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Scaling dimensions from the mirror TBA

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Abstract

The mirror TBA equations proposed by Arutyunov, Suzuki and the author are solved numerically up to 't Hooft's coupling $\lambda \approx 2340$ for several two-particle states dual to $\mathcal{N}=4$ SYM operators from the $\mathfrak{sl}(2)$ sector. The data obtained for states with mode numbers $n=1,2,3,4$ is used to propose a general charge J dependent formula for the first nonvanishing subleading coefficient in the strong coupling expansion of scaling dimensions. In addition we find that the first critical and subcritical values of the coupling for the $J=4, n=1$ operator are at $\lambda \approx 133$ and $\lambda \approx 190$, respectively.

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Contents

1 Introduction and Summary

The mirror Thermodynamic Bethe Ansatz (TBA) is an efficient tool to analyze energies of the light-cone $\mathrm{AdS}_5 \times \mathrm{S}^5$ string states, and through the AdS/CFT correspondence [1], scaling dimensions of dual primary operators in planar $\mathcal{N} = 4$ SYM. The mirror TBA originally proposed to determine the finite-size spectrum of two-dimensional relativistic integrable models [2, 3, 4] reformulates the spectral problem in terms of thermodynamics of a so-called mirror theory [5] related to the original model by a double-Wick rotation. The $\mathrm{AdS}_5 \times \mathrm{S}^5$ mirror model was studied in detail in [5] where, in particular, the bound state spectrum and the mirror form of the Bethe-Yang (BY) equations of [6], necessary to formulate the string hypothesis for the $\mathrm{AdS}_5 \times \mathrm{S}^5$ mirror model [7], were determined. This opened a way to derive the ground state TBA equations [8, 9], and to construct excited state equations for string states with real [10]-[13] and complex momenta [14]. This was done by using the contour deformation trick, a procedure inspired by [3, 4, 15], and further developed in [11, 16]. The TBA equations were used to analyze various aspects of the finite-size spectrum. It was shown in [17] that at the large 't Hooft coupling λ energies of semi-classical string states found by using the mirror TBA agree with explicit string theory calculations. The scaling dimension of the Konishi operator was determined up to five loops and shown [18, 19, 12] to agree with Lüscher's corrections [20]-[23] and at four loops with explicit field-theoretic computations [24, 25]. The TBA equations for the Konishi operator were also solved numerically for intermediate values of the coupling [26, 27] and the results obtained agree with various string theory considerations [28]-[31].

In this paper the mirror TBA equations of [11] are solved numerically for several two-particle states dual to $\mathcal{N} = 4$ SYM operators from the $\mathfrak{sl}(2)$ sector with various values of the charge J and mode number n. For operators with $n \geq 2$ the scaling dimensions are found up to $\lambda \approx 2340$. The same code as in [27] is used here with minor modifications for the $J = 4, n = 1$ state beyond its (sub)critical value. The mode number n of a primary operator coincides with the string level of the dual string state [32], and at large values of λ one can expand the energy of the state in an asymptotic series in powers of $1/\sqrt[4]{n^2\lambda}$

$$E_{(J,n)}(\lambda) = c_{-1}\sqrt[4]{n^2\lambda} + c_0 + \frac{c_1}{\sqrt[4]{n^2\lambda}} + \frac{c_2}{\sqrt{n^2\lambda}} + \frac{c_3}{(n^2\lambda)^{3/4}} + \frac{c_4}{n^2\lambda} + \frac{c_5}{(n^2\lambda)^{5/4}} + \cdots\,, \quad (1)$$

where the coefficients c_i are in general nontrivial functions of J and n. The coefficient c_{-1} of the leading term should be in fact equal to 2 as follows from the spectrum of string theory in flat space [33] and asymptotic Bethe ansatz considerations [32]. The constant term c_0 is believed to vanish for two-particle states because it does in the free fermion model [34] describing the $\mathfrak{su}(1|1)$ sector in the semi-classical approximation and one does not expect getting quantum corrections to c_0 [35]. The subleading coefficients c_{2k-1} are supposed to have the following structure [34]

$$c_{2k-1} = -\frac{\left(-\frac{1}{4}\right)^k \Gamma\left(k-\frac{1}{2}\right)}{\sqrt{\pi}k!} J^{2k} + b_{2k-1}(J,n)\,, \quad (2)$$

where the J^{2k} term is fixed by the flat space spectrum, and b_{2k-1} is a polynomial of degree $2k-1$ (or less) in J with n dependent coefficients. In particular the first nonvanishing subleading coefficient c_1 is of the form $c_1 = J^2/4 + b_1(J,n)$ where b_1 may be a linear function of J. The mirror TBA prediction for the Konishi state with $J=2, n=1$ is $b_1(2,1) = 1$ [26], and it agrees with (incompletely justified) string computations [28]-[31]. In addition in the free fermion model one finds that b_1 is independent of both J and n and equal to 1/2 [34]. The formulae derived in the framework of the free fermion model definitely get quantum corrections but assuming that the J dependence of b_1 remains unchanged one immediately concludes that for any J one should have $b_1(J,1) = 1$. Our data for the $J=3, n=1$ and $J=4, n=1$ states indeed confirms the conclusion. Fitting the data for the $J=4, n=2$ and $J=5, n=2$ states we find that the formula also works fine: $b_1(J,2) = 1$. It is tempting to assume that the same formula might be valid for any (J,n) state. The analysis of the $J=6, n=3$ and $J=7, n=3$ states shows however that for $n=3$ the formula is different: $b_1(J,3) = 0$. Thus, the coefficient c_1 has a nontrivial n dependence. Assuming that $b_1(J,n)$ is independent of J and fitting the data for the $J=8, n=4$ state we find $b_1(J,4) = -2$. All these values of b_1 can be obtained from one simple formula $b_1(J,n) = n(3-n)/2$, and therefore

$$c_1 = \frac{J^2}{4} + \frac{n(3-n)}{2}\,, \quad n = 1,2,3,4\,. \quad (3)$$

It is clear that only an analytic derivation of c_1 can determine if this formula is valid for any J and n.

Coming back to the series (1), it was argued in [36] that the coefficient c_2 should vanish due to the high degree of supersymmetry of the model, and our data supports this. Recently by using a conjecture from [37] a formula for the coefficient c_3 was proposed in [38]. Our data for $n=1$ and $n=2$ states agrees with the formula. It would be interesting to see if there is a simple modification of the formula for higher values of n.

Nothing is known about other coefficients in (1). It is even unclear if the series expansion is in powers of $1/\sqrt[4]{\lambda}$ as follows from the free fermion model [34], or up to an overall factor of $\sqrt[4]{\lambda}$ it is in powers of $1/\sqrt{\lambda}$, that is $c_{2k} = 0$, as was assumed in [26]. If it is in powers of $1/\sqrt{\lambda}$ then this would imply that quantum corrections to the free fermion model expressions drastically change the structure of the strong coupling expansion. The precision of our computation is insufficient to come to a definite conclusion. Nevertheless using the formula of [38] and our data we find some evidence in favor of vanishing c_4 for the $n = 1$ states.

It is known [11] that two-particle states from the $\mathfrak{sl}(2)$ sector are divided into infinitely-many classes which differ by analytic properties of exact Y-functions and therefore by driving terms in the TBA equations. The analytic properties of Y-functions depend on the coupling and at critical values of λ a state moves from one class to another one. At weak coupling all the states we analyzed belong to the simplest Konishi-like class. The states with $n \geq 2$ remain in the class up to the largest value of λ the TBA equations were solved. The $J = 3, n = 1$ and $J = 4, n = 1$ states however have first critical values at $\lambda \approx 950$ and $\lambda \approx 133$, respectively. The values were obtained by interpolating the data because the iterations stopped to converge for λ's close to the critical values. For the $J = 3, n = 1$ state we could solve the equations only up to $\lambda \approx 540$ which is pretty far from its critical value. For the $J = 4, n = 1$ state the equations were solved up to $\lambda \approx 105$, and then in accordance with [11] we changed the TBA equations, jumped beyond the subcritical value to $\lambda \approx 191$ and resumed the iterations. The iterations however stopped to converge at $\lambda \approx 483$, and we do not really understand a reason.

The paper is organized as follows. In the next section we present the analysis of the results of the numerical solution of the TBA equations for the states. In section 3 our data for the states is collected.

2 Fitting the numerical data

2.1 $J = 3, n = 1$ operator

In the table (45) we present the data for the $J = 3, n = 1$ state. Since we could solve the TBA equations only up to $g = 3.7$ to fit the data we should make assumptions about the structure of the strong coupling expansion. Assuming that $c_{-1} = 2$ and $c_0 = 0$ and fitting the data in the interval $g \in [g_0, 3.7]$ we get

g_0	λ_0	Fit
1.2	56.8489	$2\sqrt[4]{\lambda} + \frac{3.1943}{\sqrt[4]{\lambda}} + \frac{0.831811}{\sqrt{\lambda}} - \frac{6.99044}{\lambda^{3/4}} + \frac{13.106}{\lambda} - \frac{2.08406}{\lambda^{5/4}}$
1.3	66.7185	$2\sqrt[4]{\lambda} + \frac{3.21575}{\sqrt[4]{\lambda}} + \frac{0.508957}{\sqrt{\lambda}} - \frac{5.18593}{\lambda^{3/4}} + \frac{8.66792}{\lambda} + \frac{1.96869}{\lambda^{5/4}}$
1.4	77.3777	$2\sqrt[4]{\lambda} + \frac{3.21158}{\sqrt[4]{\lambda}} + \frac{0.572881}{\sqrt{\lambda}} - \frac{5.55021}{\lambda^{3/4}} + \frac{9.58264}{\lambda} + \frac{1.11474}{\lambda^{5/4}}$
1.5	88.8264	$2\sqrt[4]{\lambda} + \frac{3.16939}{\sqrt[4]{\lambda}} + \frac{1.23048}{\sqrt{\lambda}} - \frac{9.36522}{\lambda^{3/4}} + \frac{19.3462}{\lambda} - \frac{8.1862}{\lambda^{5/4}}$

(4)

For the $J = 3, n = 1$ state we expect $c_1 = J^2/4 + 1 = 3.25$ and indeed the values we get from the fitting are very close to it. Fixing then $c_1 = 3.25$ one gets

g_0	λ_0	Fit
1.2	56.8489	$2\sqrt[4]{\lambda} + \frac{3.25}{\sqrt[4]{\lambda}} + \frac{0.0297414}{\sqrt{\lambda}} - \frac{2.71216}{\lambda^{3/4}} + \frac{3.08535}{\lambda} + \frac{6.61514}{\lambda^{5/4}}$
1.3	66.7185	$2\sqrt[4]{\lambda} + \frac{3.25}{\sqrt[4]{\lambda}} + \frac{0.00509108}{\sqrt{\lambda}} - \frac{2.43564}{\lambda^{3/4}} + \frac{2.06532}{\lambda} + \frac{7.85264}{\lambda^{5/4}}$
1.4	77.3777	$2\sqrt[4]{\lambda} + \frac{3.25}{\sqrt[4]{\lambda}} - \frac{0.00358391}{\sqrt{\lambda}} - \frac{2.33641}{\lambda^{3/4}} + \frac{1.6914}{\lambda} + \frac{8.31688}{\lambda^{5/4}}$
1.5	88.8264	$2\sqrt[4]{\lambda} + \frac{3.25}{\sqrt[4]{\lambda}} - \frac{0.00135757}{\sqrt{\lambda}} - \frac{2.36233}{\lambda^{3/4}} + \frac{1.79102}{\lambda} + \frac{8.19057}{\lambda^{5/4}}$

(5)

We see that the coefficient c_2 becomes very small and we set it to 0: $c_2 = 0$

g_0	λ_0	Fit
1.2	56.8489	$2\sqrt[4]{\lambda} + \frac{3.25}{\sqrt[4]{\lambda}} - \frac{2.39691}{\lambda^{3/4}} + \frac{1.99012}{\lambda} + \frac{7.86361}{\lambda^{5/4}}$
1.3	66.7185	$2\sqrt[4]{\lambda} + \frac{3.25}{\sqrt[4]{\lambda}} - \frac{2.38036}{\lambda^{3/4}} + \frac{1.8682}{\lambda} + \frac{8.08376}{\lambda^{5/4}}$
1.4	77.3777	$2\sqrt[4]{\lambda} + \frac{3.25}{\sqrt[4]{\lambda}} - \frac{2.37618}{\lambda^{3/4}} + \frac{1.83666}{\lambda} + \frac{8.14215}{\lambda^{5/4}}$
1.5	88.8264	$2\sqrt[4]{\lambda} + \frac{3.25}{\sqrt[4]{\lambda}} - \frac{2.37771}{\lambda^{3/4}} + \frac{1.8484}{\lambda} + \frac{8.11991}{\lambda^{5/4}}$

(6)

According to the conjecture of [38] the coefficient c_3 for the states with $n = 1$ is equal to

$$c_3 = -\frac{J^4}{64} + \frac{3J^2}{8} - 3\zeta(3) - \frac{3}{4}\,, \tag{7}$$

and therefore for $J = 3$, $c_3 \approx -2.2468$. We see that the number is indeed very close to the one we get from the fit. Fixing the coefficient to this value and adding more terms to the expansion, one gets

g_0	λ_0	Fit
1.2	56.8489	$2\sqrt[4]{\lambda} + \frac{3.25}{\sqrt[4]{\lambda}} - \frac{2.2468}{\lambda^{3/4}} + \frac{0.363256}{\lambda} + \frac{13.8122}{\lambda^{5/4}} - \frac{7.89879}{\lambda^{3/2}} + \frac{1.96167}{\lambda^{7/4}}$
1.3	66.7185	$2\sqrt[4]{\lambda} + \frac{3.25}{\sqrt[4]{\lambda}} - \frac{2.2468}{\lambda^{3/4}} + \frac{0.0493902}{\lambda} + \frac{17.2428}{\lambda^{5/4}} - \frac{20.231}{\lambda^{3/2}} + \frac{16.5518}{\lambda^{7/4}}$
1.4	77.3777	$2\sqrt[4]{\lambda} + \frac{3.25}{\sqrt[4]{\lambda}} - \frac{2.2468}{\lambda^{3/4}} - \frac{0.0985827}{\lambda} + \frac{18.8979}{\lambda^{5/4}} - \frac{26.3304}{\lambda^{3/2}} + \frac{23.9612}{\lambda^{7/4}}$
1.5	88.8264	$2\sqrt[4]{\lambda} + \frac{3.25}{\sqrt[4]{\lambda}} - \frac{2.2468}{\lambda^{3/4}} - \frac{0.0771042}{\lambda} + \frac{18.6526}{\lambda^{5/4}} - \frac{25.4057}{\lambda^{3/2}} + \frac{22.8107}{\lambda^{7/4}}$

(8)

The outcome of this fitting is very interesting because the coefficient c_4 becomes very small and this implies that up to the overall $\sqrt[4]{\lambda}$ the expansion may indeed be in powers of $1/\sqrt{\lambda}$.[1] Finally setting $c_{2k} = 0$ one gets the following fitting for the $J = 3, n = 1$ state

g_0	λ_0	Fit
1.2	56.8489	$1.99982\sqrt[4]{\lambda} + \frac{3.25805}{\sqrt[4]{\lambda}} - \frac{2.30759}{\lambda^{3/4}} + \frac{13.7801}{\lambda^{5/4}} - \frac{11.3876}{\lambda^{7/4}}$
1.3	66.7185	$1.9999\sqrt[4]{\lambda} + \frac{3.25325}{\sqrt[4]{\lambda}} - \frac{2.20648}{\lambda^{3/4}} + \frac{12.8714}{\lambda^{5/4}} - \frac{8.44329}{\lambda^{7/4}}$
1.4	77.3777	$1.99988\sqrt[4]{\lambda} + \frac{3.25475}{\sqrt[4]{\lambda}} - \frac{2.23931}{\lambda^{3/4}} + \frac{13.1794}{\lambda^{5/4}} - \frac{9.49031}{\lambda^{7/4}}$
1.5	88.8264	$1.9997\sqrt[4]{\lambda} + \frac{3.2658}{\sqrt[4]{\lambda}} - \frac{2.48916}{\lambda^{3/4}} + \frac{15.6166}{\lambda^{5/4}} - \frac{18.1459}{\lambda^{7/4}}$

(9)

[1] One can check by using the data from [27] that a similar phenomenon also happens for the Konishi operator. For example fitting the data in the interval $g \in [1.8, 5.]$ with $c_{-1} = c_1 = 2, c_0 = c_2 = 0, c_3 = \frac{1}{2} - 3\zeta(3) \approx 3.10617$, one gets

$$E_K = 2\sqrt[4]{\lambda} + \frac{2}{\sqrt[4]{\lambda}} - \frac{3.10617}{\lambda^{3/4}} + \frac{0.0175158}{\lambda} + \frac{19.2403}{\lambda^{5/4}} - \frac{39.3787}{\lambda^{3/2}} + \frac{36.5992}{\lambda^{7/4}}\,,$$

which gives additional evidence in favor of the $1/\sqrt{\lambda}$ expansion at least for the $n = 1$ operators.

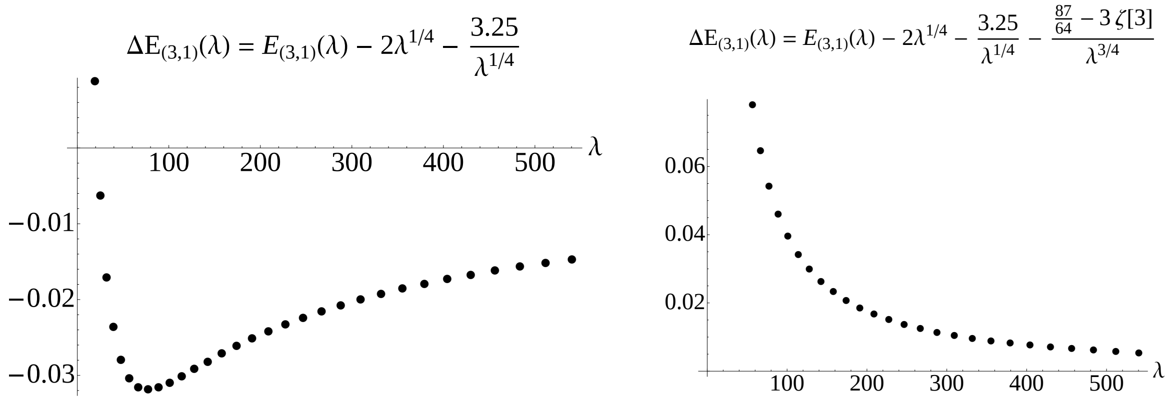

Figure 1: Black dots represent the difference between the numerical solution and conjectured asymptotic expansions.

In Figure 1 we plot the difference between the numerical solution and its large λ asymptotics $2\lambda^{1/4} + \frac{3.25}{\lambda^{1/4}}$ and $2\lambda^{1/4} + \frac{3.25}{\lambda^{1/4}} - \frac{2.2468}{\lambda^{3/4}}$.

2.2 $J = 4, n = 1$ operator

In the table (46) we present the data for the $J = 4, n = 1$ state. For this operator the data is known only up to $g = 3.4$ with a gap between $g = 1.63$ and $g = 2.2$. The operator has a critical point at $g \approx 1.83$ and a subcritical one at $g \approx 2.19$ and our iterations did not converge for $g \in [1.64, 2.1]$. By this reason we fit the data in the interval $g \in [g_0, 3.4]$ with the assumption that $c_{2k} = 0$

g_0	λ_0	Fit
0.9	31.9775	$2.00114\sqrt[4]{\lambda} + \frac{4.96572}{\sqrt[4]{\lambda}} - \frac{2.24203}{\lambda^{3/4}} + \frac{21.2561}{\lambda^{5/4}} - \frac{52.8464}{\lambda^{7/4}}$
1.	39.4784	$2.00071\sqrt[4]{\lambda} + \frac{4.98755}{\sqrt[4]{\lambda}} - \frac{2.63256}{\lambda^{3/4}} + \frac{24.1832}{\lambda^{5/4}} - \frac{60.6194}{\lambda^{7/4}}$
1.1	47.7689	$1.9995\sqrt[4]{\lambda} + \frac{5.05089}{\sqrt[4]{\lambda}} - \frac{3.81607}{\lambda^{3/4}} + \frac{33.5122}{\lambda^{5/4}} - \frac{86.8559}{\lambda^{7/4}}$
1.2	56.8489	$1.99575\sqrt[4]{\lambda} + \frac{5.25258}{\sqrt[4]{\lambda}} - \frac{7.71075}{\lambda^{3/4}} + \frac{65.4442}{\lambda^{5/4}} - \frac{180.907}{\lambda^{7/4}}$

(10)

We see that c_{-1} is indeed close to 2, and $c_1 \approx 5$ as expected for the $J = 4, n = 1$ state. Let us then fix $c_{-1} = 2, c_1 = 5$

g_0	λ_0	Fit
0.9	31.9775	$2\sqrt[4]{\lambda} + \frac{5}{\sqrt[4]{\lambda}} - \frac{2.51332}{\lambda^{3/4}} + \frac{21.4871}{\lambda^{5/4}} - \frac{50.4063}{\lambda^{7/4}}$
1.	39.4784	$2\sqrt[4]{\lambda} + \frac{5}{\sqrt[4]{\lambda}} - \frac{2.49341}{\lambda^{3/4}} + \frac{21.0556}{\lambda^{5/4}} - \frac{48.2863}{\lambda^{7/4}}$
1.1	47.7689	$2\sqrt[4]{\lambda} + \frac{5}{\sqrt[4]{\lambda}} - \frac{2.47161}{\lambda^{3/4}} + \frac{20.5437}{\lambda^{5/4}} - \frac{45.5455}{\lambda^{7/4}}$
1.2	56.8489	$2\sqrt[4]{\lambda} + \frac{5}{\sqrt[4]{\lambda}} - \frac{2.44817}{\lambda^{3/4}} + \frac{19.9559}{\lambda^{5/4}} - \frac{42.1681}{\lambda^{7/4}}$

(11)

According to [38] the coefficient c_3 for the $J = 4, n = 1$ state should be equal to $c_3 = \frac{5}{4} - 3\zeta(3) \approx -2.35617$, and it indeed agrees well with the one we get from the fit. In Figure 2 we plot the difference between the numerical solution and its large λ asymptotics $2\lambda^{1/4} + \frac{5}{\lambda^{1/4}}$ and $2\lambda^{1/4} + \frac{5}{\lambda^{1/4}} - \frac{2.35617}{\lambda^{3/4}}$.

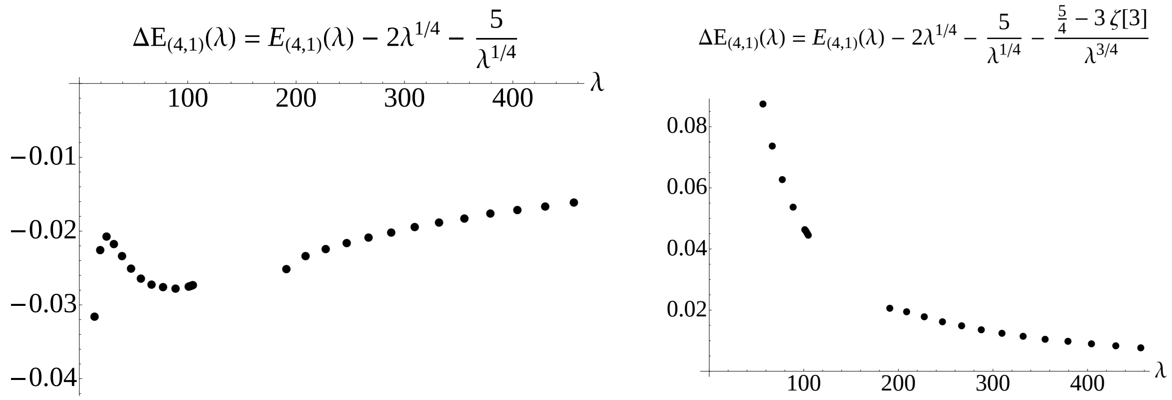

Figure 2: Black dots represent the difference between the numerical solution and conjectured asymptotic expansions.

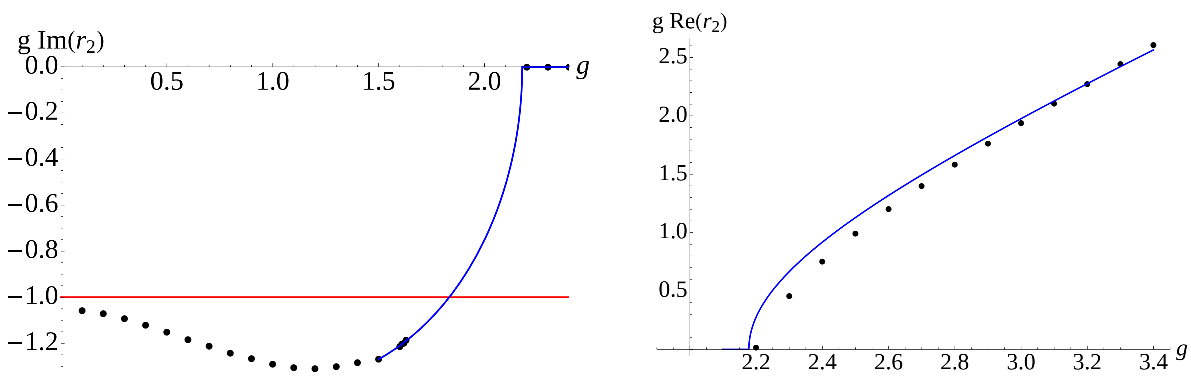

Figure 3: On the left and right figures the graphs of the imaginary and real parts of the rescaled root r_2 together with the fitting function (blue curves) $-1.25844\, i\, \sqrt{g(2.17804-g)}$ are shown.

To determine the values of the critical g_{cr} and subcritical $\bar{g}_{cr}$ points we found the solutions to the equation $Y_{1|vw}(r_2 + \frac{i}{g}) = -1$, where r_2 is also a root of $Y_{2|vw}$. The results are collected in Table 12 and shown in Figure 3.

g	r_2	g	r_2	g	r_2	g	r_2	g	r_2
0.1	$-10.562i$	0.8	$-1.55116i$	1.5	$-0.84559i$	2.2	0.0087264	2.9	0.608164
0.2	$-5.34994i$	0.9	$-1.40522i$	1.6	$-0.75899i$	2.3	0.199508	3.	0.646168
0.3	$-3.63868i$	1.	$-1.29021i$	1.61	$-0.74687i$	2.4	0.314217	3.1	0.679729
0.4	$-2.7981i$	1.1	$-1.18521i$	1.62	$-0.739759i$	2.5	0.397187	3.2	0.711265
0.5	$-2.30147i$	1.2	$-1.09082i$	1.63	$-0.727045i$	2.6	0.463024	3.3	0.740638
0.6	$-1.97299i$	1.3	$-1.00062i$			2.7	0.519	3.4	0.767281
0.7	$-1.73166i$	1.4	$-0.91655i$			2.8	0.565948		

(12)

The root r_2 is purely imaginary for $g \leq 2.1$ and real for $g \geq 2.2$. It vanishes at $g = \bar{g}_{cr}$

Fitting our data in the interval $[1.5, 16.3]$ to the function $c\sqrt{\frac{g-\bar{g}_{cr}}{g}}$, we get

$$r_2(g) \sim -1.25844\, i\sqrt{\frac{2.17804-g}{g}}\,. \tag{13}$$

The subcritical value obtained from this fitting agrees very well with the data in Table 12 which shows that it should be $\bar{g}_{cr} \approx 2.19$. To find the critical value we use the fitting function and solve the equation $r_2(g) = -i/g$. This gives $g_{cr} \approx 1.83$.

2.3 $J = 4, n = 2$ operator

In the table (47) we present the results of the computation of the energy of the $J = 4, n = 2$ state. Fitting the data in the interval $g \in [g_0, 7.7]$ we get

g_0	λ_0	Fit
1.6	101.065	$2.01324\sqrt[4]{\Lambda} - 0.599655 + \frac{16.0334}{\sqrt[4]{\Lambda}} - \frac{106.352}{\sqrt{\Lambda}} + \frac{535.582}{\Lambda^{3/4}} - \frac{1548.42}{\Lambda} + \frac{1884.37}{\Lambda^{5/4}}$
1.7	114.093	$2.00765\sqrt[4]{\Lambda} - 0.364641 + \frac{11.9675}{\sqrt[4]{\Lambda}} - \frac{69.3126}{\sqrt{\Lambda}} + \frac{348.21}{\Lambda^{3/4}} - \frac{1049.34}{\Lambda} + \frac{1337.51}{\Lambda^{5/4}}$
1.8	127.91	$2.00248\sqrt[4]{\Lambda} - 0.144622 + \frac{8.11028}{\sqrt[4]{\Lambda}} - \frac{33.6705}{\sqrt{\Lambda}} + \frac{165.145}{\Lambda^{3/4}} - \frac{553.8}{\Lambda} + \frac{785.161}{\Lambda^{5/4}}$
1.9	142.517	$1.99906\sqrt[4]{\Lambda} + 0.00296822 + \frac{5.49016}{\sqrt[4]{\Lambda}} - \frac{9.13276}{\sqrt{\Lambda}} + \frac{37.2992}{\Lambda^{3/4}} - \frac{202.443}{\Lambda} + \frac{387.196}{\Lambda^{5/4}}$

(14)

where the expansion parameter $\Lambda = n^2\lambda = 4\lambda$ is introduced. One sees that c_{-1} is very close to 2 as expected, and fixing $c_{-1} = 2$ one gets

g_0	λ_0	Fit
1.6	101.065	$2\sqrt[4]{\Lambda} - 0.0636918 + \frac{7.12558}{\sqrt[4]{\Lambda}} - \frac{28.5421}{\sqrt{\Lambda}} + \frac{158.802}{\Lambda^{3/4}} - \frac{589.219}{\Lambda} + \frac{881.06}{\Lambda^{5/4}}$
1.7	114.093	$2\sqrt[4]{\Lambda} - 0.0505662 + \frac{6.66846}{\sqrt[4]{\Lambda}} - \frac{22.2713}{\sqrt{\Lambda}} + \frac{116.455}{\Lambda^{3/4}} - \frac{448.43}{\Lambda} + \frac{696.68}{\Lambda^{5/4}}$
1.8	127.91	$2\sqrt[4]{\Lambda} - 0.0413324 + \frac{6.34268}{\sqrt[4]{\Lambda}} - \frac{17.7387}{\sqrt{\Lambda}} + \frac{85.3727}{\Lambda^{3/4}} - \frac{343.379}{\Lambda} + \frac{556.661}{\Lambda^{5/4}}$
1.9	142.517	$2\sqrt[4]{\Lambda} - 0.0367516 + \frac{6.17907}{\sqrt[4]{\Lambda}} - \frac{15.4318}{\sqrt{\Lambda}} + \frac{69.3241}{\Lambda^{3/4}} - \frac{288.293}{\Lambda} + \frac{482.019}{\Lambda^{5/4}}$

(15)

We see that the coefficient c_0 becomes small and we set it to 0: $c_0 = 0$

g_0	λ_0	Fit
1.6	101.065	$2\sqrt[4]{\Lambda} + \frac{5.0046}{\sqrt[4]{\Lambda}} - \frac{0.795942}{\sqrt{\Lambda}} - \frac{19.4612}{\Lambda^{3/4}} - \frac{26.4872}{\Lambda} + \frac{182.368}{\Lambda^{5/4}}$
1.7	114.093	$2\sqrt[4]{\Lambda} + \frac{4.95929}{\sqrt[4]{\Lambda}} + \frac{0.454694}{\sqrt{\Lambda}} - \frac{32.1551}{\Lambda^{3/4}} + \frac{29.665}{\Lambda} + \frac{90.9888}{\Lambda^{5/4}}$
1.8	127.91	$2\sqrt[4]{\Lambda} + \frac{4.92584}{\sqrt[4]{\Lambda}} + \frac{1.39108}{\sqrt{\Lambda}} - \frac{41.8075}{\Lambda^{3/4}} + \frac{73.0916}{\Lambda} + \frac{19.0113}{\Lambda^{5/4}}$
1.9	142.517	$2\sqrt[4]{\Lambda} + \frac{4.90238}{\sqrt[4]{\Lambda}} + \frac{2.05656}{\sqrt{\Lambda}} - \frac{48.7676}{\Lambda^{3/4}} + \frac{104.906}{\Lambda} - \frac{34.6303}{\Lambda^{5/4}}$

(16)

This fitting shows that $c_1 \approx 5$, and assuming that b_1 is independent of J one concludes that for any two-particle $n = 2$ state we should expect the same formula as for the $n = 1$ states: $c_1(J, 2) = J^2/4 + 1$. Setting $c_1 = 5$, one gets

g_0	λ_0	Fit
1.6	101.065	$2\sqrt[4]{\Lambda} + \frac{5}{\sqrt[4]{\Lambda}} - \frac{0.675854}{\sqrt{\Lambda}} - \frac{20.6105}{\Lambda^{3/4}} - \frac{21.7064}{\Lambda} + \frac{175.066}{\Lambda^{5/4}}$
1.7	114.093	$2\sqrt[4]{\Lambda} + \frac{5}{\sqrt[4]{\Lambda}} - \frac{0.626357}{\sqrt{\Lambda}} - \frac{21.6187}{\Lambda^{3/4}} - \frac{15.0422}{\Lambda} + \frac{160.75}{\Lambda^{5/4}}$
1.8	127.91	$2\sqrt[4]{\Lambda} + \frac{5}{\sqrt[4]{\Lambda}} - \frac{0.608926}{\sqrt{\Lambda}} - \frac{21.9793}{\Lambda^{3/4}} - \frac{12.6155}{\Lambda} + \frac{155.435}{\Lambda^{5/4}}$
1.9	142.517	$2\sqrt[4]{\Lambda} + \frac{5}{\sqrt[4]{\Lambda}} - \frac{0.614803}{\sqrt{\Lambda}} - \frac{21.856}{\Lambda^{3/4}} - \frac{13.4596}{\Lambda} + \frac{157.318}{\Lambda^{5/4}}$

(17)

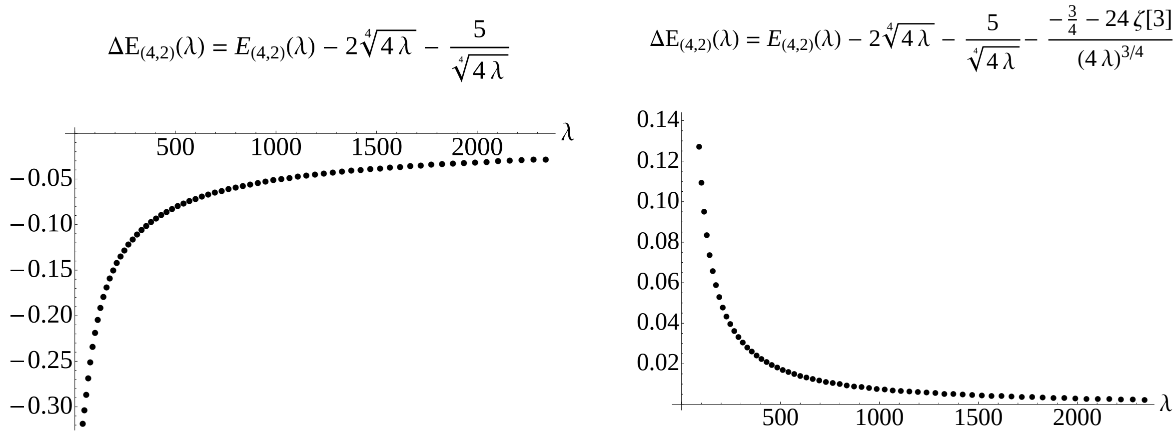

Figure 4: Black dots represent the difference between the numerical solution and conjectured asymptotic expansions.

The coefficient c_2 is not really small and we cannot reliably conclude that it vanishes. The contribution of the corresponding term is however smaller than the contribution of the next term and we believe that increasing the precision of the computation one would show that $c_2 = 0$.

Assuming that $c_{2k} = 0$ one gets the following fitting

g_0	λ_0	Fit
1.6	101.065	$2.00005\sqrt[4]{\Lambda} + \frac{4.96322}{\sqrt[4]{\Lambda}} - \frac{26.0062}{\Lambda^{3/4}} + \frac{136.487}{\Lambda^{5/4}} + \frac{65.3371}{\Lambda^{7/4}}$
1.7	114.093	$1.99999\sqrt[4]{\Lambda} + \frac{4.97512}{\sqrt[4]{\Lambda}} - \frac{26.8411}{\Lambda^{3/4}} + \frac{160.533}{\Lambda^{5/4}} - \frac{175.699}{\Lambda^{7/4}}$
1.8	127.91	$1.99995\sqrt[4]{\Lambda} + \frac{4.98407}{\sqrt[4]{\Lambda}} - \frac{27.4881}{\Lambda^{3/4}} + \frac{179.837}{\Lambda^{5/4}} - \frac{377.218}{\Lambda^{7/4}}$
1.9	142.517	$1.99992\sqrt[4]{\Lambda} + \frac{4.99048}{\sqrt[4]{\Lambda}} - \frac{27.9635}{\Lambda^{3/4}} + \frac{194.497}{\Lambda^{5/4}} - \frac{536.145}{\Lambda^{7/4}}$

, (18)

which also agrees very well with $c_1 = 5$. The coefficient c_3 for the states with $n = 2$ is conjectured [38] to be equal to

$$c_3 = -\frac{J^4}{64} + \frac{3J^2}{8} - 24\zeta(3) - \frac{11}{4}\,, \tag{19}$$

and for $J = 4$, $c_3 \approx -29.5994$. We see that the number is indeed close to the one we get from the fit. The agreement with the conjectured value of c_3 becomes even more evident if one sets $c_{-1} = 2, c_1 = 5$

g_0	λ_0	Fit
1.6	101.065	$2\sqrt[4]{\Lambda} + \frac{5}{\sqrt[4]{\Lambda}} - \frac{29.8997}{\Lambda^{3/4}} + \frac{269.391}{\Lambda^{5/4}} - \frac{1346.11}{\Lambda^{7/4}}$
1.7	114.093	$2\sqrt[4]{\Lambda} + \frac{5}{\sqrt[4]{\Lambda}} - \frac{30.0467}{\Lambda^{3/4}} + \frac{281.386}{\Lambda^{5/4}} - \frac{1561.66}{\Lambda^{7/4}}$
1.8	127.91	$2\sqrt[4]{\Lambda} + \frac{5}{\sqrt[4]{\Lambda}} - \frac{30.1745}{\Lambda^{3/4}} + \frac{292.273}{\Lambda^{5/4}} - \frac{1767.33}{\Lambda^{7/4}}$
1.9	142.517	$2\sqrt[4]{\Lambda} + \frac{5}{\sqrt[4]{\Lambda}} - \frac{30.2888}{\Lambda^{3/4}} + \frac{302.41}{\Lambda^{5/4}} - \frac{1967.94}{\Lambda^{7/4}}$

(20)

In Figure 4 we plot the difference between the numerical solution and its large λ asymptotics $2(4\lambda)^{1/4} + \frac{5}{(4\lambda)^{1/4}}$ and $2(4\lambda)^{1/4} + \frac{5}{(4\lambda)^{1/4}} - \frac{29.6}{(4\lambda)^{3/4}}$.

2.4 $J = 5, n = 2$ operator

In the table (48) we present the results of the computation of the energy of the $J = 5, n = 2$ state. Fitting the data in the interval $g \in [g_0, 7.7]$ we get

g_0	λ_0	Fit
1.6	101.065	$2.00445\sqrt[4]{\Lambda} - 0.20912 + \frac{11.1966}{\sqrt[4]{\Lambda}} - \frac{38.8728}{\sqrt{\Lambda}} + \frac{178.449}{\Lambda^{3/4}} - \frac{577.756}{\Lambda} + \frac{881.1}{\Lambda^{5/4}}$
1.7	114.093	$2.00546\sqrt[4]{\Lambda} - 0.251818 + \frac{11.9353}{\sqrt[4]{\Lambda}} - \frac{45.6022}{\sqrt{\Lambda}} + \frac{212.492}{\Lambda^{3/4}} - \frac{668.43}{\Lambda} + \frac{980.454}{\Lambda^{5/4}}$
1.8	127.91	$2.00297\sqrt[4]{\Lambda} - 0.145733 + \frac{10.0755}{\sqrt[4]{\Lambda}} - \frac{28.4169}{\sqrt{\Lambda}} + \frac{124.224}{\Lambda^{3/4}} - \frac{429.497}{\Lambda} + \frac{714.13}{\Lambda^{5/4}}$
1.9	142.517	$1.99987\sqrt[4]{\Lambda} - 0.0122334 + \frac{7.70547}{\sqrt[4]{\Lambda}} - \frac{6.22177}{\sqrt{\Lambda}} + \frac{8.58421}{\Lambda^{3/4}} - \frac{111.684}{\Lambda} + \frac{354.159}{\Lambda^{5/4}}$

(21)

where $\Lambda = n^2\lambda = 4\lambda$ is the same expansion parameter as for the $J = 4, n = 2$ state. One sees that c_{-1} is very close to 2 as expected, and fixing $c_{-1} = 2$ one gets

g_0	λ_0	Fit
1.6	101.065	$2\sqrt[4]{\Lambda} - 0.0290362 + \frac{8.20356}{\sqrt[4]{\Lambda}} - \frac{12.7288}{\sqrt{\Lambda}} + \frac{51.8515}{\Lambda^{3/4}} - \frac{255.467}{\Lambda} + \frac{543.99}{\Lambda^{5/4}}$
1.7	114.093	$2\sqrt[4]{\Lambda} - 0.0275291 + \frac{8.15107}{\sqrt[4]{\Lambda}} - \frac{12.0088}{\sqrt{\Lambda}} + \frac{46.9889}{\Lambda^{3/4}} - \frac{239.301}{\Lambda} + \frac{522.818}{\Lambda^{5/4}}$
1.8	127.91	$2\sqrt[4]{\Lambda} - 0.0221103 + \frac{7.95989}{\sqrt[4]{\Lambda}} - \frac{9.34883}{\sqrt{\Lambda}} + \frac{28.7486}{\Lambda^{3/4}} - \frac{177.652}{\Lambda} + \frac{440.648}{\Lambda^{5/4}}$
1.9	142.517	$2\sqrt[4]{\Lambda} - 0.0175866 + \frac{7.79832}{\sqrt[4]{\Lambda}} - \frac{7.07073}{\sqrt{\Lambda}} + \frac{12.9004}{\Lambda^{3/4}} - \frac{123.254}{\Lambda} + \frac{366.939}{\Lambda^{5/4}}$

(22)

We see that the coefficient c_0 is even smaller than it was for the $J = 4, n = 2$ state, and setting it to 0 one gets

g_0	λ_0	Fit
1.6	101.065	$2\sqrt[4]{\Lambda} + \frac{7.23663}{\sqrt[4]{\Lambda}} - \frac{0.0797212}{\sqrt{\Lambda}} - \frac{29.4165}{\Lambda^{3/4}} + \frac{1.07508}{\Lambda} + \frac{225.465}{\Lambda^{5/4}}$
1.7	114.093	$2\sqrt[4]{\Lambda} + \frac{7.22057}{\sqrt[4]{\Lambda}} + \frac{0.363655}{\sqrt{\Lambda}} - \frac{33.9167}{\Lambda^{3/4}} + \frac{20.9822}{\Lambda} + \frac{193.07}{\Lambda^{5/4}}$
1.8	127.91	$2\sqrt[4]{\Lambda} + \frac{7.20197}{\sqrt[4]{\Lambda}} + \frac{0.884427}{\sqrt{\Lambda}} - \frac{39.2849}{\Lambda^{3/4}} + \frac{45.134}{\Lambda} + \frac{153.039}{\Lambda^{5/4}}$
1.9	142.517	$2\sqrt[4]{\Lambda} + \frac{7.18739}{\sqrt[4]{\Lambda}} + \frac{1.29794}{\sqrt{\Lambda}} - \frac{43.6097}{\Lambda^{3/4}} + \frac{64.9025}{\Lambda} + \frac{119.708}{\Lambda^{5/4}}$

(23)

This fitting shows that c_1 agrees with $c_1(J, 2) = J^2/4 + 1$. Setting $c_1 = 7.25$, one gets

g_0	λ_0	Fit
1.6	101.065	$2\sqrt[4]{\Lambda} + \frac{7.25}{\sqrt[4]{\Lambda}} - \frac{0.428891}{\sqrt{\Lambda}} - \frac{26.0747}{\Lambda^{3/4}} - \frac{12.8258}{\Lambda} + \frac{246.699}{\Lambda^{5/4}}$
1.7	114.093	$2\sqrt[4]{\Lambda} + \frac{7.25}{\sqrt[4]{\Lambda}} - \frac{0.417828}{\sqrt{\Lambda}} - \frac{26.3}{\Lambda^{3/4}} - \frac{11.3363}{\Lambda} + \frac{243.5}{\Lambda^{5/4}}$
1.8	127.91	$2\sqrt[4]{\Lambda} + \frac{7.25}{\sqrt[4]{\Lambda}} - \frac{0.410937}{\sqrt{\Lambda}} - \frac{26.4426}{\Lambda^{3/4}} - \frac{10.3768}{\Lambda} + \frac{241.398}{\Lambda^{5/4}}$
1.9	142.517	$2\sqrt[4]{\Lambda} + \frac{7.25}{\sqrt[4]{\Lambda}} - \frac{0.415359}{\sqrt{\Lambda}} - \frac{26.3498}{\Lambda^{3/4}} - \frac{11.0121}{\Lambda} + \frac{242.815}{\Lambda^{5/4}}$

(24)

The coefficient c_2 is again smaller than the one for the $J = 4, n = 2$ case, and moreover the contribution of the corresponding term is much smaller than the contribution of the next one. It is possible that one cannot see that c_2 vanishes because of the exponentially suppressed corrections at large λ. These corrections decrease with J increasing and this would explain why for the $J = 5, n = 2$ case the coefficients c_0 and c_2 are closer to 0 than the ones for the $J = 4, n = 2$ state.

Next setting $c_{2k} = 0$ one gets the following fitting

g_0	λ_0	Fit
1.6	101.065	$2.00001\sqrt[4]{\Lambda} + \frac{7.23042}{\sqrt[4]{\Lambda}} - \frac{29.6628}{\Lambda^{3/4}} + \frac{231.828}{\Lambda^{5/4}} - \frac{27.3344}{\Lambda^{7/4}}$
1.7	114.093	$1.99999\sqrt[4]{\Lambda} + \frac{7.23469}{\sqrt[4]{\Lambda}} - \frac{29.9624}{\Lambda^{3/4}} + \frac{240.459}{\Lambda^{5/4}} - \frac{113.852}{\Lambda^{7/4}}$
1.8	127.91	$1.99997\sqrt[4]{\Lambda} + \frac{7.23966}{\sqrt[4]{\Lambda}} - \frac{30.3219}{\Lambda^{3/4}} + \frac{251.185}{\Lambda^{5/4}} - \frac{225.82}{\Lambda^{7/4}}$
1.9	142.517	$1.99995\sqrt[4]{\Lambda} + \frac{7.24362}{\sqrt[4]{\Lambda}} - \frac{30.6156}{\Lambda^{3/4}} + \frac{260.241}{\Lambda^{5/4}} - \frac{323.995}{\Lambda^{7/4}}$

, (25)

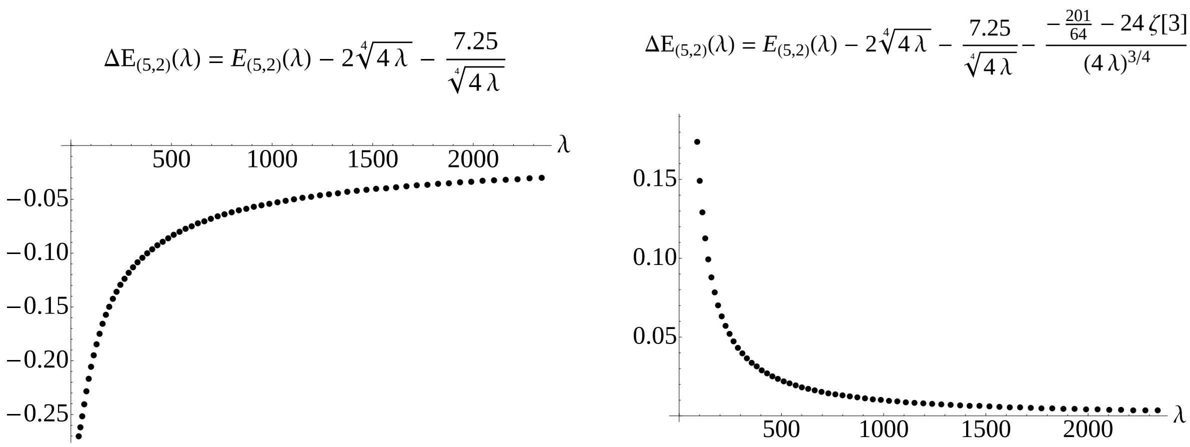

Figure 5: Black dots represent the difference between the numerical solution and conjectured asymptotic expansions.

which agrees very well with $c_1 = 7.25$ and shows that the coefficient c_3 is close to the conjectured value $c_3 = -\frac{201}{64} - 24\zeta(3) \approx -31.99$. Setting $c_{-1} = 2, c_1 = 7.25$

g_0	λ_0	Fit
1.6	101.065	$2\sqrt[4]{\Lambda} + \frac{7.25}{\sqrt[4]{\Lambda}} - \frac{31.9121}{\Lambda^{3/4}} + \frac{310.838}{\Lambda^{5/4}} - \frac{878.22}{\Lambda^{7/4}}$
1.7	114.093	$2\sqrt[4]{\Lambda} + \frac{7.25}{\sqrt[4]{\Lambda}} - \frac{32.}{\Lambda^{3/4}} + \frac{318.014}{\Lambda^{5/4}} - \frac{1007.16}{\Lambda^{7/4}}$
1.8	127.91	$2\sqrt[4]{\Lambda} + \frac{7.25}{\sqrt[4]{\Lambda}} - \frac{32.0823}{\Lambda^{3/4}} + \frac{325.019}{\Lambda^{5/4}} - \frac{1139.5}{\Lambda^{7/4}}$
1.9	142.517	$2\sqrt[4]{\Lambda} + \frac{7.25}{\sqrt[4]{\Lambda}} - \frac{32.1572}{\Lambda^{3/4}} + \frac{331.669}{\Lambda^{5/4}} - \frac{1271.1}{\Lambda^{7/4}}$

(26)

makes the agreement even more impressive.

In Figure 5 we plot the difference between the numerical solution and its large λ asymptotics $2(4\lambda)^{1/4} + \frac{7.25}{(4\lambda)^{1/4}}$ and $2(4\lambda)^{1/4} + \frac{7.25}{(4\lambda)^{1/4}} - \frac{31.99}{(4\lambda)^{3/4}}$.

2.5 $J = 6, n = 3$ operator

In the table (49) we present the results of the computation of the energy of the $J = 6, n = 3$ state. Fitting the data in the interval $g \in [g_0, 7.7]$ we get

g_0	λ_0	Fit
2.2	191.076	$2.02122\sqrt[4]{\Lambda} - 1.1605 + \frac{35.1776}{\sqrt[4]{\Lambda}} - \frac{312.4}{\sqrt{\Lambda}} + \frac{1974.72}{\Lambda^{3/4}} - \frac{7098.66}{\Lambda} + \frac{10856.1}{\Lambda^{5/4}}$
2.3	208.841	$2.01305\sqrt[4]{\Lambda} - 0.711293 + \frac{24.9684}{\sqrt[4]{\Lambda}} - \frac{189.655}{\sqrt{\Lambda}} + \frac{1151.37}{\Lambda^{3/4}} - \frac{4177.2}{\Lambda} + \frac{6572.03}{\Lambda^{5/4}}$
2.4	227.396	$2.00352\sqrt[4]{\Lambda} - 0.182115 + \frac{12.8223}{\sqrt[4]{\Lambda}} - \frac{42.0904}{\sqrt{\Lambda}} + \frac{150.596}{\Lambda^{3/4}} - \frac{584.867}{\Lambda} + \frac{1239.93}{\Lambda^{5/4}}$
2.5	246.74	$1.99454\sqrt[4]{\Lambda} + 0.320466 + \frac{1.17703}{\sqrt[4]{\Lambda}} + \frac{100.809}{\sqrt{\Lambda}} - \frac{828.784}{\Lambda^{3/4}} + \frac{2969.68}{\Lambda} - \frac{4097.42}{\Lambda^{5/4}}$

(27)

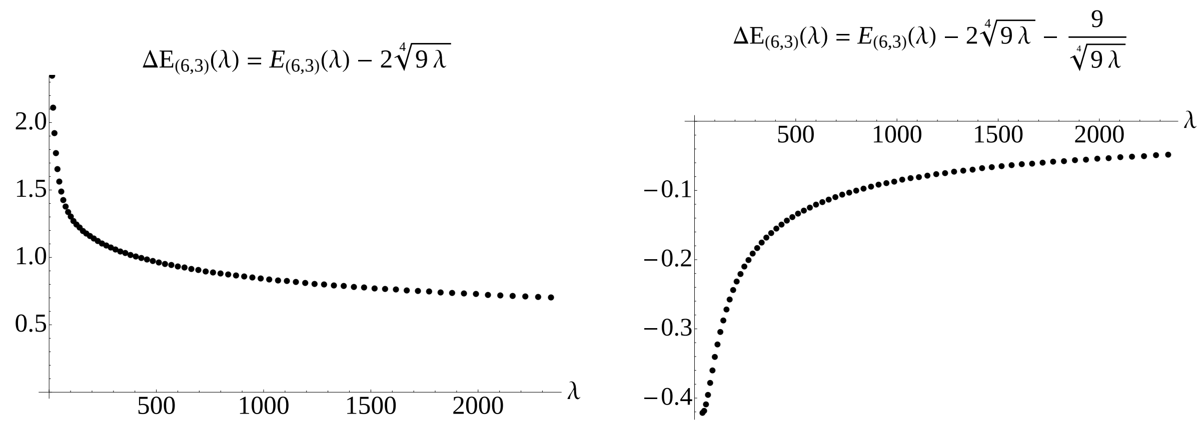

Figure 6: Black dots represent the difference between the numerical solution and conjectured asymptotic expansions.

where the expansion parameter $\Lambda = n^2\lambda = 9\lambda$ is introduced. One sees that as expected c_{-1} is close to 2, and fixing $c_{-1} = 2$ one gets

g_0	λ_0	Fit
2.2	191.076	$2\sqrt[4]{\Lambda} - 0.0280675 + \frac{10.2297}{\sqrt[4]{\Lambda}} - \frac{21.9871}{\sqrt{\Lambda}} + \frac{90.6627}{\Lambda^{3/4}} - \frac{639.338}{\Lambda} + \frac{1711.89}{\Lambda^{5/4}}$
2.3	208.841	$2\sqrt[4]{\Lambda} - 0.00754532 + \frac{9.29219}{\sqrt[4]{\Lambda}} - \frac{5.02319}{\sqrt{\Lambda}} - \frac{61.3087}{\Lambda^{3/4}} + \frac{34.676}{\Lambda} + \frac{527.865}{\Lambda^{5/4}}$
2.4	227.396	$2\sqrt[4]{\Lambda} + 0.00938827 + \frac{8.51111}{\sqrt[4]{\Lambda}} + \frac{9.25714}{\sqrt{\Lambda}} - \frac{190.658}{\Lambda^{3/4}} + \frac{615.119}{\Lambda} - \frac{504.505}{\Lambda^{5/4}}$
2.5	246.74	$2\sqrt[4]{\Lambda} + 0.0202856 + \frac{8.00379}{\sqrt[4]{\Lambda}} + \frac{18.6248}{\sqrt{\Lambda}} - \frac{276.408}{\Lambda^{3/4}} + \frac{1004.23}{\Lambda} - \frac{1204.79}{\Lambda^{5/4}}$

(28)

We see that the coefficient c_0 becomes small and we set it to 0: $c_0 = 0$

g_0	λ_0	Fit
2.2	191.076	$2\sqrt[4]{\Lambda} + \frac{8.99159}{\sqrt[4]{\Lambda}} - \frac{0.390111}{\sqrt{\Lambda}} - \frac{95.5746}{\Lambda^{3/4}} + \frac{154.709}{\Lambda} + \frac{372.41}{\Lambda^{5/4}}$
2.3	208.841	$2\sqrt[4]{\Lambda} + \frac{8.95568}{\sqrt[4]{\Lambda}} + \frac{0.91621}{\sqrt{\Lambda}} - \frac{113.173}{\Lambda^{3/4}} + \frac{258.775}{\Lambda} + \frac{144.476}{\Lambda^{5/4}}$
2.4	227.396	$2\sqrt[4]{\Lambda} + \frac{8.93423}{\sqrt[4]{\Lambda}} + \frac{1.70451}{\sqrt{\Lambda}} - \frac{123.911}{\Lambda^{3/4}} + \frac{323.035}{\Lambda} + \frac{1.91857}{\Lambda^{5/4}}$
2.5	246.74	$2\sqrt[4]{\Lambda} + \frac{8.9273}{\sqrt[4]{\Lambda}} + \frac{1.96172}{\sqrt{\Lambda}} - \frac{127.452}{\Lambda^{3/4}} + \frac{344.466}{\Lambda} - \frac{46.2066}{\Lambda^{5/4}}$

(29)

This fitting shows that $c_1 \approx 9$, and assuming that b_1 is independent of J one concludes that for any two-particle $n = 3$ state we should expect the formula: $c_1(J, 3) = J^2/4$. This is different from the $n = 1$ and $n = 2$ cases and therefore c_1 shows a nontrivial dependence of the string level n. Then setting $c_1 = 9$, one gets

g_0	λ_0	Fit
2.2	191.076	$2\sqrt[4]{\Lambda} + \frac{9}{\sqrt[4]{\Lambda}} - \frac{0.683102}{\sqrt{\Lambda}} - \frac{91.8011}{\Lambda^{3/4}} + \frac{133.416}{\Lambda} + \frac{416.847}{\Lambda^{5/4}}$
2.3	208.841	$2\sqrt[4]{\Lambda} + \frac{9}{\sqrt[4]{\Lambda}} - \frac{0.646888}{\sqrt{\Lambda}} - \frac{92.7809}{\Lambda^{3/4}} + \frac{142.105}{\Lambda} + \frac{391.588}{\Lambda^{5/4}}$
2.4	227.396	$2\sqrt[4]{\Lambda} + \frac{9}{\sqrt[4]{\Lambda}} - \frac{0.641557}{\sqrt{\Lambda}} - \frac{92.9268}{\Lambda^{3/4}} + \frac{143.414}{\Lambda} + \frac{387.73}{\Lambda^{5/4}}$
2.5	246.74	$2\sqrt[4]{\Lambda} + \frac{9}{\sqrt[4]{\Lambda}} - \frac{0.659812}{\sqrt{\Lambda}} - \frac{92.422}{\Lambda^{3/4}} + \frac{138.829}{\Lambda} + \frac{401.412}{\Lambda^{5/4}}$

(30)

Even though the coefficient c_2 is not really small the contribution of the corresponding term is much smaller than the contribution of the next term, and we believe that this supports $c_2 = 0$. Setting $c_{2k} = 0$ one gets the following fitting

$$\begin{array}{|c|c|c|}\hline g_0 & \lambda_0 & \text{Fit} \\ \hline 2.2 & 191.076 & 2.00009\sqrt[4]{\Lambda}+\frac{8.93337}{\sqrt[4]{\Lambda}}-\frac{85.2279}{\Lambda^{3/4}}+\frac{1023.82}{\Lambda^{5/4}}-\frac{3934.94}{\Lambda^{7/4}} \\ 2.3 & 208.841 & 2.00006\sqrt[4]{\Lambda}+\frac{8.94417}{\sqrt[4]{\Lambda}}-\frac{86.5518}{\Lambda^{3/4}}+\frac{1092.42}{\Lambda^{5/4}}-\frac{5204.87}{\Lambda^{7/4}} \\ 2.4 & 227.396 & 2.00004\sqrt[4]{\Lambda}+\frac{8.95121}{\sqrt[4]{\Lambda}}-\frac{87.4341}{\Lambda^{3/4}}+\frac{1139.29}{\Lambda^{5/4}}-\frac{6097.}{\Lambda^{7/4}} \\ 2.5 & 246.74 & 2.00003\sqrt[4]{\Lambda}+\frac{8.95445}{\sqrt[4]{\Lambda}}-\frac{87.8483}{\Lambda^{3/4}}+\frac{1161.82}{\Lambda^{5/4}}-\frac{6537.3}{\Lambda^{7/4}} \\ \hline \end{array} \tag{31}$$

which agrees very well with $c_1 = 9$ but it shows that the coefficient c_3 is different from the conjectured value $c_3 = -\frac{87}{8} - 81\zeta(3) \approx -108.242$. Setting $c_{-1} = 2, c_1 = 9$

$$\begin{array}{|c|c|c|}\hline g_0 & \lambda_0 & \text{Fit} \\ \hline 2.2 & 191.076 & 2\sqrt[4]{\Lambda}+\frac{9}{\sqrt[4]{\Lambda}}-\frac{96.7473}{\Lambda^{3/4}}+\frac{1737.24}{\Lambda^{5/4}}-\frac{18361.}{\Lambda^{7/4}} \\ 2.3 & 208.841 & 2\sqrt[4]{\Lambda}+\frac{9}{\sqrt[4]{\Lambda}}-\frac{96.9946}{\Lambda^{3/4}}+\frac{1774.91}{\Lambda^{5/4}}-\frac{19670.4}{\Lambda^{7/4}} \\ 2.4 & 227.396 & 2\sqrt[4]{\Lambda}+\frac{9}{\sqrt[4]{\Lambda}}-\frac{97.2184}{\Lambda^{3/4}}+\frac{1809.99}{\Lambda^{5/4}}-\frac{20931.5}{\Lambda^{7/4}} \\ 2.5 & 246.74 & 2\sqrt[4]{\Lambda}+\frac{9}{\sqrt[4]{\Lambda}}-\frac{97.4245}{\Lambda^{3/4}}+\frac{1843.16}{\Lambda^{5/4}}-\frac{22163.}{\Lambda^{7/4}} \\ \hline \end{array} \tag{32}$$

does not help and one has to conclude that the formula conjectured in [38] is correct only for the $n = 1$ and $n = 2$ operators. It is not really surprising because [38] used $c_1 = 10$ while the mirror TBA predicts $c_1 = 9$.

In Figure 6 we plot the difference between the numerical solution and its large λ asymptotics $2(9\lambda)^{1/4}$ and $2(9\lambda)^{1/4} + \frac{9}{(9\lambda)^{1/4}}$.

2.6 $J = 7, n = 3$ operator

In the table (50) we present the results of the computation of the energy of the $J = 7, n = 3$ state. Fitting the data in the interval $g \in [g_0, 7.7]$ we get

$$\begin{array}{|c|c|c|}\hline g_0 & \lambda_0 & \text{Fit} \\ \hline 2.2 & 191.076 & 2.02727\sqrt[4]{\Lambda}-1.46037+\frac{44.517}{\sqrt[4]{\Lambda}}-\frac{377.119}{\sqrt{\Lambda}}+\frac{2352.97}{\Lambda^{3/4}}-\frac{8336.68}{\Lambda}+\frac{12906.3}{\Lambda^{5/4}} \\ 2.3 & 208.841 & 2.03163\sqrt[4]{\Lambda}-1.70019+\frac{49.9676}{\sqrt[4]{\Lambda}}-\frac{442.652}{\sqrt{\Lambda}}+\frac{2792.56}{\Lambda^{3/4}}-\frac{9896.42}{\Lambda}+\frac{15193.5}{\Lambda^{5/4}} \\ 2.4 & 227.396 & 2.03791\sqrt[4]{\Lambda}-2.04842+\frac{57.9602}{\sqrt[4]{\Lambda}}-\frac{539.755}{\sqrt{\Lambda}}+\frac{3451.11}{\Lambda^{3/4}}-\frac{12260.3}{\Lambda}+\frac{18702.2}{\Lambda^{5/4}} \\ 2.5 & 246.74 & 2.04761\sqrt[4]{\Lambda}-2.59153+\frac{70.5447}{\sqrt[4]{\Lambda}}-\frac{694.179}{\sqrt{\Lambda}}+\frac{4509.47}{\Lambda^{3/4}}-\frac{16101.5}{\Lambda}+\frac{24470.}{\Lambda^{5/4}} \\ \hline \end{array} \tag{33}$$

where $\Lambda = n^2\lambda = 9\lambda$. One sees that c_{-1} is close to 2, and fixing $c_{-1} = 2$ one gets

$$\begin{array}{|c|c|c|}\hline g_0 & \lambda_0 & \text{Fit} \\ \hline 2.2 & 191.076 & 2\sqrt[4]{\Lambda}-0.00494946+\frac{12.4534}{\sqrt[4]{\Lambda}}-\frac{3.87439}{\sqrt{\Lambda}}-\frac{68.4585}{\Lambda^{3/4}}-\frac{35.0256}{\Lambda}+\frac{1153.99}{\Lambda^{5/4}} \\ 2.3 & 208.841 & 2\sqrt[4]{\Lambda}+0.00564328+\frac{11.9695}{\sqrt[4]{\Lambda}}+\frac{4.88171}{\sqrt{\Lambda}}-\frac{146.9}{\Lambda^{3/4}}+\frac{312.873}{\Lambda}+\frac{542.839}{\Lambda^{5/4}} \\ 2.4 & 227.396 & 2\sqrt[4]{\Lambda}+0.0162742+\frac{11.4791}{\sqrt[4]{\Lambda}}+\frac{13.8469}{\sqrt{\Lambda}}-\frac{228.106}{\Lambda^{3/4}}+\frac{677.277}{\Lambda}-\frac{105.285}{\Lambda^{5/4}} \\ 2.5 & 246.74 & 2\sqrt[4]{\Lambda}+0.0264211+\frac{11.0067}{\sqrt[4]{\Lambda}}+\frac{22.5695}{\sqrt{\Lambda}}-\frac{307.95}{\Lambda^{3/4}}+\frac{1039.59}{\Lambda}-\frac{757.345}{\Lambda^{5/4}} \\ \hline \end{array} \tag{34}$$

Since the coefficient c_0 becomes small we set it to 0: $c_0 = 0$

$$\begin{array}{|c|c|c|}\hline g_0 & \lambda_0 & \text{Fit} \\ \hline 2.2 & 191.076 & 2\sqrt[4]{\Lambda}+\frac{12.2351}{\sqrt[4]{\Lambda}}-\frac{0.065948}{\sqrt{\Lambda}}-\frac{101.3}{\Lambda^{3/4}}+\frac{104.997}{\Lambda}+\frac{917.781}{\Lambda^{5/4}} \\ 2.3 & 208.841 & 2\sqrt[4]{\Lambda}+\frac{12.2212}{\sqrt[4]{\Lambda}}+\frac{0.439529}{\sqrt{\Lambda}}-\frac{108.11}{\Lambda^{3/4}}+\frac{145.265}{\Lambda}+\frac{829.583}{\Lambda^{5/4}} \\ 2.4 & 227.396 & 2\sqrt[4]{\Lambda}+\frac{12.2126}{\sqrt[4]{\Lambda}}+\frac{0.754727}{\sqrt{\Lambda}}-\frac{112.403}{\Lambda^{3/4}}+\frac{170.959}{\Lambda}+\frac{772.582}{\Lambda^{5/4}} \\ 2.5 & 246.74 & 2\sqrt[4]{\Lambda}+\frac{12.2096}{\sqrt[4]{\Lambda}}+\frac{0.866542}{\sqrt{\Lambda}}-\frac{113.942}{\Lambda^{3/4}}+\frac{180.276}{\Lambda}+\frac{751.66}{\Lambda^{5/4}} \\ \hline \end{array} \tag{35}$$

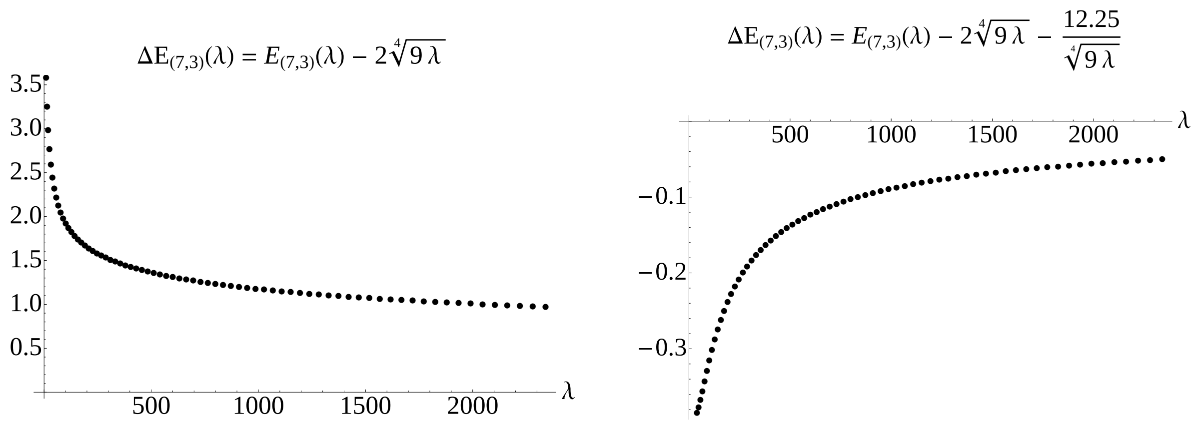

Figure 7: Black dots represent the difference between the numerical solution and conjectured asymptotic expansions.

This fitting shows that $c_1 \approx 12.25$ in agreement with the formula $c_1(J,2) = J^2/4$ we obtained analyzing the $J = 6, n = 3$ state. Then setting $c_1 = 12.25$, one gets

g_0	λ_0	Fit
2.2	191.076	$2\sqrt[4]{\Lambda} + \frac{12.25}{\sqrt[4]{\Lambda}} - \frac{0.586725}{\sqrt{\Lambda}} - \frac{94.5925}{\Lambda^{3/4}} + \frac{67.1513}{\Lambda} + \frac{996.766}{\Lambda^{5/4}}$
2.3	208.841	$2\sqrt[4]{\Lambda} + \frac{12.25}{\sqrt[4]{\Lambda}} - \frac{0.577563}{\sqrt{\Lambda}} - \frac{94.8404}{\Lambda^{3/4}} + \frac{69.3494}{\Lambda} + \frac{990.376}{\Lambda^{5/4}}$
2.4	227.396	$2\sqrt[4]{\Lambda} + \frac{12.25}{\sqrt[4]{\Lambda}} - \frac{0.57989}{\sqrt{\Lambda}} - \frac{94.7767}{\Lambda^{3/4}} + \frac{68.7779}{\Lambda} + \frac{992.06}{\Lambda^{5/4}}$
2.5	246.74	$2\sqrt[4]{\Lambda} + \frac{12.25}{\sqrt[4]{\Lambda}} - \frac{0.59125}{\sqrt{\Lambda}} - \frac{94.4626}{\Lambda^{3/4}} + \frac{65.9249}{\Lambda} + \frac{1000.57}{\Lambda^{5/4}}$

(36)

Since the contribution of the c_2 term is much smaller than the contribution of the next term the fitting supports $c_2 = 0$. Setting $c_{2k} = 0$ one gets the following fitting

g_0	λ_0	Fit
2.2	191.076	$2.00005\sqrt[4]{\Lambda} + \frac{12.2061}{\sqrt[4]{\Lambda}} - \frac{93.0595}{\Lambda^{3/4}} + \frac{1340.05}{\Lambda^{5/4}} - \frac{2462.57}{\Lambda^{7/4}}$
2.3	208.841	$2.00004\sqrt[4]{\Lambda} + \frac{12.2106}{\sqrt[4]{\Lambda}} - \frac{93.6089}{\Lambda^{3/4}} + \frac{1368.52}{\Lambda^{5/4}} - \frac{2989.6}{\Lambda^{7/4}}$
2.4	227.396	$2.00003\sqrt[4]{\Lambda} + \frac{12.2136}{\sqrt[4]{\Lambda}} - \frac{93.9924}{\Lambda^{3/4}} + \frac{1388.89}{\Lambda^{5/4}} - \frac{3377.36}{\Lambda^{7/4}}$
2.5	246.74	$2.00003\sqrt[4]{\Lambda} + \frac{12.2152}{\sqrt[4]{\Lambda}} - \frac{94.1927}{\Lambda^{3/4}} + \frac{1399.79}{\Lambda^{5/4}} - \frac{3590.34}{\Lambda^{7/4}}$

(37)

which agrees with $c_1 = 12.25$ but disagrees with the conjectured value of c_3: $c_3 = -\frac{1489}{64} - 81\zeta(3) \approx -120.632$. Finally setting $c_{-1} = 2, c_1 = 12.25$ one gets

g_0	λ_0	Fit
2.2	191.076	$2\sqrt[4]{\Lambda} + \frac{12.25}{\sqrt[4]{\Lambda}} - \frac{100.961}{\Lambda^{3/4}} + \frac{1838.01}{\Lambda^{5/4}} - \frac{12626.6}{\Lambda^{7/4}}$
2.3	208.841	$2\sqrt[4]{\Lambda} + \frac{12.25}{\sqrt[4]{\Lambda}} - \frac{101.136}{\Lambda^{3/4}} + \frac{1864.62}{\Lambda^{5/4}} - \frac{13551.4}{\Lambda^{7/4}}$
2.4	227.396	$2\sqrt[4]{\Lambda} + \frac{12.25}{\sqrt[4]{\Lambda}} - \frac{101.298}{\Lambda^{3/4}} + \frac{1890.13}{\Lambda^{5/4}} - \frac{14468.5}{\Lambda^{7/4}}$
2.5	246.74	$2\sqrt[4]{\Lambda} + \frac{12.25}{\sqrt[4]{\Lambda}} - \frac{101.452}{\Lambda^{3/4}} + \frac{1914.79}{\Lambda^{5/4}} - \frac{15384.2}{\Lambda^{7/4}}$

(38)

In Figure 7 we plot the difference between the numerical solution and its large λ asymptotics $2(9\lambda)^{1/4}$ and $2(9\lambda)^{1/4} + \frac{12.25}{(9\lambda)^{1/4}}$.

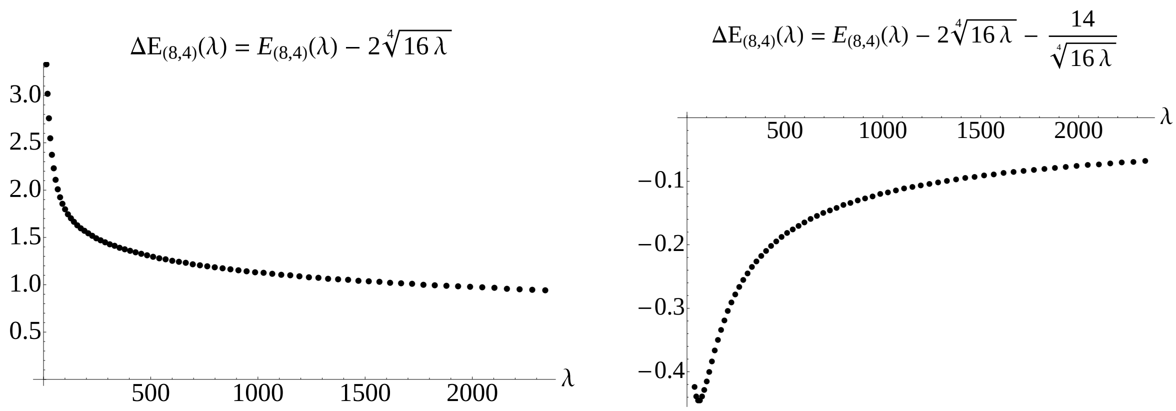

Figure 8: Black dots represent the difference between the numerical solution and conjectured asymptotic expansions.

2.7 $J = 8, n = 4$ operator

In the table (51) we present the results of the computation of the energy of the $J = 8, n = 4$ state. The precision for this operator is higher than for the other operators and should be about 0.00002. Fitting the data in the interval $g \in [g_0, 7.7]$ we get

g_0	λ_0	Fit
2.4	227.396	$2.02587\sqrt[4]{\Lambda} - 1.71713 + \frac{61.1202}{\sqrt[4]{\Lambda}} - \frac{685.038}{\sqrt{\Lambda}} + \frac{5322.77}{\Lambda^{3/4}} - \frac{23175.3}{\Lambda} + \frac{43004.2}{\Lambda^{5/4}}$
2.5	246.74	$2.0207\sqrt[4]{\Lambda} - 1.38273 + \frac{52.1731}{\sqrt[4]{\Lambda}} - \frac{558.265}{\sqrt{\Lambda}} + \frac{4319.5}{\Lambda^{3/4}} - \frac{18970.8}{\Lambda} + \frac{35714.1}{\Lambda^{5/4}}$
2.6	266.874	$2.02039\sqrt[4]{\Lambda} - 1.36242 + \frac{51.6249}{\sqrt[4]{\Lambda}} - \frac{550.422}{\sqrt{\Lambda}} + \frac{4256.81}{\Lambda^{3/4}} - \frac{18705.2}{\Lambda} + \frac{35248.6}{\Lambda^{5/4}}$
2.7	287.798	$2.0272\sqrt[4]{\Lambda} - 1.81055 + \frac{63.8305}{\sqrt[4]{\Lambda}} - \frac{726.645}{\sqrt{\Lambda}} + \frac{5679.17}{\Lambda^{3/4}} - \frac{24790.5}{\Lambda} + \frac{46029.7}{\Lambda^{5/4}}$

(39)

where $\Lambda = n^2\lambda = 16\lambda$. One sees that c_{-1} is close to 2, and fixing $c_{-1} = 2$ one gets

g_0	λ_0	Fit
2.4	227.396	$2\sqrt[4]{\Lambda} - 0.0901497 + \frac{18.8267}{\sqrt[4]{\Lambda}} - \frac{103.386}{\sqrt{\Lambda}} + \frac{859.119}{\Lambda^{3/4}} - \frac{5051.13}{\Lambda} + \frac{12580.9}{\Lambda^{5/4}}$
2.5	246.74	$2\sqrt[4]{\Lambda} - 0.0684271 + \frac{17.659}{\sqrt[4]{\Lambda}} - \frac{78.4877}{\sqrt{\Lambda}} + \frac{595.952}{\Lambda^{3/4}} - \frac{3672.19}{\Lambda} + \frac{9715.32}{\Lambda^{5/4}}$
2.6	266.874	$2\sqrt[4]{\Lambda} - 0.0558615 + \frac{16.9775}{\sqrt[4]{\Lambda}} - \frac{63.8179}{\sqrt{\Lambda}} + \frac{439.324}{\Lambda^{3/4}} - \frac{2842.69}{\Lambda} + \frac{7972.04}{\Lambda^{5/4}}$
2.7	287.798	$2\sqrt[4]{\Lambda} - 0.0516738 + \frac{16.7484}{\sqrt[4]{\Lambda}} - \frac{58.8418}{\sqrt{\Lambda}} + \frac{385.679}{\Lambda^{3/4}} - \frac{2555.67}{\Lambda} + \frac{7362.33}{\Lambda^{5/4}}$

(40)

Since the coefficient c_0 becomes small we set it to 0: $c_0 = 0$

g_0	λ_0	Fit
2.4	227.396	$2\sqrt[4]{\Lambda} + \frac{14.1353}{\sqrt[4]{\Lambda}} - \frac{6.68802}{\sqrt{\Lambda}} - \frac{127.649}{\Lambda^{3/4}} - \frac{64.996}{\Lambda} + \frac{2598.41}{\Lambda^{5/4}}$
2.5	246.74	$2\sqrt[4]{\Lambda} + \frac{14.0619}{\sqrt[4]{\Lambda}} - \frac{3.54403}{\sqrt{\Lambda}} - \frac{177.626}{\Lambda^{3/4}} + \frac{284.294}{\Lambda} + \frac{1692.72}{\Lambda^{5/4}}$
2.6	266.874	$2\sqrt[4]{\Lambda} + \frac{14.0123}{\sqrt[4]{\Lambda}} - \frac{1.39941}{\sqrt{\Lambda}} - \frac{212.063}{\Lambda^{3/4}} + \frac{527.596}{\Lambda} + \frac{1054.52}{\Lambda^{5/4}}$
2.7	287.798	$2\sqrt[4]{\Lambda} + \frac{13.9798}{\sqrt[4]{\Lambda}} + \frac{0.0198075}{\sqrt{\Lambda}} - \frac{235.074}{\Lambda^{3/4}} + \frac{691.858}{\Lambda} + \frac{618.868}{\Lambda^{5/4}}$

(41)

This fitting shows that $c_1 \approx 14$ which suggests the formula $c_1(J, 2) = J^2/4 - 2$. Then setting $c_1 = 14$, one gets

$$
\begin{array}{|c|c|c|}
\hline
g_0 & \lambda_0 & \text{Fit} \\
\hline
2.4 & 227.396 & 2\sqrt[4]{\Lambda} + \frac{14}{\sqrt[4]{\Lambda}} - \frac{1.11641}{\sqrt{\Lambda}} - \frac{212.617}{\Lambda^{3/4}} + \frac{503.771}{\Lambda} + \frac{1187.75}{\Lambda^{5/4}} \\
2.5 & 246.74 & 2\sqrt[4]{\Lambda} + \frac{14}{\sqrt[4]{\Lambda}} - \frac{0.966912}{\sqrt{\Lambda}} - \frac{217.391}{\Lambda^{3/4}} + \frac{553.832}{\Lambda} + \frac{1015.24}{\Lambda^{5/4}} \\
2.6 & 266.874 & 2\sqrt[4]{\Lambda} + \frac{14}{\sqrt[4]{\Lambda}} - \frac{0.881553}{\sqrt{\Lambda}} - \frac{220.144}{\Lambda^{3/4}} + \frac{583.027}{\Lambda} + \frac{913.412}{\Lambda^{5/4}} \\
2.7 & 287.798 & 2\sqrt[4]{\Lambda} + \frac{14}{\sqrt[4]{\Lambda}} - \frac{0.839038}{\sqrt{\Lambda}} - \frac{221.528}{\Lambda^{3/4}} + \frac{597.866}{\Lambda} + \frac{861.055}{\Lambda^{5/4}} \\
\hline
\end{array}
\qquad (42)
$$

Since the contribution of the c_2 term is much smaller than the contribution of the next term the fitting supports $c_2 = 0$. Setting $c_{2k} = 0$ one gets the following fitting

$$
\begin{array}{|c|c|c|}
\hline
g_0 & \lambda_0 & \text{Fit} \\
\hline
2.4 & 227.396 & 2.00022\sqrt[4]{\Lambda} + \frac{13.8411}{\sqrt[4]{\Lambda}} - \frac{180.225}{\Lambda^{3/4}} + \frac{3375.94}{\Lambda^{5/4}} - \frac{13371.7}{\Lambda^{7/4}} \\
2.5 & 246.74 & 2.00017\sqrt[4]{\Lambda} + \frac{13.8627}{\sqrt[4]{\Lambda}} - \frac{183.91}{\Lambda^{3/4}} + \frac{3643.1}{\Lambda^{5/4}} - \frac{20334.4}{\Lambda^{7/4}} \\
2.6 & 266.874 & 2.00014\sqrt[4]{\Lambda} + \frac{13.8782}{\sqrt[4]{\Lambda}} - \frac{186.603}{\Lambda^{3/4}} + \frac{3842.8}{\Lambda^{5/4}} - \frac{25671.9}{\Lambda^{7/4}} \\
2.7 & 287.798 & 2.00012\sqrt[4]{\Lambda} + \frac{13.8893}{\sqrt[4]{\Lambda}} - \frac{188.555}{\Lambda^{3/4}} + \frac{3990.66}{\Lambda^{5/4}} - \frac{29719.7}{\Lambda^{7/4}} \\
\hline
\end{array}
\qquad (43)
$$

which agrees with $c_1 = 14$. The agreement becomes even better if one sets $c_{-1} = 2$

$$
\begin{array}{|c|c|c|}
\hline
g_0 & \lambda_0 & \text{Fit} \\
\hline
2.4 & 227.396 & 2\sqrt[4]{\Lambda} + \frac{13.9364}{\sqrt[4]{\Lambda}} - \frac{195.091}{\Lambda^{3/4}} + \frac{4356.69}{\Lambda^{5/4}} - \frac{36551.8}{\Lambda^{7/4}} \\
2.5 & 246.74 & 2\sqrt[4]{\Lambda} + \frac{13.9395}{\sqrt[4]{\Lambda}} - \frac{196.166}{\Lambda^{3/4}} + \frac{4473.31}{\Lambda^{5/4}} - \frac{40538.4}{\Lambda^{7/4}} \\
2.6 & 266.874 & 2\sqrt[4]{\Lambda} + \frac{13.9418}{\sqrt[4]{\Lambda}} - \frac{196.979}{\Lambda^{3/4}} + \frac{4563.51}{\Lambda^{5/4}} - \frac{43705.8}{\Lambda^{7/4}} \\
2.7 & 287.798 & 2\sqrt[4]{\Lambda} + \frac{13.9435}{\sqrt[4]{\Lambda}} - \frac{197.604}{\Lambda^{3/4}} + \frac{4634.52}{\Lambda^{5/4}} - \frac{46263.4}{\Lambda^{7/4}} \\
\hline
\end{array}
\qquad (44)
$$

In Figure 8 we plot the difference between the numerical solution and its large λ asymptotics $2(16\lambda)^{1/4}$ and $2(16\lambda)^{1/4} + \frac{14}{(16\lambda)^{1/4}}$.

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3 Numerical data

Here we collect our numerical data for the energy of the two-paticle states or, equivalently, the conformal dimension of the dual $\mathcal{N} = 4$ SYM operators as a function of the effective string tension g related to 't Hooft's coupling λ as $\lambda = 4\pi^2 g^2$.

An interested reader can download a Mathematica file with this data and the data for Bethe roots from the Arxiv page of this paper: http://arXiv.org/abs/arXiv:12??.????

$J = 3, n = 1$ operator

g	$E_{(3,1)}$	g	$E_{(3,1)}$	g	$E_{(3,1)}$	g	$E_{(3,1)}$	g	$E_{(3,1)}$	g	$E_{(3,1)}$
0	5	0.7	5.75288	1.4	6.9957	2.1	8.13347	2.8	9.14362	3.5	10.0564
0.1	5.01985	0.8	5.92734	1.5	7.16701	2.2	8.28485	2.9	9.27936	3.6	10.1802
0.2	5.07773	0.9	6.10559	1.6	7.33539	2.3	8.4337	3.	9.41322	3.7	10.3025
0.3	5.16917	1.	6.2862	1.7	7.50075	2.4	8.58018	3.1	9.54522		
0.4	5.28828	1.1	6.46621	1.8	7.66321	2.5	8.72423	3.2	9.67549		
0.5	5.42891	1.2	6.64492	1.9	7.82271	2.6	8.86614	3.3	9.80404		
0.6	5.5854	1.3	6.82157	2.	7.97949	2.7	9.00592	3.4	9.93099		

(45)

$J = 4, n = 1$ operator

g	$E_{(4,1)}$	g	$E_{(4,1)}$	g	$E_{(4,1)}$	g	$E_{(4,1)}$	g	$E_{(4,1)}$	g	$E_{(4,1)}$
0	6	0.6	6.42681	1.2	7.28623	1.62	7.92062	2.6	9.29982	3.2	10.0659
0.1	6.01375	0.7	6.55594	1.3	7.43822	1.63	7.93556	2.7	9.43134	3.3	10.1884
0.2	6.05416	0.8	6.6934	1.4	7.58999	2.2	8.75551	2.8	9.56136	3.4	10.3096
0.3	6.11897	0.9	6.83685	1.5	7.74085	2.3	8.89485	2.9	9.68974		
0.4	6.20502	1.	6.98458	1.6	7.89072	2.4	9.03165	3.	9.81658		
0.5	6.30878	1.1	7.13475	1.61	7.90572	2.5	9.16659	3.1	9.94203		

(46)

$J = 4, n = 2$ operator

g	$E_{(4,2)}$	g	$E_{(4,2)}$	g	$E_{(4,2)}$	g	$E_{(4,2)}$	g	$E_{(4,2)}$	g	$E_{(4,2)}$
0	6	1.3	9.05125	2.6	12.1841	3.9	14.643	5.2	16.7364	6.5	18.5923
0.1	6.03583	1.4	9.32919	2.7	12.3912	4.	14.8147	5.3	16.8865	6.6	18.7273
0.2	6.13947	1.5	9.60001	2.8	12.5948	4.1	14.9845	5.4	17.0353	6.7	18.8614
0.3	6.30103	1.6	9.8638	2.9	12.795	4.2	15.1523	5.5	17.1828	6.8	18.9945
0.4	6.50828	1.7	10.1208	3.	12.9921	4.3	15.3182	5.6	17.3289	6.9	19.1267
0.5	6.74951	1.8	10.3713	3.1	13.1861	4.4	15.4824	5.7	17.4739	7.	19.258
0.6	7.01484	1.9	10.6157	3.2	13.3771	4.5	15.6447	5.8	17.6176	7.1	19.3883
0.7	7.29635	2.	10.8543	3.3	13.5654	4.6	15.8054	5.9	17.7601	7.2	19.5177
0.8	7.58773	2.1	11.0876	3.4	13.7511	4.7	15.9644	6.	17.9016	7.3	19.6464
0.9	7.88397	2.2	11.3158	3.5	13.9342	4.8	16.1218	6.1	18.0418	7.4	19.7742
1.	8.18112	2.3	11.5393	3.6	14.1148	4.9	16.2777	6.2	18.181	7.5	19.9012
1.1	8.47607	2.4	11.7583	3.7	14.2931	5.	16.4321	6.3	18.3191	7.6	20.0273
1.2	8.7666	2.5	11.9732	3.8	14.4691	5.1	16.585	6.4	18.4562	7.7	20.1526

(47)

$J = 5, n = 2$ operator

g	$E_{(5,2)}$	g	$E_{(5,2)}$	g	$E_{(5,2)}$	g	$E_{(5,2)}$	g	$E_{(5,2)}$	g	$E_{(5,2)}$
0	7	1.3	9.63657	2.6	12.576	3.9	14.9615	5.2	17.0124	6.5	18.8393
0.1	7.02974	1.4	9.88812	2.7	12.7755	4.	15.1292	5.3	17.1599	6.6	18.9725
0.2	7.11606	1.5	10.1356	2.8	12.9719	4.1	15.2951	5.4	17.3061	6.7	19.1048
0.3	7.25139	1.6	10.3787	2.9	13.1653	4.2	15.4592	5.5	17.4511	6.8	19.2361
0.4	7.42611	1.7	10.6173	3.	13.3559	4.3	15.6215	5.6	17.5949	6.9	19.3666
0.5	7.63068	1.8	10.8513	3.1	13.5438	4.4	15.7822	5.7	17.7375	7.	19.4962
0.6	7.85687	1.9	11.081	3.2	13.7291	4.5	15.9412	5.8	17.879	7.1	19.6249
0.7	8.09799	2.	11.3063	3.3	13.912	4.6	16.0986	5.9	18.0194	7.2	19.7527
0.8	8.3488	2.1	11.5274	3.4	14.0924	4.7	16.2545	6.	18.1586	7.3	19.8798
0.9	8.60527	2.2	11.7445	3.5	14.2705	4.8	16.4089	6.1	18.2968	7.4	20.006
1.	8.86434	2.3	11.9577	3.6	14.4464	4.9	16.5619	6.2	18.434	7.5	20.1314
1.1	9.12364	2.4	12.1673	3.7	14.6201	5.	16.7134	6.3	18.5701	7.6	20.2561
1.2	9.38146	2.5	12.3734	3.8	14.7918	5.1	16.8636	6.4	18.7052	7.7	20.3799

(48)

$J = 6, n = 3$ operator

g	$E_{(6,3)}$	g	$E_{(6,3)}$	g	$E_{(6,3)}$	g	$E_{(6,3)}$	g	$E_{(6,3)}$	g	$E_{(6,3)}$
0	8	1.3	11.3223	2.6	15.0857	3.9	18.0764	5.2	20.6266	6.5	22.8892
0.1	8.03765	1.4	11.6469	2.7	15.3374	4.	18.2855	5.3	20.8094	6.6	23.0539
0.2	8.14655	1.5	11.9667	2.8	15.5848	4.1	18.4922	5.4	20.9908	6.7	23.2174
0.3	8.31636	1.6	12.2807	2.9	15.8282	4.2	18.6966	5.5	21.1705	6.8	23.3798
0.4	8.53443	1.7	12.5884	3.	16.0677	4.3	18.8987	5.6	21.3487	6.9	23.541
0.5	8.78889	1.8	12.8897	3.1	16.3036	4.4	19.0986	5.7	21.5254	7.	23.7011
0.6	9.06998	1.9	13.1845	3.2	16.536	4.5	19.2964	5.8	21.7007	7.1	23.8602
0.7	9.37014	2.	13.4729	3.3	16.765	4.6	19.4921	5.9	21.8745	7.2	24.0182
0.8	9.68362	2.1	13.7553	3.4	16.9909	4.7	19.6858	6.	22.0469	7.3	24.1751
0.9	10.006	2.2	14.0319	3.5	17.2136	4.8	19.8776	6.1	22.2179	7.4	24.331
1.	10.3337	2.3	14.3029	3.6	17.4335	4.9	20.0675	6.2	22.3876	7.5	24.4859
1.1	10.6639	2.4	14.5687	3.7	17.6505	5.	20.2556	6.3	22.5561	7.6	24.6398
1.2	10.9941	2.5	14.8296	3.8	17.8647	5.1	20.4419	6.4	22.7232	7.7	24.7927

(49)

$J = 7, n = 3$ **operator**

g	$E_{(7,3)}$	g	$E_{(7,3)}$	g	$E_{(7,3)}$	g	$E_{(7,3)}$	g	$E_{(7,3)}$	g	$E_{(7,3)}$
0	9	1.3	12.0182	2.6	15.5507	3.9	18.4526	5.2	20.9521	6.5	23.1806
0.1	9.03383	1.4	12.3149	2.7	15.7929	4.	18.6569	5.3	21.132	6.6	23.3431
0.2	9.13189	1.5	12.6083	2.8	16.0315	4.1	18.859	5.4	21.3103	6.7	23.5045
0.3	9.28531	1.6	12.8979	2.9	16.2667	4.2	19.059	5.5	21.4871	6.8	23.6647
0.4	9.483	1.7	13.1833	3.	16.4984	4.3	19.2568	5.6	21.6625	6.9	23.8239
0.5	9.71427	1.8	13.4643	3.1	16.727	4.4	19.4525	5.7	21.8365	7.	23.9821
0.6	9.97009	1.9	13.7407	3.2	16.9524	4.5	19.6463	5.8	22.009	7.1	24.1391
0.7	10.2434	2.	14.0124	3.3	17.1748	4.6	19.8382	5.9	22.1802	7.2	24.2952
0.8	10.5286	2.1	14.2796	3.4	17.3944	4.7	20.0282	6.	22.3501	7.3	24.4502
0.9	10.8216	2.2	14.5423	3.5	17.6112	4.8	20.2164	6.1	22.5187	7.4	24.6043
1.	11.1193	2.3	14.8006	3.6	17.8253	4.9	20.4029	6.2	22.686	7.5	24.7574
1.1	11.4192	2.4	15.0546	3.7	18.0369	5.	20.5876	6.3	22.852	7.6	24.9095
1.2	11.7193	2.5	15.3046	3.8	18.2459	5.1	20.7707	6.4	23.0169	7.7	25.0607

(50)

$J = 8, n = 4$ **operator**

g	$E_{(8,4)}$	g	$E_{(8,4)}$	g	$E_{(8,4)}$	g	$E_{(8,4)}$	g	$E_{(8,4)}$	g	$E_{(8,4)}$
0	10	1.3	13.4355	2.6	17.6319	3.9	21.0493	5.2	23.9733	6.5	26.5717
0.1	10.0384	1.4	13.7839	2.7	17.9186	4.	21.2889	5.3	24.1833	6.6	26.761
0.2	10.1495	1.5	14.1306	2.8	18.2007	4.1	21.5257	5.4	24.3914	6.7	26.9489
0.3	10.3227	1.6	14.4743	2.9	18.4784	4.2	21.7598	5.5	24.5978	6.8	27.1355
0.4	10.5452	1.7	14.814	3.	18.7518	4.3	21.9915	5.6	24.8024	6.9	27.3208
0.5	10.8049	1.8	15.1491	3.1	19.0213	4.4	22.2206	5.7	25.0053	7.	27.5049
0.6	11.0922	1.9	15.4789	3.2	19.2868	4.5	22.4474	5.8	25.2065	7.1	27.6877
0.7	11.3996	2.	15.8032	3.3	19.5486	4.6	22.6718	5.9	25.4061	7.2	27.8693
0.8	11.7217	2.1	16.1218	3.4	19.8068	4.7	22.894	6.	25.6041	7.3	28.0497
0.9	12.0542	2.2	16.4347	3.5	20.0617	4.8	23.114	6.1	25.8006	7.4	28.2289
1.	12.3942	2.3	16.742	3.6	20.3132	4.9	23.3318	6.2	25.9955	7.5	28.407
1.1	12.739	2.4	17.0438	3.7	20.5616	5.	23.5476	6.3	26.189	7.6	28.584
1.2	13.0867	2.5	17.3404	3.8	20.8069	5.1	23.7615	6.4	26.3811	7.7	28.7598

(51)

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