

Progress in lattice algorithms

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The development of Monte Carlo algorithms for generating gauge field configurations with dynamical fermions and methods for extracting the most information from ensembles are summarised.

1. INTRODUCTION

This conference illustrated that most large collaborations actively investigating QCD physics are performing simulations of the theory with dynamical quark fields. Dynamical staggered fermion simulations have been pursued for many years now, however it is clear that the community is now also firmly in the era of full Wilson fermion studies. These calculations will address one of the biggest systematic error in many non-perturbative QCD predictions from the lattice; use of the “quenched approximation”. At present, most simulations are not being performed at realistic values of the quark masses, or with the correct number of flavours of light fermions. A technically challenging extrapolation of data to the up and down quark masses must subsequently be performed.

At the conference, statements were given [1] by the biggest collaborations. Estimated costs for reproducing the state-of-the-art in quenched spectrum data in the full theory were in the range 10-100 Tflop-years. In this review, efforts to develop better algorithms to simulate closer to the physical parameters of QCD are discussed.

2. ALGORITHMS FOR WILSON FERMION SIMULATIONS

In this section, the main features of the techniques for simulating dynamical Wilson fermions are summarised. The simulations of dynamical staggered fermions are more mature; the principle difficulty at present is one of controlling the choice of discretisation [2]. The new challenge is to find

efficient methods for including recently proposed improved discretisations. Developing algorithms incorporating “fat-link” gauge fields, designed to enhance the flavour symmetry of the action are presented later. It is worth noting that the exact algorithms developed for odd-flavour simulations with Wilson fermions are being applied to staggered fermion simulations.

The Hybrid Monte Carlo (HMC) algorithm [3] is still the work-horse powering most large Wilson fermion simulations. Over the past few years there have been further advances in our understanding of the issue of finite precision arithmetic and the reversibility and stability of the molecular dynamics (MD) integrator component. HMC is an exact algorithm if the MD integrator is reversible and area preserving. It has long been understood that the Hamiltonian evolution of QCD is chaotic, and the Liapunov exponents were studied in detail [4,5]. More recently, the stability of the MD leap-frog integrator [4] has been investigated in detail by the UKQCD collaboration [6]. Their findings were presented to the conference by Joó [7]. Analysis of a leap-frog integrator of the simple harmonic oscillator reveals the solution diverges once the step-size exceeds a critical value. The problem is exacerbated for higher-order integrators. This same behaviour is observed in QCD, and the critical step-size is seen to fall as the quark mass, $m_q \rightarrow 0$. The suggestion is that the problem of simulating light Wilson fermions with HMC will amount to managing this stability problem.

There has been progress in implementations of the local boson algorithm (LBA) [8]. This algorithm relies on first constructing a polynomial

approximation to the inverse of the fermion matrix, then expressing the polynomial as a product of roots. The determinant is expressed as a local bosonic partition function involving a large number (the degree of the polynomial) of auxiliary fields. The beauty of the method is that a local change in the gauge fields leads to a local computation for the change in the action. Unfortunately, the simplest implementations of this method suffer from long autocorrelations [9]; a large number of auxiliary fields are coupled to the gauge fields, and so only small changes in these degrees of freedom can be made in each sweep. The problem can be circumvented by using lower-order polynomials, and correcting for any discrepancy between the approximation and the true inverse using a noisy accept-reject step [10]. This idea led to the development of the two-step multi-boson (TSMB) algorithm [11], where two polynomials are constructed, $\mathcal{P}_1(x)$ (low-order) and $\mathcal{P}_2(x)$ (high-order) such that

$$\mathcal{P}_1(x)\mathcal{P}_2(x) \approx \frac{1}{x^{N_f/2}}. \quad (1)$$

$\mathcal{P}_1(x)$ is replaced by a partition function of a small number of boson fields, and $\mathcal{P}_2(x)$ is used in the acceptance test.

Polynomial approximations have been incorporated into other techniques. The Polynomial Hybrid Monte Carlo (PHMC) algorithm [12] proposed by Frezzotti and Jansen is one such development. A single auxiliary field, ϕ is introduced, rather than the large number of LBA, and the action is

$$S_\phi = S_G[U] + \phi^* \mathcal{P}(M^\dagger M) \phi \quad (2)$$

This avoids the stiff update dynamics, however the action is now non-local, so HMC is used to update the gauge fields. The algorithm can be made exact either via an accept-reject step or by a re-weighting of expectation values by a correction factor:

$$\langle \mathcal{O} \rangle_{\text{true}} = \frac{\langle \mathcal{O} \det Q^2 \mathcal{P}(Q^2) \rangle_\phi}{\langle \det Q^2 \mathcal{P}(Q^2) \rangle_\phi}. \quad (3)$$

Frezzotti and Jansen advocate this re-weighting and the break-down of the approximation close to zero is now regarded as advantageous. The

polynomial weight generates more configurations with low eigenvalues (which are then assigned a lower weight in the ensemble average), and this over-sampling should lead to a more reliable determination of quantities that are sensitive to the details of the lowest eigenvalues.

The ALPHA collaboration have made a comparison [13] between PHMC and pseudofermionic HMC. In the range of parameters of interest, they find there is a slight advantage to PHMC and favour this method for future simulations, on account of its over-sampling property. Schroers presented results to the conference [14] that suggest the performance of the tuned LBA is faster than the SESAM production HMC code. Performance comparisons between TSMB and HMC in compact QED were presented to the conference by Zverev [15]. The study finds the two algorithms are equally competitive, although the authors emphasise their TSMB implementation can still be optimised by changing the local gauge update scheme and tuning the polynomials $\mathcal{P}_1$ and $\mathcal{P}_2$ more carefully.

One remarkable feature of these comparisons is that the results are, in fact, so close. de Forcrand emphasises that the LBA algorithm turns into a standard local heatbath/over-relaxation algorithm as the fermions are made heavier, and the expectation there is that the over-relaxation algorithms should out-perform (global) HMC significantly.

3. ODD-FLAVOUR SIMULATIONS

Since QCD has three flavours of light fermions, there is a natural incentive to study algorithms for simulating an odd number of fermion flavours. The standard pseudofermion formulation is unsuitable, since the Wilson matrix can have eigenvalues with negative real parts, meaning there are regions of configuration space where the gaussian integral is not defined. One extremely useful development arising from the local boson method has been the possibility of performing these odd-flavour simulations, even maintaining an exact update algorithm. Since these simulations still rely on gaussian integrals, they in fact generate configurations with probability measure defined

from the modified partition function,

$$Z_+ = \int \mathcal{D}U |\det M| e^{-S_G}. \quad (4)$$

The sign of the determinant can be included in a *post-hoc* reweighting of observables:

$$\langle \mathcal{O} \rangle_{\text{true}} = \frac{\langle \mathcal{O} \text{sgn}(\det M) \rangle_+}{\langle \text{sgn}(\det M) \rangle_+}. \quad (5)$$

This may lead to a “sign problem”, where statistical estimates of reweighted observables fluctuate wildly if a mixture of configurations with both positive and negative sign determinants occur in the ensemble. Empirical studies of this problem for new simulations of QCD with three quarks (with a target of $m_q \approx m_s/4$) were presented to the conference by Gebert [16]. They expect the sign problem to be very mild in this region.

The local boson algorithm can approximate the partition function of Eqn. 4; starting from a polynomial approximation to $1/\sqrt{x}$ in the range $[\epsilon, 1]$ leads directly to the one-flavour partition function. Polynomials of the non-hermitian fermion matrix M also present a solution, as suggested in Ref. [10] and developed in Ref. [17,18]. A polynomial approximation to M^{-1} of order $2n$ is constructed,

$$M^{-1} \approx \mathcal{P}(M) = \prod_{k=1}^{2n} (M - z_k). \quad (6)$$

For a suitably chosen polynomial, the roots, z_k come in complex-conjugate pairs, so the polynomial can be written

$$\mathcal{P}(M) = \prod_{k=1}^n (M - z_k)(M - z_k^*). \quad (7)$$

The γ_5 hermiticity of M means

$$\det(M - z_k^*) = \det(M^\dagger - z_k^*) \quad (8)$$

so $|\det M|$ can be represented approximately by a gaussian integral,

$$|\det M| \approx \int \mathcal{D}\phi \mathcal{D}\phi^* \exp \{ -\phi^* T_n^\dagger(M) T_n(M) \phi \} \quad (9)$$

with

$$T_n(M) = \prod_{k=1}^n (M - z_k). \quad (10)$$

The scheme can be made exact with a Metropolis test; two alternatives were proposed in Refs [17,18], both of which rely on a noisy Kennedy-Kuti [19] acceptance test. The three-flavour simulations of QCD are being carried out by the JLQCD collaboration and results were presented to the conference [20].

4. ACCELERATING HMC

HMC remains the most popular technique for generating ensembles of gauge fields with two flavours of dynamical Wilson (and Sheikholeslami-Wohlert improved) fermions. Most of the simulations use this well-established method, so it is useful to find simple schemes to enhance the method.

4.1. ILU preconditioning

Replacing the fermion determinant by a gaussian integral couples extra stochastic degrees of freedom to the gauge fields. This in turn leads to a lower Metropolis acceptance rate in HMC. It has been known for a long time that coupling the gauge fields and pseudofermions with the even-odd preconditioned fermion matrix (whose determinant is equal to that of the original fermion matrix) leads to a higher acceptance rate for a given molecular dynamics step-size [21]. The SESAM collaboration [22] emphasised that the even-odd preconditioning scheme is an example of an incomplete LU factorisation technique for matrix preconditioning. ILU preconditions a matrix by left and right multiplication by two readily invertible matrices;

$$\tilde{M} = (I - L)^{-1} M (I - U)^{-1} \quad (11)$$

where L and U are the lower and upper sections of the matrix. Defining L and U first requires sites on the lattice, $\mathbf{x}$ are assigned an index, $s(\mathbf{x})$ then site ordering is defined as

$$\mathbf{x} > \mathbf{y} \quad \text{if} \quad s(\mathbf{x}) > s(\mathbf{y}). \quad (12)$$

SESAM developed a site ordering that is well-suited to accelerate matrix inversion on parallel computers. Since this preconditioning leaves the determinant of the matrix invariant, $\tilde{M}$ can be used to couple the pseudofermions to the gauge

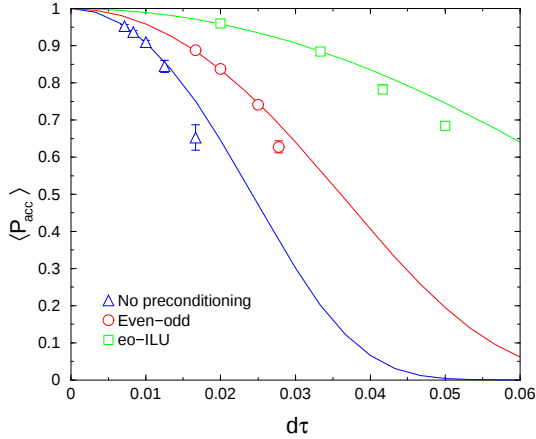


Figure 1. Schwinger model HMC Metropolis acceptance probability for pseudofermions coupled via different preconditioned matrices [23].

fields. This holds for any choice of ordering function, $s(\mathbf{x})$. In Ref. [23], different site orderings were tested in HMC simulations of the two-flavour Schwinger model. An unusual pattern emerged; the optimal ordering for speeding up matrix inversion was the global lexicographic scheme, while the best ordering for improving Metropolis acceptance was the usual even-odd preconditioner. In Ref. [24], it was noted that the even-odd matrix itself can be ILU factorised again, and this both accelerates matrix inversion and improves the Metropolis acceptance rate. Ref. [23] tested a range of two step (eo-ILU) preconditioned matrices (defined now by different choices of an indexing function, $s(\mathbf{x}_e)$ on the even sub-lattice only). These simulations found the same pattern emerged; the optimal ordering for inversion is a globally lexicographic one, while using a “locally ordered” scheme is best for improving Metropolis acceptance. Using an eo-ILU ordered interaction allows a three times larger step-size for a given Metropolis acceptance rate. The naive expectation that the autocorrelations of the update scheme are independent of the integrator step-size at a fixed acceptance is supported by simulation data.

4.2. Splitting the pseudofermions

Hasenbusch [25] recently proposed modifying the pseudofermion sector of HMC to improve the acceptance rate for a fixed molecular dynamics step-size. The key observation is (as above) that coupling the gauge fields to pseudofermions via a well-conditioned matrix allows a larger molecular dynamics step-size to be taken for a fixed Metropolis acceptance. The fermion determinant is first split into two pieces, namely

$$\det M = \det M \bar{M}^{-1} \det \bar{M}. \quad (13)$$

The two-flavour determinant is then represented by a gaussian integral,

$$\det M^2 = \int \mathcal{D}[\phi, \phi^* \psi, \psi^*] \exp \{-S_\phi - S_\psi\} \quad (14)$$

with

$$S_\phi = |\bar{M} M^{-1} \phi|^2 \quad \text{and} \quad S_\psi = |\bar{M}^{-1} \psi|^2. \quad (15)$$

Hasenbusch then chooses $\bar{M} = I - \tilde{\kappa} \Delta$ and recognises that for $0 < \tilde{\kappa} < \kappa$, the two matrices $M \bar{M}^{-1}$ and $\bar{M}$ are better conditioned than M . This leads to improved acceptance. In Ref [25], Hasen-

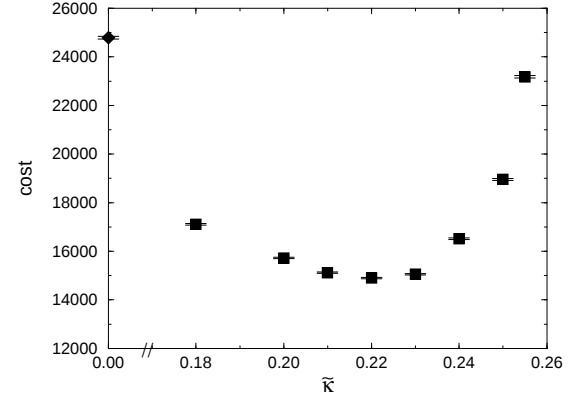


Figure 2. Schwinger model simulation cost vs $\tilde{\kappa}$ for the split pseudofermion scheme [25,26].

busch tested the idea in the two-flavour Schwinger model and at the conference [26], results for preliminary studies in QCD were presented. The cost of HMC simulations (defined in units of matrix $\times$ vector operations) as a function of the

splitting parameter, $\tilde{\kappa}$ is summarised in Fig. 2 In the Schwinger model study, simulations are accelerated by a factor of 1.7 when the pseudoscalar meson mass, $m_P = 0.210(3)$, rising to a speed-up of 2 when $m_P = 0.124(5)$. This increase at lighter fermion mass is an encouraging result. The studies in QCD [26] are more exploratory, but effects of similar magnitude are seen.

4.3. Multiple time-scale integration

Based on the idea of separating the infra-red and ultra-violet sectors of the fermion matrix, Jim Sexton and I are working on a practical implementation of a multiple time-scale molecular dynamics integrator. A low-order polynomial approximation to the fermion matrix inverse, which captures the short distance scale effects of quark loops is used, along with a pseudofermion action to reproduce long-range physics. The force on the gauge field from the polynomial term in the action is integrated (along with the easy-to-compute force from the Yang-Mills discretisation) using a finer time-step, and the long-range modes, contained in the pseudofermion action are integrated using a larger time step [27].

4.4. Are these schemes compatible?

If these ideas could be combined, the possibility of an order-of-magnitude acceleration of Wilson fermion simulations would open up. ILU preconditioned pseudofermions (Sec. 4.1) can be used straightforwardly in the multiple time-scale scheme of Sec. 4.3. This idea is under investigation, and it seems likely the benefits from each component will be combined. Hasenbusch demonstrated directly that his scheme works with even-odd preconditioned pseudofermions.

5. NEW DEVELOPMENTS

The Kentucky group [28] are developing a new dynamical fermion updater based on a Kennedy-Kuti noisy acceptance test on changes in the determinant. The ratio of the determinants evaluated on the new and old configurations is computed using a Z_2 stochastic estimate of the trace of a Padé approximation to the logarithm of the fermion matrix. The scheme is still being developed. One interesting feature [29] is the possibility

of using the algorithm to simulate finite density QCD, by including a projection onto states with definite particle number.

Bakeyev [30] has proposed a new scheme for performing exact, finite updates of the gauge fields interacting with dynamical fermions. The update involves finding a new configuration, $\tilde{U}$ that solves

$$M(\tilde{U})\eta = \omega M(U)\eta \quad (16)$$

for a stochastic parameter, ω and gaussian noise source, η . The idea is in its infancy as yet, and may prove to be most useful as an additional step in an HMC simulation.

The flavour-symmetry breaking of staggered fermions can be reduced by using a “fat link” to induce quark-gluon interactions. Simulating these staggered actions poses a new problem in algorithm design. The fat link is constructed following a similar procedure to APE smearing, which involves summing the six staples around a link, then projecting the resulting matrix back into the gauge group. The fattening step is repeated iteratively. The challenge for simulation is that the force term on the underlying gauge fields is difficult to compute. New algorithms are being developed and tested to circumvent this problem. Knechtli [31] presented these ideas to the conference. HMC fat-link simulations with four flavours when the lightest pion has mass $r_0 m_\pi = 2.0$ are seven times more costly than runs where the quarks interact directly with the link variables. This factor is reduced to about 3 once $r_0 m_\pi$ reaches 1.6 This extra overhead is more than offset by the much better flavour symmetry of the fat-link actions, and the quark mass accessible to these studies should be much lighter.

6. HIERARCHICAL NOISE REDUCTION IN YANG-MILLS THEORIES

Recently, Lüscher and Weisz [32] proposed a novel algorithm for reducing the stochastic noise in large Wilson loop expectation values by a factor that grows exponentially with the loop size. The scheme involves building a hierarchy of sub-domains within the lattice, and performing a re-

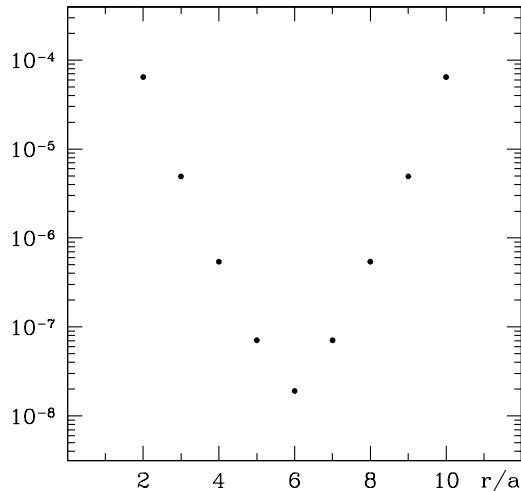


Figure 3. The Polyakov loop correlator on a $12^4, \beta = 5.7$ lattice, computed using the hierarchical noise reduction method. Statistical errors are smaller than the plot symbols [32].

duced Monte-Carlo integration over the degrees of freedom inside each domain. The results at each level are passed up the hierarchy to construct variance reduced averages of larger and larger objects. The scheme is a natural extension of the temporal link “multi-hit” method [33]. They illustrate their proposal with a calculation of the correlator of two spatially-separated Polyakov lines (see Fig. 3), and show a reduction of many orders of magnitude in the statistical error over the standard technique.

The scheme can be extended to include matter fields. If the fermion determinant is included using the LBA, the action is localised within a small neighbourhood. With Wilson fermions (and the Wilson parameter set to unity) the hierarchical measurement scheme can be incorporated without modification. The difficulty is including the stochastic accept-reject schemes usually employed with the LBA to make the method exact. These rely on lattice-wide observations, involving all the gauge degrees of freedom and the domain decompositions of Ref [32] would no longer apply.

It remains an unsolved challenge to include

the highly developed tools for optimising ground-state overlap into the hierarchy scheme. Methods such as APE smearing and Teper blocking have proved crucial in the accurate calculations of the inter-quark potential and its excitations in Yang-Mills theory. Variational techniques, which build a large basis sets of creation operators for the states of interest, then find an optimal ground-state operator by diagonalisation have also dramatically improved these calculations (for an example, see Ref. [34]). While the theoretical aspect of including these ideas is readily resolved, it is not obvious how well this toolkit can be used within the hierarchy method in practice. An empirical test seems the only way to resolve this issue. If the noise reduction of the method of Ref. [32] can be married to the excellent ground-state construction of the smoothing and operator basis methods, then extremely precise calculations of the detailed nature of the confining string spectrum at large separations could be performed.

7. ALL-TO-ALL FERMION PROPAGATORS

Having expended such a considerable number of cycles on some of the world’s largest supercomputers in generating an ensemble of gauge field configurations, it is clear we are duty bound to make the most of the information they contain.

For many observables, believed to be sensitive to vacuum fluctuations in the quark fields, traditional point-source propagator methods are inefficient. To extract a useful signal from the ensemble, the propagator from all (or many) points to all points on the lattice is needed. There has been a good deal of interest in improving these techniques in recent years.

7.1. Gaussian and Z_2 stochastic estimators

A gaussian representation of a fermion matrix entry can be written [35]:

$$Q_{ij} = \int \mathcal{D}\phi \mathcal{D}\phi^* \phi_i (\phi^* Q)_j \exp \{-\phi^* Q^2 \phi\}. \quad (17)$$

Thus a propagator from any point on the lattice to any other can be estimated by performing a sub-Monte Carlo simulation on each configuration.

At the conference, Duncan [36] presented simulation data using a gaussian all-to-all scheme. He emphasised that for many interesting physics applications, this simplest of methods is adequate to ensure the dominant source of statistical error is from fluctuations within the gauge ensemble, rather than the stochastic estimator. The stochastic degrees of freedom can be updated in a global heatbath step, but this requires a matrix inversion. Ref. [37] gives a scheme to accelerate the process; the iterative matrix solver is stopped after a small number of steps, and a Metropolis test is used to correct for the error in this step.

The Kentucky group [38] noted that other stochastic variables can be used to compute matrix traces, and exploited random elements of Z_2 . These estimators have been employed in a range of calculations involving disconnected diagrams.

7.2. Improving stochastic estimators

Ref. [39] suggests reducing the noise in a gaussian estimator by dividing the lattice into two segments, then computing the independent sets of integrals in each segment exactly in a fixed background of the degrees of freedom on the interface between the segments. Conceptually, the new Lüscher-Weisz hierarchy method, described in Sec. 6 is reminiscent of this scheme. The method has one restriction; the source and sink points of the propagator can not be in the same domain, so closed fermion loops can not be computed. In Ref. [40] a further enhancement of the method is presented.

At the conference, a method was presented by Wilcox [41] to mix gaussian and Z_2 noise, allowing an approximate heatbath with Metropolis correction to be used.

The SESAM collaboration [42] improved Z_2 estimators by breaking the vector space into sub-spaces (in their study, the 4 sub-spaces corresponding to the spin indices) and performing separate stochastic estimators in each sub-space. The idea was tested further in Ref [43]. While this means more matrix inversion must be performed for a fixed stochastic ensemble size, the resulting estimators can have a significantly smaller variance. This idea of thinning can be extended beyond just spin and colour indices to spatial sites

as well. If the scheme were to be carried to the extreme, where the N dimensional vector space is decomposed into N 1-dimensional spaces, a Z_2 estimator would reproduce the exact result for any ensemble size. This is not such a surprising result, since this computation would require N inversions to be performed and it is a trivial result that the trace of M^{-1} can be computed in N inversions. Without thinning however the error falls off, as any statistical estimate should, like $1/\sqrt{N}$ so eventually thinning must become more efficient.

The thinning procedure has not been explored fully for computations of all-to-all propagators in QCD, and it would be worthwhile to investigate this more completely.

7.3. Eigenvalue decomposition

If all the eigenvectors (and eigenvalues) of the fermion matrix are known, then any propagator entry can be computed straightforwardly. It is natural to expect that the long-range physics of QCD are contained in the lowest-lying eigenvectors, and these will then dominate in the spectral representation of a mesonic correlator (for example). The importance of understanding the physics of the lowest modes of the fermion matrix was emphasised in the plenary talk of Edwards [44].

Neff presented the conference [45] with an investigation of methods for computing the low-lying eigenvectors efficiently. The acceleration comes from using a polynomial to isolate the region of eigenvalues of interest, and this leads to more rapid converges of the Arnoldi method. At the conference, Schilling [46,47] discussed the dominance of the π and η correlators by the low-lying eigenmodes. Truncating the spectral expansion introduces a systematic error, but this can be avoided by computing all-to-all propagators in a hybrid scheme (as noted by Edwards [44]).

7.4. Hybrid schemes

The two schemes can be combined straightforwardly to construct a hybrid all-to-all estimator method [44]. A number of low-lying eigenvalues of the fermion matrix are computed and incorporated exactly and the effect of all the remain-

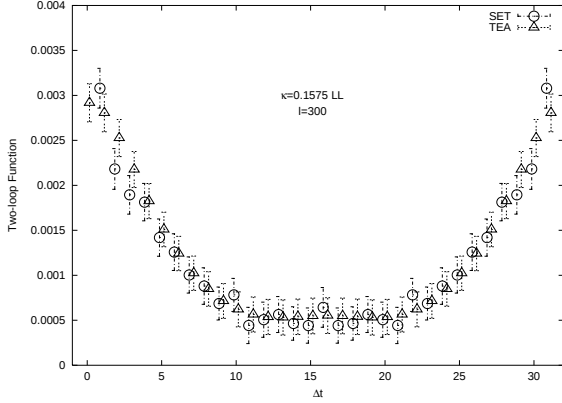


Figure 4. The disconnected (hairpin) correlator for the η' , computed using both an eigenvalue decomposition (TEA) and stochastic estimators (SET) [46].

ing eigenvalues is estimated stochastically. This involves building a projection operator onto the reduced sub-space spanned by the explicitly computed eigenvectors.

For illustration, consider the observable

$$\text{Tr } Q^{-1} \Gamma = \sum_i^N \frac{1}{\lambda_i} v_i^* \Gamma v_i. \quad (18)$$

If the lowest m ($m \ll N$) eigenvectors of the hermitian matrix $Q = \gamma_5 M$ are computed, then it is natural break the the spectral representation of Q into two parts,

$$Q = \sum_{i=1}^m \lambda_i v_i v_i^* + \sum_{i=m+1}^N \lambda_i v_i v_i^*. \quad (19)$$

Defining the (sub-space) inverses and projectors,

$$\bar{Q}_{(0)} = \sum_{i=1}^m \frac{1}{\lambda_i} v_i v_i^*, \quad \mathcal{P}_{(0)} = \sum_{i=1}^m v_i v_i^* \quad (20)$$

the trace becomes

$$\text{Tr } Q^{-1} \Gamma = \text{Tr } \bar{Q}_{(0)} \Gamma + \text{Tr } \bar{Q}_{(1)} \Gamma.$$

The first term is the truncated estimate, and the remaining term can be estimated stochastically;

$$\begin{aligned} \text{Tr } \bar{Q}_{(1)} \Gamma &= \langle \eta^* \Gamma \bar{Q}_{(1)} \eta \rangle \\ &= \langle \eta^* \Gamma (Q^{-1} - \bar{Q}_{(0)}) \eta \rangle \\ &= \langle \eta^* \Gamma Q^{-1} (I - \mathcal{P}_{(0)}) \eta \rangle. \end{aligned} \quad (21)$$

The operation of $I - \mathcal{P}_{(0)}$ is an orthogonalisation of the noise vector, η with respect to all m known eigenvectors. The matrix inversion in this last step is accelerated, since the eigenvectors corresponding to the low-lying eigenvalues have been removed from η . It will be interesting to see hybrid schemes tested in practical applications. The developments of GMRES algorithms presented to the conference [48] look particularly interesting in this context.

8. CONCLUSIONS

For the next few years, most large-scale simulations of full QCD will use the Wilson or staggered fermion formulation, while the chiral lattice actions [49] are beyond the reach of current computing resources; Ginsparg-Wilson fermions are a factor of 100 times more expensive. As pointed out by Karl Jansen, the Hybrid Monte-Carlo algorithm is not far off its twentieth birthday and remains (largely unmodified) the most popular tool for dynamical Wilson fermion simulations.

Current predictions [1] are that resources available in the next few years are an order of magnitude below the requirements for reliable QCD simulations. Including the newest ideas presented, it might be hoped to get close to the required factor of ten improvement in performance.

Realistic comparative tests of HMC vs LBA have been carried out in recent years, leading to the observation that the two methods have identical costs (for all practical purposes). These comparisons are always very difficult to make. It would be helpful to see a standard benchmark emerge on which direct comparisons of algorithms can be made. Defining a standard is difficult but with this in place, fair races of new methods against “thoroughbred” algorithms rather than the simplest implementations could be made.

Excitement at the conference generated by the recent idea of Lüscher and Weisz suggests reconsidering the traditional Monte-Carlo analysis pathway; generating configurations then subsequently making measurements. Their hierarchy scheme, which knits these two processes together can lead to orders of magnitude increases in efficiency. The challenge is to apply this philosophy

to a wider range of applications.

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