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Reliablity of Dynamic Structural Systems

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ABSTRACT: In this paper we derive the exact expression for the reliability of a series system of components of a dynamic structure subjected to a stochastic loading process. The reliability is formulated in terms of the failure probability of this series system, where the failure for each component is defined as exceedance of a critical level during a predefined period of time. It will be shown that the failure probability of such a series system is given in terms of a multidimensional extreme value distribution. Due to the availability of the recently developed ACER methodology, which allows a precise estimation of multidimensional extreme value distributions, it is possible to estimate this reliability.

1. INTRODUCTION

The estimation of the reliability of dynamic structural systems subjected to environmental loads, meets with several challenges. One is the long term perspective. In the case of e.g. a jacket platform used for oil production at an offshore location, this platform has to be designed to withstand the environmental loads for a period of several decades. To properly deal with this challenge ideally requires long term modelling. This is often avoided by introducing a simplifying procedure known as a design storm, which dramatically reduces the amount of analysis needed. This simplifying procedure is nowadays typically justified by using an environmental contour method, cf. e.g. (Haver and Winterstein, 2008). The next challenge is how to formulate the reliability problem itself. In this paper we shall assume that the reliability problem can be formulated as a series system of a few critical dynamic components, and that failure of the system is related to first passage of critical response levels over designated periods of time for the components of the system.

We shall derive the exact expression for the reliability of such a series system of components of a dynamic structure subjected to stochastic loading processes. The reliability is formulated in terms of the failure probability of this series system, where the failure for each component is defined as exceedance of a critical level during a predefined period of time. It will be shown that the failure probability of such a series system is given in terms of a multidimensional extreme value distribution. Due to the availability of the recently developed ACER methodology, which allows a precise estimation of multidimensional extreme value distributions, cf. (Naess, 2011, 2023) and (Naess and Karpa, 2015), it is in principle possible to estimate this reliability.

Relatively little work has been published on the specific problem raised in this paper. The main reason would seem to be the lack of effective methods for calculation or estimation of the reliability of dynamic systems. Indeed, even the classical time-invariant system reliability problems formulated in terms of random variables, have yet to be solved satisfactorily. Only approximate numerical procedures have been developed, cf. e.g. (Naess et al., 2009). It is therefore to be expected that the time-variant system reliability problem, which from a theoretical point of view would require much more sophisticated methodologies, is still a long way off from reaching satisfactory solution strategies, in spite of recent efforts, cf. e.g. (Qian et al., 2023). Hence, the purpose of this paper is not to claim the establishment of a fully workable solution, but only to point out certain possibilities for obtaining approximate numerical estimates of the failure probability of dynamic systems applying the multidimensional ACER method, cf. (Naess, 2011, 2023).

2. PROBLEM FORMULATION

Our problem formulation will be in terms of a stochastic vector response process whose components represent a set of selected structural responses of a dynamic structure subjected to random environmental loads like e.g. wave and wind loads. To simplify the exposition, we have chosen to focus on the bivariate case instead of the general multivariate. The reason is mainly one of expediency, because the general multivariate case would require a more heavy machinery of notation without contributing to a deeper understanding of the issues involved. And, in fact, the extension of the bivariate to the general multivariate case is conceptually rather straight forward.

Consider a bivariate stochastic response process $Z(t) = \big(X(t), Y(t)\big)$ with dependent component processes, which has been observed over a time interval, $(0,T)$ say. Since a continuous in time stochastic process has to be sampled in order to be stored in a computer, assume that the sampled values $(X_1,Y_1),\ldots,(X_N,Y_N)$ are allocated to the (usually equidistant) discrete times $t_1,\ldots,t_N$ in $(0,T)$. The selection of sampling frequency depends on the purpose of the study. The reliability problem to be studied in this paper can now be formulated in terms of the following two extreme values over the period $(0,T)$: $M_{x,N} = \max\big\{X_j\,;\, j=1,\ldots,N\big\}$ and $M_{y,N} = \max\big\{Y_j\,;\, j=1,\ldots,N\big\}$. The reliability of the dynamic system is expressed in terms of the following failure probability relative to two first passage failure events,

$$
\begin{aligned}
P_f =& \mathrm{Prob}\big(M_{x,N} > \xi_c \cup M_{y,N} > \eta_c\big)\\
=& 1 - \mathrm{Prob}\big(M_{x,N} \le \xi_c \cap M_{y,N} \le \eta_c\big),
\end{aligned}
\tag{1}
$$

where ξ_c and η_c denote the critical response levels.

Therefore, the failure probability can be written in terms of the joint probability distribution of the extreme value vector $(M_{x,N}, M_{y,N})$: $P(\xi,\eta) = \mathrm{Prob}\big(M_{x,N} \le \xi \cap M_{y,N} \le \eta\big) = \mathrm{Prob}\big(M_{x,N} \le \xi, M_{y,N} \le \eta\big)$.

Our goal in this paper is to construct a methodology that allows us to accurately determine empirically the joint distribution function of the extreme value vector $(M_{x,N}, M_{y,N})$, Specifically, we want to estimate $P(\xi,\eta)$ accurately for large values of ξ and η, providing an estimate of the failure probability by,

$$
P_f = 1 - P(\xi_c, \eta_c). \tag{2}
$$

For notational convenience, it is expedient to introduce the non-exceedance event $\mathscr{C}_{kj}(\xi,\eta) = \big\{X_{j-1} \le \xi, Y_{j-1} \le \eta, \ldots, X_{j-k+1} \le \xi, Y_{j-k+1} \le \eta\big\}$ for $1 \le k \le j \le N+1$. Then, from the definition of $P(\xi,\eta)$ it follows that,

$$
\begin{aligned}
P(\xi,\eta) =& \mathrm{Prob}\big(\mathscr{C}_{N+1,N+1}(\xi,\eta)\big)\\
&= \mathrm{Prob}\big(X_N \le \xi, Y_N \le \eta \,|\, \mathscr{C}_{NN}(\xi,\eta)\big)\\
&\quad \cdot \mathrm{Prob}\big(\mathscr{C}_{NN}(\xi,\eta)\big)\\
&= \prod_{j=2}^{N} \mathrm{Prob}\big(X_j \le \xi, Y_j \le \eta \,|\, \mathscr{C}_{jj}(\xi,\eta)\big)\\
&\quad \cdot \mathrm{Prob}\big(\mathscr{C}_{22}(\xi,\eta)\big).
\end{aligned}
\tag{3}
$$

Based on Eq. (3) and the properties of conditional probability, a sequence of approximations may be introduced which converges to the target distribution $P(\xi,\eta)$, cf. (Naess, 2011, 2023). In practice, it is therefore assumed that the following representation applies for a suitably chosen k: For $\xi, \eta \to \infty$,

$$
\begin{aligned}
&P(\xi,\eta)\\
&\approx \exp\Big\{-\sum_{j=k}^{N}\big(\alpha_{kj}(\xi;\eta) + \beta_{kj}(\eta;\xi) - \gamma_{kj}(\xi,\eta)\big)\Big\},
\end{aligned}
\tag{4}
$$

where we have used the notation $\alpha_{kj}(\xi;\eta) = \mathrm{Prob}\big(X_j > \xi \,|\, \mathscr{C}_{kj}(\xi,\eta)\big)$, $\beta_{kj}(\eta;\xi) = \mathrm{Prob}\big(Y_j > \eta \,|\, \mathscr{C}_{kj}(\xi,\eta)\big)$ and $\gamma_{kj}(\xi,\eta) = \mathrm{Prob}\big(X_j > \xi, Y_j > \eta \,|\, \mathscr{C}_{kj}(\xi,\eta)\big)$. Note that Eq. (4) applies equally well to stationary and nonstationary time series. This is due to the fact that the possible time dependence of the conditional exceedance probabilities $\alpha_{kj}(\xi;\eta)$, $\beta_{kj}(\eta;\xi)$ and $\gamma_{kj}(\xi,\eta)$ has been retained, which is reflected in the presence of the time parameter j.

From Eq. (4) it emerges that for the estimation of the bivariate extreme value distribution it is necessary and sufficient to estimate the sequence of functions $\{\alpha_{kj}(\xi;\eta)+\beta_{kj}(\eta;\xi)-\gamma_{kj}(\xi,\eta)\}_{j=k}^{N}$. To get a more compact representation, it is expedient to introduce the concept of a k'th order bivariate average conditional exceedance rate (ACER) function as follows: For $k=1,2,\ldots$,

$$\mathscr{E}_k(\xi,\eta) = \frac{1}{N-k+1}\sum_{j=k}^{N}\left(\alpha_{kj}(\xi;\eta)+\beta_{kj}(\eta;\xi)-\gamma_{kj}(\xi,\eta)\right) \tag{5}$$

Hence, when $N \gg k$, we may write: For $\xi,\eta \to \infty$,

$$P(\xi,\eta) \approx \exp\{-(N-k+1)\,\mathscr{E}_k(\xi,\eta)\}\,. \tag{6}$$

The numerical estimation of the bivariate ACER function for the observed stationary or non-stationary time series consists in counting of the appropriate exceedance events. The estimation procedure is derived in more details in (Naess and Karpa, 2015) or (Naess, 2023). Recently, a simplified method has been proposed to replace the exact formulation of the failure probability by constructing a one-dimensional stream of peak events obtained by coalescing the corresponding ones for all components, cf. (Gaidai et al., 2022). In spite of statements made by the authors, the proposed method may give accurate results only for cases when the components are nearly independent and have weak memory effects, implying that k may have values 1 or 2, depending on how the data have been sampled. If higher k-values are needed for accurate results, the conditional counting will give different results for the exact and the proposed simplified method, which implies different reliability estimates.

3. TAIL REPRESENTATION OF THE EMPIRICALLY ESTIMATED ACER FUNCTIONS

It is evident, that, in general, the empirically estimated k-th order bivariate ACER function does not provide enough information for estimation of high quantiles of the joint extreme value distribution relevant for reliability estimation. In addition, the exact behavior of the bivariate ACER as a continuous function of two variables cannot be decided using available statistical data. Therefore, the sub-asymptotic functional form of the ACER surface $\mathscr{E}_k(\xi,\eta)$ can possibly be obtained approximately by the copula representation of a bivariate extreme value distribution.

From the result by Sklar (1959), for any pair of random variables (X,Y) with marginal distribution functions $F_x(\xi)$ and $G_y(\eta)$, the joint distribution function $H_{xy}(\xi,\,\eta)=\text{Prob}(X\le\xi, Y\le\eta)$ can be presented by a bivariate copula $C(u,v)$ as follows: $H_{xy}(\xi,\,\eta)=C\big(F_x(\xi),\ G_y(\eta)\big)$, cf. e.g. (Nelsen, 2006), (Balakrishnan and Lai, 2009). This result applies to any bivariate extreme value distribution as well.

Considering the above, and using the result of Pickands (1981), any bivariate extreme value distribution $H_{xy}(\xi,\,\eta)$ with marginal univariate extreme value distributions $F_x(\xi)$ and $G_y(\eta)$ is given by the formula,

$$H_{xy}(\xi,\,\eta)=\exp\Big\{\log\big(F_x(\xi)G_y(\eta)\big) \cdot\mathscr{D}\Big(\frac{\log\big(F_x(\xi)\big)}{\log\big(F_x(\xi)G_y(\eta)\big)}\Big)\Big\}, \tag{7}$$

where the Pickands dependence function $\mathscr{D}(\cdot)$ is a convex function and satisfies $\mathscr{D}(x):[0,1]\longmapsto[\max(x,\,1-x),\ 1]$, cf. (Gudendorf and Segers, 2010).

We assume that asymptotically consistent marginal extreme value distributions $F_x(\xi)$ and $G_y(\eta)$ are represented by the corresponding univariate ACER functions, that is,

$$\begin{aligned} F_x(\xi) &\approx \exp\{-(N-k+1)\varepsilon_k^x(\xi)\}\,,\ \xi\ge\xi_1\,,\\ G_y(\eta) &\approx \exp\{-(N-k+1)\varepsilon_k^y(\eta)\}\,,\ \eta\ge\eta_1\,, \end{aligned} \tag{8}$$

where the sub-asymptotic functional form of the univariate ACER function is represented as $\varepsilon_k^x(\xi)=q_k^x\exp\{-a_k^x(\xi-b_k^x)^{c_k^x}\}$ with a similar definition of $\varepsilon_k^y(\eta)$, cf. (Naess and Gaidai, 2009).

Now, substituting Eq. (8) into Eq. (7) the following representation of the bivariate extreme value distribution applies:

$$H_{xy}(\xi,\eta) = \exp\Big\{ -(N-k+1)\Big(\varepsilon_k^x(\xi)+\varepsilon_k^y(\eta)\Big) \cdot \mathscr{D}\left(\frac{\varepsilon_k^x(\xi)}{\varepsilon_k^x(\xi)+\varepsilon_k^y(\eta)}\right)\Big\}. \quad (9)$$

On the other hand, as we have discovered before in Eq. (6), the bivariate extreme value distribution can be expressed through the bivariate ACER function: $H_{xy}(\xi,\eta) = \exp\{-(N-k+1)\mathscr{E}_k(\xi,\eta)\}$. Thereby, the functional form of the bivariate ACER surface can possibly be obtained by:

$$\mathscr{E}_k(\xi,\eta) = \Big(\varepsilon_k^x(\xi)+\varepsilon_k^y(\eta)\Big)\mathscr{D}\left(\frac{\varepsilon_k^x(\xi)}{\varepsilon_k^x(\xi)+\varepsilon_k^y(\eta)}\right). \quad (10)$$

Consequently, our aim now is to find the dependence function $\mathscr{D}(\cdot)$ that would provide optimal fit of the parametrical surface defined by Eq. (10) to the empirical bivariate ACER $\hat{\mathscr{E}}_k(\xi,\eta)$.

Subject to the form of the dependence function $\mathscr{D}(\cdot)$, different parametric differentiable and non-differentiable models can be considered. A differentiable model suitable for our purpose is acquired by setting $\mathscr{D}(x) = [x^m + (1-x)^m]^{1/m}$ for $m \geq 1$. This is the Gumbel logistic (GL) model (Gumbel, 1961; Hougaard, 1986). The functional form of the bivariate ACER surface in the GL case becomes,

$$\mathscr{G}_k(\xi,\eta) = \Big[\big(\varepsilon_k^x(\xi)\big)^m + \big(\varepsilon_k^y(\eta)\big)^m\Big]^{\frac{1}{m}}, \quad (11)$$

which is a form that can easily be extended to any dimension. A good discussion of possible extensions to higher dimensional forms is provided by Beirlant et al. (2004).

In the dependence function $\mathscr{D}(\cdot)$ for the GL model, Tawn (1988) added extra parameters ϕ and θ to get further flexibility. This leads to the Asymmetric logistic (AL) model, which sets $\mathscr{D}(x) = [\phi^m x^m + \theta^m(1-x)^m]^{1/m} + (\theta-\phi)x + 1 - \theta$ with $0 \leq \theta \leq 1$, $0 \leq \phi \leq 1$, $m \geq 1$. The functional form of the bivariate ACER surface in the AL case is obtained as,

$$\mathscr{A}_k(\xi,\eta) = \Big[\big(\phi\varepsilon_k^x(\xi)\big)^m + \big(\theta\varepsilon_k^y(\eta)\big)^m\Big]^{\frac{1}{m}} + (1-\phi)\varepsilon_k^x(\xi) + (1-\theta)\varepsilon_k^y(\eta). \quad (12)$$

The optimal parameters m^* for $\mathscr{G}_k(\xi,\eta)$ and θ^*, ϕ^* and m^* for $\mathscr{A}_k(\xi,\eta)$, can be found by minimizing a mean square error function. We discuss this only for the case of $\mathscr{A}_k(\xi,\eta)$, since the case of $\mathscr{G}_k(\xi,\eta)$ then follows easily. Specifically, the mean square error function for $\mathscr{A}_k(\xi,\eta)$ is defined as,

$$F(m,\theta,\phi) = \sum_{j=1}^{N_\eta}\sum_{i=1}^{N_\xi} w'_{ij}\Big(\log\hat{\mathscr{E}}_k(\xi_i,\eta_j) - \log\mathscr{A}_k(\xi_i,\eta_j)\Big)^2, \quad (13)$$

where N_ξ, N_η are numbers of levels ξ and η, respectively, at which the ACER function have been empirically estimated, and $w'_{ij} = w_{ij}/\sum\sum w_{ij}$ with,

$$w_{ij} = \big(\log CI^+(\xi_i,\eta_j) - \log CI^-(\xi_i,\eta_j)\big)^{-2}, \quad (14)$$

denoting normalized weight factors that put more emphasis on the more reliable estimates.

The constrained optimization problem with the objective function F defined in Eq. (13), is written as,

$$\begin{cases} F(m,\theta,\phi) \to \min, \\ \{m,\theta,\phi\} \in \mathscr{S}, \end{cases} \quad (15)$$

with a constraints domain,

$$\mathscr{S} = \big\{\{m,\theta,\phi\} \in \mathbb{R}^3 \;\big|\; \theta,\phi \in [0,\,1];\ m \in [1,+\infty)\big\}. \quad (16)$$

A simple and usually effective procedure for estimating the reliability without having to construct multidimensional copulas, is obtained by using the enhanced Monte Carlo method, cf. (Naess et al., 2009). Adapted to the present scenario, one would proceed to estimate $P_f(\lambda) = P(\lambda\xi_c, \lambda\eta_c)$ for a range of λ values for $0 < \lambda < 1$ directly from the multidimensional empirical ACER surface using Eq. (6). Since the empirical ACER function typically does not allow a direct estimate of $P_f = P_f(1)$, it would be obtained by extrapolation of the available estimated $P_f(\lambda)$ values as explained in (Naess et al., 2009).

4. NUMERICAL EXAMPLE

Since the purpose of this paper is one of demonstrating a possible procedure for deep tail estimation of multivariate extreme value distributions by applying the ACER method, we have chosen to utilize data easily available to us. Our example will be using the results obtained by Naess and Karpa (2015). The simultaneous wind speed data measured along the Norwegian coast at Sula and Nordøyan Fyr weather stations (station numbers are 65940 and 75410, respectively), were analyzed to obtain numerical estimates of bivariate extreme wind speeds.

The hourly maximum of the three seconds wind gust (10 meters above the ground) were recorded during 13 years (1999 - 2012). Figure 1 demonstrates the plot of the observed data. This plot reveals a rather strong dependence between the two time series.

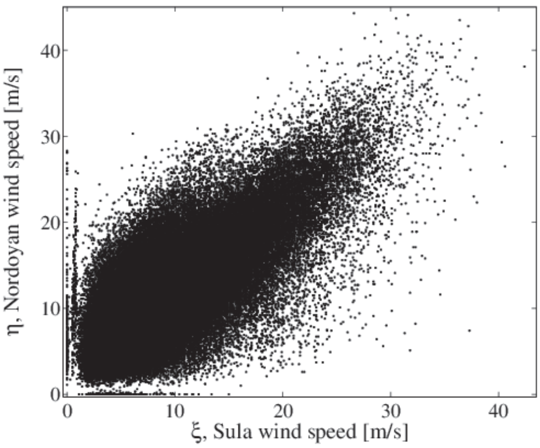

Figure 1: Coupled observations of Wind speed data observed at the Sula station (ξ axis) and at the Nordøyan Fyr station (η axis).

The Pearson product-moment correlation coefficient is found to be $\rho = 0.73$. The Kendall's rank correlation coefficient is $\tau = 0.5$ and Spearman's ρ equals to 0.68, which also indicates a nonlinear relationship between Sula and Nordøyan wind speeds.

It was decided to divide the data series into 13 one-year records for the analysis. By this, also the standard deviation of the ACER function estimates can be calculated fairly accurately.

The univariate ACER functions were estimated first, using the Matlab-based standalone downloadable application (Karpa, 2012). In Figures 2 - 3 the cascade of $\hat{\varepsilon}_1 \ldots \hat{\varepsilon}_{96}$ are plotted versus different wind speed levels. Both figures reveal that there is significant temporal dependence between consecutive data. It is also seen that this dependence effect is largely accounted for by $k = 24$ since there is a marked degree of convergence in the tail of $\hat{\varepsilon}_k$ for $k \geq 24$ in both cases. Here, for $k = 96$, which corresponds to conditioning on data recorded up to 4 days earlier, $\hat{\varepsilon}_{96}$ is considered to represent the final converged results, since $\hat{\varepsilon}_{96} \approx \hat{\varepsilon}_k$ for $k > 96$ in the tail. Therefore, there is no need to consider conditioning of an even higher order than 96. Also note that 4 days is a typical separation of wind speed data adopted in the declustering process to achieve independence between the data used in e.g. a peaks-over-threshold analysis. Figures 2 - 3 also demonstrate that for extreme value estimation $\hat{\varepsilon}_1$ can be used since the ACER functions all coalesce in the far tail. This makes it possible to choose $k = 1$, which makes much more data available for estimation, with a possible reduction of uncertainty in estimation as a result.

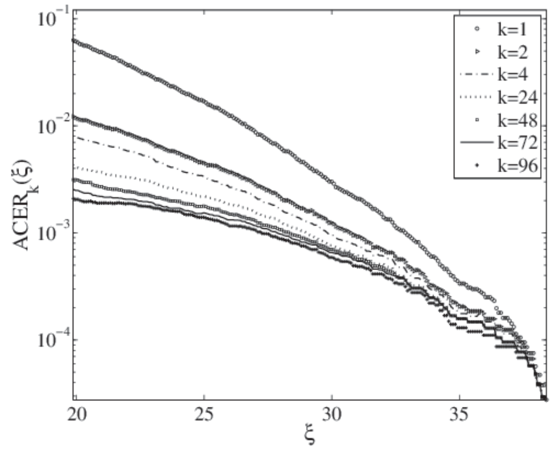

Figure 2: Comparison between ACER estimates for different degrees of conditioning. Wind speed data from the Sula station.

The cascade of estimated bivariate ACER surfaces $\hat{\mathscr{E}}_k(\xi, \eta)$ is shown in Figure 4. Matlab programs for ACER 2D analyses are available, cf. (Karpa, 2014). $\hat{\mathscr{E}}_k(\xi, \eta)$ with $k = 1$ is the uppermost. As it is seen from the figure, the cross-section

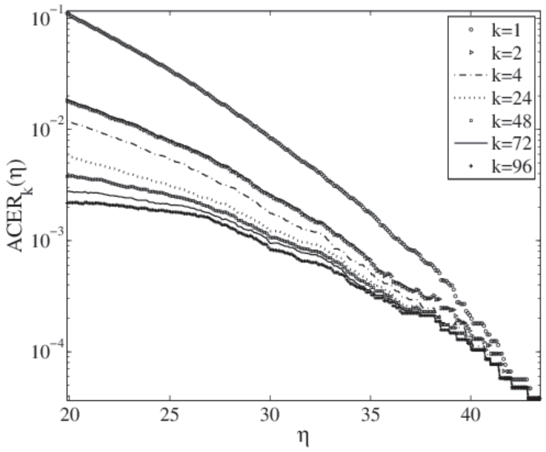

Figure 3: Comparison between ACER estimates for different degrees of conditioning. Wind speed data from the Nordøyan Fyr station.

of the surfaces at the high value of wind level η gives the univariate ACER functions of the wind speed data from the Sula station, while the cross-section at a high level of ξ represents the univariate ACER of the time series from the Nordøyan Fyr.

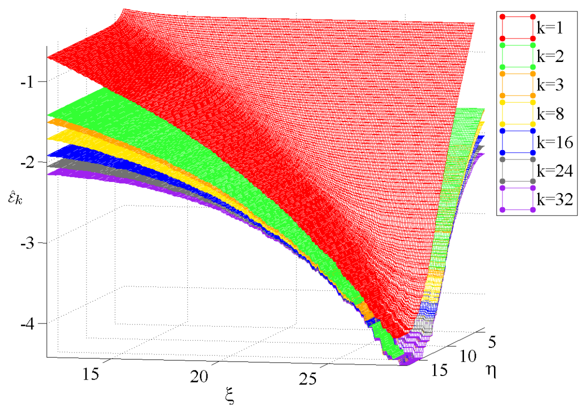

Figure 4: Comparison between Bivariate ACER surface estimates for different degrees of conditioning. $\hat{\mathscr{E}}_k(\xi,\eta)$ surfaces are plotted on a logarithmic scale.

Parameters of the optimal AL fit are $m_A = 2.44, \theta = 0.86, \phi = 0.92$ and of the GL fit is $m_G = 2.01$.

Figure 5 shows the contour plot of the optimized AL $\mathscr{A}_k(\xi,\eta)$ and the optimized GL $\mathscr{G}_k(\xi,\eta)$ fits to the data for $\hat{\mathscr{E}}_k(\xi,\eta)$ surface for $k=1$ and $k=96$, respectively. The contour lines of three surfaces are plotted for those levels of ξ and η, where the bivariate ACER surface $\hat{\mathscr{E}}_k(\xi,\eta)$ have been empirically estimated.

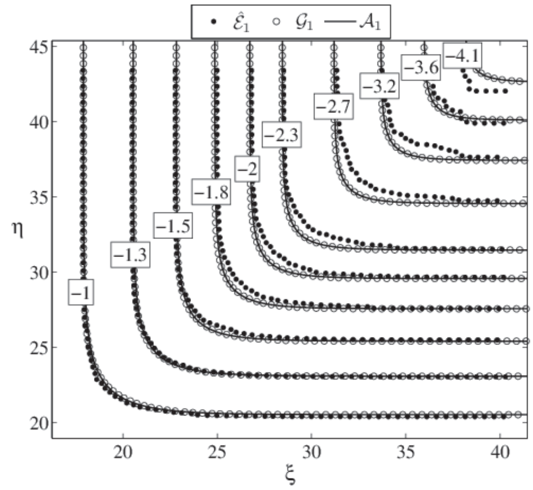

Figure 5: Contour plot of the empirically estimated $\hat{\mathscr{E}}_1$ surface, and the optimized GL $\mathscr{G}_1$ and optimized AL $\mathscr{A}_1$ surfaces based on marginal univariate ACER. Boxes indicate levels on a logarithmic scale.

The figure reveals that the empirical bivariate ACER surface $\hat{\mathscr{E}}_k$ captures high correlation between the data, and so do the optimally fitted $\mathscr{G}_k$ and $\mathscr{A}_k$ surfaces.

It is noticeable that the level of agreement between the estimated bivariate ACER and both the optimized AL and the GL surfaces is equally significant. It is also important to keep in mind that the empirical bivariate ACER $\hat{\mathscr{E}}_k$ is the only discrete surface. The MATLAB (2009) built-in routine contourc that has been used to obtain the figures, calculates the contour lines by producing a regularly spaced grid determined by the dimensions of a surface. Therefore, it evidently generates a certain spacing inaccuracy of the ACER $\hat{\mathscr{E}}_k$ surface level lines plot. In addition, the figures ascertain that the optimized AL and GL surfaces conform at a level sufficient to affirm that they actually coincide.

Thereby, the optimized GL model with asymptotically consistent marginals obtained from the optimized univariate ACER, can be used as the parametric representative of the bivariate ACER surface estimated from the given data set. This is significant since the extension of the GL model to higher dimensions is straightforward, cf. Eq. (11). However, to what extent that simple extension is accurate enough for higher dimensions, has yet to be verified. Unfortunately, computer programs for calculating the ACER functions has so far not been developed for dimensions higher than two.

5. CONCLUSIONS

The main purpose of this paper has been to demonstrate that the ACER method allows the estimation of the failure probability of a series system of dynamic response processes where failure is given by a first passage event. The specific demonstration has been for the case of two response processes. The main reason for this is that there are at this time no computer programs available for estimation of ACER functions for dimensions greater than two. However, the bivariate case clearly illustrates that the ACER method offers an interesting approach to the estimation of the failure probability of a dynamic series system. Future efforts should be directed towards establishing computer programs to estimate higher dimensional ACER functions from measured or simulated data.

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