

An Effective Theory of Superfluid Turbulence from Local Scale Invariance

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ABSTRACT

An effective locally scale invariant model is constructed using Weyl's method starting from a free Schroedinger equation. The model requires additional gauge and gravitational degrees of freedom. It is suggested that this scale invariant model is an effective theory for superfluid turbulence. The additional degrees of freedom introduced can then be identified with filament excitations or zeros of a Gross-Pitaevski equation which are used to describe superfluid turbulence. Qualitative estimates of the way filaments separate after collision are made which agree with observations.

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1 Introduction

Turbulence in superfluid liquid Helium [1] has recently been demonstrated experimentally [2] to exhibit Kolmogorov scaling [1,3]. However the distribution functions are different from those of classical fluid turbulence. For instance, the velocity distribution function is not Gaussian and has a power law tail. Another distinctive feature of superfluid turbulence is the appearance of quantum filaments which “interact” by joining and separating. The velocity of separation of filaments, a characteristic feature of superfluid turbulence, has been measured.

In view of the three-dimensional scaling behaviour of a turbulent superfluid system it seems worthwhile to construct a three-dimensional local scale invariant version of a bosonic theory which allows superfluidity. We find that such a theory with scale invariance built in possesses features that make it an attractive candidate for a minimal effective theory of superfluid turbulence. Historically, a locally scale-invariant system was constructed as a gauge theory by Weyl [4] in an attempt to constructing a unified theory of gravitation and electromagnetism. Requiring Einstein’s theory of general relativity to be locally scale invariant in four dimensions necessitated the introduction of a gauge field with an action resembling that of Maxwell’s theory. He identified this term as electromagnetism. Thus, by introducing an additional degree of freedom to effect local changes in the length scale Weyl showed that a unified theory of gravitation and an electromagnetism-like interaction ensued. Weyl’s approach based on local scaling symmetry was criticised by Einstein [5] as being incompatible with the observed discrete spectra of atoms. The idea was thus relinquished as a means to producing a unified field theory. However, it was resurrected afterwards with the local scaling of lengths replaced by local change of phase of a quantum wave function [6]. This construction, which came to be known as “gauge theory”, turned out to be extremely fruitful and has since become a cornerstone of fundamental physics.

Here we use Weyl’s original idea as the objection due to Einstein does not pertain to a superfluid turbulent system. We start with a Bose-Einstein system described by a free Schroedinger field theory. This theory is then rendered locally scale invariant only in the spatial directions by introducing a time-independent metric for the spatial part and a gauge field. This allows incorporation of a term in the action depending on the Ricci scalar corresponding to the spatial metric. The resulting theory allows Bose-Einstein condensation and is thus an appropriate starting point for an effective theory of superfluid turbulence.

Let us point out that the local scale invariance is rather restrictive. A Maxwellian term quadratic in the field strength corresponding to the gauge field alluded to above can not be made scale invariant in a polynomial action in three dimensions and thus can not be added to the action. A Chern-Simons-like term is similarly prohibited. Three-dimensional scale invariance is implemented through cancellation of variations between the Ricci tensor term and the expressions involving the gauge field that appear. Implementing gauge invariance in this fashion has been called Ricci gauging [7]. Furthermore, dynamics for the Schroedinger field is scale invariant only if the space and time scaling rules are different. There is similarly no scale invariant that can be introduced for the gauge field. Time dependence breaks scale invariance. Dynamics breaks scale invariance too. However, the scale invariance in three dimensions identifies the degrees of free-

dom. As superfluid turbulence does break scale invariance through the production of filaments which have a length scale this feature of the effective theory is acceptable.

Thus the effective theory is a simple one involving a gauge field, a gravitational field or metric and the Schroedinger field. In this picture excitations of the gauge and the gravitational field are absent when the quantum fluid is in its superfluid ground state. The effective collective wave function of the superfluid is a constant proportional to $\sqrt{N}$, where N is the total number of Helium atoms. When energy is injected into the system, say by heating, excitations in the gauge field and the metric appear. This is in agreement with the experimental results where superfluid turbulence with vortex filaments appear only when the system is an excited state.

There are two popular theoretical approaches to superfluid turbulence. In one approach the Gross-Pitaevski (GP) equation [8] is used to model the system with the wave function thought of as describing a superfluid condensate. The non-linear GP equation can be understood as arising from a quantum field theory for Helium which has four point field interactions. Kolmogorov spectrum in superfluid turbulence has been numerically studied using the GP equation [9]. The zeros of the wave function describing the condensate are identified as the observed vortex filaments. In an alternative approach an effective theory of interacting filaments is considered and their dynamics is studied [10].

Degrees of freedom of both kinds, namely the condensate and the filaments, can be dealt with in the simple effective locally scale invariant theory presented here. Assuming the spatial metric to be flat and choosing an ansatz that the gauge field is vortex-like, say, with several cores all parallel between two plates modelled by a sum of delta functions for simplicity, we find that the modulus of the Schroedinger field is a non-zero constant everywhere excepting the support of the delta functions, where it vanishes. Thus the Schroedinger field and the loci of its zeros lend themselves to the interpretation as the superfluid condensate and the vortex filaments, respectively.

In order to make contact with the two approaches mentioned above, we then proceed to obtain effective theories of the Schroedinger field and the gauge field separately by integrating out the other, choosing the metric to be flat in both cases. Integrating out the gauge field from the theory produces the action for a GP system in a quadratic potential with corrections higher order in the wave function, while integrating out the Schroedinger field yields an effective theory for the gauge field, modelling interacting filaments. Using an approximate form of this interacting filament action the velocity of separation between a pair of filaments is estimated as a function of time. The result we obtain, valid for small times, is in agreement with experimental results [2].

In section 2 we present the Ricci-gauged action for the Schroedinger field. A solution to the equations of motion for a special arrangement of filaments is presented in section 3. In section 4 we obtain two effective theories by integrating out the Schroedinger field and the gauge field. We briefly discuss about dynamics in section 5 before concluding in section 6.

2 Scale-invariant Schroedinger equation

We shall construct a scale-invariant action starting from the action for a free Schroedinger field. The free Schroedinger equation, in operator form, allows Bose-Einstein condensation and is thus an appropriate starting point for a theory of quantum turbulence. We restrict ourselves to constructing a scale invariant model only for the spatial coordinates. First, the free system is rendered globally scale invariant by introducing a time-independent metric for the three spatial directions. It is then rendered invariant under local scaling by introducing a gauge field.

The action of the Schroedinger field in $\mathbf{R}^1 \times \mathbf{R}^3$, with the first factor designating time, t , and the second one corresponding to the spatial coordinates $\mathbf{x} = (x^1, x^2, x^3) = (x, y, z)$ is

$$\mathcal{S}(\psi, g) = i \int \psi^* \partial_t \psi \sqrt{g} dt d^3x - \frac{1}{2m} \int g^{ij} \partial_i \psi^* \partial_j \psi \sqrt{g} dt d^3x, \quad (1)$$

where we have introduced a metric g on $\mathbf{R}^3$ and ∂_i designates the derivative with respect to x^i . The second term of the action is invariant under the global scaling transformation of the field ψ and the metric

$$\begin{aligned} \psi &\longmapsto e^{-\Lambda/4} \psi, \\ g_{ij} &\longmapsto e^\Lambda g_{ij}, \end{aligned} \quad (2)$$

where Λ is a constant. Let us point out that the scale invariance could not be effected without the metric. Moreover, as mentioned before, we do not impose scale invariance on the first term involving temporal derivative of the Schroedinger field. We now promote this global scaling symmetry to a spatially local symmetry by allowing spatial dependence of Λ [11]. For this we need to introduce a gauge field A_i and define covariant derivatives of the field ψ and the metric g as [12]

$$\begin{aligned} D_i \psi &= \partial_i \psi - \alpha A_i \psi, \\ D_i g_{km} &= \partial_i g_{km} + 4\alpha A_i g_{km}, \end{aligned} \quad (3)$$

where α is a real parameter, to be restricted later. Then under the gauge transformation

$$\begin{aligned} \psi &\longmapsto e^{-\Lambda(\mathbf{x})/4} \psi, \\ g_{ij} &\longmapsto e^{\Lambda(\mathbf{x})} g_{ij}, \\ A_i &\longmapsto A_i - \frac{1}{4\alpha} \partial_i \Lambda(\mathbf{x}), \end{aligned} \quad (4)$$

with space-dependent Λ , the covariant derivatives of the scalar field ψ and the metric transform as

$$\begin{aligned} D\psi &\longmapsto e^{-\Lambda(\mathbf{x})/4} \psi, \\ D_i g_{jk} &\longmapsto e^{\Lambda(\mathbf{x})} D_i g_{jk}. \end{aligned} \quad (5)$$

Hence replacing the derivatives with respect to the spatial coordinates in the second term of (1) by covariant derivatives we obtain the action

$$\mathcal{S}(\psi, A, g) = i \int \psi^* \partial_t \psi \sqrt{g} dt d^3x - \frac{1}{2m} \int g^{ij} D_i \psi^* D_j \psi \sqrt{g} dt d^3x, \quad (6)$$

which is invariant under the gauge transformations (4). One can add one more gauge-invariant term to the above action involving the curvature and the gauge field [7]. To this end let us define Christoffel symbols [12] as

$$\tilde{\Gamma}_{jk}^i = \frac{1}{2}g^{im}(D_j g_{mk} + D_k g_{mj} - D_m g_{jk}). \quad (7)$$

By (5), the Christoffel symbol is invariant under the local scaling transformations (4). Then the Ricci tensor ensuing from this Christoffel symbol defined as

$$\tilde{R}_{jkl}^i = \partial_l \tilde{\Gamma}_{jk}^i - \partial_k \tilde{\Gamma}_{jl}^i + \tilde{\Gamma}_{ml}^i \tilde{\Gamma}_{jk}^m - \tilde{\Gamma}_{mk}^i \tilde{\Gamma}_{jl}^m \quad (8)$$

is also invariant under the gauge transformation (4). The resulting scalar curvature defined as

$$\tilde{R} = g^{jl} \tilde{R}_{jil} \quad (9)$$

then transforms as $\tilde{R} \mapsto e^{-\Lambda} \tilde{R}$ under (4). Hence,

$$\int |\psi|^2 \tilde{R} \sqrt{g} dt d^3x \quad (10)$$

is invariant under the gauge transformation. It can be checked that no other term involving curvature tensors or derivatives of A or their combinations can be made gauge invariant in this fashion to yield a local polynomial action. In particular the term F_{ij}^2 constructed from the gauge field A_i is not scale invariant in three dimensions, nor can it be made gauge invariant in a polynomial action.

The Ricci scalar $\tilde{R}$ defined above can be related to the Ricci scalar corresponding to the metric g by expanding $\tilde{\Gamma}_{jk}^I$ using (3) [12], resulting into

$$\begin{aligned} \tilde{R} &= R + 8\alpha \nabla_i A^i + 8\alpha^2 A^2, \\ &= R + \frac{8\alpha}{\sqrt{g}} \partial_i (\sqrt{g} A^i) + 8\alpha^2 A^2, \end{aligned} \quad (11)$$

where we used $A^2 = g^{ij} A_i A_j = A^i A_i$, ∇_i and R denote, respectively, the covariant derivative with respect to x^i and the scalar curvature corresponding to the metric g .

Thus, using (11) in (10) and adding to (6) the unique spatially scale-invariant action in 1 + 3 dimensions is given by

$$\begin{aligned} \mathcal{S}(\psi, A, g) &= \int \sqrt{g} dt d^3x \left(i\psi^* \partial_t \psi - \frac{1}{2m} g^{ij} (\partial_i \psi^* \partial_j \psi - \alpha A_i \partial_j |\psi|^2 + \alpha^2 A_i A_j |\psi|^2) \right) \\ &\quad + \beta \int dt d^3x |\psi|^2 (\sqrt{g} R + 8\alpha \partial_i (\sqrt{g} A^i) + 8\alpha^2 \sqrt{g} A^2), \end{aligned} \quad (12)$$

where β is another real parameter. We proceed to study the properties of this unique locally scale invariant three-dimensional system.

3 Solutions for special geometrical arrangement

Physical configurations are obtained as solutions to the Euler-Lagrange equations ensuing from the action (12) by variation of the fields. The equation obtained upon varying the metric g is

$$\begin{aligned} & -\frac{1}{2}g_{ij} \left(i\psi^* \partial_t \psi - \frac{1}{2m}(\partial_i \psi^* \partial^i \psi - \alpha A^i \partial_i |\psi|^2 + \alpha^2 A^2 |\psi|^2) + \beta |\psi|^2 (R + 8\alpha \nabla_i A^i + 8\alpha^2 A^2) \right) \\ & - \frac{1}{2m}(\partial_i \psi^* \partial_j \psi - \alpha A_i \partial_j |\psi|^2 + \alpha^2 A_i A_j |\psi|^2) + \beta |\psi|^2 (R_{ij} + 8\alpha \nabla_i A_j + 8\alpha^2 A_i A_j) = 0 \end{aligned} \quad (13)$$

Equations from the variations of the gauge field A and the Schroedinger field ψ^* are, respectively,

$$\partial_i |\psi|^2 = \alpha A_i |\psi|^2, \quad (14)$$

$$i\sqrt{g}\partial_t \psi + \frac{1}{2m}\partial_i(\sqrt{g}g^{ij}\partial_j \psi) + \beta\sqrt{g}R\psi + \alpha(8\beta - \frac{1}{2m}) (\partial_i(\sqrt{g}A^i) + \alpha\sqrt{g}A^2) \psi = 0, \quad (15)$$

where $\partial^2 = \partial^i \partial_i$ and we assumed

$$16m\beta \neq 1. \quad (16)$$

Assuming the metric to be flat, $g_{ij} = \eta_{ij}$, as is expected for the physical system under consideration, and using (14), the equation (13) simplifies to

$$3i\psi^* \partial_t \psi - \frac{1}{2m}|\partial\psi|^2 + 8\beta|\psi|^2 (\partial^2 \log |\psi|^2 + (\partial \log |\psi|^2)^2) = 0. \quad (17)$$

Similarly, (15) reduces to

$$i\partial_t \psi + \frac{1}{2m}\partial^2 \psi + (8\beta - \frac{1}{2m}) (\partial^2 \log |\psi|^2 + (\partial \log |\psi|^2)^2) \psi = 0. \quad (18)$$

For a flat metric we thus need to solve equation (14) and the simplified equations (17) and (18). Now, equation (14) is solved with

$$|\psi|^2 = |\psi_0|^2 \exp \left(\alpha \int A_i dx^i \right), \quad (19)$$

where ψ_0 is a constant. We set $\psi_0 = \sqrt{N/V}$, V denoting the volume of the superfluid.

We now relate the field ψ to the superfluid condensate and the gauge field A to the filaments by choosing the gauge field to be localised on filaments or strings. Let us consider a set of strings, represented in $\mathbf{R}^3$ as $\mathbf{x}^\perp = \mathbf{x}_a^\perp(z)$ where $\mathbf{x}^\perp = (x, y)$ and a runs over the number of strings. Choosing the gauge field to be supported on a set of strings as

$$A_i = \sum_a \rho_{ia}(z) \delta^{(2)}(\mathbf{x}^\perp - \mathbf{x}_a^\perp(z, t)), \quad (20)$$

and $\alpha < 0$, $|\psi|^2$ vanishes on the support of the gauge field, that is the strings, and equals the constant $|\psi_0|^2$ everywhere else.

This also solves equations (17) and (18) since on the support of the delta functions ψ vanishes, while A vanishes outside the support. The constant wave function can be interpreted as representing a Bose-Einstein condensate. The filaments then represent excitations of the system corresponding to injection of energy. Thus the simple model has features which suggest that excitations can be described either as filaments or as zeros of the condensate. In view of this in the next section we proceed to construct an effective action for the system in terms of the wave function by integrating out the gauge field and in terms of the gauge field by integrating out the wave function.

4 Effective theories

As mentioned in the introduction, two important approaches in studying superfluid turbulence are in terms of a Bose-Einstein condensate and in terms of filaments or strings. In the previous section we found that the local scale invariant theory allows configurations of both type. The gauge field can be so chosen as to model filaments resulting into solutions for the Schroedinger field modelling a Bose-Einstein condensate. In the present section we write down effective theories starting from the action (12). Integrating out the gauge field by means of a path-integral results into an effective theory for the condensate, while integrating out the Schroedinger field yields a theory for the strings. We now proceed to present these two theories in the next subsections. Throughout this section we set, for simplicity, the metric to be flat, $g_{ij} = \eta_{ij}$ and define $S(\psi, A) = S(\psi, A, \eta)$.

4.1 Effective theory for the condensate

First let us integrate out the gauge field from the action (12) with a flat metric to obtain the effective action for the condensate ψ defined by the path integral

$$e^{iS_{\text{eff}}(\psi)} = \frac{1}{\sqrt{\pi}} \int \mathcal{D}A e^{iS(\psi, A)}. \quad (21)$$

We shall focus only on the terms of (12) involving the gauge field. Let us also introduce fluctuation in the Schroedinger field ψ over the classical condensate represented by ψ_0 as

$$\psi = \psi_0 + \chi, \quad (22)$$

where χ denotes the fluctuation. Indeed, we shall obtain an effective action for the fluctuation χ . Writing

$$\Theta = \left(\frac{1}{2m} - 8\beta \right) \alpha^2 \psi_0^2 (1 + (\chi + \chi^*)/\psi_0 + |\chi|^2/\psi_0^2), \quad (23)$$

$$J_i = \alpha \left(\frac{1}{2m} - 8\beta \right) \partial_i (|\chi|^2 + \psi_0(\chi + \chi^*)), \quad (24)$$

the relevant part of the path integral assumes the form

$$(\det \Theta)^{-1/2} e^{(J, \Theta^{-1} J)/4}. \quad (25)$$

Let us further simplify the situation by ignoring the interaction term involving J up to order $\mathcal{O}(\psi_0^2)$. Then the only contribution to the effective action comes from the determinant. The overall factor in Θ contributes but a constant to the action and may thus be ignored. The rest gives

$$\begin{aligned} & \det(1 + (\chi + \chi^*)/\psi_0 + |\chi|^2/\psi_0^2)^{-1/2}, \\ & = e^{-\frac{1}{2}\text{Tr}\log(1+(\chi+\chi^*)/\psi_0+|\chi|^2/\psi_0^2)}, \end{aligned} \quad (26)$$

where the trace is given by the volume integral in $\mathbf{R} \times \mathbf{R}^3$, $\text{Tr} = \int dt d^3x$. Expanding the logarithm we obtain the potential in the effective action. In addition, if we assume that the fluctuation χ to be odd under spatial reflection, then up to fourth order in the fluctuation, the effective potential becomes

$$-\frac{1}{2\psi_0^2}(|\chi|^2 - \frac{1}{2}(\chi + \chi^*)^2) + \frac{1}{4\psi_0^4}(|\chi|^4 + \frac{1}{2}(\chi + \chi^*)^4), \quad (27)$$

containing only terms with even powers of χ . Non-linear Schroedinger equation with higher order terms in the potential has been considered earlier for describing Bose-Einstein condensate [13, 14]. The quartic term gives rise to a cubic term in the equation of motion for χ , as in the GP equation. The effective action contains corrections to it as well as a harmonic oscillator term, which can be used to describe a trapped Bose-Einstein condensate.

4.2 Effective theory for the filaments

Next let us integrate out the Schroedinger field ψ from the action (12) to obtain an effective theory for the gauge field A . In view of the classical solution obtained in the previous section, the effective theory for the gauge field describes the superfluid system in terms of filaments. Description of superfluids in terms of dynamics and interaction of filaments is an important approach. The effective action is defined by the path integral

$$e^{iS_{\text{eff}}(A)} = \frac{1}{\sqrt{\pi}} \int \mathcal{D}\psi e^{iS(\psi, A)}, \quad (28)$$

with a flat metric, as before. In order to integrate out ψ we rewrite the action by setting $g_{ij} = \eta_{ij}$ in (12) as

$$S(\psi, A) = \int dt d^3x \psi^* \left(i\partial_t + \frac{1}{2m}\partial^2 + \alpha(8\beta - \frac{1}{2m})(\partial_i A^i + \alpha A^2) \right) \psi \quad (29)$$

up to boundary terms. The effective action is then given by

$$S_{\text{eff}}(A) = -\frac{1}{2}\text{Tr}\log\left(i\partial_t + \frac{1}{2m}\partial^2 + \alpha(8\beta - \frac{1}{2m})(\partial_i A^i + \alpha A^2)\right), \quad (30)$$

where the trace is defined as $\text{Tr} = \int dt d^3x$ as before. To the lowest order the effective action is then given by

$$S_{\text{eff}}(A) = -\frac{1}{2}\alpha(8\beta - \frac{1}{2m}) \int dt d^3x \frac{1}{i\partial_t + \frac{1}{2m}\partial^2} (\partial_i A^i + \alpha A^2), \quad (31)$$

obtained by expanding the logarithm. Let us concentrate on the term

$$-\frac{1}{2}\alpha^2(8\beta - \frac{1}{2m}) \int dt d^3x \frac{1}{i\partial_t + \frac{1}{2m}\partial^2} A^2. \quad (32)$$

The Green's function for the operator

$$i\partial_t + \frac{1}{2m}\partial^2 \quad (33)$$

is given by

$$G(\Delta x, \Delta t) = e^{i\pi/4} \frac{m^{3/2}}{\sqrt{2}(\Delta t)^{3/2}} e^{6i(m\Delta x)^2/\Delta t}. \quad (34)$$

Using the Green's function we can rewrite (32) as

$$-\frac{1}{2}\alpha^2(8\beta - \frac{1}{2m}) \int dt d^3x \int dt' d^3x' A_i(\mathbf{x}) G(\mathbf{x} - \mathbf{x}', t - t') A_i(\mathbf{x}') \delta^{(3)}(\mathbf{x} - \mathbf{x}'). \quad (35)$$

Using a regulated form of the delta function, namely,

$$\delta(x) = \frac{1}{\sqrt{\pi}\ell} e^{-x^2/\ell^2} \quad (36)$$

we can write down the effective interaction between strings as

$$-\frac{1}{2}\alpha^2(8\beta - \frac{1}{2m}) \int dt d^3x \int dt' d^3x' A_i(\mathbf{x}) \hat{G}(\mathbf{x} - \mathbf{x}', t - t') A_i(\mathbf{x}'), \quad (37)$$

in terms of the approximate Green's function

$$\hat{G}(\mathbf{x} - \mathbf{x}', t - t') = \frac{1}{\pi^{3/2}\ell^3} G(\mathbf{x} - \mathbf{x}', t - t') e^{-|\mathbf{x} - \mathbf{x}'|^2/\ell^2}. \quad (38)$$

Certain qualitative features of the model can be studied based on these expressions. The effective interaction term can be considered to represent the scattering of two filaments. The variables $\mathbf{x}$, $\mathbf{x}'$ represent different points on a string at different times. If we think of this as due to scattering then the way the points change with time will give the velocity of separation between strings after collision. For small times if we set the time difference equal to Δt and the corresponding spatial separation as $\Delta \mathbf{x}$, then the Green's function tells us how these variables are related. Writing $\Delta \mathbf{x} = \mathbf{v}\Delta t$, where $\mathbf{v}$ represents the average separation speed between two points we get a term

$$e^{6imv^2\Delta t} \times e^{-(\mathbf{x} - \mathbf{x}')^2/\ell^2} \quad (39)$$

This term falls off exponentially as $\Lambda(\mathbf{x} - \mathbf{x}')^2$ unless we set $v^2\Delta t = K$, with K/ℓ^2 a constant less than unity, in which case it falls off as $e^{-K\Delta t/\ell^2}$. The term also gets suppressed by the oscillating first term unless the velocity and time are correlated as suggested. Such a correlation is the observed behaviour of the velocity of separation between two strings. From this, as shown, a power law for the velocity distribution function follows as observed [2].

5 Putting in dynamics

We have neglected introducing an explicit term in the action to describe the dynamics between filaments. This can be done, but will, as mentioned before, lead to breaking of scaling invariance. Without a principle to guide us such a term will be ad hoc. A natural choice would be a Chern-Simons-like term as it is most appropriate in the low energy region. In our qualitative discussion of the filament term we have thus refrained from exploring this possibility. One reason for doing this is the fact that for the condensate wave function we do know the dynamical term and we do get the GP equation which is known to work well.

6 Conclusions

A locally scale invariant Schroedinger system has been shown to generate a GP equation for describing superfluid turbulence. The effective theory requires gauge fields. We have seen that for a low-density filament system there is a correlation between the zeros of the GP equation and the location of the filaments. This identification is standard but it is nice to see it emerge in a natural way from scale invariance. The velocity of separation result for filaments that we obtain is rather speculative and depends on a specific regularisation but, again it is nice to see a result in agreement with observations emerge. Our main result is thus to show that Weyl's original gauge invariance idea does lead to a sensible model for superfluid turbulence as it leads to the GP equation. For the filament model we need to introduce dynamics between filaments before we can use the effective theory properly. It would be of interest to introduce, for instance, a Chern-Simons term and study filament dynamics.

We find it satisfying that the degrees of freedom for superfluid turbulence naturally emerge from Weyl's idea of local scale invariance and the approach even leads to an effective theory, the Gross Pitaevski equation, that is widely used.

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References

- [1] M. Tsubota, "Quantum Turbulence", J. Phys. Soc. Jpn. **77**, 111006.(2008). [arXiv:0806.2737(cond-mat)]
- [2] M. Paoletti *et al*, "Velocity Statistics Distinguish Quantum Turbulence from Classical Turbulence", Phys. Rev. Lett. **101**.154501(2008).

- [3] D. Kivotides *et. al.*, “Velocity spectra of superfluid turbulence”, *Europhys. Lett.*, **57**.845 (2002).
- [4] H. Weyl, “Gravitation and Electricity”, *Sitzungsber. Preuss. Akad. Berlin* (1918) 465. Collected in [5].
- [5] L. O’Raifeartaigh, “The Dawning of Gauge Theory”, *Princeton Series in Physics*, 1997.
- [6] F. London, “Quantum-mechanical interpretation of Weyl’s theory”, *Zeit. F. Phys.*, **42**:375,1927.
- [7] A. Iorio, L. O’Raifeartaigh, I. Sachs *et al.*, “Weyl gauging and conformal invariance,” *Nucl. Phys.* **B495**, 433 (1997). [hep-th/9607110].
- [8] E. Gross, “Structure of a quantized vortex in boson systems”, *Il Nuovo Cimento* **20**.(1961).454.
L. Pitaevskii, “Vortex Lines in an Imperfect Bose Gas”, *Soviet Physics JETP-USSR* (Woodbury, New York: American Institute of Physics) **13**.(1961).451.
- [9] M. Kobayashi and M. Tsubota, “Kolmogorov Spectrum of Superfluid Turbulence: Numerical Analysis of the Gross-Pitaevskii Equation with a Small-Scale Dissipation” *Phys. Rev. Lett.* **94**, 065302 (2005)
- [10] K. Schwarz, “Three-dimensional vortex dynamics in superfluid ^4He : Line-Line and line-boundary interactions, *Phys. Rev.* **B31**(1985)5782.
- [11] C. Connaughton and S. Sen, “Local scale invariance and weak wave Turbulence”, (Unpublished).
- [12] T. Padmanabhan, “Conformal invariance, gravity and massive gauge theories”, *Class. Quantum Grav.* **2**(1985)L105.
- [13] I. V. Barashnikov and V. G. Makhankov, “Soliton-like bubbles in a system of interacting bosons”, *Phys. Lett.* **A128**(1988)52.
- [14] A. Farina, “From Ginzburg-Landau to Gross-Pitaevskii”, *Monatsh. Math.* **139**, 265269 (2003)