_{j'j} \delta_{\beta'\beta} f(\Omega_j) + \delta_{kj} \delta_{\alpha\beta} f(\Omega_k) \delta_{k'j'} \delta_{\alpha'\beta'} [1 - f(\Omega_j)] \quad (3.16)$$

The first term in this expression is equivalent to the product of the means and will cancel with the contribution from $\langle \hat{n} \rangle_x^2$ in Eq. (3.14). The second term in equation (3.16) is unique to the four-point correlation function, and its contribution to $\langle \hat{\zeta}^\dagger(\omega) \hat{\zeta}(\omega') \hat{\zeta}^\dagger(\nu) \hat{\zeta}(\nu') \rangle$ is

$$4\pi^2 \sum_k \sum_j |g_{k\alpha}|^2 |g_{j\beta}|^2 f_\alpha(\Omega_k) [1 - f_\beta(\Omega_j)] \delta(\omega + \nu') \delta(\omega' + \nu) \delta(\omega - \Omega_{k\alpha}) \delta(\omega' - \Omega_{j\beta}) \quad (3.17)$$

Using the same method as the two-point correlation function, the contribution to

$\langle \hat{c}^\dagger(\omega) \hat{c}(\omega') \hat{c}^\dagger(\nu) \hat{c}(\nu') \rangle$ is therefore

$$A(\omega) A(\omega') \kappa_\alpha(\omega) \kappa_\beta(\nu) f_\alpha(\omega) [1 - f_\beta(\nu)] \delta(\omega - \nu') \delta(\omega' - \nu). \quad (3.18)$$

Performing a Fourier transform, change of variables and letting $s = s' = 0$ and $t = t'$ one arrives at the result

$$\begin{aligned} \langle \hat{n}(t) \hat{n}(0) \rangle_x - \langle \hat{n} \rangle_x^2 &= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega d\omega' e^{-i\omega t} A(\omega') A(\omega' + \omega) \\ &\quad \times \kappa_\alpha(\omega') \kappa_\beta(\omega' + \omega) f_\alpha(\omega') [1 - f_\beta(\omega + \omega')]. \end{aligned} \quad (3.19)$$

Inserting this into Eq. (3.14) one finds

$$S_x(\omega) = F^2 \sum_{\alpha, \beta} \int_{-\infty}^{\infty} \frac{d\omega'}{4\pi^2} A(\omega') A(\omega + \omega') \kappa_\alpha(\omega') \kappa_\beta(\omega + \omega') f_\alpha(\omega') [1 - f_\beta(\omega + \omega')]. \quad (3.20)$$

The diffusion coefficient is therefore

$$D(x) = F^2 \sum_{\alpha, \beta \in L, R} \int \frac{d\omega'}{2\pi} A(\omega')^2 \kappa_\alpha(\omega') \kappa_\beta(\omega') f_\alpha(\omega') [1 - f_\beta(\omega')], \quad (3.21)$$

while the dissipation rate for the system is

$$m\gamma(x) = F^2 \sum_{\alpha, \beta} \int \frac{d\omega}{2\pi} A(\omega) \kappa_\alpha(\omega) f_\alpha(\omega) \frac{d}{d\omega'} \left(\kappa_\beta(\omega + \omega') A(\omega + \omega') [1 - f_\beta(\omega + \omega')] \right) \Big|_{\omega'=0}. \quad (3.22)$$

One can thus perform these calculations for any system given the spectral density $\kappa_\alpha(\omega)$.

3.1.1 Energy Independent Coupling

When the coupling to the leads is energy-independent, the spectral density is given by

$$\kappa_\alpha(\omega) = \Gamma_\alpha, \quad (3.23)$$

the coupling to the leads is the same regardless of the dot's energy. The spectral function in the wide band limit is

$$A(\omega) = \frac{\Gamma}{(\omega - \epsilon_x)^2 + \Gamma^2/4} \quad (3.24)$$

where $\Gamma = \Gamma_L + \Gamma_R$ is the sum of the coupling strengths to the leads. The occupation of the quantum dot for a given position is

$$\langle \hat{c}^\dagger \hat{c} \rangle_x = \frac{1}{2\pi} \sum_{\alpha \in \{L, R\}} \int \frac{\Gamma_\alpha f_\alpha(\omega)}{(\omega - \epsilon_x)^2 + \Gamma^2/4} d\omega \quad (3.25)$$

the diffusion coefficient

$$D(x) = F^2 \Gamma_L \Gamma_R \sum_{\alpha, \beta \in \{L, R\}} \int \frac{f_\alpha(\omega) [1 - f_\beta(\omega)]}{((\omega - \epsilon_x)^2 + \Gamma^2/4)^2} d\omega, \quad (3.26)$$

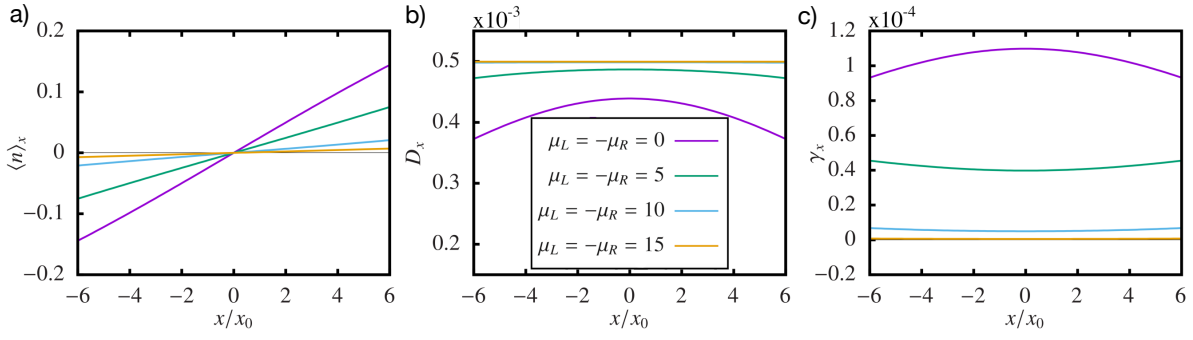


Figure 3.1: Excess occupation, fluctuation and dissipation curves for various voltages as a function of the oscillator's position for a wideband spectral density. a) The quantum dot's excess occupation. The excess occupation is monotonic regardless of the bias across the leads. b) The fluctuation curve at various voltages. c) The dissipation curve at various voltages. When the occupation curve is monotonic, no negative damping occurs. Parameters: $\omega_0 = 0.2$, $m = 1$, $\lambda = 0.1$, $\Gamma_L = \Gamma_R = 2$, $\beta_L = \beta_R = 0.5$.

and the damping coefficient

$$\gamma(x) = \frac{F^2 \Gamma_L \Gamma_R}{m} \sum_{\alpha, \beta \in \{L, R\}} \int \frac{d\omega}{2\pi} \frac{f_\alpha(\omega)}{(\omega - \epsilon_x)^2 + \Gamma^2/4} \left[\frac{2(1 - f_\beta(\omega))(\omega - \epsilon_x)}{((\omega - \epsilon_x)^2 + \Gamma^2/4)^2} + \frac{f_\beta(\omega) \frac{1}{T} e^{\frac{\omega - \mu}{T}}}{(\omega - \epsilon_x)^2 + \Gamma^2/4} \right]. \quad (3.27)$$

The dissipation function was calculated for various parameters $\Gamma_R, \Gamma_L, \mu_R, \mu_L$ and β but negative damping was never obtained. This corresponds nicely with the works of Clerk and Bennett [164, 165]. They describe the backaction on the oscillator in an intuitive, classical form. Assuming the average backaction force is proportional to the oscillator position and the back-action force does not respond instantaneously to the changes in the oscillator's motion (hence a quasi-adiabatic rather than an adiabatic approach), the back-action on the oscillator can be expressed through a Taylor expansion

$$f_{\text{avg}}[x(t - \tau_{\text{resp}})] \approx \frac{df_{\text{avg}}}{dx} x(t - \tau_{\text{resp}}) \approx \frac{df_{\text{avg}}}{dx} [x(t) - \tau_{\text{resp}} \dot{x}(t)], \quad (3.28)$$

here f_{avg} denotes the average backaction force and τ_{resp} denotes the time delay of the response. In this expansion, the damping term is given by

$$m\gamma = \frac{df_{\text{avg}}}{dx} \tau_{\text{resp}}, \quad (3.29)$$

as the average back-action on the oscillator is proportional to the occupation of the dot, we would expect to see negative damping occur in a region where the occupation function is non-monotonous in position. Instead of analysing the difficult Eq. 3.27 we can look at the

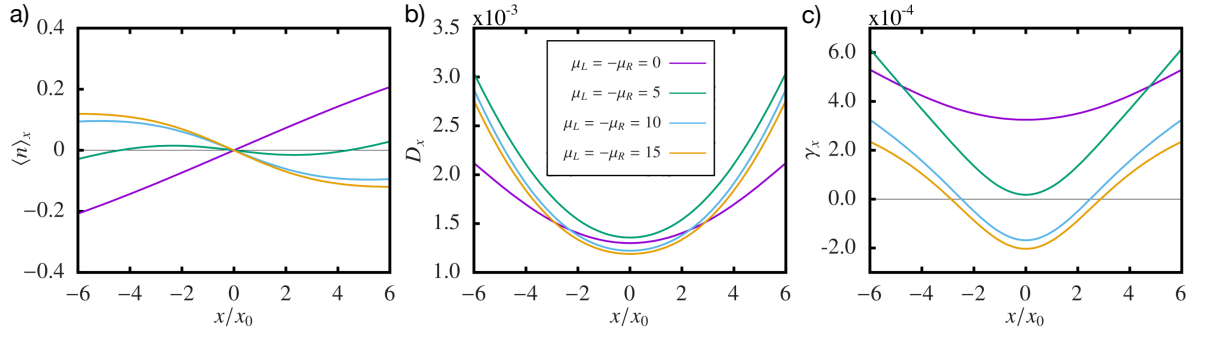


Figure 3.2: Excess occupation, fluctuation and dissipation curves for various voltages as a function of the oscillator's position for a Lorentzian spectral density. a) The quantum dot's excess occupation. At zero voltage the occupation curve is monotonous with the oscillator position, as the voltage across the quantum dot is increased the curve becomes non-monotonic. b) The fluctuation curve at various voltages. c) The dissipation curve at various voltages. When the occupation curve is monotonic, no negative damping occurs however once the voltage reaches a threshold and the occupation becomes nonmonotonic, the damping coefficient becomes negative. Parameters: $\omega_0 = 0.2$, $m = 1$, $\lambda = 0.1$, $\omega_L = 0.5$, $\omega_R = -0.5$, $\delta_L = \delta_R = 1$, $\Gamma_L = \Gamma_R = 2$, $\beta_L = \beta_R = 0.5$.

much nicer Eq. 3.25. The occupation function comprises of a decreasing Fermi function multiplied by a Lorentzian function. The effect of changing the energy of the dot is to shift the Lorentzian function, as the Fermi function is a decreasing function; the overlap between the Fermi function and the Lorentzian will be monotonic as the Lorentzian function is shifted, thus causing a monotonic occupation function.

From this reasoning and the exploration done in the parameter space, we can see that to achieve negative damping, we must move to an energy-dependent spectral density.

3.1.2 Energy Dependent Coupling

To incorporate energy dependence in the spectral density, we introduce Lorentzian spectral densities of the form

$$\kappa_\alpha(\omega) = \frac{\Gamma\delta^2}{(\omega - \omega_\alpha)^2 + \delta^2}, \quad (3.30)$$

which describe electronic bands centred around frequency ω_α with effective bandwidth δ and overall coupling strength Γ . For the Lorentzian spectral density, the electrode self-energy is given by

$$\tilde{\chi}_\alpha(\omega) = \frac{\Gamma_\alpha\delta_\alpha(\omega - \omega_{0\alpha})}{2[(\omega - \omega_{0\alpha})^2 + \delta_\alpha^2]} - i\frac{\Gamma_\alpha\delta_\alpha^2}{2[(\omega - \omega_{0\alpha})^2 + \delta_\alpha^2]}. \quad (3.31)$$

The explicit expressions for the occupation, dissipation, and diffusion terms can be calcu-

lated through Eqs. 3.13, 3.21 and 3.22. The explicit forms are not given here as they are complex and unnecessary. With energy-dependent coupling and when the leads are out of equilibrium with each other, one can find parameter regimes where the dissipation becomes negative. This is shown in Fig. 3.2, on panel three. As the bias across the leads increases, the damping term becomes negative.

3.2 Self-Sustained Oscillations

The Langevin equation, Eq. 2.40, was solved by using a stochastic Euler method [71, 70]. A time-step Δt was selected, the system was evolved to a time $t + \Delta t$ using the values of the oscillator at the position at time t . The evolution step from time t to time $t + \Delta t$ is given by

$$v(t + \Delta t) = v(t) + \left(-\omega_0^2 x(t) - \gamma[x(t)]v(t) + \langle n \rangle[x(t)] \right) \Delta t + \sqrt{D[x(t)]} \Delta W, \quad (3.32)$$

$$x(t + \Delta t) = x(t) + v(t) \Delta t, \quad (3.33)$$

where ΔW is a Gaussian variable with zero mean and variance Δt . Once the initial state of the oscillator is selected, an array of velocity and position values can be generated. The Wigner function for the oscillator can then be constructed for the steady state as shown in Figure 3.3. For each bias, 2.5×10^8 time steps were used. Here, we assume that the system is ergodic - that we can sample the steady-state probability density via a single trajectory calculation and thus only perform one iteration for each calculation.

Fig. 3.3 shows the steady state Wigner function for three applied voltages. At zero applied voltage, the oscillator is in equilibrium with the two electrodes, and the Wigner function forms a thermal blob near the origin [left panel]. This qualitative structure persists for all voltages below a certain threshold, defined in the following section. Above this threshold, however, the damping rate $\gamma(x)$ becomes negative for small x , leading to amplification of the mechanical motion in direct analogy with a lasing transition. This results in an annular Wigner function [central panel], indicating the presence of population inversion. The radius of the annulus increases with V until a saturation voltage is reached, beyond which any further voltage increase has no effect.

For further analysis, we numerically transform the steady-state Wigner function into a den-

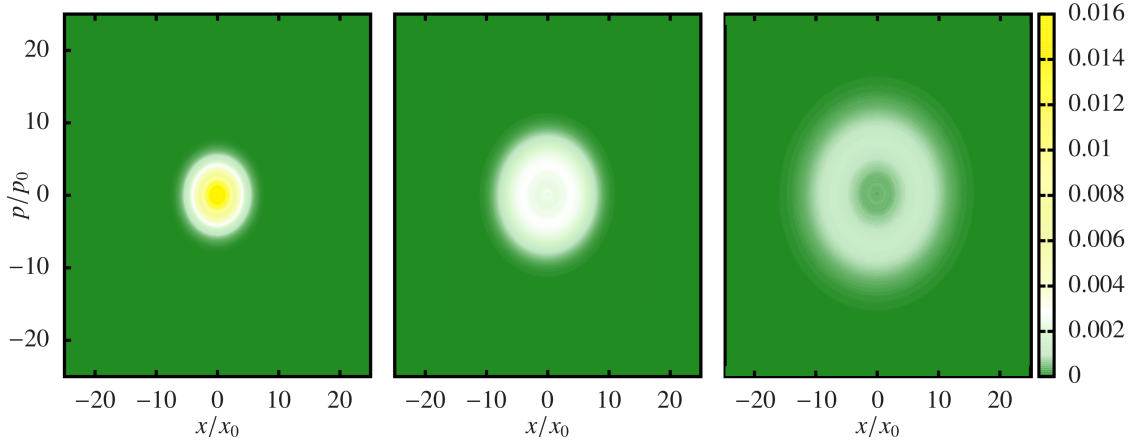


Figure 3.3: Wigner functions of the oscillator at various biases. The left panel shows the Wigner function at zero applied bias showing the Wigner blob, the central panel shows the Wigner function at 6 applied bias just after the threshold voltage, the right panel shows the Wigner function at 16 bias where the voltage has reached its saturation point. Parameters: $\omega_0 = 0.2, m = 1, \lambda = 0.1, \omega_L = 0.5, \omega_R = -0.5, \delta_L = \delta_R = 1, \Gamma_L = \Gamma_R = 2, \beta_L = \beta_R = 0.5, x_0 = (2/m\omega_0)^{1/2}$, and $p_0 = (2m\omega_0)^{1/2}$.

sity matrix within a truncated basis. Note that the Wigner functions are rotationally symmetric in phase space [see Fig. 3.4], yielding density matrices that are diagonal in the energy eigenbasis of the oscillator Hamiltonian $\hat{H}_b = \omega_0 \hat{a}^\dagger \hat{a}$. It follows that the mode quadratures vanish on average, $\langle \hat{a} \rangle = 0$, in direct analogy with the quantum state of a laser field, which approximates a phase-averaged coherent state [43] in the absence of explicit symmetry breaking. The coherent nature of the steady state above threshold is instead conveyed through the second-order coherence function [43]

$$g^{(2)}(0) = \frac{\langle \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \rangle}{\langle \hat{a}^\dagger \hat{a} \rangle^2}, \quad (3.34)$$

where $\hat{a} = \hat{x}/x_0 + i\hat{p}/p_0$ denotes the annihilation operator for the vibrational mode. Given the diagonal steady-state density matrix $\hat{\rho}_b = \sum_n p_n |n\rangle \langle n|$, one has $g^{(2)}(0) = (\sum_n n(n-1)p_n) / (\sum_n np_n)^2$. In Fig. 3.4, we plot $g^{(2)}(0)$ in the steady state as a function of voltage. At no applied bias, $g^{(2)}(0) \approx 2$, as expected for a thermal state. Increasing the voltage, the oscillator is pushed away from thermal equilibrium and the second-order coherence function decreases, reaching a plateau at $g^{(2)}(0) \approx 1.5$ above threshold. This indicates the sub-thermal amplitude noise associated with lasing, but is nonetheless greater than the value of $g^{(2)}(0) = 1$ for an ideal coherent state, highlighting the importance of fluctuations in this nanoscale non-equilibrium system.

The system described here is analogous to quantum mechanical models describing lasers [43, 184]. Such models are derived from a two-level system coupled to a bath with some inverse

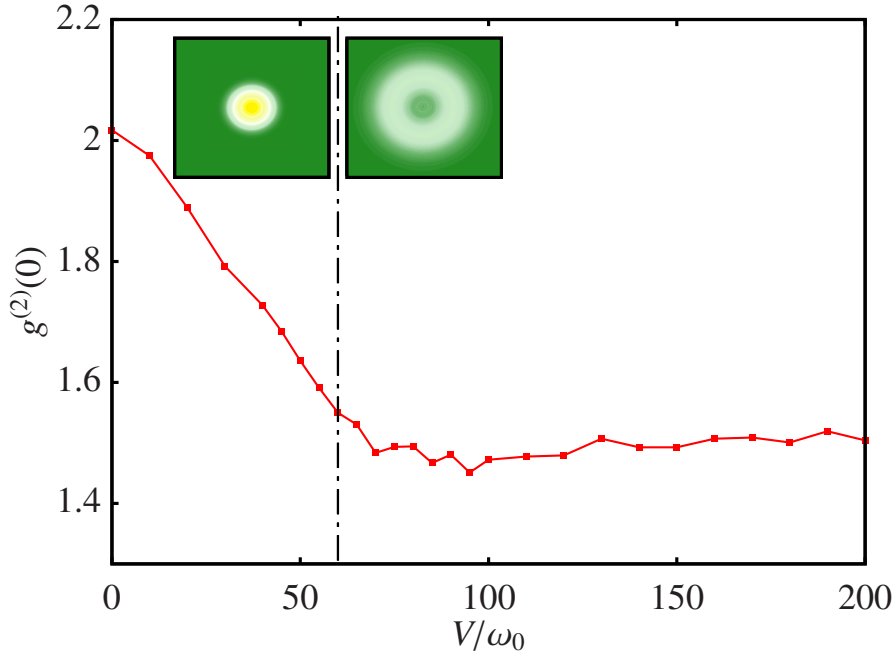


Figure 3.4: Second-order coherence function of the oscillator as a function of voltage, for the same parameters as in Fig. 3.3

temperature, generating a population inversion in the atomic degrees of freedom. Using a similar adiabatic elimination procedure, a Fokker-Planck equation describing the P-function can be generated. One can generate a Langevin equation for the quadrature modes [43]

$$\frac{d\alpha}{dt} = -\kappa\alpha \left(1 - \frac{C}{1 + |\alpha|^2/\tilde{n}_0} \right) + \frac{1}{\sqrt{N}}\tilde{\Gamma}(t), \quad (3.35)$$

where C is a pump parameter, n_0 is the saturation photon number, κ is the emission rate, N is the number of two level atoms and $\tilde{\Gamma}(t)$ is a noise term. In this system, negative damping and a population inversion occur when the terms inside the bracket become negative. The $1/\sqrt{N}$ prefactor to the noise term highlights the fluctuations in the nanoscale systems — as the size of the system reaches the thermodynamic limit the noise vanishes. While having a similar result the cause of the population inversion is different in both models. In this simplistic laser model the population inversion arises due to the coupled bath having a population inversion. In the electromechanical model the population occurs due to the backaction of electrons hopping on and off the central island.

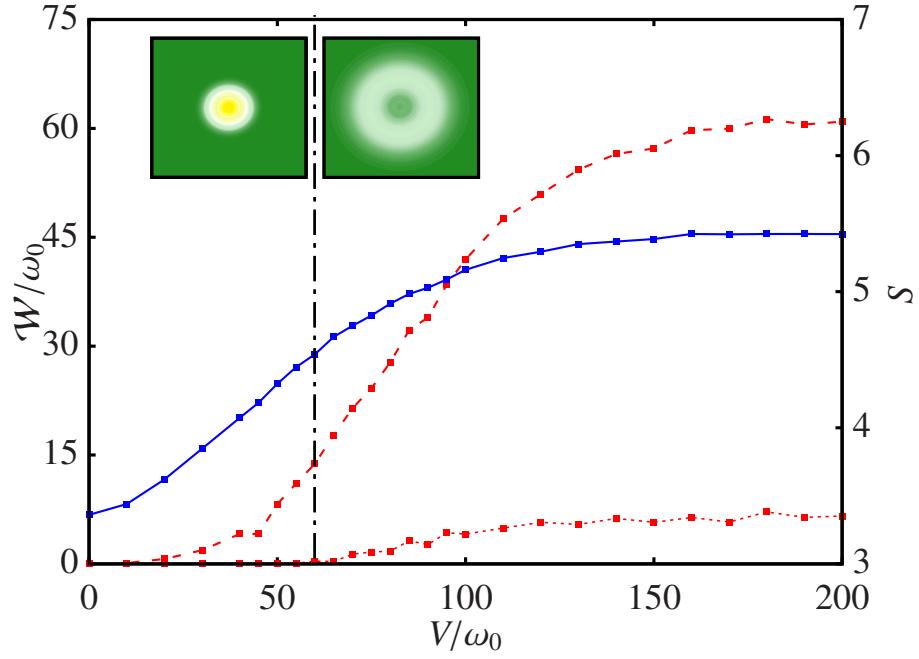


Figure 3.5: Total work extractable from the steady state of the vibrational mode via a unitary transformation ($\mathcal{W}_E$, red, dotted) and a thermal operation ($\mathcal{W}_F$, red, dashed). The threshold voltage is indicated by the vertical dot-dashed line and the von Neumann entropy of the oscillator is shown in blue. The same parameters as Fig. 3.3 are used.

3.3 Extractable Work

This electromechanical setup is an ideal platform for studying work storage and extraction. The non-equilibrium steady state of the device gives rise to a non-thermal state in the vibrational mode with a total energy that depends on the voltage applied. The vibrational degree of freedom thus serves as a battery or flywheel that stores energy deposited by the electronics. However, the division of this energy into useful work and waste — as per the first law of thermodynamics — is far from obvious, since some of the deposited energy merely contributes to increasing the entropy of the oscillator, i.e. heating.

We consider two different approaches to quantifying the deposited work. The first is the ergotropy, which gives the maximum work that can be extracted from a quantum battery by a cyclic unitary transformation [22] as discussed in section 1.2. Unitary work extraction is idealised since the requisite protocol is often challenging to implement. We thus consider a second approach to work extraction based on the non-equilibrium free energy [23], as discussed in 1.2. Since thermal operations may be non-unitary, we have $\mathcal{W}_E \leq \mathcal{W}_F$.

In our case, the battery is a vibrational mode with Hamiltonian $\hat{H}_b = \omega_0 \hat{a}^\dagger \hat{a}$, and we consider the work extractable from its steady state $\hat{\rho}_b$. The results of this analysis are presented in Fig. 3.5. The energy of the oscillator increases monotonically with voltage, as expected. This is reflected in the behaviour of the von Neumann entropy, which grows rapidly below threshold and eventually saturates far above threshold.

We now turn to the ergotropy (red, dotted line) and the result is striking. Below threshold, the ergotropy is exactly zero, meaning the non-thermal state of the oscillator is passive. In contrast, above threshold the state immediately becomes non-passive and $\mathcal{W}_E$ increases monotonically until saturation. Remarkably, therefore, we conclude that the ergotropy serves as an order parameter for the phonon lasing transition — unlike, for example, the mode quadratures themselves — so that the electromechanical battery only stores useful work above threshold. This finding motivates us to *define* the threshold as the voltage at which the ergotropy becomes non-zero. The appearance of non-zero ergotropy above threshold corresponds to the onset of population inversion in the mechanical state, and we expect similar behaviour to arise in other quantum systems exhibiting self-sustained oscillations.

The non-equilibrium free energy difference $\mathcal{W}_F$ is also plotted in Fig. 3.5 (red, dashed line). This follows a similar trend to the ergotropy but with one exception: even the passive oscillator steady states below threshold may yield some work output under thermal operations due to their distance from thermal equilibrium. However, the value of $\mathcal{W}_F$ below threshold is relatively small, demonstrating the importance of self-sustained oscillations for effective battery charging.

3.4 Summary and Outlook

In this chapter, we have quantified the conversion of electrical power into mechanical work stored in the quantum state of a NEMS device. Our results demonstrate that the self-sustained oscillations that were recently observed in Refs. [20, 124] can be interpreted as an electromechanical battery, thus bridging a gap between quantum thermodynamic theory and mesoscopic device physics. We found that typical battery states in the quasi-adiabatic regime are diagonal in the energy eigenbasis, but it may be possible to engineer

configurations that generate mechanical coherence — from which additional work can be extracted [185, 56] — for example, by working in the strong electromechanical coupling regime [156].

A pressing question for future work is to devise concrete protocols for transferring energy from the non-equilibrium battery states back into the electronic circuit [5], opening up the exciting prospect of on-demand energy storage and extraction using mechanical degrees of freedom. Such electromechanical energy conversion could also be performed using heat rather than charge currents [186].

Quantum Electromechanical Clock

Measuring time is perhaps the most primitive and also the most important of all physical measurements. As discussed in Chapter 1, clocks act as a foundation of the modern world, underpinning satellite navigation, magnetometry, gravimetry, telecommunications and financial markets [187, 188, 96]. Despite the spectacular precision now routinely achieved with atomic clocks, fundamental gaps in our understanding of time at the quantum level remain.

Previously, many simple theoretical toy models were constructed to study the fundamental thermodynamic limits of timekeeping. However, the connection between the abstract theory of quantum clocks and concrete experimental implementations remains lacking. We seek to remedy this by analysing an autonomous quantum clock based on nanoelectromechanical self-oscillations. We flip the analysis performed in the previous chapter by examining the electronic degrees of freedom for the NEMS model defined in Chapter 2. The nanomechanical oscillations are driven by a time-independent bias, providing an ideal setting to realise an autonomous clockwork.

In this model, the electromechanical system encapsulates the clockwork, while ticks are registered by monitoring the induced oscillations in the electronic current. As in previous studies, the autonomous nature of the clockwork dynamics allows for a self-contained analysis of the thermodynamic resources needed to measure time [8]. Our work goes beyond the existing literature on clock thermodynamics in at least two important respects.

First, our electromechanical clockwork is based on the oscillations of a continuous degree of freedom, reminiscent of a pendulum, see Sect. 1.6. Yet, unlike a pendulum clock, or indeed any other clock considered in the literature to our knowledge, these oscillations are driven by measurement backaction associated with the tick readout itself. Previous theoretical models of quantum clocks consider distinct irreversible processes that drive tick generation and tick readout [8, 25, 110, 109, 12, 10, 118]. Experimental work on the thermodynamics of timekeeping has either lain in a purely classical regime where measurement backaction is negligible [9] or focused purely on the role of measurement backaction in a non-autonomous setting [189]. Our setup is also distinct from Ref. [190], which considered a quantum clock powered by measurements that are unrelated to tick readout.

Second, we go beyond the performance measures introduced in previous works [8, 110] by evaluating the Allan variance [191], which quantifies clock accuracy over different integration times. At intermediate timescales, our results corroborate the general expectation that higher accuracy entails more dissipation. However, at longer timescales, this trend reverses. To explain this crossover in the Allan variance, we show that the non-trivial dynamics of the clockwork generates persistent temporal correlations between the clock's ticks. Our results thus call for further research to understand both the origin and consequences of such temporal correlations, which have been largely neglected in previous studies of autonomous quantum clocks.

The outline of this chapter is as follows: first, we will derive formulae to calculate the current and current noise in this setup. We will then calculate current trajectories and analyse the correlation function at different biases, identifying a separation of timescales between phase and amplitude fluctuations. To understand this separation of timescales, we introduce an Ornstein Uhlenbeck stochastic model and a bistable stochastic model. We then calculate the waiting time distribution of the clock to examine the tick distribution of the clock and find a trade-off between the accuracy, resolution and entropy production of the clock. Finally, we show that the Allan variance exhibits a non-trivial dependence on integration time and applied voltage, which can be understood in terms of correlations between the clock's ticks that we quantify explicitly using the mutual information and the relative entropy. This chapter is adapted from the paper “Powering an autonomous clock with quantum electromechanics” [123].

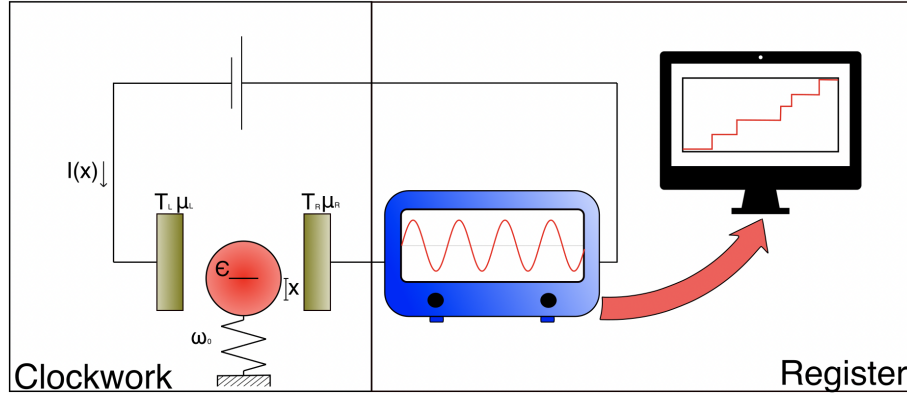


Figure 4.1: Schematic of the clock setup. Identical to 2.2, a quantum dot is coupled to two electrodes, each at a temperature T , with the left electrode having a chemical potential of μ_L and the right a chemical potential of μ_R . The quantum dot is coupled to simple harmonic oscillator mode. Under an appropriate static voltage bias, the current flowing through the system drives self-sustained mechanical oscillations, which in turn induces an oscillatory component in the current. A series of ticks are recorded by counting oscillations in the output current.

4.1 Electrical Current and Current Noise

The schematic of this electromechanical setup is outlined in Fig. 4.1. The clockwork consists of the electromechanical model outlined in Chapter 2. Throughout this work, we will use a Lorentzian spectral density defined in Eq. 3.30, with the same parameters shown in 3.2. This is done to explore whether the lasing transition has any notable effects on the clock's performance. The dynamics of the clockwork depend qualitatively on the voltage bias. At zero bias, the leads act as an equilibrium reservoir, with no net current flowing through the system.

Under a voltage bias, electrical current flows continuously through the system. Mechanical oscillations induce variations of this current that can be monitored to generate a series of ticks, as depicted in Fig. 4.1; this acts as the register for our clock. We assume that the current signal through the electromechanical system is effectively classical, i.e. it can be read out without additional measurement backaction beyond that already contained in the damping and diffusion terms of Eq. (2.40). This is justified in the quasi-adiabatic limit, because many electrons tunnel through the device within any given oscillation period. In practice, oscillations in the current can be continuously monitored by using a resonant tank circuit to filter frequency components near the relevant frequency, together with quantum-limited amplification [20, 19].

To understand how the oscillator motion modifies the current, we use the fact that $\hat{\rho}(x, p) \approx \mathcal{P}\hat{\rho}(x, p)$ to lowest order in small quantities [122], i.e. within the quasi-adiabatic limit the electronic state conditioned on the oscillator coordinates can be approximated by its stationary state. The electronic state at long times is therefore given to lowest order by

$$\hat{\rho}_{\text{el}} \approx \int dx \int dp W(x, p) \hat{\rho}_x = \int dx P(x) \hat{\rho}_x. \quad (4.1)$$

where $W(x, p)$ is the steady-state Wigner function and $P(x) = \int dp W(x, p)$ is the corresponding marginal probability distribution for the oscillator's position. Equation (4.1) can be used to compute observables of the electronic system. In particular, the average of the electronic current operator, $\hat{I}$, is given to lowest order by

$$\langle \hat{I} \rangle = \text{Tr} \{ \hat{I} \hat{\rho}_{\text{el}} \} \approx \int dx P(x) \text{Tr} \{ \hat{I} \hat{\rho}_x \} = \int dx P(x) \langle \hat{I} \rangle_x. \quad (4.2)$$

Here we have defined $\langle \hat{I} \rangle_x \equiv \text{Tr} \{ \hat{I} \hat{\rho}_x \}$ as the conditional expectation value of the current operator, given the oscillator's position x .

While the ensemble-averaged current tends to the stationary value (4.2) at long times, individual trajectories of the measured current will show periodic oscillations. This is because fluctuations of the oscillator position are transduced into current fluctuations, as characterised by the symmetrised autocorrelation function [192]

$$C(t) = \frac{1}{2} \langle \{ \hat{I}(t), \hat{I}(0) \} \rangle - \langle \hat{I} \rangle^2 = \text{Re} \left(\text{Tr} \{ e^{i\hat{H}t} \hat{I} e^{-i\hat{H}t} \hat{I} \hat{\rho} \} \right) - \langle \hat{I} \rangle^2, \quad (4.3)$$

where all averages are taken with respect to the steady-state density matrix, $\hat{\rho}$. Let us define an auxiliary density operator

$$\hat{K}(t) = e^{-i\hat{H}t} \hat{I} \hat{\rho} e^{i\hat{H}t}, \quad (4.4)$$

such that the correlation function can be found from the real part of

$$\langle \hat{I}(t) \hat{I}(0) \rangle = \text{Tr} \{ \hat{I} \hat{K}(t) \} = \int dx \int dp \text{Tr} \{ \hat{I} \hat{K}(x, p, t) \}, \quad (4.5)$$

where $\hat{K}(x, p, t)$ is the Wigner transform of $\hat{K}(t)$, defined analogously to Eq. (2.9). Since

$\hat{K}(t)$ obeys the Liouville-von Neumann equation

$$\frac{d\hat{K}}{dt} = -i[\hat{H}, \hat{K}], \quad (4.6)$$

we have that $\hat{K}(x, p, t)$ obeys the same equation of motion as $\hat{\rho}(x, p)$: namely,

$$\frac{\partial}{\partial t} \hat{K}(x, p, t) = (\mathcal{L}_x + \mathcal{H} + \mathcal{V}) \hat{K}(x, p, t), \quad (4.7)$$

where the superoperators on the right-hand side are defined in Eqs. (2.11)–(2.13). We can therefore approximate the solution to this equation following an adiabatic elimination procedure analogous to Sec. 2.5, but where now both $\mathcal{P}$ and $\mathcal{Q}$ projections are needed to reconstruct the correlation function.

Like the derivation in Sec. 2.5, we make use of the quasi-adiabatic limit. The electronic degrees of freedom can be assumed to relax quickly to the stationary state of $\mathcal{L}_x$, while $\mathcal{H}$ and $\mathcal{V}$ can be treated perturbatively. We define the compact notation $\hat{r}(t) = \mathcal{P} \hat{K}(x, p, t)$ and $\hat{q}(t) = \mathcal{Q} \hat{K}(x, p, t)$, suppressing phase-space arguments for concision. Inserting appropriate factors of the identity $1 = \mathcal{Q} + \mathcal{P}$ on both sides of Eq. (4.7), and using the properties (2.16), one has

$$\frac{\partial \hat{r}}{\partial t} = \mathcal{P} \mathcal{H} \hat{r} + \mathcal{P} (\mathcal{H} + \mathcal{V}) \hat{q}, \quad (4.8)$$

$$\frac{\partial \hat{q}}{\partial t} = \mathcal{L}_x \hat{q} + \mathcal{Q} (\mathcal{H} + \mathcal{V}) \hat{q} + \mathcal{Q} (\mathcal{H} + \mathcal{V}) \hat{r}. \quad (4.9)$$

Formally integrating Eq. (4.9) yields

$$\hat{q}(t) = \mathcal{G}(t) \hat{q}(0) + \int_0^t dt' \mathcal{G}(t-t') \mathcal{Q} (\mathcal{H} + \mathcal{V}) \hat{r}(t'), \quad (4.10)$$

$$\mathcal{G}(t) = \exp \{ [\mathcal{L}_x + \mathcal{Q} (\mathcal{H} + \mathcal{V})] t \}. \quad (4.11)$$

Substituting this back into Eq. (4.8), we obtain the closed equation of motion

$$\frac{\partial \hat{r}}{\partial t} = \mathcal{P} \mathcal{H} \hat{r}(t) + \int_0^t dt' \mathcal{P} (\mathcal{H} + \mathcal{V}) \mathcal{G}(t-t') \mathcal{Q} (\mathcal{H} + \mathcal{V}) \hat{r}(t') + \mathcal{P} (\mathcal{H} + \mathcal{V}) \mathcal{G}(t) \hat{q}(0). \quad (4.12)$$

The first two terms on the right-hand side (RHS) of Eq. (4.12) are identical to Eq. 2.25, and

can be treated by identical manipulations. We define the Fokker-Planck generator

$$\mathcal{L}_{\text{FP}}K = \frac{\partial}{\partial p} \left(m\omega_0^2 x - F \langle \hat{n} \rangle_x + \gamma_x p \right) K - \frac{p}{m} \frac{\partial K}{\partial x} + \frac{D_x}{2} \frac{\partial^2 K}{\partial p^2}, \quad (4.13)$$

where $K(x, p, t) = \text{Tr} \{ \hat{K}(x, p, t) \}$. The full equation for $K(x, p, t)$ is given by

$$\frac{\partial K}{\partial t} = \mathcal{L}_{\text{FP}}K + \text{Tr} \{ \mathcal{P} (\mathcal{H} + \mathcal{F}) \mathcal{G}(t) \hat{q}(0) \}. \quad (4.14)$$

This equation has the solution

$$K(t) = e^{\mathcal{L}_{\text{FP}}t} K(0) + \int_0^t dt' e^{\mathcal{L}_{\text{FP}}(t-t')} \text{Tr} [\mathcal{P} (\mathcal{H} + \mathcal{F}) \mathcal{G}(t') \hat{q}(0)]. \quad (4.15)$$

Since $\mathcal{P} \hat{K}(x, p, t) = K(x, p, t) \hat{\rho}_x$ and $\mathcal{P} \hat{K}(x, p, t)$ is given by Eq. (4.10), we have a formal solution for both components of $\hat{K}(x, p, t)$. We are considering the steady-state current correlation function, the initial condition of Eq. (4.7) is

$$\hat{K}(x, p, 0) = \hat{I} \hat{\rho}(x, p) \approx \hat{I} \hat{\rho}_x W(x, p), \quad (4.16)$$

to lowest order in small quantities, where $\hat{\rho}(x, p)$ and $W(x, p)$ correspond to the stationary state. Applying the projection operators, we get at lowest order

$$\mathcal{P} \hat{K}(x, p, 0) \approx \hat{\rho}_x \langle \hat{I} \rangle_x W(x, p), \quad (4.17)$$

$$\mathcal{Q} \hat{K}(x, p, 0) \approx (\hat{I} - \langle \hat{I} \rangle_x) \hat{\rho}_x W(x, p). \quad (4.18)$$

To the same order, the second term on the right-hand side of both Eqs. (4.10) and (4.15) can be neglected and we can write $\mathcal{G}(t) \approx e^{\mathcal{L}_x t}$. Therefore, we finally obtain the approximate solution

$$\begin{aligned} \hat{K}(x, p, t) &= \mathcal{P} \hat{K}(x, p, t) + \mathcal{Q} \hat{K}(x, p, t) \\ &\approx \hat{\rho}_x e^{\mathcal{L}_{\text{FP}}t} [\langle W(x, p) \rangle] + e^{\mathcal{L}_x t} [(\hat{I} - \langle \hat{I} \rangle_x) \hat{\rho}_x] W(x, p). \end{aligned} \quad (4.19)$$

Plugging this into Eq. (4.5) yields

$$\begin{aligned} \langle \hat{I}(t) \hat{I}(0) \rangle &= \int dx \int dp \langle \hat{I} \rangle_x e^{\mathcal{L}_{\text{FP}} t} [\langle \hat{I} \rangle_x W(x, p)] \\ &\quad + \int dx \int dp W(x, p) \left[\langle e^{\mathcal{L}_x^\dagger t} (\hat{I}) \hat{I} \rangle_x - \langle \hat{I} \rangle_x^2 \right]. \end{aligned} \quad (4.20)$$

The first term in Eq. (4.20) can be interpreted as an autocorrelation function of the stochastic variable $\langle \hat{I} \rangle_{x(t)}$ with respect to the Langevin evolution of the oscillator. To see this, note that $e^{\mathcal{L}_{\text{FP}} t} [\langle \hat{I} \rangle_x W(x, p)]$ is the solution of the Fokker-Planck equation with initial condition $\langle \hat{I} \rangle_x W(x, p)$. We define the Fokker-Planck propagator $\Pi(x', p', t | x, p, 0)$ as the conditional probability to find the oscillator at the phase space point (x', p') at time t' given that it was at (x, p) at time t . We may therefore write the first term in Eq. (4.20) as

$$\begin{aligned} \int dx \int dp \langle \hat{I} \rangle_x e^{\mathcal{L}_{\text{FP}} t} [\langle \hat{I} \rangle_x W(x, p)] &= \int dx' \int dp' \int dx \int dp \langle \hat{I} \rangle_x \Pi(x, p, t | x', p', 0) \langle \hat{I} \rangle_{x'} W(x', p') \\ &= \int dx' \int dp' \int dx \int dp \langle \hat{I} \rangle_x \langle \hat{I} \rangle_{x'} P(x, p, t; x', p', 0) \\ &\equiv \mathbb{E} [\langle \hat{I} \rangle_{x(t)} \langle \hat{I} \rangle_{x(0)}], \end{aligned} \quad (4.21)$$

where $P(x, p, t; x', p', 0) = \Pi(x, p, t | x', p', 0) W(x', p')$ is the joint probability to find the system initially (time $t = 0$) at (x', p') and finally (time t) at (x, p) , while $\mathbb{E}[\bullet]$ denotes an average over the joint distribution which can be computed by sampling Langevin trajectories.

The second term in Eq. (4.20) represents the instantaneous current fluctuations conditioned on the oscillator position x . Within the quasi-adiabatic limit, this correlation decays very quickly compared to the slow timescales of the oscillator. Thus, on timescales relevant for the oscillator, the real part of this correlation function can be approximated by a Dirac delta function:

$$\text{Re} \left(\langle e^{\mathcal{L}_x^\dagger t} (\hat{I}) \hat{I} \rangle_x - \langle \hat{I} \rangle_x^2 \right) \approx \Delta_x \delta(t), \quad (4.22)$$

where Δ_x is the steady-state current noise conditioned on x , i.e.

$$\Delta_x = \int_{-\infty}^{\infty} dt \text{Re} \left(\langle e^{\mathcal{L}_x^\dagger t} (\hat{I}) \hat{I} \rangle_x - \langle \hat{I} \rangle_x^2 \right). \quad (4.23)$$

Putting everything together we have generated the current autocorrelation function $C(t) =$

$C_{\mathcal{Q}}(t) + C_{\mathcal{P}}(t)$, with

$$C_{\mathcal{Q}}(t) = \mathbb{E}[\Delta_{x(0)}]\delta(t), \quad (4.24)$$

$$C_{\mathcal{P}}(t) = \mathbb{E}[\langle \hat{I} \rangle_{x(t)} \langle \hat{I} \rangle_{x(0)}] - \mathbb{E}[\langle \hat{I} \rangle_{x(0)}]^2. \quad (4.25)$$

For our model, both the conditional average current $\langle I \rangle_x$ and current noise Δ_x can be found using Landauer-Büttiker theory [47], as

$$\langle \hat{I} \rangle_x = \frac{1}{\pi} \int dE \tau_x(E) [f_L(E) - f_R(E)], \quad (4.26)$$

$$\begin{aligned} \Delta_x = & \frac{2}{\pi} \int dE \tau_x(E) [f_L(E)(1 - f_L(E)) + f_R(E)(1 - f_R(E))] \\ & + \frac{2}{\pi} \int dE \tau_x(E) [1 - \tau_x(E)] (f_L(E) - f_R(E))^2. \end{aligned} \quad (4.27)$$

Here, $f_\alpha(E) = [e^{\beta(E - \mu_\alpha)} + 1]^{-1}$ is the Fermi-Dirac distribution for lead α , while $\tau_x(E)$ is the transmission function of the resonant-level model with the position-dependent energy $\epsilon - Fx$ [47]:

$$\tau_x(E) = \frac{\kappa_L(E)\kappa_R(E)}{|E - \epsilon + Fx - \tilde{\chi}_L(E) - \tilde{\chi}_R(E)|^2}, \quad (4.28)$$

where $\kappa_\alpha(E)$ is the spectral density of lead $\alpha = L, R$ and $\tilde{\chi}_\alpha$ the self-energy. The above equations are valid, assuming coherent electron transport through the device so that we can neglect Coulomb interactions and inelastic scattering. However, our conclusions are not strongly dependent on the choice of transport model. We have checked explicitly that qualitatively similar results are obtained using a sequential tunnelling (i.e. rate equation [193]) description of transport.

4.2 Results

4.2.1 Analysis of current fluctuations

Current trajectories and timescale separation

As described in Sec. 4.1, the mechanical motion induces changes in the measured current. Therefore, mechanical self-oscillations will lead to a periodic current signal, which can be used to measure the passage of time. The accuracy of such timekeeping is then limited by stochastic fluctuations in the current signal, as characterised by the autocorrelation func-

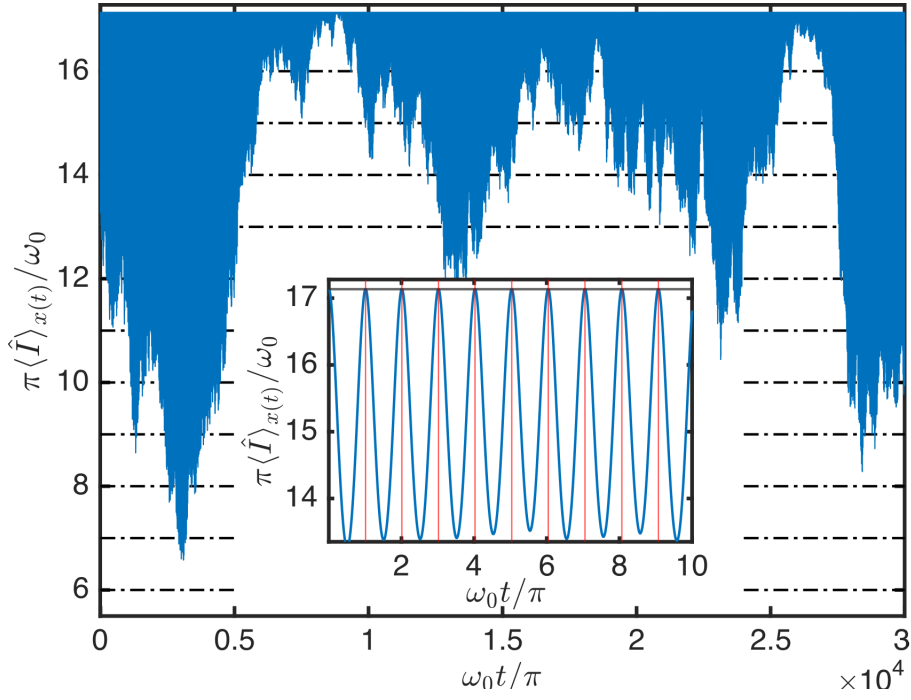


Figure 4.2: Portion of a single current trajectory above the self-oscillation voltage threshold, with $V = 100\omega_0$. The inset shows the oscillations on a shorter time scale where the oscillations appear more sinusoidal. The red vertical lines indicate times when a tick is recorded. Parameters: $\lambda = 0.5\omega_0$, $\Gamma = 10\omega_0$, $\beta\omega_0 = 0.1$, $\omega_L = -\omega_R = 2.5\omega_0$, and $\delta = 5\omega_0$.

tion $C(t)$. The most relevant fluctuations for timekeeping are encoded in the slow part of the correlation function $C_\phi(t)$ [Eq. (4.25)] because this varies on the same timescale as the clockwork dynamics. The fast fluctuations described by $C_\Omega(t)$ merely add a constant to the current noise spectrum, as discussed at the end of this section.

To characterise the properties of the measured current signal, we solve the Langevin equation 2.40 using the stochastic Euler method used in Chapter 3 and convert trajectories $x(t)$ into a corresponding current signal $\langle \hat{I} \rangle_{x(t)}$ via Eq. (4.26). Fig. 4.2 shows a sample current trajectory within the parameter regime where self-oscillations occur. We use the Lorentzian spectral density and the same parameters as chapter 2 for these calculations. On shorter timescales, we observe near-periodic oscillations of the current (Fig. 4.2 inset), with a peak-to-peak amplitude that varies significantly over longer timescales (Fig. 4.2 main panel). We also see that the oscillation frequency is approximately twice the natural frequency of the mechanical oscillator, and that the maximum oscillation amplitude has a fixed value that does not fluctuate. Both of these observations can be understood by the fact that the maximum current is obtained whenever the oscillator's position crosses $x = 0$. A natural way to measure time is, therefore, to register a tick of the clock whenever the current reaches its

maximum value, as illustrated in the inset of Fig. 4.2. Note that while the precise position x at which the current is maximised depends on the specific choice of parameters, other parameters would yield qualitatively similar results.

Figure 4.3 shows the correlation function $C(t)$ for three different applied voltages, spanning the transition from thermal noise at low bias to self-sustained oscillations at high bias. At low voltage, we observe decaying oscillations, as expected for the thermal fluctuations of an underdamped harmonic oscillator [72]. Meanwhile, two distinct timescales can be discerned at voltages above the threshold for self-sustained oscillations. The first, fast timescale, governs the decay of oscillations, which results from phase diffusion on the limit cycle. The second, slow timescale, dictates the asymptotic decay of the correlation function: for example, in the lower panel of Fig. 4.3, oscillations decay over a few hundred oscillation periods, while the correlation function itself does not decay completely until approximately 10^4 periods have elapsed. This latter decay is monotonic rather than oscillatory and can, therefore, be understood as a consequence of long-lived amplitude correlations.

Modelling amplitude and phase fluctuations

To elucidate the different roles of amplitude and phase fluctuations, in Appendix A we derive an effective equation of motion describing small fluctuations of the oscillator trajectory away from the limit cycle. We express the results in polar coordinates, i.e. $x = A \cos(\phi)$ and $\dot{x}/\omega_0 = -A \sin(\phi)$. An ideal limit cycle would have a constant amplitude $A = A_0$ and a linearly increasing phase $\phi = \omega_0 t$. Taking into account fluctuations away from the limit cycle up to first order, we obtain an Ornstein-Uhlenbeck process describing amplitude variations and a biased diffusion process describing the phase evolution, viz.

$$dA = -\gamma_A(A - A_0)dt + \sqrt{D_A}dw_A, \quad (4.29)$$

$$d\phi = \omega_0 dt + \sqrt{D_\phi}dw_\phi. \quad (4.30)$$

Here, γ_A is the damping rate governing the regression of amplitude fluctuations back to the limit cycle A_0 , dw_A and dw_ϕ are independent Wiener increments describing amplitude and phase fluctuations, respectively, and D_A and D_ϕ are the corresponding diffusion coefficients. Our simplified model assumes that the limit-cycle amplitude is large and that the timescale of phase and amplitude diffusion is small compared to the oscillation frequency. This jus-

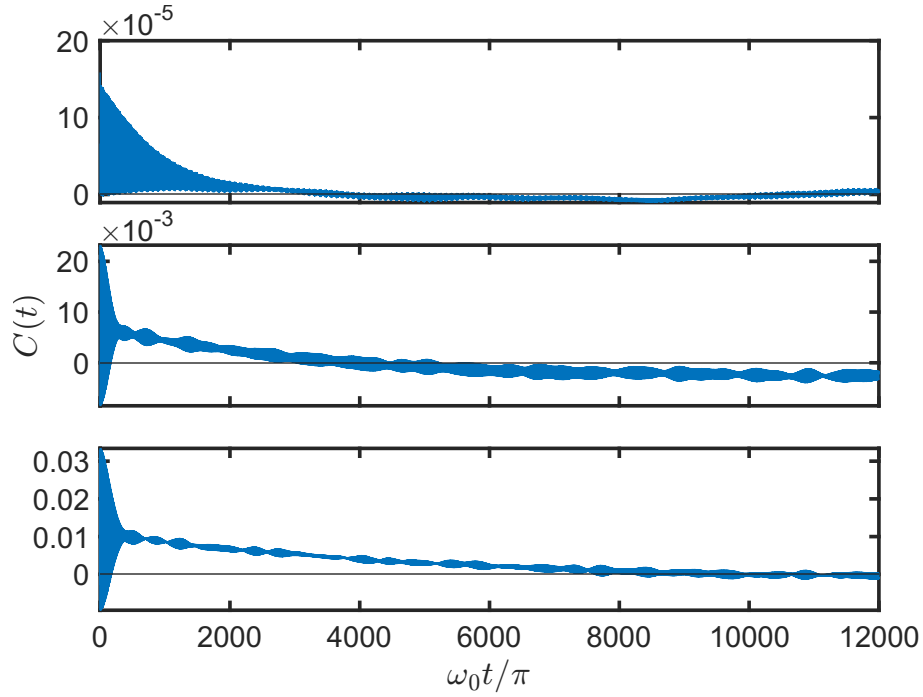


Figure 4.3: Auto-correlation function of the current for the same parameters as Fig. 4.2 and three different voltages: $V = 5\omega_0$ (top), $V = 50\omega_0$ (middle), $V = 100\omega_0$ (bottom). Two timescales emerge under high voltage bias: a fast-decaying component due to phase fluctuations and a longer-lived component associated with amplitude noise.

tifies approximating the amplitude and phase noise as independent even though they derive from the same Wiener process. The explicit expressions for the above parameters are cumbersome (see Appendix A for details), but a straightforward numerical evaluation shows that $\gamma_A \gg D_\phi$ in the high-bias regime where Eqs. (4.29) and (4.30) are valid. This indicates that small amplitude variations evolve significantly faster than phase decoherence, and therefore, this model alone cannot explain the timescale separation evident in Fig. 4.3.

We thus turn to an alternative toy model that captures the large-scale amplitude variations seen in Fig. 4.2. Specifically, the trajectory shown in Fig. 4.2 exhibits rare periods where the oscillation amplitude reduces dramatically, so the current remains close to its maximum value for a short time before returning to its previous oscillatory state. We model this behaviour as a bistable process, where the stochastic current switches between two states labelled by $i = 1, 2$, with corresponding currents $I_1(t)$ and $I_2(t)$. This kind of bistable behaviour was predicted for electromechanical self-oscillations in Ref. [160], and electromechanical bistabilities have recently been observed experimentally [194]. We assume that these two states have different steady-state averages $\bar{I}_i = \mathbb{E}[I_i(t)]$, and that they are statistically independent from each other and from the switching process itself. Under these

assumptions, we show in Appendix A that the steady-state autocorrelation function takes the form

$$C_{\mathcal{P}}(t) = \sum_{i=1}^2 p_i p_{i|i}(t) C_i(t) + C_{1 \leftrightarrow 2}(t), \quad (4.31)$$

$$C_{1 \leftrightarrow 2}(t) = \sum_{i,j=1}^2 p_i [p_{j|i}(t) - p_j] \bar{I}_i \bar{I}_j, \quad (4.32)$$

where p_i is the steady-state probability to be in state i , $p_{j|i}(t)$ is the probability to transition $i \rightarrow j$ over a time t , and $C_i(t) = \mathbb{E}[I_i(t)I_i(0)] - \bar{I}_i^2$ is the auto-correlation function of $I_i(t)$.

The first term in Eq. (4.31) clearly quantifies correlations within each state. The second term, $C_{1 \leftrightarrow 2}(t)$, is the contribution from the bistability itself: its time dependence arises only from the statistics of the switching process, while it depends on the current only through the conditional averages, $\bar{I}_i$. Therefore, the autocorrelation function will exhibit a separation of timescales whenever the switching process is much slower than phase and amplitude fluctuations of the current within each state. In Appendix A, we demonstrate this explicitly, showing that this simple bistable model recovers the features of the auto-correlation function above threshold in Fig. 4.3. Crucially, using the conservation of probability, $\sum_i p_i = \sum_j p_{j|i}(t) = 1$, one can show that $C_{1 \leftrightarrow 2}(t) \propto \bar{I}_1 - \bar{I}_2$. Therefore, it is important that the cycle-averaged current depends on the oscillation amplitude, which is indeed the case in our setup (cf. Fig. 4.2).

Current noise power spectrum

Finally, it is instructive to analyse the current noise in the Fourier domain via the power spectrum

$$\mathfrak{S}(\omega) = \int dt e^{i\omega t} C(t). \quad (4.33)$$

In addition to a flat white-noise contribution from the quantum shot-noise term $C_{\mathcal{Q}}(t)$, we observe two distinct peaks in the power spectrum at $\omega = 0$ and $\omega \approx 2\omega_0$, reflecting the separation of timescales inherent to $C_{\mathcal{P}}(t)$. The height of both peaks grows with voltage bias but for different reasons. The zero-frequency peak is associated with amplitude noise and its height scales with the lifetime of amplitude correlations (see Appendix A), which becomes very large well above threshold. This growth of the zero-frequency noise is well-known in other non-equilibrium systems with long-lived metastable states [160, 195]. Meanwhile,

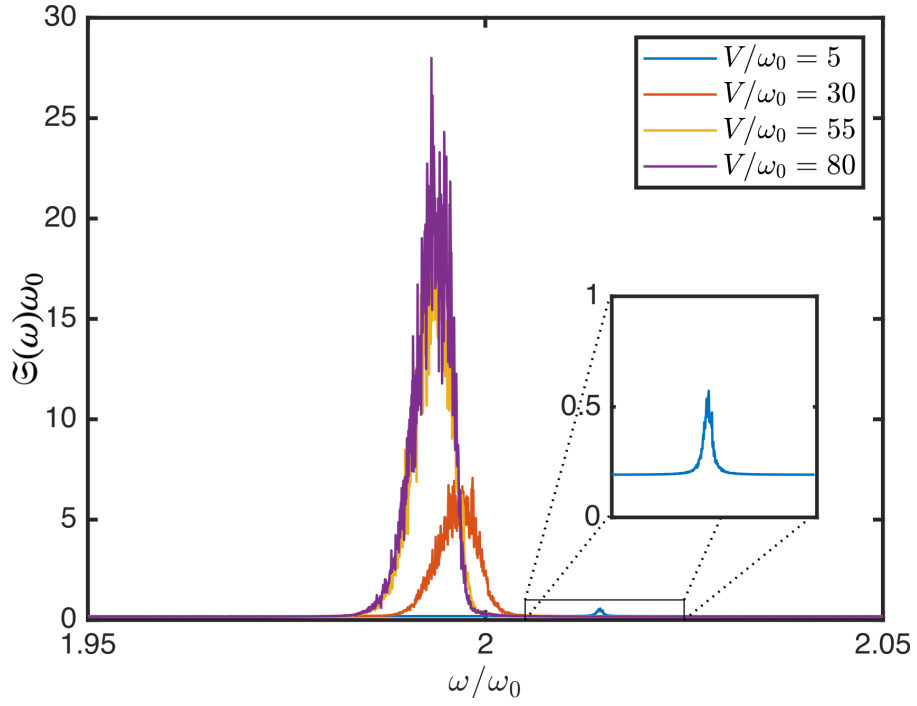


Figure 4.4: Power spectrum of the output current near the oscillation frequency $\omega \approx 2\omega_0$, for four different voltage biases; all other parameters are the same as Fig. 4.2. The frequency-independent contribution from $C_Q(t)$ is small and only makes a significant contribution for smaller biases as shown in the inset.

the peak in $\mathfrak{S}(\omega)$ at $\omega \approx 2\omega_0$ is related to electromechanical oscillations, and is plotted in Fig. 4.4. The height of this peak scales with the oscillation amplitude, which is, of course, larger in the self-oscillating regime.

As the voltage increases, the position of the finite-frequency peak shifts slightly downwards in frequency while its width becomes narrower. This implies that the clock's ticks become more regular in time while their frequency slightly decreases. As discussed in Sec. 4.2.2, these results indicate an increase in the clock's accuracy as the rate of dissipation increases, accompanied by a slight decrease in the clock's resolution. This is consistent with previous studies of autonomous clocks [8, 25, 9, 10].

4.2.2 Accuracy and resolution from the tick distribution

We now turn to a more detailed analysis of the individual ticks of the clock. As discussed in chapter 1, The tick distribution or waiting-time distribution (WTD) is defined as the probability, $W(\tau)$, of a time delay τ between two consecutive ticks. The clock's resolution

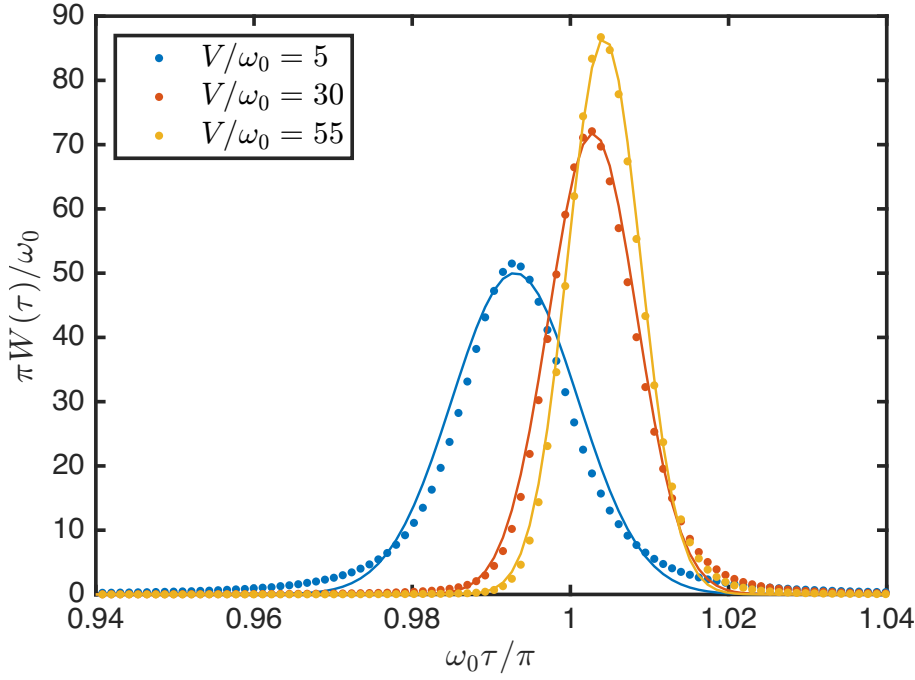


Figure 4.5: Waiting time distributions of the ticks for different voltage biases, with other parameters the same as Fig. 4.2. Dots are the data from the numeric calculations, the solid curves are the lines of best fit to the inverse Gaussian distribution (4.36), with fitting parameters μ and σ .

ν and accuracy N are defined as [8]

$$\nu = \mu^{-1}, \quad (4.34)$$

$$N = \frac{\mu^2}{\sigma^2}. \quad (4.35)$$

The resolution is simply the rate at which the ticks are generated, while the accuracy is the expected number of ticks before the clock's reading is off by one tick. This definition of accuracy assumes that the waiting time between ticks are independent and identically distributed (i.i.d.) random variables.

We numerically construct the WTD for our model as a normalised histogram of the waiting times between all pairs of consecutive ticks in the time series. The results are plotted in Fig. 4.5 for three different voltages. Above threshold, we find that the WTD is well approximated by an inverse Gaussian (Wald) distribution

$$W(\tau) = \sqrt{\frac{\mu^3}{2\pi\sigma^2\tau^3}} \exp\left(-\frac{\mu(\mu - \tau)^2}{2\sigma^2\tau}\right). \quad (4.36)$$

As shown in Ref. [196], Eq. (4.36) can be derived by considering the distribution of the

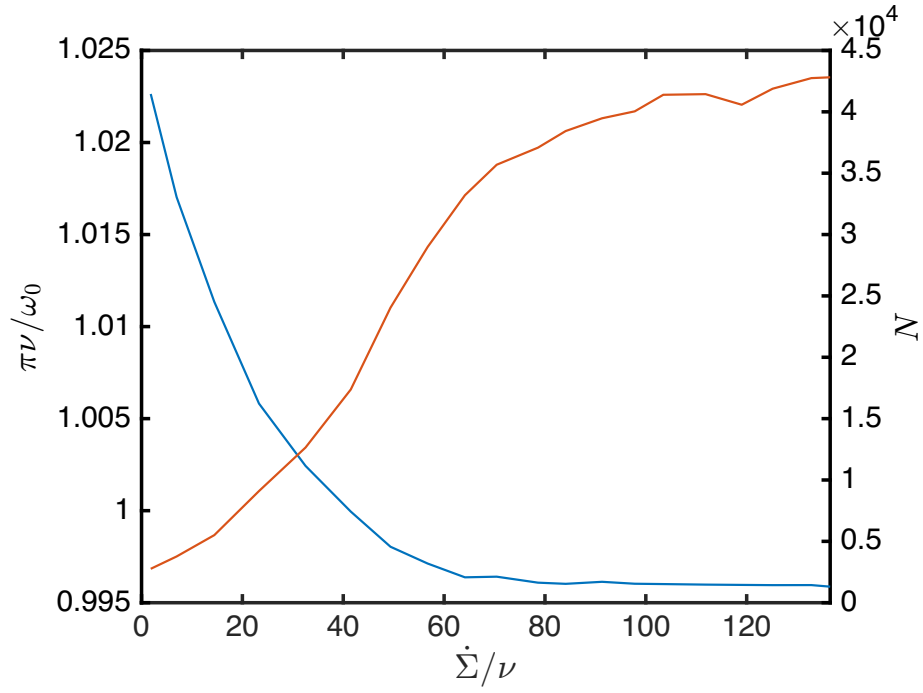


Figure 4.6: Accuracy (red line), and resolution (blue line) for increasing bias voltage, plotted against the entropy produced per tick of the clock. Parameters are the same as Fig. 4.2.

first-passage time τ for the oscillation phase on the limit cycle to complete one full 2π rotation. This result holds when the phase is described by a continuous-time biased random walk, as in Eq. (4.30). The solid curves in Fig. 4.5 show the best fits of the numerical WTDs to Eq. (4.36), with μ and σ as fitting parameters.

Notably, as the bias across the system is increased, the distribution's mean increases and the distribution's variance is decreased. This indicates that the clock's accuracy increases as the bias across the system is increased while its resolution decreases slightly. Increasing the bias also increases the average rate of entropy production, which is proportional to the power dissipated in the circuit:

$$\dot{\Sigma} = \beta \langle \hat{I} \rangle V. \quad (4.37)$$

Figure 4.6 shows the accuracy and resolution of the electromechanical clock against the entropy produced per tick, $\Delta S_{\text{tick}} = \dot{\Sigma}/\nu$. We observe an approximately linear relationship between accuracy and entropy production during the onset of self-sustained oscillations, showing that accurate timekeeping comes at the cost of increased dissipation. However, as the voltage is increased further, the growth of accuracy slows and begins to saturate to a value of around 4×10^4 : at this point, additional dissipation gives little further advantage to timekeeping.

Meanwhile, the clock's resolution decreases and eventually saturates with increasing dissipation. However, the change in resolution is minimal for this model, which is unsurprising given that the physical oscillator frequency fundamentally fixes the tick period. The small decrease in resolution is probably due to effective non-linearities associated with the position dependence mean-field force $\propto F \langle \hat{n} \rangle_x$ and damping rate γ_x , whose effect will change as the oscillator explores a larger region of phase space.

4.2.3 Allan variance

The measures of accuracy and resolution considered in Sec. 4.2.2 are only strictly applicable when the waiting times are i.i.d. random variables. The latter condition is equivalent to the statement that the total number of ticks is a renewal process [197]. However, this assumption does not precisely hold because the limit cycle is mesoscopic in scale and subject to strong amplitude fluctuations. While the amplitude and phase noise can be approximated as independent for very small deviations from the limit cycle [see Sec. 4.2.1], this conclusion generally does not hold. Larger amplitude fluctuations will become correlated with phase noise because they both derive from the same underlying stochastic force induced by electron tunnelling. Therefore, differences in the state of the system after each tick may cause the next tick to be correlated with the previous ones.

Given that the ticks of our mesoscopic electromechanical clock are correlated, we need a more sophisticated measure of performance than the accuracy and resolution defined in Eqs. (4.34) and (4.35). We focus instead on the Allan variance [191], which measures the frequency stability of a stochastic process and is widely used to test the stability of sensors, clocks and accelerometers [198–200]. The Allan variance quantifies the deviation of a measured clock signal θ , from a second reference clock. For simplicity, we take this reference to be an ideal clock whose reading is the coordinate time, t , so that the measured deviation is

$$X(t) = \theta(t) - t. \quad (4.38)$$

Then, $y(t) = dX/dt$ is the instantaneous frequency difference between the measured clock and the reference clock at a time t . This frequency deviation is estimated by observing the

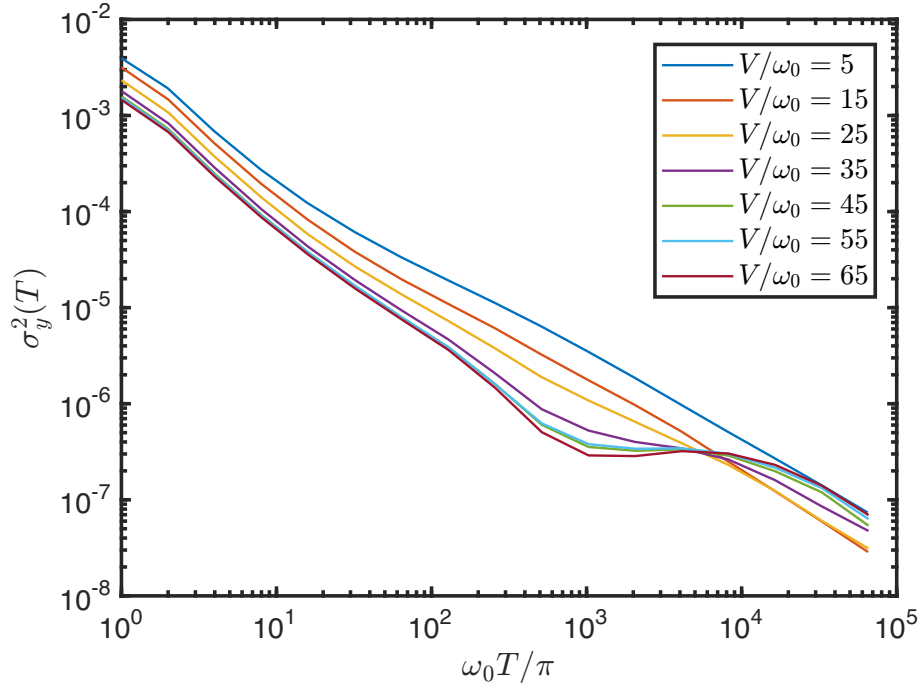


Figure 4.7: Allan variance as a function of integration time T for various voltage biases. Parameters are the same as Fig. 4.2.

clock's reading over a finite integration time T , leading to the average fractional frequency difference

$$\bar{y}(t, T) = \frac{1}{T} \int_0^T dt' y(t + t') = \frac{X(t + T) - X(t)}{T}. \quad (4.39)$$

The Allan variance is then proportional to the mean-square deviation of successive average fractional frequency differences

$$\sigma_y^2(T) = \frac{1}{2} \mathbb{E} \left\{ [\bar{y}(t + T, T) - \bar{y}(t, T)]^2 \right\}, \quad (4.40)$$

thus quantifying how instability in the average frequency scales with integration time. In the case of deterministic signals, where the frequency remains constant, the anticipated frequency deviation is zero. Consequently, a clock that closely adheres to deterministic behaviour — and is, therefore, more accurate — would exhibit a smaller Allan variance.

We compute the Allan variance for our electromechanical clock by partitioning the stochastic current trajectories into non-overlapping regions of equal duration T . The measured deviation pertaining to the n^{th} region, $X_n = X(n\tau)$, is evaluated by counting the number of

registered ticks up to time $t = n\tau$. Finally, the Allan variance is computed as

$$\sigma_y^2(T) = \frac{1}{2T^2} \mathbb{E} \left[(X_{n-1} - 2X_n + X_{n+1})^2 \right]. \quad (4.41)$$

The behaviour of the Allan variance as a function of T may be sensitive to the precise choice of T values — we have repeated our calculations for different values of T to ensure that the results are robust.

The results for the Allan variance are plotted in Fig. 4.7, for a range of different voltages. In general, the Allan variance decreases systematically with T : evidently, integrating the clock signal over a longer time yields greater precision. This dynamic highlights the trade-off between the clock's timekeeping accuracy and resolution [10]. Here, σ_y^2 decreases indefinitely with T , whereas real experiments usually find that the Allan variance grows beyond a certain integration time. This is typically due to the influence of low-frequency fluctuations, e.g. due to temperature variations or the gradual degradation of the apparatus over time, which are absent in our theoretical model.

At short and intermediate timescales, we observe from Fig. 4.7 that higher voltage leads to smaller values of σ_y^2 , demonstrating a quantitative link between accuracy and the rate of energy dissipation. It follows that the relationship between timekeeping precision and entropy production extends over multiple clock ticks, beyond the simple accuracy measure of Eq. (4.35) that only accounts for the single-tick WTD. At long times, all curves in Fig. 4.7 tend to the approximate scaling $\sigma_y^2 \sim T^{-1}$. This is consistent with known results for the long-time Allan variance for a renewal process [118]:

$$\sigma_y^2 \rightarrow \frac{\mu}{NT}, \quad (4.42)$$

which is fully determined by the clock's accuracy N defined in Eq. (4.35). This result can be derived by considering the mean and variance of a ticking process $n(t)$,

$$\mathbb{E} [n(t)] = \frac{t}{\mu} + \mathcal{O}(t^0), \quad (4.43)$$

$$\mathbb{E} [n(t)^2] - \mathbb{E} [n(t)]^2 = \frac{\sigma^2 t}{\mu^3} + \mathcal{O}(t^0), \quad (4.44)$$

where μ and σ are the mean and variance of the waiting time distribution, and t^0 is a finite

time effect from sampling the initial tick. The fractional frequency deviation averaged over a time T is

$$\bar{y}(t, T) = \frac{\mu}{T} [n(t + T) - m(t)]. \quad (4.45)$$

Using this to calculate the Allan variance

$$\begin{aligned} \sigma_y^2(T) &= \frac{1}{2} \mathbb{E} \{ [\bar{y}(t + T, T) - \bar{y}(t, T)]^2 \} \\ &= \frac{1}{2} \mathbb{E} [\bar{y}(t + T, T)^2] + \frac{1}{2} \mathbb{E} [\bar{y}(t, T)^2] - \mathbb{E} [\bar{y}(t + T, T) \bar{y}(t, T)] \\ &\xrightarrow{T \rightarrow \infty} \mathbb{E} [\bar{y}(t, T)^2] - \mathbb{E} [\bar{y}(t, T)]^2. \end{aligned} \quad (4.46)$$

The first two terms are equal on the second line as the process is stationary. The third line then follows in the limit of large T because $\bar{y}(t + T, T)$ and $\bar{y}(t, T)$ become statistically independent, i.e. the number of renewals in time interval $[t, t + T]$ becomes independent of the number of renewals in the following time interval $[t + T, t + 2T]$. Therefore, the Allan variance for large T is (up to scale factors) nothing but the variance of $n(t + T) - n(t)$, i.e. the, number of renewals within a time interval of duration T . Analogously to Eq. (4.44), this result is known asymptotically to be $\text{Var} [n(t + T) - n(t)] = \sigma^2 T / \mu^3 + \mathcal{O}(T^0)$ [197]. Finally, this yields the asymptotic Allan variance for a renewal process

$$\sigma_y^2(T) \longrightarrow \frac{\sigma^2}{\mu T}, \quad (4.47)$$

A similar $1/T$ scaling is observed in Fig. 4.7 across essentially all times for voltages below the threshold for self-oscillations. Interestingly, however, the Allan variance initially decreases slightly faster than T^{-1} for voltages above threshold. This trend continues up to an intermediate timescale on the order of a few thousand ticks, after which the slope of $\sigma_y^2(T)$ changes dramatically and the advantage in timekeeping precision due to self-oscillations is eventually lost altogether. The non-trivial behaviour of the Allan variance as a function of T indicates that temporal correlations play a significant role in determining the clock's performance over multiple ticks. The exact cause for the curve crossing that appears at a timescale at around 10^4 isn't fully understood. While the performance of the clock on short timescales for higher biases can be accounted for through an increase in entropy dissipation and correlations between ticks - see the following section. The long-time behaviour of the Allan variances can be explained as renewal processes; the intermediate behaviour needs further analysis, especially the non-monotonicity of the higher bias curves and the crossings

seen.

4.2.4 Quantifying correlations between ticks

We turn to explicitly quantifying correlations between the ticks of the clock, which are responsible for the faster decrease of $\sigma_y^2(T)$ above threshold. We consider two different measures of statistical correlations. First, we consider the clock reading after n ticks, $t_n = \sum_{i=1}^n \tau_i$, which is the sum of n consecutive waiting times $\tau_1, \tau_2, \dots, \tau_n$. Let $P_n(t_n)$ be the corresponding probability distribution. If the ticks were generated by a renewal process, so that the individual waiting times were i.i.d., this distribution would be given by the n -fold convolution of single-tick WTDs, i.e.

$$\left[\overset{n}{\circledast} W \right] (t_n) = \int_0^{t_n} dt_1 \int_{t_1}^{t_n} dt_2 \cdots \int_{t_{n-2}}^{t_n} dt_{n-1} W(t_1) \times \\ W(t_2 - t_1) \times \cdots \times W(t_n - t_{n-1}). \quad (4.48)$$

We use the relative entropy (Kullback-Leibler divergence) to measure the distance between $P_n(t_n)$ and the i.i.d. result in Eq. (4.48):

$$D_{\text{KL}}(P_n \| \overset{n}{\circledast} W) = \int_0^\infty dt P_n(t) \ln \left(\frac{P_n(t)}{\left[\overset{n}{\circledast} W \right] (t)} \right). \quad (4.49)$$

Since we are considering a stationary process, if the waiting times are not i.i.d. then they are not independent. Equation (4.49) can thus be understood as a measure of the total statistical correlations among all n ticks.

The inset of Fig. 4.8 demonstrates that the ticks are indeed correlated: the relative entropy grows as a function of n . Moreover, this growth is faster for higher voltages, indicating that correlations between ticks become more significant in the self-oscillation regime. Since these correlations should have a finite range in time, we expect the total correlations to eventually saturate as n increases. Unfortunately, we are unable to reliably extend our calculation of $D_{\text{KL}}(P_n \| \overset{n}{\circledast} W)$ to higher values of n : the results become very noisy due to the limited quantity of available numerical data and the relative complexity of the n -fold convolution.

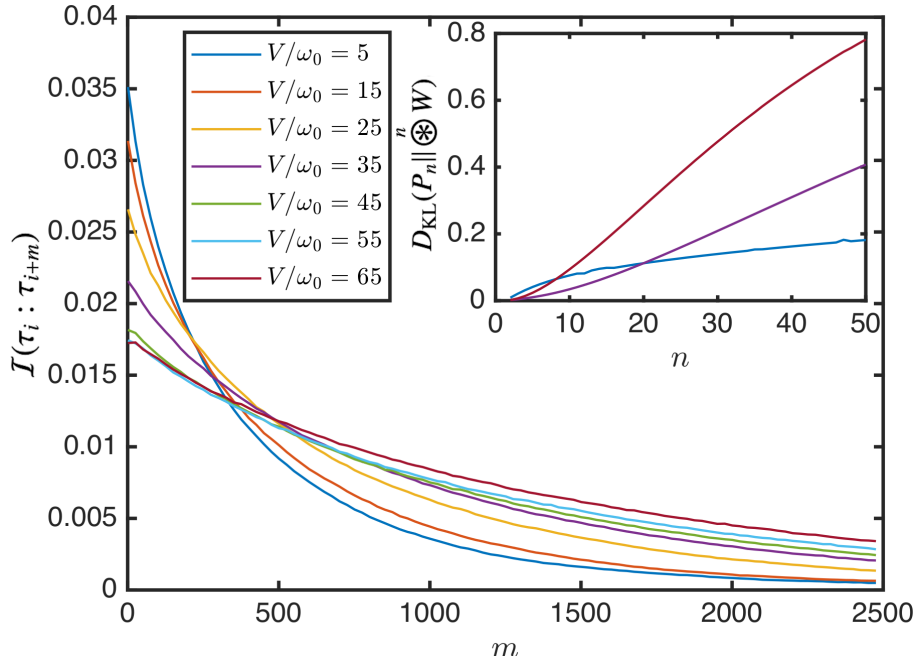


Figure 4.8: The main panel shows the mutual information [Eq. (4.50)] between a tick and the m^{th} subsequent tick, i.e. the pairwise correlations between the waiting times τ_i and τ_{i+m} . The inset shows the relative entropy [Eq. (4.49)] between the distribution of the n -tick clock reading $P(t_n)$ and the n -fold convolution of single-tick WTD, which measures the total correlations among all n ticks. Parameters are the same as Fig. 4.2.

To estimate the timescale over which correlations decay, we focus on a simpler measure of correlations: the mutual information between pairs of ticks that are m ticks apart. This is defined as

$$\mathcal{I}(\tau_i : \tau_{i+m}) = 2H[W(\tau_i)] - H[W(\tau_i, \tau_{i+m})], \quad (4.50)$$

where $W(\tau_i, \tau_{i+m})$ is the joint distribution of the waiting times τ_i and τ_{i+m} , $W(\tau_i)$ is the WTD (identical for τ_i and τ_{i+m}) and $H[W(\tau)] = -\int d\tau W(\tau) \ln W(\tau)$ denotes the Shannon entropy. Unlike the relative entropy, which measures the total correlations among all ticks, Eq. (4.50) quantifies only correlations between specific pairs of ticks and can be computed reliably over long timescales using our numerical data. The mutual information is plotted in the main panel of Fig. 4.8. This plot demonstrates that, indeed, the correlations are short-ranged in time: $\mathcal{I}(\tau_i : \tau_{i+m})$ is largest for small m and at lower voltages. However, these correlations persist over long timescales on the order of several thousand ticks, especially at voltages above threshold. This is the same timescale at which the non-monotonic behaviour of the Allan variance in Fig. 4.7 appears.

We thus conclude that σ_y^2 decreases more rapidly above threshold because of temporal

correlations between the ticks. These correlations provide an advantage in timekeeping precision, so long as the integration time T is not much greater than the correlation time. By contrast, when T far exceeds the range of temporal correlations, the integrated signals in two neighbouring time periods become effectively independent, and the behaviour of the Allan variance reduces to that of a renewal process, i.e. $\sigma_y^2 \sim T^{-1}$.

These correlations are likely dictated by several factors acting in tandem. The timescale over which they persist appears to be far longer than the timescale of pure phase decoherence apparent in Fig. 4.5, indicating that amplitude fluctuations also play a role. This can be understood from the fact that amplitude and phase fluctuations are generally not independent, meaning that a variation in amplitude after one cycle can influence the time taken for the next cycle to be completed. The timescale over which tick correlations persist will then be determined by a combination of phase decoherence, amplitude fluctuations near the limit cycle, and large-scale amplitude variations due to bistability in the current signal.

4.3 Summary and Outlook

Through this chapter, we have introduced a self-contained model of an autonomous clock, where the clockwork is an electromechanical self-oscillator, and the ticks of the clock are read out from the electrical current passing through the system. A peculiar and novel feature of our setup is that the clockwork is driven by the same irreversible process that is used for tick readout — the tunnelling of electrons through the device. We have confirmed that the clock's accuracy increases with the rate of entropy production, as the current drives the oscillator into a limit cycle with long-lived phase coherence. We have identified that both phase and amplitude fluctuations play an important role in determining the statistical properties of the measured current.

A key finding of our work is that the clock's ticks are generally correlated and that these correlations can improve the clock's accuracy, as quantified by the Allan variance. It remains an open problem to understand under what conditions such correlations can improve clock accuracy in general. We note that temporal correlations have recently been considered in the context of (classical) parameter estimation, where they may either help or hinder de-

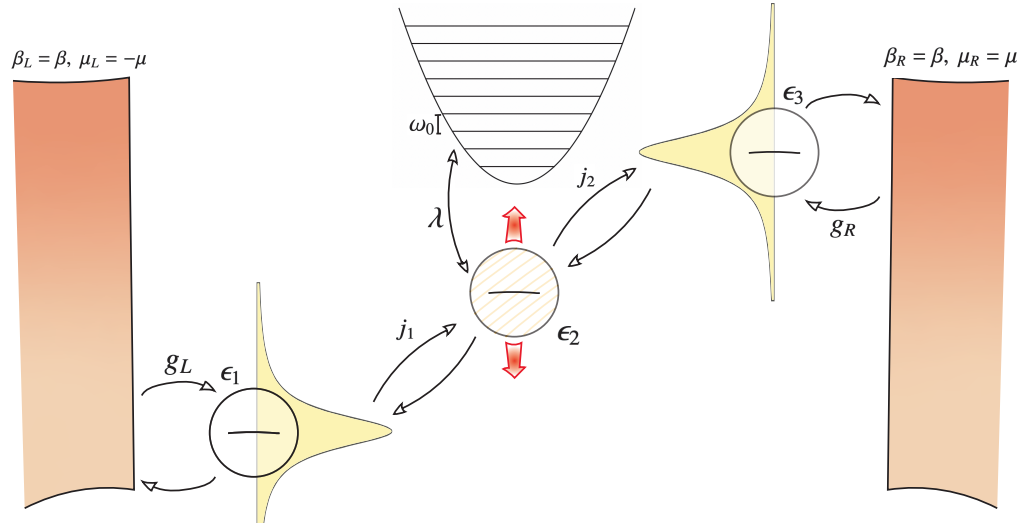


Figure 4.9: Cartoon sketch outlining a simple mesoscopic leads approach. The central dot is sandwiched between two auxiliary quantum dots. The auxiliary quantum dots are weakly coupled to two reservoirs. The effective coupling between the central dot is described by an effective Lorentzian spectral density, incorporating energy dependence into the coupling.

pending on their nature [201]. Yet far more work is needed to unify the theory of quantum ticking clocks with the standard framework of quantum parameter estimation, especially beyond the asymptotic regime of infinitely many samples (or in this case, ticks).

Our model is strongly motivated by modern electromechanical platforms, meaning our predictions could be directly probed in state-of-the-art experiments. In the present work, we have limited ourselves to the quasi-adiabatic regime, which is often the most experimentally relevant and the simplest regime to treat theoretically. However, the quasi-adiabatic regime is very far from thermodynamically optimal since many electrons tunnel through the device per clock tick. This regime is also sub-optimal from the perspective of readout, since the mechanical oscillations are “oversampled” by the tunnelling events [10]. It is thus important to explore the complementary regime where each oscillation is sampled by only a handful of electrons, approaching the lower sensitivity limit of electrical readout and entailing thus a far smaller rate of dissipation. In this regime, fluctuations in the tunnelling time of individual electrons (e.g. see Ref. [202] and references therein) may become relevant for the clock’s performance. At the level of theory and modelling, going beyond the quasi-adiabatic limit requires a non-perturbative methodology capable of tackling situations where the electromechanical coupling and the tunnelling rates are comparable.

To move away from the limit of many tunnelling events there are many options available,

one could apply the mesoscopic-leads/extended-reservoir approach [203, 204]. One could also model the electromechanical system using a Lindblad master equation approach, modifying the Hamiltonian in Eqs. 2.1-2.4

$$\hat{H}_S = \sum_{i=0}^2 \epsilon_i \hat{c}_i^\dagger \hat{c}_i + \sum_{i=0}^1 j_i (\hat{c}_i^\dagger \hat{c}_{i+1} + \hat{c}_{i+1}^\dagger \hat{c}_i), \quad (4.51)$$

$$\hat{H}_B = \sum_k \left(\Omega_{kL} \hat{d}_{kL}^\dagger \hat{d}_{kL} + \Omega_{kR} \hat{d}_{kR}^\dagger \hat{d}_{kR} \right), \quad (4.52)$$

$$\hat{H}_T = \sum_k \left(g_L \left[\hat{c}_0^\dagger \hat{d}_{kL} + \hat{d}_{kL}^\dagger \hat{c}_0 \right] + g_R \left[\hat{c}_2^\dagger \hat{d}_{kL} + \hat{d}_{kL}^\dagger \hat{c}_2 \right] \right), \quad (4.53)$$

$$\hat{H}_V = \frac{\hat{p}^2}{2m} + \frac{m\omega_0^2 \hat{x}^2}{2} - \lambda \hat{c}_1^\dagger \hat{c} (\hat{a} + \hat{a}^\dagger). \quad (4.54)$$

Here, we have introduced two additional dots coupled between the central dot and the electrodes with energies ϵ_i and the coupling strength between dots j_i . The coupling between the outer dots and leads is characterised by an energy-independent spectral density $\kappa_\alpha = g_\alpha$; however, for the central dot, the effective spectral density is given as

$$\kappa_\alpha^{\text{eff}}(\omega) = \sum_{k=1}^L \frac{|j_\alpha|^2 \kappa_\alpha}{(\omega - \epsilon_\alpha)^2 + \left(\frac{\kappa_\alpha}{2}\right)^2}. \quad (4.55)$$

This is equivalent to the mesoscopic lead approach using a single auxiliary mode per bath [203]. This description incorporates the energy-dependent spectral density key to observing self-sustained oscillations. So long as the coupling between the leads and dots is small, this model can be solved using the Lindblad master equation approach outlined in Sect. 1.1. One can then unravel the dynamics, monitoring the exchange of electrons from the leads to the system [40, 205]. Such unravellings can be used to probe the waiting time distribution, trade-offs between accuracy, resolution and entropy production and the correlations between ticks.

It is interesting to note that our model predicts clock accuracies (Fig. 4.6) that significantly surpass thermodynamic performance bounds that apply to classical clocks with a discrete state space [13]. In particular, the thermodynamic uncertainty relation [111, 112] limits the accuracy of such clocks to one half of the entropy production per tick, i.e. $N \leq \dot{S}/2\nu$ (assuming the ticks are generated by a renewal process). The fact that our electromechanical clock can surpass these bounds is unsurprising: the dynamics of our clock are governed

by an underdamped classical Langevin equation, which does not necessarily obey these bounds [119, 206]. Nevertheless, the approximations inherent to our model do neglect some potentially relevant contributions, which leads us to overestimate the accuracy and underestimate the entropy production.

First, our definition of the ticks is based on the quasi-adiabatic current fluctuations quantified by $C_{\varphi}(t)$. We have neglected the shot-noise contribution due to the discrete nature of the electronic charge, i.e. $C_Q(t)$. This additional source of white noise would partially obscure the precise time at which ticks occur. To improve tick identification in the presence of such noise, one could redefine the tick time to occur when the time derivative of the current, rather than the current itself, is at its maximum [9]. Second, reading out relatively small currents would require amplification in practice, and we have ignored the additional dissipation associated with this measurement step. We note that the thermodynamic cost of the measurement has also been neglected in previous work on autonomous clocks [8, 25, 109, 118]. Understanding how this additional source of dissipation enters the balance of entropy production for timekeeping is an important open question for future work.

Finally, it would be interesting to explore the thermodynamics of timekeeping for other kinds of self-oscillators in the quantum regime. Conditions for self-oscillations to emerge from open quantum dynamics have only recently been established [207–210], within the broader context of research on broken time-translation symmetry [211–213]. However, a general recipe for exploiting such symmetry-breaking for accurate timekeeping remains lacking.

The Thermodynamics of the Quantum Mpemba Effect

The Mpemba effect is a paradoxical phenomenon understood in pop culture as “hot water freezing faster than cold water”. It was first detailed by Aristotle over two millennia ago when observing ancient Greek fishermen in Pontus repairing fishing rods using hot water rather than cold [214]. The effect was recorded multiple times throughout the past millennium by various writers, including René Descartes [215] and Francis Bacon [216].

A Tanzanian secondary school student - Erasto Mpemba - brought the effect into the modern scientific world. Whilst preparing ice cream in class, Mpemba put his cream and sugar mixture into the freezer while still hot. Returning to the freezer later, Mpemba noticed that his ice cream had crystallised faster than his classmates, who had placed theirs in the freezer at room temperature. Perplexed by this observation, Mpemba would later perform a more systematic study with Denis Osbourne [217]. Mpemba would later become eponymous with the effect he rediscovered. It is worth noting that at around the same time, another physicist, G.S. Kell, also wrote an article about the Mpemba effect [218]. Kell garnered inspiration from Canadian folklore. However, this story was less charming than Mpemba's, and, unfortunately for Kell, he did not get the eponym.

Since its rediscovery, the effect has generated friction between proponents and detractors of the phenomenon in macroscopic systems [219, 220]. The effect was the focus of a competition held by the Royal Society of Chemistry in 2012 [221]. Seemingly defying Newton's law of cooling, part of this commotion is due to the broad definition of the effect, the sub-

stance being cooled, cooling vs freezing, etc. Experimentally, observation of the effect (or lack thereof) can depend on the shape of the container, cooling mechanism, and placement of the thermometer, among other factors [222]. Various candidate mechanisms, including hydrogen bonds and supercooling, have been proposed to enable the effect.

The rest of this chapter will steer clear of the controversial effect in macroscopic systems and focus on the Mpemba effect in microscopic systems. We briefly introduce Davies maps, a Lindblad master equation describing the thermalisation of quantum systems. We demonstrate how a unitary transformation on the initial state can generate an exponential speed-up, which will always occur if a complex eigenpair defines the spectral gap of the generator. This transformation also increases the state's non-equilibrium free energy, thus constituting what we coin a genuine Mpemba effect. We illustrate this finding in single-qubit and many-qubit examples. This chapter is adapted from the paper “Thermodynamics of the Quantum Mpemba Effect” [223].

5.1 The Mpemba Effect

For this thesis, we need to define the Mpemba effect explicitly (or, perhaps less controversially — a Mpemba-like effect). We define some distance from the stationary point as $D[\hat{\rho}(t); \hat{\tau}]$, where $\hat{\rho}(t)$ is the state of the system at a time t and $\hat{\tau}$ is the steady state of the system, this distance function could be the L_1 distance, the relative entropy or even temperature in the case of Mpemba's original work. A Mpemba effect occurs when two distinct initial states, ρ_i^1 and ρ_i^2 , evolve under the same dynamical map with the same fixed point. If ρ_i^1 is initially further away from the steady state than ρ_i^2 but at some later time t , $\rho_1(t)$ is closer to the steady state than $\rho_2(t)$, then we can say a Mpemba effect has occurred. More succinctly, a Mpemba effect occurs if

$$D[\hat{\rho}_i^1; \hat{\tau}] > D[\hat{\rho}_i^2; \hat{\tau}], \quad \exists t : D[\hat{\rho}^1(t); \hat{\tau}] < D[\hat{\rho}^2(t); \hat{\tau}]. \quad (5.1)$$

This definition, of course, ignores Mpemba's effects in phase transitions. In Mpemba's original observation, this definition would only consider the cooling of the cream and sugar mixture and not the ice cream crystallisation. However, this is a sufficient definition of the effect for our purposes.

A breakthrough in understanding this anomalous effect came from analysing the classical stochastic thermodynamics of Markovian systems [14, 15]. Given a system with some discrete probability distribution $\vec{p}$, coupled to an ideal thermal bath, the dynamics of $\vec{p}$ is Markovian and follows the master equation

$$\frac{d\vec{p}(t)}{dt} = \mathcal{L}\vec{p}(t); \quad (5.2)$$

where $\mathcal{L}$ is the generator for the dynamics, we will not delve into the specific structure of $\mathcal{L}$ here. However, the generator can be decomposed into its eigenbasis with eigenvalues $0 < \lambda_2 \leq \dots \leq \lambda_n$ and corresponding eigenvectors $\vec{\pi}, \vec{v}_2, \dots, \vec{v}_n$ where n is the dimension of the system. The initial state of the system can be decomposed into the eigenbasis of the generator $\vec{p}(0) = \vec{\pi} + a_2\vec{v}_2 + \dots + a_n\vec{v}_n$. The evolution of the probability distribution is given by

$$\vec{p}(t) = \vec{\pi} + e^{\lambda_2 t} a_2 \vec{v}_2 + \dots + e^{\lambda_n t} a_n \vec{v}_n. \quad (5.3)$$

At long times, the dynamical evolution of the state is a linear combination of the stationary state and the system's slowest decaying mode. A Mpemba effect can occur if the initial state with higher temperature has a smaller amplitude with the slowest decaying mode; this is known as a weak Mpemba effect. A strong Mpemba effect occurs if the overlap amplitude is zero, resulting in an exponential speedup to equilibrium [15].

Here, we see the problem of using temperature difference as a measure of the distance to the fixed point. First, the initial state and the fixed point may not be an equilibrium state and thus may not have an associated temperature. Second, even if the initial and final states are in thermal equilibrium, the evolution of the state for most non-equilibrium processes will not follow the locus of temperatures between the initial and final states. Therefore, a system initially at some temperature T_i does not necessarily relax to some middle temperature T_m before reaching a final temperature T_f .

The eigenbasis decomposition framework has been used to experimentally probe the Mpemba effect, finding a Mpemba effect in colloidal systems [224]. The inverse [18] Mpemba effect, where a speed-up to heating rather than cooling occurs, has also been observed under controlled conditions.

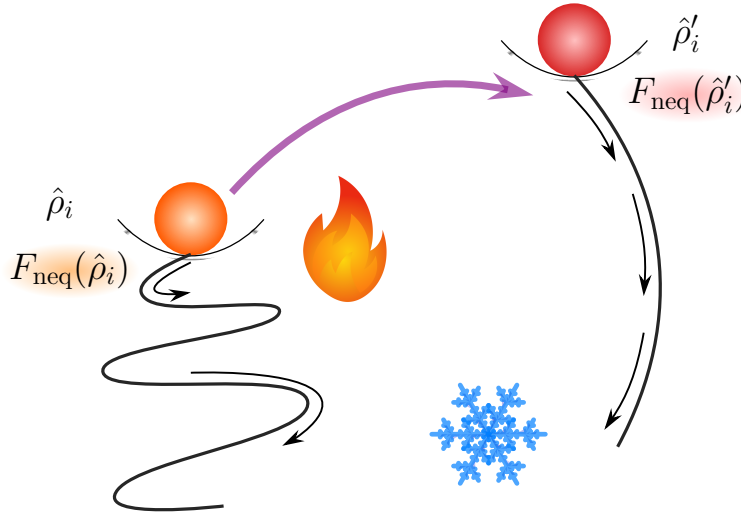


Figure 5.1: For a general quantum state, one can construct a unitary transformation, which increases the state's non-equilibrium free energy and eliminates the overlap with a set of slow-decaying modes of the generator. This leads to exponentially faster thermalisation and a genuine quantum Mpemba effect.

Recently, work has been undertaken to generalise the Mpemba effect on quantum dynamics. The phenomenon and the framework generalise naturally to open quantum systems [225, 16, 226, 227, 17, 228–233] and this has recently been explored experimentally in ion traps [234, 235]. However, it should be stated that very few of these studies focus specifically on thermalisation but more on anomalous relaxation to general fixed points. Another series of works have investigated a quantum Mpemba-like effect in isolated systems related to symmetry restoration in quenched dynamics starting from states that break the symmetry of the generating Hamiltonian [32–36].

5.2 Davies Maps

Consider a quantum Markovian master equation that evolves a density matrix $\hat{\rho}(t)$

$$\frac{d\hat{\rho}(t)}{dt} = \mathcal{L}\hat{\rho}(t). \quad (5.4)$$

Here $\mathcal{L}$ is the generator, known as the Lindbladian, consisting of a unitary part and a dissipative part such that $\mathcal{L} = i\delta + \mathcal{D}$ with $\delta(\cdot) = -[\hat{H}, \cdot]$, $\hat{H}$ being the system Hamiltonian and $\mathcal{D}(\cdot) = \sum_l \hat{L}_l(\cdot)\hat{L}_l^\dagger - \frac{1}{2}\{\hat{L}_l^\dagger\hat{L}_l, (\cdot)\}$, where $\hat{L}_k$ are the so-called jump operators. Similar to

Eq. 5.3, we can write down the general evolution of an initial density matrix $\hat{\rho}_i$ as

$$\hat{\rho}(t) = e^{\mathcal{L}t} \hat{\rho}_i = \hat{\tau} + \sum_{k=2}^{D^2} \text{Tr} \{ \hat{l}_k \hat{\rho}_i \} \hat{r}_k e^{\lambda_k t}, \quad (5.5)$$

where $D = \dim(\hat{H})$ and we assume that $\hat{\tau}$ is the unique steady state, which is the right eigenoperator of the generator $\mathcal{L}$ associated with the zero eigenvalue. Here $\hat{l}_k$ and $\hat{r}_k$ are the left and right eigenoperators corresponding to the eigenvalue λ_k such that $\mathcal{L}[\hat{r}_k] = \lambda_k \hat{r}_k$ and $\mathcal{L}^\dagger[\hat{l}_k] = \lambda_k \hat{l}_k$ where $\mathcal{L}^\dagger$ is the adjoint generator that acts on observables rather than states. The superoperator $\mathcal{L}$ preserves the hermiticity of $\hat{\rho}(t)$, which implies that if λ_k is a complex eigenvalue then λ_k^* is also. We order the eigenvalues in ascending order according to the modulus of their real part such that $0 = \lambda_1 < |\text{Re}(\lambda_2)| \leq |\text{Re}(\lambda_3)| \leq \dots$. The spectral gap is then $|\text{Re}(\lambda_2)|$ and defines the longest timescale in the system such that $|\hat{\rho}(t) - \hat{\tau}| \propto \exp(\text{Re}(\lambda_2)t)$. When the lowest eigenmodes of the generator form a complex conjugate pair at long times one has

$$\hat{\rho}(t) \propto \hat{\tau} + e^{\text{Re}(\lambda_2)t} \left(\text{Tr} \{ \hat{l}_2 \hat{\rho}_i \} \hat{r}_2 e^{i \text{Im}(\lambda_2)t} + \text{Tr} \{ \hat{l}_2^\dagger \hat{\rho}_i \} \hat{r}_2^\dagger e^{-i \text{Im}(\lambda_2)t} \right), \quad (5.6)$$

We consider Davies maps [46, 236], which are defined by two properties. First, the unitary and dissipative part of the generator commute, and second they obey quantum detailed balance [237] with respect to the thermal state $\hat{\tau}_\beta = e^{-\beta \hat{H}}/Z$, with $Z = \text{Tr} \{ \exp(-\hat{H}\beta) \}$. Mathematically quantum detailed balance means that given two arbitrary operators $\hat{A}$ and $\hat{B}$, the unitary and dissipative parts of the generator obey $\langle \hat{A}, \mathcal{D}^\dagger(\hat{B}) \rangle_{\hat{\tau}_\beta} = \langle \mathcal{D}^\dagger(\hat{A}), \hat{B} \rangle_{\hat{\tau}_\beta}$ and $[\hat{H}, \hat{\tau}_\beta] = 0$, where $\langle \hat{A}, \hat{B} \rangle_{\hat{\tau}_\beta} = \text{Tr} \{ \hat{\tau}_\beta \hat{A}^\dagger \hat{B} \}$ is the KMS inner product.

The eigenmatrices of the unitary part δ are the elemental matrix units $|n\rangle\langle m|$ with eigenvalues $\omega_{nm} = E_n - E_m$. Since this is a Davies map, the dissipative part $\mathcal{D}$ shares a common eigenspace with δ . Furthermore, if the Hamiltonian is non-degenerate, then the overall Davies map $e^{\mathcal{L}t}$ is block diagonal in the energy eigenbasis. We can write $\mathcal{L} = \mathcal{L}_P \oplus \mathcal{L}_C$ where $\mathcal{L}_P = \mathcal{D}_P$ is the population subblock, whose right eigenmatrices are diagonal in the energy eigenbasis $|n\rangle\langle n|$ and $\mathcal{G}_C = i\delta + \mathcal{D}_C$ is the coherence subblock. The slowest decaying modes shown in Eq. 5.6 and indeed more generally $\hat{r}_i$ for $i \geq 2$ will be off-diagonal matrices in the energy eigenbasis. This ensures that they are traceless which guarantees that the dynamics preserve normalisation at all times. However not only does δ commute with

$\mathcal{D}$, it also commutes with $\mathcal{D}^\dagger$. This means that the left eigenvectors of $\mathcal{G}_c$ are also purely off-diagonal matrices in the energy eigenbasis since $\mathcal{L}^\dagger[\hat{l}_k] = \lambda_k \hat{l}_k$.

Davies maps are characterised by the following jump operators:

$$\begin{aligned}\hat{L}_{nm}^{(1)}(\beta) &= \underbrace{\gamma (1 \mp f^\pm(\beta, h_m - h_n))^{1/2}}_{\alpha_{nm}^{(1)}} |n\rangle \langle m| \\ \hat{L}_{nm}^{(2)}(\beta) &= \underbrace{\gamma (f^\pm(\beta, h_m - h_n))^{1/2}}_{\alpha_{nm}^{(2)}} |m\rangle \langle n|, \quad \text{with } m < n,\end{aligned}\tag{5.7}$$

where h_n are the eigenvalues of the Hamiltonian, γ is the system-environment coupling, henceforth will be set to 1, and $f^\pm(\beta, h_n) = 1/(\exp(\beta h_n) \pm 1)$ are the Fermi and the Bose distribution functions.

To perform calculations, Davies maps are restructured into a non-Hermitian matrix via a vectorisation procedure [238, 239] as

$$\hat{\mathcal{L}} = -i\hat{H} \otimes \hat{1} + i\hat{1} \otimes \hat{H}^T + \sum_l \hat{L}_l \otimes (\hat{L}_l^\dagger)^T - \frac{1}{2} \hat{L}_l^\dagger \hat{L}_l \otimes \hat{1} - \frac{1}{2} \hat{1} \otimes (\hat{L}_l^\dagger \hat{L}_l)^T. \tag{5.8}$$

As noted previously, if the Hamiltonian is non-degenerate, populations and coherences evolve independently thus $\hat{\mathcal{L}}$ can be recast into a block-diagonal form $\hat{\mathcal{L}} = \hat{\mathcal{L}}_p \oplus \hat{\mathcal{L}}_c$, where $\hat{\mathcal{L}}_p$ describing the system's populations and $\hat{\mathcal{L}}_c$ the system's coherences

$$\hat{\mathcal{L}} = \left(\begin{array}{cc|c} \text{populations} & 2^L & 4^L - 2^L \\ 2^L & & \\ \hline & & \text{coherences} \\ & 4^L - 2^L & \end{array} \right), \tag{5.9}$$

where L is the number of fermions, hard-core bosons, or spin-1/2 particles in the system. Moreover, the left eigenmatrices $\hat{l}_k$ associated with the system's populations are diagonal

and correspond to real eigenvalues λ_k , while those associated with the system's coherences are strictly off-diagonal, and the relative eigenvalues come in complex conjugate pairs.

Instead of plugging the jump operators 5.7 into the general expression for the vectorised Lindbladian 5.8, one can also construct the two blocks of the Davies map individually, saving significant computational resources. For this purpose, we define the matrix $\hat{J}$, collecting all the jump operators. For concreteness, for two qubits it reads:

$$\hat{J} = \begin{pmatrix} 0 & \alpha_{01}^{(2)} & \alpha_{02}^{(2)} & \alpha_{03}^{(2)} \\ \alpha_{10}^{(1)} & 0 & \alpha_{12}^{(2)} & \alpha_{13}^{(2)} \\ \alpha_{20}^{(1)} & \alpha_{21}^{(1)} & 0 & \alpha_{23}^{(2)} \\ \alpha_{30}^{(1)} & \alpha_{31}^{(1)} & \alpha_{32}^{(1)} & 0 \end{pmatrix}. \quad (5.10)$$

The coefficients $\alpha_{nm}^{(1)}$ and $\alpha_{nm}^{(2)}$ are obtained from Eq. 5.7. It can be shown that the populations block $\hat{\mathcal{L}}_p$ reads

$$\hat{\mathcal{L}}_p = \hat{J}_2 + \hat{J}_s, \quad (5.11)$$

where $\hat{J}_2$ is obtained by squaring $\hat{J}$ elementwise and $\hat{J}_s$ is a diagonal matrix given by $\hat{J}_{s,[k,k]} = -\sum_{i=0}^{2^L-1} (\hat{J}_{[i,k]})^2$, where we have used the subscript notation $[i, k]$ to denote the corresponding matrix element.

Let us now construct the coherences block $\hat{\mathcal{L}}_c$, which is diagonal in the energy eigenbasis. For this one has to pick all possible $(4^L - 2^L)/2$ couples of columns $\hat{J}_{[:,n]}$ and $\hat{J}_{[:,m]}$ with $n = 0, 1, \dots, 2^L$ and $m > n$, where $[:, n]$ denotes the n th column. Each couple of columns (n, m) identifies two elements $\hat{\mathcal{L}}_{c,[k,k]}$ and $\hat{\mathcal{L}}_{c,[l,l]}$ which are the complex conjugate of one another. The indices of these two elements are $k = m \cdot 2^L - m + n - 1$ and $l = n \cdot 2^L + m - n$. Their real part reads

$$\text{Re}(\hat{\mathcal{L}}_{c,[k,k]}) = \text{Re}(\hat{\mathcal{L}}_{c,[l,l]}) = -\sum_{i=0}^{2^L-1} \left(\frac{(\hat{J}_{[i,n]})^2}{2} + \frac{(\hat{J}_{[i,m]})^2}{2} \right), \quad (5.12)$$

and their imaginary part is

$$\text{Im}(\hat{\mathcal{L}}_{c,[k,k]}) = \text{Im}(\hat{\mathcal{L}}_{c,[l,l]}^*) = i(h_n - h_m). \quad (5.13)$$

5.3 Speedups to Thermalisation

We follow a similar path to Carollo et al. in Ref. [16]. The authors generate an exponential speed up to the steady state by constructing a unitary acting on the initial state such that

$$\text{Tr} \{ \hat{L}_2 \hat{U} \hat{\rho}_i \hat{U}^\dagger \} = 0, \quad (5.14)$$

so that $|\hat{\rho}(t) - \hat{\tau}| \propto \exp(\text{Re} \lambda_3 t)$ at long times. Carollo's result makes two key assumptions. First, a pure state is required for the transformation to generate an exponential speedup; second the generator needs to have a spectral gap defined by a real eigenvalue. Moreover, the rotated initial state has no guarantee to be further from the fixed point than the initial state; in these cases, a Mpemba effect is not observed. Stressing an essential point: an exponential speedup alone does not constitute a genuine Mpemba effect. The rotated state must be further from the steady state than the initial unrotated state. We wish to construct a unitary that achieves an exponential speedup for Davies maps valid for both mixed and pure states and performs when a complex eigenpair defines the spectral gap. To ensure a true Mpemba effect is observed, this transformation will always increase the state's distance from equilibrium.

As we are dealing with non-equilibrium quantum states with possible coherences in the energy eigenbasis we do not have any initial notion of temperature. We have one point of equilibrium, namely the fixed point $\hat{\tau}_\beta$. For quantum dynamics, we propose using the non-equilibrium free energy discussed in Chapter 1.2. We consider the initial scenario $F_{\text{neq}}(\hat{\rho}'(t_0)) > F_{\text{neq}}(\hat{\rho}(t_0)) > F_{\text{eq}}$ where $\hat{\rho}'(t_0) = \hat{U} \hat{\rho}(t_0) \hat{U}^\dagger$ is the state following unitary transformation and $F_{\text{eq}} = -\beta^{-1} \ln Z$ is the equilibrium free energy of the fixed point. A quantum Mpemba effect occurs if some time t_m exists such that $F_{\text{neq}}(\hat{\rho}'(t)) < F_{\text{neq}}(\hat{\rho}(t))$ for all times $t > t_m$. We call this a *genuine* quantum Mpemba effect to differentiate it from exponential speedups achieved by other transformations. We emphasise that for the speedup to be a quantum Mpemba effect the non-equilibrium free energy curves must cross in time. The non-equilibrium free energy can be written as

$$F_{\text{neq}}(\hat{\rho}(t)) = \beta^{-1} D(\hat{\rho}(t) \| \hat{\tau}_\beta) + F_{\text{eq}}(\hat{\tau}_\beta), \quad (5.15)$$

where $D(\hat{\rho} \| \hat{\sigma}) = \text{Tr} \{ \hat{\rho} (\ln \hat{\rho} - \ln \hat{\sigma}) \}$ is the quantum relative entropy. The Klein inequal-

ity [240] guarantees the positivity of the relative entropy, ensuring that $F_{\text{neq}}(\hat{\rho}(t)) \geq F_{\text{eq}}(\hat{\tau}_\beta) \forall t$. The quantum relative entropy is a very stringent measure of the distinguishability of two quantum states. While not itself a metric, it still upper bounds the trace distance via Pinsker's inequality [241] $D(\hat{\rho}||\hat{\sigma}) \geq \|\hat{\rho} - \hat{\sigma}\|_1^2/2$, which captures the optimal distinguishability of quantum states with a single measurement.

To construct the rotation, we consider a unitary operator divided into two separate unitary operations

$$\hat{U} = \hat{U}_2 \hat{U}_1, \quad (5.16)$$

where $\hat{U}_1$ achieves the exponential, asymptotic speed up to relaxation and $\hat{U}_2$ ensures that the rotated state will have a higher non-equilibrium free energy than the initial state. The unitary $\hat{U}_1$ diagonalises the density matrix in the energy eigenbasis;

$$\hat{U}_1 \hat{\rho}_i \hat{U}_1^\dagger = \hat{\Lambda}, \quad (5.17)$$

where $\hat{\Lambda}$ is the diagonal matrix of eigenvalues of the density operator. Given an initial state $\hat{\rho}_i$ which has coherences in the energy eigenbasis, this diagonalisation removes overlaps with all coherent modes, achieving an exponential speed-up if the real part of a complex eigenpair defines the spectral gap. Since all $\hat{l}_m$ with $\lambda_m \in \mathbb{C}$ are purely off-diagonal matrices in the energy eigenbasis then all associated amplitudes will be simultaneously eliminated $\text{Tr}\{\hat{l}_m \hat{U} \hat{\rho}_i \hat{U}^\dagger\} = 0$. We define the unitary $\hat{U}_2$ to order the eigenvalues of $\hat{\Lambda}$ in decreasing value, generating a state with a complete population inversion. The full unitary $\hat{U} = \hat{U}_2 \hat{U}_1$ is unique and has been decomposed into two elements to highlight that $\hat{U}_1$ is associated with the speed-up in relaxation and $\hat{U}_2$ with the increase in non-equilibrium free energy.

To ensure this is a genuine Mpemba effect, we need to ensure that the non-equilibrium free energy of the rotated state $\hat{\rho}'_i$ greater than any arbitrary initial state $\hat{\rho}_i$:

$$\begin{aligned} F_{\text{neq}}(\hat{\rho}'_i) &\geq F_{\text{neq}}(\hat{\rho}_i) \\ \beta^{-1} D(\hat{\rho}(t)||\hat{\tau}_\beta) + F_{\text{eq}}(\hat{\tau}_\beta) &\geq \beta^{-1} D(\hat{\rho}(t)||\hat{\tau}_\beta) + F_{\text{eq}}(\hat{\tau}_\beta) \\ D(\hat{\rho}'||\hat{\tau}_\beta) &\geq D(\hat{\rho}_i||\hat{\tau}_\beta) \\ \text{Tr}\{\hat{\rho}'_i \ln(\hat{\rho}'_i)\} - \text{Tr}\{\hat{\rho}'_i \ln(\hat{\tau}_\beta)\} &\geq \text{Tr}\{\hat{\rho}_i \ln(\hat{\rho}_i)\} - \text{Tr}\{\hat{\rho}_i \ln(\hat{\tau}_\beta)\} \\ \text{Tr}\{\hat{\rho}'_i \ln(\hat{\tau}_\beta)\} &\leq \text{Tr}\{\hat{\rho}_i \ln(\hat{\tau}_\beta)\}, \end{aligned} \quad (5.18)$$

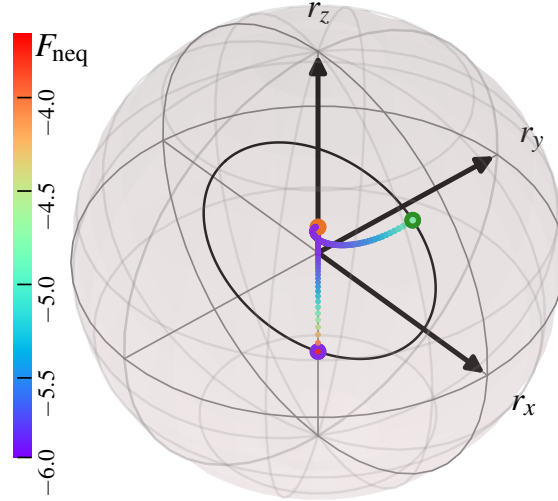


Figure 5.2: Cartoon showing the Mpemba transformation outlined in the text. The Bloch sphere representation of the equilibration of a random state $\hat{\rho}_i$ (green dot) and the transformed state $\hat{\rho}'_i = \hat{U}\hat{\rho}\hat{U}^\dagger$ (purple dot) towards the thermal steady state (orange dot) is shown. The Hamiltonian is given as $\hat{H} = -J\hat{\sigma}_z$. The non-equilibrium free energy F_{neq} for the two thermalisation processes is represented by the colour scheme. The transformation diagonalises the state in the energy eigenbasis and increases the non-equilibrium free energy.

where we have used the fact the Von Neumann entropy of both states are equal due to being unitarily connected. Denote the probability distribution generated by taking the diagonal of the density matrices $\hat{\rho}_i$, $\hat{\rho}'_i$ and $\hat{\tau}_\beta$ as p , p' and r with elements p_k , p'_k and r_k respectively. τ_β is diagonal in the energy eigenbasis, allowing us to rewrite Eq. 5.18 as

$$\sum_k p'_k \ln(r_k) \leq \sum_k p_k \ln(r_k). \quad (5.19)$$

As the elements r_k come from a thermal distribution, they are decreasing, $r_{k+1} < r_k \forall k$, and less than one, $r_k < 1$, assuming a finite β . Because of this the values of $\ln(r_k)$ are decreasing negative values $\ln(r_{k+1}) < \ln(r_k)$, any probability distribution, q that is increasing in k will minimise $\sum_k q'_k \ln(r_k)$ relative to all other permutations of q . Thus, the population inverted state will maximise the non-equilibrium free energy relative to any other permutation.

But what about relative to any other unitarily connected state? To demonstrate this, we use the concept of majorisation [242]. Majorisation has recently been used as a tool to understand the Mpemba effect in classical systems [243]. Denote $q^\downarrow$ as the permutation of the distribution q with values in descending order, i.e. $q_{k+1}^\downarrow \leq q_k^\downarrow \forall k$. One distribution q

majorises another distribution s , $q \succ s$ if

$$\sum_{k=1}^j q_k^\downarrow \geq \sum_{k=1}^j s_k^\downarrow \quad \forall j. \quad (5.20)$$

The probability distribution q majorises s if it is “less spread out”. Due to $\ln(r_k)$ decreasing with k if a distribution p' majorises a distribution p then $\sum_k p'_k \ln(r_k) \leq \sum_k p_k \ln(r_k)$. To verify p' majorises p we consider the diagonal elements of any unitary transformation of the diagonal matrix $\hat{\rho}'$

$$\langle k | \hat{U} \hat{\rho}' \hat{U}^\dagger | k \rangle = p_k, \quad (5.21)$$

expanding $\hat{\rho}'$ into its eigenbasis, $\hat{\rho}' = p_r |r\rangle \langle r|$:

$$\sum_r \langle k | \hat{U} | r \rangle p_r \langle r | \hat{U}^\dagger | k \rangle = \sum_r |\langle k | \hat{U} | r \rangle|^2 p_r = \sum_r S_{kr} p_r = p_k, \quad (5.22)$$

here $\hat{S}$ denotes a doubly stochastic matrix that maps the probability distribution p_r to the probability distribution p_σ . From the Hardy-Littlewood-Polya inequality [244] such a doubly stochastic matrix exists if and only if the distribution p_r majorises the distribution p_σ . Thus, this final state has a higher non-equilibrium free energy than any other state connected by a unitary rotation with equality if and only if the initial state is diagonal in the energy eigenbasis with a full population inversion.

5.4 Toy Models

Here, we demonstrate this Mpemba effect using two toy models: a single-qubit example and a multi-qubit example.

5.4.1 Single Qubit System

The qubit is prepared in a random state $\hat{\rho}_i = 1/2(1 + \mathbf{r} \cdot \hat{\boldsymbol{\sigma}})$, where $\mathbf{r}$ is a Bloch vector, a vector in $\mathbb{R}^3$ with $|\mathbf{r}| \leq 1$ that fully describes the state of the qubit (see Figure 5.2 for a visualisation of the Bloch Sphere — the space of all possible Bloch vectors). $\hat{\boldsymbol{\sigma}}$ is the vector containing the three Pauli matrices. The thermalisation is described by a unitary part governed by a random Hamiltonian $\hat{H} = J(h_0 \mathbb{1} + \mathbf{h}_{\text{rand}} \cdot \hat{\boldsymbol{\sigma}})$ and a dissipative part $\mathcal{D}$ defined according to the Davies map for a bath at temperature T_b .

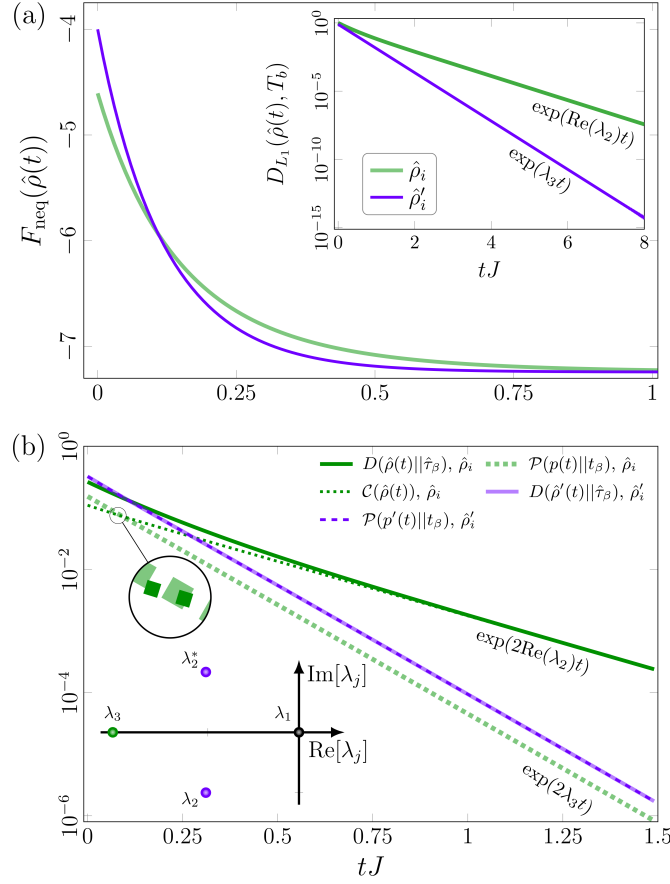


Figure 5.3: The genuine Mpemba effect in a single qubit quantum system. a) F_{neq} is initially higher for the optimized state (purple line) but drops below the random state's one (green line) at a later time. This crossing shows the occurrence of a genuine quantum Mpemba effect. Inset: The L_1 -distances from the steady state indicate that $\hat{\rho}'_i$ thermalises exponentially faster than $\hat{\rho}_i$. b) Dynamics of the total entropy production D and its classical P and coherent C contributions. The rotation $\hat{U}$ yields an exponential speed up. Inset: spectrum of the Davies generator G for a single qubit. Here $\omega = 5J$, $r_i = (0.276, 0.359, 0.303)$, $r'_i = (0., 0., 0.545)$, and $T_b = 10J$.

We perform a unitary $\hat{U}$ on $\hat{\rho}_i$, which rotates the Bloch vector to $\mathbf{r}'_i$ and brings it to a new state $\hat{\rho}'_i$ diagonal in the energy eigenbasis with the eigenvalues in ascending order. The evolution of the two initial states are shown in Fig. 5.3. Panel (a) shows the evolution of the non-equilibrium free energy for the two initial states. The rotated state has a much larger initial non-equilibrium free energy than the non-rotated state. The crossing of the two lines indicates a genuine quantum Mpemba effect. The inset shows the L_1 -distance from the steady state $D_{L_1}[\hat{\rho}(t), T_b] = \sum_{i,j} |\hat{\rho}^{ij}(t) - \hat{\tau}_\beta^{ij}|$, indicating that the thermalisation for the transformed state $\hat{\rho}'_i$ is exponentially accelerated. We now study the division of the entropy production defined by Eq. 1.26 in Fig. 5.3(b).

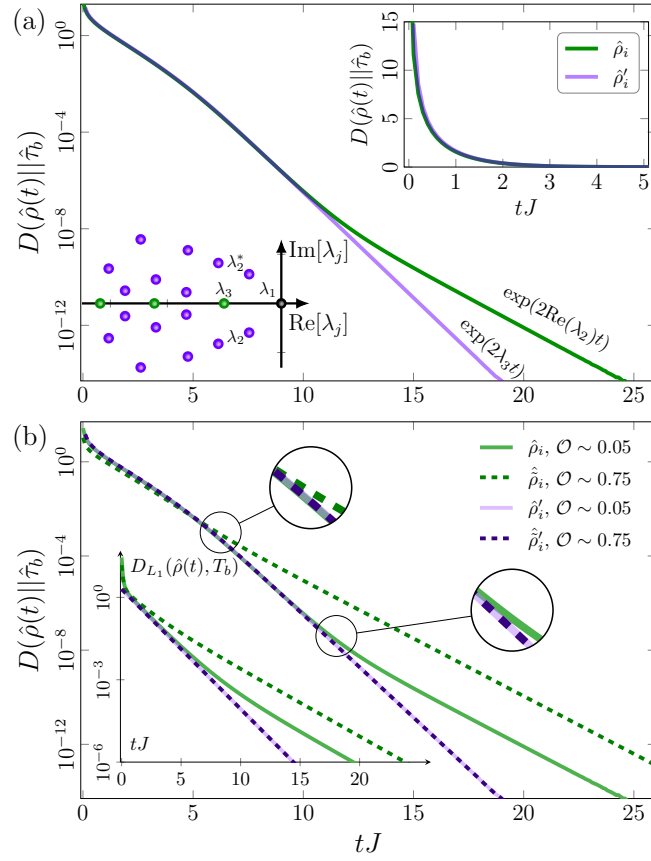


Figure 5.4: The genuine quantum Mpemba effect in a many-qubit system manifests at different timescales. (a) We study the total entropy production D for the transverse field Ising model with anisotropy at $h = J/2$ with $L = 5$ spins and bath temperature $T_b = 0.1J$. The rotation $\hat{U}$ allows for an exponential speedup governed by λ_3 that shows up at timescales much larger than for the single qubit case. The upper inset shows D in linear scale while the lower inset displays the first 20 eigenvalues of the spectrum of the Davies generator $\mathcal{L}$. (b) An initial state with a large overlap with the two slowest decaying modes $\mathcal{O} = \text{Tr}[\hat{l}_2^\dagger \hat{\rho}_i] + \text{Tr}[\hat{l}_2 \hat{\rho}_i]$ achieves the speedup sooner than one with a small overlap. For $\mathcal{O} \sim 0.05$ (full green line), the curves for the total entropy production cross around $t/J \sim 10$, while with $\mathcal{O} \sim 0.75$ (dashed green line), they coalesce at half of this time.

It can be seen that for the transformed state $\hat{\rho}'_i$ the coherent contribution is zero (since the overlap with the coherent modes has been eliminated) and the classical entropy production rate is exponentially accelerated from $\text{Re}(\lambda_2)$ to λ_3 . The spectrum of the Davies map is shown as an inset. Note that for a single qubit, a complex eigenpair defines the spectral gap at any temperature T_b .

5.4.2 Multi-qubit Systems

Shifting focus to a more complex system, we consider a Transverse Field Ising Model with open boundary conditions

$$\hat{H} = -J \sum_{j=1}^{L-1} \hat{\sigma}_j^z \hat{\sigma}_{j+1}^z + h \sum_{j=1}^L \hat{\sigma}_j^x, \quad (5.23)$$

where J represents the nearest neighbour interaction strength, h is the transverse field strength, and $\hat{\sigma}_i^{z[y]}$ is the Pauli $z[y]$ operator acting on site i . We construct a random mixed state $\hat{\rho}_i$ by averaging over 1000 random pure states. Then, as for the single qubit case, we obtain the transformed state $\hat{\rho}'_i$ by applying $\hat{U}$.

We compute the irreversible entropy production in panel (a) of 5.4. The main difference to the single qubit case is that since the overlap of the random state $\hat{\rho}_i$ with the slowest mode, $\hat{l}_2$, is small, the difference with the transformed state manifests only at later a time. The situation changes when one considers an initial state $\hat{\rho}_i$ that has a larger overlap with $\hat{l}_2$. The dashed green line in panel (b) shows that similarly to the single qubit case, the exponential difference in the thermalisation speed of $\hat{\rho}_i$ and $\hat{\rho}'_i$ is evident already at short times.

5.5 Summary and outlook

This chapter describes a procedure to generate a Mpemba effect in systems whose dynamics are described by a Davies map. One can derive an effective Davies map description for many physically relevant scenarios, starting from the microscopic Hamiltonian for system and environment and following the standard procedure [42]. Such systems include quantum optics setups [245] and mesoscopic heat engines [246]. Importantly, Davies maps can be used to accurately estimate the impact of noise on single-qubit and two-qubit operations in superconducting transmon architectures [247]. We discuss this in more detail in Appendix B. Note also that the classical and quantum contributions to the entropy production (1.26) have recently been measured in a tabletop optical experiment described by a Davies map [248].

We have demonstrated that a quantum Mpemba effect can be engineered by exploiting the

block structure of the generator in thermalising open Markovian systems. We highlight the non-equilibrium free energy as a central thermodynamic object to both define and study the effect in open quantum systems. When the spectral gap of a Davies generator is defined by a complex eigenpair, a unitary operation can always be found on an initial state which simultaneously raises the free energy and provides exponentially quicker thermalisation. The work presented here enhances our understanding of the effect in quantum systems and should inspire further studies from the perspective of quantum thermodynamics, and could prove useful in dissipative state engineering protocols [249–251].

Mpemba Meets Electromechanics

In realistic scenarios, quantum systems are somewhat coupled to an environment. Thus, any stored work will slowly dissipate. For example, in the electromechanical system considered in Chapter 3, if the bias across the leads were switched off, the oscillator’s state would eventually return to equilibrium with the leads. Protecting the charge stored in quantum batteries is vital for developing quantum batteries. Various protocols have been developed to stabilise the charge of a quantum battery coupled to an environment. One technique uses dark states - quantum states that do not relax when coupled to the reservoir. These dark states can be generated using decoherence-free subspaces [252, 253] or spontaneously created through interactions between the charger and reservoir [254]. The ergotropy can also be stabilised through measurements of the battery [177, 255]. While the majority of works in QBs deal with discrete systems, recent works have extended the framework to continuous variable (CV) settings [256–259].

In this chapter, we will use the Mpemba effect described in Chapter 5 to preserve the extractable work of an already charged system. Instead of using the Mpemba effect to speed up relaxation, we will use the effect to slow down the thermalisation of a charged state, utilising the ergotropy as the distance measure. We will first investigate the effect in a classical harmonic oscillator coupled to a thermal bath, finding that symmetric initial states (with respect to position and momentum) will relax faster than asymmetric initial states. We will then analyse quantum oscillators whose dynamics are described by Davies maps. We will focus specifically on Gaussian states and ergotropy preservation. We find that storing extractable work via displacements provides an exponential slowdown in dissipation relative

to squeezed states. Much of this chapter has been adapted from the paper “Anomalous discharging of quantum batteries: the ergotropic Mpemba effect” [260].

6.1 Electromechanics Eigenmode Decomposition

In Chapter 3, we considered the evolution of an electromechanical system generated by the Fokker-Planck equation

$$\frac{\partial W}{\partial t} = \frac{\partial}{\partial p} \left(m\omega_0^2 x - F\langle \hat{n} \rangle_x + \gamma(x)p \right) W - \frac{p}{m} \frac{\partial W}{\partial x} + \frac{1}{2} D(x) \frac{\partial^2 W}{\partial p^2}. \quad (6.1)$$

When the oscillator is coupled to reservoirs at equal temperature and potential, the fluctuation-dissipation theorem is satisfied:

$$D(x) = 2T\gamma(x)m, \quad (6.2)$$

where T is the temperature of the equilibrium reservoirs. To further simplify the evolution of the oscillator, we will make two approximations. First, we will assume the coupling force acting on the oscillator, F , is negligible. Second, we will assume that $\gamma(x)$ weakly depends on the oscillator position in the interest regime. These two approximations reduce the evolution of the oscillator reducing to the classical Fokker-Planck equation of an oscillator coupled to a thermal reservoir

$$\frac{\partial W}{\partial t} = \left(\frac{\partial}{\partial v} \left(\omega_0^2 x + \gamma v \right) - v \frac{\partial}{\partial x} + \frac{\gamma T}{m} \frac{\partial^2}{\partial v^2} \right) W = L_{\text{FP}} W, \quad (6.3)$$

where we have made the substitution $v = p/v$ and have defined the Fokker-Planck generator L_{FP} . To investigate whether a Mpemba effect occurs in this system, we will undertake a similar analysis to that done in Chapter 5, namely, performing an eigenvalue decomposition of the equation of motion. A detailed derivation can be found in Appendix C; here, we will outline a brief sketch of the derivation. An eigenfunction decomposition finds solutions to a Fokker-Planck equation of the form

$$W(x, v, t) = W_{\text{ss}} + \sum_n c_n \phi_n(x, v) e^{\lambda_n t}. \quad (6.4)$$

Here W_{ss} is the steady state solution, $\phi(x, v)_n$ is the n th eigenfunction of the Fokker-Planck generator, λ_n is the corresponding eigenvalue and c_n is the corresponding overlap function with the initial state $W(x, p, 0)$. Once the overlap function is constructed, different initial

systems can be investigated to minimise or maximise overlaps with the Fokker-Planck generator. The Fokker-Planck equation can be mapped onto an equivalent Schrödinger-like equation. For a classical oscillator, one can transform the Fokker-Planck equation

$$\bar{L}_{\text{FP}} = \exp[\Phi(x, v)] L_{\text{FP}} \exp[-\Phi(x, v)], \quad (6.5)$$

where we denote

$$\Phi(x, v) = \frac{1}{4} \left[\left(\frac{v}{v_{\text{th}}} \right)^2 + \left(\frac{x\omega_0}{v_{\text{th}}} \right)^2 \right], \quad (6.6)$$

where $v_{\text{th}} = \sqrt{T/m}$ is the thermal velocity. The operator can be converted

$$\bar{L}_{\text{FP}} = -\gamma \hat{b}^\dagger \hat{b} - \omega_0 (\hat{a} \hat{b}^\dagger - \hat{a}^\dagger \hat{b}), \quad (6.7)$$

where we have denoted the operators $\hat{a}$ and $\hat{b}$ that are of the form

$$\hat{a} = \frac{v_{\text{th}}}{\omega_0} \frac{\partial}{\partial x} + \frac{\omega_0 x}{2v_{\text{th}}}, \quad (6.8)$$

$$\hat{b} = v_{\text{th}} \frac{\partial}{\partial v} + \frac{1}{2} \frac{v}{v_{\text{th}}}, \quad (6.9)$$

These operators satisfy the bosonic commutation relations, $[\hat{a}, \hat{a}^\dagger] = 1$ and $[\hat{b}, \hat{b}^\dagger] = 1$. The eigenvalues of the system are given by

$$\lambda_{n_1, n_2} = \frac{1}{2} \gamma (n_1 + n_2) + \frac{1}{2} \sqrt{\gamma^2 - 4\omega_0^2} (n_1 - n_2). \quad (6.10)$$

Here n_1 and n_2 denote independent mode numbers. The eigenfunctions of the Fokker-Planck equation are given in Appendix C. To proceed, we consider the case of the overdamped oscillator, when $\gamma > 2\omega_0$, the eigenvalue corresponding to the slowest decaying mode is

$$\lambda_{0,1} = \frac{1}{2} \left(\gamma - \sqrt{\gamma^2 - 4\omega_0^2} \right). \quad (6.11)$$

The corresponding eigenfunction is

$$c_{0,1} = \int \int dx dv \left(-\sqrt{\eta_+} \frac{v}{v_{\text{th}}} + \sqrt{\eta_-} \frac{\omega_0 x}{v_{\text{th}}} \right) W(x, v, 0), \quad (6.12)$$

where $\eta_\pm = 1/2(\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2})$. If the initial distribution is symmetric wrt x and v , then the integrand in Eq. 6.12 is anti-symmetric, and the overlap with the slowest decaying

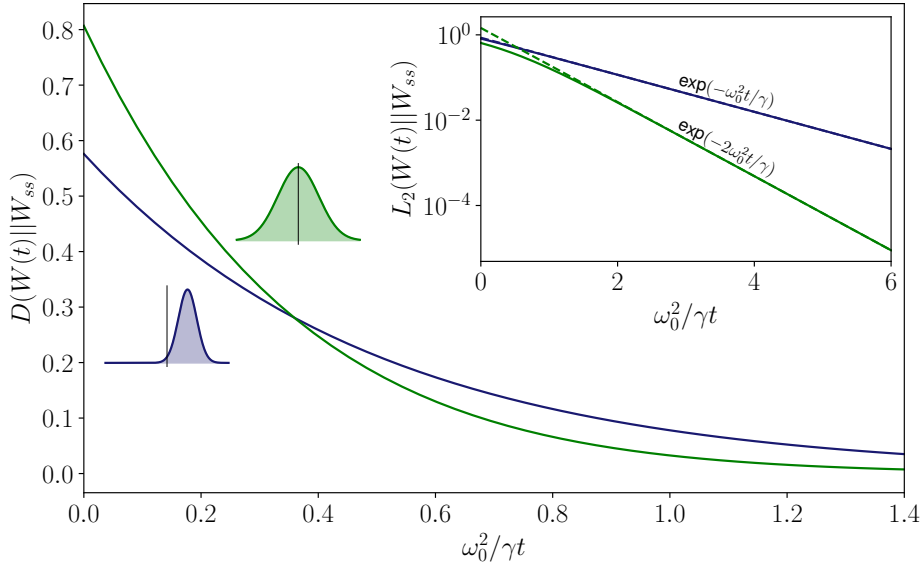


Figure 6.1: Mpemba effect in a classical oscillator system. $D(W(t)||W_{ss})$ is initially higher for the symmetric state (green), but drops below the asymmetric state (purple) at a later time. The crossing demonstrates a Mpemba effect in a continuous system. Gaussian insets exhibit the difference between the symmetric and asymmetric states. Inset shows the decay of the L_2 distance between the relaxing and thermal states. The symmetric state relaxes exponentially faster than the asymmetric state. The parameters are $\omega_0^2/\gamma = 0.2$ and $T/(\gamma m) = 0.5$.

mode is zero. To substantiate this claim, we numerically solve Eq. 6.3 in the strongly overdamped limit, $\gamma \gg 2\omega_0$. In this regime, the velocity of the oscillator decays on a timescale much faster than the relaxation of the position. The velocity of the oscillator can then be adiabatically eliminated to obtain a one-dimensional Fokker-Planck equation describing the oscillator position [71]:

$$\frac{\partial W(x, t)}{\partial t} = \frac{1}{\gamma} \frac{\partial}{\partial x} \left[\left(\omega_0^2 x + \frac{T}{m} \frac{\partial}{\partial x} W(x, t) \right) \right]. \quad (6.13)$$

The eigenvalues in Eq. 6.10 are approximately given by

$$\lambda_n = \frac{\omega_0^2}{\gamma} n. \quad (6.14)$$

In Fig. 6.1, the relative entropy and L_2 distance between two relaxing Gaussian distributions and the thermal steady-state distribution are shown. One Gaussian state is symmetric with respect to position and has a higher initial relative entropy to the fixed point. In comparison, the second Gaussian distribution is asymmetric, with a lower initial relative entropy. In the long time limit, the asymmetric state relaxes with a rate $\frac{\omega_0^2}{\gamma}$ — the slowest decaying

eigenvalue, while the symmetric state relaxes with rate $2\frac{\omega_0^2}{\gamma}$ — the second slowest decaying eigenvalue. This generates a Mpemba effect similar to the effect seen in Chapter 5 and can be attributed to the overlap function, Eq. 6.12, vanishing for symmetric initial states.

The equation of motion in the strongly overdamped limit is known as the Smoluchowski equation. One can perform an eigenmode expansion on this equation — a much easier task than performing the expansion on Eq. 6.3. The eigenvalues are exactly those generated in Eq. 6.14 while the overlap functions are given by [71]

$$c_n = \int dx W(x, 0) H_n \left(x \sqrt{\frac{m\omega_0^2}{2T}} \right) (2^n n!)^{-1/2}, \quad (6.15)$$

where $H_n(\bullet)$ is the n th Hermite polynomial. Similar to Eq. 6.12, if the initial state is symmetric with respect to position, then the integrand is anti-symmetric, and the overlap with the slowest decaying eigenmode vanishes. Due to the structure of Hermite polynomials, any symmetric initial state will have no overlap with all odd eigenmodes.

6.2 Gaussian States

One can consider merging the analysis used in chapter 5 with the analysis performed in the previous section. Consider a quantum oscillator weakly coupled to a thermal bath; the master equation describing the evolution of the oscillator is

$$\frac{d\hat{\rho}}{dt} = -i[\hat{H}, \hat{\rho}] + \frac{\gamma}{2}(\bar{n} + 1)(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a}) + \frac{\gamma}{2}\bar{n}(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger), \quad (6.16)$$

where $\bar{n} = (\exp(\beta\omega_0) - 1)^{-1}$ is the Bose-Einstein distribution, $\hat{a}^{(\dagger)}$ is the annihilation (creation) operator of the bosonic mode, and γ is a coupling constant between the bath and oscillator. To simplify this analysis, we will only consider Gaussian states — states whose Wigner distributions are Gaussian distributed. Thus, they can be fully characterised by the first and second moments of the quadrature, reducing the infinite-dimensional Hilbert space associated with continuous variable systems to a space of dimension $2N$, where N is the number of modes. Gaussian states and operations represent a significant restriction on the set of all possible states and transformations; however, the simplification comes with the benefit of significantly easier analytic and numeric calculations. Gaussian states and

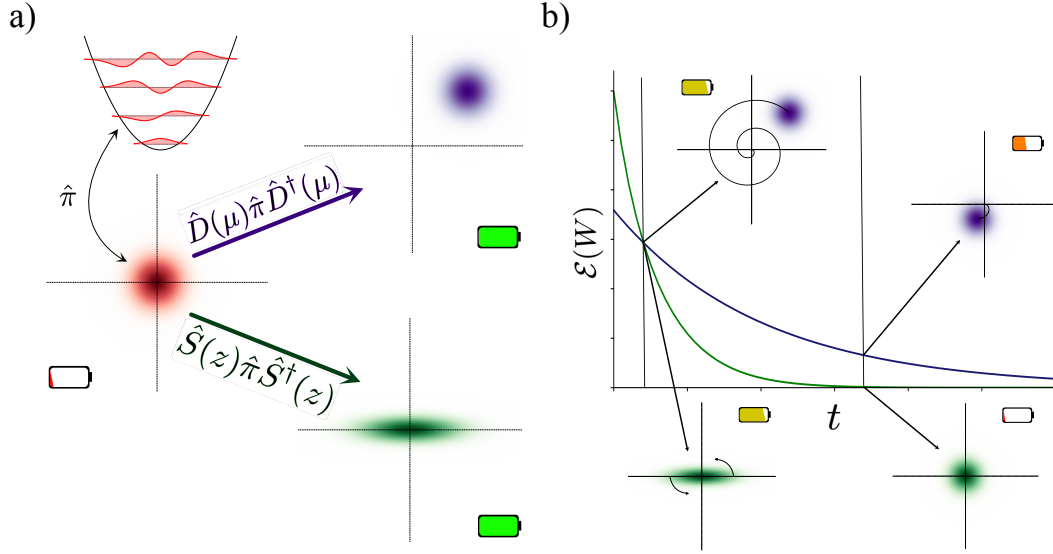


Figure 6.2: a) A single-mode thermal-state quantum oscillator with no stored ergotropy can be charged via squeezing and displacement operations, preserving the system's Gaussianity. b) When weakly coupled to a thermal bath, the charge is lost to the environment. Charge generated via squeezing operations is lost faster than charge stored via displacement operations, leading to an ergotropic Mpemba effect.

operations are often employed in physics as the coupling terms between subsystems often have linear dependence. This simplicity allows for an analytic analysis of work storage and extraction [256, 257]. For single mode oscillators, one can define the vector of mean values as $\vec{\mu}' = [\langle \hat{q} \rangle, \langle \hat{p} \rangle]$. The covariance matrix of the system is described by

$$\Theta' = \begin{pmatrix} \langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2 & \frac{1}{2} (\langle \hat{x} \hat{p} \rangle + \langle \hat{p} \hat{x} \rangle) - \langle \hat{x} \rangle \langle \hat{p} \rangle \\ \frac{1}{2} (\langle \hat{p} \hat{x} \rangle + \langle \hat{x} \hat{p} \rangle) - \langle \hat{p} \rangle \langle \hat{x} \rangle & \langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2 \end{pmatrix}. \quad (6.17)$$

The Wigner function can be written as

$$W(x, p) = \frac{1}{2\pi|\Theta'|^{1/2}} \exp \left[-\frac{1}{2} (\vec{r}' - \vec{\mu}')^T \Theta'^{-1} (\vec{r}' - \vec{\mu}') \right], \quad (6.18)$$

where $\vec{r}' = [x, p]$. To constitute a valid quantum state, the covariance matrix must be real, symmetric and positive definite. The canonical commutation relations also imply that [261]

$$2\Theta + i\Omega \geq 0, \quad \Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (6.19)$$

The covariance matrix and vector of means can be rotated into the mode basis via the following transforms

$$\mu(\alpha, \alpha^*) = \hat{L}_c \mu'(x, p), \quad (6.20)$$

$$\Theta(\alpha, \alpha^*) = \hat{L}_c \Theta'(x, p) \hat{L}_c^\dagger, \quad (6.21)$$

$$\hat{L} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}. \quad (6.22)$$

Where we have included the mode and quadrature arguments for clarity. The rotated vector of means is $\mu = [\langle \hat{a} \rangle, \langle \hat{a}^\dagger \rangle]$ and the rotated covariance matrix

$$\Theta = \begin{pmatrix} \frac{1}{2} (\langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a} \hat{a}^\dagger \rangle) - \langle \hat{a}^\dagger \rangle \langle \hat{a} \rangle & \langle \hat{a}^2 \rangle - \langle \hat{a} \rangle^2 \\ \langle (\hat{a}^\dagger)^2 \rangle - \langle \hat{a}^\dagger \rangle^2 & \frac{1}{2} (\langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a} \hat{a}^\dagger \rangle) - \langle \hat{a}^\dagger \rangle \langle \hat{a} \rangle \end{pmatrix}. \quad (6.23)$$

Due to Eq. 1.74, a Gaussian state in the mode basis is given by

$$W(\alpha, \alpha^*) = \frac{1}{\pi |\Theta|^{1/2}} \exp \left[-\frac{1}{2} (\vec{r} - \vec{\mu})^\dagger \Theta^{-1} (\vec{r} - \vec{\mu}) \right], \quad (6.24)$$

where $\vec{r} = [\alpha, \alpha^*]$. Thermal states, which are examples of Gaussian states, are particularly interesting for this analysis. Thermal states have a first moment that completely vanishes, $\langle \hat{a} \rangle = \langle \hat{a}^\dagger \rangle = 0$ and a variance matrix that is given by $\Theta = \frac{1}{2} \coth(\beta\omega/2) \mathbb{1}$. In mode representation, the thermal state is given as [43]¹

$$W_{\text{th}} = \frac{2}{\pi} \tanh\left(\frac{\omega_0\beta}{2}\right) \exp\left(-2|\alpha|^2 \tanh\left(\frac{\omega_0\beta}{2}\right)\right) = \frac{1}{\pi} f(\bar{n}) \exp\left(-|\alpha|^2 f(\bar{n})\right), \quad (6.25)$$

where $f(\bar{n}) = (\bar{n} + 1/2)^{-1}$ and $\bar{n} = (e^{\beta\omega} - 1)^{-1}$ is the Bose-Einstein distribution function. We now introduce two Gaussian preserving unitaries; the first is the displacement operator

$$\hat{D}(d) = \exp(d\hat{a}^\dagger - d^*\hat{a}), \quad (6.26)$$

¹A quick note on the presence of the -2 coefficient in the exponential as it has confused me many times. The vector $r = [\alpha, \alpha^*]$, thus the inner product $\vec{r}^\dagger \vec{r} = 2|\alpha|^2$, this factor of 2 cancels out with the factor of $\frac{1}{2}$ in the square root and thus gives the result Eq. 6.25.

this operator displaces the Gaussian state, shifting the vector of means without affecting the covariance matrix:

$$\hat{\rho} \Rightarrow \hat{D}^\dagger(d)\hat{\rho}\hat{D}(d) : \quad \vec{\mu} = \begin{pmatrix} \alpha \\ \alpha^* \end{pmatrix} \Rightarrow \vec{\mu}_d = \begin{pmatrix} \alpha + d \\ \alpha^* + d^* \end{pmatrix}, \quad \Theta \Rightarrow \Theta. \quad (6.27)$$

The second Gaussian preserving unitary we will use in our analysis is the squeezing operator,

$$\hat{S}(\zeta) = \exp\left(\frac{1}{2}(\zeta^* \hat{a}^2 - \zeta \hat{a}^{\dagger 2})\right), \quad (6.28)$$

the squeezing operator is parametrised with $\zeta = r \exp(i\theta)$. The squeezing operator transforms the modes according to

$$\hat{S}^\dagger(\zeta)\hat{a}\hat{S}(\zeta) = \hat{a} \cosh(r) - \hat{a}^\dagger e^{i\theta} \sinh(r), \quad (6.29)$$

$$\hat{S}^\dagger(\zeta)\hat{a}^\dagger\hat{S}(\zeta) = \hat{a}^\dagger \cosh(r) - \hat{a} e^{-i\theta} \sinh(r). \quad (6.30)$$

The squeezing operator transforms the vector of means according to

$$\vec{\mu} = \begin{pmatrix} \alpha \\ \alpha^* \end{pmatrix} \Rightarrow \vec{\mu}_d = \begin{pmatrix} \alpha \cosh(r) - \alpha^* e^{i\theta} \sinh(r) \\ \alpha^* \cosh(r) - \alpha e^{-i\theta} \sinh(r) \end{pmatrix} \quad (6.31)$$

If the initial state $\hat{\rho}$ is a thermal state, the mean vector is given by the null vector and the transformation won't displace the state. However, the squeezing operation will transform the covariance matrix according to

$$\begin{aligned} \hat{\rho}_{\text{th}} \Rightarrow \hat{S}^\dagger(\zeta)\hat{\rho}_{\text{th}}\hat{S}(\zeta) : \quad \vec{\mu} \Rightarrow \vec{\mu} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\ \Theta = \frac{1}{2} \coth\left(\frac{\beta\omega_0}{2}\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \Theta_s = \frac{1}{2} \coth\left(\frac{\beta\omega_0}{2}\right) \begin{pmatrix} \cosh(2r) & -e^{i\theta} \sinh(2r) \\ -e^{-i\theta} \sinh(2r) & \cosh(2r) \end{pmatrix}. \end{aligned} \quad (6.32)$$

The effect of the squeezing and displacement operators on thermal states is visualised in Fig. 6.2.

As the Hamiltonian, $\hat{H}$ in the Davies Lindblad generator, Eq. 6.16, is quadratic, and the jump operators, $\hat{L}_k$ are linear, the Davies map is Gaussian preserving [262]. Due to this property, for Gaussian states, one need only solve for the evolution of the mean vector and

covariance matrix to model the relaxation of the entire system. Solving for the operator expectation values:

$$\frac{d\langle\hat{a}\rangle}{dt} = \left(\frac{d\langle\hat{a}^\dagger\rangle}{dt}\right)^\dagger = -(\gamma/2 - i\omega_0)\langle\hat{a}\rangle, \quad (6.33a)$$

$$\frac{d\langle\hat{a}^\dagger\hat{a}\rangle}{dt} = \gamma(\bar{n}_0 - \langle\hat{a}^\dagger\hat{a}\rangle), \quad (6.33b)$$

$$\frac{d\langle\hat{a}\hat{a}^\dagger\rangle}{dt} = \gamma((\bar{n}_0 + 1) - \langle\hat{a}\hat{a}^\dagger\rangle), \quad (6.33c)$$

$$\frac{d\langle\hat{a}\hat{a}\rangle}{dt} = \left(\frac{d\langle\hat{a}^\dagger\hat{a}^\dagger\rangle}{dt}\right)^\dagger = -(\gamma - 2i\omega_0)\langle\hat{a}\hat{a}\rangle. \quad (6.33d)$$

The evolution of the vector of means is

$$\frac{d\vec{\mu}}{dt} = W\vec{\mu}, \quad (6.34)$$

where

$$W = \begin{pmatrix} -\left(\frac{\gamma}{2} - i\omega_0\right) & 0 \\ 0 & -\left(\frac{\gamma}{2} + i\omega_0\right) \end{pmatrix}, \quad (6.35)$$

with solution

$$\vec{\mu}(t) = e^{-\gamma t/2} \begin{pmatrix} \alpha_0 e^{-i\omega_0 t} \\ \alpha_0^* e^{i\omega_0 t} \end{pmatrix}. \quad (6.36)$$

The covariance matrix evolves according to a Lyapunov equation [263]

$$\frac{d\Theta}{dt} = \Lambda\Theta + \Theta\Lambda^\dagger + F, \quad (6.37)$$

$$\Lambda = -\frac{1}{2} \begin{pmatrix} \gamma - 2i\omega & 0 \\ 0 & \gamma + 2i\omega \end{pmatrix}, \quad F = \frac{\gamma}{2} \coth\left(\frac{\omega_0\beta}{2}\right) \mathbb{1}. \quad (6.38)$$

A general covariance matrix evolves according to

$$\Theta_0 = \begin{pmatrix} C_{11} & C_{12} \\ C_{12}^* & C_{11} \end{pmatrix},$$

$$\Theta(t) = \begin{pmatrix} C_{11}e^{-\gamma t} + \frac{1}{2} \coth\left(\frac{\omega_0\beta}{2}\right)(1 - e^{-\gamma t}) & C_{12}e^{-(\gamma - 2i\omega_0)t} \\ C_{12}^*e^{-(\gamma + 2i\omega_0)t} & C_{11}e^{-2\gamma t} + \frac{1}{2} \coth\left(\frac{\omega_0\beta}{2}\right)(1 - e^{-\gamma t}) \end{pmatrix}, \quad (6.39)$$

which restores the thermal state in the long time limit. The evolution of a squeezed and displaced state under a Davies map is visualised in Fig. 6.2.

6.3 The Ergotropic Mpemba Effect

To proceed, we will derive an expression for the entropy and Kullbeck-Lieber divergence. We define the Wigner entropy as the classical entropy of the Gaussian Wigner function [264]

$$S(W(\alpha, \alpha^*)) = - \int d^2\alpha W(\alpha, \alpha^*) \ln [W(\alpha, \alpha^*)]. \quad (6.40)$$

For brevity, we will drop the arguments for the Wigner function. The Wigner entropy is a valid entropy measure for Gaussian distributions and can be shown to be equivalent to the Rényi-2 entropy up to an additive constant [265]

$$S_2 = - \ln(\text{Tr} \{\rho^2\}). \quad (6.41)$$

To derive an expression for the Wigner entropy, we note that the Wigner entropy is equivalent to finding the expectation value of $\ln [W]$, defined in Eq. 6.24:

$$\begin{aligned} S(W) &= -\mathbb{E} [\ln(W)] \\ &= -\mathbb{E} [\ln(\pi)^{-1}] - \mathbb{E} [\ln(|\Theta|^{-1/2})] - \mathbb{E} \left[\ln \left(\exp \left(-\frac{1}{2} (\vec{r} - \vec{\mu})^\dagger \Theta^{-1} (\vec{r} - \vec{\mu}) \right) \right) \right] \\ &= -\ln(\pi) - \ln(|\Theta|^{-1/2}) - \mathbb{E} \left[\left(-\frac{1}{2} (\vec{r} - \vec{\mu})^\dagger \Theta^{-1} (\vec{r} - \vec{\mu}) \right) \right] \\ &= \ln(\pi) + \frac{1}{2} \ln(|\Theta|) + \frac{1}{2} \mathbb{E} \left[(\vec{r} - \vec{\mu})^\dagger \Theta^{-1} (\vec{r} - \vec{\mu}) \right]. \end{aligned} \quad (6.42)$$

We denote $\mathbb{E} [\bullet]$ as the expectation value of the function $\bullet$ on the probability distribution W . To understand the final term of the RHS we use the cyclic property of the trace

$$\begin{aligned} \mathbb{E} \left[(\vec{r} - \vec{\mu})^\dagger \Theta^{-1} (\vec{r} - \vec{\mu}) \right] &= \mathbb{E} \left[\text{Tr} \left[(\vec{r} - \vec{\mu})^\dagger \Theta^{-1} (\vec{r} - \vec{\mu}) \right] \right] \\ &= \mathbb{E} \left[\text{Tr} \left[(\Theta^{-1} (\vec{r} - \vec{\mu}) (\vec{r} - \vec{\mu})^\dagger) \right] \right] \\ &= \text{Tr} \left[(\Theta^{-1} \mathbb{E} [(\vec{r} - \vec{\mu}) (\vec{r} - \vec{\mu})^\dagger]) \right] \\ &= \text{Tr} [\Theta^{-1} \Theta] = 2. \end{aligned} \quad (6.43)$$

where we have used the definition of the covariance matrix, $\Theta = \mathbb{E} [(\vec{v} - \vec{\mu})(\vec{v} - \vec{\mu})^\dagger]$. Inserting this expression into Eq. 6.42, the Wigner entropy is expressed as

$$S(W) = [\ln(\pi) + 1] + \frac{1}{2} \ln(|\Theta|). \quad (6.44)$$

One can also define a distance measure between two Gaussian states W_1 and W_2 with mean vectors and covariance matrices $\vec{\mu}_1, \vec{\mu}_2$ and Θ_1, Θ_2 respectively. The distance measure is given by the relative entropy, also known as the Kullbeck-Lieber divergence of the Gaussian probability distribution

$$K[W_1||W_2] = \int d\alpha^2 W_1 \ln\left(\frac{W_1}{W_2}\right) = -S(W_1) - \int d\alpha^2 W_1 \ln(W_2). \quad (6.45)$$

To calculate the second term on the RHS, we introduce new notation, $\mathbb{E}_{W_1}[\bullet]$ — the expectation value of the function $\bullet$ with respect to the Wigner function W_1 . Rewriting the second term:

$$\begin{aligned} \int W_1 \ln(W_2) d|\alpha|^2 &= \mathbb{E}_{W_1} \left[\log(\pi)^{-1} + \ln(|\Theta_2|^{-1/2}) + \ln \left(\exp \left(-\frac{1}{2} (\vec{r} - \vec{\mu}_2)^\dagger \Theta_2^{-1} (\vec{r} - \vec{\mu}_2) \right) \right) \right] \\ &= -\ln(\pi) - \frac{1}{2} \ln(|\Theta_2|) - \frac{1}{2} \mathbb{E}_{W_1} \left[(\vec{r} - \vec{\mu}_2)^\dagger \Theta_2^{-1} (\vec{r} - \vec{\mu}_2) \right]. \end{aligned} \quad (6.46)$$

We can use the same trick as before to calculate the last term on the right-hand side

$$\mathbb{E}_{W_1} \left[(\vec{r} - \vec{\mu}_2)^\dagger \Theta_2^{-1} (\vec{r} - \vec{\mu}_2) \right] = \text{Tr} \left(\Theta_2^{-1} \mathbb{E}_{W_1} \left[(\vec{r} - \vec{\mu}_2)(\vec{r} - \vec{\mu}_2)^T \right] \right). \quad (6.47)$$

The final term is tricky, but by multiplying out terms, one finds the following

$$\mathbb{E}_{W_1} \left[\vec{r} \vec{r}^\dagger \right] = \Theta_1 + \vec{\mu}_1 \vec{\mu}_1^\dagger, \quad (6.48a)$$

$$\mathbb{E}_{W_1} \left[\vec{r} \vec{\mu}_2^\dagger \right] = \vec{\mu}_1 \vec{\mu}_2^\dagger, \quad (6.48b)$$

$$\mathbb{E}_{W_1} \left[\vec{\mu}_2 \vec{r}^\dagger \right] = \vec{\mu}_2 \vec{\mu}_1^\dagger, \quad (6.48c)$$

$$\mathbb{E}_{W_1} \left[\vec{\mu}_2 \vec{\mu}_2^\dagger \right] = \vec{\mu}_2 \vec{\mu}_2^\dagger. \quad (6.48d)$$

Plugging these identities into Eq. 6.47 one gets

$$\mathbb{E}_{W_1} \left[(\vec{r} - \vec{\mu}_2)^T \Theta_2^{-1} (\vec{r} - \vec{\mu}_2) \right] = \text{Tr} \left(\Theta_2^{-1} \Theta_1 \right) + (\vec{\mu}_1 - \vec{\mu}_2)^T \Theta_2^{-1} (\vec{\mu}_1 - \vec{\mu}_2). \quad (6.49)$$

Combining everything

$$\int W_1 \ln(W_2) d\alpha^2 = -\ln(\pi) - \frac{1}{2} \ln(|\Theta_2|) - \frac{1}{2} \left(\text{Tr} \left(\Theta_2^{-1} \Theta_1 \right) + (\vec{\mu}_1 - \vec{\mu}_2)^T \Theta_2^{-1} (\vec{\mu}_1 - \vec{\mu}_2) \right). \quad (6.50)$$

Substituting this and the entropy of the Gaussian distribution into the equation for the relative entropy

$$K(W_1||W_2) = -1 + \frac{1}{2} \left(\ln \left(\frac{|\Theta_2|}{|\Theta_1|} \right) + \text{Tr}(\Theta_2^{-1} \Theta_1) + (\vec{\mu}_1 - \vec{\mu}_2)^\dagger \Theta_2^{-1} (\vec{\mu}_1 - \vec{\mu}_2) \right). \quad (6.51)$$

Now we can define the ergotropy of Gaussian states, taking the definition of the relative entropy between a Gaussian state and a thermal state defined by Eq. 6.25 and adding and subtracting the Wigner entropy of the equilibrium state

$$\begin{aligned} K[W||W_{\text{th}}] &= - \int d^2\alpha W \ln W_{\text{th}} - S(W) + S(W_{\text{th}}) - S(W_{\text{th}}) \\ &= - \int d^2\alpha W \ln W_{\text{th}} + \int d^2\alpha W_{\text{th}} \ln W_{\text{th}} + S(W_{\text{th}}) - S(W) \\ &= - \int d^2\alpha (W - W_{\text{th}}) \ln W_{\text{th}} + S(W_{\text{th}}) - S(W). \end{aligned} \quad (6.52)$$

Inserting the definition of the equilibrium state into Eq. 6.52

$$\begin{aligned} K[W||W_{\text{th}}] &= - \int d^2\alpha (W - W_{\text{th}}) \left(\ln \left(\frac{2}{\pi} \tanh \left(\frac{\omega_0 \beta}{2} \right) \right) - 2|\alpha|^2 \tanh \left(\frac{\omega_0 \beta}{2} \right) \right) + S(W_{\text{th}}) - S(W) \\ &= 2 \tanh \left(\frac{\omega_0 \beta}{2} \right) \int d^2\alpha (W - W_{\text{th}}) |\alpha|^2 + S(W_{\text{th}}) - S(W) \end{aligned} \quad (6.53)$$

For a single-mode bosonic operator, the Hamiltonian is defined as $\hat{H} = \omega_0 (\hat{a}^\dagger \hat{a} + 1/2)$. Rewriting this in symmetric ordering — $\hat{H} = \omega_0/2 (\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger)$ — allows the calculation of the expectation energy through the Wigner function

$$\langle \hat{H} \rangle_W = \omega_0 \int d^2\alpha |\alpha|^2 W, \quad (6.54)$$

which can be inserted into the expression for the relative entropy

$$K[W||W_{\text{eq}}] = \frac{2 \tanh \left(\frac{\omega_0 \beta}{2} \right)}{\omega_0} [\langle \hat{H} \rangle_W - \langle \hat{H} \rangle_{W_{\text{th}}}] + S(W_{\text{th}}) - S(W). \quad (6.55)$$

Now, focusing on the ergotropy, the ergotropy is defined by the energy difference between a quantum state and its associated passive state

$$\mathcal{W}_E(W) = \langle H \rangle_W - \langle H \rangle_{W_p}. \quad (6.56)$$

Adding $2 \tanh\left(\frac{\omega_0\beta}{2}\right) \omega_0^{-1} [\langle \hat{H} \rangle_{W_p} - \langle \hat{H} \rangle_{W_p}]$ to the righthand side of Eq.6.55 and that the entropy of a quantum state is equal to the entropy of the corresponding passive state, the ergotropy of a Gaussian system can be expressed as

$$\mathcal{W}_E(W) = \frac{\omega_0}{2} \coth\left(\frac{\omega_0\beta}{2}\right) (K[W||W_{th}] - K[W_p||W_{th}]). \quad (6.57)$$

Eq. 6.57 is valid for any thermal state; however, as the passive state of a Gaussian state is a thermal state, we can select $W_{th} = W_p$ and the term $K[W_p||W_{th}]$ will vanish, simplifying the expression. The ergotropy can then be written as

$$\mathcal{W}_E(W) = \frac{\omega_0}{2} \coth\left(\frac{\omega_0\beta_W}{2}\right) K[W||W_{th}]. \quad (6.58)$$

Where we have defined the inverse temperature of the passive state as $\beta_W = 2 \coth^{-1}(2|\Theta|^{1/2})/\omega_0$. Using the expression for the relative entropy and the properties of thermal states, Eq. 6.25, this expression can be written as

$$\begin{aligned} \mathcal{W}_E(W) &= \frac{\omega_0}{2} \coth\left(\frac{\omega_0\beta_W}{2}\right) K[W||W_{th}] \\ &= \frac{\omega_0}{2} \coth\left(\frac{\omega_0\beta_W}{2}\right) \left(-1 + \frac{1}{2} \left(\ln\left(\frac{|\Theta_W|}{|\Theta|}\right) + \text{Tr}(\Theta_W^{-1}\Theta) + (\vec{\mu} - \vec{\mu}_W)^\dagger \Theta_W^{-1} (\vec{\mu} - \vec{\mu}_W) \right)\right) \\ &= \frac{\omega_0}{2} \coth\left(\frac{\omega_0\beta_W}{2}\right) \left(-1 + \frac{1}{2} \left(\text{Tr}(\Theta_W^{-1}\Theta) + (\vec{\mu})^\dagger \Theta_W^{-1} (\vec{\mu}) \right)\right) \\ &= \frac{\omega_0}{2} \left[-\coth\left(\frac{\omega_0\beta_W}{2}\right) + \text{Tr}(\Theta) + 2|\langle\alpha\rangle|^2 \right] \\ &= \frac{\omega_0}{2} \left[-2|\Theta|^{1/2} + \text{Tr}(\Theta) + 2|\langle\alpha\rangle|^2 \right]. \end{aligned} \quad (6.59)$$

On line three, we have used the fact that the passive state and the current state are unitarily connected, i.e. $|\Theta_W| = |\Theta|$, and the mean displacement of the thermal state is zero. We can separate this expression into separate terms, one relating to the displacement of the Gaussian and another relating to the covariance of the Gaussian

$$\mathcal{W}_E^\mu(W) = \omega_0 |\langle\alpha\rangle|^2, \quad (6.60)$$

$$\mathcal{W}_E^\Theta(W) = \frac{\omega_0}{2} (\text{Tr}(\Theta) - 2|\Theta|^{1/2}). \quad (6.61)$$

Thus, the ergotropy related to the displacement of the Gaussian is independent of the passive state. In contrast, the ergotropy related to the relaxation of the covariance matrix is dependent on the related passive state.

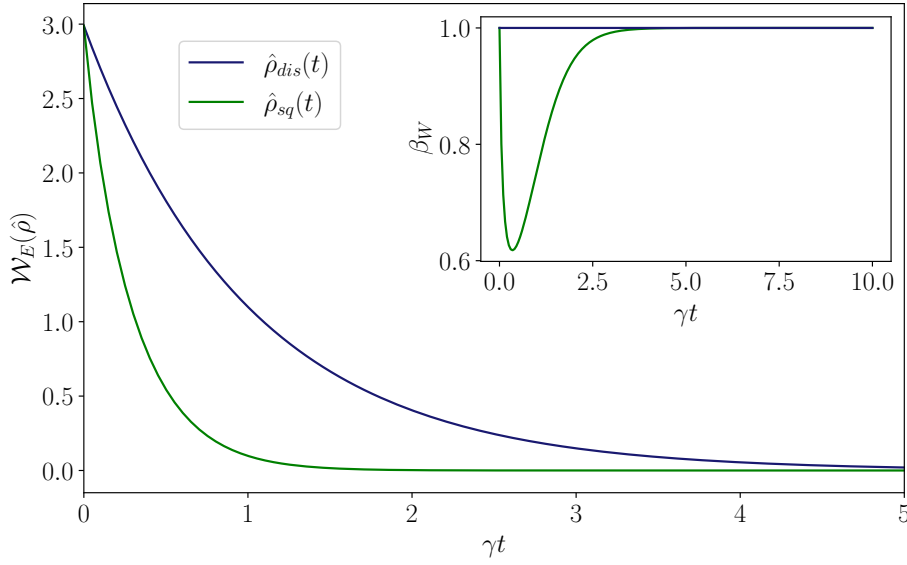


Figure 6.3: Decay of the ergotropy of a displaced thermal state (blue) vs a squeezed thermal state (green) for a quantum oscillator weakly coupled to a thermal bath. Despite initially having equal ergotropy, the decay of the ergotropy for the displaced state is slower than the squeezed state. The displaced state can be seen as a better resource for storing useful ergotropy in the oscillator. *Inset:* Plot showing the inverse temperature of the passive state associated with the state evolution for both the squeezed and thermal state. The associated passive state temperature is constant for the displaced state. The squeezed state is much different; despite the temperature of the associated passive state initially being equal to that of the bath, at long times, the transient behaviour is highly non-monotonic. The parameters are $\omega_0 = 1$, $T = 1$, $r = 5$, $\gamma = 0.5$, the displacement amplitude $|d|$ is selected to give the same initial displaced state ergotropy as the squeezed state.

A Mpemba effect can occur in these systems due to a separation of relaxation timescales. Consider a system initially in equilibrium with a heat bath and two options to store work — a displacement operator or a squeezing operator. Which transformation should be chosen to protect the stored work from the influence of the bath?

Considering the displacement operator. After applying the displacement operator, $\hat{\rho}'_{\text{dis}}(0) = \hat{D}^\dagger(d)\hat{\rho}\hat{D}(d)$, the initial ergotropy of the system is entirely contained in the displacement contribution:

$$\mathcal{W}_E(\hat{\rho}'_{\text{dis}}(0)) = \omega_0|d|^2 \quad (6.62)$$

From Eq.6.36 the evolution of the ergotropy is given as

$$\mathcal{W}_E(\hat{\rho}'_{\text{dis}}(t)) = e^{-\gamma t} \omega_0|d|^2 \quad (6.63)$$

Now consider an initially squeezed state - $\hat{\rho}'_{\text{sq}}(0) = \hat{S}^\dagger(\zeta)\hat{\rho}\hat{S}(\zeta)$ where $\zeta = re^{i\theta}$. Counter-

intuitively, even in the case without a thermal quench, the relaxation dynamics affect the temperature of the passive thermal state associated with the Gaussian state. The square root of the determinant of the covariance matrix is given by

$$|\Theta|^{1/2} = \frac{1}{2} \coth\left(\frac{\beta\omega_0}{2}\right) [e^{-\gamma t}(1 - e^{-\gamma t})(2 \cosh(2r) - 1) + 1]^{1/2}. \quad (6.64)$$

At the initial time, $t = 0$ and in the long time limit $t \rightarrow \infty$, the temperature of the passive state associated with the time-evolved squeezed state is equal to the bath temperature. At intermediate times, the temperature of the associated passive state is non-monotonic. A plot of the inverse temperature evolution of both an initially squeezed state and an initially displaced state is shown in Fig. 6.3. The initial ergotropy of the squeezed is contained purely in the covariance contribution:

$$\mathcal{W}_E(\hat{\rho}'_{\text{sq}}(0)) = \frac{\omega_0}{2} \coth\left(\frac{\omega_0\beta}{2}\right) (\cosh(2r) - 1). \quad (6.65)$$

From Eq. 6.39 the evolution of the ergotropy of the squeezed state is given as

$$\mathcal{W}_E(\hat{\rho}'_{\text{sq}}(t)) = \frac{\coth\left(\frac{\omega_0\beta}{2}\right)\omega_0}{2} \left(e^{-\gamma t} [\cosh(2r) - 1] + 1 - [e^{-\gamma t}(1 - e^{-\gamma t})(2 \cosh(2r) - 1) + 1]^{1/2} \right). \quad (6.66)$$

In Fig. 6.3, we show the discharging of a quantum oscillator for a state which has had ergotropy stored via a displacement operation and another state which has had its ergotropy stored via a squeezing operation. Despite having initially equal ergotropy, the squeezed state relaxed much faster than the displaced thermal state. Thus, the displacement operator can be seen as a better resource for storing useful ergotropy in the oscillator.

6.4 Non-Gaussian Eigenvalue Expansion

One can also perform a similar analysis to the quantum oscillator as done in Section 6.1. Using the operator-phase space correspondences 1.77 the master equation 6.16 can be converted to a Fokker-Planck equation acting in Wigner space, see Appendix D for details:

$$\begin{aligned} \frac{dW(x, p)}{dt} = & \left(-\frac{p}{m} \frac{\partial}{\partial x} + m\omega_0^2 x \frac{\partial}{\partial p} \right) W(x, p) + \frac{\gamma}{2} \left[\frac{\partial}{\partial x} x + \frac{\partial}{\partial p} p \right] W(x, p) \\ & + \frac{\gamma}{2} \left(\bar{n} + \frac{1}{2} \right) \left[m\omega_0 \frac{\partial^2}{\partial p^2} + \frac{1}{m\omega_0} \frac{\partial^2}{\partial x^2} \right] W(x, p). \end{aligned} \quad (6.67)$$

Although similar to the classical Fokker-Planck equation, this equation differs in two respects. First, there is an inclusion of damping and diffusion terms acting on the oscillator position. This incorporates a position-based damping term, which is highly non-classical. This term originates from the rotating wave approximation, the equation is only valid for large ω_0 . Because of this, Eq. 6.67 is known as the optical master equation where large oscillator frequencies are met. The second difference between Eq. 6.67 and the classical Fokker-Planck equation is the presence of the $\bar{n} + \frac{1}{2}$ term in the diffusion coefficient. This term arises due to zero temperature fluctuations that are not present in the classical limit — in the classical Fokker-Planck at zero temperature, the evolution of the oscillator is deterministic. The bosonic occupation function is given as $\bar{n} = \left(e^{\beta\omega_0} - 1\right)^{-1}$, one can restore the fluctuation-dissipation relation by taking the high-temperature limit $(\bar{n} + 1/2) \approx (\beta\omega_0)^{-1}$.

We now perform a similar analysis as in Section 6.1. A detailed derivation is given in Appendix E. Once again, the Fokker-Planck equation is converted into a Schrödinger-like equation. The eigenvalues of the Fokker-Planck equation are given by

$$\lambda_{n_1, n_2} = n_1 \lambda_1 + n_2 \lambda_2, \quad (6.68)$$

where

$$\lambda_1 = -\gamma - \sqrt{2}i\omega_0, \quad \lambda_2 = -\gamma + \sqrt{2}i\omega_0. \quad (6.69)$$

The eigenfunctions are constructed by applying ladder operators to the steady-state eigenvalue; see Appendix E for details. The steady state of the system is given as

$$\psi_{0,0} = \frac{\omega_0}{2\pi v_{\text{th}}^2} \exp \left[-\frac{1}{2} \left(\left(\frac{v}{v_{\text{th}}} \right)^2 + \left(\frac{\omega_0 x}{v_{\text{th}}} \right)^2 \right) \right], \quad (6.70)$$

while the slowest decaying eigenmodes are

$$\psi_{0,1}(x, v) = \frac{\sqrt{2}}{2} \left(-i \frac{\omega_0}{v_{\text{th}}} x + \frac{v}{v_{\text{th}}} \right) \psi_{0,0}(x, v), \quad (6.71)$$

$$\psi_{1,0}(x, v) = \frac{\sqrt{2}}{2} \left(\frac{\omega_0}{v_{\text{th}}} x - i \frac{v}{v_{\text{th}}} \right) \psi_{0,0}(x, v), \quad (6.72)$$

with overlap functions given by

$$c_{0,1} = \frac{1}{2} \sqrt{\frac{\omega_0}{\pi v_{\text{th}}^2}} \int W_0 \left(i \frac{\omega_0}{v_{\text{th}}} x + \frac{v}{v_{\text{th}}} \right) dx dv, \quad (6.73)$$

$$c_{1,0} = \frac{1}{2} \sqrt{\frac{\omega_0}{\pi v_{\text{th}}^2}} \int W_0 \left(\frac{\omega_0}{v_{\text{th}}} x + i \frac{v}{v_{\text{th}}} \right) dx dv. \quad (6.74)$$

Thus, any symmetric initial state will have no overlap with the slowest decaying mode, generating an exponential speedup to thermalisation. While this analysis is insightful in probing the system's relaxation dynamics, it is unclear how to relate the decomposition to the ergotropy of the system, as discussed in the previous section. It is an open question of how to relate this analysis with the ergotropy production; however, some interesting analysis can be done simply by considering the energy of the relaxing oscillator. In Appendix E.2, we show that the energy expectation value of the system will always evolve without any contribution from the slowest decaying mode due to symmetries in the slowest decaying eigenmode.

6.5 Summary and Outlook

In this chapter, we have extended the analysis of the Mpemba effect to both a classical oscillator and a quantum oscillator coupled to thermal baths. An eigenmode decomposition was performed on the Fokker-Planck equation, describing the evolution of the classical oscillator. The slowest decaying eigenmode was symmetric with respect to the position and momentum variables. Thus any symmetric initial state showed an exponential speed up to relaxation over asymmetric states. The equations of motion for Gaussian states evolving under a Davies map were studied for the quantum oscillator. The Mpemba effect was explored through the lens of the ergotropy of the state. It was found that displaced thermal states relax exponentially slower than squeezed thermal states.

An eigenmode decomposition was performed on the Davies map describing the relaxation of a quantum oscillator. Similar to the classical case, it was found that symmetric states have no overlap with the slowest decaying mode, generating an exponential speedup to relaxation. An open question is how to relate this decomposition analysis to the analysis of the ergotropy of the system, as it is unclear if there is a simple connection between the unitary generating the passive state and a transformation in phase space. While the analysis

of this work focused on a single-mode system, it would be interesting to extend to a multi-mode system with interactions between the modes. Another interesting avenue would be to explore non-Gaussian states. However, this may be a trickier task in Wigner space as the Wigner entropy is only well-defined for positive distributions. One could instead use the Q-function as it is always positive. However, this approach comes with its drawbacks, notably that the Wehrl entropy

$$S(Q) = - \int d\vec{\alpha} Q(\vec{\alpha}) \ln Q(\vec{\alpha}) \quad (6.75)$$

is not invariant under unitary transformations [266]. One can also extend the analysis of the Mpemba effect to more general dynamics, including quantum systems coupled to squeezed thermal reservoirs [37].

Finally, it is worth noting that the Davies map description relaxation is only valid in the regime of large ω_0 . Quantum Brownian motion is an example of a system where the rotating wave approximation is inapplicable. The evolution of a harmonic oscillator is given by the Fokker-Planck equation [267]

$$\frac{\partial W}{\partial t} = - \{H_{\text{ren}}(t), W\}_{PB} + \gamma(t) \frac{\partial}{\partial p} (pW) + D(t) \frac{\partial^2}{\partial p^2} W - f(t) \frac{\partial}{\partial p \partial x} W. \quad (6.76)$$

Here $\{A, B\}_{PB}$ denotes a Poisson bracket, H_{ren} denotes the renormalised Hamiltonian, γt is a damping term, $D(t)$ a diffusion term and $f(t)$ an anomalous diffusion term. The evolution is not Markovian — the time dependence of the coefficients encodes some memory of the initial state. While an analytic eigenmode decomposition may be prohibitive due to the time dependence on the coefficients, in the asymptotic limit, the coefficients may be replaced with their asymptotic values [40], making an analysis of the eigenmodes plausible.

Conclusions

Through this thesis, we have investigated the thermodynamics of two fascinating phenomena: self-sustained oscillations and the Mpemba effect. These topics are at the forefront of research, experimentally [20, 18] and theoretically [16, 153], providing fertile ground to explore new areas of physics.

The field of mesoscopic physics has been used to test the limits of thermodynamics, showcasing energy harvesting [126], tunable heat currents [127] and self-sustained oscillations [20]. These systems act as interesting platforms to further our understanding and serve as an inspiration for much of this work. Taking inspiration from the self-sustained oscillations, we constructed a model that qualitatively reproduces this phenomenon, acting as a bridge between thermodynamic physics and mesoscopic devices.

We focused our analysis of the self-sustained oscillations through the lens of extractable work. We characterised the lasing threshold through the ergotropy. States below the lasing threshold were found to be passive with respect to the oscillator Hamiltonian. In contrast, states above threshold were found to be non-passive, with a population inversion in the energy eigenbasis. An interesting follow-up to this work would be to move away from the quasi-adiabatic limit [156]. The typical battery states seen through this work were diagonal in the energy eigenbasis; through stronger coupling, it may be possible to generate mechanical coherences; from these coherent states, additional work may be stored and extracted [56, 185].

While our analysis explored the maximum amount of extractable work stored in the mechanical degrees of freedom, we did not consider work extraction in these systems. An

interesting question for future work is how the work stored in the battery state can be discharged into the electronics, generating an electric current against a potential bias [5]. This conversion does not need to use charge currents, instead heat currents may be used to extract the work [186], opening up the prospect of on-demand energy storage and extraction through mechanical degrees of freedom, which could be demonstrated on experimental platforms.

These mesoscopic devices also act as candidates to probe our understanding of timekeeping and test the fundamental thermodynamics behind our ability to measure time. In electromechanical setups, the coupling between electronic and mechanical degrees of freedom allows for the construction of autonomous clocks. While these clocks may never reach the precision of modern optical clocks, they provide an ideal candidate to test the fundamental limits of the laws of thermodynamics [8].

Our analysis goes beyond previous analyses of quantum clocks by incorporating a continuous degree of freedom into the system of interest. Our analysis focused solely on the quasi-adiabatic regime, which is often the accessible regime in state-of-the-art experiments. However, there are more optimal regimes than the quasi-adiabatic regime for studying and understanding the thermodynamics of timekeeping due to the assumption of a large electron flow through the system per tick cycle, producing large amounts of entropy per tick. Further work could push the analysis into the strong coupling regime where fundamental tradeoffs between precision and entropy production could be further analysed. Other works generate bounds on the precision of currents; for example, thermodynamic uncertainty relations bound the currents by the entropy production [111]. Our work also opens up new unexplored avenues of quantum clocks, namely temporal correlations between ticks. Such correlations are often ignored in the quantum clocks community, instead studying models where the ticks are considered i.i.d. The presence of correlations between ticks warrants the exploration of other methods to quantify the precision of quantum clocks as the accuracy defined in [8] assumes the presence of i.i.d. ticks.

The second aspect of this thesis focused on the Mpemba effect. An idea permeated through generations in the folklore of many societies but has only recently had the spotlight of physics [217]. Inspired by experimental and theoretical works, we investigated the Mpemba

effect in quantum systems, placing the quantum effect on a thermodynamic foundation. We have demonstrated that a quantum Mpemba effect can be engineered by exploiting the sub-block structure of the generator in thermalising open Markovian systems. We highlight the non-equilibrium free energy as a central thermodynamic object to define and study the effect in open quantum systems. When the spectral gap of a Davies generator is defined by a complex eigenpair, a unitary operation can always be found on an initial state, simultaneously raising the free energy and providing exponentially quicker thermalisation.

The work presented here not only enhances our understanding of the Mpemba effect in quantum systems but should also inspire further studies from the perspective of quantum thermodynamics. The Mpemba effect could also prove useful in dissipative state engineering protocols [249–251].

We have also analysed the Mpemba effect in continuous variable systems, using the effect to preserve the work stored in an oscillator, similar to the oscillator in the electromechanical model. Through an eigenvalue expansion of the Fokker-Planck equation, we found that a Mpemba effect can be generated by breaking the phase-space symmetry of a classical oscillator. Investigating a quantum oscillator, whose relaxation dynamics are described by a Davies map, we found that the relaxation dynamics for a displaced thermal state is exponentially slower than the relaxation dynamics for a squeezed thermal state. This leads to a Mpemba effect in the ergotropy of an oscillator, where work stored in a displaced state will be preserved longer than a squeezed state. In an upcoming publication, we plan on exploring the eigenfunction decomposition of the Davies map to understand the role of symmetry in these maps. While many questions are left unanswered, the work outlined in this thesis acts as a starting point to probe further questions in the fields of work extraction in mesoscopic systems, the fundamental limits on the precision of timekeeping and the Mpemba effect in quantum systems.

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Appendices

A Models used to describe quantum clocks with continuous degrees of freedom

In this appendix, we derive two simple descriptions of phase and amplitude fluctuations on the oscillator's limit cycle. One description is based on a modified Ornstein-Uhlenbeck process, and the second on a telegraph process.

We will begin with the modified Ornstein-Uhlenbeck process. Our starting point is the Langevin Eq. 2.40 governing the evolution of the oscillator. We first transform the equation to polar coordinates. Using the substitution $z\omega_0 = v$ and converting the system to polar coordinates, $x = A\cos\phi$, $z = A\sin\phi$ the equations of motion become

$$dA = \cos\phi dx + \sin\phi dz + \frac{\cos^2\phi}{2A} dz^2, \quad (\text{A.1})$$

$$d\phi = -\frac{\sin\phi}{A} dx + \frac{\cos\phi}{A} dz - \frac{\cos\phi\sin\phi}{2A^2} dz^2. \quad (\text{A.2})$$

Here we have used Itô's lemma, Eq. 1.34, where $dz^2 = \frac{D_x}{\omega_0^2} dt$, $dx^2 = 0$ and $dx dz = 0$. Focusing first on the amplitude equation, the equation of motion is given as

$$\begin{aligned} dA = & -m\gamma_{(A,\phi)} A \sin^2(\phi) dt + \frac{F}{\omega_0} \langle n \rangle_{(A,\phi)} \sin(\phi) dt \\ & + \frac{\sqrt{D_{(A,\phi)}}}{\omega_0} \sin(\phi) dW + \frac{\cos^2(\phi)}{2A} \frac{D_{(A,\phi)}}{\omega_0^2} dt. \end{aligned} \quad (\text{A.3})$$

Here dW is a Wiener increment, and the subscript (A, ϕ) is used to indicate dependence on both amplitude and the phase. Assuming the amplitude change over a single cycle is small,

we can replace the deterministic terms with their mean over a single oscillation and the stochastic term with its RMS over a single oscillation, see Section 1.6, for a more in-depth discussion;

$$\begin{aligned} dA = & -\frac{A}{2\pi} \oint (\gamma_{(A,\phi)} \sin^2(\phi)) d\phi dt + \frac{F}{2\pi\omega_0} \oint (\langle n \rangle_{(A,\phi)} \sin(\phi)) d\phi dt \\ & + \frac{1}{\sqrt{2\pi}\omega_0} \sqrt{\oint (\sqrt{D_{(A,\phi)}} \sin(\phi))^2 d\phi} dW_A + \frac{1}{4\pi A \omega_0^2} \oint (D_{(A,\phi)} \cos^2(\phi)) d\phi dt. \end{aligned} \quad (\text{A.4})$$

Here dW_A is an independent Wiener increment. Due to symmetry in the $\langle \hat{n} \rangle_x$ term, its average over a single cycle will be zero. We can find the limit cycle of the system by taking the expectation value of both sides of the equation and setting $\frac{\Delta \mathbb{E}[A]}{\Delta t} = 0$. The limit cycle is given by the solution of the transcendental equation

$$1 = \frac{1}{2\mathbb{E}[A]^2 \omega_0^2} \frac{\mathbb{E} \left[\oint \cos^2(\phi) D_{(A,\phi)} d\phi \right]}{\mathbb{E} \left[\oint \sin^2(\phi) \gamma_{(A,\phi)} d\phi \right]}. \quad (\text{A.5})$$

We can also truncate Eq. A.4 to lowest order for the dissipation and the fluctuation terms around the limit cycle. Using the substitutions

$$\gamma_A = \frac{d}{dA} \left(-\frac{A}{2\pi} \oint (\gamma_{(A,\phi)} \sin^2(\phi)) d\phi + \frac{1}{4\pi A \omega_0^2} \oint (D_{(A,\phi)} \cos^2(\phi)) d\phi \right) \Big|_{A=A_0}, \quad (\text{A.6})$$

$$\sqrt{D_A} = \frac{1}{\sqrt{2\pi}\omega_0} \sqrt{\oint (\sqrt{D_{(A,\phi)}} \sin(\phi))^2 d\phi} \Big|_{A=A_0}. \quad (\text{A.7})$$

Substituting these into the Eq. A.4, one arrives at the much simpler equation

$$dA = -\gamma_A (A - A_0) dt + \sqrt{D_A} dW_A. \quad (\text{A.8})$$

This equation describes the amplitude of the system as an Ornstein-Uhlenbeck process [71].

We can perform a similar analysis on the equation of motion for the phase:

$$d\phi = -\omega_0 dt - \gamma_x \cos(\phi) \sin(\phi) dt + \frac{\cos \phi F}{\omega_0 A} dt + \frac{\cos(\phi) \sqrt{D_x}}{A \omega_0} dW - \frac{\cos(\phi) \sin(\phi) D_x}{2A^2 \omega_0^2} dt. \quad (\text{A.9})$$

We will assume that the oscillator's natural frequency is the dominant term in the deterministic part of the equation. The phase evolution is given by

$$d\phi = -\omega_0 dt + \frac{\cos(\phi) \sqrt{D_x}}{A \omega_0} dW. \quad (\text{A.10})$$

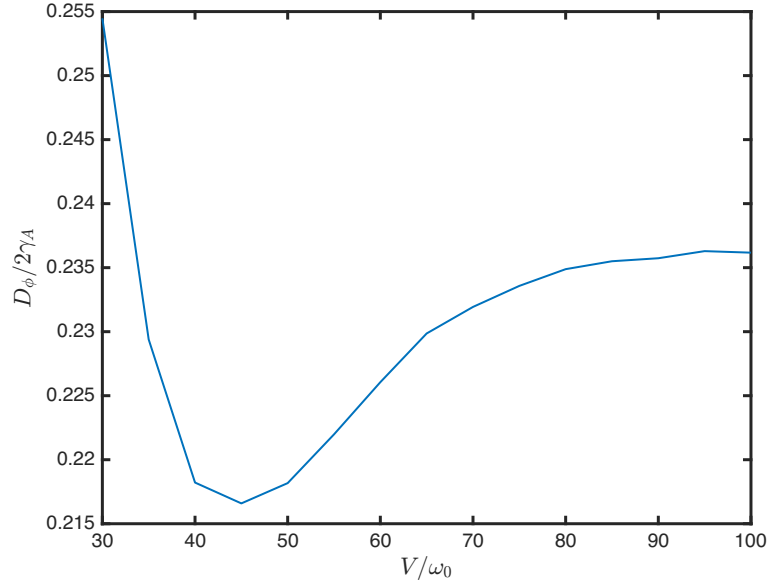


Figure A12: Ratio between the phase diffusion coefficient D_ϕ and the amplitude damping rate γ_A computed for small fluctuations away from the ideal limit cycle. The relatively small values indicate that phase coherence on the limit cycle is long-lived compared to the rate of amplitude variation. Parameters are the same as Fig. 4.2.

Assuming sufficiently large $\omega_0^2 A / \sqrt{D_x}$ we can linearise Eq. (A.10) by replacing the stochastic term prefactor with its RMS over a single cycle so that

$$d\phi = -\omega_0 dt + \sqrt{D_\phi} dW_\phi, \quad (\text{A.11})$$

here dW_ϕ is the effective Wiener increment originating from the linearisation. The diffusion term is given by

$$\sqrt{D_\phi} = \frac{1}{A\omega_0 \sqrt{2\pi}} \sqrt{\oint [\cos^2(\phi) D_x] d\phi} \Big|_{A=A_0}. \quad (\text{A.12})$$

Note that in the main text near Eqs. (4.29) and (4.30) we have replaced $\phi \rightarrow -\phi$ merely for clarity of presentation. Since the amplitude and phase noise are independent within our approximations, it is straightforward to compute the steady-state position autocorrelation function [71]

$$\begin{aligned} \mathbb{E}[x(t)x(0)] - \mathbb{E}[x(0)]^2 &= \mathbb{E}[A(t)A(0)]\mathbb{E}[\cos \phi(t) \cos \phi(0)] \\ &= \left(A_0^2 + \frac{D_A e^{-2\gamma_A t}}{2\gamma_A} \right) \frac{\cos(\omega_0 t) e^{-D_\phi t/2}}{2}. \end{aligned} \quad (\text{A.13})$$

This function exhibits oscillatory decay at a rate that is ultimately dictated by the phase decoherence time, D_ϕ^{-1} . This conclusion holds irrespectively of the scale, D_A , and regression

rate, γ_A , of amplitude fluctuations.

To compare the characteristic timescales of amplitude and phase fluctuations that result from this description, we numerically compute the quantities γ_A and D_ϕ from Eqs. (A.6) and (A.12), respectively. Their ratio is plotted in Fig. A12 as a function of the voltage bias, for values of V above and beyond the threshold for self-oscillations. We see that we have $D_\phi/4\gamma_A \ll 1$ for all voltages considered, implying that phase coherence lasts longer than the characteristic timescale of amplitude variations.

We will now move on to the second model, we derive the autocorrelation function (4.31) for a bistable current that switches between two states, finding that it reproduces the features seen in Fig. 4.3 when the switching process is sufficiently slow. Concretely, consider a stochastic process $I(t)$ that switches randomly between two different oscillatory states labelled by $i = 1, 2$, giving rise to stochastic sub-signals $I_1(t)$ and $I_2(t)$. We are interested in the steady-state correlation function

$$C(t) = \mathbb{E}[I(t)I(0)] - \mathbb{E}[I(0)]^2. \quad (\text{A.14})$$

The average signal is

$$\mathbb{E}[I(0)] = p_1 \bar{I}_1 + p_2 \bar{I}_2, \quad (\text{A.15})$$

where p_i is the steady-state probability to be in state i and $\bar{I}_i = \mathbb{E}[I_i(0)]$ is the steady-state average of the corresponding sub-signal. Assuming that switching between the two states occurs independently of the value of $I_i(t)$, we can write

$$\mathbb{E}[I(t)I(0)] = \sum_{i,j=1}^2 p_{ji}(t) p_i \mathbb{E}[I_j(t)I_i(0)], \quad (\text{A.16})$$

where $p_{ji}(t)$ is the conditional probability to be in state j at time t , given state i at $t = 0$. Here we take $t > 0$ without loss of generality, since the stationary correlation function obeys $C(t) = C(-t)$. We further assume that the two sub-signals are statistically independent from each other, i.e.

$$\mathbb{E}[I_1(t)I_2(t')] = \mathbb{E}[I_1(t)]\mathbb{E}[I_2(t')]. \quad (\text{A.17})$$

Now, splitting the sum in Eq. (A.16) into diagonal ($i = j$) and off-diagonal ($i \neq j$) terms,

and subtracting the square of Eq. (A.15), we arrive at Eqs. (4.31) and (4.32), i.e.

$$C(t) = \sum_{i=1}^2 p_i p_{i|i}(t) C_i(t) + C_{1 \leftrightarrow 2}(t), \quad (\text{A.18})$$

$$C_{1 \leftrightarrow 2}(t) = \sum_{i,j=1}^2 p_i [p_{j|i}(t) - p_j] \bar{I}_i \bar{I}_j, \quad (\text{A.19})$$

where $C_i(t) = \mathbb{E}[I_i(t)I_i(0)] - \bar{I}_i^2$ is the auto-correlation function of $I_i(t)$.

The switching term $C_{1 \leftrightarrow 2}(t)$ has the properties expected of an autocorrelation function: in particular, it vanishes as $t \rightarrow \infty$ since $p_{j|i}(t) \rightarrow p_j$ at long times. Using the conservation of probability, $\sum_i p_i = \sum_j p_{j|i}(t) = 1$, one can also show that

$$C_{1 \leftrightarrow 2}(t) = (\bar{I}_2 - \bar{I}_1) (p_1 \bar{I}_1 [p_{2|1}(t) - p_2] - p_2 \bar{I}_2 [p_{1|2}(t) - p_1]). \quad (\text{A.20})$$

Therefore, the switching term is non-zero only if $\bar{I}_2 \neq \bar{I}_1$, i.e. the two sub-signals have a different mean value. If this is the case, and if the switching probabilities $p_{j|i}(t)$ evolve on a slow timescale compared to $C_i(t)$, then Eq. (A.18) will exhibit two well-separated timescales as in Fig. 4.3.

To demonstrate this explicitly, we assume that the switching occurs via a simple telegraph process described by the master equation

$$\dot{p}_1(t) = \lambda_2 p_2(t) - \lambda_1 p_1(t) = -\dot{p}_2(t), \quad (\text{A.21})$$

where λ_i are transition rates between the sub-signals. The steady-state probabilities are given by

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \frac{1}{\lambda_1 + \lambda_2} \begin{pmatrix} \lambda_2 \\ \lambda_1 \end{pmatrix}, \quad (\text{A.22})$$

The transition probability of switching from state i to state $j \neq i$ after a time $t > 0$ is given by the solution of Eq. (A.21) with $p_i(0) = 1$, i.e.

$$p_{j|i}(t) = \frac{\lambda_i}{\lambda_1 + \lambda_2} (1 - e^{-(\lambda_1 + \lambda_2)t}). \quad (\text{A.23})$$

Substituting these results into Eq. (A.20) gives the two-point correlation function of the

telegraph process

$$C_{1\leftrightarrow 2}(t) = \frac{(\overline{I_2} - \overline{I_1})^2 \lambda_1 \lambda_2 e^{-(\lambda_1 + \lambda_2)|t|}}{(\lambda_1 + \lambda_2)^2}. \quad (\text{A.24})$$

Therefore, the total correlation function [Eq. (A.18)] is the sum of two terms: an oscillatory function determined by the $C_i(t)$, which decays due to amplitude and phase fluctuations within each state, and a pure exponential decay given by Eq. (A.24). So long as the total switching rate $\lambda_1 + \lambda_2$ is slow compared to the decay of the $C_i(t)$, the latter contribution will survive after the oscillations have died out. This is precisely the qualitative behaviour above threshold seen in Fig. 4.3.

This model also predicts a large zero-frequency component to the current noise, consistent with our observations in Sec. 4.2.1. From Eq. (4.33) one has [263]

$$\Xi(0) = 2 \int_0^\infty dt C(t). \quad (\text{A.25})$$

Therefore the switching term contributes a term that scales as

$$\Xi(0) \sim \frac{\lambda_1 \lambda_2}{(\lambda_1 + \lambda_2)^3}, \quad (\text{A.26})$$

which diverges as $\lambda_i \rightarrow 0$. For example, if $\lambda_1 = \lambda_2 = \lambda$ then we simply have $\Xi(0) \sim \lambda^{-1}$, where λ^{-1} is the lifetime of each state.

Note that other models of slow amplitude fluctuations give rise to similar behaviour. However, it is crucial that the cycle-averaged signal depends on the amplitude, which is reflected here by the condition that $\overline{I_1} \neq \overline{I_2}$. For example, the model of independent amplitude and phase fluctuations developed for the Ornstein Uhlenbeck process yields a position autocorrelation function given by Eq. (A.13). This does not show a separation of timescales, even when the amplitude fluctuations are much slower than phase fluctuations, i.e. $D_A, \gamma_A \ll D_\phi$. This is because the average of $x(t) = A(t) \cos \phi(t)$ around one cycle is zero, independently of the value of $A(t)$.

To illustrate the importance of an amplitude-dependent cycle average, let us instead consider a model of stochastic oscillations with an offset:

$$I(t) = A(t) \{\cos[\phi(t)] - b\} + c, \quad (\text{A.27})$$

where b and c are constants, and $A(t)$ and $\phi(t)$ are described by an Ornstein-Uhlenbeck

process [Eq. (4.29)] and a biased diffusion process [Eq. (4.30)], respectively. The resulting stochastic dynamics is qualitatively similar to the behaviour seen in Fig. 4.2 in regions where the peak-to-peak amplitude is large, with c representing the maximum current and the scale factor b determining how much oscillations reduce the current on average. In particular, the cycle average of this signal is now amplitude-dependent and given by $\mathbb{E}_\phi[I(t)] = c - bA(t)$. We find the autocorrelation function for this model is given by

$$C(t) = \left(A_0^2 + \frac{D_A e^{-2\gamma_A t}}{2\gamma_A} \right) \frac{\cos(\omega_0 t) e^{-D_\phi t/2}}{2} + \frac{b^2 D_A e^{-2\gamma_A t}}{2\gamma_A}. \quad (\text{A.28})$$

The first term above is identical to Eq. (A.13), but now another term appears that depends only on the amplitude fluctuations. This latter term can be understood as a continuous analogue of Eq. (A.19), where now b instead of $\bar{I}_1 - \bar{I}_2$ determines how amplitude fluctuations affect the cycle-averaged signal.

B Physical Systems Realised by Davies Maps

In this appendix, we consider two physical setups described by the Davies map to which the description of the quantum Mpemba effect in Chapter 5 can be applied.

A two-level atom in a photonic environment First, we consider a two-level atom interacting with a photonic environment. This system is characterized by the Jaynes-Cummings Hamiltonian that, in the rotating wave approximation, reads [245]:

$$\begin{aligned} \hat{H} = \hat{H}_s + \hat{H}_e + \hat{H}_{\text{int}} = \epsilon \hat{\sigma}^+ \hat{\sigma}^- + \sum_k \omega_k \hat{a}_k^\dagger \hat{a}_k \\ + \sum_k g \left(\hat{\sigma}^- \hat{a}_k^\dagger e^{-i(\epsilon - \omega_k)t} + \hat{\sigma}^+ \hat{a}_k e^{+i(\epsilon - \omega_k)t} \right). \end{aligned} \quad (\text{B.1})$$

Here ϵ is the energy difference between the two states of the atom, $\hat{\sigma}^+$ and $\hat{\sigma}^-$ are the ladder operators for the two-level atom, $\hat{a}^\dagger$ and $\hat{a}$ are the bosonic creation and annihilation operators, respectively, and the homogeneous coupling g describes the interaction strength between the two-level atom and the bosonic modes. We now consider the photons $\hat{H}_e$ to be in thermal equilibrium at inverse temperature β . Then, following the standard steps [42] (Born, Markov, and secular approximation), a Lindblad equation of Davies type for the

two-level atom can be derived [245]

$$\begin{aligned} \frac{d\hat{\rho}_s(t)}{dt} = & -i[\hat{H}_s, \hat{\rho}(t)] + \gamma n_\beta (\hat{\sigma}^+ \hat{\rho}(t) \hat{\sigma}^- - \frac{1}{2} \{\hat{\sigma}^- \hat{\sigma}^+, \hat{\rho}(t)\}) \\ & + \gamma (n_\beta + 1) (\hat{\sigma}^- \hat{\rho}(t) \hat{\sigma}^+ - \frac{1}{2} \{\hat{\sigma}^+ \hat{\sigma}^-, \hat{\rho}(t)\}), \end{aligned} \quad (\text{B.2})$$

with n_β being the occupation of the two-level atom at the inverse temperature β and $\gamma \propto g$ the damping coefficient. A similar approach considering a two-level system and a resonator was studied in [268] for a transmon qubit. The parameters used in [268] were $\epsilon = 2\pi \times 4\text{GHz}$, $\gamma = 2\pi \times 1.41\text{MHz}$, and $T_b = 0.1\text{K} = 2.08\text{GHz} \gg \gamma$. In Fig. B2a we show that the thermalisation of the two-level atom obeying Eq. B.2 can be exponentially accelerated by applying the exact unitary transformation outlined in the main text.

A spinful fermionic quantum dot As a second example, we study a spinful fermionic quantum dot interacting with a fermionic reservoir. This setup can be described by [246]

$$\begin{aligned} \hat{H} = \hat{H}_s + \hat{H}_e + \hat{H}_{\text{int}} = & \sum_{\sigma=\uparrow,\downarrow} \epsilon \hat{n}_\sigma + E_c \hat{n}_\uparrow \hat{n}_\downarrow \\ & + \sum_{k,\sigma} \omega_k \hat{c}_{k,\sigma}^\dagger \hat{c}_{k,\sigma} + \sum_{k,\sigma} g \hat{d}_\sigma^\dagger \hat{c}_{k,\sigma} + \text{h.c.}, \end{aligned} \quad (\text{B.3})$$

where ϵ is the dot's energy, E_c is the Coulomb repulsion, ω_k is the energy of the reservoir-fermions, g describes the coupling between the dot and the reservoir-fermions, $\hat{d}$ ($\hat{d}^\dagger$) and $\hat{c}$ ($\hat{c}^\dagger$) are the annihilation (creation) operators for the quantum dot and the reservoir, respectively and $\hat{n}_{\uparrow(\downarrow)} = \hat{d}_{\uparrow(\downarrow)}^\dagger \hat{d}_{\uparrow(\downarrow)}$. In analogy to B.3 considering a weak coupling g and a reservoir in thermal equilibrium, one can derive the Davies map for the quantum dot [40]

$$\begin{aligned} \frac{d\hat{\rho}_s(t)}{dt} = & -i[\hat{H}_s, \hat{\rho}(t)] + \gamma n_\beta \sum_{\sigma=\uparrow,\downarrow} (\hat{d}_\sigma^\dagger \hat{\rho}(t) \hat{d}_\sigma - \frac{1}{2} \{\hat{d}_\sigma \hat{d}_\sigma^\dagger, \hat{\rho}(t)\}) \\ & + \gamma (1 - n_\beta) \sum_{\sigma=\uparrow,\downarrow} (\hat{d}_\sigma \hat{\rho}(t) \hat{d}_\sigma^\dagger - \frac{1}{2} \{\hat{d}_\sigma^\dagger \hat{d}_\sigma, \hat{\rho}(t)\}). \end{aligned} \quad (\text{B.4})$$

Following the experimental parameters outlined in [268] we set $E_c = 1189\text{GHz}$, $\gamma = 1\text{GHz}$, $T_b = 2\text{K} = 41.67\text{GHz} \gg \gamma$. The dot's energy can be tuned by applying a voltage and we set it to $\epsilon = 242\text{GHz}$. In Fig. B2b we illustrate the exponential acceleration of the thermalisation of the spinful quantum dot dictated by Eq. B.4. Since the eigenvalue λ_2 is real, the speedup is obtained via a Metropolis-based numerical algorithm outlined in [223]. Moreover, since the difference between $\text{Re}(\lambda_2)$ and $\text{Re}(\lambda_3)$ is small, the obtained speedup

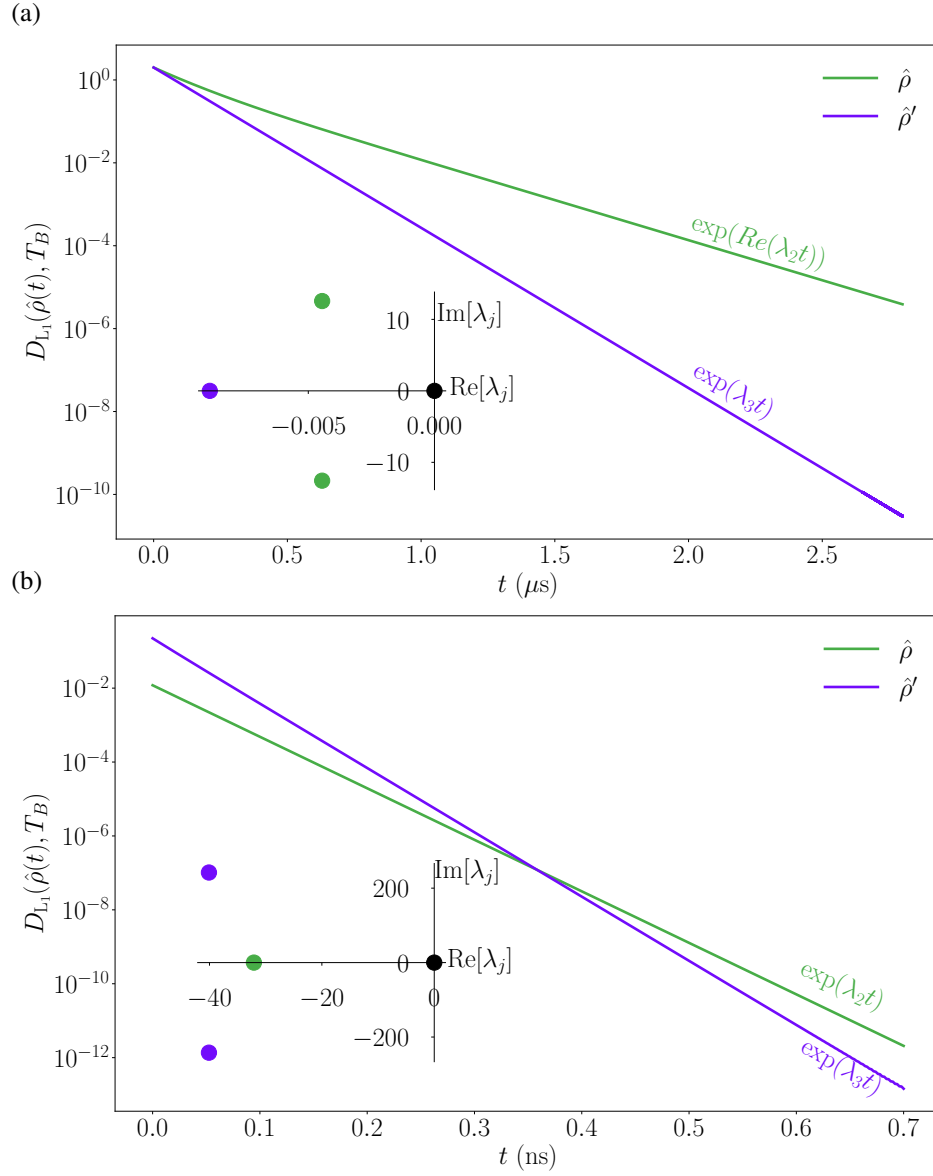


Figure B2: Exponentially accelerating the thermalisation of a two-level atom in a photonic environment and of a spinful fermionic quantum dot. (a) The two-level atom is initialized in state $\hat{\rho}(t=0) = |+\rangle\langle+|$ with $|+\rangle = 1/\sqrt{2}(|0\rangle + |1\rangle)$. Since the eigenvalue λ_2 of the Davies map is complex, the state $\hat{\rho}'(t=0)$ having zero overlap with the slowest-decaying mode $\hat{l}_2$ is obtained by applying the transformation $\hat{U}$ introduced in the main text. *Inset*: the eigenvalues of the Davies map. (b) The spinful fermionic quantum dot is initialized in the thermal state at $T_i = 0.1\text{K}$. For this system the eigenvalue λ_2 of the Davies map is real, and thus a transformed state $\hat{\rho}'(t=0)$ with overlap $\Theta = |\text{Tr}(\hat{l}_2\hat{\rho}')| + |\text{Tr}(\hat{l}_2^*\hat{\rho}')| < 2 \cdot 10^{-5}$ is obtained using the Metropolis-based algorithm outlined in [223]. *Inset*: the four eigenvalues of the Davies map with the largest real part.

is less significant than for the two-level atom (see Fig. B2a) and the spin chain considered in the main text.

C Eigenfunction Decomposition of a Classical Harmonic Oscillator Coupled to a Thermal Bath

In this appendix, we detail the derivation of the eigenmode decomposition used in Section 6.1 of the main text; the derivation is outlined in Risken [72]. Our starting point is the Fokker-Planck equation for a harmonic oscillator coupled to a thermal bath

$$\frac{\partial W}{\partial t} = \frac{\partial}{\partial v} (\omega_0^2 x + \gamma v) W - v \frac{\partial W}{\partial x} + \frac{\gamma T}{m} \frac{\partial^2 W}{\partial v^2}, \quad (\text{C.1})$$

here W is the phase space probability distribution of the classical oscillator, with position x , momentum v , mass m and natural frequency ω_0 . γ denotes the damping coefficient, and T is the temperature of the coupled bath. We denote the Fokker-Planck generator as

$$L_{\text{FP}} = \frac{\partial}{\partial v} (\omega_0^2 x + \gamma v) - v \frac{\partial}{\partial x} + \frac{T\gamma}{2} \frac{\partial^2}{\partial v^2}, \quad (\text{C.2})$$

We are interested in solutions of the form

$$W(x, v, t) = W(x, v, \infty) + \sum_n c_n \phi_n(x, v) e^{\lambda_n t}. \quad (\text{C.3})$$

Here $W(x, v, \infty)$ is the steady state solution, $\phi_n(x, v)$ is the n th eigenfunction of the Fokker-Planck generator, λ_n is the corresponding eigenvalue and c_n is the corresponding overlap function with the initial state. The eigenfunctions satisfy the relation

$$L_{\text{FP}} \phi_n(x, v) = \lambda_n \phi_n(x, v). \quad (\text{C.4})$$

The generator L_{FP} is not a hermitian operator due to the dissipation and fluctuation terms present. We transform the Fokker-Planck equation to using the following transformation:

$$\bar{L}_{\text{FP}} = \exp [\Phi(x, v)] L_{\text{FP}} \exp [-\Phi(x, v)] \quad (\text{C.5})$$

where we denote

$$\Phi(x, v) = -\frac{1}{4} \left[\left(\frac{v}{v_{th}} \right)^2 + \left(\frac{x\omega_0}{v_{th}} \right)^2 \right], \quad (C.6)$$

where v_{th} is the thermal velocity. The eigenvalues of the transformed operator $\bar{L}_{FP}$ are equal to L_{FP} but the eigenfunctions are transformed

$$\bar{\psi}(x, v) = \exp[\Phi(x, v)] \psi(x, v). \quad (C.7)$$

The generator $\bar{L}_{FP}$ can be rewritten in terms of bosonic operators

$$\bar{L}_{FP} = -\gamma \hat{b}^\dagger \hat{b} - \omega_0 (\hat{a} \hat{b}^\dagger - \hat{a}^\dagger \hat{b}) \quad (C.8)$$

where the bosonic operators are given as

$$\hat{a} = \frac{v_{th}}{\omega_0} \frac{\partial}{\partial x} + \frac{\omega_0}{2v_{th}} x, \quad (C.9a)$$

$$\hat{a}^\dagger = -\frac{v_{th}}{\omega_0} \frac{\partial}{\partial x} + \frac{\omega_0}{2v_{th}} x, \quad (C.9b)$$

$$\hat{b} = v_{th} \frac{\partial}{\partial v} + \frac{1}{2v_{th}} v, \quad (C.9c)$$

$$\hat{b}^\dagger = -v_{th} \frac{\partial}{\partial v} + \frac{1}{2v_{th}} v. \quad (C.9d)$$

The stationary solution of Eq. C.8 is given by

$$\hat{a} \bar{\phi}_{0,0}(x, v) = \hat{b} \bar{\phi}_{0,0}(x, v) = 0, \quad (C.10)$$

which gives the ground-state solution

$$\bar{\phi}_{0,0}(x, v) = \sqrt{\frac{\omega_0}{2\pi v_{th}^2}} \exp \left[-\frac{1}{4} \left(\left(\frac{v}{v_{th}} \right)^2 + \left(\frac{x\omega_0}{v_{th}} \right)^2 \right) \right]. \quad (C.11)$$

This solution can be quickly verified by transforming back to the original Fokker-Planck operator

$$\phi_{0,0}(x, v) = \sqrt{\frac{\omega_0}{2\pi v_{th}^2}} \exp \left[-\frac{1}{2} \left(\left(\frac{v}{v_{th}} \right)^2 + \left(\frac{x\omega_0}{v_{th}} \right)^2 \right) \right], \quad (C.12)$$

which is a normalised distribution and is the steady state of the Fokker-Planck generator, i.e. $L_{FP}\phi_{0,0} = 0$. We will now introduce new operators to convert Eq. C.8 to a simpler form:

$$\hat{c}_{1+} = \delta^{-1/2} \left(\sqrt{\lambda_1} \hat{b}^\dagger - \sqrt{\lambda_2} \hat{a}^\dagger \right), \quad (\text{C.13a})$$

$$\hat{c}_{1-} = \delta^{-1/2} \left(\sqrt{\lambda_1} \hat{b} + \sqrt{\lambda_2} \hat{a} \right), \quad (\text{C.13b})$$

$$\hat{c}_{2+} = \delta^{-1/2} \left(-\sqrt{\lambda_2} \hat{b}^\dagger - \sqrt{\lambda_1} \hat{a}^\dagger \right), \quad (\text{C.13c})$$

$$\hat{c}_{2-} = \delta^{-1/2} \left(\sqrt{\lambda_2} \hat{b} + \sqrt{\lambda_1} \hat{a} \right). \quad (\text{C.13d})$$

where λ_1 , λ_2 and δ are defined as

$$\lambda_1 = \frac{\gamma + \delta}{2} \quad \lambda_2 = \frac{\gamma - \delta}{2}, \quad (\text{C.14a})$$

$$\delta = \sqrt{\gamma^2 - 4\omega_0^2}. \quad (\text{C.14b})$$

The transformed Fokker-Planck operator is given as

$$\bar{L}_{\text{FP}} = -\lambda_1 \hat{c}_{1+} \hat{c}_{1-} - \lambda_2 \hat{c}_{2+} \hat{c}_{2-}, \quad (\text{C.15})$$

with commutation relations

$$[\hat{c}_{1-}, \hat{c}_{1+}] = [\hat{c}_{2-}, \hat{c}_{2+}] = 1, \quad (\text{C.16a})$$

$$[\hat{c}_{1-}, \hat{c}_{2+}] = [\hat{c}_{2-}, \hat{c}_{1+}] = [\hat{c}_{1-}, \hat{c}_{2-}] = [\hat{c}_{1+}, \hat{c}_{2+}] = 0, \quad (\text{C.16b})$$

$$\left[(-\bar{L}_{\text{FP}}), \hat{c}_{i,\pm} \right] = \pm \lambda_i \hat{c}_{i,\pm} \quad (i = 1, 2). \quad (\text{C.16c})$$

The eigenfunctions of $\bar{L}_{\text{FP}}$ can now be constructed in the same way as for the harmonic oscillator

$$\bar{\phi}_{n_1, n_2}(x, v) = (n_1! n_2!)^{-1/2} (\hat{c}_{1+})^{n_1} (\hat{c}_{2+})^{n_2} \bar{\phi}_{0,0}(x, v), \quad (\text{C.17})$$

with eigenvalues

$$\lambda_{n_1, n_2} = \lambda_1 n_1 + \lambda_2 n_2 = \frac{1}{2} \gamma (n_1 + n_2) + \frac{1}{2} \delta (n_1 - n_2). \quad (\text{C.18})$$

The eigenfunctions of the Fokker-Planck operator are thus given by

$$\phi_{n_1, n_2}(x, v) = \sqrt{\frac{\omega_0}{2\pi v_{\text{th}}^2}} \exp[\Phi(x, v)] (n_1! n_2!)^{-1/2} (\hat{c}_{1+})^{n_1} (\hat{c}_{2+})^{n_2} \exp[\Phi(x, v)]. \quad (\text{C.19})$$

To calculate the overlap functions we first calculate the eigenfunctions of the adjoint operator, $\bar{L}_{\text{FP}}^\dagger$:

$$\bar{L}_{\text{FP}}^\dagger = \lambda_1 \hat{c}_{1-} \hat{c}_{1+} - \lambda_2 \hat{c}_{2-} \hat{c}_{2+}. \quad (\text{C.20})$$

The eigenfunctions of $\bar{L}_{\text{FP}}^\dagger$ are given by:

$$\bar{\phi}_{n_1, n_2}^\dagger(x, v) = (n_1! n_2!)^{-1/2} (\hat{c}_{1-})^{n_1} (\hat{c}_{2-})^{n_2} \bar{\phi}_{0,0}^\dagger(x, v). \quad (\text{C.21})$$

The eigenfunctions of the rotated and adjoint rotated operator form a biorthogonal set [72]:

$$\int \bar{\phi}_{n_1, n_2}(x, v) \bar{\phi}_{n'_1, n'_2}^\dagger(x, v) dx dv = \delta_{n_1, n'_1} \delta_{n_2, n'_2}, \quad (\text{C.22})$$

with eigenfunctions related through:

$$\bar{\phi}_{n_1, n_2}^\dagger(x, v) = (-1)^{n_1} \bar{\phi}_{n_1, n_2}(x, v)(x, -v). \quad (\text{C.23})$$

The overlap function is given as

$$\begin{aligned} c_{n_1, n_2} \phi_{n_1, n_2} &= W_0 \\ c_{n_1, n_2} \phi_{n_1, n_2} \bar{\phi}_{n'_1, n'_2}^\dagger &= \exp[\Phi(x, v)] c_{n_1, n_2} \bar{\phi}_{n_1, n_2} \bar{\phi}_{n'_1, n'_2}^\dagger = W_0 \bar{\phi}_{n'_1, n'_2}^\dagger \\ c_{n_1, n_2} \int \bar{\phi}_{n_1, n_2} \bar{\phi}_{n'_1, n'_2}^\dagger dx dv &= \int W_0 \exp[-\Phi(x, v)] \bar{\phi}_{n'_1, n'_2}^\dagger dx dv \\ c_{n_1, n_2} &= \int W_0 \exp[-\Phi(x, v)] \bar{\phi}_{n_1, n_2}^\dagger dx dv. \end{aligned} \quad (\text{C.24})$$

We have used the notation $W_0 = W(x, v, 0)$ to denote the initial state of the oscillator. For the overdamped oscillator, the slowest decaying eigenfunction is

$$\bar{\phi}_{0,1} = \delta^{-1/2} \left(-\sqrt{\lambda_2} \frac{v}{v_{th}} + \sqrt{\lambda_1} \frac{x\omega_0}{v_{th}} \right) \exp[\Phi(x, v)]. \quad (\text{C.25})$$

The adjoint eigenfunction is

$$\bar{\phi}_{0,1}^\dagger = \delta^{-1/2} \left(\sqrt{\lambda_2} \frac{v}{v_{th}} + \sqrt{\lambda_1} \frac{x\omega_0}{v_{th}} \right) \exp[\Phi(x, v)]. \quad (\text{C.26})$$

Inserting this into Eq. E.39 gives the symmetric overlap function for the slowest decaying mode:

$$c_{0,1} = \int W(x, v, 0) \delta^{-1/2} \left(\sqrt{\lambda_2} \frac{v}{v_{th}} + \sqrt{\lambda_1} \frac{x\omega_0}{v_{th}} \right) dx dv. \quad (\text{C.27})$$

D Converting the Master Equation to Wigner Space

In this appendix, we will outline the derivation of the Fokker-Planck equation describing the evolution of the Wigner function, Eq. E.3. Starting from the Davies map describing the thermalisation, Eq. 6.16. Using the Hamiltonian $\hat{H} = \omega_0 (\hat{a}^\dagger \hat{a} + 1/2)$:

$$\frac{d\hat{\rho}}{dt} = -i\omega_0 (\hat{a}^\dagger \hat{a} \hat{\rho} - \hat{\rho} \hat{a}^\dagger \hat{a}) + \frac{\gamma}{2} (\bar{n} + 1) (2\hat{a} \hat{\rho} \hat{a}^\dagger - \hat{a}^\dagger \hat{a} \hat{\rho} - \hat{\rho} \hat{a}^\dagger \hat{a}) + \frac{\gamma}{2} \bar{n} (2\hat{a}^\dagger \hat{\rho} \hat{a} - \hat{a} \hat{a}^\dagger \hat{\rho} - \hat{\rho} \hat{a} \hat{a}^\dagger). \quad (\text{D.1})$$

This master equation can be recast into quadrature operator representation

$$\begin{aligned} \frac{d\hat{\rho}}{dt} = & -i \left(\left(\frac{\hat{p}^2}{2m} + \frac{m\omega_0^2 \hat{x}^2}{2} \right) \hat{\rho} - \hat{\rho} \left(\frac{\hat{p}^2}{2m} + \frac{m\omega_0^2 \hat{x}^2}{2} \right) \right) + \frac{\gamma}{2} (\bar{n} + 1) \frac{m\omega_0}{2} \left(2 \left(\hat{x} + \frac{i}{m\omega_0} \hat{p} \right) \hat{\rho} \left(\hat{x} - \frac{i}{m\omega_0} \hat{p} \right) \right. \\ & - \left(\hat{x} - \frac{i}{m\omega_0} \hat{p} \right) \left(\hat{x} + \frac{i}{m\omega_0} \hat{p} \right) \hat{\rho} - \hat{\rho} \left(\hat{x} - \frac{i}{m\omega_0} \hat{p} \right) \left(\hat{x} + \frac{i}{m\omega_0} \hat{p} \right) \left. \right) + \frac{\gamma}{2} \bar{n} \frac{m\omega_0}{2} \\ & \left(2 \left(\hat{x} - \frac{i}{m\omega_0} \hat{p} \right) \hat{\rho} \left(\hat{x} + \frac{i}{m\omega_0} \hat{p} \right) - \left(\hat{x} + \frac{i}{m\omega_0} \hat{p} \right) \left(\hat{x} - \frac{i}{m\omega_0} \hat{p} \right) \hat{\rho} \right. \\ & \left. - \hat{\rho} \left(\hat{x} + \frac{i}{m\omega_0} \hat{p} \right) \left(\hat{x} - \frac{i}{m\omega_0} \hat{p} \right) \right). \end{aligned} \quad (\text{D.2})$$

Using the correspondence relations, Eq. 1.77, each term can be converted to Wigner space, the unitary evolution:

$$-i \left(\left(\frac{\hat{p}^2}{2m} + \frac{m\omega_0^2 \hat{x}^2}{2} \right) \hat{\rho} - \hat{\rho} \left(\frac{\hat{p}^2}{2m} + \frac{m\omega_0^2 \hat{x}^2}{2} \right) \right) \leftrightarrow \left(m\omega_0^2 x \frac{\partial}{\partial x} - \frac{p}{m} \frac{\partial}{\partial p} \right) W(x, p). \quad (\text{D.3})$$

The emission term

$$\begin{aligned} & \frac{\gamma}{2} (\bar{n} + 1) \frac{m\omega_0}{2} \left(2 \left(\hat{x} + \frac{i}{m\omega_0} \hat{p} \right) \hat{\rho} \left(\hat{x} - \frac{i}{m\omega_0} \hat{p} \right) - \left(\hat{x} - \frac{i}{m\omega_0} \hat{p} \right) \left(\hat{x} + \frac{i}{m\omega_0} \hat{p} \right) \hat{\rho} - \right. \\ & \left. \hat{\rho} \left(\hat{x} - \frac{i}{m\omega_0} \hat{p} \right) \left(\hat{x} + \frac{i}{m\omega_0} \hat{p} \right) \right) \leftrightarrow \frac{\gamma}{4} (\bar{n} + 1) \left(m\omega_0 \frac{\partial^2}{\partial p^2} + 2 \frac{\partial}{\partial x} x + 2 \frac{\partial}{\partial p} p \frac{1}{m\omega_0} \frac{\partial^2}{\partial p^2} \right) W(x, p) \end{aligned} \quad (\text{D.4})$$

The absorption term:

$$\begin{aligned} & \frac{\gamma}{2} \bar{n} \frac{m\omega_0}{2} \left(2 \left(\hat{x} - \frac{i}{m\omega_0} \hat{p} \right) \hat{\rho} \left(\hat{x} + \frac{i}{m\omega_0} \hat{p} \right) - \left(\hat{x} + \frac{i}{m\omega_0} \hat{p} \right) \left(\hat{x} - \frac{i}{m\omega_0} \hat{p} \right) \hat{\rho} - \right. \\ & \left. \hat{\rho} \left(\hat{x} + \frac{i}{m\omega_0} \hat{p} \right) \left(\hat{x} - \frac{i}{m\omega_0} \hat{p} \right) \right) \leftrightarrow \frac{\gamma}{4} \bar{n} \left(m\omega_0 \frac{\partial^2}{\partial p^2} - 2 \frac{\partial}{\partial x} x - 2 \frac{\partial}{\partial p} p \frac{1}{m\omega_0} \frac{\partial^2}{\partial p^2} \right) W(x, p). \end{aligned} \quad (\text{D.5})$$

Combining the three terms, one arrives at the Fokker-Planck equation, Eq. E.3.

E Eigenfunction Decomposition of a Quantum Oscillator Evolving via a Davies Map

We start with the equation of motion for the state in the quadrature basis

$$\begin{aligned} \frac{dW(x, p)}{dt} = & \left(-\frac{p}{m} \frac{\partial}{\partial x} + m\omega_0^2 x \frac{\partial}{\partial p} \right) W(x, p) + \frac{\gamma}{2} \left[\frac{\partial}{\partial x} x + \frac{\partial}{\partial p} p \right] W(x, p) \\ & + \frac{\gamma}{2} \left(\bar{n} + \frac{1}{2} \right) \left[m\omega_0 \frac{\partial^2}{\partial p^2} + \frac{1}{m\omega_0} \frac{\partial^2}{\partial x^2} \right] W(x, p). \end{aligned} \quad (\text{E.1})$$

we define the Fokker-Planck generator as

$$L_{\text{FP}} = \left(-v \frac{\partial}{\partial x} + \omega_0^2 x \frac{\partial}{\partial v} \right) + \frac{\gamma}{2} \left[\frac{\partial}{\partial x} x + \frac{\partial}{\partial v} v \right] + \frac{\gamma}{2m} \left(\bar{n} + \frac{1}{2} \right) \left[\omega_0 \frac{\partial^2}{\partial v^2} + \frac{1}{\omega_0} \frac{\partial^2}{\partial x^2} \right]. \quad (\text{E.2})$$

where we have converted from momentum to velocity space $p = vm$. We will now denote an effective temperature of the environment — $T_{\text{eff}} = \left(\bar{n} + \frac{1}{2} \right) \omega_0$ and will denote the thermal velocity $v_{\text{th}}^2 = T_{\text{eff}}/m$ -

$$L_{\text{FP}} = \left(-v \frac{\partial}{\partial x} + \omega_0^2 x \frac{\partial}{\partial v} \right) + \frac{\gamma}{2} \left[\frac{\partial}{\partial x} x + \frac{\partial}{\partial v} v \right] + \frac{\gamma v_{\text{th}}^2}{2} \left[\frac{\partial^2}{\partial v^2} + \frac{1}{\omega_0^2} \frac{\partial^2}{\partial x^2} \right]. \quad (\text{E.3})$$

We are interested in a solution of the form

$$W(x, v, t) = W_{\text{ss}}(x, v) + \sum_n c_n \phi_n(x, v) e^{\lambda_n t}, \quad (\text{E.4})$$

where W_{ss} is the steady state Wigner function, $\phi_n(x, v)$ is the n th eigenfunction of the Fokker-Planck generator, λ_n is the eigenvalue associated with the n th eigenfunction and c_n is the overlap between the n th eigenmode and the initial state. We transform the generator

$$\bar{L}_{\text{FP}} = \exp [\Phi(x, v)] L_{\text{FP}} \exp [-\Phi(x, v)] \quad (\text{E.5})$$

where

$$\Phi(x, v) = \frac{1}{4} \left(\left(\frac{v}{v_{\text{th}}} \right)^2 + \left(\frac{x\omega_0}{v_{\text{th}}} \right)^2 \right) \quad (\text{E.6})$$

The eigenfunctions are transformed according to

$$\bar{\phi}_n(x, v) = \exp (\Phi(x, v)) \phi_n(x, v), \quad (\text{E.7})$$

and the eigenvalues are unchanged. The transformed Fokker-Planck operator is

$$\bar{L}_{FP} = -v \frac{\partial}{\partial x} + \omega_0^2 x \frac{\partial}{\partial v} + \frac{\gamma}{2} \left(1 - \left[\frac{x\omega_0}{2v_{th}} \right]^2 + \left[\frac{v}{2v_{th}} \right]^2 + \frac{v_{th}^2}{\omega_0^2} \frac{\partial^2}{\partial x^2} + v_{th}^2 \frac{\partial^2}{\partial v^2} \right). \quad (E.8)$$

We now introduce the following operators

$$\hat{a} = \frac{v_{th}}{\omega_0} \frac{\partial}{\partial x} + \frac{\omega_0}{2v_{th}} x, \quad (E.9)$$

$$\hat{a}^\dagger = -\frac{v_{th}}{\omega_0} \frac{\partial}{\partial x} + \frac{\omega_0}{2v_{th}} x, \quad (E.10)$$

$$\hat{b} = v_{th} \frac{\partial}{\partial v} + \frac{1}{2v_{th}} v, \quad (E.11)$$

$$\hat{b}^\dagger = -v_{th} \frac{\partial}{\partial v} + \frac{1}{2v_{th}} v. \quad (E.12)$$

These operators satisfy the bosonic commutation relations $[\hat{a}, \hat{a}^\dagger] = [\hat{b}, \hat{b}^\dagger] = 1$ and $[\hat{a}\hat{b}] = [\hat{a}^\dagger\hat{b}] = 0$. We can rewrite the transformed Fokker-Planck equation in terms of the bosonic operators:

$$\bar{L}_{FP} = -\gamma (\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b}) - \omega_0 (\hat{a}\hat{b}^\dagger - \hat{a}^\dagger \hat{b}). \quad (E.13)$$

We will now introduce the adjoint operator

$$\int (A^\dagger \phi) \psi dx dv = \int \phi A \psi dx dv. \quad (E.14)$$

We note that the inner product isn't taken using conjugation as is used in quantum mechanics. The rotated Fokker-Planck equation can be decomposed into Hermitian and anti-Hermitian components

$$\bar{L}_{FP} = \bar{L}_{FP}^H + \bar{L}_{FP}^{AH}, \quad (E.15)$$

where $\bar{L}_{FP}^H = -\gamma (\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b})$ and $\bar{L}_{FP}^{AH} = -\omega_0 (\hat{a}\hat{b}^\dagger - \hat{a}^\dagger \hat{b})$ where

$$(\bar{L}_{FP}^H)^\dagger = \bar{L}_{FP}^H \quad (E.16)$$

$$(\bar{L}_{FP}^{AH})^\dagger = -\bar{L}_{FP}^{AH}, \quad (E.17)$$

from these relations it is straightforward to show that the rotated Fokker Planck equation and its adjoint are related via

$$\bar{L}_{FP}(x, -v) = \bar{L}_{FP}^\dagger(x, v). \quad (E.18)$$

This relation will be useful later on. The steady-state solution is given by

$$\hat{a}\bar{\phi}_{0,0} = \hat{b}\bar{\phi}_{0,0} = 0 \quad (\text{E.19})$$

which gives the ground-state solution

$$\bar{\phi}_{0,0}(x, v) = \sqrt{\frac{\omega_0}{2\pi v_{\text{th}}^2}} \exp \left[-\frac{1}{4} \left(\left(\frac{v}{v_{\text{th}}} \right)^2 + \left(\frac{x\omega_0}{v_{\text{th}}} \right)^2 \right) \right]. \quad (\text{E.20})$$

This solution can be quickly verified by transforming back to the original Fokker-Planck operator

$$\phi_{0,0}(x, v) = \sqrt{\frac{\omega_0}{2\pi v_{\text{th}}^2}} \exp \left[-\frac{1}{2} \left(\left(\frac{v}{v_{\text{th}}} \right)^2 + \left(\frac{x\omega_0}{v_{\text{th}}} \right)^2 \right) \right], \quad (\text{E.21})$$

which is the steady state of the Fokker-Planck generator, i.e. $L_{\text{FP}}\phi_{0,0} = 0$; however, it is not normalised. We will now introduce new operators to convert Eq. C.8 to a simpler form:

$$\hat{c}_{1+} = \frac{\sqrt{2}}{2} (\hat{a}^\dagger - i\hat{b}^\dagger) \quad (\text{E.22})$$

$$\hat{c}_{1-} = \frac{\sqrt{2}}{2} (\hat{a} + i\hat{b}) \quad (\text{E.23})$$

$$\hat{c}_{2+} = \frac{\sqrt{2}}{2} (-i\hat{a}^\dagger + \hat{b}^\dagger) \quad (\text{E.24})$$

$$\hat{c}_{2-} = \frac{\sqrt{2}}{2} (i\hat{a} + \hat{b}) \quad (\text{E.25})$$

where the transformed operators satisfy the commutation relations $[\hat{c}_{1-}, \hat{c}_{1+}] = [\hat{c}_{2-}, \hat{c}_{2+}] = 1$ and $[\hat{c}_{1-}, \hat{c}_{2+}] = [\hat{c}_{2-}, \hat{c}_{1+}] = [\hat{c}_{1-}, \hat{c}_{1-}] = [\hat{c}_{1-}, \hat{c}_{2+}] = 0$. Note however that $\hat{c}_{1-} \neq \hat{c}_{1+}^\dagger$. The inverse relations are

$$\hat{a} = \frac{\sqrt{2}}{2} (\hat{c}_{1-} - i\hat{c}_{2-}) \quad (\text{E.26})$$

$$\hat{a}^\dagger = \frac{\sqrt{2}}{2} (\hat{c}_{1+} + i\hat{c}_{2+}) \quad (\text{E.27})$$

$$\hat{b} = \frac{\sqrt{2}}{2} (-i\hat{c}_{1-} + \hat{c}_{2-}) \quad (\text{E.28})$$

$$\hat{b}^\dagger = \frac{\sqrt{2}}{2} (i\hat{c}_{1+} + \hat{c}_{2+}) \quad (\text{E.29})$$

Two independent oscillators can now describe the transformed Fokker-Planck generator

$$\bar{F}_{\text{FP}} = \lambda_1 \hat{c}_{1+} \hat{c}_{1-} + \lambda_2 \hat{c}_{2+} \hat{c}_{2-} \quad (\text{E.30})$$

where

$$\lambda_1 = -\gamma - \sqrt{2}i\omega_0, \quad \lambda_2 = -\gamma + \sqrt{2}i\omega_0. \quad (\text{E.31})$$

The eigenvalues of the Fokker-Planck equation are thus given as

$$\lambda_{n_1, n_2} = n_1 \lambda_1 + n_2 \lambda_2. \quad (\text{E.32})$$

The eigenfunctions for Eq. E.30 can be found in the same manner as the quantum harmonic oscillator. The eigenfunctions are constructed as

$$\bar{\psi}_{n_1, n_2}(x, v) = (n_1! n_2!)^{-1/2} (c_{1+})^{n_1} (c_{2+})^{n_2} \bar{\psi}_{0,0}(x, v) \quad (\text{E.33})$$

We now look at the eigenfunctions of the adjoint operator

$$\bar{L}_{\text{FP}}^+ = -\lambda_1 \hat{c}_{1-}^+ \hat{c}_{1+}^+ - \lambda_2 \hat{c}_{2-}^+ \hat{c}_{2+}^+ \quad (\text{E.34})$$

With the same eigenvalues, but with eigenfunctions

$$\bar{\psi}_{n_1, n_2}^+ = (n_1! n_2!)^{-1/2} (c_{1-}^+)^{n_1} (c_{2-}^+)^{n_2} \bar{\psi}_{0,0}(x, v). \quad (\text{E.35})$$

The adjoint eigenfunctions form a biorthogonal set with the eigenfunctions of the rotated Fokker-Planck equation

$$\int \bar{\psi}_{n'_1, n'_2}^+ \bar{\psi}_{n_1, n_2} dx dv = \delta_{n_1, n'_1} \delta_{n_2, n'_2} \quad (\text{E.36})$$

This can quickly be verified by inserting the definition of the eigenfunctions into Eq. E.36, using commutation relations and using the relation

$$\hat{c}_{1-} \bar{\psi}_{0,0} = \hat{c}_{2-} \bar{\psi}_{0,0} = \hat{c}_{1+}^+ \bar{\psi}_{0,0} = \hat{c}_{2+}^+ \bar{\psi}_{0,0} = 0 \quad (\text{E.37})$$

As $\hat{c}_{1-}^+(x, v) = \hat{c}_{1+}(x, -v)$ and $\hat{c}_{2-}^+(x, v) = -\hat{c}_{2+}(x, -v)$ the eigenfunctions are related by

$$\bar{\psi}_{n_1, n_2}^+(x, v) = (-1)^{n_2} \bar{\psi}_{n_1, n_2}(x, -v) \quad (\text{E.38})$$

To calculate the overlap functions we use the fact that at time $t = 0$:

$$\begin{aligned}
 \sum_{n_1, n_2} c_{n_1, n_2} \phi_{n_1, n_2} &= W_0 \\
 \sum_{n_1, n_2} c_{n_1, n_2} \phi_{n_1, n_2} \bar{\phi}_{n'_1, n'_2}^\dagger &= \sum_{n_1, n_2} \exp[-\Phi(x, v)] c_{n_1, n_2} \bar{\phi}_{n_1, n_2} \bar{\phi}_{n'_1, n'_2}^\dagger = W_0 \bar{\phi}_{n'_1, n'_2}^\dagger \\
 \sum_{n_1, n_2} c_{n_1, n_2} \int \bar{\phi}_{n_1, n_2} \bar{\phi}_{n'_1, n'_2}^\dagger dx dv &= \int W_0 \exp[\Phi(x, v)] \bar{\phi}_{n'_1, n'_2}^\dagger dx dv \\
 c_{n_1, n_2} &= \int W_0 \exp[\Phi(x, v)] \bar{\phi}_{n_1, n_2}^\dagger dx dv.
 \end{aligned} \tag{E.39}$$

We can now start calculating the slowest decaying eigenmodes. The slowest decaying eigenvalues are given as

$$\lambda_{0,1} = -\gamma - \sqrt{2}i\omega_0 \quad \lambda_{1,0} = -\gamma + \sqrt{2}i\omega_0 \tag{E.40}$$

The corresponding eigenfunctions are given by

$$\bar{\psi}_{0,1}(x, v) = \hat{c}_{2+} \bar{\psi}_{0,0}(x, v) = \frac{\sqrt{2}}{2} (-i\hat{a}^\dagger + \hat{b}^\dagger) \bar{\psi}_{0,0}(x, v) = \frac{\sqrt{2}}{2} \left(-i\frac{\omega_0}{v_{\text{th}}}x + \frac{v}{v_{\text{th}}} \right) \bar{\psi}_{0,0}(x, v) \tag{E.41}$$

$$\bar{\psi}_{1,0}(x, v) = \hat{c}_{1+} \bar{\psi}_{0,0}(x, v) = \frac{\sqrt{2}}{2} (\hat{a}^\dagger - i\hat{b}^\dagger) \bar{\psi}_{0,0}(x, v) = \frac{\sqrt{2}}{2} \left(\frac{\omega_0}{v_{\text{th}}}x - i\frac{v}{v_{\text{th}}} \right) \bar{\psi}_{0,0}(x, v) \tag{E.42}$$

where

$$\hat{a}^\dagger \bar{\psi}_{0,0}(x, v) = \frac{x\omega_0}{v_{\text{th}}} \bar{\psi}_{0,0}(x, v) \tag{E.43}$$

$$\hat{b}^\dagger \bar{\psi}_{0,0}(x, v) = \frac{v}{v_{\text{th}}} \bar{\psi}_{0,0}(x, v). \tag{E.44}$$

The slowest decaying eigenfunction of the original Fokker Planck operator is

$$\psi_{0,1}(x, v) = \frac{\sqrt{2}}{2} \left(-i\frac{\omega_0}{v_{\text{th}}}x + \frac{v}{v_{\text{th}}} \right) \psi_{0,0}(x, v), \tag{E.45}$$

$$\psi_{1,0}(x, v) = \frac{\sqrt{2}}{2} \left(\frac{\omega_0}{v_{\text{th}}}x - i\frac{v}{v_{\text{th}}} \right) \psi_{0,0}(x, v). \tag{E.46}$$

The adjoint eigenfunction is

$$\bar{\psi}_{0,1}^+(x, v) = \frac{\sqrt{2}}{2} \left(i \frac{\omega_0}{v_{\text{th}}} x + \frac{v}{v_{\text{th}}} \right) \bar{\psi}_{0,0}(x, v) \quad (\text{E.47})$$

$$\bar{\psi}_{1,0}^+(x, v) = \frac{\sqrt{2}}{2} \left(\frac{\omega_0}{v_{\text{th}}} x + i \frac{v}{v_{\text{th}}} \right) \bar{\psi}_{0,0}(x, v) \quad (\text{E.48})$$

Thus the overlap for the slowest decaying modes are given by

$$c_{0,1} = \frac{1}{2} \sqrt{\frac{\omega_0}{\pi v_{\text{th}}^2}} \int W_0 \left(i \frac{\omega_0}{v_{\text{th}}} x + \frac{v}{v_{\text{th}}} \right) dx dv \quad (\text{E.49})$$

$$c_{1,0} = \frac{1}{2} \sqrt{\frac{\omega_0}{\pi v_{\text{th}}^2}} \int W_0 \left(\frac{\omega_0}{v_{\text{th}}} x + i \frac{v}{v_{\text{th}}} \right) dx dv \quad (\text{E.50})$$

Thus, the slowest decaying mode will vanish if the initial state is symmetric with respect to x and v . A squeezed state is initially symmetric and thus will decay with the second slowest eigenmode, whereas the displaced state will decay with the slowest mode.

E.1 Overlap with the steady state

The steady-state eigenfunction is

$$\bar{\psi}_{0,0}(x, v) = \bar{\psi}_{0,0}^+(x, v) = \sqrt{\frac{\omega_0}{2\pi v_{\text{th}}^2}} \exp \left[-\frac{1}{4} \left(\left(\frac{v}{v_{\text{th}}} \right)^2 + \left(\frac{\omega_0 x}{v_{\text{th}}} \right)^2 \right) \right] \quad (\text{E.51})$$

The overlap function is given by

$$c_{0,0} = \sqrt{\frac{\omega_0}{2\pi v_{\text{th}}^2}} \int W_0 dx dv = \sqrt{\frac{\omega_0}{2\pi v_{\text{th}}^2}}. \quad (\text{E.52})$$

This overlap function thus gives the desired steady-state solution

$$W_{\text{ss}}(x, v) = \frac{\omega_0}{2\pi v_{\text{th}}^2} \exp \left[-\frac{1}{2} \left(\left(\frac{v}{v_{\text{th}}} \right)^2 + \left(\frac{\omega_0 x}{v_{\text{th}}} \right)^2 \right) \right]. \quad (\text{E.53})$$

E.2 Average Energy does not Relax with the Slowest Eigenmode

Consider using the expectation energy of an oscillator relaxing according to a Davies map. The expectation energy of the system evolves according to

$$\begin{aligned} \langle \hat{H} \rangle_{\hat{\rho}(t)} = & \langle \hat{H} \rangle_{\hat{\rho}_{ss}} + \frac{m}{2} c_{1,0} \int \psi_{1,0} (v^2 + \omega_0^2 x^2) dx dv e^{-\gamma - \sqrt{2}i\omega_0} + \\ & \frac{m}{2} c_{0,1} \int \psi_{0,1} (v^2 + \omega_0^2 x^2) dx dv e^{-\gamma + \sqrt{2}i\omega_0} + \mathcal{O}(e^{-2\lambda}). \end{aligned} \quad (\text{E.54})$$

As $\psi_{1,0}(x, v)$ and $\psi_{0,1}(x, v)$ are antisymmetric wrt to x and v the slowest decaying mode vanishes regardless of the initial state. Thus, the slowest decaying mode evolution will always vanish. The relaxation of the energy will evolve according to

$$\langle \hat{H} \rangle_{\hat{\rho}(t)} = \langle \hat{H} \rangle_{\hat{\rho}_{ss}} + \mathcal{O}(e^{-2\lambda}). \quad (\text{E.55})$$