

Bayesian Inference and Maximum Likelihood Estimation for fitting Distribution Functions of Road Traffic Load Effects

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ABSTRACT: More and more bridges reach the end of their design life. Traffic loads have changed substantially over the last few decades since the design of these bridges. To prevent unnecessary costs and material use in redesign or renovation of existing bridges, it is necessary to consider realistic, location-specific traffic loads. In this paper, bridge-specific design traffic load effects are derived from load effect simulations based on Weigh-In-Motion (WIM)-data. Two methods of statistical inference, Maximum Likelihood Estimation (MLE) and Bayesian Inference (BI), are compared for deriving design values for extreme traffic load effects, following extreme value theory. Statistical uncertainties arising in fitting parametric models are addressed for both methods, specifically uncertainty in parameters of the distribution models due to limited amount of data. It was shown that the assumptions in the MLE approach are not valid. The influence of statistical uncertainties on the computed design value are therefore underestimated in the MLE approach. BI results in higher design value estimates even though unrealistic tail shape parameter values are bounded through definition of the prior distribution. BI is therefore the preferred approach since, in contrary to MLE, it provides explicit information on statistical uncertainties through the posterior distribution function. Both methods were found to be highly sensitive to the choice of threshold value.

1. INTRODUCTION

Traffic load is the governing variable load for bridge assessment (Gao et al. 2021) and therefore an accurate description of the probabilistic traffic load effect model is required. Multiple studies have shown that road traffic is highly dependent on the site characteristics (Obrien et al. 2015). For a bridge assessment, it is necessary to consider realistic, location-specific traffic load effects. This information can be obtained e.g. by strain measurements or by performing simulations with Weigh-In-Motion (WIM) data (axle weights and distances) over a bridge-specific influence line. Since measurements only cover a limited period of time, extrapolation to the extreme load effect relevant for assessment (and design) of a bridge

with a very small probability of exceedance, is necessary.

In this process of fitting distribution functions and using extrapolation methods, statistical uncertainties are inherent. Statistical uncertainties could have a big influence on the extrapolated extreme traffic load effects and therefore should be accounted for in a reliability assessment of a bridge (Obrien et al. 2015). Thus, it is important to use a probabilistic approach for determining the traffic load effects and to address the statistical uncertainties in the extreme traffic load effect model explicitly and accurately.

Statistical inference methods are generally used to extrapolate from a relatively short measurement period to the return period relevant for bridge design or assessment. In literature the

extreme load effects are usually extrapolated to a return period belonging to the characteristic value e.g. 75 (Leahy et al. 2015) or 1000 years (Zhou et al. 2012) and a partial safety factor is applied. Another option is to extrapolate directly to the load effect relevant for design: the design value. In both cases a reliability based calibration is needed since the approach should lead to design values consistent with the target reliability levels. Additionally, the influence of statistical uncertainties is more dominant at larger return periods relevant for the design value. Therefore, in this paper, we extrapolate extreme traffic load effects to the design value, using a reliability based approach explicitly accounting for the statistical uncertainties.

For this purpose a distribution function is fitted to the measured data and extrapolated to the required return period. In this paper the extreme value theory is applied using the generalized extreme value (GEV)-distribution, which is a popular choice in literature. Statistical inference methods are applied to infer the model parameters of the GEV-distribution. In this paper two different approaches to statistical inference are compared for deriving the design values of location-specific traffic load effects: Maximum Likelihood Estimation (MLE) and Bayesian Inference (BI). MLE is often used for deriving extreme values of traffic load effects (Nowak et al. 2019), but BI shows great prospects in its application (Leahy et al. 2015).

The application of MLE and BI and their approach of addressing statistical uncertainties are compared in this paper. Statistical uncertainties arising in fitting parametric models are addressed for both methods, specifically uncertainty in parameters of the distribution models due to limited availability of the extreme data. The sensitivity of the obtained design values to the probabilistic approach of distribution fitting is the main focus of this paper.

2. METHODOLOGY

2.1. Extreme value analysis – Block maximum

The methods compared in this work use the block maxima method based on the extreme value theorem, proved by Gnedenko (1943) and based on the work of Fisher and Tippet (1928). This theorem states that for large sample sizes, when the sample maxima are identically distributed and independent, these can be approximated by a generalized extreme value distribution:

$$F_X(x; \theta) = \begin{cases} \exp \left\{ - \left[1 + \xi \left(\frac{x - \mu}{\sigma} \right) \right]^{\frac{1}{\xi}} \right\}; k \neq 0 \\ \exp \left\{ - \exp \left[- \left(\frac{x - \mu}{\sigma} \right) \right] \right\}; k = 0 \end{cases} \quad (1)$$

with θ a three-parameter space of shape ξ , location μ and scale parameter σ . The GEV is a family of three distributions: Gumbel (type I), Frechet (type II) and Weibull (type III). When $\xi = 0$, the GEV resolves to a Gumbel distribution. The Frechet type distribution is characterized by a positive ($\xi > 0$) and the Weibull type distribution by a negative ($\xi < 0$) shape parameter.

Considering this extreme value theory, the distribution of variables with reference period T can be derived from the distribution of the block extremes with block duration t with $t < T$. This holds as long as these t -extremes are independent and identically distributed. This is described by the univariate theorem:

$$F_{X,T}(x) = F_{X,t}(x)^{T/t} \quad (2)$$

Single realizations of extreme traffic load effects have been shown to result from a mixture of different loading event types and therefore distributions, rather than from a homogeneous distribution (Caprani et al. 2008). For traffic load effect data e.g. from strain measurements these individual load event types cannot always be identified. We work towards a single approach for deriving extreme traffic load effects for both load

effect (strain) measurements and from simulations with WIM-data. The Fisher-Tippet-Gnedenko theorem therefore does not hold for the entire set of extreme load effects.

To account for this issue of non-identically distributed extremes, we follow a strategy at reducing the data to the part that best satisfies these assumptions, either by increasing the block duration or fitting the upper tail of the distribution. In this paper, the amount of block maxima used for statistical inference is maximised to minimise statistical uncertainties by choosing a smaller block duration, while still satisfying the necessary conditions for applying the extreme value theory by fitting to the upper tail.

Two methods are used to infer the parameter estimates of the distribution function of random variable with true parameters θ_0 from the observed (upper tail) extremes x_i : maximum likelihood estimation and Bayesian inference.

2.2. Maximum likelihood estimation

Statistical inference based on the maximum likelihood method entails that the likelihood function given the data is maximized to obtain model parameter estimates θ_{ML} . For computational reasons, the negative log-likelihood function $-l(\theta)$ is minimized. The traffic load effect extremes below the threshold of the upper tail are treated as censored data (partial observation). The MLE approach can have numerical instability problems when searching for a minimum or maximum. Under irregularity conditions, a local maximum or minimum can be found instead of a global one. This should therefore be checked.

The statistical uncertainties of the maximum likelihood estimate θ_{ML} are considered using the Delta method (Coles 2001). This method makes use of the approximation by asymptotic theory that, if the data $x_1, \dots, x_n$ are independent realizations of the random variable X having distribution $F_X(\theta_0)$ with d -dimensional model parameter θ_0 , under regularity conditions for large number of extremes, the distribution of the

parameter estimates is given by a multivariate normal distribution (MVN):

$$f_{\theta}(\theta_{0,ML}) \sim \text{MVN}_d(\mu_{\theta}, I_E(\theta)^{-1}) \quad (3)$$

with $\mu_{\theta} = \theta_{ML}$ since the true parameters θ_0 are unknown and $I_E(\theta_{ML})$ the Fisher information matrix whose elements are the second order partial derivatives of the log-likelihood function, approximated at the parameter estimates θ_{ML} .

The maximum likelihood estimate of the distribution function is conditional on θ_{ML} . The unconditional distribution function of extreme traffic load effects Y considering the statistical uncertainties can be derived by:

$$F_Y(y) = \int_{\Theta} F_Y(y | \theta_{ML}) \cdot f_{\theta}(\theta_{ML}) d\theta \quad (4)$$

This can be approximated numerically by a Monte Carlo simulation on the parameter values θ and summation over the realizations by:

$$F_Y(y) \approx \left[\sum_{i=1}^N F_Y(y | \theta_i) \right] / N \quad (5)$$

2.3. Bayesian inference

The main difference between the Bayesian and the maximum likelihood (frequentist) approach is that the uncertainty in parameters is quantified through a probability distribution directly, which makes the uncertainty of the fitted parameters explicit. The posterior distribution of the model parameters follows from the inference and it is therefore not necessary to make assumptions of this distribution as was the case for MLE (Coles 2001). BI thereby has the advantages of considering prior information and in a structured way combining it with the (often limited) data.

It is assumed that the data $x_1, \dots, x_n$ are generated by a definite statistical model, which is fully specified with a set of (unknown) parameters θ . The core of Bayesian approach is the Bayes theorem, where the (posterior) distribution of the parameters can be derived from:

$$f(\theta | x) = f(x | \theta) f(\theta) / f(x) \quad (6)$$

where $f(x|\theta)$ is the likelihood function of the model and $f(\theta)$ is the so-called prior distribution, which represents the prior knowledge of the parameters by the analyst before the data is considered. This aspect will be addressed in the following paragraph. The denominator $f(x)$ is calculated as an integral of the numerator taken over the parameter space.

The probability distribution of extreme traffic load effects Y including statistical uncertainties, the so-called predictive distribution, is then defined as:

$$F_Y(y) = \int_{\Theta} F_Y(y|\theta) \cdot f(\theta|x) d\theta \quad (7)$$

with x the previous observations used for inference, and $F_Y(y|\theta)$ the distribution function of Y given parameter values θ .

Computation of the integral in the denominator of Eq. (7) can be an obstacle in BI when considering complex statistical models, like the GEV-distribution. Therefore a simulation technique considering ensemble Markov chain Monte Carlo samplers with affine invariance according to Goodman and Weare (2010) is utilized to estimate the posterior distribution. This has the advantage that after deletion of the values generated in the so-called settling-in period, the procedure leads to s samples for θ that can be considered observations from distribution $f(\theta|x)$ (Coles 2001). The predictive distribution can then be estimated by:

$$F_Y(y|x) \approx \frac{1}{s} \sum_{i=1}^s F_Y(y|\theta_i) \quad (8)$$

Additionally, in this paper we present the mean posterior distribution function which is the mean value of $F_Y(y|\theta)$ for all s samples of θ .

2.3.1. Prior definition for GEV-parameters

Bayesian inference requires the definition of a prior distribution, often considered the main disadvantage of the method. The choice of prior distribution is usually based on expert judgement and/or prior information. If no specific prior

information exist a highly uninformative prior should be used. Additionally, with prior information it is also possible to bound the admissible domain for the parameters so parameter values that for a certain reason are judged to be unrealistic, are not considered or considered with small probability, being a great advantage of the method. We therefore follow this approach in this paper. The sensitivity of the posterior distribution to the choice of the prior is checked, as we do not want to enforce any behavior not backed up by prior information.

For the location parameter μ a uniform prior is adopted. As we consider maximum traffic load effects $\mu > 0$ (for minimum traffic load effects $\mu < 0$). The upper bound for μ should be uninformative, but at the same time should be bounded because of practical reasons. However, the posterior distribution should not be sensitive to the upper bound, meaning the upper bound for μ should not be taken too low.

The scale parameter σ should always be positive $\sigma > 0$ and decreasing probabilities for higher values are expected. Therefore the prior is defined for the natural logarithm of σ , $\ln(\sigma)$, further called log-scale.

The shape parameter ξ has a large influence on the shape of the tail of the distribution and therefore on the calculated design value. The prior distribution should not be taken too narrow, which enforces certain tail behavior not backed up by prior information. However, the prior for the shape parameter can be bounded considering unrealistic values as for $\xi > 0.5$ the variance of the GEV-distribution becomes infinitive. The latter is not realistic for extreme traffic load effects as these are bounded by physics; vehicle weight and a maximum number of vehicles on the bridge. Therefore a prior distribution is adopted with mean at $\xi = 0$ and with a variance such that $P(\xi > 0.5) \approx 0$ (and $P(\xi < -0.5) \approx 0$).

The prior distributions for the GEV-parameters for the use case presented in this paper are given in Table 1.

Table 1: Prior distribution GEV-parameters extreme traffic load effects.

Parameter	Distribution-type	Distribution
μ	Uniform	U(0,50 [MPa])
$\ln(\sigma)$	Normal	N(0,0.5)
ξ	Normal	N(0,0.2)

3. EXTREME VALUE ANALYSIS OF TRAFFIC LOAD EFFECTS

3.1. Traffic load effect data

For this research, traffic load effects are simulated with a year of data from two WIM-databases in the Netherlands (highway A50 and A27, from February 2015 until January 2016). Furthermore, a bridge specific influence line is used considering the load effects in the cables of a cable-stayed bridge loaded by four heavy traffic lanes (two main and two parallel lanes). The timeseries of WIM-data is run directly over the case-study bridge influence line. This results in a full year of continuous time history data of simulated traffic load effects (cable stresses).

3.2. Extremes – block duration

The time history traffic load effect data is divided into intervals of a chosen block duration. The maximum of each block is determined, generating a series of block maxima. A block duration should be defined which ensures identically distributed and independent block maxima. For traffic load effects often a block duration of one week or one day is used (Nowak et al. 2019), or to increase the number of data for statistical inference a duration of one hour is sometimes applied (Zhou et al. 2012). Here, one year of time history data is available, providing 52 weekly maxima and 250 daily maxima (considering only working days).

Maximizing the number of data has significant advantages considering the statistical uncertainties. Therefore it was checked whether the daily extremes could be used. This was done by checking whether the univariate theorem holds and the conditions of homogeneity for extreme value analysis therefore are satisfied. In Figure 1

it is shown that the shape of the upper tail of the shifted (under assumption of independence) daily maxima empirical distribution indeed matches well with the weekly maxima distribution. Therefore analyses are performed using, by good approximation, daily maxima traffic load effects.

Figure 2 shows that the full set of daily maxima is not identically distributed as the probability of exceedance of the empirical distribution function shows a kink starting from stresses around 20-25 MPa. Therefore an upper tail approach is applied for the GEV fit with a threshold stress of at least 20 MPa.

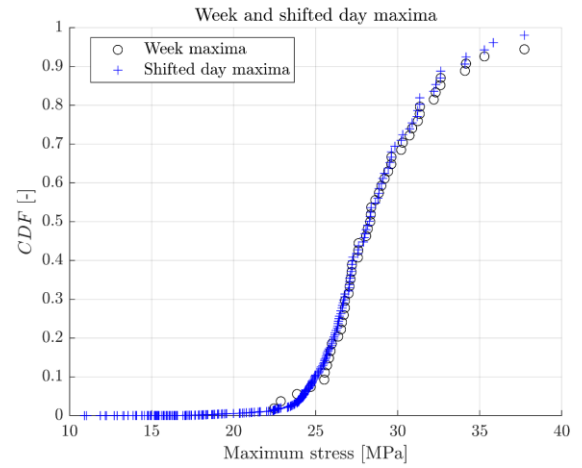


Figure 1: Cumulative distribution function (CDF) of weekly and shifted daily maxima cable stresses.

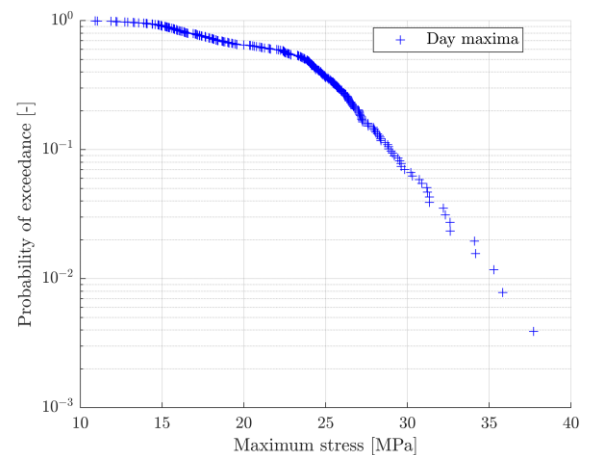


Figure 2: Probability of exceedance of daily maxima cable stresses.

3.3. Maximum likelihood estimation

The GEV-distribution is fitted to the daily extreme traffic load effects for multiple upper tail threshold values u_i using MLE, from a threshold stress of 20 MPa where a kink in the distribution function is noticed suggesting a change in the dominant subpopulation (Figure 2). Figure 3 shows the fitted GEV maximum likelihood parameters (circles) and their uncertainty $\pm 2\sigma$ (crosses), showing that from a threshold of 24 MPa statistical uncertainties start to increase considerably, especially on the shape parameter.

The threshold choice requires balancing two aspects. On the one hand, a higher threshold leads to a better fit of the tail. On the other hand, fewer extremes, thus fewer information, results in larger statistical uncertainties, suggesting 23 MPa as an upper bound threshold value. Therefore both 20 and 23 MPa are considered to study the sensitivity of the design value to the threshold choice.

To compare the distributions at the design value relevant for assessment, the fitted GEV-distribution including the effect of the statistical uncertainties is shown in Figure 4 and Figure 5 for a threshold of 20 and 23 MPa respectively. The horizontal lines represent the probability of exceedance of the design level calculated by:

$$P(E > E_d) = 1 - (1 - \Phi(\alpha_E \cdot \beta))^{1/numblocks} \quad (9)$$

with $\beta = 3.3$ for a 30 year reference period as a typical value for existing structures (Steenbergen and Vrouwenvelder 2010) and $\alpha_E = -0.7$, the standard sensitivity factor for the dominant load parameter proposed in EN1990 (dotted line) and $\alpha_E = -0.28$ for the non-dominant case (dashed line). $numblocks$ is the number of days within the reference period, with 250 working days per year.

The difference between the calculated design value stresses is 10 MPa (25%), which is significant. The choice of an optimal threshold is therefore very important. Even though a lower threshold of 20 MPa seems to be preferred for the number of extremes, a higher threshold seems necessary to provide a good fit of the tail.

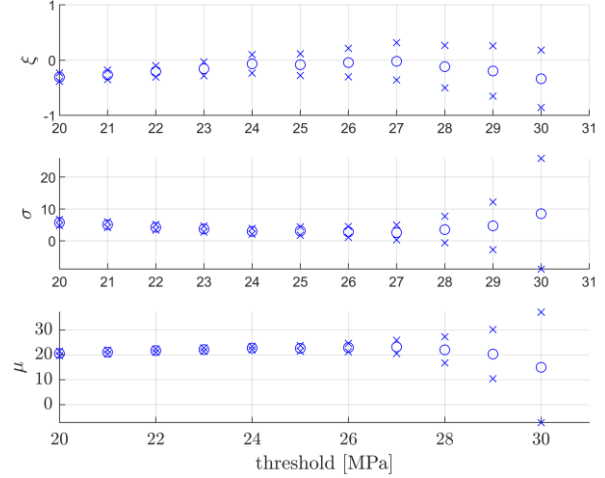


Figure 3: Fitted MLE parameters GEV distribution $\pm 2\sigma$ for multiple thresholds.

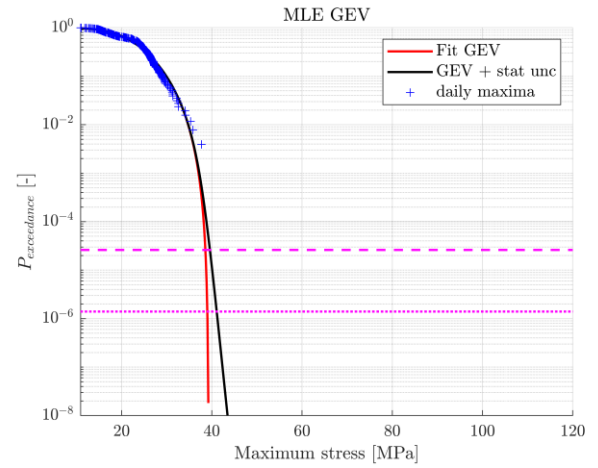


Figure 4: Through MLE GEV fitted distribution function for daily maxima traffic load effects above a threshold of 20 MPa.

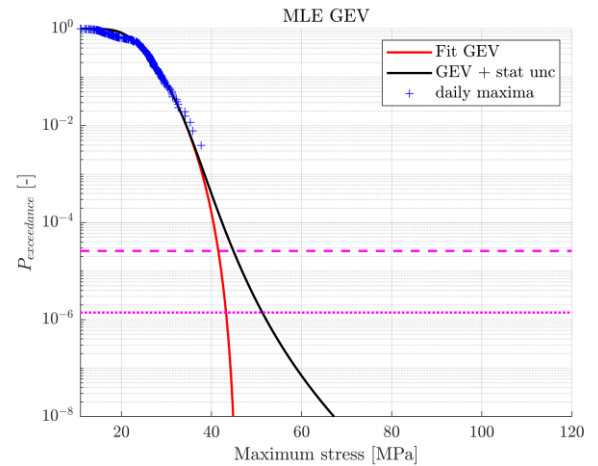


Figure 5: Through MLE GEV fitted distribution function for daily maxima traffic load effects above a threshold of 23 MPa.

3.4. Bayesian inference

The daily extreme traffic load effects are also fitted with a GEV-distribution applying BI for multiple upper tail thresholds u_i . The results for a threshold stress of 20 and 23 MPa are compared.

Figure 6 shows the posterior distribution function for the parameters θ using a threshold of 23 MPa. The posterior distributions clearly show a slight non-normality, with a heavier upper tail for the shape parameter and heavier lower tail for the log-scale and location parameter. The assumption of normality for the statistical uncertainties in parameter estimations for the MLE approach is therefore not valid.

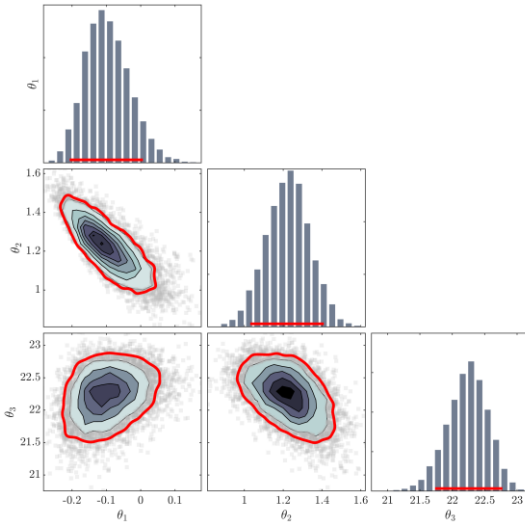


Figure 6: Posterior distribution function for parameters of the fitted GEV-distribution for daily maxima traffic load effects above a threshold of 23 MPa ($\theta_1 = \xi, \theta_2 = \ln(\sigma), \theta_3 = \mu$).

In Figure 7 and Figure 8 the fitted mean posterior and predictive GEV-distribution are shown for a threshold of 20 and 23 MPa respectively to compare the distributions at the design value relevant for assessment. The choice of prior distributions for the shape and log-scale parameters were found to have no influence on the posterior distributions and design values, while the prior for the shape parameter was found to only have a negligible influence.

The predictive posteriors in Figure 7 and Figure 8 show that the statistical uncertainties

have a larger contribution to the computed design value than was the case for the MLE results in Figure 4 and Figure 5 respectively. One of the reasons is that the assumption of normality in the MLE approach underestimates the statistical uncertainties.

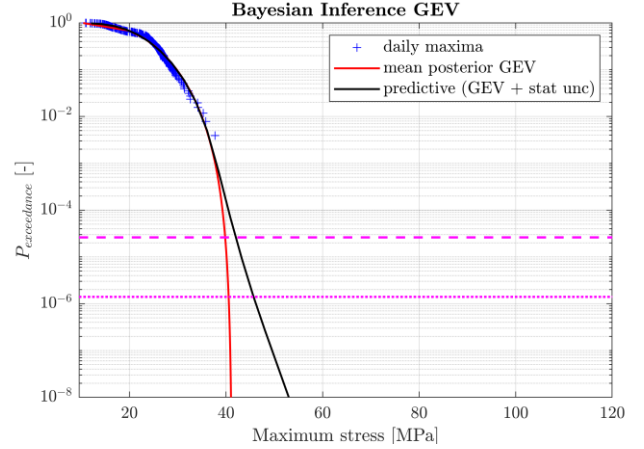


Figure 7: Through BI GEV fitted mean posterior and predictive distribution function for daily maxima traffic load effects above a threshold of 20 MPa.

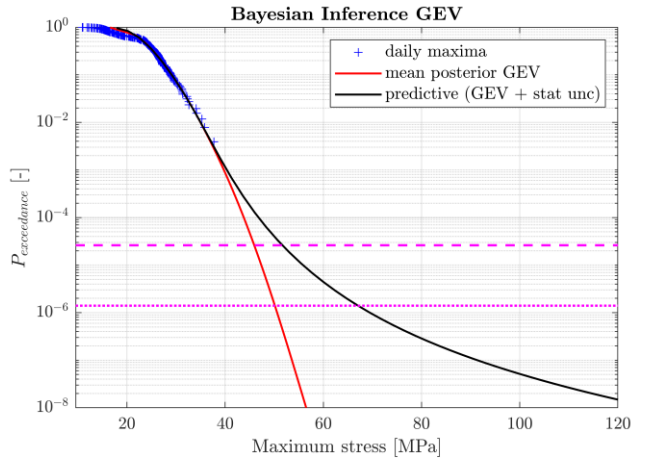


Figure 8: Through BI GEV fitted mean posterior and predictive distribution function for daily maxima traffic load effects above a threshold of 23 MPa.

4. CONCLUSIONS

In this paper, a year of data was used to simulate traffic load effects for a specific case-study bridge. Both maximum likelihood estimation and Bayesian inference were used to fit distribution functions and extrapolate the extreme traffic load

effects to the design value considering statistical uncertainties explicitly.

The BI results showed a slight non-normal tail behavior for the uncertainties in parameter estimates and therefore non valid assumption of normality for the MLE approach. The influence of statistical uncertainties on the computed design value are therefore underestimated in the MLE approach. BI results in higher design value estimates even though unrealistic tail shape parameter values are bounded through definition of the prior distribution. BI is therefore the preferred approach since it considers statistical uncertainties directly through the posterior distribution.

Both methods were found to be sensitive to the choice of threshold value. Developing methods to make a well-considered choice of a threshold value should be the topic for further research.

5. ACKNOWLEDGEMENTS

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