First Excursion Probability Sensitivity Estimation by means of Multidomain Line Sampling

Marcos A. Valdebenito
Chief Engineer, Chair for Reliability Engineering, TU Dortmund University, Dortmund, Germany

Matthias G.R. Faes
Professor, Chair for Reliability Engineering, TU Dortmund University, Dortmund, Germany

Mauricio A. Misraji
Graduate Student, Chair for Reliability Engineering, TU Dortmund University, Dortmund, Germany

Danko J. Jerez
Graduate Student, Institute for Risk and Reliability, Leibniz University Hannover, Hannover, Germany

Michael Beer
Professor, Institute for Risk and Reliability, Leibniz University Hannover, Hannover, Germany

ABSTRACT: This paper develops a framework for calculating the sensitivity of the first excursion probability of linear dynamical structural systems subject to a Gaussian excitation. An approach based on multidomain Line Sampling is developed for estimating both the first excursion probability and its gradient with respect to parameters of the structural system. The results obtained suggest that the proposed framework may outperform a similar in terms of accuracy and efficiency.

1. INTRODUCTION

The first excursion probability offers the means for measuring the chances that one or more dynamic responses of a structural system subject to stochastic loading exceed a prescribed threshold level within a certain time duration (Soong and Grigoriu, 1993). Such probability allows quantifying the level of safety and is dependent on the properties of the structure, such as dimensions of structural elements. The sensitivity of that probability with respect to such properties offers valuable information for optimal design under uncertain conditions and for pinpointing influential parameters (Lu et al., 2008). Nonetheless, the calculation of both first excursion probability and its sensitivity remains a challenging issue, as they must be estimated by means of simulation approaches for cases of practical interest, which can become numerically demanding (Papaioannou et al., 2018). Thus, this contribution proposes a framework for calculating the first excursion probability and its sensitivity (in terms of a gradient) of a class of structural dynamical systems, namely deterministic linear structures subject to Gaussian loading. The framework is developed within the context of multidomain Line Sampling (Valdebenito et al., 2021b), which allows estimating first excursion probabilities with a reduced number of samples. Moreover, the sought sensitivity is obtained as a byproduct of the reliability analysis, and its numerical evaluation demands performing a sensitivity analysis of the mode shapes and natural frequencies (Lee and Jung, 1997). A numerical example illustrates the performance of the proposed framework, indicating that the application of multidomain Line Sampling may outperform a comparable simulation approach (Valdebenito et al., 2021a).

2. PROBLEM SETTING

2.1. Dynamic Gaussian Loading

Assume that the dynamic loading acting over a structural system is modelled as a Gaussian process. This stochastic process possesses a duration T and is represented using the Karhunen-Loève expansion. Thus, the loading is expressed as $p(t,\boldsymbol{z})$, where p represents the load magnitude, t denotes time and $\boldsymbol{z}$ is a vector of dimension n_z. This last vector is a realisation of a n_z-dimensional standard normal distribution with probability distribution $f_{\boldsymbol{Z}}(\boldsymbol{z})$.

2.2. Linear Structural System

Consider a structural system whose behaviour is modelled as linear elastic and classically damped (Chopra, 1995). Using an appropriate numerical scheme (see e.g. Bathe (1996)), the set of differential equations that govern the behaviour of the structure is:

$$\begin{gathered}\boldsymbol{M}(\boldsymbol{y})\ddot{\boldsymbol{x}}(t,\boldsymbol{y},\boldsymbol{z})+\boldsymbol{C}(\boldsymbol{y})\dot{\boldsymbol{x}}(t,\boldsymbol{y},\boldsymbol{z})+\\ \boldsymbol{K}(\boldsymbol{y})\boldsymbol{x}(t,\boldsymbol{y},\boldsymbol{z})=\boldsymbol{g}(\boldsymbol{y})p(t,\boldsymbol{z}),\ t\in[0,T]\end{gathered} \tag{1}$$

where $\boldsymbol{x}$, $\dot{\boldsymbol{x}}$ and $\ddot{\boldsymbol{x}}$ are vectors representing the displacement, velocity and acceleration, respectively, and whose dimension is $n_d \times 1$; $\boldsymbol{M}(\boldsymbol{y})$, $\boldsymbol{C}(\boldsymbol{y})$, and $\boldsymbol{K}(\boldsymbol{y})$ are the mass, damping and stiffness matrices, respectively, of dimension $n_d \times n_d$, which are dependent on the parameter vector $\boldsymbol{y}$; vector $\boldsymbol{g}(\boldsymbol{y})$, of dimension $n_d \times 1$, couples the stochastic loading $p(t,\boldsymbol{z})$ with the corresponding degrees-of-freedom of the structure. Vector $\boldsymbol{y}$ collects parameters that affect the structural behaviour and which may vary due to practical design decisions (Haftka and Gürdal, 1992). That is, $\boldsymbol{y}$ may contain quantities such as thicknesses of structural members, material properties, etc.

It is assumed that due to practical design reasons, there are a total of n_r responses from the structural systems which are of interest. These responses are denoted as $r_i, i=1,\ldots,n_r$ and due to the linearity of the structural system, they are calculated by means of a convolution integral between the corresponding unit impulse response functions $h_i, i=1,\ldots,n_r$ and the stochastic loading, that is:

$$r_i(t,\boldsymbol{y},\boldsymbol{z})=\int_0^t h_i(t-\tau,\boldsymbol{y})\,p(t,\boldsymbol{z})\,d\tau. \tag{2}$$

Note that impulse response functions can be calculated by means of modal analysis (Chopra, 1995).

2.3. First Excursion Probability

For practical design reasons, the responses of interest of the structural system should remain below a prescribed threshold level $b_i,\ i=1,\ldots,n_r$ during the duration T of the stochastic excitation. However, note that the responses of interest are dependent on both the parameter vector $\boldsymbol{y}$ and the realisation $\boldsymbol{z}$ of the random variable vector associated with the Gaussian loading. This implies that, due to the uncertainty associated with $\boldsymbol{z}$, there are chances that the responses of interest exceed their prescribed thresholds within the duration T of the Gaussian loading. These chances can be quantified in terms of the first excursion probability p_F, which is expressed by means of the classical integral:

$$p_F(\boldsymbol{y})=\int_{\boldsymbol{z}\in\mathbb{R}^{n_z}} I_F(\boldsymbol{y},\boldsymbol{z})\,f_{\boldsymbol{Z}}(\boldsymbol{z})d\boldsymbol{z} \tag{3}$$

where $I_F(\cdot,\cdot)$, is an indicator function whose value is equal to one in case any of the responses of interest exceed their prescribed thresholds. For cases of practical interest, the probability integral in Eq. (3) must be evaluated with advanced simulation methods (Schuëller and Pradlwarter, 2007), as its dimensionality is usually large (that is, n_z is in the order of hundreds or even thousands) and the indicator function is known point-wise only and must be evaluated numerically.

2.4. *Gradient of First Excursion Probability*

The failure probability integral in Eq. (3) depends on the particular values that the parameter vector $\boldsymbol{y}$ assumes. This makes sense from a physical viewpoint, as by changing the properties $\boldsymbol{y}$ of a structural system, one may alter the failure probability value. For practical design purposes, it is of interest to determine the sensitivity of this failure probability with respect to those parameters. A possible means for quantifying the sensitivity is calculating the gradient of the failure probability, that is:

$$\frac{\partial p_F(\boldsymbol{y})}{\partial y_q} = \frac{\partial}{\partial y_q}\left(\int_{\boldsymbol{z}\in\mathbb{R}^{n_z}} I_F\left(\boldsymbol{y},\boldsymbol{z}\right) f_{\boldsymbol{Z}}\left(\boldsymbol{z}\right) d\boldsymbol{z}\right), \quad q = 1,\ldots,n_y \tag{4}$$

where n_y is the dimensionality of vector $\boldsymbol{y}$. Evaluating this gradient is usually as challenging as evaluating the failure probability.

3. MULTIDOMAIN LINE SAMPLING

3.1. *Multidomain Line Sampling for Evaluation of the First Excursion Probability*

The linearity of the responses of interest r_i with respect to the input Gaussian loading $p(t,\boldsymbol{z})$ as cast in Eq. (2) brings important consequences from a practical viewpoint. Indeed, as recognised and discussed in Der Kiureghian (2000), the realisations of $\boldsymbol{z}$ for which the responses of interest exceed their prescribed thresholds (the so-called failure domain) are contained in a region limited by a series of hyperplanes, as illustrated schematically in Figure 1. Several simulation methods have taken advantage of the geometry of the failure domain in order to calculate first excursion probabilities, see e.g. Au and Beck (2001); Katafygiotis and Cheung (2006); Misraji et al. (2020). Multidomain Line Sampling (mLS; Valdebenito et al., 2021b) also exploits such geometry by expressing the failure probability p_F as a summation of *effective* contributions $p_{i,k}$, that is:

$$p_F(\boldsymbol{y}) = \sum_{i=1}^{n_r}\sum_{k=1}^{n_t} p_{i,k}(\boldsymbol{y}) \tag{5}$$

where each effective contribution is defined as:

$$p_{i,k}(\boldsymbol{y}) = \int_{\boldsymbol{z}\in F_{i,k}} \frac{1}{\sum_{i_1=1}^{n_r}\sum_{k_1=1}^{n_t} I_{F_{i_1,k_1}}(\boldsymbol{y},\boldsymbol{z})} f_{\boldsymbol{Z}}(\boldsymbol{z}) d\boldsymbol{z} \tag{6}$$

where $F_{i,k}$ denotes the event where the i-th response of interest exceeds its prescribed threshold b_i at the k-th time instant; and $I_{F_{i,k}}(\boldsymbol{y},\boldsymbol{z})$ is an indicator function which is equal to 1 in case that $\boldsymbol{z} \in F_{i,k}$. Note that $F_{i,k}$ is actually bounded by two hyperplanes, as explained in detail in Der Kiureghian (2000).
The effective contributions $p_{i,k}$ can be calculated most efficiently by resorting to Line Sampling (Koutsourelakis et al., 2004). This is illustrated schematically in Figure 2, where it is assumed for simplicity that $n_r = 1$ and $n_t = n_z = 2$. In this Figure, $\boldsymbol{\alpha_1}$ represents the so-called important direction, which points in the direction of the event $F_{1,1}$. The coordinates $z_1^{\parallel}$ and $\boldsymbol{z}_1^{\perp}$ point in the direction parallel and orthogonal to the important direction $\boldsymbol{\alpha}_1$. Line Sampling estimates the integral in Eq. (6) by generating a random sample in the coordinate set $\boldsymbol{z}_1^{\perp}$ (denoted with a circle in Figure 2) and calculating the values of the responses of interest along a line (denoted with two arrowheads in Figure 2) that passes through the aforementioned sample and which is parallel to $\boldsymbol{\alpha}_1$. As discussed in detail in Valdebenito et al. (2021b), the calculation of the responses of interest along a line demands performing two dynamic analyses, so it can be performed with negligible numerical efforts.

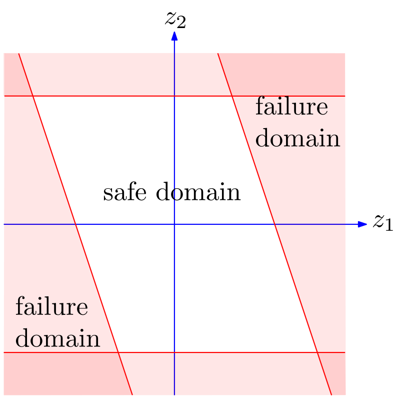

Figure 1: Schematic representation of failure domain.

For the practical implementation of multidomain Line Sampling, one does not calculate explicitly all effective contributions $p_{i,k}$ that appear in Eq. (5).

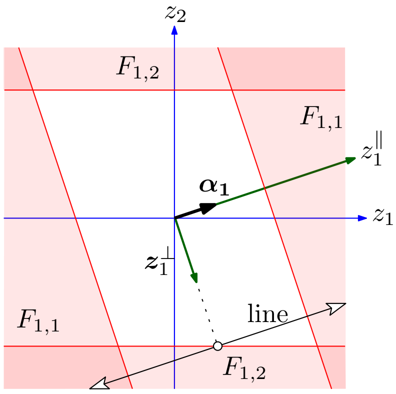

Figure 2: Application of Line Sampling for estimating effective contribution associated with $p_{1,1}$.

Instead, the summation is estimated resorting to simulation by introducing weights $\omega_{i,k}$, that is:

$$p_F(\mathbf{y}) = \sum_{i=1}^{n_r} \sum_{k=1}^{n_t} \frac{1}{\omega_{i,k}} p_{i,k}(\mathbf{y}) \omega_{i,k}. \qquad (7)$$

The weights $\omega_{i,k}$ are chosen to be proportional to the probability of occurrence of the event $F_{i,k}$, as suggested in Schuëller and Stix (1987); Au and Beck (2001). Thus, Eq. (7) is calculated by first randomly sampling a pair (i,k) with probability proportional to the weights and then by estimating the effective contribution $p_{i,k}$ by means of Eq. (6) evaluated for a single random line. Repeating the procedure N times and averaging over the results obtained for the associated N lines, one can estimate both the failure probability and its coefficient of variation, as shown in detail in (Valdebenito et al., 2021b).

3.2. *Multidomain Line Sampling for Evaluation of the Gradient of the First Excursion Probability*

The gradient of the failure probability can be calculated by differentiating Eq. (7) with respect to y_q, $q = 1, \ldots, n_y$. That is:

$$\frac{\partial p_F(\mathbf{y})}{\partial y_q} = \sum_{i=1}^{n_r} \sum_{k=1}^{n_t} \frac{1}{\omega_{i,k}} \frac{\partial p_{i,k}(\mathbf{y})}{\partial y_q} \omega_{i,k}. \qquad (8)$$

Naturally, the challenge is to differentiate the effective contribution $p_{i,k}$ shown in Eq. (6) with respect to y_q. Such differentiation can be calculated with the help of Leibniz' rule, as explained in detail in Flanders (1973). After carrying out such differentiation, the problem reduces to determining the sensitivity of the responses of interest with respect to y_q. As the responses of interest are expressed in terms of a convolution integral (see Eq. (2)), one requires calculating the gradient of the impulse response functions h_i, $i = 1, \ldots, n_r$. As such functions are calculated using modal analysis, the task of calculating the gradient of the failure probability ultimately reduces to calculating the sensitivity of the spectral properties (that is, mode shapes and natural frequencies) of the structural system with respect to y_q. The latter is a problem which has been studied in depth in the literature, see e.g. Friswell and Mottershead (1995). In particular, in this contribution, the sensitivity of the spectral properties is calculated using the approach proposed in Lee and Jung (1997). Such an approach is quite advantageous from a numerical viewpoint, as it does not demand performing additional spectral analysis but just solving systems of linear equations associated with the structural matrices and its derivatives.

4. EXAMPLE

The application of multidomain Line Sampling for estimating the gradient of the first excursion probability is illustrated by means of an example involving a linear elastic structural system of two degrees-of-freedom, as depicted in Figure 3. The system possesses masses $m_1 = m_2 = 30000$ [kg], stiffnesses $k_1 = k_2 = 18 \times 10^6$ [N/m] and modal damping $d_1 = d_2 = 4\%$. A stochastic ground acceleration modelled as a discrete white noise process acts over the structure. Its spectral intensity, duration and time step are $S = 10^{-4}$ [m^2/s^3], $T = 15$ [s] and $\Delta t = 0.01$ [s], respectively; this leads to a problem involving $n_t = 1501$ time instants. The objective is estimating the sensitivity of the failure probability with respect to the stiffnesses k_1 and k_2. The failure event is described as the displacement of mass m_1 or m_2 or the relative displacement between masses m_1 and m_2 exceeding a threshold of 0.006 [m].

The estimates of the gradients of the first excursion probability obtained applying multidomain Line Sampling (mLS) are shown in Figure 4. This Figure presents in addition the results obtained with

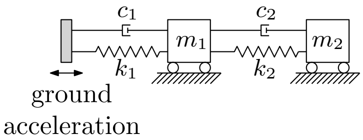

Figure 3: Two-degree-of-freedom system.

Directional Importance Sampling (DIS), which is another simulation approach that allows estimating the sought gradient (Valdebenito et al., 2021a). The obtained results show the evolution of the probability sensitivity estimators as a function of the number of samples of the stochastic ground acceleration. The results produced with mLS are superior to those produce with DIS. Indeed, the estimators produced with mLS are much more stable than those produced with DIS. Validation calculations show that the results obtained are in good agreement with reference results produced considering a large number of simulations.

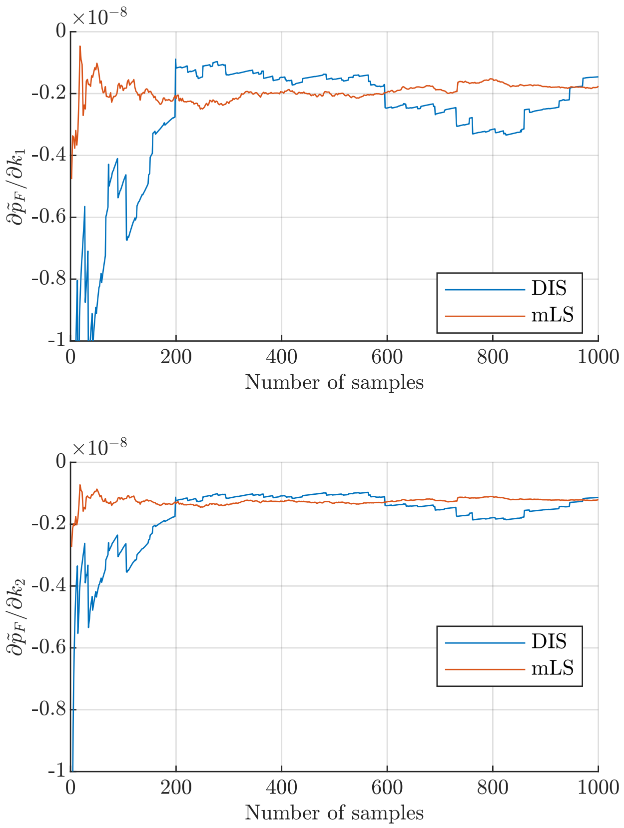

Figure 4: Estimators for the gradient of the first excursion probability.

5. CONCLUSIONS

The results shown in this paper suggest that the sensitivity of the first excursion probability can be calculated by means of multidomain Line Sampling (mLS). Furthermore, mLS seems to outperform Directional Importance Sampling (DIS). Future research efforts aim at validating the latter assertion in more general cases, involving general types of Gaussian excitation (other than discrete white noise processes) as well as more complex structural systems. These issues are currently being investigated by the authors.

6. ACKNOWLEDGMENTS

The research reported here was partially supported by ANID (National Agency for Research and Development, Chile) and DAAD (German Academic Exchange Service) under CONICYT-PFCHA/Doctorado Acuerdo Bilateral DAAD Becas Chile/2018-62180007. This support is gratefully acknowledged by the authors.

7. REFERENCES

Au, S. and Beck, J. (2001). "First excursion probabilities for linear systems by very efficient importance sampling." *Probabilistic Engineering Mechanics*, 16(3), 193–207.

Bathe, K. (1996). *Finite Element Procedures*. Prentice Hall, New Jersey.

Chopra, A. (1995). *Dynamics of structures: theory and applications to earthquake engineering*. Prentice Hall.

Der Kiureghian, A. (2000). "The geometry of random vibrations and solutions by FORM and SORM." *Probabilistic Engineering Mechanics*, 15(1), 81–90.

Flanders, H. (1973). "Differentiation under the integral sign." *The American Mathematical Monthly*, 80(6), 615–627.

Friswell, M. and Mottershead, J. (1995). *Finite element model updating in structural dynamics*. Kluwer Academic Publishers.

Haftka, R. and Gürdal, Z. (1992). *Elements of Structural Optimization*. Kluwer, Dordrecht, The Netherlands, 3rd edition.

Katafygiotis, L. and Cheung, S. (2006). "Domain decomposition method for calculating the failure probability of linear dynamic systems subjected to Gaussian stochastic loads." *Journal of Engineering Mechanics*, 132(5), 475–486.

Koutsourelakis, P., Pradlwarter, H., and Schuëller, G. (2004). "Reliability of structures in high dimensions, part I: Algorithms and applications." *Probabilistic Engineering Mechanics*, 19(4), 409–417.

Lee, I.-W. and Jung, G.-H. (1997). "An efficient algebraic method for the computation of natural frequency and mode shape sensitivities - part I. distinct natural frequencies." *Computers & Structures*, 62(3), 429–435.

Lu, Z., Song, S., Yue, Z., and Wang, J. (2008). "Reliability sensitivity method by line sampling." *Structural Safety*, 30(6), 517–532.

Misraji, M., Valdebenito, M., Jensen, H., and Mayorga, C. (2020). "Application of directional importance sampling for estimation of first excursion probabilities of linear structural systems subject to stochastic Gaussian loading." *Mechanical Systems and Signal Processing*, 139, 106621.

Papaioannou, I., Breitung, K., and Straub, D. (2018). "Reliability sensitivity estimation with sequential importance sampling." *Structural Safety*, 75, 24 – 34.

Schuëller, G. and Pradlwarter, H. (2007). "Benchmark study on reliability estimation in higher dimensions of structural systems – An overview." *Structural Safety*, 29, 167–182.

Schuëller, G. and Stix, R. (1987). "A critical appraisal of methods to determine failure probabilities." *Structural Safety*, 4(4), 293–309.

Soong, T. and Grigoriu, M. (1993). *Random Vibration of Mechanical and Structural Systems*. Prentice Hall, Englewood Cliffs, New Jersey.

Valdebenito, M., Misraji, M., Jensen, H., and Mayorga, C. (2021a). "Sensitivity estimation of first excursion probabilities of linear structures subject to stochastic gaussian loading." *Computers & Structures*, 248, 106482.

Valdebenito, M., Wei, P., Song, J., Beer, M., and Broggi, M. (2021b). "Failure probability estimation of a class of series systems by multidomain Line Sampling." *Reliability Engineering & System Safety*, 213, 107673.