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Novel Observations in Internal Conical Diffraction

Rónán Darcy

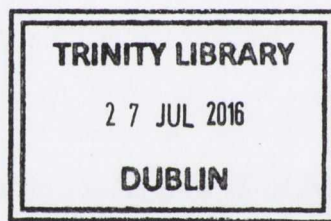
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Abstract

Internal conical diffraction occurs when a beam of light is incident along an optic axis direction in a biaxial material. The beam transforms into a hollow skewed cone inside the material and refracts upon exiting.

The phenomenon has been studied extensively for the case of laser beams with Gaussian spatial intensity profiles incident along an optic axis of a biaxial crystal. When a top-hat incident beam is used, the resultant intensity profile beyond the crystal is observed to be distinct from that generated using a Gaussian beam. The evolution of a conically diffracted top-hat beam has many intensity oscillations along its centre, and these can be observed with a suitable experimental apparatus.

Dispersion in biaxial crystals results in a complex beam structure when using a broadband light source. A transition between conical diffraction and double refraction is observed over the full spectrum of the incident beam. The theoretical model describing this transition may be expanded to contain an explicit wavelength dependency. Simulations of the intensity profile may be produced which match experimental observations closely. Dispersion compensation may be realised in biaxial crystals using a suitable experimental arrangement, resulting in a white ring of light at the focal image plane.

When a spatially Gaussian monochromatic laser beam is misaligned with the optic axis, interference patterns are predicted to occur. These low intensity features represent a qualitatively new phenomenon in conical diffraction. Observation of these features is difficult due to the high intensity range observed in the beam, but it may be accomplished through the use of a high dynamic range imaging technique. Polarisation in misaligned beams produces distinct intensity profiles which match well with theoretical simulations.

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List of symbols and initialisms

CCD	Charge-coupled device
FIP	Focal image plane
ϵ	Permittivity tensor
ϵ_i	Principal permittivity values
n	Refractive index
n_i	Principal refractive indices
E	Electric field
D	Electric displacement field
k	Wave vector
k_i	Wavenumber in direction i
ω	Angular frequency of light
μ	Magnetic permeability
H	Magnetising field
η	Unit vector in the direction of k
η_i	Unit vector in direction of k_i
S	Poynting vector
χ	Azimuthal angle of D
ϵ^{-1}	Rotated permittivity tensor
θ_{OA}	Angle between η_3 and optic axis
A	Semi-angle of cone of conical diffraction
R_0	Ring radius at the exit face of the crystal
l	Length of crystal
R	Radial vector
ρ	Dimensionless radial vector

w	$1/e$ intensity radius of incident Gaussian beam
$\mathbf{k}_\perp$	Orthogonal wavevector
$\boldsymbol{\kappa}$	Dimensionless orthogonal wavevector
$\mathbf{D}_0(\boldsymbol{\rho})$	Electric displacement field of incident beam
$\mathbf{d}_0$	Polarisation vector of incident beam
$a(\boldsymbol{\kappa})$	Fourier transform of $\mathbf{D}_0(\boldsymbol{\rho})$
ϕ	Azimuthal angle of $\mathbf{d}_0$
J_0	The 0 th -order Bessel function of the first kind
J_1	The 1 st -order Bessel function of the first kind
$\mathbf{p}$	Transverse momentum
k_0	Vacuum wavenumber of incident beam
λ	Vacuum wavelength of incident beam
$\mathbf{d}_\pm$	Orthogonally polarised eigenmodes
$\mathcal{H}$	Hamiltonian of the system
$\mathcal{H}_{\text{crystal}}$	Hamiltonian of beam inside biaxial crystal
$\mathcal{I}$	2×2 identity matrix
$\Theta(x)$	Unit step function
$\boldsymbol{\sigma}_1$	First Pauli matrix
$\boldsymbol{\sigma}_3$	Third Pauli matrix
$\boldsymbol{\Sigma}$	$\{\boldsymbol{\sigma}_3, \boldsymbol{\sigma}_1\}$
ζ	Dimensionless propagation distance measured from FIP
B_0	Fundamental integral of order zero
B_1	Fundamental integral of order one
I	Intensity of the beam
$ L\rangle$	Left-circularly polarised beam
$ R\rangle$	Right-circularly polarised beam
z_{FIP}	Location of the focal image plane
D_+	Dawson function
erfi	Imaginary error function
P_λ	λ -degree Legendre function of the first kind
Q_λ	λ -degree Legendre function of the second kind
κ_{max}	Maximal transverse wavevector permitted in crystal
R_{lim}	Radius of iris between crystal and imaging device

z_{lim}	Distance between entrance face of crystal and iris
μ	Dimensionless misalignment parameter
ξ	First coordinate of $\boldsymbol{\rho}$
η	Second coordinate of $\boldsymbol{\rho}$
$\tilde{\boldsymbol{\rho}}$	Complex radial vector for misaligned beam
$\tilde{\zeta}$	Complex propagation parameter for misaligned beam
$\tilde{\phi}$	Complex azimuthal angle for misaligned beam
$\delta\theta$	Angle between optic axis and incident beam
n'_2	Corrected value of n_2 to reconcile theory and experiment
I_{LED}	Intensity produced by LED
P	Power produced by LED
ϕ_q	Photon flux produced by LED
E	Energy of photon
h	Planck constant
c	Speed of light in a vacuum
u	Misalignment parameter without wavelength dependence
$A_{\pm}$	Sum and difference integrals
Abs	Absorbance of crystal

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Chapter 1

Introduction and Theory

1.1 History

In 1832 William Rowan Hamilton (Fig. 1.1a) of Trinity College Dublin made a prediction based on Fresnel's theory of refraction [1]. A mathematical singularity which emerges in certain types of dielectric media, known as biaxial materials, gives rise to a degeneracy in the Poynting vector direction. A single ray incident along a specific direction, the optic axis, will refract into an infinity of rays tracing out a hollow skewed cone inside the material [2], a process called *internal conical refraction*. A related effect, known as *external conical refraction* was also predicted, however we will concentrate exclusively on the internal case in this thesis.

Hamilton's colleague Humphrey Lloyd (Fig. 1.1b) was tasked with experimentally confirming the prediction, and although initially confounded by the low quality biaxial materials available to him, he observed the phenomenon in December 1832 [3, 4] using aragonite, a form of calcite (CaCO_3) with orthorhombic crystal symmetry, as opposed to trigonal.

"This phenomenon was exceedingly striking. It looked like a small ring of gold viewed upon a dark background; and the sudden and almost magical change of the appearance from two luminous points to a perfect luminous ring, contributed not a little to enhance the interest."

—Humphrey Lloyd

James MacCullagh (Fig. 1.1c) is sometimes overlooked in the history of conical diffraction. Upon the publication of Hamilton's theory, MacCullagh issued a note in Philosophical Magazine [5]:

"When Professor Hamilton announced his discovery of Conical Refraction, he does not seem to have been aware that it is an obvious and immediate consequence of the theorems published by me, three years ago..."

—James MacCullagh

Furthermore, MacCullagh insisted he had spoken with Hamilton about his intention to publish a supplemental essay on Fresnel's theory, something

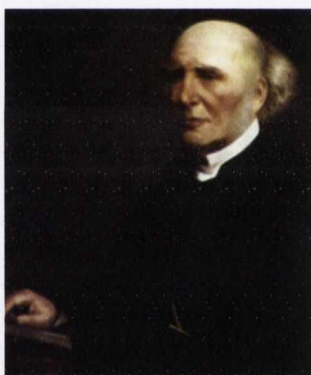
which Hamilton claimed had “quite escaped [his] recollection”. Following some correspondence, occasionally with Lloyd acting as an intermediary, the pair reached an amicable conclusion on the matter, with MacCullagh conceding in a letter to Transactions of the Royal Irish Academy in 1834 [6] that the theory of conical refraction was Hamilton’s:

“The curves of contact on biaxal surfaces and the conical intersections and nodes were lately discovered by Professor Hamilton, who deduced from these properties a theory of conical refraction which has been verified by the experiments of Professor Lloyd. . . The indeterminate cases of circular section—at least the case of the nodes—had occurred to me long ago; but having neglected to examine the matter attentively, I did not perceive the properties involved in it.”

—James MacCullagh



(a) William Hamilton



(b) Humphrey Lloyd



(c) James MacCullagh

Figure 1.1: The physicists associated with the discovery of conical refraction.

Further observations of the phenomenon were made by Poggendorff in 1839 [7], using a narrower iris restricting the range of angles of the incident rays, in which he noticed a dark ring separating two bright annuli. This observation was made again in 1841 by Potter [8], confirming that the emergent cylinder of light was in fact two coaxial cylinders separated by a dark region. This was explained by Voigt who refined Hamilton’s theory [9–13], realising that rays inclined at small angles to the optic axis direction form the two

distinct cones.

Raman performed several experiments with biaxial materials [14, 15], observing that the double ring structure was restricted to a range of positions beyond the material, after which the beam evolved, eventually converging to a high-intensity region at the centre of the beam, the ‘axial spike’. As Raman made the first observation of this feature, it is sometimes referred to as the ‘Raman spike’.

There have been several attempts to construct a complete theoretical model describing conical refraction. Portugal [16] and later Schell and Bloembergen [17] were successful in developing theories which described the beam at the exit face of the crystal, but omitted any dependence on location in the propagation direction. Bel’skii and Khapalyuk successfully included this dependence in their theory [18, 19], although the derivation was omitted and the model was quite complicated. Using this theory, predictions of secondary rings were made by Warnick and Arnold [20], which should be observed beyond the exit face of the crystal as the beam evolved while propagating. Berry reformulated the Bel’skii-Khapalyuk theory producing a highly accurate and useful model of internal conical refraction [21].

There have been many publications on conical refraction in the past decade, from fundamental theory [22] to potential applications [23–26]. It has proven to be a useful method for the generation of Bessel beams [27–30] and radially polarised beams [31], the creation of a novel optical trap [32, 33], and even the notion of a conical refraction laser [34]. Placing n biaxial crystals in sequence results in cascade conical refraction, which can be used to generate vortex beams of maximum charge n [35–39].

1.2 Thesis overview

The goal of the work presented in this thesis was to further contribute to the understanding of conical refraction. This thesis will concentrate primarily on the fundamental theory and observations of beams undergoing conical refraction rather than the potential myriad applications.

Chapter 1: Having already introduced the history behind conical refrac-

tion and mentioning some of the more recent developments and applications, we will present a theoretical overview of the phenomenon. This will begin with a ray theory consideration, similar to Hamilton's formulation, which explains the behaviour of a single ray of light along an optic axis in a biaxial material. It will be observed that the Poynting vector traces out a hollow skewed cone inside a biaxial material. We will then present a recapitulated form of the theory developed by Berry, whereafter we will refer to the phenomenon as conical *diffraction*. This will involve a paraxial approximation, deriving the Hamiltonian of the system, and finding an expression for the intensity at any position beyond the biaxial material, for an incident Gaussian beam. This expression will be used to plot simulated intensity profiles at certain locations in the beam, and an evolution profile will also be presented. The key features of the conically refracted beam will be highlighted, and the polarisation of the incident beam will also be considered.

Chapter 2: Having already considered a Gaussian beam incident on a biaxial crystal, in this chapter we will consider the intensity profile resulting from the conical diffraction of a top-hat beam. The theoretical model derived in Chapter 1 will be used to predict this profile. Beyond the focal image plane, the beam is predicted to evolve with a distinct and interesting profile, when compared with the case of a conically refracted Gaussian beam. An experiment was performed which observed these profiles, matching well with the theoretical predictions. Experimental limitations are also incorporated into the theoretical model which further improves the agreement with experimental data. Another incident beam profile is also considered. The work presented in this chapter resulted in the publication [40].

Chapter 3: In this chapter, we consider dispersion in biaxial crystals which arises when using polychromatic light sources. We use the Sellmeier equation to describe the behaviour of each of the principal refractive indices as a function of wavelength, and predict that the optic axis direction will vary with wavelength. Hence, a single broadband beam of light incident on a biaxial crystal may only be aligned with the optic axis at one wavelength. For all other colours, the beam undergoes double refraction to some extent, with a transition to conical diffraction for colours close to the aligned wavelength.

The model presented by Berry and Jeffrey [41] will be used to derive an explicit wavelength dependent theoretical intensity profile. Combined with a suitable representation of colour, this will be used to generate theoretical profiles for an incident white light beam. These will be compared to experimental images. Having developed an understanding of white light conical diffraction, we will design and perform an experiment to compensate for dispersion, generating a symmetric white ring of light. The work in this chapter resulted in the publication [42].

Chapter 4: Building on the theoretical model outlined by Berry and Jeffrey [41] which was presented in Chapter 3, we observe that interference in beams which have been misaligned with the optic axis direction leads to the prediction of low intensity regions in the conically diffracted beam. Observation of these features would represent a qualitatively new effect in conical diffraction, building on those effects observed and predicted by Poggendorff (the Poggendorff dark ring), Raman (the Raman spike), and Warnick and Arnold (the Warnick-Arnold rings), among others. We present experimental images of misaligned beams, and using a suitable imaging technique, we successfully record a large enough intensity range to observe these weak features. We also consider the effect of the polarisation of the misaligned incident beam. The work presented in this chapter resulted in the publication [43].

Chapter 5: We will present a short piece of primarily experimental work involving a biaxial crystal doped with a rare earth metal. The strong absorption peaks observed in the doped material will have an effect on the radius of the conically diffracted ring, and anomalous dispersion close to the absorption peaks may also be observed. This chapter will then summarise the previous chapters and conclude the thesis, offering ideas on potential future work.

1.3 The wave surface

In a dielectric material there exists a frame in which the permittivity tensor is diagonal [44]:

$$\boldsymbol{\epsilon} = \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix} \quad (1.1)$$

where ϵ_i are the principal permittivity values, which are assumed to be real, and $\epsilon_i = n_i^2$, where n_i are the principal refractive indices. If $\epsilon_1 = \epsilon_2 = \epsilon_3$, the material is isotropic. For $\epsilon_1 < \epsilon_2 = \epsilon_3$ or $\epsilon_1 = \epsilon_2 < \epsilon_3$, the material is uniaxial and the phenomenon of double refraction arises. We are interested in materials with a suitable anisotropy in the crystal structure, namely the orthorhombic, monoclinic or, triclinic classes, in which these values are distinct [45] with $\epsilon_1 < \epsilon_2 < \epsilon_3$ by convention. We assume all $\epsilon_i > 0$, although the interesting case of conical refraction in negative index materials $\epsilon_i < 0$ has been investigated [46].

The electric field $\mathbf{E}$ is related to the electric displacement vector $\mathbf{D}$ in an anisotropic crystal by

$$\mathbf{E} = \boldsymbol{\epsilon} \cdot \mathbf{D}, \quad (1.2)$$

and $\mathbf{D}$ is always orthogonal to the incident wave vector $\mathbf{k}$. Maxwell's equations give rise to the following:

$$\mathbf{k} \times \mathbf{E} = \omega\mu\mathbf{H}, \quad \mathbf{k} \times \mathbf{H} = -\omega\mathbf{D}, \quad (1.3)$$

where $\mathbf{H}$ is the magnetising field and μ is the magnetic permeability. Defining $\boldsymbol{\eta}$ to be the unit vector in the direction of $\mathbf{k}$, *i.e.* $\boldsymbol{\eta} \equiv \mathbf{k}/|\mathbf{k}|$, and combining Eqs. (1.3) results in the expression for $\mathbf{D}$:

$$\mu c^2 \mathbf{D} = \boldsymbol{\eta} \times (\mathbf{E} \times \boldsymbol{\eta}) = \eta^2 \mathbf{E} - (\boldsymbol{\eta} \cdot \mathbf{E})\boldsymbol{\eta}. \quad (1.4)$$

Using Eq. (1.2), the i^{th} component can be expressed in terms of $\mathbf{E}$:

$$\frac{E_i}{n^2} - \boldsymbol{\eta} \cdot \mathbf{E} \frac{\eta_i}{n^2 - \epsilon_i} = 0. \quad (1.5)$$

Multiplying by η_i , summing over i , and dividing by $\boldsymbol{\eta} \cdot \mathbf{E} n^{-2}$ gives

$$n^2 \sum_{i=1}^3 \frac{\eta_i^2}{n^2 - \epsilon_i} = 1. \quad (1.6)$$

Since $\boldsymbol{\eta}$ is a unit vector, subtracting $\eta_1^2 + \eta_2^2 + \eta_3^2 = 1$ from both sides gives

$$\sum_{i=1}^3 \eta_i^2 \left(\frac{n^2}{n^2 - \epsilon_i} - 1 \right) = 0, \quad (1.7)$$

which can be rearranged to yield one form of the Fresnel equation:

$$\sum_{i=1}^3 \frac{\epsilon_i \eta_i^2}{n^2 - \epsilon_i} = 0. \quad (1.8)$$

Solving this equation yields two solutions for n^2 for a given direction $\boldsymbol{\eta}$, which together form a two-sheeted surface as seen in Fig. 1.2. There is no universally accepted terminology for this surface, but it is known as the isofrequency surface, the refractive index surface, or the wave surface which we will use. The sheets give the phase velocities in the medium for a given wave vector direction. Hence, in general there are two distinct phase velocities for a single incident ray—this is the phenomenon of double refraction. Furthermore, the Poynting vector is orthogonal to the wave surface at all points, an exception being the degenerate points which we will discuss below.

The solutions for Eq. (1.8) are degenerate at four distinct points where the two sheets touch [47]. A magnified view of one of these points and the wave surface surrounding it is shown in Fig. 1.3. Solving for these degeneracies yields physical solutions

$$\eta_1 = \pm \sqrt{\frac{\epsilon_3(\epsilon_2 - \epsilon_1)}{\epsilon_2(\epsilon_3 - \epsilon_1)}}, \quad \eta_2 = 0, \quad \eta_3 = \pm \sqrt{\frac{\epsilon_1(\epsilon_3 - \epsilon_2)}{\epsilon_2(\epsilon_3 - \epsilon_1)}}. \quad (1.9)$$

These points occur in four distinct quadrants of the wave surface in the η_1 - η_3 plane, as seen in Fig. 1.2, and the lines joining opposing degenerate points define the two optic axis directions. Thus it is trivial to calculate the angle

θ_{OA} between the η_3 -axis and the optic axis:

$$\theta_{\text{OA}} = \arctan \frac{\eta_1}{\eta_3} = \arctan \sqrt{\frac{\epsilon_3(\epsilon_2 - \epsilon_1)}{\epsilon_1(\epsilon_3 - \epsilon_2)}}. \quad (1.10)$$

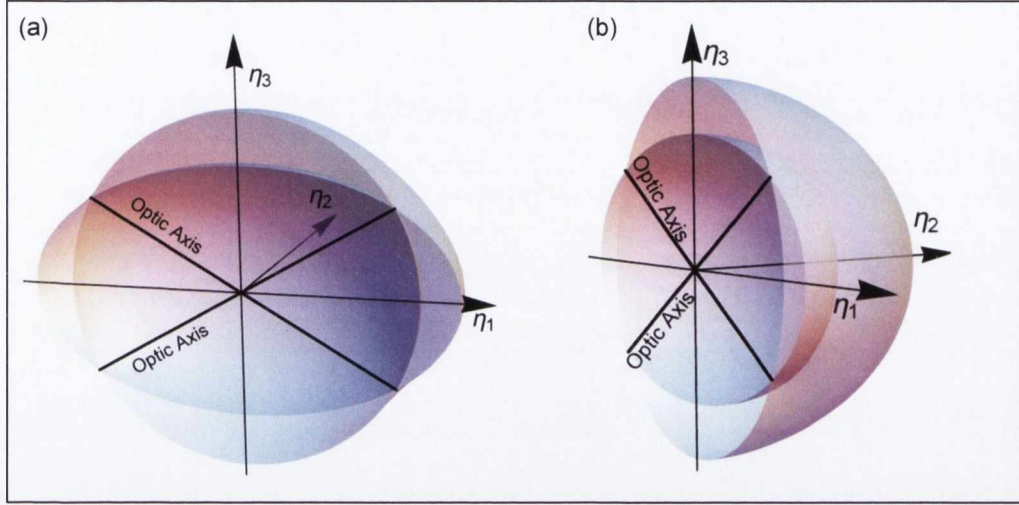


Figure 1.2: The wave surface for a biaxial material at two different viewing angles, cut through the η_1 - η_3 plane. The four degenerate points occur in this plane where the sheets touch, and opposing degenerate points define the two optic axis directions.

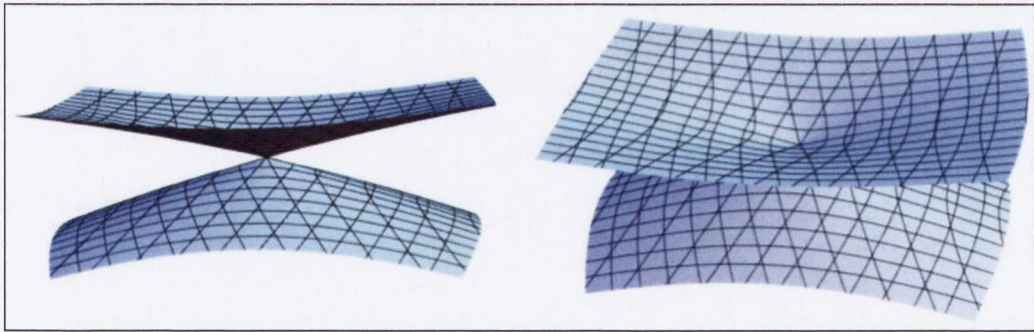


Figure 1.3: Two views of the wave surface close to a degenerate point. The shape of the surface around these points bears a resemblance to a diaboloid giving rise to another term for the degenerate points—the *diabolical points*.

1.4 Conical refraction

To understand the phenomenon of conical refraction, it is useful to observe how the Poynting vector $\mathbf{S}$ behaves in a biaxial material, where

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}. \quad (1.11)$$

Although $\mathbf{S}$ is always orthogonal to $\mathbf{E}$, we usually work with the electric displacement vector $\mathbf{D}$ in such materials, because $\mathbf{D}$ is always orthogonal to the wave vector $\mathbf{k}$. These vectors are shown in Fig. 1.4.

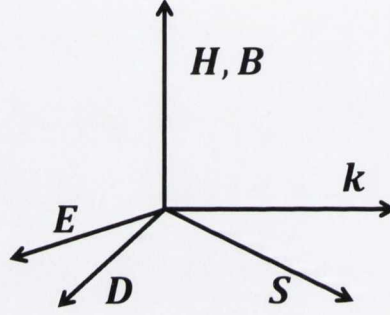


Figure 1.4: In a biaxial material, the magnetic field $\mathbf{B}$ and magnetising field $\mathbf{H}$ point in the same direction. They are orthogonal to the electric field vector $\mathbf{E}$, which is itself orthogonal to the Poynting vector $\mathbf{S}$. The displacement field vector $\mathbf{D}$ is orthogonal to the wave vector $\mathbf{k}$, and both are orthogonal to $\mathbf{H}$ and $\mathbf{B}$. In general, $\mathbf{E}$ and $\mathbf{D}$ do not coincide.

Let us define a new coordinate system xyz in which the z -axis corresponds to the optic axis, and the y and η_2 directions coincide, as seen in Fig. 1.5. In this frame a beam directed along the optic axis, *i.e.* in the positive z direction, may be described by

$$\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} \cos \chi \\ \sin \chi \\ 0 \end{pmatrix}, \quad \mathbf{H} = \begin{pmatrix} -\sin \chi \\ \cos \chi \\ 0 \end{pmatrix}, \quad (1.12)$$

where χ is the angle between the $\mathbf{D}$ vector and the x -axis. In order to map $\mathbf{D}$ to $\mathbf{E}$ using (1.2) we must first change to the xyz coordinate system. This

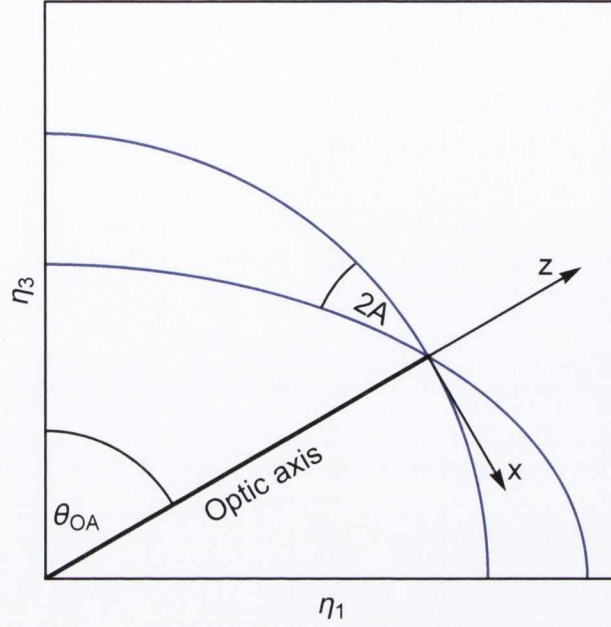


Figure 1.5: The rotated coordinate frame. z is aligned with the optic axis direction, and x is orthogonal to z . The y and η_2 directions (out of page) coincide.

amounts to a rotation about the η_2 -axis by θ_{OA} :

$$\mathbf{E} = \epsilon_R^{-1} \mathbf{D}, \quad (1.13)$$

where

$$\epsilon_R^{-1} = \mathbf{R} \cdot \epsilon^{-1} \cdot \mathbf{R}^{-1} \quad (1.14)$$

and the rotation matrix $\mathbf{R}$ is given by

$$\mathbf{R} = \begin{pmatrix} \cos \theta_{OA} & 0 & -\sin \theta_{OA} \\ 0 & 1 & 0 \\ \sin \theta_{OA} & 0 & \cos \theta_{OA} \end{pmatrix}. \quad (1.15)$$

Evaluating Eq. (1.13) explicitly results in

$$\mathbf{E} = \begin{pmatrix} \cos \chi \left(\frac{\cos^2 \theta_{OA}}{\epsilon_1} + \frac{\sin^2 \theta_{OA}}{\epsilon_3} \right) \\ \frac{\sin \chi}{\epsilon_2} \\ \cos \chi \left(\cos \theta_{OA} \sin \theta_{OA} \frac{\epsilon_3 - \epsilon_1}{\epsilon_1 \epsilon_3} \right) \end{pmatrix} \quad (1.16)$$

The first term may be simplified by approximating $\epsilon_1\epsilon_3 \approx \epsilon_2^2$, which is true for $\epsilon_3 - \epsilon_2 \approx \epsilon_2 - \epsilon_1$, a condition observed in many biaxial materials. Hence,

$$\frac{\cos^2 \theta_{OA}}{\epsilon_1} + \frac{\sin^2 \theta_{OA}}{\epsilon_3} \approx \frac{1}{\epsilon_2}. \quad (1.17)$$

Observing that

$$\cos \theta_{OA} \sin \theta_{OA} = \frac{\tan \theta_{OA}}{1 + \tan^2 \theta_{OA}}, \quad (1.18)$$

and making the substitution for $\tan \theta_{OA}$ given by Eq. (1.10), we can rearrange the third term to find

$$\cos \theta_{OA} \sin \theta_{OA} \frac{\epsilon_3 - \epsilon_1}{\epsilon_1 \epsilon_3} = \sqrt{\frac{(\epsilon_3 - \epsilon_2)(\epsilon_2 - \epsilon_1)}{\epsilon_1 \epsilon_2^2 \epsilon_3}}. \quad (1.19)$$

Defining

$$A \equiv \frac{1}{2} \sqrt{\frac{(\epsilon_3 - \epsilon_2)(\epsilon_2 - \epsilon_1)}{\epsilon_1 \epsilon_3}}, \quad (1.20)$$

and combining Eqs. (1.16), (1.17), and (1.19) the electric field vector can now be written

$$\mathbf{E} = \frac{1}{\epsilon_2} \begin{pmatrix} \cos \chi \\ \sin \chi \\ 2A \cos \chi \end{pmatrix} \quad (1.21)$$

The Poynting vector Eq. (1.11) may now be calculated:

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} \propto \begin{pmatrix} -2A \cos^2 \chi \\ -2A \cos \chi \sin \chi \\ 1 \end{pmatrix} = \begin{pmatrix} -A(1 + \cos 2\chi) \\ -A \sin 2\chi \\ 1 \end{pmatrix}. \quad (1.22)$$

This is the equation of a skewed cone, with the Poynting vector $\mathbf{S}$ rotating at twice the rate of the electric displacement vector $\mathbf{D}$. The cone traced out by $\mathbf{S}$ is shown in Fig. 1.6.

The reason for the parameter combination in Eq. (1.20) now becomes apparent since A is the semi-angle of the skewed cone. If $A \ll 1$, which is true for many biaxial materials, then the radius of the cone at the exit face



Figure 1.6: The skewed cone traced out by the Poynting vector $\mathbf{S}$ for a single ray aligned with the optic axis (solid black line). The skewed cone has semi-angle A as defined in Eq. (1.20). The colour gradient is for clarity only.

of the crystal R_0 may be approximated as

$$R_0 = \tan Al \approx Al \quad (1.23)$$

where l is the length of the crystal, as seen in Fig. 1.7.

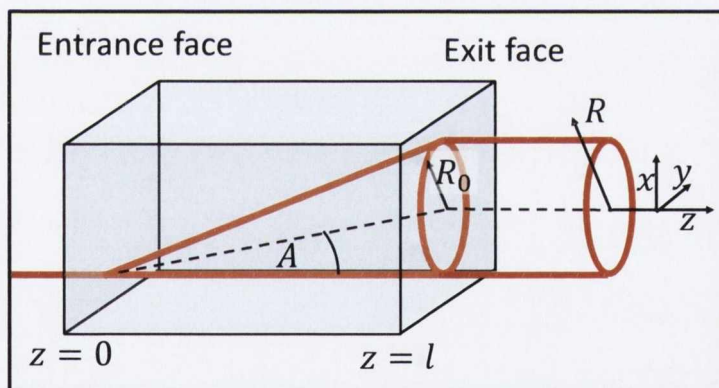


Figure 1.7: A beam undergoing conical refraction inside a biaxial crystal of length l . The semi-angle of the skewed cone is A resulting in a cone radius of $R_0 \approx Al$ at the exit face of the crystal.

While this ray argument gives an insight into the transformation of a single ray into a cone of rays, *i.e.* conical refraction, it fails to capture the intricacies of the phenomenon. As we will see in the following sections, rather than a single ring as suggested by the ray theory, there are in fact two concentric rings separated by a dark region—the *Poggendorff dark ring*.

1.5 Conical diffraction

While the method outlined in Section 1.4 is useful for an intuitive understanding of the process of conical refraction, it fails to capture some of the intricacies observed in experiments. In order to model the fine structure and evolution of the beam, we will recapitulate some of the theoretical work developed by Bel'skiĭ and Khapalyuk [18], and later reformulated by Berry [21], which incorporates diffraction theory. This leads to a change in nomenclature—conical *refraction* becomes conical *diffraction*.

Define the dimensionless radial vector $\boldsymbol{\rho}$ such that $\rho = 0$ defines the centre of the skewed cone and resultant emergent cylinder:

$$\boldsymbol{\rho} \equiv \frac{1}{w} \mathbf{R} = \frac{1}{w} \{x + Az, y\} = \{\rho \cos \phi, \rho \sin \phi\} = \{\xi, \eta\}, \quad (1.24)$$

where w is the $1/e$ radius of the incident beam, typically of Gaussian spatial profile. The radius of the cone R_0 at the exit face of the crystal defines the parameter

$$\rho_0 \equiv \frac{R_0}{w} = \frac{Al}{w}, \quad (1.25)$$

with R_0 given by Eq. (1.23). $\boldsymbol{\rho}$ and ρ_0 are shown in Fig. 1.8. A dimensionless wave vector $\boldsymbol{\kappa}$ may be defined in a similar way:

$$\boldsymbol{\kappa} \equiv w \mathbf{k}_\perp = w \{k_x, k_y\} = \{\kappa \cos \phi_\kappa, \kappa \sin \phi_\kappa\}. \quad (1.26)$$

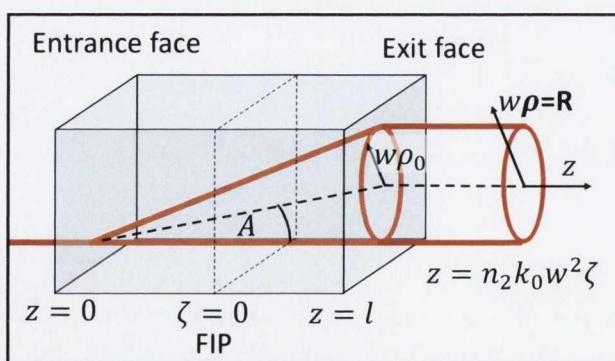


Figure 1.8: A schematic of conical refraction showing the dimensionless parameters ρ and ρ_0 .

1.5.1 Plane wave representation

If the incident beam has a spatial profile $\mathbf{D}_0(\boldsymbol{\rho})$ and polarisation vector $\mathbf{d}_0$, it may be written as a superposition of plane waves:

$$\mathbf{D}_0(\boldsymbol{\rho}) = \frac{1}{2\pi} \iint d\boldsymbol{\kappa} a(\boldsymbol{\kappa}) \exp[i\boldsymbol{\kappa} \cdot \boldsymbol{\rho}] \mathbf{d}_0, \quad (1.27)$$

where $a(\boldsymbol{\kappa})$ is the Fourier transform of $\mathbf{D}_0(\boldsymbol{\rho})$. Assuming the beam is circularly symmetric allows us to evaluate the integral over ϕ_κ to find

$$\mathbf{D}_0(\boldsymbol{\rho}) = D_0 \mathbf{d}_0 = \int_0^\infty d\kappa \kappa a(\kappa) J_0(\kappa \rho) \mathbf{d}_0, \quad (1.28)$$

where J_0 is the 0th-order Bessel function of the first kind. Thus, the Fourier transform of the incident beam may be expressed as

$$a(\kappa) = \int_0^\infty d\rho \rho D_0(\rho) J_0(\kappa \rho). \quad (1.29)$$

In real space coordinates, and in terms of the wave vector $\mathbf{k} = \{k_x, k_y, k_z\}$, the evolution of the incident beam inside a biaxial crystal may be described by the sum of plane wave components:

$$\mathbf{D} = \frac{1}{2\pi} \iint dk_x dk_y a(k_x, k_y) \exp[i(k_x x + k_y y + k_z z)] \mathbf{d}_0. \quad (1.30)$$

The z component of the wave vector k_z may be written

$$k_z = \sqrt{k_T^2 - k_\perp^2}, \quad (1.31)$$

where k_T is the total wave vector magnitude and $k_\perp$ is the transverse wave vector magnitude. In order to express the refractive index in terms of the transverse momentum $\mathbf{p} = \mathbf{k}_\perp/(n_2 k_0)$, allow the wave vector to deviate by a small angle θ from the optic axis, as seen in Fig. 1.9.

Hence, in the paraxial approximation [21],

$$n(\mathbf{p}) = n_2(1 + \mu(\mathbf{p})), \quad (1.32)$$

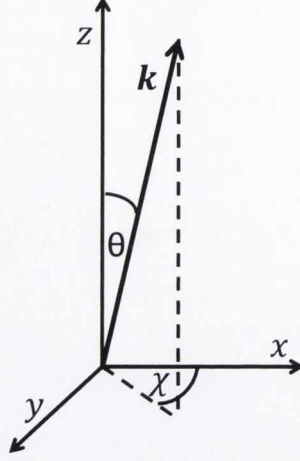


Figure 1.9: The incident wave vector $\mathbf{k}$ displaced from the optic axis by a small angle θ .

where, expanding the solution to the Fresnel equation Eq. (1.8), we get

$$\mu_{\pm}(\mathbf{p}) = A(p_x \mp p). \quad (1.33)$$

Therefore, $k_T^2 = n^2 k_0^2 = k^2 (1 + \mu(\mathbf{p}))$ and

$$\begin{aligned} k_z &= k \sqrt{(1 + \mu(\mathbf{p}))^2 - p^2} \\ &= k \sqrt{1 + 2\mu(\mathbf{p}) + \mu(\mathbf{p})^2 - p^2}. \end{aligned} \quad (1.34)$$

As $\mu(\mathbf{p}) \ll 1$ we take to first order, and using the Taylor expansion we get

$$\begin{aligned} k_z &\approx k \left(1 + \mu(\mathbf{p}) - \frac{1}{2}p^2 \right) \\ &= k \left(1 + A(p_x \mp p) - \frac{1}{2}p^2 \right). \end{aligned} \quad (1.35)$$

Inserting this into Eq. (1.30) gives

$$\begin{aligned} \mathbf{D}_{\pm} &= \\ \frac{1}{2\pi} \iint dk_x dk_y a(k_x, k_y) \exp \left[ik \left(p_x x + p_y y + z + z A p_x \mp z A p - z \frac{1}{2} p^2 \right) \right] \mathbf{d}_{\pm}, \end{aligned} \quad (1.36)$$

where $\mathbf{d}_\pm$ are the orthogonally polarised eigenmodes:

$$\mathbf{d}_+ = \begin{pmatrix} \cos \frac{1}{2}\phi_\kappa \\ \sin \frac{1}{2}\phi_\kappa \end{pmatrix}, \quad \mathbf{d}_- = \begin{pmatrix} \sin \frac{1}{2}\phi_\kappa \\ -\cos \frac{1}{2}\phi_\kappa \end{pmatrix}. \quad (1.37)$$

Combining the p_x terms in Eq. (1.36) to account for the skewed cone allows us to switch to the radial coordinates defined in Eq.(1.24):

$$\begin{aligned} \mathbf{D}_\pm &= \frac{1}{2\pi} \iint dk_x dk_y a(k_x, k_y) \exp [ik (\mathbf{p} \cdot \mathbf{R} + z - z (\frac{1}{2}p^2 \pm Ap))] \mathbf{d}_\pm \\ &= \frac{\exp [ikz]}{2\pi} \iint d\mathbf{k}_\perp a(\mathbf{k}_\perp) \exp [ik (\mathbf{p} \cdot \mathbf{R} - z (\frac{1}{2}p^2 \pm Ap))] \mathbf{d}_\pm \\ &= \frac{\exp [ikz]}{2\pi} \iint d\mathbf{k}_\perp a(\mathbf{k}_\perp) \exp \left[i \left(k\mathbf{p} \cdot \mathbf{R} - \frac{z}{kw^2} \left(\frac{1}{2}k^2w^2p^2 \pm Ak^2w^2p \right) \right) \right] \mathbf{d}_\pm, \end{aligned} \quad (1.38)$$

where we can now drop the $\exp [ikz]$ phase term which does not contribute to the integral. Observing that $kw\mathbf{p} = \boldsymbol{\kappa}$ and $\mathbf{R}/w = \boldsymbol{\rho}$ allows us to write this equation in terms of the dimensionless parameters:

$$\mathbf{D}_\pm = \frac{1}{2\pi} \iint d\boldsymbol{\kappa} a(\boldsymbol{\kappa}) \exp \left[i \left(\boldsymbol{\kappa} \cdot \boldsymbol{\rho} - \frac{z}{kw^2} \left(\frac{1}{2}\kappa^2 \pm kwA\kappa \right) \right) \right] \mathbf{d}_\pm. \quad (1.39)$$

1.5.2 The beam Hamiltonian

Equation (1.39) describes the electric displacement vector of a plane wave propagating in a biaxial material in the z direction, and it must satisfy the Hamiltonian form of the paraxial wave equation

$$\mathcal{H}(\boldsymbol{\kappa})\mathbf{D} = ikw^2 \frac{\partial \mathbf{D}}{\partial z}. \quad (1.40)$$

Thus, using Eq. (1.40) and the orthogonally polarised eigenmodes given by Eq. (1.37) the Hamiltonian inside the material may be constructed:

$$\begin{aligned} \mathcal{H}_{\text{crystal}}(\boldsymbol{\kappa}) &= \mathbf{d}_+^\dagger \left(\frac{1}{2}\kappa^2 + kwA\kappa \right) \mathbf{d}_+ + \mathbf{d}_-^\dagger \left(\frac{1}{2}\kappa^2 - kwA\kappa \right) \mathbf{d}_- \\ &= \frac{1}{2}\kappa^2 \mathcal{I} + kwA\kappa \begin{pmatrix} \cos \phi_{\boldsymbol{\kappa}} & \sin \phi_{\boldsymbol{\kappa}} \\ \sin \phi_{\boldsymbol{\kappa}} & -\cos \phi_{\boldsymbol{\kappa}} \end{pmatrix}, \end{aligned} \quad (1.41)$$

where $\mathcal{I}$ is the 2×2 identity matrix.

Upon entering the biaxial crystal at $z = 0$, the beam spreads out into a hollow skewed cone until reaching the exit face at $z = l$, at which point it undergoes refraction due to the medium interface. Hence, the Hamiltonian $\mathcal{H}$ of the evolving beam has two components: the region inside the crystal $z \leq l$ given by Eq. (1.41), which has a ‘potential’ causing the expansion of the hollow cone; and the region outside the crystal $z > l$, which only contains momentum. The region outside the crystal also has a factor of n_2 to account for the change of medium. The full Hamiltonian of the evolving beam described by Eq. (1.39) may be expressed in terms of the step function

$$\Theta(x) \equiv \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0, \end{cases} \quad (1.42)$$

which switches on the free-space Hamiltonian at the exit face $z = l$. The full Hamiltonian of the evolution of (1.39) with z is then

$$\mathcal{H}(\boldsymbol{\kappa}) = \left[\frac{1}{2}\kappa^2 \mathcal{I} + kwA\boldsymbol{\kappa} \cdot \boldsymbol{\Sigma} \right] \Theta(l - z) + \frac{1}{2}n_2\kappa^2 \mathcal{I} \Theta(z - l), \quad (1.43)$$

conveniently expressed in terms of the Pauli matrices

$$\Sigma = \{\sigma_3, \sigma_1\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}. \quad (1.44)$$

The full evolution of the plane wave can be found by integrating along the path length of the Hamiltonian:

$$\mathbf{D} = \frac{1}{2\pi} \iint d\boldsymbol{\kappa} a(\boldsymbol{\kappa}) \exp \left[-i \left(-\boldsymbol{\kappa} \cdot \boldsymbol{\rho} + \frac{1}{kw^2} \int_0^z dz \mathcal{H}(\boldsymbol{\kappa}) \right) \right] \mathbf{d}_0, \quad (1.45)$$

with $\mathcal{H}(\boldsymbol{\kappa})$ given by Eq. (1.43). Integrating over the optical path length gives

$$\begin{aligned} \int_0^z dz \mathcal{H}(\boldsymbol{\kappa}) &= \int_0^l dz \left[\frac{1}{2} \kappa^2 \mathcal{I} + kw A \boldsymbol{\kappa} \cdot \Sigma \right] + \int_l^z dz \frac{1}{2} n_2 \kappa^2 \mathcal{I} \\ &= \frac{1}{2} \kappa^2 \mathcal{I} (l + (z - l) n_2) + Al kw \boldsymbol{\kappa} \cdot \Sigma. \end{aligned} \quad (1.46)$$

From Eq. (1.25), we can substitute $Al = w\rho_0$, and using $k = n_2 k_0$ we find

$$\begin{aligned} \frac{1}{kw^2} \int_0^z dz \mathcal{H}(\boldsymbol{\kappa}) &= \frac{1}{2} \kappa^2 \left(\frac{l + (z - l) n_2}{n_2 k_0 w^2} \right) + \rho_0 \boldsymbol{\kappa} \cdot \Sigma \\ &= \frac{1}{2} \kappa^2 \zeta + \rho_0 \boldsymbol{\kappa} \cdot \Sigma, \end{aligned} \quad (1.47)$$

where we have made the parameter combination

$$\zeta \equiv \frac{l + (z - l) n_2}{n_2 k_0 w^2}. \quad (1.48)$$

This dimensionless propagation distance is measured from the focus of the incident beam, where the $1/e$ radius w is measured, the location called the focal image plane (FIP). It is measured in units of $n_2 k_0 w^2$, called the Fresnel length. For an incident Gaussian beam, this is related to the diffraction length $k_0 w^2$.

1.5.3 The fundamental integrals

Combining Eqs. (1.45) and (1.47), $\mathbf{D}(\boldsymbol{\rho}, \zeta)$ may be expressed as

$$\mathbf{D}(\boldsymbol{\rho}, \zeta) = \frac{1}{2\pi} \iint d\boldsymbol{\kappa} \exp \left[-i \left(-\boldsymbol{\kappa} \cdot \boldsymbol{\rho} + \frac{1}{2} \kappa^2 \zeta + \rho_0 \boldsymbol{\kappa} \cdot \boldsymbol{\Sigma} \right) \right] a(\boldsymbol{\kappa}) \mathbf{d}_0, \quad (1.49)$$

or, more concisely,

$$\mathbf{D} = \frac{1}{2\pi} \iint d\boldsymbol{\kappa} a(\boldsymbol{\kappa}) \exp [i \boldsymbol{\kappa} \cdot \boldsymbol{\rho}] \mathbf{U}(\boldsymbol{\kappa}) \mathbf{d}_0, \quad (1.50)$$

where $\mathbf{U}(\boldsymbol{\kappa})$ is the unitary operator

$$\mathbf{U}(\boldsymbol{\kappa}) = \exp \left[-i \left(\frac{1}{2} \kappa^2 \zeta + \rho_0 \boldsymbol{\kappa} \cdot \boldsymbol{\Sigma} \right) \right]. \quad (1.51)$$

The double integral in Eq. (1.50) may be evaluated numerically, although this is computationally expensive. A closed form expression cannot be obtained, but we can reduce it to a single integral, greatly simplifying the calculations. This is achieved by assuming $a(\boldsymbol{\kappa})$ is circularly symmetric, which is true for any incident beam with a circularly symmetric spatial profile. Switching to polar coordinates, Eq. (1.49) becomes

$$\mathbf{D} = \frac{1}{2\pi} \int_0^\infty d\kappa \int_0^{2\pi} d\phi_{\boldsymbol{\kappa}} a(\boldsymbol{\kappa}) \kappa \exp \left[i \left(-\frac{1}{2} \kappa^2 \zeta + \boldsymbol{\kappa} \cdot \boldsymbol{\rho} - \rho_0 \boldsymbol{\kappa} \cdot \boldsymbol{\Sigma} \right) \right] \mathbf{d}_0. \quad (1.52)$$

Since the first term inside the exponent does not depend on $\phi_{\boldsymbol{\kappa}}$, nor does the circularly symmetric $a(\boldsymbol{\kappa})$, the integral may be rearranged:

$$\mathbf{D} = \frac{1}{2\pi} \int_0^\infty d\kappa \kappa a(\kappa) \exp \left[-\frac{1}{2} i \kappa^2 \zeta \right] \int_0^{2\pi} d\phi_{\boldsymbol{\kappa}} \exp [i (\boldsymbol{\kappa} \cdot \boldsymbol{\rho} - \rho_0 \boldsymbol{\kappa} \cdot \boldsymbol{\Sigma})] \mathbf{d}_0. \quad (1.53)$$

For the second integral, evaluating the matrix exponential and integrating over $\phi_{\boldsymbol{\kappa}}$ results in a 2×2 matrix

$$2\pi J_0(\kappa\rho) \cos(\kappa\rho_0) \mathcal{I} + 2\pi J_1(\kappa\rho) \sin(\kappa\rho_0) \begin{pmatrix} \cos \phi & \sin \phi \\ \sin \phi & -\cos \phi \end{pmatrix}, \quad (1.54)$$

where J_1 is the 1st-order Bessel function of the first kind. Defining the fundamental integrals

$$B_0 \equiv \int_0^\infty d\kappa \kappa a(\kappa) \exp \left[-\frac{1}{2}i\kappa^2\zeta \right] J_0(\kappa\rho) \cos(\kappa\rho_0), \quad (1.55)$$

$$B_1 \equiv \int_0^\infty d\kappa \kappa a(\kappa) \exp \left[-\frac{1}{2}i\kappa^2\zeta \right] J_1(\kappa\rho) \sin(\kappa\rho_0), \quad (1.56)$$

allows us to write Eq. (1.53) as

$$\mathbf{D} = \begin{pmatrix} B_0 + B_1 \cos \phi & B_1 \sin \phi \\ B_1 \sin \phi & B_0 - B_1 \cos \phi \end{pmatrix} \mathbf{d}_0. \quad (1.57)$$

It is more useful to find an expression for the intensity $I = \mathbf{D}^* \cdot \mathbf{D}$ of the beam since this can be measured directly, for example using a charge-coupled device (CCD). Using Eq. (1.57), we write

$$I = \mathbf{D}^* \cdot \mathbf{D} = \mathbf{d}_0^* \cdot \mathbf{M} \cdot \mathbf{d}_0 \quad (1.58)$$

where

$$\mathbf{M} = \begin{pmatrix} |B_0|^2 + |B_1|^2 + 2\text{Re}[B_0^* B_1] \cos \phi & 2\text{Re}[B_0^* B_1] \sin \phi \\ 2\text{Re}[B_0^* B_1] \sin \phi & |B_0|^2 + |B_1|^2 - 2\text{Re}[B_0^* B_1] \cos \phi \end{pmatrix}. \quad (1.59)$$

The intensity of a conically diffracted beam may now be found for any beam parameter ρ_0 and at any position $\{\boldsymbol{\rho}, \zeta\}$, via the fundamental integrals Eqs. (1.55) and (1.56), assuming the incident beam is circularly symmetric.

1.6 Incident beam polarisation

The incident beam polarisation $\mathbf{d}_0$ appears explicitly in the equation for the intensity of a conically diffracted beam, Eq. (1.58), allowing us to investigate the effect polarisation has on the beam profile [48]. Consider a normalised polarisation vector $\mathbf{d}_0$:

$$\mathbf{d}_0 = \begin{pmatrix} d_{0x} \\ d_{0y} \end{pmatrix} \quad (1.60)$$

with $|d_{0x}|^2 + |d_{0y}|^2 = 1$. Eq. (1.58) may then be expressed as

$$I = |B_0|^2 + |B_1|^2 + 2\text{Re}[B_0^* B_1] \{ \cos \phi (|d_{0x}|^2 - |d_{0y}|^2) + 2 \sin \phi \text{Re}[d_{0x}^* d_{0y}] \}. \quad (1.61)$$

A left-hand circularly polarised beam $|L\rangle$ or right-hand circularly polarised beam $|R\rangle$ are given by the Jones vectors

$$\mathbf{d}_{0,|L\rangle} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad \mathbf{d}_{0,|R\rangle} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}. \quad (1.62)$$

Inserting these into Eq. (1.61) reveals that for both cases,

$$I_{|L\rangle} = I_{|R\rangle} = I_{\text{circ}} = |B_0|^2 + |B_1|^2. \quad (1.63)$$

Consider now a linearly polarised beam $|LP\rangle$ whose polarisation direction makes an angle χ with the ξ -axis, as given in Eq. (1.24). The Jones vector for such a beam is

$$\mathbf{d}_{0,|LP\rangle} = \begin{pmatrix} \cos \chi \\ \sin \chi \end{pmatrix}, \quad (1.64)$$

which, when inserted into Eq. (1.61), yields

$$\begin{aligned} I_{|LP\rangle} &= I_{\text{lin}}(\chi) = |B_0|^2 + |B_1|^2 + 2\text{Re}[B_0^* B_1] \cos(2\chi - \phi) \\ &= I_{\text{circ}} + 2\text{Re}[B_0^* B_1] \cos(2\chi - \phi). \end{aligned} \quad (1.65)$$

The polarisation of the ring is linear at every point, although the orientation rotates by π for every 2π revolution around the centre of the beam, as seen in Fig. 1.10. Due to this phase rotation, the conically diffracted beam is a type of vortex beam [49, 50], and this had lead to some interesting publications regarding the orbital and spin angular momentum carried by the beam [29, 37, 51, 52].

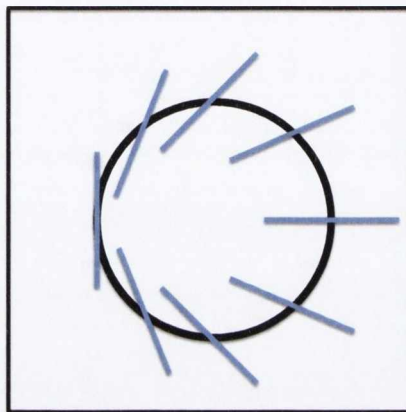


Figure 1.10: A visualisation of the polarisation distribution about the conically diffracted ring. Each point on the ring has linear polarisation, with the orientation orthogonal to that at the antipodal point.

For unpolarised beams, such as those produced using light-emitting diodes (LEDs), we may consider the diffracted beam to be the average of two orthogonally linearly polarised beams, with polarisation angles χ and $\chi + \frac{\pi}{2}$. This is essentially a statement that the contribution from each orthogonal polarisation direction is equal. From Eq. (1.65), we find

$$\begin{aligned} I_{\text{unpol}} &= \frac{1}{2} (I_{\text{lin}}(\chi) + I_{\text{lin}}(\chi + \frac{\pi}{2})) \\ &= |B_0|^2 + |B_1|^2 = I_{\text{circ}}. \end{aligned} \tag{1.66}$$

Hence, an unpolarised beam will produce an identical intensity pattern to a left- or right-hand circularly polarised beam.

1.7 Intensity profile at the focal image plane

If $z = 0$ corresponds to the focus of the incident beam, then the point $\zeta = 0$ corresponds to the most focussed image of the source, the focal image plane (FIP), which in turn corresponds to the sharpest ring profile in the conically diffracted beam. This location is found by solving $\zeta = 0$ from Eq. (1.48):

$$z_{\text{FIP}} = l \left(1 - \frac{1}{n_2} \right), \quad (1.67)$$

where l is the length of the crystal. For the following examples, we will consider an incident beam with a Gaussian spatial profile

$$D_G(\rho) = \exp \left(-\frac{1}{2}\rho^2 \right), \quad (1.68)$$

whose Fourier transform is calculated using Eq. (1.29):

$$a(\kappa) = \exp \left(-\frac{1}{2}\kappa^2 \right). \quad (1.69)$$

The intensity for a circularly polarised (or unpolarised) beam may then be explicitly evaluated using Eq. (1.63) and Eqs. (1.55) and (1.56) with $\zeta = 0$. The parameter ρ_0 determines the sharpness of the ring structure, with two distinct rings only becoming apparent for $\rho_0 > 2$. Intensity profiles for various ρ_0 values are shown in Fig. 1.11, and the corresponding two-dimensional plots are shown in Fig. 1.12 .

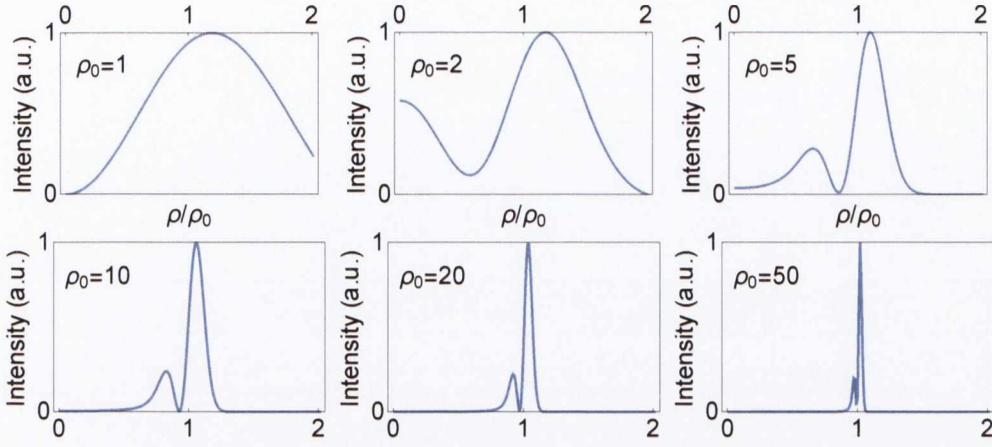


Figure 1.11: Intensity plots at the focal image plane (FIP) for various values of ρ_0 (inset), showing how the double ring structure becomes sharper as ρ_0 increases.

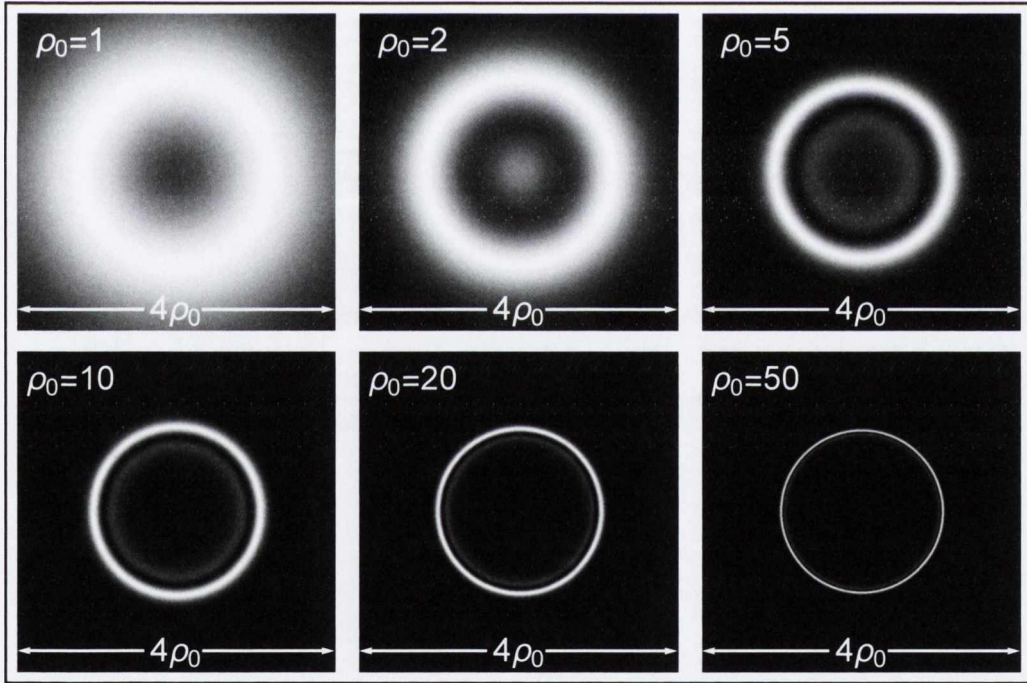


Figure 1.12: Two-dimensional plots of the intensity profiles given in Fig. 1.11, demonstrating how the double ring structure only becomes apparent for $\rho_0 > 2$. For $\rho_0 \approx 1$, the beam is a single annulus.

For a linearly polarised beam, the intensity is given by Eq. (1.65). Using the Bessel function approximation

$$J_\nu(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right), \quad x \gg 1, \quad (1.70)$$

we get approximations for B_0 and B_1 sufficiently far from the centre of the beam:

$$B_0 \approx \sqrt{\frac{2}{\pi\rho}} \int_0^\infty d\kappa \sqrt{\kappa} a(\kappa) \exp\left[-\frac{1}{2}i\kappa^2\zeta\right] \cos\left(\kappa\rho - \frac{\pi}{4}\right) \cos(\kappa\rho_0), \quad (1.71)$$

$$B_1 \approx \sqrt{\frac{2}{\pi\rho}} \int_0^\infty d\kappa \sqrt{\kappa} a(\kappa) \exp\left[-\frac{1}{2}i\kappa^2\zeta\right] \sin\left(\kappa\rho - \frac{\pi}{4}\right) \sin(\kappa\rho_0). \quad (1.72)$$

Using the trigonometric identities

$$2 \cos A \cos B = \cos(A - B) + \cos(A + B), \quad (1.73)$$

$$2 \sin A \sin B = \cos(A - B) - \cos(A + B), \quad (1.74)$$

allows us to express Eq. (1.71) as the sum of two integrals and Eq. (1.72) as the difference of two integrals. Assuming $\rho + \rho_0 \gg \rho - \rho_0$, which is true for $\rho_0 \gg 0$ and $\rho \approx \rho_0$, the contribution from $\cos(\kappa(\rho + \rho_0) + \frac{\pi}{4})$ under integration is negligible compared to that from $\cos(\kappa(\rho - \rho_0) + \frac{\pi}{4})$ and so $B_0 \approx B_1$. Hence,

$$\begin{aligned} I_{\text{lin}} &\approx |B_0|^2 + |B_1|^2 + (|B_0|^2 + |B_1|^2) \cos(2\chi - \phi) \\ &= I_{\text{circ}}(1 + \cos(2\chi - \phi)) \\ &= 2I_{\text{circ}} \cos^2\left(\chi - \frac{\phi}{2}\right), \end{aligned} \quad (1.75)$$

where, since the average of $\cos^2 = 0.5$, the factor of 2 arises so that the same energy is observed in both the linearly polarised case and the circularly polarised case. From the form of Eq. (1.75), it is apparent that the intensity around the ring will vary from zero at $\phi = \pi - 2\chi$ to maximum intensity at $\phi = 2\pi - 2\chi$, the antipodal point. This may be seen from Fig. 1.10, where each point on the ring has orthogonal polarisation to the antipodal point. Thus, a linearly polarised incident beam will produce a zero intensity

point on the ring at the location which has orthogonal polarisation. It is also apparent that if the incident polarisation angle χ is rotated at some rate, the resultant intensity profile will rotate at twice the rate. The intensity profile generated by Eq. (1.75) is shown in Fig. 1.13.



Figure 1.13: The intensity profile at the FIP produced by a linearly polarised incident Gaussian beam, found using Eq. (1.75) with $\phi = 0$. The parameter $\rho_0 = 20$, and the image has dimensions $3\rho_0 \times 3\rho_0$.

1.8 Evolution of the beam

Beyond the FIP, the beam evolves as it propagates. For a circularly polarised (or unpolarised) incident beam, consider the centre of the resultant conically diffracted beam where $\rho = 0$. Using this value in Eqs. (1.55) and (1.56) eliminates B_1 , since $J_1(0) = 0$. As $J_0(0) = 1$, we have from Eq. (1.63)

$$I(\rho = 0, \zeta) = |B_0(\rho = 0, \zeta)|^2 = \left| \int_0^\infty d\kappa \kappa a(\kappa) \exp \left[-\frac{1}{2} i \kappa^2 \zeta \right] \cos(\kappa \rho_0) \right|^2. \quad (1.76)$$

Assuming an incident beam with a Gaussian spatial profile, $a(\kappa)$ is given by Eq. (2.6) and the integral may be evaluated explicitly, as given by Gradshteyn and Ryzhik [53], resulting in

$$I(\rho = 0, \zeta) = \left| \frac{1}{1 + i\zeta} \left\{ 1 - \sqrt{\frac{2}{1 + i\zeta}} \rho_0 D_+ \left(\frac{\rho_0}{\sqrt{2 + 2i\zeta}} \right) \right\} \right|^2 \quad (1.77)$$

where D_+ is the Dawson function given by

$$\begin{aligned} D_+(x) &= \exp[-x^2] \int_0^x dy \exp[y^2] \\ &= \frac{1}{2} \sqrt{\pi} \exp[-x^2] \operatorname{erfi}(x), \end{aligned} \quad (1.78)$$

and erfi is the imaginary error function.

We may derive an approximation for this axial intensity for any circularly symmetric incident beam using the method of stationary phase outlined in Appendix A. We first rewrite (1.55) in exponential form as

$$B_0(\rho = 0) = \int_0^\infty d\kappa \frac{1}{2} \kappa a(\kappa) \exp \left[-\frac{1}{2} i \kappa^2 \zeta \right] (\exp[i\kappa \rho_0] + \exp[-i\kappa \rho_0]). \quad (1.79)$$

Assuming $\kappa a(\kappa)$ varies slowly in comparison to the exponent, the use of the stationary phase method is justified and by inspection we have

$$F(\kappa) = \frac{1}{2} \kappa a(\kappa), \quad \Phi_\pm(\kappa) = -\frac{1}{2} \zeta \kappa^2 \pm \rho_0 \kappa, \quad \kappa_{0\pm} = \pm \frac{\rho_0}{\zeta}, \quad (1.80)$$

for the terms defined in Appendix A. Noting that the integral range over

$B_0, \{0, \infty\}$, requires a factor of $1/2$ to compensate for the integral range of $\{-\infty, \infty\}$ in Appendix A, and that $a(-\kappa) = a(\kappa)$, we have

$$B_0(\rho = 0) \approx \left| \frac{\rho_0}{2\zeta} a\left(\frac{\rho_0}{\zeta}\right) \right| \sqrt{\frac{2\pi}{-i\zeta}} \exp\left[-i\frac{\rho_0^2}{2\zeta}\right]. \quad (1.81)$$

Hence, the intensity along $\rho = 0$ is given by

$$\begin{aligned} I(\rho = 0) &= |B_0|^2 \approx \frac{\pi\rho_0^2}{2\zeta^3} \left| a\left(\frac{\rho_0}{\zeta}\right) \right|^2 \exp \operatorname{Im} \left[\frac{\rho_0^2}{\zeta} \right] \\ &= \frac{\pi\rho_0^2}{2\zeta^3} \left| a\left(\frac{\rho_0}{\zeta}\right) \right|^2, \end{aligned} \quad (1.82)$$

since both ρ_0 and ζ are real. For an incident beam with a Gaussian spatial profile, $a(\kappa)$ is given by Eq. (2.6) and

$$I(\rho = 0) \approx \frac{\pi\rho_0^2}{2\zeta^3} \exp\left[-\frac{\rho_0^2}{\zeta^2}\right]. \quad (1.83)$$

A plot of this approximation is shown in Fig. 1.14 where it is compared to the exact form given by Eq. (1.77).

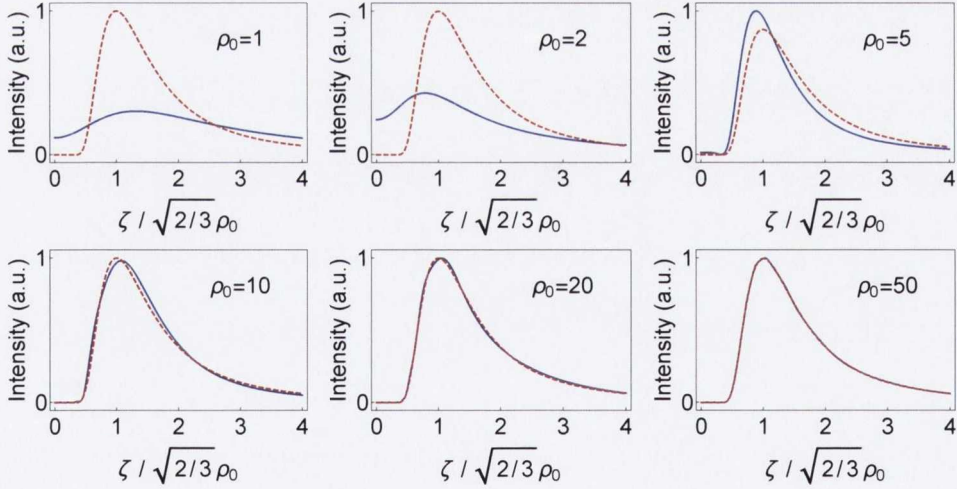
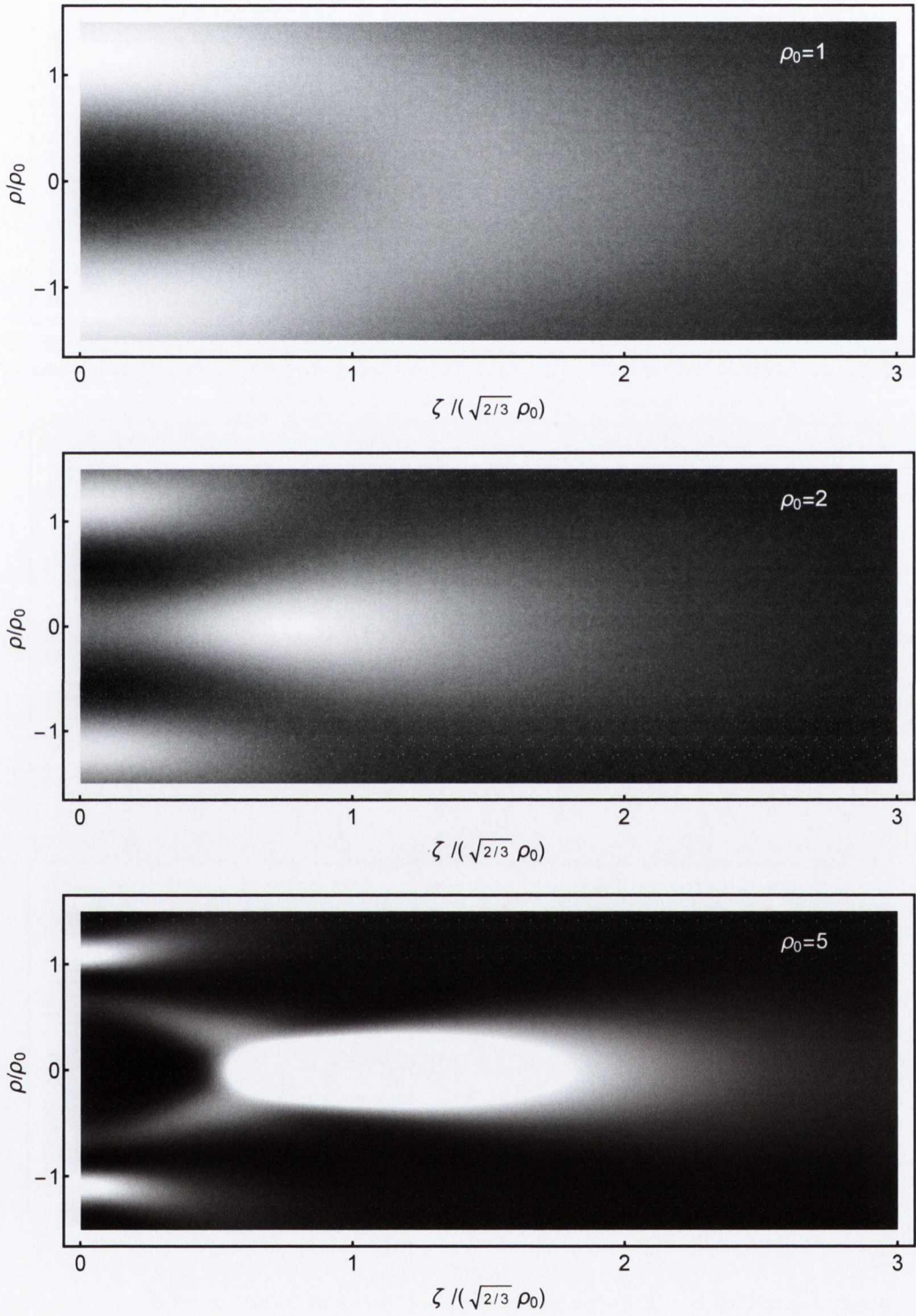


Figure 1.14: Comparison of the exact (blue line) and approximate (dashed red line) axial intensities along $\rho = 0$ for various ρ_0 values (inset). The exact intensity is given by Eq. (1.77), while the approximate intensity is given by Eq. (1.83) and shows very good agreement for $\rho_0 \gg 1$.

To get the point of maximal axial intensity, we can find the root of the derivative of Eq. (1.83) with respect to ζ , which gives

$$\begin{aligned} \frac{d}{d\zeta} I(\rho = 0) &= (2\rho_0^2 - 3\zeta^2) \frac{\pi\rho_0^2}{2\zeta^6} \exp\left[-\frac{\rho_0^2}{\zeta^2}\right] = 0 \\ \Rightarrow \zeta &= \sqrt{\frac{2}{3}}\rho_0. \end{aligned} \tag{1.84}$$

To observe the full evolution of the conically diffracted Gaussian beam, it is necessary to evaluate Eq. (1.63) over a range of ρ and ζ values. The resultant intensity profiles are shown in Fig. 1.15. As the beam propagates beyond the focal image plane, the inner ring begins to converge and the outer ring diverges. Modulations in the intensity of the converging ring are clearly visible in Fig. (1.15) for $\rho_0 \geq 10$, and can also be seen in Fig. 1.16. These modulations are known as the Warnick-Arnold rings [20]. As the inner ring continues to converge, the axial spike or Raman spike appears at $\rho = 0$. This bright feature dominates the beam evolution structure for large values of ζ and eventually tapers off, as seen in Fig. 1.14.



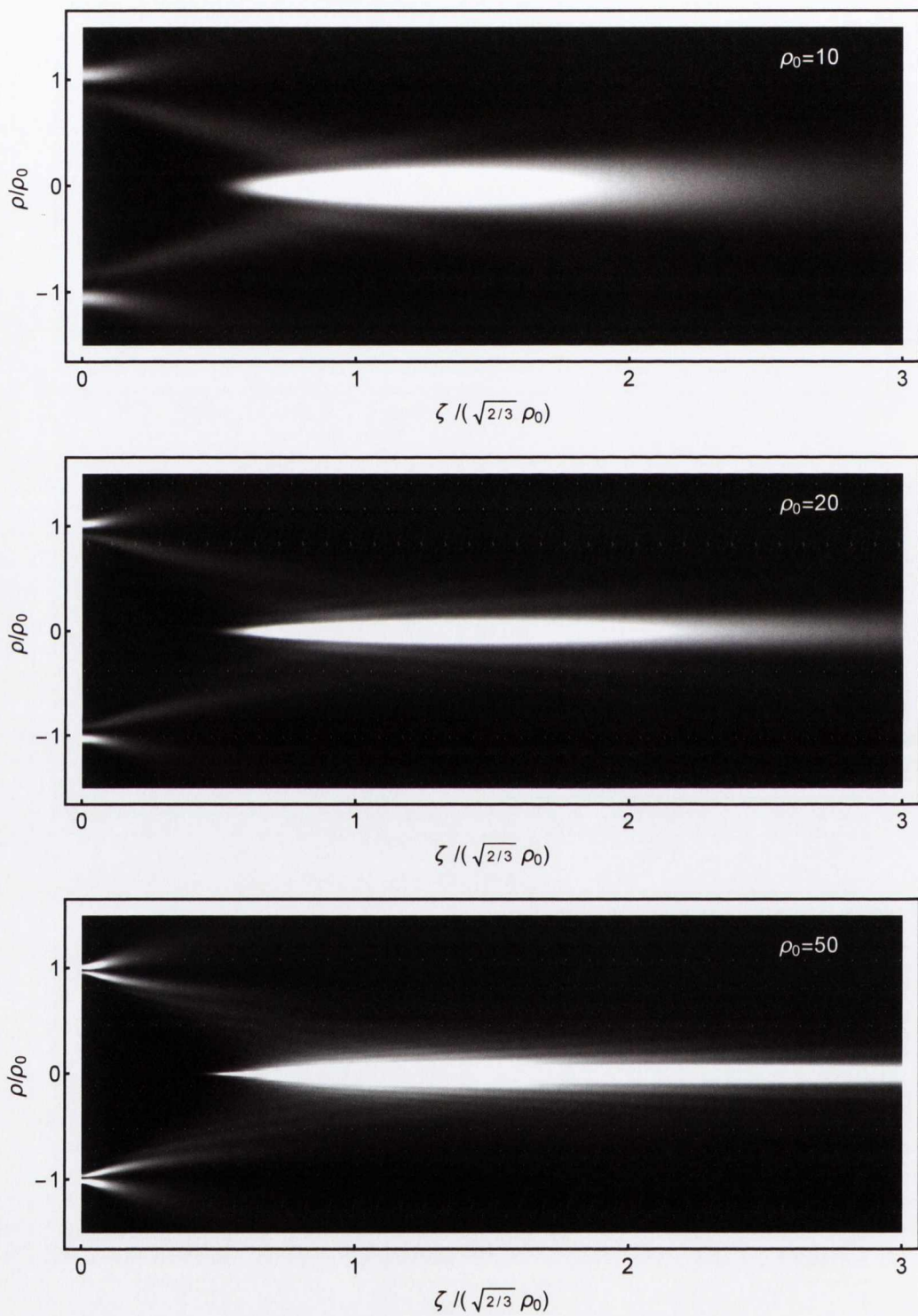


Figure 1.15: (Begins on previous page). The evolution of conically diffracted beams with various ρ_0 values (inset). The figures are two-dimensional plots on a ρ - ζ plane which contains $\rho = 0$. In some of the plots, the axial spike has been saturated to allow the lower intensity Warnick-Arnold rings to be seen.

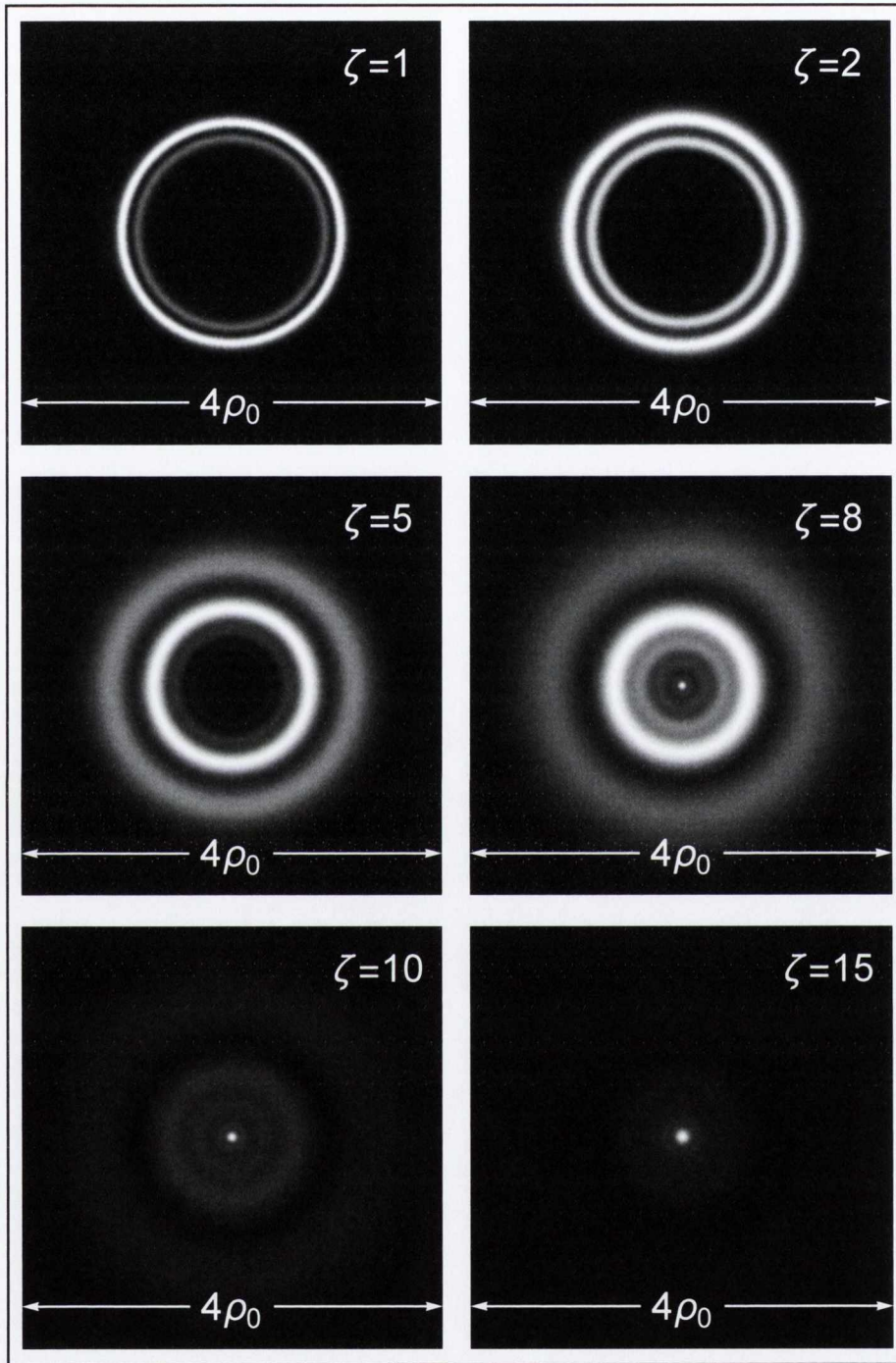


Figure 1.16: Two-dimensional plots in the ρ -plane at various ζ values (inset) for a beam with $\rho_0 = 20$. The Warnick-Arnold intensity oscillations are clear for $\zeta = 8$ and $\zeta = 10$.

Chapter 2

Conical diffraction of a top-hat beam

2.1 Motivation and theory

Recall that the fundamental integrals for the internal conical diffraction of a circularly symmetric incident beam with spatial profile $D_0(\rho)$ are

$$B_0 = \int_0^\infty d\kappa \kappa a(\kappa) \exp\left(-\frac{1}{2}i\zeta\kappa^2\right) J_0(\kappa\rho) \cos(\kappa\rho_0), \quad (2.1)$$

$$B_1 = \int_0^\infty d\kappa \kappa a(\kappa) \exp\left(-\frac{1}{2}i\zeta\kappa^2\right) J_1(\kappa\rho) \sin(\kappa\rho_0), \quad (2.2)$$

which combine to give the intensity distribution for a circularly polarised (or unpolarised) incident beam:

$$I = |B_0|^2 + |B_1|^2. \quad (2.3)$$

$a(\kappa)$ is the Fourier transform of the spatial profile of the incident beam:

$$a(\kappa) = \int_0^\infty d\rho \rho D_0(\rho) J_0(\kappa\rho). \quad (2.4)$$

Modern experimental observations of conical diffraction typically use an incident beam with a Gaussian spatial profile, which is smoothly differentiable, described by

$$D_{0,G}(\rho) = \exp\left(-\frac{1}{2}\rho^2\right) \quad (2.5)$$

which yields a Fourier transform

$$a_G(\kappa) = \exp\left(-\frac{1}{2}\kappa^2\right). \quad (2.6)$$

The equations (2.1) and (2.2) then become

$$B_0 = \int_0^\infty d\kappa \kappa \exp\left[-\frac{1}{2}\kappa^2(1+i\zeta)\right] J_0(\kappa\rho) \cos(\kappa\rho_0), \quad (2.7)$$

$$B_1 = \int_0^\infty d\kappa \kappa \exp\left[-\frac{1}{2}\kappa^2(1+i\zeta)\right] J_1(\kappa\rho) \sin(\kappa\rho_0). \quad (2.8)$$

Using Eq. (2.3) yields the intensity distributions previously discussed in Sections 1.7 and 1.8.

Intuitively, we may expect a beam with a non-smoothly differentiable profile to exhibit interesting features when directed through a biaxial material. We use the example of a top-hat beam whose spatial profile contains discontinuities as seen in Fig. 2.1(a). This profile is essentially a step function, which we may write as

$$D_{0,\text{TH}}(\rho) = \Theta(1 - |\rho|), \quad (2.9)$$

where Θ is defined by

$$\Theta(x) \equiv \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}. \quad (2.10)$$

The Fourier transform of Eq. (2.9) is found using Eq. (2.4) to be

$$a_{\text{TH}}(\kappa) = \frac{J_1(\kappa)}{\kappa}. \quad (2.11)$$

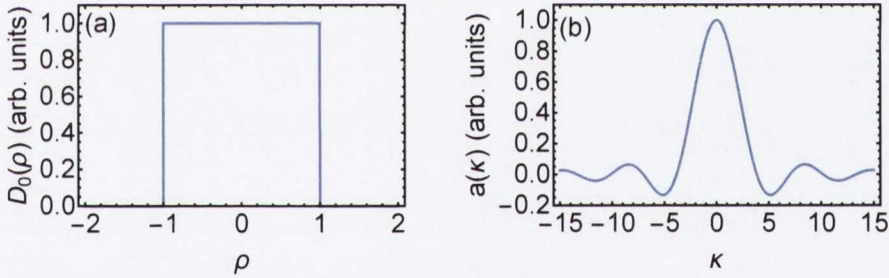


Figure 2.1: (a) The normalised profile of a top-hat beam $D_{0,\text{TH}}$ given by Eq. (2.9). (b) The normalised Fourier transform of a top-hat beam a_{TH} given by Eq. (2.11).

Inserting Eq. (2.11) into Eqs. (2.1) and (2.2) yields functions containing products of Bessel functions:

$$B_0 = \int_0^\infty d\kappa \exp\left(-\frac{1}{2}i\zeta\kappa^2\right) J_1(\kappa) J_0(\kappa\rho) \cos(\kappa\rho_0), \quad (2.12)$$

$$B_1 = \int_0^\infty d\kappa \exp\left(-\frac{1}{2}i\zeta\kappa^2\right) J_1(\kappa) J_1(\kappa\rho) \sin(\kappa\rho_0), \quad (2.13)$$

which combine to give the intensity distribution, as in Eq. (2.3).

Consider the focal image plane (FIP) where $\zeta = 0$ meaning the exponents in Eqs. (2.12) and (2.13) are unity. There is an exact form for the following integral [53]:

$$\begin{aligned}
 \int_0^\infty dx J_\nu(ax) J_\nu(bx) \sin(cx) \\
 &= 0 & 0 < c < b - a, \\
 &= \frac{1}{2\sqrt{ab}} P_{\nu-1/2} \left(\frac{b^2 + a^2 - c^2}{2ab} \right) & b - a < c < b + a, \\
 &= -\frac{\cos(\nu\pi)}{\pi\sqrt{ab}} Q_{\nu-1/2} \left(-\frac{b^2 + a^2 - c^2}{2ab} \right) & b + a < c,
 \end{aligned} \tag{2.14}$$

where $\text{Re}\nu > -1$ and $0 < a < b$. P_λ is the λ -degree Legendre function, while Q_λ is the λ -degree Legendre function of the second kind. Substituting $\nu = 1$, $a = 1$, $b = \rho$, and $c = \rho_0 > 0$ gives the exact solution for Eq. (2.13) at the FIP:

$$\begin{aligned}
 \int_0^\infty d\kappa J_1(\kappa) J_1(\rho\kappa) \sin(\rho_0\kappa) \\
 &= 0 & \rho > \rho_0 + 1, \\
 &= \frac{1}{2\sqrt{\rho}} P_{1/2} \left(\frac{\rho^2 - \rho_0^2 + 1}{2\rho} \right) & \rho_0 - 1 < \rho < \rho_0 + 1, \\
 &= \frac{1}{\pi\sqrt{\rho}} Q_{1/2} \left(\frac{\rho_0^2 - \rho^2 - 1}{2\rho} \right) & 1 \leq \rho < \rho_0 - 1.
 \end{aligned} \tag{2.15}$$

Note that this does not provide a solution for $\rho < 1$, a case which we will deal with later. This result is similar to that obtained in Berry [54], although there the electric displacement field is expressed in terms of elliptic integrals. Eq. (2.12) does not have an exact solution, however we may make an approximation which will allow us to attain an expression for the intensity. Using the approximation for a ν^{th} order Bessel function of the first kind:

$$J_\nu(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4} - \frac{\nu\pi}{2}\right), \quad x \gg 1, \tag{2.16}$$

allows us to approximate $B_0 \approx B_1$ for $\rho_0 \gg 1$ and $\rho \approx \rho_0$, as shown in Eqs.

(1.71), (1.72), and (1.75) Hence, at the FIP we have

$$I(\rho \approx \rho_0, \zeta = 0) \approx 2|B_1(\rho \approx \rho_0, \zeta = 0)|^2 \quad (2.17)$$

for which we have the exact solution in Eq. (2.15). An interesting feature of this function is that it takes a zero value for $\rho > \rho_0 + 1$, but for $\rho_0 - 1 < \rho < \rho_0 + 1$ as $\rho \rightarrow \rho_0 + 1$, we get

$$\begin{aligned} \frac{1}{2\sqrt{\rho}} P_{1/2} \left(\frac{\rho^2 - \rho_0^2 + 1}{2\rho} \right) &\rightarrow \frac{1}{2\sqrt{\rho_0 + 1}} P_{1/2} \left(\frac{\rho_0^2 + 2\rho_0 + 1 - \rho_0^2 + 1}{2(\rho_0 + 1)} \right) \\ &= \frac{1}{2\sqrt{\rho_0 + 1}} P_{1/2} \left(\frac{2\rho_0^2 + 2}{2\rho_0 + 2} \right) \\ &= \frac{1}{2\sqrt{\rho_0 + 1}} P_{1/2}(1). \end{aligned} \quad (2.18)$$

$P_{1/2}(1)$ is unity, and although the fraction approaches zero as ρ_0 approaches infinity, overall the expression will be non-zero for finite values of ρ_0 . This results in a jump from a non-zero value to zero at the point $\rho = \rho_0 + 1$.

Another interesting point in the function is $\rho \rightarrow \rho_0 - 1$:

$$\begin{aligned} \lim_{\rho \rightarrow (\rho_0 - 1)^+} \frac{1}{2\sqrt{\rho}} P_{1/2} \left(\frac{\rho^2 - \rho_0^2 + 1}{2\rho} \right) &= \\ \lim_{\rho \rightarrow (\rho_0 - 1)^-} \frac{1}{\pi\sqrt{\rho}} Q_{1/2} \left(\frac{\rho_0^2 - \rho^2 - 1}{2\rho} \right) &= \frac{1}{\pi\sqrt{\rho_0 - 1}} Q_{1/2} \left(\frac{\rho_0^2 - \rho_0^2 + 2\rho_0 - 1 - 1}{2\rho_0 - 2} \right) \\ &= \frac{1}{\pi\sqrt{\rho_0 - 1}} Q_{1/2}(1) = \infty. \end{aligned} \quad (2.19)$$

This is a term with indeterminate argument but infinite modulus, and so the intensity of the beam at this point is infinite.

Figure 2.2 contains a plot of the $1/2$ -degree Legendre function $P_{1/2}(x)$ showing that it changes sign at $x \approx -0.6522$. Thus Eq. (2.15) has a root at

$$\begin{aligned} \frac{\rho^2 - \rho_0^2 + 1}{2\rho} &\approx -0.6522 \\ \Rightarrow \rho^2 + 1.3045\rho &\approx \rho_0^2 - 1 \\ \Rightarrow \rho &\approx -0.6522 + \sqrt{\rho_0^2 - 0.5746}, \end{aligned} \quad (2.20)$$

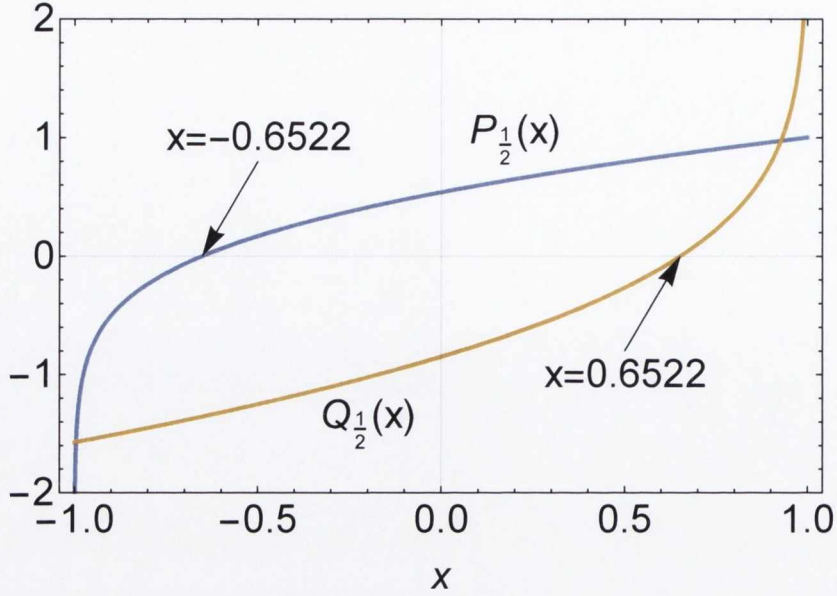


Figure 2.2: The Legendre functions $P_{1/2}(x)$ (blue) and $Q_{1/2}(x)$ (orange) and their roots.

discarding the unphysical negative root. This zero-intensity region of the profile occurs between the discontinuity (2.18) and the infinite spike (2.19), and for example for a beam with $\rho_0 = 20$ occurs at $\rho \approx 19.3334$. The profile of a conically diffracted top-hat beam is shown in Fig. 2.3 with the features discussed above labelled. Integrating Eq. (2.17) from $\rho = 0$ to the zero intensity given by Eq. (2.20) reveals the infinitely intense spike contains 28.79% of the energy of the beam.

It was previously noted that Eq. (2.15) is only valid for $\rho \geq 1$. However, for $0 < \rho < 1$, Eq. (2.14) may be recast with $a = \rho$ and $b = 1$, resulting in identical terms to those in Eq. (2.15) but with different conditions. The first two conditions $0 < \rho_0 < 1 - \rho$ and $1 - \rho < \rho_0 < 1 + \rho$ are forbidden for $\rho_0 > 2$, which is required for high quality intensity profiles, leaving only the final condition $\rho < \rho_0 - 1$. Thus, the third option in Eq. (2.15) (containing the $Q_{1/2}$ Legendre function) is also valid for $0 < \rho < 1$, although in this region the approximation $B_0 \approx B_1$ fails.

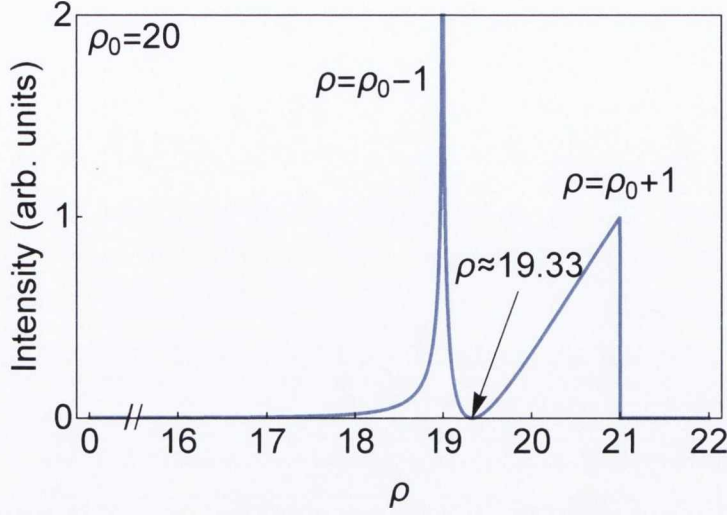


Figure 2.3: The intensity profile of a conically diffracted top-hat beam generated using Eq. (2.17) with $\rho_0 = 20$, normalised with the intensity of the feature at $\rho = \rho_0 + 1$. The infinitely intense spike at $\rho = \rho_0 - 1$ and the zero intensity given by Eq. (2.20) are also labelled. The profile is featureless and approximately zero for $\rho < 16$ and so the ρ -axis has been broken.

2.2 Limiting transverse wave vector

The unique and interesting profile predicted by Eq. (2.17) warranted experimental investigation, particularly as it had not yet been observed. However, the plot shown in Fig. 2.3 can of course never be realised experimentally owing to the presence of the infinitely intense spike and the discontinuous step. Firstly, a true top-hat beam is unattainable; there will always be some smoothing of the profile. Secondly, the biaxial crystal has finite dimensions meaning the transverse wave vector components κ are limited to within the material. These effects can be incorporated into the model by modifying the range over which Eqs. (2.12) and (2.13) are integrated, as shown in Jeffrey [55], where we approximate the shape of the biaxial crystal to be a cylinder to preserve circular symmetry:

$$B_0 = \int_0^{\kappa_{\max}} d\kappa \exp\left(-\frac{1}{2}i\zeta\kappa^2\right) J_1(\kappa) J_0(\kappa\rho) \cos(\kappa\rho_0), \quad (2.21)$$

$$B_1 = \int_0^{\kappa_{\max}} d\kappa \exp\left(-\frac{1}{2}i\zeta\kappa^2\right) J_1(\kappa) J_1(\kappa\rho) \sin(\kappa\rho_0), \quad (2.22)$$

where $\kappa_{\max}$ is the maximal transverse wave vector component permitted in the biaxial crystal. The physical interpretation of this limiting wave vector will be discussed further in Section 2.4. We may observe what this low-pass spatial-filter effect will have on the conically diffracted profile at the FIP by generating a three-dimensional plot of the intensity as a function of both ρ and $\kappa_{\max}$, as seen in Fig. 2.4. As $\kappa_{\max}$ increases, the profile tends to that seen in Fig. 2.3. For low values of $\kappa_{\max}$, both the wedge-shaped feature and the high intensity spike broaden and their profiles become smooth. Oscillations are also visible in the profile.

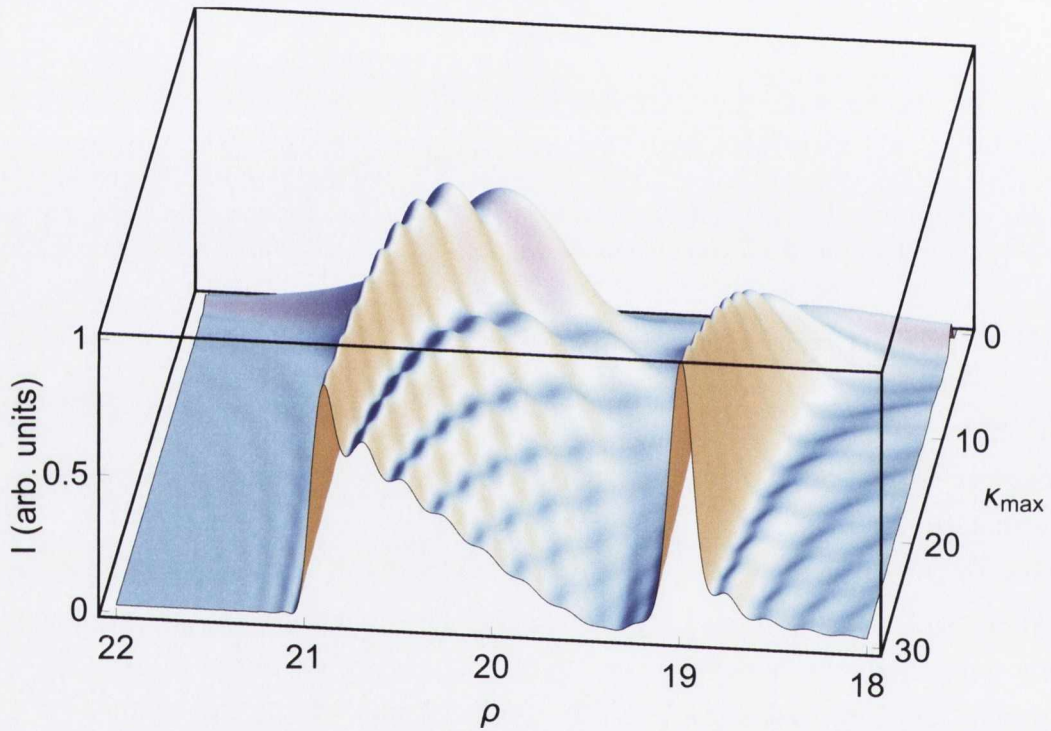


Figure 2.4: Intensity of the conically diffracted profile with $\rho_0 = 20$ at the FIP as a function of ρ and $\kappa_{\max}$. As $\kappa_{\max}$ increases, the profile tends to that seen in Fig. 2.3. The number of oscillations in the profile also increases with $\kappa_{\max}$, although their relative amplitudes decay. The colour gradient is for clarity only.

2.3 Beyond the FIP

The conically diffracted beam evolves as it propagates beyond the focal image plane [30, 54, 56]. This evolution has been discussed in Section 1.8 for the case of an incident beam with a Gaussian spatial profile. In this case the rings which form at the FIP spread out as ζ increases and eventually the inner ring converges to produce a high intensity region in the centre of the beam $\rho = 0$, known as the axial spike. When $\rho_0 \gg 1$, the peak intensity of this axial spike occurs at $\zeta \approx \rho_0 \sqrt{2/3}$. A theoretical plot of the evolution over the range $0 \leq \zeta \leq 10$ is shown in Fig. 2.5 generated using Eq. (2.3) with $\rho_0 = 7.2$ and $a(\kappa)$ given by Eq. (2.6).

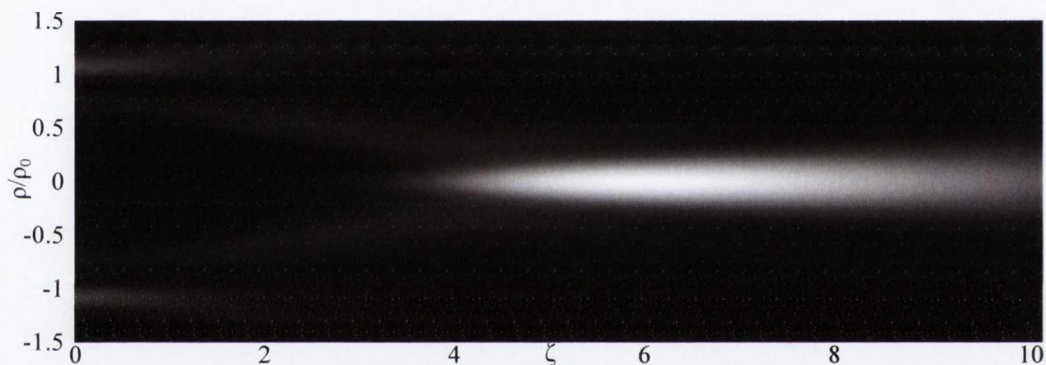


Figure 2.5: Theoretical plot of the far-field evolution of a conically diffracted Gaussian beam generated using Eq. (2.3) with $a(\kappa)$ given by Eq. (2.6).

The evolution of a conically diffracted top-hat has not previously been investigated. To examine this evolution, we will make use of the stationary phase approximation for the intensity of a conically diffracted beam at $\rho = 0$, which we derived in Section 1.8:

$$I(\rho = 0) \approx \frac{\pi \rho_0^2}{2\zeta^3} \left| a\left(\frac{\rho_0}{\zeta}\right) \right|^2. \quad (2.23)$$

Using the Fourier transform of a top-hat beam $a_T(\kappa)$ as given in Eq. (2.11) allows us to approximate the axial intensity for a conically diffracted top-hat beam as

$$I(\rho = 0) \approx \frac{\pi \rho_0^2}{2\zeta^3} \left| \frac{\zeta}{\rho_0} J_1\left(\frac{\rho_0}{\zeta}\right) \right|^2 = \frac{\pi}{2\zeta} \left| J_1\left(\frac{\rho_0}{\zeta}\right) \right|^2. \quad (2.24)$$

It will now be useful to use the approximation for a ν^{th} order Bessel function of the first kind, which is valid when $x \gg 1$, as given in Eq. (2.16):

$$\begin{aligned}
 J_\nu(x) &\approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4} - \frac{\nu\pi}{2}\right), \quad x \gg 1, \\
 \rightarrow J_1\left(\frac{\rho_0}{\zeta}\right) &\approx \sqrt{\frac{2\zeta}{\pi\rho_0}} \cos\left(\frac{\rho_0}{\zeta} - \frac{3\pi}{4}\right), \quad \zeta \ll \rho_0, \\
 \Rightarrow I(\rho = 0) &\approx \frac{\pi}{2\zeta} \frac{2\zeta}{\pi\rho_0} \cos^2\left(\frac{\rho_0}{\zeta} - \frac{3\pi}{4}\right) \\
 &= \frac{1}{\rho_0} \cos^2\left(\frac{\rho_0}{\zeta} - \frac{3\pi}{4}\right). \tag{2.25}
 \end{aligned}$$

The form of the expression obtained in Eq. (2.25) suggests the intensity along $\rho = 0$ oscillates as a function of ζ as a consequence of the $\cos^2$ term. Furthermore, an interesting observation is the absence of ζ in the factor before the oscillatory $\cos^2$ term meaning the maxima along $\rho = 0$ have equal intensity when $\zeta \ll \rho_0$. The minima will of course also have equal intensity, where $\cos^2 = 0$. These predictions are confirmed in the plots of Eqs. (2.24) and (2.25) in Fig. 2.6.

We may find the extrema of Eq. (2.25) by taking the derivative and finding the values of ζ which yield zero:

$$\begin{aligned}
 \frac{\partial}{\partial \zeta} I(\rho = 0) &= \frac{2}{\zeta^2} \cos\left(\frac{\rho_0}{\zeta} + \frac{\pi}{4}\right) \sin\left(\frac{\rho_0}{\zeta} + \frac{\pi}{4}\right) = 0, \\
 \Rightarrow \zeta_{\pm} &= \frac{\rho_0}{\pi\left(n \pm \frac{1}{4}\right)}, \quad n \in \mathbb{Z}^+. \tag{2.26}
 \end{aligned}$$

Taking the second derivative of (2.25)

$$\begin{aligned}
 \frac{\partial^2}{\partial \zeta^2} I(\rho = 0) &= \\
 &= \frac{2\rho_0 \sin^2\left(\frac{\rho_0}{\zeta} + \frac{\pi}{4}\right)}{\zeta^4} - \frac{2\rho_0 \cos^2\left(\frac{\rho_0}{\zeta} + \frac{\pi}{4}\right)}{\zeta^4} - \frac{4 \sin\left(\frac{\rho_0}{\zeta} + \frac{\pi}{4}\right) \cos\left(\frac{\rho_0}{\zeta} + \frac{\pi}{4}\right)}{\zeta^3} \tag{2.27}
 \end{aligned}$$

allows us to determine whether the extrema correspond to local maxima or

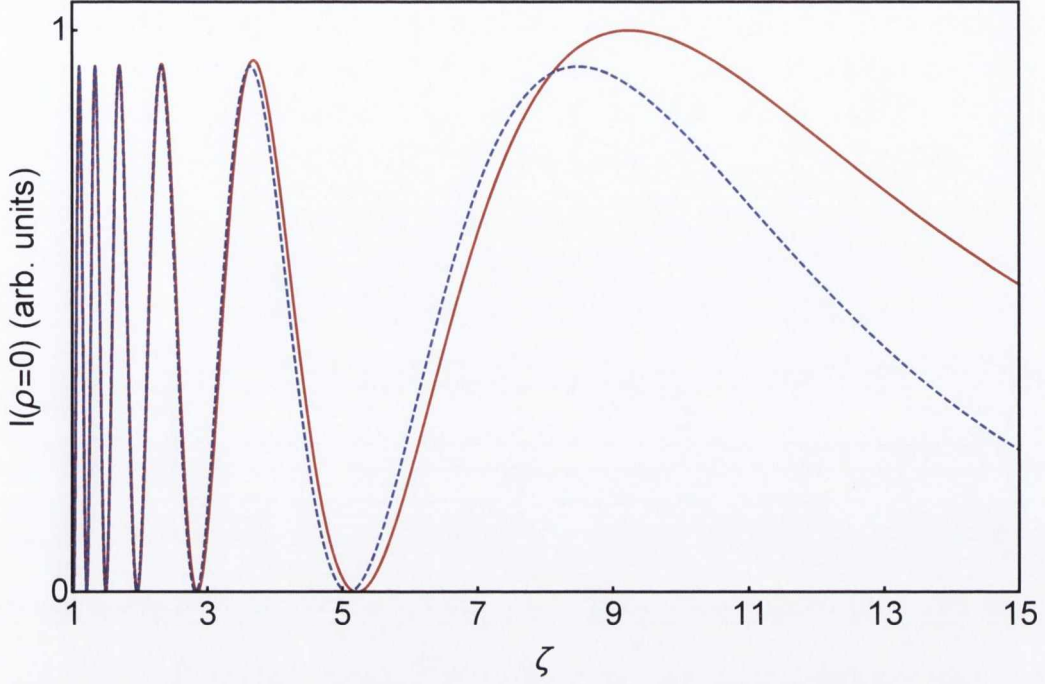


Figure 2.6: Normalised intensity along the centre ($\rho = 0$) of a conically diffracted top-hat beam with $\rho_0 = 20$ from $\zeta = 1$ to $\zeta = 15$. The solid red line is the exact intensity, while the dashed blue line is the approximate intensity given by Eq. (2.25). It is clear that the approximation is very good provided $\zeta \ll \rho_0$. The region $0 \leq \zeta < 1$ has been omitted since the function is highly oscillatory in this region. There are no more maxima beyond $\rho \approx 9$, instead the intensity slowly tapers off to zero.

minima. Picking $n = 1$, we find Eq. (2.27) gives

$$\frac{\partial^2}{\partial \zeta^2} I(\rho = 0, \zeta_+) = \frac{625\pi^4}{128\rho_0^3} > 0, \quad \frac{\partial^2}{\partial \zeta^2} I(\rho = 0, \zeta_-) = -\frac{81\pi^4}{128\rho_0^3} < 0. \quad (2.28)$$

Thus, ζ_+ correspond to local minima while ζ_- correspond to local maxima. Examining Eq. (2.26) reveals that as $n \rightarrow \infty$, *i.e.* as $\zeta \rightarrow 0$, the separation between adjacent $\zeta_{\pm}$ values decreases approximately at the rate of $1/n$. This means the intensity along $\rho = 0$ is an oscillatory function whose frequency increases rapidly as $\zeta \rightarrow 0$. As $\zeta \rightarrow \rho_0$, the approximation in Eq. (2.25) breaks down and instead we must examine Eq. (2.24) to find that as $\zeta \rightarrow \infty$, the axial intensity trails off slowly to zero since $J_1(\rho_0/\infty) = J_1(0) = 0$.

The unusual structure of the evolution was examined more closely by generating a two dimensional logarithmic plot of the intensity for a beam with $\rho_0 = 7.2$ as shown in Fig. 2.7. It becomes apparent that the structure is very complex, with many rings which converge as the beam propagates. Indeed it appears that there are a multitude of intensity maxima where the rings converge along the $\rho = 0$ line for $0 < \zeta < 0.4$ which appear and disappear over extremely short distances as predicted by Eq. (2.25).

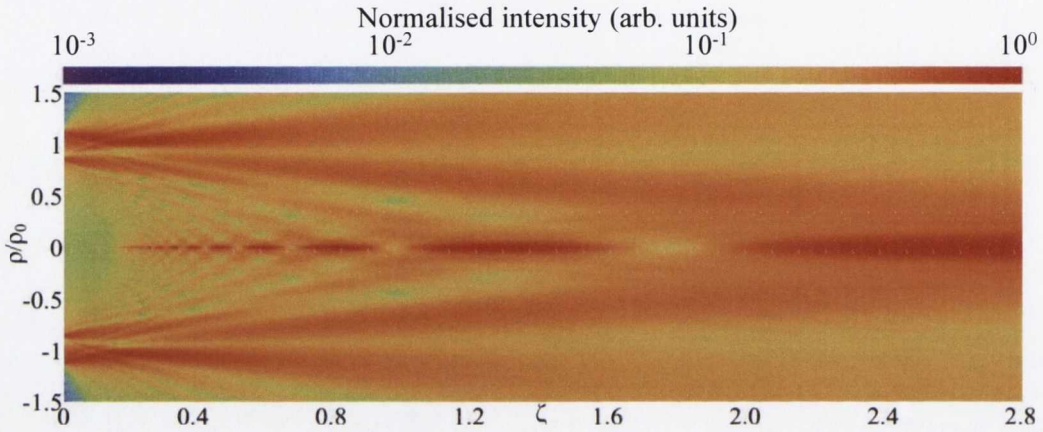


Figure 2.7: A logarithmic plot of the far-field evolution of a conically diffracted top-hat beam, calculated using Eq. (2.3) and $\rho_0 = 7.2$, in order to show the complex and intricate structure of the beam. It also demonstrates how the oscillating lobes along the $\rho = 0$ line are formed from many converging rings.

2.4 Imaging FIP radial profile

In order to study the profile of a conically diffracted top-hat beam in high detail the experimental apparatus shown in Fig. 2.8 was used. A circularly polarised Helium-Neon (HeNe) laser beam with a peak emission at 632.8 nm was directed through a 100 μm diameter pinhole. Since the $1/e$ radius of the laser beam was of the order of millimetres, the 100 μm pinhole acted to cut off enough of the beam to generate a good approximation of a top-hat profile, as can be seen in Fig. 2.9.

A 22 mm long slab of the biaxial material $\text{KGd}(\text{WO}_4)_2$ was placed as close as possible to the pinhole and was aligned so that the beam was incident along

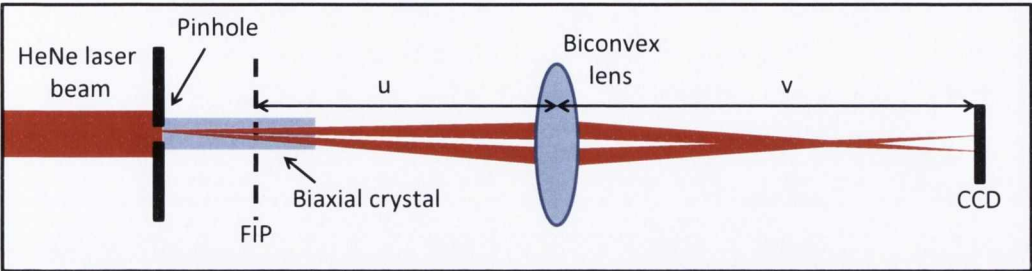


Figure 2.8: The experimental apparatus used to study the fine structure of the beam profile in the FIP which is imaged onto the CCD. The values of u and v could be adjusted to give the image a desired magnification at the correct location in the beam.

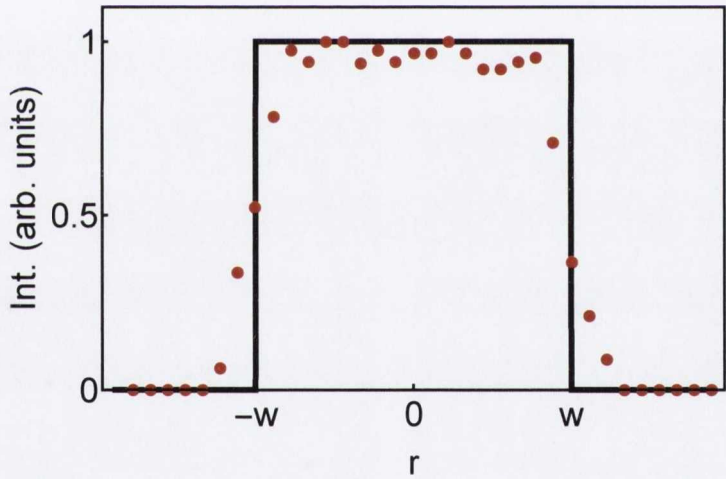


Figure 2.9: A plot of a top-hat beam of radius w (solid black) superimposed with experimental data (red dots) of the profile of the top-hat beam used during the experiments. This demonstrates that a reasonably good approximation of a top-hat beam.

an optic axis. A biconvex lens of focal length $f = 3$ cm was placed at $u = 3.11$ cm after the FIP where $\zeta = 0$. This location occurs inside the crystal, ~ 11 mm from the entrance face, as can be found using Eq. (1.67). The image of the FIP was formed 86 cm after the lens, where a ThorLabs DC310C colour charge-coupled device (CCD) of pixel size $4.65 \mu\text{m}$ was placed to record the profiles generated. The magnification produced by the lens was calculated to be $m = |v/u| = 28$ which was sufficient to almost fill the CCD chip with the entire singularity and wedge structure. The unmagnified FIP radius was determined by averaging the radial intensity distribution around the ring and counting the pixels between the centre and the dark Poggendorff region separating the two bright rings. This was found to be $R_0 = 360 \pm 10 \mu\text{m}$ which corresponds to $\rho_0 = 7.2$ when using a pinhole with a $50 \mu\text{m}$ radius. Note that this value differs from the predicted value of $R_0 = 430 \mu\text{m}$ obtained using Eq. (1.23), a discrepancy which is discussed in more detail in Section 3.4.

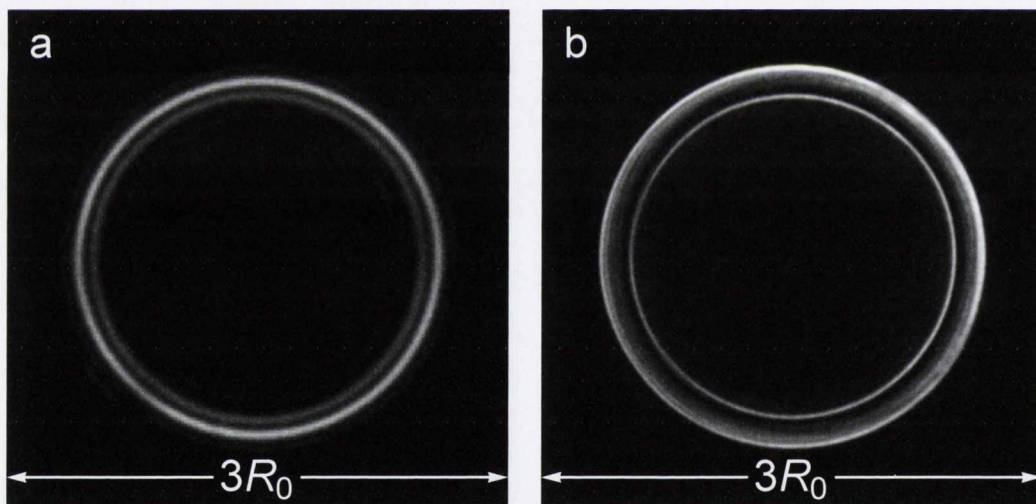


Figure 2.10: (a) An experimental image of the conically diffracted Gaussian beam in the FIP. The structure is distinct from that formed when using a top-hat input beam as seen in (b). The radius of the Poggendorff dark ring in image (b) was determined to be $R_0 = 360 \pm 10 \mu\text{m}$. Both images are $0.8 \text{ mm} \times 0.8 \text{ mm}$.

An example of the images recorded using this apparatus is shown in Fig. 2.10. Both images are taken at the FIP, with Fig. 2.10(b) demonstrating the

observed profile when using a top-hat input beam. Fig. 2.10(a) is an image generated using a Gaussian input beam.

In practice the range over which Eqs. (2.12) and (2.13) are integrated is not infinite, rather the experimental apparatus will introduce constraints on these values [55] with the maximum contribution $\kappa_{\max}$ being the maximum transverse wave vector component reaching the imaging device. Consider an iris of radius R_{lim} centred on the conically diffracted beam and placed a distance z_{iris} from the entrance face of the crystal, as seen in Fig. 2.11. The maximum transverse wave vector component $\kappa_{\max}$ passing through this iris is then

$$\kappa_{\max} \equiv n_2 k_0 w \sin \left[\arctan \left(\frac{R_{\text{lim}}}{z_{\text{iris}}} \right) \right]. \quad (2.29)$$

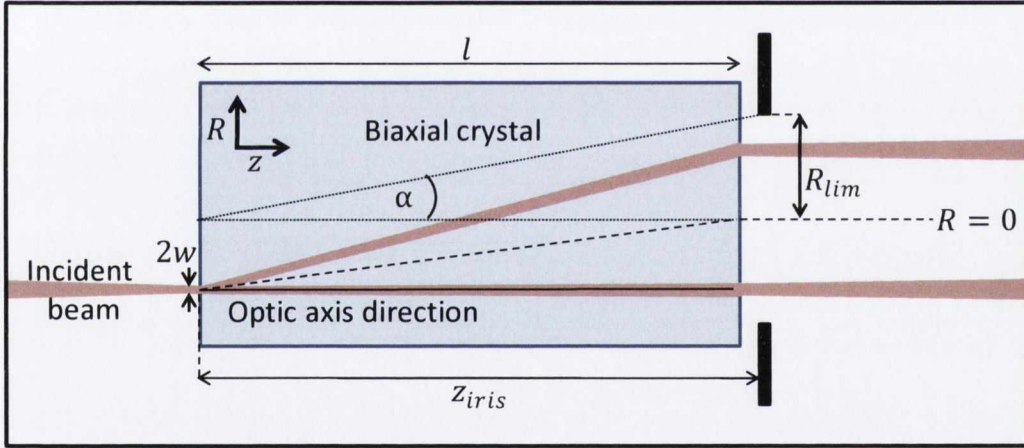


Figure 2.11: The maximum transmissible transverse wave vector component $\kappa_{\max}$ is determined by the radius of the iris R_{lim} and its position z_{iris} . This transverse wave vector is given by $\kappa_{\max} = n_2 k_0 w \sin \alpha$ where $\alpha = \arctan (R_{\text{lim}}/z_{\text{iris}})$.

This is effectively a low-pass spatial filter effect, cutting off higher transverse wave vector components reaching the imaging device while retaining lower components. The smallest iris in the system shown in Fig. 2.8 is the crystal itself which has a limiting radius of $R_{\text{lim}} = 1.5$ mm. Since the beam is free to propagate transversely in the crystal until it emerges, the value of z_{iris} is simply the length of the crystal, 22 mm. Using these values in Eq. (2.29)

gives $\kappa_{\max} = 69$ which was subsequently used in Eqs. (2.21) and (2.22) to generate the plot shown in Fig. 2.12(a). The experimental profile was generated by first finding the centre of the ring structure in the image. A radial profile was taken in the horizontal direction, then the image was rotated by a small angle and another profile was taken. This process was repeated until the image had undergone a full rotation, and the individual radial profiles were averaged to obtain an average radial intensity profile. The rapid oscillations for $0.9 \leq \rho/\rho_0 \leq 1.1$ are somewhat smoothed out in the experimental profile due to the effect of this averaging, but otherwise it agrees very well with the prediction.

A variable iris was then inserted between the crystal and the biconvex lens, a distance of $z_{\text{iris}} = 40$ mm from the entrance face of the crystal. When the iris was at its minimum setting, the limiting radius was $R_{\text{lim}} = 0.75$ mm which is smaller than the radius of the crystal, and so using Eq. (2.29) we find $\kappa_{\max} = 19$. Equations (2.21) and (2.22) were subsequently used to calculate the theoretical intensity profile shown in Fig. 2.12(b), compared with the experimentally observed average radial profile generated using the method outlined above. This again shows very good agreement. The inner ring is less intense than in the case with no iris, as seen in Fig. 2.12(a), and it is also broader, as predicted by Fig. 2.4. The oscillations of the wedge-shaped feature have become more pronounced with lower frequencies than in Fig. 2.12(a).

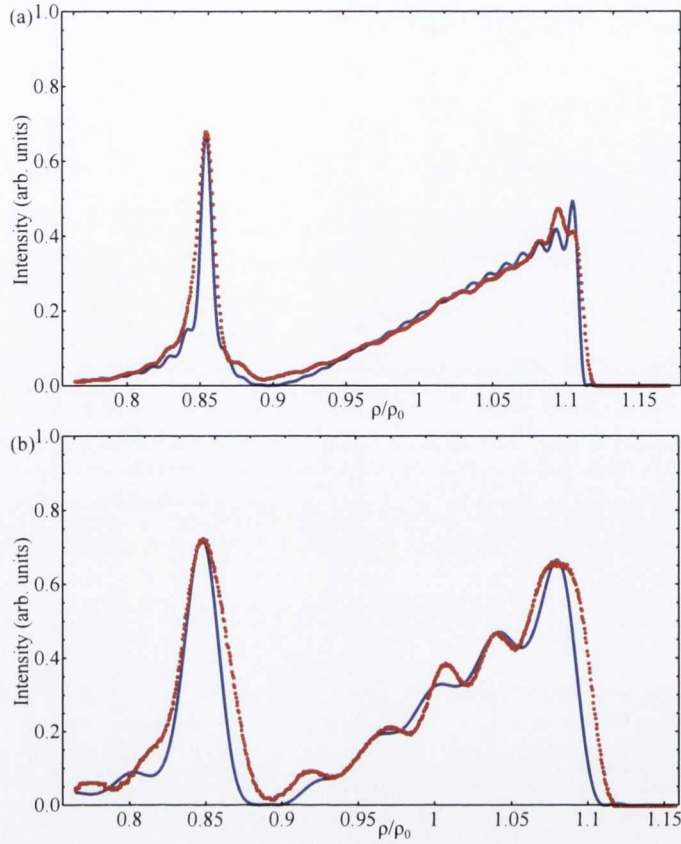


Figure 2.12: (a) Comparison of the observed intensity profile (red dots) with the theoretical profile (solid blue) where $\kappa_{\max} = 69$. (b) Comparison of the observed intensity profile (red dots) with the theoretical profile (solid blue) where $\kappa_{\max} = 19$, corresponding to an iris of diameter 1.5 mm present before the imaging lens in Fig. 2.8.

2.5 Experimental imaging of the beam evolution

The propagation of a conically diffracted top-hat beam beyond the FIP was examined using the experimental arrangement in Fig. 2.13. A HeNe laser beam was directed through a 100 μm diameter pinhole and a biconvex lens of focal length $f = 10$ cm was placed a distance $u = 20$ cm after the pinhole. This produced an unmagnified image of the pinhole at a distance of $v = 20$ cm beyond the lens. When a 22 mm long slab of $\text{KGd}(\text{WO}_4)_2$ was inserted into the beam between the lens and $z = 0$, the FIP occurred outside the

crystal as determined by Eq. (1.67), rather than inside the crystal. A CCD with a pixel size of $4.65 \mu\text{m}$ was mounted on a rail allowing movement in the ζ direction. This experimental arrangement allowed the ρ -plane to be viewed for a large range of ζ values without introducing a ζ -dependent magnification. However, this came at a cost—the value of κ_{max} was determined to be 12, much lower than the value of 69 for the experimental arrangement in Fig. 2.8.

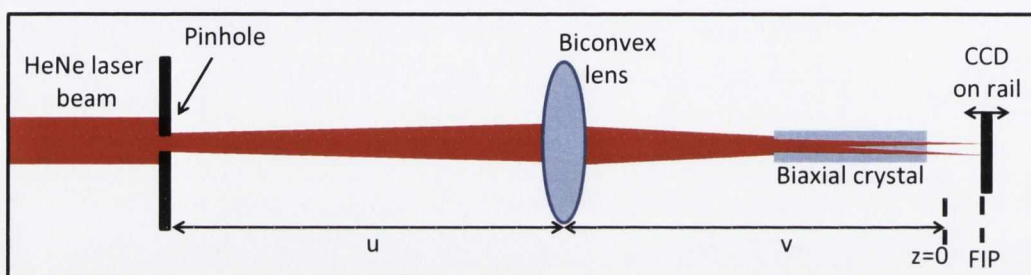


Figure 2.13: The experimental apparatus used to study how the conically diffracted top-hat beam evolves in space. The pinhole is imaged using a biconvex lens of focal length f to a point $z = 0$ beyond the crystal, thus using Eq. (1.67) the FIP occurs outside the crystal at a distance $z_{\text{FIP}} = v + l(1 - 1/n_2)$ from the lens.

The system was used to record a series of images beginning at $z = z_{\text{FIP}} \Rightarrow \zeta = 0$ and moving away from the crystal in 1 mm increments to $z = z_{\text{FIP}} + 7$ cm corresponding to $\zeta = 2.8$. The images were then stitched together using numerical interpolation software which allowed the beam propagation to be viewed in a plane which bisects the ring structure and is orthogonal to the propagation direction, i.e. the ρ - ζ plane which contains $\rho = 0$. The resultant intensity profile is shown in Fig. 2.14(b). A theoretical profile was generated to show how the beam evolves in this region using Eq. (2.3) with $a(\kappa)$ given by Eq. (2.11), $\kappa_{\text{max}} = 12$, and $\rho_0 = 7.2$. The theoretical profile is shown in Fig. 2.14(a). There is good agreement between theory and experiment and the presence of the predicted oscillatory axial spike is obvious; a feature which does not occur for the Gaussian input beam as seen in Fig. 2.5. The experimental profile is slightly skewed owing to a small inclination of the CCD relative to the beam; an issue which was only discovered after image processing. Nevertheless, the experimental profile is of high quality and the

relative locations of the maxima and minima match very well with theory.

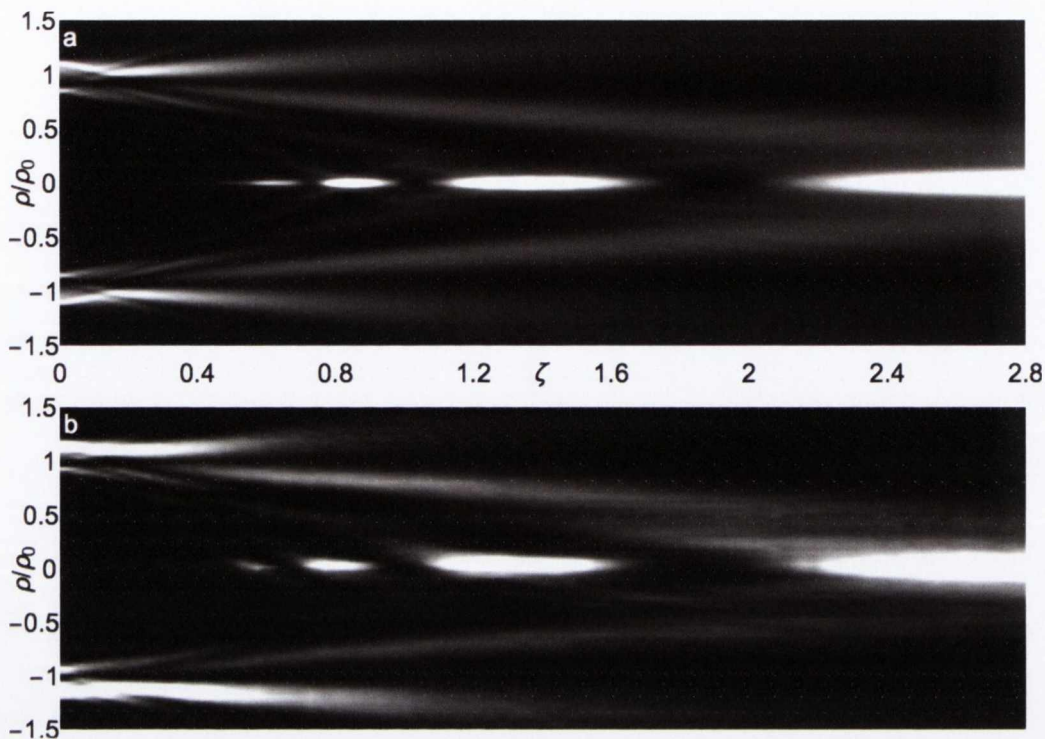


Figure 2.14: Comparison of the theoretical evolution of the beam (a) with the observed evolution (b) in the case of a 50 μm radius top-hat beam. Image (b) was generated by stitching together a series of images taken at 1 mm increments in the direction of beam propagation.

An axial profile was taken along $\rho = 0$ for both the experimental and theoretical profiles in order to compare the minima and maxima. These profiles are shown in Fig. 2.15 and demonstrate a quenching of the axial oscillations for $\zeta < 0.5$ compared to Fig. 2.7, due to the limiting κ_{max} value. With a suitable experimental arrangement, it should be possible to see more intensity maxima. Examples of the images taken are shown in Fig. 2.16 (d-f) where they are compared with the corresponding theoretical profiles (a-c), which were calculated using Eq. (2.3).

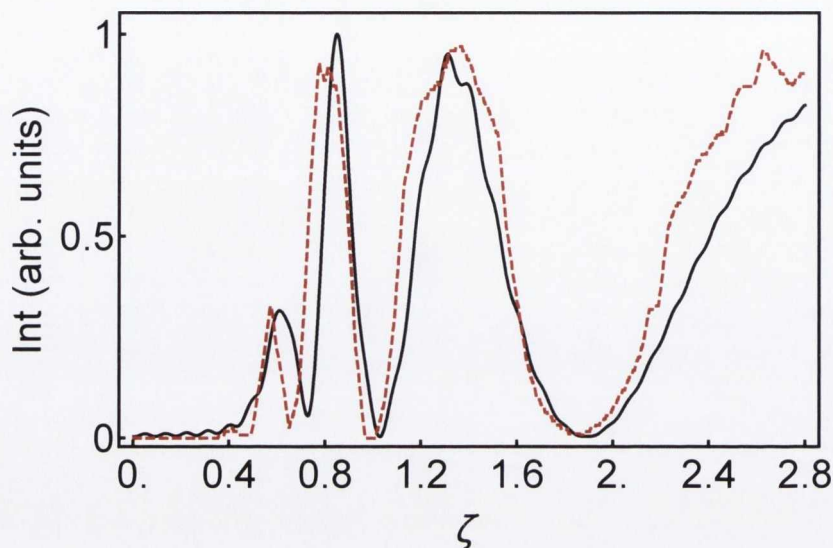


Figure 2.15: The axial intensity ($\rho = 0$) of the experimental image (dashed red line) compared to the theoretical axial intensity (solid black line) for a conically diffracted top-hat beam with $\rho_0 = 7.2$ and $\kappa_{\max} = 12$. The locations of the maxima agree quite well, as do their relative widths and intensities.

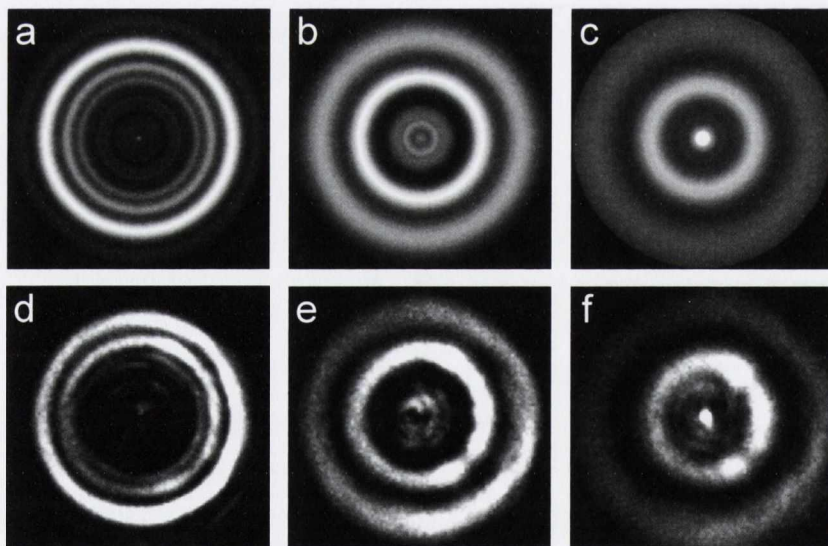


Figure 2.16: Theoretical intensity distributions in the ρ -plane (a-c) compared to the images taken using the CCD (d-f). The value of ζ is (a) and (d): 0.5, (b) and (e): 1, (c) and (f): 1.5, and $\rho_0 = 7.2$ with $\kappa_{\max} = 12$. All images are $3R_0 \times 3R_0$ in size.

2.6 Incident Bessel beams

Using the theory outlined in this chapter, for any circularly symmetric incident beam profile D_0 , we may predict the intensity pattern of the conically diffracted beam. Although Gaussian and top-hat beams are relatively common and thus of interest, other beams may also be considered [57], such as Bessel beams [58, 59]. These can be generated using axicons, or by allowing a pinhole beam to propagate producing an airy disk pattern. For most of this chapter we have considered an incident top-hat profile, whose Fourier transform has a Bessel profile, as seen in Eq. (2.11). It seems logical then to investigate the converse case—an incident Bessel beam, which produces a top-hat Fourier transform:

$$D_{0,B}(\rho) = \frac{J_1(\rho)}{\rho}, \quad a_B(\kappa) = \Theta(\kappa). \quad (2.30)$$

This effectively reduces the integral range in the fundamental integrals B_0 and B_1 from $\{0, \infty\}$ to $\{0, 1\}$, with $a(\kappa) = 1$:

$$B_0 = \int_0^1 d\kappa \kappa \exp\left(-\frac{1}{2}i\zeta\kappa^2\right) J_0(\kappa\rho) \cos(\kappa\rho_0), \quad (2.31)$$

$$B_1 = \int_0^1 d\kappa \kappa \exp\left(-\frac{1}{2}i\zeta\kappa^2\right) J_1(\kappa\rho) \sin(\kappa\rho_0). \quad (2.32)$$

Such a beam does not violate the paraxial approximation we made in Section 1.5 [60] and evaluating Eq. (2.3), we observe a distinct profile when compared to the Gaussian and top-hat cases. For $\rho_0 > 10$, there are many rings decorating the profile at the focal image plane, with two dominant rings close to $\rho = \rho_0$ as seen in Figs. 2.18 and 2.19.

Of particular interest is the non-zero intensity at $\rho = 0$ in the FIP for $\rho_0 = 20$. This may be explained by examining Eqs. (2.31) and (2.32), observing that at $\{\rho = 0, \zeta = 0\}$ the Bessel function $J_1(0) = 0$ while $J_0(0) = 1$, and the exponent goes to unity. Therefore the intensity at $\rho = 0$ may be written

$$I_{\rho=0} = \left| \int_0^1 d\kappa \kappa \cos(\kappa\rho_0) \right|^2, \quad (2.33)$$

which can be evaluated explicitly as

$$I_{\rho=0} = \left| \frac{\cos \rho_0 + \rho_0 \sin \rho_0 - 1}{\rho_0^2} \right|^2, \quad (2.34)$$

which is an oscillatory function in ρ_0 . Thus, in general there is non-zero intensity at the centre of the beam, the exceptions being $\rho_0 = 2n\pi$ where n is an integer, and all ρ_0 values which give $\cos \rho_0 + \rho_0 \sin \rho_0 = 1$. Due to the presence of ρ_0^2 in the denominator, this central non-zero intensity will diminish rapidly as ρ_0 increases, as seen in Fig. 2.17.

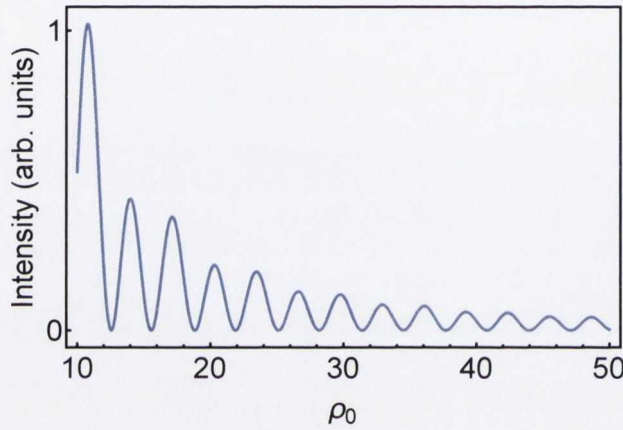


Figure 2.17: The central intensity $\rho = 0$ at the FIP $\zeta = 0$ for an incident Bessel beam as a function of ρ_0 .

Allowing the beam to propagate beyond the FIP, an axial spike is apparent for the entire beam propagation range as seen in Fig. 2.20. Many intensity oscillations are visible in the radial direction, particularly for $\rho_0 = 20$.

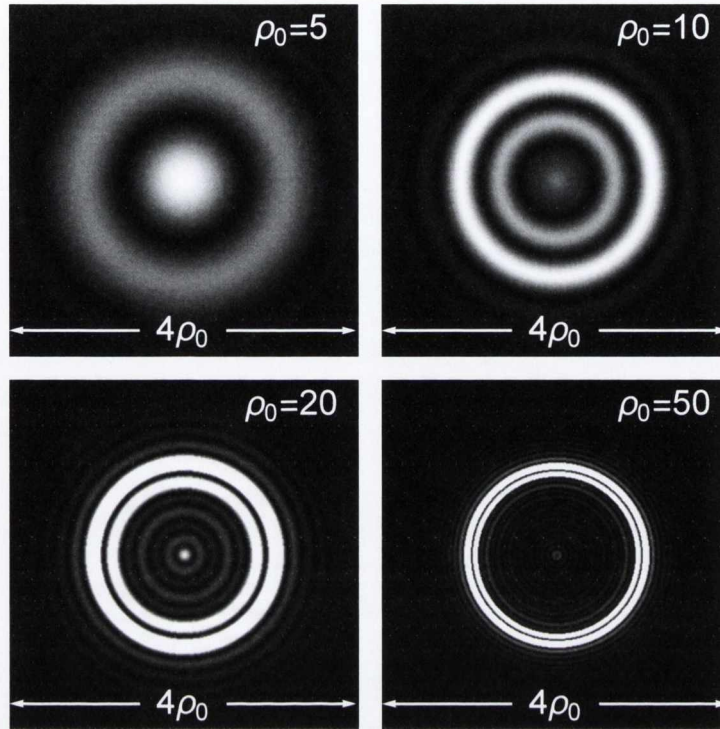


Figure 2.18: The intensity profile at the focal image plane for conically diffracted incident Bessel beams of various ρ_0 values (inset). The profiles for $\rho_0 = 20$ and $\rho_0 = 50$ have been saturated to allow observation of the lower intensity supernumerary rings.

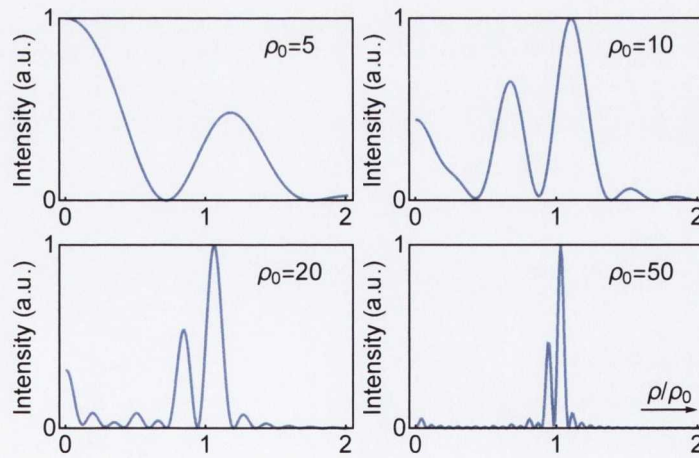


Figure 2.19: 1-dimensional intensity profiles corresponding to the 2-dimensional images in Fig. 2.18. The horizontal axes are in units of ρ/ρ_0 . The low-intensity oscillations are clearly visible for $\rho_0 > 10$.

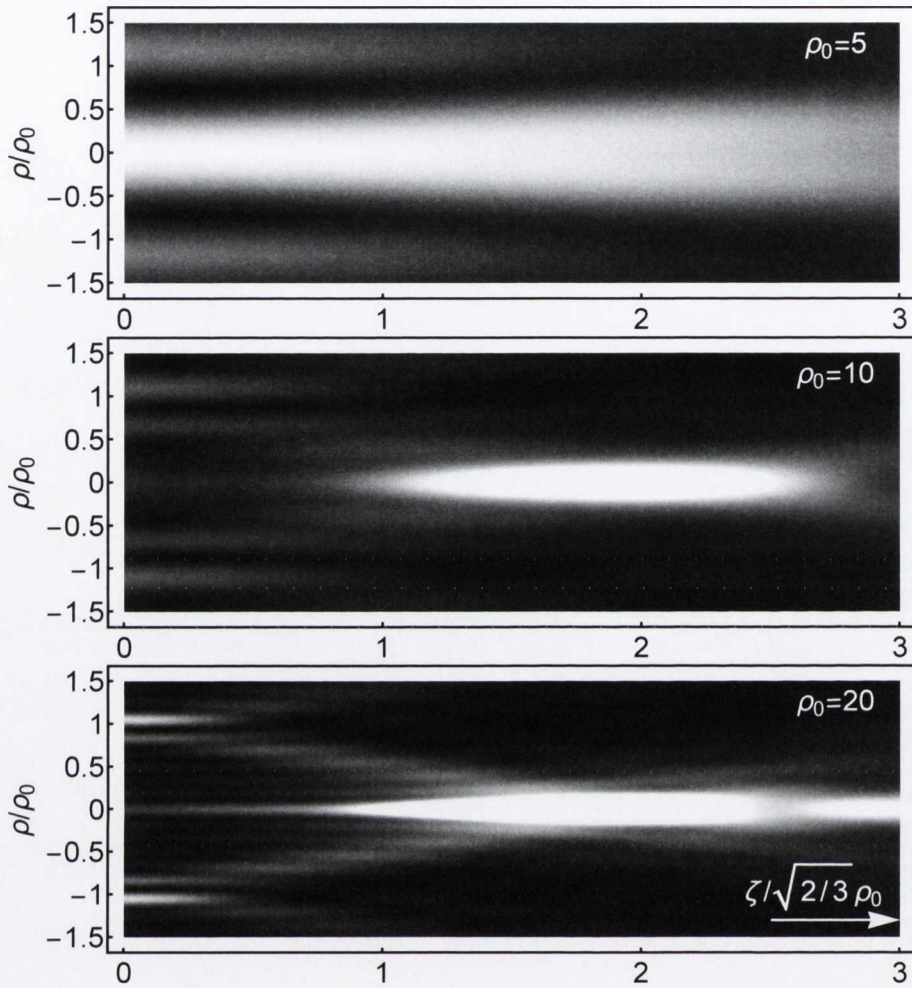


Figure 2.20: The far field evolution of a conically diffracted Bessel beam for various ρ_0 values (inset). The ζ -direction is measured in units of $\zeta/\sqrt{2/3}\rho_0$. The structure is distinct from both the Gaussian (Fig. 2.5) and top-hat (Fig. 2.14) cases, with many fringes in the ring structure.

2.7 Conclusion

The structure of conically diffracted light was shown to depend sensitively on the profile of the incident beam, with a very clear difference between the intensity distributions formed when using a top-hat input beam, as opposed to a Gaussian input beam. In particular, the interesting beam profile predicted

to occur at the FIP for a top-hat input beam was compared to experimental observations and found to match well when taking into account the physical limitations of the apparatus. The use of a variable iris to continuously vary the maximum transverse wave vector component of the beam allowed very fine control of the features of the intensity profile, such as: the width of the high-intensity inner ring, and the relative intensities of the inner ring and the wedge-shaped feature. The evolution of such a beam beyond the FIP was simulated using the theoretical model presented which yielded the unusual feature of an oscillating axial spike. This evolution was then observed experimentally and found to match extremely well with the prediction.

This unusual intensity profile may find applications in optical trapping since the strength of a trap depends on the gradient of the intensity [61]. Since in theory the intensity of the inner ring is limited by the apparatus being used, the use of high-quality components could lead to a very stiff trap indeed. It may also be used to create microstructures with very sharp features [62], or controlled using a variable iris as described in Section 2.4 to produce microstructures with tunable feature quality.

An incident Bessel beam was found to produce interesting intensity profiles, notably containing a non-zero intensity at $\rho = 0$ and $\zeta = 0$ for large ρ_0 values.

Chapter 3

White light conical diffraction

3.1 Introduction

A beam directed along an arbitrary direction in a biaxial material will, in general, undergo double refraction [63] due to the presence of two distinct sheets of the surface of wave normals [45], and hence two unique Poynting vector directions. When the beam is aligned with an optic axis direction however, the sheets intersect and the degeneracy results in the Poynting vector tracing out a skewed cone leading to the phenomenon of internal conical diffraction. There must exist a smooth transition between the cases of double refraction and conical diffraction, occurring when the beam is brought from an arbitrary direction closer to the optic axis direction. This transition has been discussed by Berry and Jeffrey [41] where a theoretical model was presented which takes both absorption and dichroism into account. This theory will be discussed in the next section.

The majority of work on conical diffraction to date has used a monochromatic laser as an incident beam, although the first experimental observation of the phenomenon by Lloyd used a rather broadband light source—the sun [3, 4]. Since the optic axis direction is uniquely defined for a given wavelength, if the incident beam is polychromatic there can only be one particular wavelength λ_0 aligned to produce a conical diffraction pattern, with all other wavelengths exhibiting double refraction. In this chapter we will examine the relationship between the incident wavelength and optic axis direction, as well as refining the model developed by Berry and Jeffrey to provide a full description of a white light beam undergoing internal conical diffraction. This will be of importance for applications such as second-harmonic generation in rare-earth doped biaxial materials [64–66].

There has been some interest in the potential applications of white light Bessel beams [67], and a conically diffracted beam possesses the properties of Bessel-like beams [30]. Once a model had been successfully developed and experimentally verified, dispersion compensation prisms were used to generate a broad-spectrum incident beam with a spread of directions dependent on wavelength. The prisms were carefully positioned to ensure each wavelength entering the biaxial crystal was aligned along its associated optic axis direc-

tion. In this way, the dispersion caused by the crystal could be overcome, resulting in true white-light conical diffraction. This method may be used to generate broad spectrum Bessel beams.

3.2 Theory

The refractive index of an optical medium depends on the wavelength of the incident light [68]. This dependence was described by Wilhelm Sellmeier [69] in a powerful empirical relationship known as the Sellmeier equation. It takes the form

$$n_i(\lambda) = A_i + \frac{B_i}{1 - \left(\frac{C_i}{\lambda}\right)^2} - D_i\lambda^2, \quad (3.1)$$

where A_i , B_i are dimensionless, C_i has units of nm, and D_i has units of nm^{-2} . These data have been measured by Pujol *et al.* [70] for $\text{KGd}(\text{WO}_4)_2$ and are presented in Table 3.1. The refractive indices $n_1(\lambda)$, $n_2(\lambda)$, and $n_3(\lambda)$ are plotted using these data in Fig. 3.1.

i	A_i	B_i	$C_i/(\text{nm})$	$D_i/(\text{nm}^{-2})$
1	1.5344	0.4360	186.18	2.0999×10^{-9}
2	1.5437	0.4541	188.91	2.1567×10^{-9}
3	1.3867	0.6573	170.02	0.2913×10^{-9}

Table 3.1: Sellmeier parameters of $\text{KGd}(\text{WO}_4)_2$ as measured by Pujol *et al.*

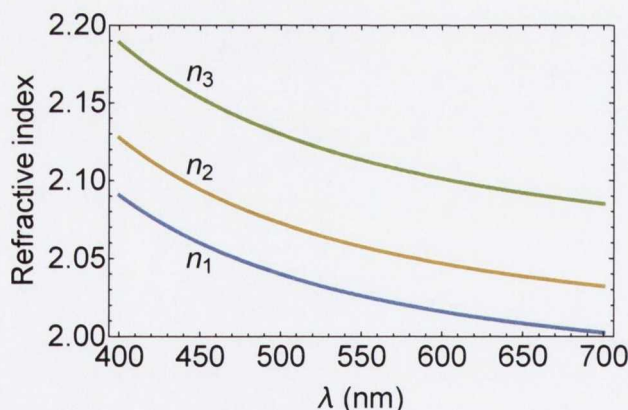


Figure 3.1: Plots of the refractive indices n_1 , n_2 , and n_3 of $\text{KGd}(\text{W}_4)_2$ as a function of wavelength λ in the visible spectrum generated using Eq. (3.1).

Since the surface of wave normals is unique for a given wavelength, we expect a unique intersection point and hence a unique pair of optic axis directions θ_{OA} for any particular wavelength. Taking Eq. (1.10) and substituting

$\epsilon_i = n_i^2$, we have

$$\theta_{\text{OA}}(\lambda) = \arctan \sqrt{\frac{n_3^2 (n_2^2 - n_1^2)}{n_1^2 (n_3^2 - n_2^2)}}. \quad (3.2)$$

where the n_i are functions of λ as given by Eq. (3.1). This suggests a biaxial material may be aligned to produce conical diffraction for a given incident beam direction at one wavelength only. To achieve conical diffraction at any other wavelength would require a reorientation of either the biaxial material or the incident beam. To examine what effect dispersion has on the optic axis direction, a plot of $\theta_{\text{OA}}(\lambda)$ was generated using Eq. (3.2) where the refractive index values are specified by Eq. (3.1) using the data given in Table 3.1.

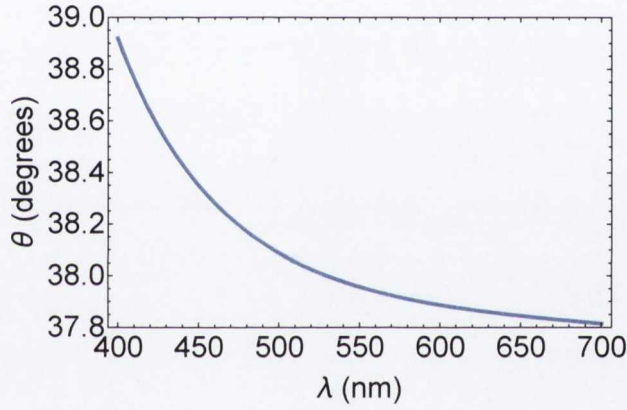


Figure 3.2: Plot of the optic axis direction θ as a function of wavelength λ in the visible spectrum generated using Eq. (3.2).

The wavelength-dependent refractive indices also have consequences for other features of the beam. For example, the semi-angle of the cone derived in Eq (1.20), in terms of n_i ,

$$A = \frac{1}{2} \sqrt{\frac{(n_3^2 - n_2^2)(n_2^2 - n_1^2)}{n_1^2 n_3^2}} \quad (3.3)$$

will result in a λ -dependent ring radius at the focal image plane, given by

$$R_0(\lambda) = A(\lambda)l. \quad (3.4)$$

The location of the FIP also depends on wavelength. Recalling that

$$\zeta \equiv \frac{l + (z - l)n_2}{n_2 k_0 w^2} \quad (3.5)$$

and solving for $\zeta = 0$ yields the location of the FIP:

$$z_{\text{FIP}}(\lambda) = l \left(1 - \frac{1}{n_2(\lambda)} \right). \quad (3.6)$$

Although most publications on conical refraction assume the beam is directed along the optic axis, it is possible to formulate a theory which allows a small deviation from this direction. Berry and Jeffrey have detailed such a theory [41] in which u is used as a parameter which encompasses both dichroism and absorption in the material. Although absorption in biaxial materials has been discussed [55, 71], we will assume no absorption and recapitulate the theory as follows. Consider a circularly symmetric incident beam whose Fourier transform is $a(\boldsymbol{\kappa})$. The transverse electric vector $\mathbf{D}$ beyond the crystal at incident polarisation $\mathbf{d}_0$ is given by

$$\mathbf{D} = \frac{1}{2\pi} \iint d\boldsymbol{\kappa} a(\boldsymbol{\kappa}) \cdot \exp(i\boldsymbol{\kappa} \cdot \boldsymbol{\rho}) \mathbf{U}(\boldsymbol{\kappa}) \mathbf{d}_0 \quad (3.7)$$

where the origin $\boldsymbol{\kappa} = 0$ represents the incident beam aligned with the optic axis. $\mathbf{U}$ is the unitary operator

$$\mathbf{U}(\boldsymbol{\kappa}) = \exp \left\{ -i \left(\frac{1}{2} \kappa^2 \zeta + \rho_0 \boldsymbol{\kappa} \cdot \boldsymbol{\Sigma} \right) \right\} \quad (3.8)$$

expressed in terms of the Pauli matrices

$$\boldsymbol{\Sigma} = \{\boldsymbol{\sigma}_3, \boldsymbol{\sigma}_1\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}. \quad (3.9)$$

Now consider an incident Gaussian beam in a direction $\boldsymbol{\kappa}_0$, whose Fourier transform may be given by

$$a(\boldsymbol{\kappa}) = \exp \left\{ -\frac{1}{2} (\boldsymbol{\kappa} - \boldsymbol{\kappa}_0)^2 \right\}. \quad (3.10)$$

The field after the crystal at $\{\boldsymbol{\rho}, \zeta\}$ may now be expressed by combining Eqs. (3.7), (3.8), and (3.10) as

$$\mathbf{D} = \frac{1}{2\pi} \iint d\boldsymbol{\kappa} \exp \{-i\mathbf{F}(\boldsymbol{\kappa})\} \mathbf{d}_0 \quad (3.11)$$

where the non-Hermitian operator in the exponent is

$$\mathbf{F}(\boldsymbol{\kappa}) = -\frac{1}{2}i(\boldsymbol{\kappa} - \boldsymbol{\kappa}_0)^2 - \boldsymbol{\kappa} \cdot \boldsymbol{\rho} + \frac{1}{2}\kappa^2\zeta + \rho_0\boldsymbol{\kappa} \cdot \boldsymbol{\Sigma}. \quad (3.12)$$

Make the following transformation

$$\tilde{\zeta} \equiv \zeta - i, \quad (3.13)$$

and define μ in the following way, determining how misaligned the incident beam is with the optic axis:

$$\boldsymbol{\mu} \equiv -\boldsymbol{\kappa}_0 \equiv \mu \mathbf{e}_\xi. \quad (3.14)$$

The misalignment occurs along the transverse ξ axis, and we define a new coordinate system by shifting from $\boldsymbol{\rho} = \{\xi, \eta\}$ to:

$$\tilde{\boldsymbol{\rho}} \equiv \{\xi + i\mu, \eta\}, \quad \tilde{\boldsymbol{\rho}} \equiv \boldsymbol{\rho} + i\mu \mathbf{e}_\xi \equiv \left\{ \tilde{\rho} \cos \tilde{\phi}, \tilde{\rho} \sin \tilde{\phi} \right\}. \quad (3.15)$$

These transformations allow us to express Eq. (3.12) as

$$\mathbf{F}(\boldsymbol{\kappa}) = \tilde{\mathbf{F}}(\boldsymbol{\kappa}) = \frac{1}{2}\kappa^2\tilde{\zeta} - \boldsymbol{\kappa} \cdot \tilde{\boldsymbol{\rho}} + \rho_0\boldsymbol{\kappa} \cdot \boldsymbol{\Sigma} + F_0 \quad (3.16)$$

where

$$F_0 \equiv -\frac{1}{2}i\kappa_0^2. \quad (3.17)$$

The reasoning behind the above transformations now becomes apparent; we may express Eq. (3.11) as

$$\mathbf{D} = \frac{1}{2\pi} \iint d\boldsymbol{\kappa} \exp \left\{ -i\tilde{\mathbf{F}}(\boldsymbol{\kappa}) \right\} \mathbf{d}_0 \quad (3.18)$$

Evaluating the exponential and integrating over κ gives

$$\mathbf{D} = \exp \{-iF_0\} \begin{pmatrix} B_0 + B_1 \cos \tilde{\phi} & B_1 \sin \tilde{\phi} \\ B_1 \sin \tilde{\phi} & B_0 - B_1 \cos \tilde{\phi} \end{pmatrix} \mathbf{d}_0 \quad (3.19)$$

with the trigonometric expressions given by

$$\cos \tilde{\phi} = \frac{\xi + i\mu}{\tilde{\rho}}, \quad \sin \tilde{\phi} = \frac{\eta}{\tilde{\rho}}, \quad (3.20)$$

where

$$\tilde{\rho} = \sqrt{(\xi + i\mu)^2 + \eta^2}. \quad (3.21)$$

The terms B_0 and B_1 are the fundamental integrals for a misaligned Gaussian beam:

$$B_0 = \int_0^\infty d\kappa \kappa \exp \left[-\frac{1}{2} i \tilde{\zeta} \kappa^2 \right] J_0(\kappa \tilde{\rho}) \cos(\kappa \rho_0), \quad (3.22)$$

$$B_1 = \int_0^\infty d\kappa \kappa \exp \left[-\frac{1}{2} i \tilde{\zeta} \kappa^2 \right] J_1(\kappa \tilde{\rho}) \sin(\kappa \rho_0), \quad (3.23)$$

where J_ν is the ν^{th} order Bessel function of the first kind. If the incident beam is circularly polarised (or unpolarised), these combine to give the intensity profile

$$I = |B_0|^2 + |B_1|^2. \quad (3.24)$$

To build on this model, we now require the ‘misalignment parameter’ μ to be expressed in terms of wavelength. We defined μ in Eq. (3.14) to be the projection of the misaligned incident beam onto the $\tilde{\rho}$ plane along the ξ axis. Hence, μ is the transverse wave vector component between the incident beam and the optic axis. If these directions are separated by an angle $\delta\theta$, then the transverse wave vector component is simply

$$\mu = \kappa_\perp = \kappa \sin \delta\theta. \quad (3.25)$$

Instead of initially having a beam misaligned with the optic axis, suppose we align an incident beam of wavelength λ_0 with the optic axis. Altering

the wavelength to λ causes a shift in the optic axis direction θ_{OA} , so we may express $\delta\theta$ explicitly in terms of λ as

$$\delta\theta(\lambda) = \theta_{\text{OA}}(\lambda) - \theta_{\text{OA}}(\lambda_0). \quad (3.26)$$

This leads to an expression for the wavelength-dependent μ :

$$\mu(\lambda) \equiv \kappa_\lambda \sin \delta\theta(\lambda) = w k_0(\lambda) n_2(\lambda) \sin \delta\theta(\lambda) \quad (3.27)$$

where $k_0(\lambda) = 2\pi/\lambda$. It is clear from Eq. (3.26) that $\mu(\lambda)$ will vanish as $\lambda \rightarrow \lambda_0$, and the intensity equation will return to that for a Gaussian beam aligned with the optic axis direction as discussed in Section 1.5.

3.3 Experimental procedure

As outlined in Section 4.2, the theoretical model we are using assumes an incident beam of Gaussian spatial profile. Since the white LED is not a good Gaussian source, a beam cleanup was required to generate a suitable profile. A 100 μm diameter pinhole was placed close to the LED and an achromatic biconvex doublet lens of focal length 10 cm was placed 50 cm from the pinhole. Upon passing through the pinhole, the beam undergoes diffraction and evolves into an Airy pattern whose electric displacement field profile $D_A(\rho)$ is given by

$$D_A(\rho) = \frac{J_1(\rho)}{\rho}. \quad (3.28)$$

This means the beam has a non-Gaussian profile upon striking the lens. To overcome this issue, the theory of Fourier optics was used. A lens transforms an incident spatial profile into its Fourier transform [72]. For a Gaussian beam, this Fourier transform is also Gaussian. This is not the case for the field profile given by Eq. (3.28) which becomes a step function under Fourier transformation, *i.e.* an image of the pinhole.

We exploit the fact that the central peak of the pinhole's Fourier transform is a good approximation for a Gaussian function as seen in Fig. 3.3. Cutting off the side lobes from the beam will produce an approximately Gaus-

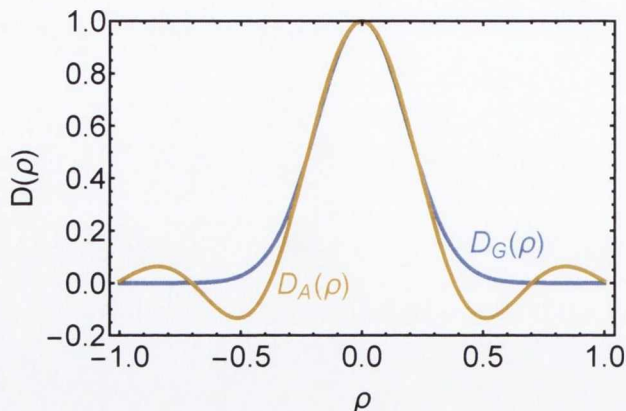


Figure 3.3: Electric displacement field profiles of a Gaussian beam $D_G(\rho)$ (blue), and of a diffracting pinhole beam $D_A(\rho)$ (orange), described by Eq. (3.28). The central peak of $D_A(\rho)$ is approximately Gaussian.

sian beam when transformed through a focussing lens. This was achieved by placing a variable iris in front of the achromatic lens, as seen in Fig. 3.4, and adjusting the opening until only the central peak was transmitted, corresponding to the central peak of the squared profile $D_A(\rho)$ in Fig. 3.3.

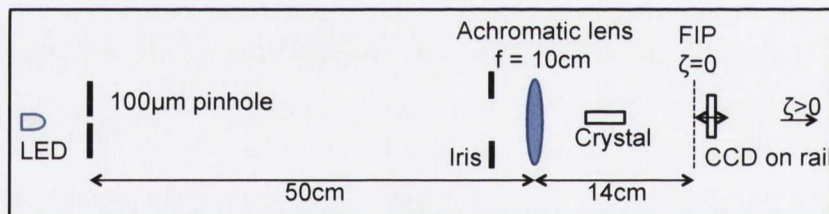


Figure 3.4: Experimental apparatus used during the white light conical diffraction experiments. The achromatic lens focussed the approximately Gaussian white light beam through the biaxial crystal onto a colour CCD.

The narrowest focus of the resulting Gaussian beam was determined to occur 12.5 cm beyond the lens, defined to be $z = 0$, and a beam profiler was placed at this location in order to measure the $1/e$ intensity radius w . This was calculated to be $15 \mu\text{m}$ for the full spectrum of the beam. To ensure that all wavelengths had approximately the same value of w , bandpass filters at 450 nm, 500 nm, 600 nm, and 650 nm were placed in the beam in turn and the value of w was determined to be $15 \mu\text{m}$ for each, within experimental error.

A 3 cm long slab of the biaxial material $\text{KGd}(\text{WO}_4)_2$ was placed in the beam such that its exit face was between the lens and $z = 0$. The FIP, which occurs at $z = l(1 - 1/n_2)$, was then in free space rather than inside the crystal. This allowed the full beam evolution to be examined without necessitating the use of an imaging lens, which may have introduced aberrations and undesired magnification. The imaging device, a ThorLabs DC310C, was mounted on a rail which allowed movement in the z direction to obtain images at various ζ values.

3.4 Ring radius

Before the white light case was examined using the experimental apparatus in Fig. 3.4, the wavelength dependence of the radius of the ring R_0 was investigated. According to Eq. (3.3), this dependence is given by

$$R_0(\lambda) = A(\lambda)l \quad (3.29)$$

where l is the length of the material. To investigate this dependence experimentally, the white LED in Fig. 3.4 was replaced with a monochromator. This is a device which uses a bright, broad-spectrum light source in conjunction with a diffraction grating and steering mirrors to select a narrow region of the visible spectrum, as seen in Fig. 3.5. A desired wavelength λ is entered into the control panel and the monochromator emits a beam with a very narrow spectral FWHM (measured to be around 3 nm) with peak emission at λ . The emission spectrum of the monochromator's light source was not required; only one wavelength was being transmitted through the crystal at any moment, and the monochromator was not used to generate white light images. The width of the beam w was also not important, since the radius of the ring R_0 is independent of w .

A similar cleanup arrangement to that described in Section 3.3 was used to ensure the beam reaching the biaxial crystal had a Gaussian spatial profile. This beam was focussed through the crystal using an achromatic biconvex doublet lens of focal length 10 cm. The monochromator was then set to

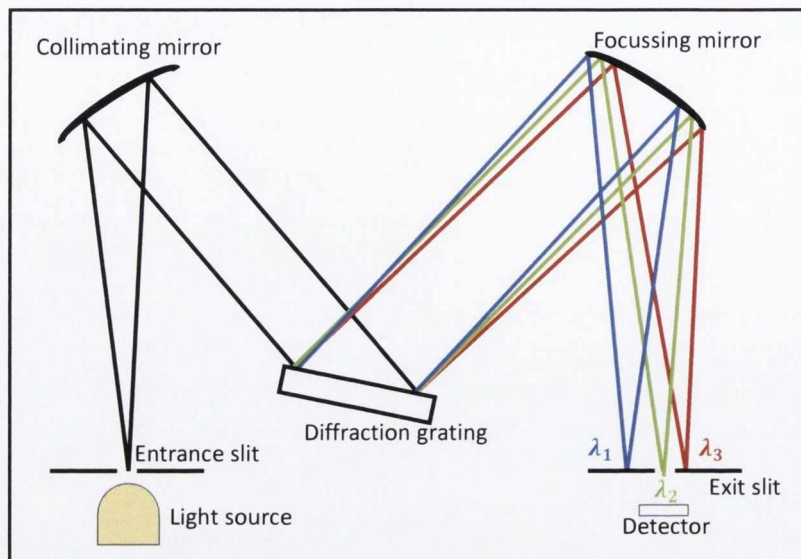


Figure 3.5: Schematic showing how a monochromator works to select certain wavelengths from a broad-spectrum light source.

390 nm and the crystal was aligned such that the optic axis direction at 390 nm and the incident beam coincided. This produced a symmetric conically diffracted beam whose focal image plane occurred beyond the crystal. The CCD was placed at this location in order to image the ring without the use of an imaging lens, which may have introduced undesired magnification. This process was repeated in increments of 10 nm to 800 nm, ensuring the crystal was reoriented and the CCD was moved to the correct FIP location for each wavelength.

The images of the rings were analysed using Mathematica. $R_0(\lambda)$ is approximately the radius of the Poggendorff dark ring [73] and so the radii of the rings were established by overlaying each image with a circle and adjusting both the centre points and radius until a match was found. This provided a very accurate result for the ring radius, to within a pixel on either side, corresponding to $\pm 4.65 \mu\text{m}$.

The predicted $R_0(\lambda)$ is compared with the measured radii in Fig. 3.6. As can be clearly seen, the experimental values vary from the theory by a large amount; well beyond experimental error. In order to explain this, refer to the form of Eq. (3.3). It has a very sensitive dependence on the values of n_1 ,

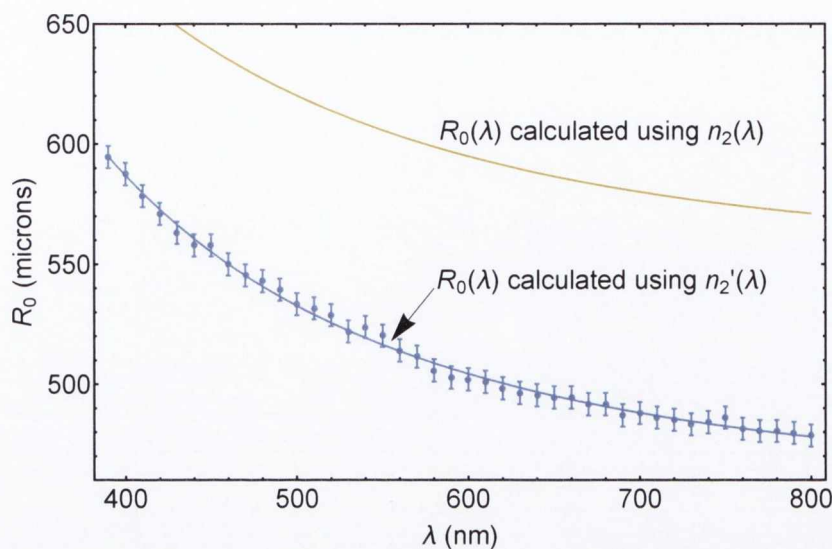


Figure 3.6: Experimental measurements of the ring radius from 390 nm to 800 nm in increments of 10 nm. The solid orange line represents the theoretical radius curve taken from Eq. (3.29). The solid blue line is also Eq. (3.29) but using $n_2'(\lambda)$.

n_2 , and n_3 , and a small variation in any of these from the data supplied by Pujol *et. al* would have a large effect on the ring radius. In order to reconcile theory and experiment, it was assumed that $n_1(\lambda)$ and $n_3(\lambda)$ were correct and solving

$$A(\lambda)l = R_{\text{measured}} \quad (3.30)$$

at each wavelength, the best fit for $n_2'(\lambda)$ was found to be

$$n_2'(\lambda) = 0.9937n_2(\lambda). \quad (3.31)$$

Plotting the new theoretical $R_0(\lambda)$ curve using Eq. (3.31) reveals that not only does the modification of $n_2'(\lambda)$ reduce the plot of $R_0(\lambda)$ to within experimental error of all the measured values, but it also alters the plot shape to fit all of the experimental points—something a multiplicative factor on $R_0(\lambda)$ would not achieve. Consequently, $n_2(\lambda)$ was replaced with $n_2'(\lambda)$ for all subsequent theoretical calculations. The variation from the data provided by Pujol [70] is understandable given that $n_2'(\lambda)$ varies from $n_2(\lambda)$ by just 0.63% over the entire visible spectrum. It is of course possible that $n_1(\lambda)$

and $n_3(\lambda)$ vary by some small amount too, but the variation of $n_2(\lambda)$ is a sufficient consideration for our work.

The location of the focal image plane also depends on wavelength, as seen in Eq. (3.6). The FIP location vs. wavelength is plotted in Fig. 3.7 assuming the incident white beam comes to a focus at the same point in space, $z = 0$.

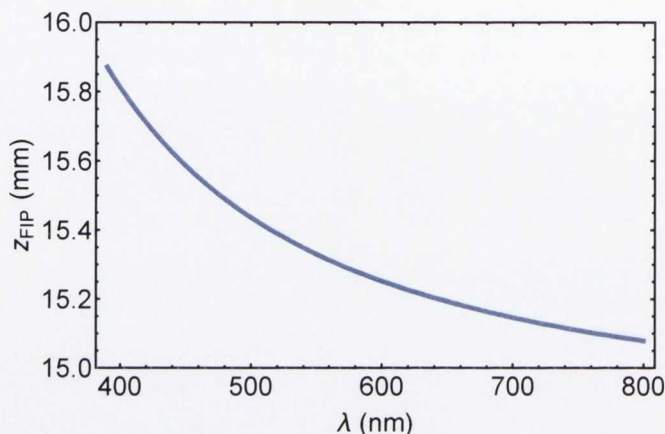


Figure 3.7: Predicted location of the focal image plane calculated using Eq. (3.6). The value of z_{FIP} gives the distance after the tightest focus of the incident Gaussian beam where the sharpest ring structure can be found.

3.5 Accurate representation of colour

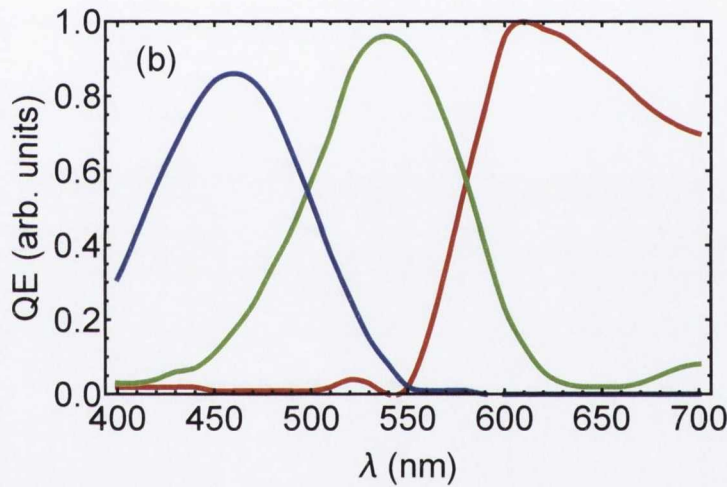
When comparing simulations and observations, it is important to ensure the simulations accurately reflect the emission spectrum of the source, as well as the sensitivity of the imaging device. The images were recorded using a Thorlabs DC310C Firewire camera, which utilises a Sony ICX204AK charge-coupled device (CCD). The CCD has a square array of 1024×768 pixels, each of which are $4.65 \mu\text{m}$ in size. Filters on the pixels allow selective sensitivity in certain regions of the visible spectrum; in total there are three types of filters — red (R), green (G), and blue (B). The filters are arranged in a Bayer pattern which is a square lattice containing twice as many green filters as red or blue, as shown below.

Each pixel is read out successively over the course of 50 ms and grouped with its neighbours to return values for red, green, and blue in the RGB colour

Gb	B	Gb	B
R	Gr	R	Gr
Gb	B	Gb	B
R	Gr	R	Gr

Table 3.2: The Bayer arrangement of pixel filters used on the CCD.

model. These values may be anything between 0 and 1 with $\{R, G, B\} = \{0, 0, 0\}$ representing black and $\{1, 1, 1\}$ representing white. Almost any colour in the visible spectrum can be represented given the appropriate R, G, and B values.

Figure 3.8: The relative quantum efficiencies of the CCD pixels coated with blue, green, and red filters as a function of wavelength λ .

The quantum efficiencies of the CCD pixels are shown in Fig. 3.8 with the information taken from the CCD manual [74]. These represent the percentage of incident photons at a given wavelength detected by the pixel. In order to obtain the response spectrum of the CCD for the white LED used, the photon flux $\phi_q(\lambda)$ from the source as a function of wavelength is required. This is possible to obtain from the intensity spectrum of the LED $I_{\text{LED}}(\lambda)$ which was recorded using a spectrophotometer, shown in Fig. 3.9. $\phi_q(\lambda)$ is found

using the relation [68]

$$I_{\text{LED}}(\lambda) = \frac{P(\lambda)}{4\pi r^2} = \frac{\phi_q(\lambda)E(\lambda)}{4\pi r^2} = \frac{\phi_q(\lambda)h\frac{c}{\lambda}}{4\pi r^2},$$

$$\Rightarrow \phi_q(\lambda) = \frac{4\pi r^2}{hc} \lambda I_{\text{LED}}(\lambda), \quad (3.32)$$

where $P(\lambda)$ is the power emitted by the LED at wavelength λ , $E(\lambda)$ is the energy carried by a photon of wavelength λ , h is the Planck constant, c is the speed of light in a vacuum, and r is the radius of the circular area used to define the intensity. Since all our results will be normalised, and since the unit area $4\pi r^2$ is invariant with wavelength, we may set the constant factor to unity, yielding the simple relation

$$\phi_q(\lambda) = \lambda I_{\text{LED}}(\lambda). \quad (3.33)$$

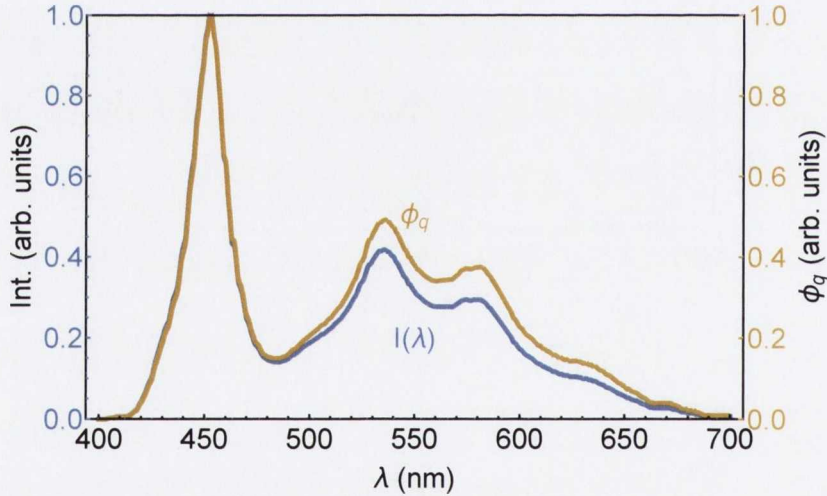


Figure 3.9: The intensity spectrum (blue) and relative photon flux (orange) ϕ_q , given by Eq. (3.33), for the white LED.

A comparison between the visible colour spectrum, which is a good approximation to the colour one would see at a given wavelength, and the ‘colour vision’ of the CCD is shown in Fig. 3.10, parts (a) and (b) respectively. Image (b) demonstrates what colour would be displayed on the

imaging device if a beam of wavelength λ fell on the CCD, calculated from the quantum efficiencies of the pixels given in Fig. 3.8. The source photon flux spectrum given in Eq. (3.33) may be convoluted with the CCD quantum efficiency spectrum to produce a true CCD response spectrum—that is the colours observed on the imaging software using the CCD and LED in question. This is demonstrated in Fig. 3.11.

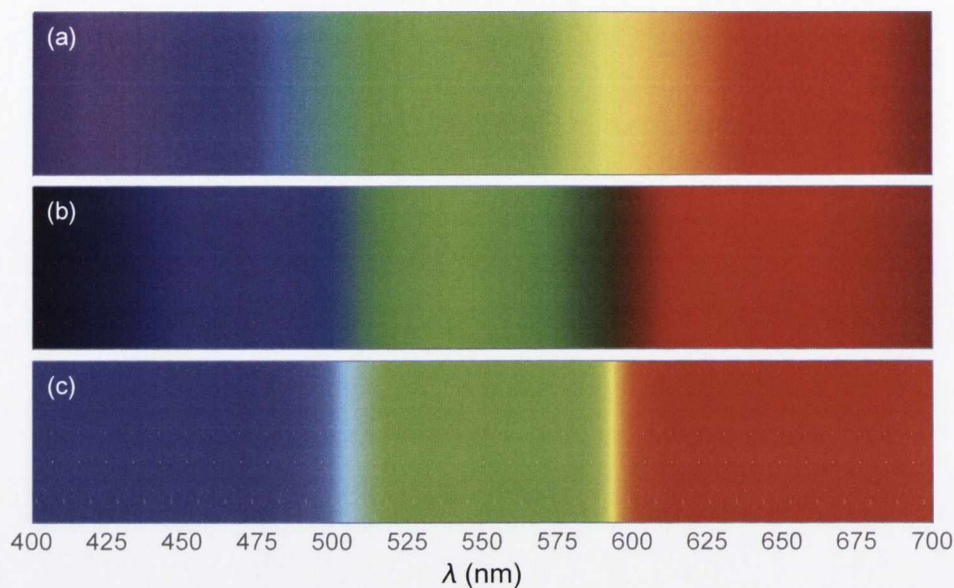


Figure 3.10: (a) Approximate human eye perception of colour as a function of wavelength, generated using the RGB colour model. (b) Colour displayed by ThorLabs DC310C as a function of wavelength, assuming a source with equal photon flux at all wavelengths. Calculated using the RGB values returned by the quantum efficiency plot in Fig. 3.8. (c) The same response from ThorLabs DC310C, but normalised at every wavelength allowing all regions of the spectrum to be clearly seen.

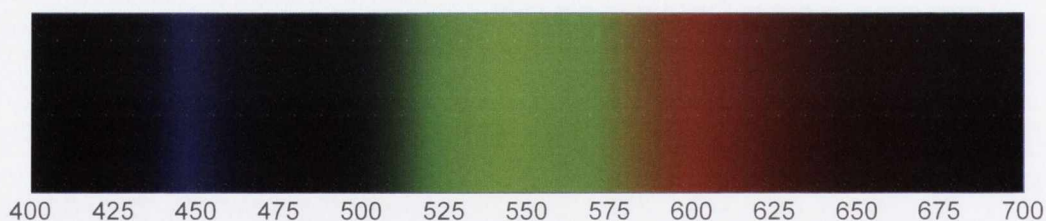


Figure 3.11: The response of the CCD combined with the LED emission spectrum, normalised to the most intense wavelength. This is the spectral signature of the CCD and white LED system used in the experiments.

3.6 Simulations

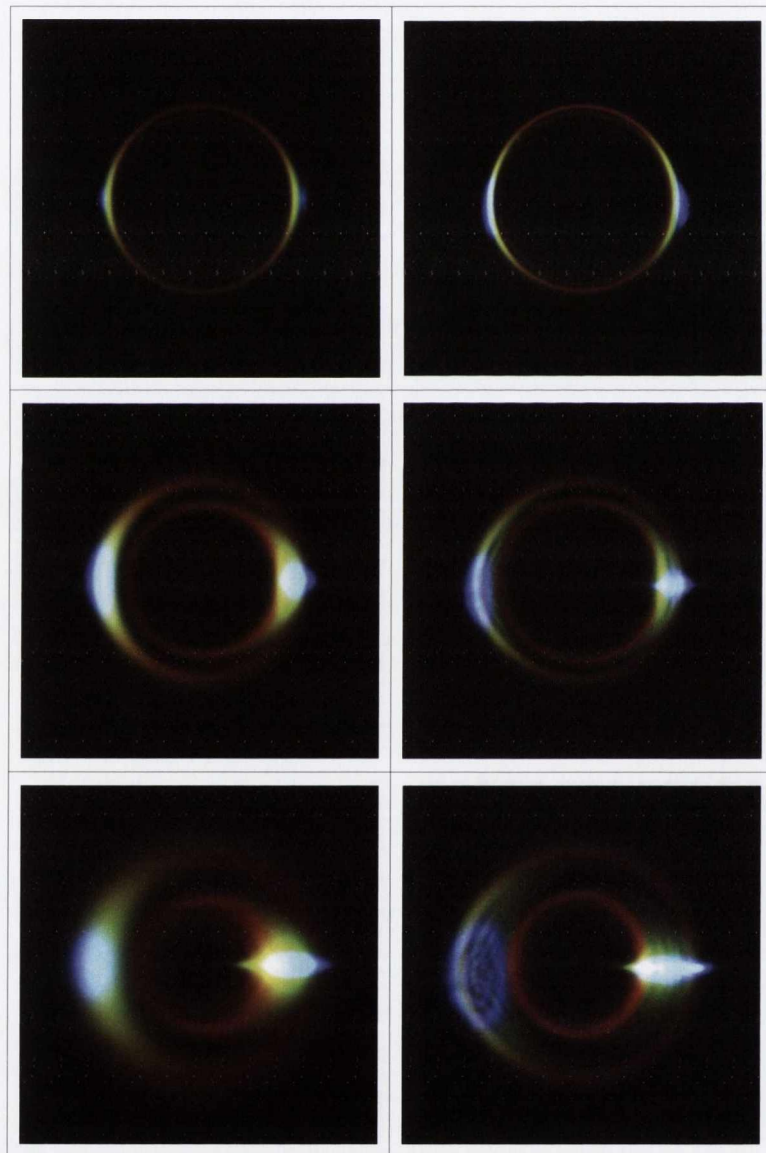
To obtain an intensity plot at a given wavelength, Eq. (3.24) was evaluated using Mathematica at discrete points $\{\xi, \eta\}$ between $\{-2\rho_0, -2\rho_0\}$ and $\{2\rho_0, 2\rho_0\}$. This was performed at suitable increments to generate a matrix of dimension of at least 100×100 to achieve comparable quality with the recorded images. In fact, the recorded images typically had dimension 450×450 pixels, but due to the computationally intensive nature of the theoretical plots, 100×100 was deemed suitable. Nevertheless, each discrete wavelength plot took anywhere between 30 minutes to 2 hours to complete. As a consequence, the spectrum was restricted to between 425 nm and 675 nm, since this is where the majority of the energy of the source is emitted, and plots were generated at increments of 25 nm, resulting in 11 plots for every white light image. It was observed that reducing this spacing had negligible effects on the resultant image.

In order to generate a theoretical colour plot, the wavelength λ was used with the quantum efficiency curves in Fig. 3.8 to return RGB values for each wavelength. The relative intensities of the individual wavelengths were encoded by the flux spectrum of the source (see Fig. 3.9) and the spectral response of the CCD, as demonstrated in Fig. 3.11. Finally, the weighted individual matrix RGB values were summed and normalised to produce a white light simulation which matches closely with observations.

3.7 Full spectrum

Using the experimental apparatus outlined in Section 3.3, many images were taken over the full spectrum of the white LED at various crystal alignments. Initially a 650 nm bandpass filter with a FWHM of 40 nm was inserted after the crystal but before the CCD. The crystal was oriented such that the image observed on the CCD was completely symmetric, aligning the incident beam with the optic axis at 650 nm inside the crystal. The CCD was then translated until the sharpest ring profile was observed, corresponding to the FIP at 650 nm. When the bandpass filter was removed, the full white

spectrum was observed and a series of images were taken at intervals of 1 cm in the z direction, starting at the FIP. These images are displayed in Fig. 3.12 where they are compared with simulations calculated from Eq. (3.24). The simulations were run at wavelength intervals of 25 nm. These individual simulations were then combined using the method outlined in Section 3.6, which incorporates the CCD sensitivity spectrum as well as the LED spectrum, and results in a simulation very similar to the observed image.



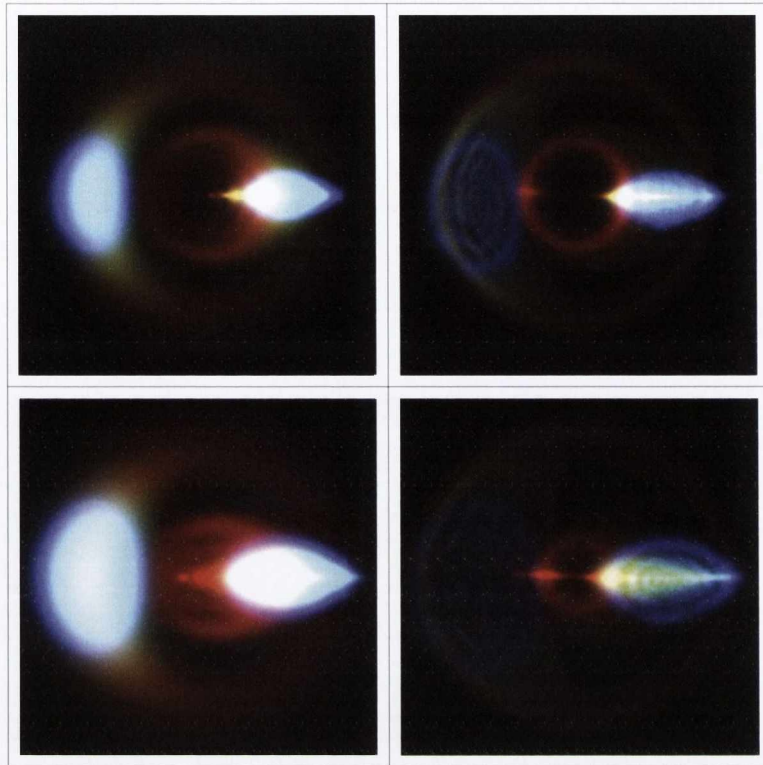


Figure 3.12: (Figure begins on previous page). Comparison of theoretical simulations (left) and experimental images (right) for a white light beam aligned with the optic axis at 650 nm inside the crystal. The first image occurs at the focal image plane where $\zeta = 0$. Each subsequent image is 1 cm beyond the previous image in the direction of beam propagation. The theoretical plots show very good agreement with the observations.

As can be seen from Fig. 3.12, the theoretical colour and intensity distributions are excellent reproductions of the experimental images. This suggests the theory proposed by Berry and Jeffrey [41], combined with the misalignment parameter $\mu(\lambda)$ defined in Eq. (3.27), is a powerful model which accurately describes the complex situation of white light conical diffraction. Furthermore, the method of replicating the colour spectrum of the observed images outlined in Section 3.5 has proven to be highly accurate.

3.8 Dispersion compensation

Having established a model which successfully describes white light conical diffraction, which arises due to the wavelength dependence of the optic axis direction, an experimental apparatus was constructed in an attempt to compensate for this dispersion. This would produce a broad spectrum conically diffracted beam at the FIP. The experimental apparatus is shown in Fig. 3.13. The essential components were the dispersion compensation prism pair, which were inserted in the collimated white light beam on a rotation-translation mount. The first prism was positioned at an angle inducing dispersion in the beam which then diverged upon exiting. This divergence was compensated using the second prism which was at a relative angle of π and laterally displaced. The result was a collimated beam with a spatial wavelength dependence. An achromatic doublet lens was inserted into this beam to generate a wavelength-dependent beam angle. All wavelengths spatially overlapped at the focal length of the lens and the biaxial crystal was placed at this point. Furthermore, the spatial beam profile at this location was approximately Gaussian in shape. The position of the lens in the beam controlled the angular spread and hence the incident beam angle as a function of wavelength. This was adjusted until the wavelength-dependent incident beam angle matched the wavelength-dependent optic axis angle $\theta(\lambda)$, as shown in Fig. 3.2.

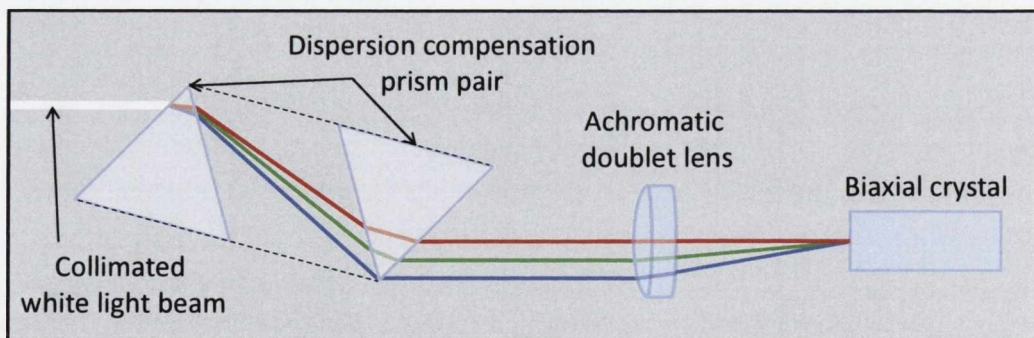


Figure 3.13: The experimental apparatus used to generate dispersion compensated incident beams. The lens is placed off-centre in the beam to generate the correct angular distribution to match that shown in Fig. 3.2.

An achromatic imaging lens was placed beyond the biaxial crystal and a CCD was placed beyond the lens at a suitable position to image the FIP. The relative angle and separation of the prism pair was adjusted until the symmetry of ring structure in the FIP was maximised. This symmetry was then improved by manipulating the position focussing lens in Fig. 3.13. The result was a ring containing the full emission spectrum of the white LED with minimal misalignment at any wavelength. This is clearly visible in Fig. 3.14. The beam structure is less well defined than those in Section 3.7, notably absent is the lower-intensity inner ring. This is due to the large incident beam width w , incurred as a result of the dispersion compensation prisms—a trade-off required to ensure a circularly-symmetric Gaussian profile at all wavelengths.



Figure 3.14: Experimental image of a dispersion-compensated conically diffracted white light beam.

While the effects of dispersion have been successfully compensated in the optic axis direction, the wavelength-dependence of the ring radius given by Eq. (3.29) is clear. The outermost region of the annulus has a blue hue, while the innermost region appears red, as expected from Fig. 3.6. To compare this result with theory, the misalignment parameter $\mu(\lambda)$ was set to zero for all wavelengths and the process discussed in Section 3.6 was repeated. The result was a symmetric ring structure at all wavelengths which combined to

provide an accurate representation of the experimental image, as seen in Fig. 3.15.



Figure 3.15: Theoretical simulation for the image shown in Fig. 3.14.

These preliminary results are encouraging, and it should be possible to improve the beam quality using a high-intensity polychromatic source, such as a super-continuum laser. This would allow narrow incident beam widths and increase the value of ρ_0 . It is possible to generate both 0th- and 1th-order Bessel beams from a conically diffracted beam as it propagates [27, 29, 30]. Combined with this method for dispersion compensation, broad-spectrum white light Bessel beams should be readily achievable.

3.9 Conclusion

Some of the properties of a conically diffracted beam generated using a white light source have been examined and successfully described using theoretical models. Both the radius of the conically diffracted ring and the position of the FIP were shown to depend on the wavelength of the input light. The radius was observed to deviate from theory by a measurable amount, and a modification of $n_2(\lambda)$ successfully reconciled theory and experiment.

The beam evolution beyond the FIP into the far-field was observed to

depend strongly on wavelength, with asymmetries arising as a result of the dispersion of the optic axis direction. A theoretical model was developed and applied to describe this beam evolution and was found to successfully predict the beam structure for the cases examined in the experiments. A method for incorporating explicit wavelength dependency in conical diffraction has been explained and this should provide a useful foundation for future work in this field.

Dispersion compensation was explored as a curiosity and was found to be a successful method of generating broadband conically diffracted beams. This has potential applications in the generation of white light Bessel beams.

Chapter 4

Features of misaligned beams

4.1 Introduction

When conical diffraction was first predicted by Hamilton in 1832, the theory was considered as a generalisation of a geometric ray optics model for describing the propagation of light in dielectric media [1, 2]. As discussed in Chapter 1, this was a simplified theory which failed to capture some of the intricacies of the beam structure. Over the following decades, these features were teased out by physicists such as: Poggendorff, who noticed a dark ring separating two rings; Raman, who made several contributions in the understanding of the evolution of the beam [14]; Belsky and Khapalyuk [75, 76] who developed a theoretical model incorporating diffraction, serving as the basis of the model refined by Berry [73]; and Warnick and Arnold, who made a prediction of intensity oscillations beyond the FIP [20].

It is appropriate then that we examine one feature of the beam predicted to occur [41], but one which had not been observed experimentally. This is manifest as a series of low intensity regions in conically diffracted beams which have been generated using an incident beam misaligned with the optic axis. In Chapter 3 we discussed polychromatic beams which, owing to dispersion, resulted in misalignment with the optic axis at certain wavelengths. However, due to the relatively low intensity of the light source and relatively noisy incident beam, those experiments were not conducive to the observation of such low intensity features. This chapter considers monochromatic beams misaligned with the optic axis and discusses an imaging technique which was employed to maximise the visibility of the features.

Also considered is a novel approach to predicting intensity profiles generated by beams misaligned with the optic axis involving ray tracing. The path traced by these rays is observed to describe the conchoid of a circle which allows highly efficient intensity modelling.

4.2 Theory

The prediction of the low intensity features arises from the theoretical model established by Berry and Jeffrey [41] and which we recapitulated in Section

3.2, in which dispersion caused a misalignment of the beam with the optic axis. We now consider a monochromatic spatially Gaussian beam, such as a laser, in an incident direction $\boldsymbol{\kappa}_0 \neq 0$ which is not coincident with the optic axis $\boldsymbol{\kappa} = 0$. It may be represented by its Fourier transform

$$a(\boldsymbol{\kappa}) = \exp \left\{ -\frac{1}{2} (\boldsymbol{\kappa} - \boldsymbol{\kappa}_0)^2 \right\}. \quad (4.1)$$

Berry and Jeffrey [41] introduce $\mathbf{u}$ as a parameter combination which, without absorption, may be expressed as

$$\mathbf{u} \equiv -\boldsymbol{\kappa}_0 \equiv u \mathbf{e}_\xi, \quad (4.2)$$

in which $\mathbf{e}_\xi$ is the unit vector in the ξ direction. Hence we define our two-dimensional coordinate system ξ - η such that the ξ -axis lies along the direction of $\mathbf{u}$. This parameter u corresponds to u_λ from Section 3.2, but without the explicit wavelength dependence. Instead, it may be expressed explicitly in terms of measurable quantities, observing that $\mathbf{u}$ is the vector difference of the optic axis direction $\boldsymbol{\kappa} = 0$ and beam direction $\boldsymbol{\kappa}_0$. For small misalignment angles $\delta\theta$, this amounts to the transverse component of $\boldsymbol{\kappa}_0$ with respect to the optic axis:

$$\begin{aligned} \boldsymbol{\kappa}_\perp = \mathbf{u} &= u \mathbf{e}_\xi = w n_2 k_0 \sin(\delta\theta) \mathbf{e}_\xi, \\ \Rightarrow u &= w n_2 k_0 \sin(\delta\theta), \end{aligned} \quad (4.3)$$

where k_0 is the vacuum wave number of the incident Gaussian beam, which has a $1/e$ radius of w . When $u = 0$, the incident beam is fully aligned with the optic axis resulting in conical diffraction. As $|u|$ is increased, the beam transitions to double refraction, as seen in Fig. 4.1.

As discussed in Section 3.2, this leads to a complexification of the radial and longitudinal parameters:

$$\tilde{\rho} = \{\xi + iu, \eta\}, \quad \tilde{\zeta} = \zeta - i, \quad (4.4)$$

and the azimuthal angles $\tilde{\phi}$ in the ξ - η plane are given by

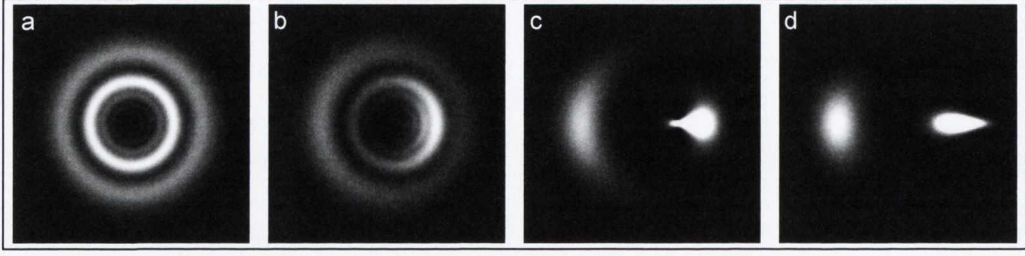


Figure 4.1: The transition between conical diffraction and double refraction for a beam with $\rho_0 = 20$, $\zeta = 6$, and (a) $u = 0$, corresponding to an incident beam aligned with the optic axis, (b) $u = 0.5$, (c) $u = 2$, and (d) $u = 5$.

$$\cos \tilde{\phi} = \frac{\xi + iu}{\tilde{\rho}}, \quad \sin \tilde{\phi} = \frac{\eta}{\tilde{\rho}}, \quad (4.5)$$

where

$$\tilde{\rho} = \sqrt{(\xi + iu)^2 + \eta^2}. \quad (4.6)$$

These terms combine to give the electric displacement field $\mathbf{D}$ beyond the crystal:

$$\mathbf{D} \propto \begin{pmatrix} B_0 + B_1 \cos \tilde{\phi} & B_1 \sin \tilde{\phi} \\ B_1 \sin \tilde{\phi} & B_0 - B_1 \cos \tilde{\phi} \end{pmatrix} \mathbf{d}_0, \quad (4.7)$$

where $\mathbf{d}_0$ is the incident beam polarisation. The terms B_0 and B_1 are the fundamental integrals derived in Section 3.2:

$$\begin{aligned} B_0(\tilde{\rho}, \tilde{\zeta}, \rho_0) &= \int_0^\infty d\kappa \kappa \exp\{-\tfrac{1}{2}i\tilde{\zeta}\kappa^2\} J_0(\kappa\tilde{\rho}) \cos(\kappa\rho_0), \\ B_1(\tilde{\rho}, \tilde{\zeta}, \rho_0) &= \int_0^\infty d\kappa \kappa \exp\{-\tfrac{1}{2}i\tilde{\zeta}\kappa^2\} J_1(\kappa\tilde{\rho}) \sin(\kappa\rho_0), \end{aligned} \quad (4.8)$$

where $J_\nu(x)$ is the ν^{th} order Bessel function of the first kind.

Noting the form of the complex radial term given by Eq. (4.6), we may find where its length will vanish:

$$\begin{aligned} \tilde{\rho} = 0 &\Rightarrow (\xi + iu)^2 + \eta^2 = 0, \\ &\Rightarrow \xi^2 - u^2 + \eta^2 + 2i\xi u = 0. \end{aligned} \quad (4.9)$$

In order to ensure a real answer, we must have $\xi = 0 \Rightarrow \eta^2 = u^2$. Therefore

the complex length vanishes at two points, the branch points, given by

$$\boldsymbol{\rho}_{b\pm} = \{0, \pm u\}. \quad (4.10)$$

$\tilde{\rho}$ changes sign around each of these branch points. Since B_0 contains J_0 , which is an even function of $\tilde{\rho}$, B_0 will not change sign around these points. However, B_1 contains J_1 which is an odd function of $\tilde{\rho}$ and so will change sign. This may seem like a problem, since of course $\mathbf{D}$ must be single valued. Observing Eqs. (4.5) however it is clear that they will both change sign around $\boldsymbol{\rho}_{b\pm}$ which compensates for B_1 in Eq. (4.7).

If the incident beam is unpolarised, we may derive using Eq. (4.7) an expression for the intensity at a position $\{\xi, \eta, \zeta\}$:

$$I = \mathbf{D}^* \cdot \mathbf{D} = |B_0|^2 + |B_1|^2 (|\cos \tilde{\phi}|^2 + |\sin \tilde{\phi}|^2). \quad (4.11)$$

This derivation may be found in Appendix B. It is important to note that because $\tilde{\phi}$ is complex, in general $|\cos \tilde{\phi}|^2 + |\sin \tilde{\phi}|^2 \neq 1$. For example, close to the branch points $\boldsymbol{\rho}_{b\pm} \approx \{0, \pm u\}$, we have $\xi \rightarrow 0$ while $\eta \rightarrow \pm u$. Thus, from Eq. (4.6), neglecting terms in ξ with order two or higher, we have

$$\tilde{\rho}_{\xi \rightarrow 0, \eta \rightarrow \pm u} \approx \sqrt{2i\xi u}. \quad (4.12)$$

Combining this with Eq. (4.5), again neglecting terms in ξ with order two or higher and setting $\eta \rightarrow \pm u$, we find that close to the branch points

$$|\cos \tilde{\phi}|^2 + |\sin \tilde{\phi}|^2 \approx \frac{2u^2}{|\sqrt{2i\xi u}|}, \quad (4.13)$$

which in general is not unity. As $\boldsymbol{\rho}$ approaches the branch points $\boldsymbol{\rho}_{b\pm}$, the complex length vanishes and Eq. (4.13) tends to infinity as $\xi \rightarrow 0$. As $\boldsymbol{\rho}$ gets farther from $\boldsymbol{\rho}_{b\pm}$, $|\cos \tilde{\phi}|^2 + |\sin \tilde{\phi}|^2$ tends to unity. Furthermore, as u approaches zero, then $\tilde{\phi}$ becomes real and Eq. (4.13) is unity everywhere, giving the intensity equation for a fully aligned circularly polarised (or unpolarised) beam, as seen in Chapter 1.

To avoid an unphysical infinite intensity in the region of the branch points

where $\tilde{\rho} \approx 0$, we first expand the Bessel function $J_1(\kappa\tilde{\rho})$ in B_1 in Eq. (4.8):

$$J_1(\kappa\tilde{\rho}) \approx \frac{1}{2}\kappa\tilde{\rho} + O(3) + \dots \quad (4.14)$$

neglecting terms of order two or higher. Since B_1 is integrated over κ , we may take $\tilde{\rho}$ outside of the equation allowing us to write Eq. (4.11) as

$$I \approx |B_0|^2 + \left| \int_0^\infty d\kappa \kappa \exp\{-\frac{1}{2}i\tilde{\zeta}\kappa^2\} \frac{1}{2}\kappa \sin(\kappa\rho_0) \right|^2 |\tilde{\rho}|^2 \left(|\cos \tilde{\phi}|^2 + |\sin \tilde{\phi}|^2 \right). \quad (4.15)$$

The term inside the parenthesis may be written as

$$|\cos \tilde{\phi}|^2 + |\sin \tilde{\phi}|^2 = \frac{|\xi + iu|^2 + |\eta|^2}{|\tilde{\rho}|^2}, \quad (4.16)$$

and hence the denominator cancels with the $|\tilde{\rho}|^2$ term in Eq. (4.15). This avoids a singularity in the intensity arising from $\tilde{\rho}$ approaching 0 as $\boldsymbol{\rho}$ approaches the branch points $\boldsymbol{\rho}_{b\pm}$.

It will now be convenient to define the sum and difference integrals

$$\begin{aligned} A_\pm &\equiv B_0 \pm B_1 \\ &= \int_0^\infty d\kappa \kappa \exp\{-\frac{1}{2}i\tilde{\zeta}\kappa^2\} (J_0(\kappa\tilde{\rho}) \cos(\kappa\rho_0) \pm J_1(\kappa\tilde{\rho}) \sin(\kappa\rho_0)) \end{aligned} \quad (4.17)$$

where B_0 and B_1 are given by Eq. (4.8). Thus Eq. (4.7) becomes

$$\mathbf{D} \propto (A_+ \mathbf{m}_+ + A_- \mathbf{m}_-) \cdot \mathbf{d}_0, \quad (4.18)$$

where

$$\mathbf{m}_\pm = \begin{pmatrix} 1 \pm \cos \tilde{\phi} & \pm \sin \tilde{\phi} \\ \pm \sin \tilde{\phi} & 1 \mp \cos \tilde{\phi} \end{pmatrix} \quad (4.19)$$

The intensity of the beam beyond the crystal is now given by

$$I = \mathbf{D}^* \cdot \mathbf{D} \propto \mathbf{d}_0^* \cdot (A_+^* \mathbf{m}_+^\dagger + A_-^* \mathbf{m}_-^\dagger) (A_+ \mathbf{m}_+ + A_- \mathbf{m}_-) \cdot \mathbf{d}_0, \quad (4.20)$$

which for an unpolarised beam gives

$$I = (|A_+|^2 + |A_-|^2)(1 + |\cos \tilde{\phi}|^2 + |\sin \tilde{\phi}|^2) + 2\text{Re}A_+^* A_- (1 - |\cos \tilde{\phi}|^2 - |\sin \tilde{\phi}|^2), \quad (4.21)$$

as discussed in Appendix B. Using the Bessel function approximation

$$J_\nu(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4} - \frac{\nu\pi}{2}\right), \quad x \gg 1, \quad (4.22)$$

we may now write a simplified expression for $A_\pm$. The term inside the parenthesis in Eq. (4.17), assuming $\kappa\tilde{\rho} \gg 1$, is:

$$\begin{aligned} & J_0(\kappa\tilde{\rho}) \cos(\kappa\rho_0) \pm J_1(\kappa\tilde{\rho}) \sin(\kappa\rho_0) \\ & \approx \sqrt{\frac{2}{\pi\tilde{\rho}}} \left[\cos\left(\kappa\tilde{\rho} - \frac{\pi}{4}\right) \cos(\kappa\rho_0) \pm \cos\left(\kappa\tilde{\rho} - \frac{3\pi}{4}\right) \sin(\kappa\rho_0) \right] \\ & = \sqrt{\frac{2}{\pi\kappa\tilde{\rho}}} \cos\left\{\kappa(\tilde{\rho} \mp \rho_0) - \frac{\pi}{4}\right\}, \end{aligned} \quad (4.23)$$

$$\Rightarrow A_\pm \approx \sqrt{\frac{2}{\pi\tilde{\rho}}} \int_0^\infty d\kappa \sqrt{\kappa} \exp\left\{-\frac{1}{2}\text{i}\tilde{\zeta}\kappa^2\right\} \cos\left\{\kappa(\tilde{\rho} \mp \rho_0) - \frac{\pi}{4}\right\}, \quad (4.24)$$

where here and hereafter we define complex roots as $\sqrt{z} = \sqrt{|z|} \exp\left[\frac{1}{2}\text{i} \arg z\right]$, $|\arg z| < \pi$.

Expanding the cosines into the component exponentials allows us to use the method of stationary phase in the integrals above, assuming the exponentials are fast-varying, as outlined in Appendix A. This method is discussed in Berry and Jeffrey [41], and in more detail in Jeffrey [55]. This yields

$$A_\pm \approx -\frac{\mu_\pm}{\tilde{\zeta}} \left(\frac{\rho_0 \mp \tilde{\rho}}{\tilde{\rho}}\right)^{1/2} \exp\left\{\frac{\text{i}}{2\tilde{\zeta}}(\tilde{\rho} \mp \rho_0)^2\right\}, \quad \begin{cases} \mu_+ = \text{sgn}\left[\text{Im}\left(\frac{\rho_0 - \tilde{\rho}}{\sqrt{1+\text{i}\tilde{\zeta}}}\right)\right] \\ \mu_- = \text{i} \end{cases} \quad (4.25)$$

The values A_+ and A_- now represent separate geometrical rays, and due to the mixing term in Eq. (4.21) these rays interfere. However, the factor of $(1 - |\cos \tilde{\phi}|^2 - |\sin \tilde{\phi}|^2)$ on this mixing term implies the interference will only be

significant close to the branch points $\rho_{b\pm}$ where $|\cos \tilde{\phi}|^2 + |\sin \tilde{\phi}|^2 \approx 2u^2/|\tilde{\rho}^2|$. Furthermore, for interference we require $A_+ = A_-$. From Eq. (4.25), we may approximate the term in the square root to be equal for both A_+ and A_- close to the branch points, since $\tilde{\rho} \approx 0$ in these regions. This also implies $\mu_+ = -$, since

$$\frac{\rho_0}{\sqrt{1+i\zeta}} = \frac{\rho_0}{\sqrt{|1+i\zeta|} \exp\left[\frac{1}{2}i\text{Arg}(1+i\zeta)\right]} = \rho_0 \exp\left[-\frac{1}{2}i \arctan \zeta\right] \quad (4.26)$$

which always has a negative imaginary part, assuming $\zeta, \rho_0 \in \mathbb{R}$. Hence, to meet the condition $A_+ = A_-$ we must have

$$\begin{aligned} \exp\left\{\frac{i}{2\tilde{\zeta}}(\tilde{\rho} - \rho_0)^2\right\} &= -i \exp\left\{\frac{i}{2\tilde{\zeta}}(\tilde{\rho} + \rho_0)^2\right\} \\ \exp\left\{-i\frac{\rho_0\tilde{\rho}}{\tilde{\zeta}}\right\} &= -i \exp\left\{i\frac{\rho_0\tilde{\rho}}{\tilde{\zeta}}\right\} \\ \Rightarrow \exp\left\{-2i\rho_0\frac{\tilde{\rho}}{\tilde{\zeta}}\right\} &= -i. \end{aligned} \quad (4.27)$$

This leads to an equality of moduli constraint:

$$\text{Im}\frac{\tilde{\rho}}{\tilde{\zeta}} = 0. \quad (4.28)$$

Thus, the term $\tilde{\rho}/\tilde{\zeta}$ must equal something real, denoted $R_{\tilde{\rho}/\tilde{\zeta}} \in \mathbb{R}$:

$$\begin{aligned} \frac{\tilde{\rho}}{\tilde{\zeta}} = R_{\tilde{\rho}/\tilde{\zeta}} &\Rightarrow \frac{\sqrt{\tilde{\rho} \cdot \tilde{\rho}}}{\tilde{\zeta}} = R_{\tilde{\rho}/\tilde{\zeta}} \\ \rightarrow \tilde{\rho} \cdot \tilde{\rho} &= R_{\tilde{\rho}/\tilde{\zeta}}^2 \tilde{\zeta}^2 = R_{\tilde{\rho}/\tilde{\zeta}}^2 \zeta^2 - R_{\tilde{\rho}/\tilde{\zeta}}^2 - 2i\zeta R_{\tilde{\rho}/\tilde{\zeta}}^2. \end{aligned} \quad (4.29)$$

Observing that $\zeta \in \mathbb{R}$, and that $\text{Re}[\tilde{\rho} \cdot \tilde{\rho}] = \xi^2 + \eta^2 - u^2$ and $\text{Im}[\tilde{\rho} \cdot \tilde{\rho}] = 2\xi u$ allows us to equate the real and imaginary parts of both sides of Eq. (4.29):

$$\begin{aligned} 2\xi u &= 2\zeta R_{\tilde{\rho}/\tilde{\zeta}}^2 \\ \Rightarrow R_{\tilde{\rho}/\tilde{\zeta}}^2 &= \frac{\xi u}{\zeta}. \end{aligned} \quad (4.30)$$

Similarly,

$$\begin{aligned}\xi^2 + \eta^2 - u^2 &= R_{\tilde{\rho}/\tilde{\zeta}}^2 (\zeta^2 - 1) \\ \Rightarrow \xi^2 + \eta^2 - u^2 &= \frac{\xi u}{\zeta} (\zeta^2 - 1).\end{aligned}\quad (4.31)$$

This quadratic equation in ξ has solutions

$$\xi = \frac{1}{2} \left(-\frac{u}{\zeta} (\zeta^2 - 1) \pm \sqrt{u^2 \left(\zeta + \frac{1}{\zeta} \right)^2 - 4\eta^2} \right). \quad (4.32)$$

Rearranging to collect the terms outside the square root we get

$$\begin{aligned}\left(\xi + \frac{u(\zeta^2 - 1)}{2\zeta} \right) &= \pm \frac{1}{2} \sqrt{u^2 \left(\zeta + \frac{1}{\zeta} \right)^2 - 4\eta^2} \\ \Rightarrow \left(\xi + \frac{u(\zeta^2 - 1)}{2\zeta} \right)^2 + \eta^2 &= \frac{1}{4} u^2 \left(\zeta + \frac{1}{\zeta} \right)^2,\end{aligned}\quad (4.33)$$

or more compactly,

$$(\boldsymbol{\rho} - \boldsymbol{\rho}_c) \cdot (\boldsymbol{\rho} - \boldsymbol{\rho}_c) = R_c^2, \quad (4.34)$$

the equation of a circle with centre $\boldsymbol{\rho}_c$ and radius R_c where

$$\boldsymbol{\rho}_c = \left\{ -\frac{u(\zeta^2 - 1)}{2\zeta}, 0 \right\}, \quad R_c = \frac{u(\zeta^2 + 1)}{2\zeta}. \quad (4.35)$$

Thus geometrical interference is confined to an arc of a circle in the $\boldsymbol{\rho}$ -plane whose equation is given by Eq. (4.34). Along these points the magnitudes of $A_{\pm}$ are equal, and on either side one of these geometrical rays increases exponentially, as can be seen from the form of Eq. (4.25). Hence, this circle is manifest as a region with exponentially lower intensity than elsewhere, the exception being the region between the two branch points $\boldsymbol{\rho}_b = \{0, \pm u\}$ which lie on this circle as seen in Fig. 4.2. These dark arcs are clearly visible in Fig. 4.3 (b) and (d).

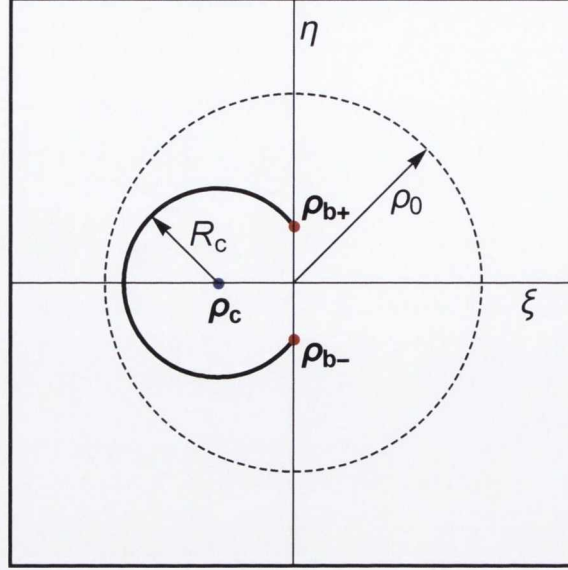


Figure 4.2: The ρ -plane in a beam with $\rho_0 = 20$, $\zeta = 3$, and $u = 6$ showing the Poggendorff ring (dashed circle), and the arc of the circle (thick arc) described by Eq. (4.34) between the two branch points $\rho_{b\pm}$, shown as red dots. The blue dot represents the centre ρ_c of the geometric interference circle, given by Eq. (4.35).

Furthermore, from Eqs. (4.27) and (4.28), we have equality of phase for

$$\begin{aligned}
 \exp \left\{ -2i\rho_0 \frac{\tilde{\rho}}{\tilde{\zeta}} \right\} &= \cos \left(2\rho_0 \frac{\tilde{\rho}}{\tilde{\zeta}} \right) - i \sin \left(-2\rho_0 \frac{\tilde{\rho}}{\tilde{\zeta}} \right) = -i \\
 &\rightarrow 2\rho_0 \operatorname{Re} \frac{\tilde{\rho}}{\tilde{\zeta}} = \arcsin 1 = 2 \left(n + \frac{1}{4} \right) \pi \\
 &\Rightarrow \operatorname{Re} \frac{\tilde{\rho}}{\tilde{\zeta}} = \frac{(n + \frac{1}{4}) \pi}{\rho_0}, \quad n \in \mathbb{Z}. \quad (4.36)
 \end{aligned}$$

When this condition is satisfied, we have total destructive interference between A_+ and A_- resulting in a series of low intensity spots decorating the dark arc described by Eq. (4.34), as seen in Fig. 4.4.

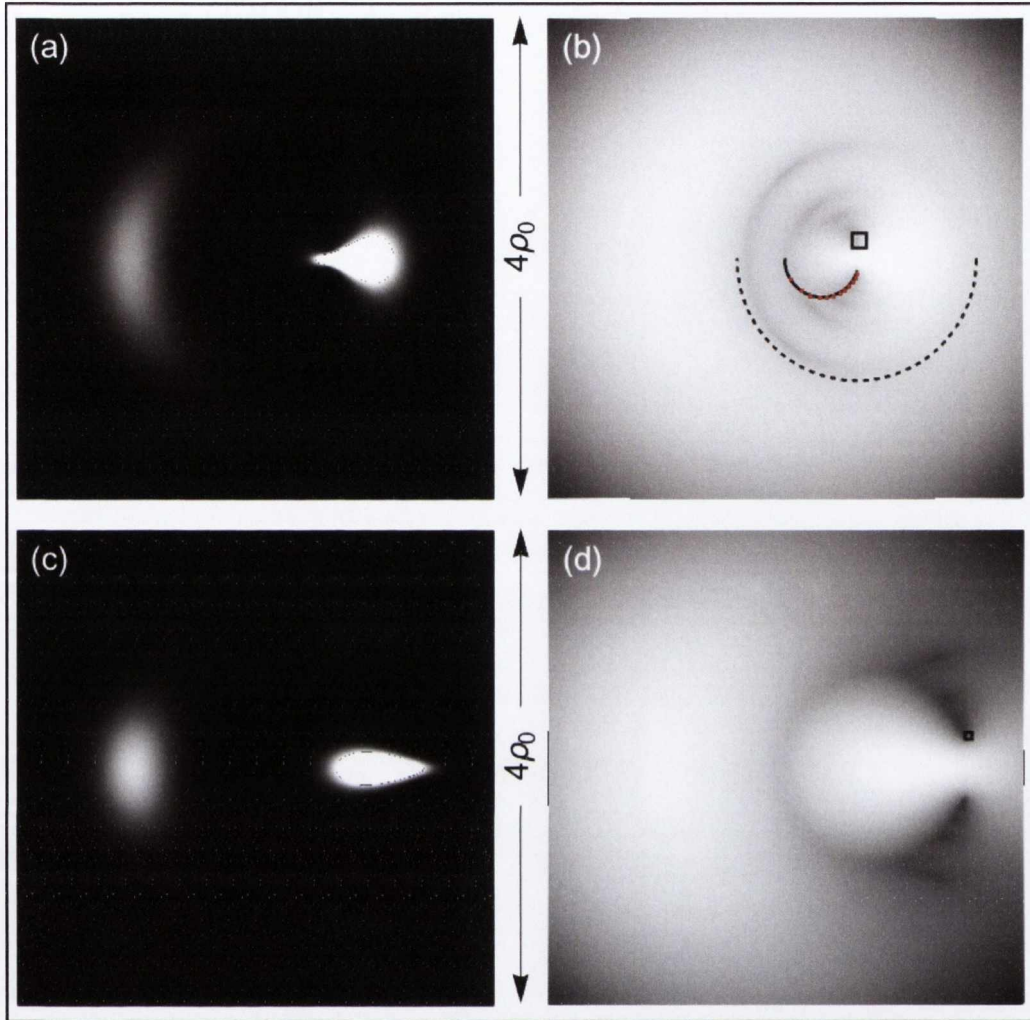


Figure 4.3: Intensity profiles for $\rho_0 = 20$, $\zeta = 6$, and (a)-(b) $u = 2$, (c)-(d) $u = 5$. Images (a) and (c) have been saturated to allow the faint left crescents to be seen. (b) and (d) are logarithmic intensity profiles. In (b), the dashed semicircle represents the Poggendorff ring with a radius of ρ_0 , while the solid black semicircle is that described by Eq. (4.34) representing the arc of exponentially lower intensity. Red dots along this arc are solutions to Eq. (4.36) and correspond to geometric interference. The small squares in (b) and (c) represent the areas shown in Fig. 4.4.

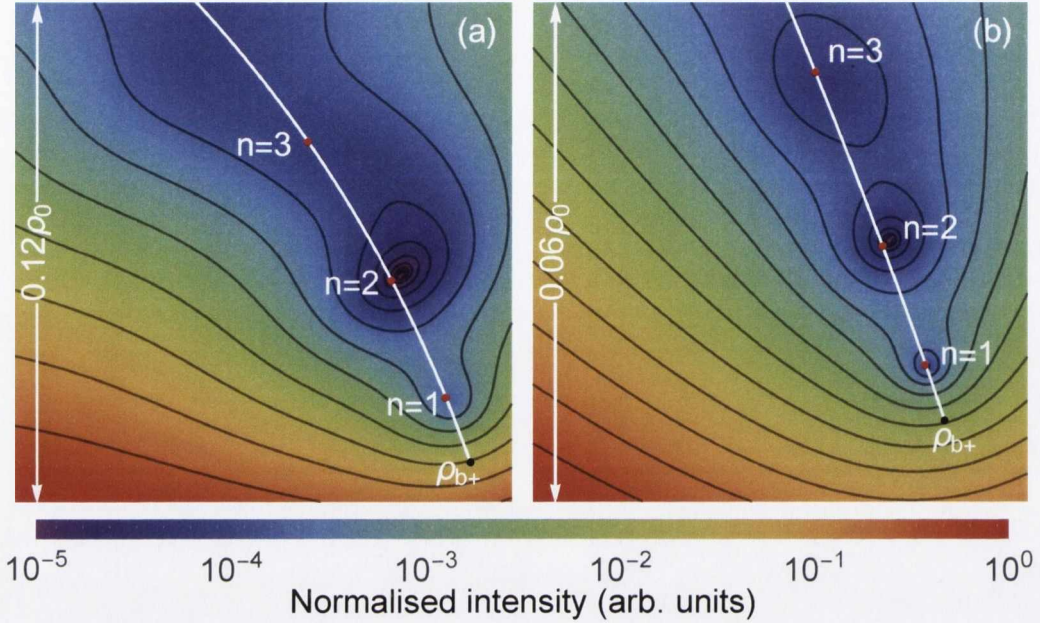


Figure 4.4: Logarithmic intensity profiles for $\rho_0 = 20$, $\zeta = 6$, and (a) $u = 2$, (b) $u = 5$. Spatially, these correspond to the squares shown in Fig. 4.3 (c) and (d) respectively. The contours shown are iso-intensity contours. In both plots, the circles described by Eq. (4.34) are marked by the white arcs. It is clear the intensity along these arcs is lower than the surrounding regions. Furthermore, regions of lower intensity are clearly visible and denoted by the red dots which correspond to solutions of Eq. (4.36) for $n = 1, 2, 3$. The solutions for $n = 0$ correspond to the branch points $\rho_{b\pm}$, two of which are marked by the black dots.

4.3 Experimental method

Due to the small size and low intensity nature of the features predicted to occur, an experimental method was required which maximised the resolution and intensity contrast of the images of the conically diffracted beam. Since the ‘quality’ of the beam is given by the ratio of the radius of the ring at the FIP and the $1/e$ intensity of the incident beam ρ_0 , a high value of ρ_0 should ensure an optimal chance to observe the features. To achieve this, it was necessary to generate a very narrow incident beam waist w , ideally $w < 7 \mu\text{m}$ to give $\rho_0 > 50$. Given an initial collimated beam of wavelength λ and radius r , upon directing it through a focussing lens of focal length f

it reaches a narrowest radius w following the equation

$$w = \frac{\lambda f}{\pi r}. \quad (4.37)$$

A Helium Neon (HeNe) laser beam with a peak emission at $\lambda = 632.8$ nm was expanded and collimated using a variable beam expander to have a $1/e$ intensity radius of $r = 990$ μm . It was subsequently directed through a biconvex lens l_1 of focal length $f = 30$ mm. Using Eq. (4.37), the minimum beam waist should occur 3 cm after l_1 , at $z = 0$, and have a value of $w = 6.14$ μm . Using the modified value of n_2 as discussed in Chapter 3, the radius of the ring generated by a 20.63 mm long slab of $\text{KGd}(\text{WO}_4)_2$ is predicted to be $R_0 = 343$ μm giving $\rho_0 = 56.2 \pm 0.8$.

In order to confirm this, the biaxial crystal was placed in the focussed beam with the entrance face 1 cm beyond the lens at $z = -2$ cm. The optic axis was aligned with the incident beam and a colour CCD was placed at $z = 1.03$ cm, corresponding to the FIP. The radius of the conically diffracted ring profile was determined to be 345 ± 4.65 μm , which matches the predicted value within experimental error. A 10x objective was then used to image the focal image plane onto the CCD, giving a magnification of 3.97. The ring profile was examined by taking intensity profiles along radii at many angles and averaging them. This averaged profile was then compared directly to theoretical intensity plots generated using Eq. (4.11) at various values of ρ_0 as shown in Fig. 4.5. The value which best matched the experimental average was found to be $\rho_0 = 56$, which is in excellent agreement with the predicted value of 56.2 using $w = 6.14$ μm .

The objective was replaced with an achromatic biconvex lens l_2 of focal length $f_2 = 150$ mm for all subsequent experiments as show in Fig. 4.6. The lens l_1 and the biaxial crystal were fixed together on a translation stage with a separation of 3 cm, so that the narrowest beam radius coincided with the crystal's entrance face at $z = 0$ cm. This allowed the relative distance between l_1 and l_2 to be varied, corresponding to imaging the beam at various ζ values.

The biaxial crystal was gradually reoriented so the incident beam was no

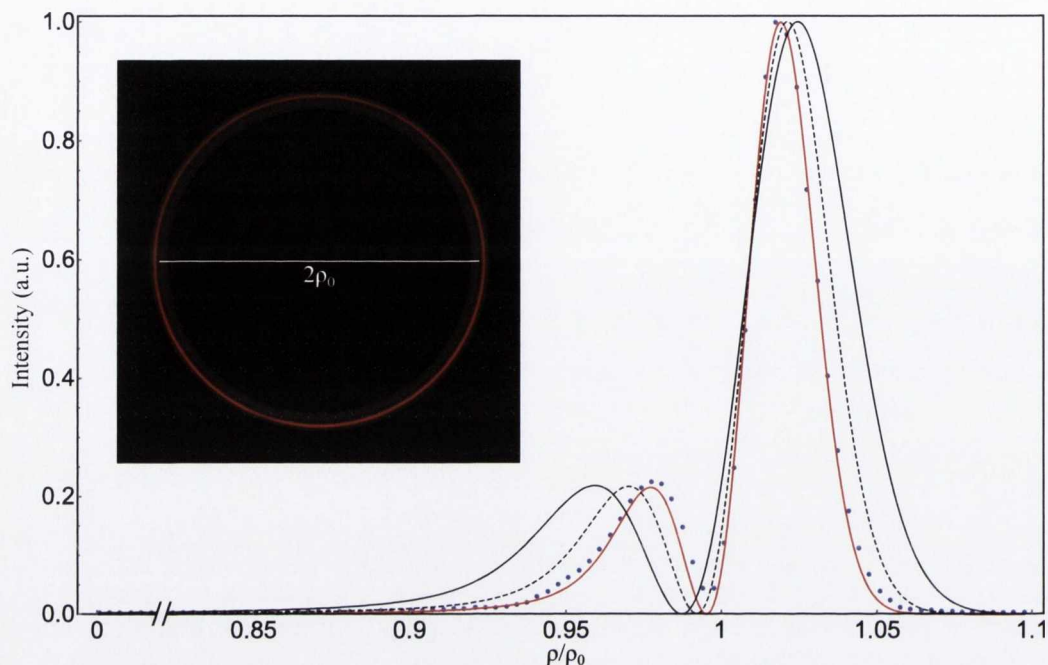


Figure 4.5: Averaged radial intensity profile (blue dots) taken from an experimental image (inset) compared to theoretical intensity plots for $\rho_0 = 36$ (solid black line), $\rho_0 = 46$ (dashed black line), and $\rho_0 = 56$ (solid red line).

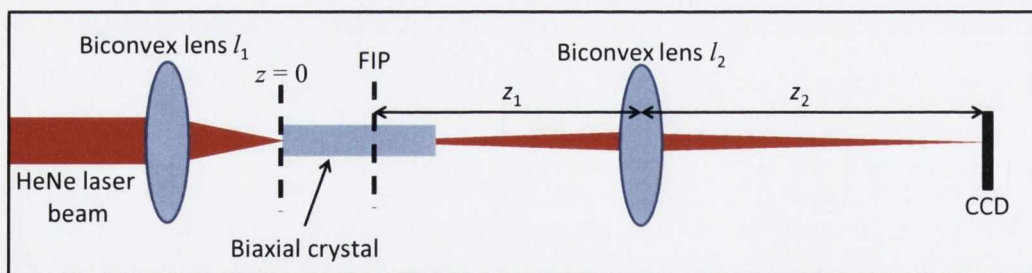


Figure 4.6: The experimental apparatus used to observe low intensity features in the conically diffracted pattern when $u \neq 0$. The distances z_1 and z_2 could be varied to achieve different magnifications. In the diagram, the FIP ($\zeta = 0$) is being imaged, but any ζ value could be selected by moving lens l_1 using the translation stage.

longer aligned to an optic axis ($u \neq 0$). This produced the characteristic transition between conical diffraction and double refraction as seen in Fig. 4.1. The value of u could be calculated explicitly using Eq. (4.3) given a reorientation of θ from the optic axis direction. However this angle was very small, typically less than 1° making measurement difficult, and so it was more useful to infer the value of u by direct comparison with theoretical plots.

4.4 High dynamic range images

Having designed an appropriate experiment to generate the low intensity features, the next issue to be considered was the imaging technique. The CCD pixels return values in quanta meaning there is a minimum non-zero value each pixel can return. This poses a problem for any attempt to image low intensity features—if the intensities of the features are lower than this threshold then nothing will be recorded. It is possible to decrease the ‘shutter speed’ on the CCD, *i.e.* increase the duration over which the pixels will record to produce an image. This effectively decreases the photon flux required to produce a reading, allowing the lower intensity features to be seen. There is a trade-off however—the image will be overexposed in the high intensity regions where the pixels have reached the maximum photon flux they can respond to.

A method known as high dynamic range (HDR) imaging [77, 78] was utilised to generate images which maximised the visible range of the interference fringes. For every combination of u and ζ , several images were taken at various camera exposure times. The first exposure time was set so that the maximum pixel value in the image was just below 1. The exposure time was then increased in increments which corresponded to multiplying the non-saturated pixel values in the previous image by a constant factor. This was performed until pixels which had read 10^{-3} in the first image were just becoming saturated, *i.e.* just reaching a value of 1.

These individual images were then combined in order to produce a HDR image which allowed very weak features to be observed while not saturating the highest intensity regions. Each image was transformed into a matrix of

pixel values and multiplied by a weighting factor to correct for the various exposure levels. For example, the image with the shortest exposure time t_s would resolve the highest intensity region of the beam, normalised to a value of 1. The image with the longest exposure time t_l was then multiplied by a factor of t_s/t_l . The images with intermediary exposure times were also processed in this way.

The resultant matrices were then summed and normalised to provide a single matrix which effectively allowed up to three orders of magnitude in the intensity to be viewed on one image. Taking the logarithmic intensity allowed the low intensity regions of profile to be seen more clearly.

4.5 Results

To examine the low intensity features of the HDR images generated using the experimental method outlined in Sec. 4.3, a logarithmic intensity plot was created for each image and compared to theoretical logarithmic intensity plots generated using Eq. (4.11). In Fig. 4.7 many fringes are visible along the low intensity arc of the circle described by Eq. (4.34). These fringes are not the low intensity points we seek however, they are remnants of the Warnick-Arnold rings [20], features which are visible as symmetric annuli in conically diffracted beams with $u = 0$. The location of the low intensity arc of the circle agrees well with theory.

In Fig. 4.8 it is possible to see the very faint remnants of the Warnick-Arnold rings on the theoretical plot (a) and on the upper half of the experimental image (b). They are visible as weak intensity oscillations and their positions are indicated on the upper half of the theoretical plot using white arcs. The clearer rapid intensity oscillations are as a result of the interference described in Eq. (4.36), an effect only observed in beams with $u \neq 0$. These are clearly visible in both the theoretical plot (a) and the lower half of the experimental image (b). The asymmetry between the upper and lower halves of the experimental image is due to the high sensitivity of the interference pattern to the alignment of the crystal. Although the beam was of course misaligned from the optic axis to achieve $u \neq 0$, varying the crystal orien-

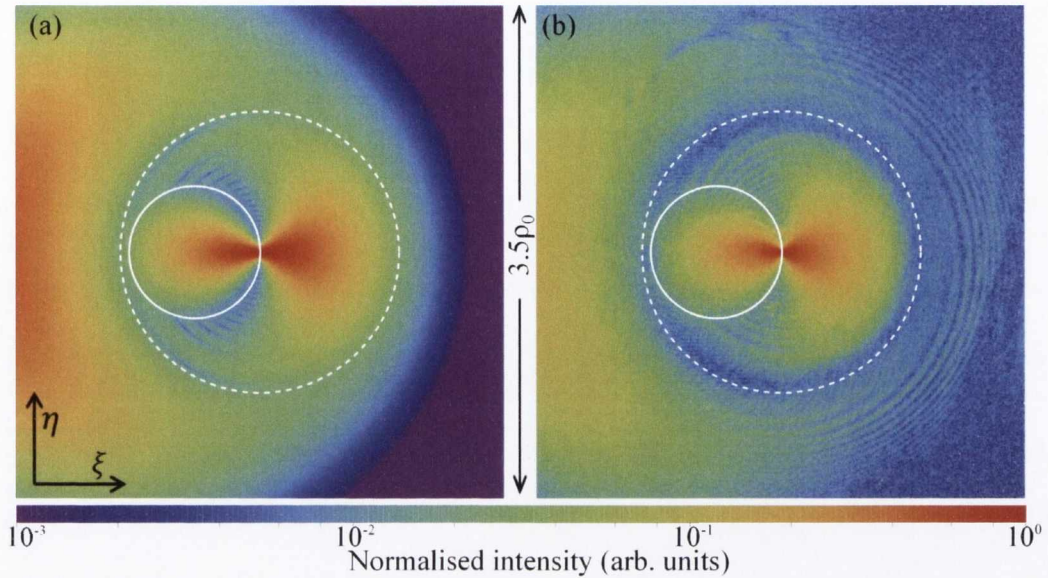


Figure 4.7: Comparison of (a) theoretical and (b) experimental logarithmic intensity plots with $\rho_0 = 56.2$, $u = 3.7$, and $\zeta = 14.2$. The dashed circles are centred about $\boldsymbol{\rho} = \{0, 0\}$ with radius ρ_0 . The solid circles are centred about $\boldsymbol{\rho}_c$ with radius R_c as given by Eq. (4.35).

tation slightly was observed to cause the interference fringes to appear and disappear rapidly. The image displayed was the clearest result attained for the parameters specified.

To examine the interference fringes further, in particular those close to the branch points $\boldsymbol{\rho}_b = \{0, \pm u\}$, the image was magnified by a factor of 13 for $u = 1.8$ and $\zeta = 42$. The results are shown in Fig. 4.9. The low-intensity fringes are visible in the experimental image (b), and are marked in the lower half of the image by white dots which are solutions of Eq. (4.36).

Although the intensity minima in the experimental image are not as clearly defined as those in the theoretical plot due to the large intensity contrast required, and due to saturation of the experimental image, the locations match very well and it is possible to discern the locations of all of the minima. The intensity contrast between the minima and the area surrounding them is approximately one order of magnitude in both the theoretical and experimental images.

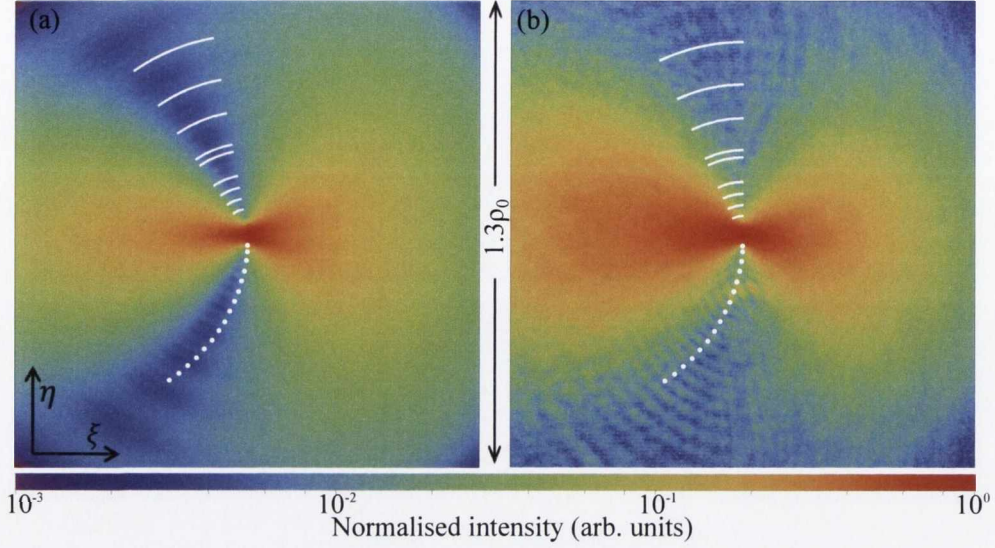


Figure 4.8: Comparison of (a) theoretical, and (b) experimental logarithmic intensity plots with $\rho_0 = 56.2$, $u = 1.8$, and $\zeta = 31$. The white arcs in the upper part of the images show the remnants of the Warnick-Arnold rings, visible as faint intensity maxima at the locations shown. The white dots in the lower part of the images are the predicted locations of intensity minima from Eq. (4.36) for $0 \leq n \leq 15$, which occur along an arc of the circle given by Eq. (4.34).

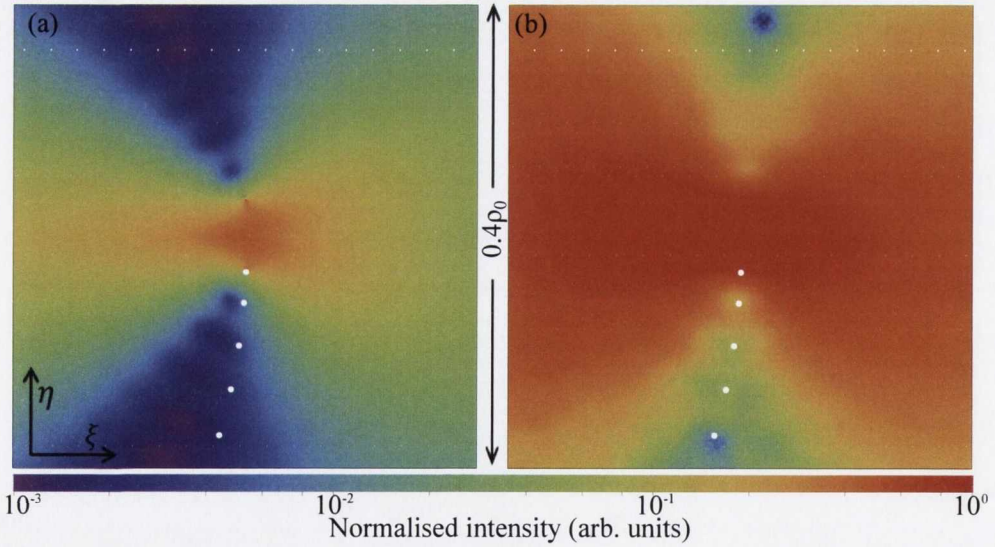


Figure 4.9: Comparison of (a) theoretical, and (b) experimental logarithmic intensity plots with $\rho_0 = 56.2$, $u = 1.8$, and $\zeta = 42$. The white dots in the lower part of the images are the predicted locations of intensity minima from Eq. (4.36) for $0 \leq n \leq 4$, which occur along an arc of the circle in Eq. (4.34).

4.6 Polarisation in misaligned beams

So far we have only considered unpolarised incident beams, but the form of Eq. (4.7) allows us to find the intensity distribution for any incident beam polarisation, as derived in Jeffrey [55] in terms of A_+ and A_- . The derivation for the intensity in terms of B_0 and B_1 is included in Appendix B, where we find that for a linearly polarised incident beam whose polarisation is inclined at an angle χ to the ξ -axis, we have

$$I_{\text{lin}} = |B_0|^2 + |B_1|^2 \left(|\cos \tilde{\phi}|^2 + |\sin \tilde{\phi}|^2 \right) + 2\text{Re} \left[B_0^* B_1 \cos \left(\tilde{\phi} - 2\chi \right) \right]. \quad (4.38)$$

To experimentally confirm the predicted intensity profiles a HeNe laser beam was directed through a linear polariser and a half-wave plate, which was placed in a rotation mount. By rotating the half-wave plate, the incident polarisation angle χ was varied. Using a similar experimental configuration to that shown in Fig. 4.6, several images of the beam were taken at various χ values. These images are shown in Fig. 4.11, and they match very closely with the theoretical predictions in Fig. 4.10.

The features of the beam generated with a linearly polarised laser are quite interesting. For $\chi = 0$, none of the left lobe of the beam is visible, similar to the case of double refraction where the ordinary and extraordinary rays have orthogonal polarisation. As the angle varies however, the evolution of the intensity profile is quite complex. The left lobe appears asymmetrically, while a discontinuity in the right lobe emerges, as seen in Figs. 4.10 and 4.11 for $\chi = \pi/4$ and $\chi = 3\pi/4$. When $\chi = \pi/2$, the right lobe almost vanishes, the only remnants being a double ‘prong’ structure, most clearly seen in Fig. 4.11. The entire evolution of the intensity pattern occurs over a span of π for χ , although there is an axis of symmetry about the ξ -axis for $0 < \chi < \pi/2$.

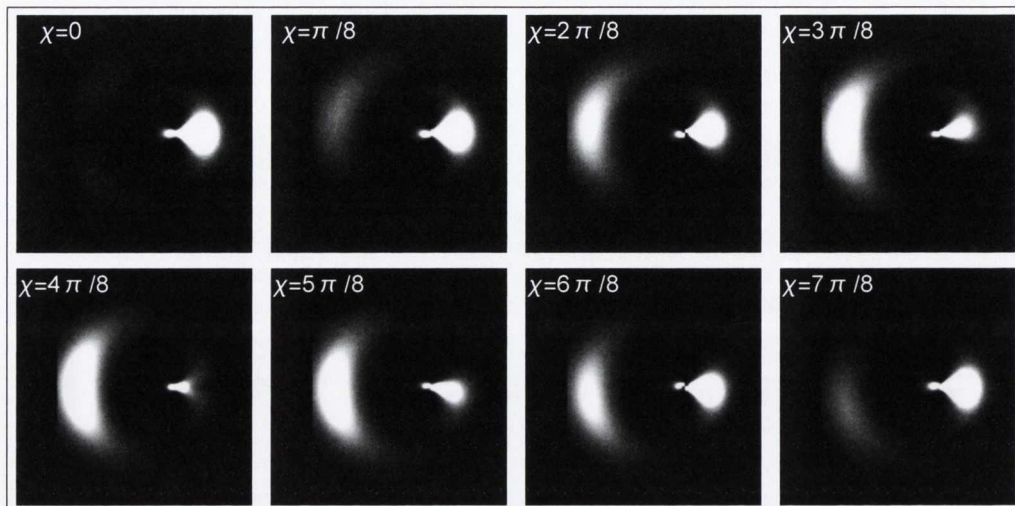


Figure 4.10: Theoretical intensity calculations for a conically diffracted beam with $\rho_0 = 20$, $\zeta = 6$, and $u = 2$. The incident beam is linearly polarised, with the polarisation vector making an angle χ with the ξ -axis. Some parts of the image are saturated to allow the low intensity regions to be seen.

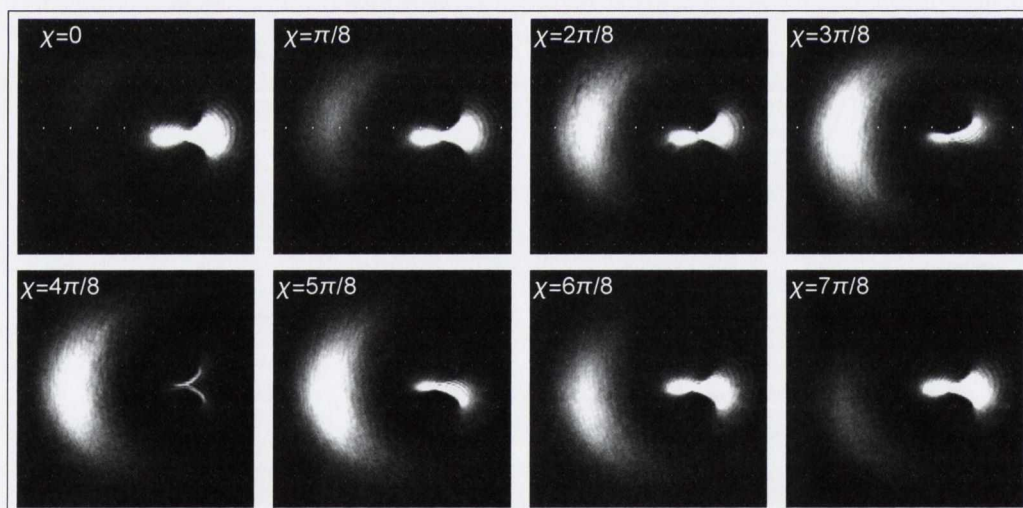


Figure 4.11: Experimental images corresponding to the simulations in Fig. 4.10 with the same beam parameters. The intensity patterns match very well with the simulations. Some parts of the image are saturated to allow the low intensity regions to be seen.

4.7 Ray tracing and the conchoid of a circle

Reverting to a simplified consideration of how light behaves, it is possible to construct a model which successfully predicts the intensity distribution of a conically diffracted beam which has been misaligned with the optic axis. We must consider the incident beam as a pencil of rays falling on the surface of wave normals, and the dual-sheeted surface results in double refraction for each ray—one for each orthogonal polarisation direction. Hence, we may take a particular incident ray and draw two refracted rays. In this way, a full picture of the refracted beam may be build up by rotating a ray around the centre of the incident beam and tracing out the end of the refracted rays on the orthogonal plane beyond the crystal. A ray optics method of deriving the intensity of a fully aligned conically diffracted beam is presented in Section 7 of Berry [21], and although some of the arguments presented here are similar, it is a novel method.

The ξ - η plane is orthogonal to the beam propagation direction. Hence, to trace a ray, we need only be concerned with the orthogonal component of the wave vector. This originates at a point on the incident beam $\mathbf{p}_i$ and undergoes double refraction, with each refracted ray orthogonally polarised. If $\mathbf{p}_i$ is a distance r from the centre of the incident beam $\mathbf{c}_i$, then rotating about $\mathbf{c}_i$ will trace out the refracted rays encompassing equal energy. For example, if $r = w$, the $1/e$ intensity width of the Gaussian beam, then the refracted rays will enclose 68.27% of the energy.

The refracted rays themselves are found by placing two equidistant points $\mathbf{s}_1$ and $\mathbf{s}_2$, representing the ends of the refracted ray vectors, on a line with $|\mathbf{s}_1\mathbf{p}_i| = |\mathbf{s}_2\mathbf{p}_i| = s$. The line on which these points sit is defined to contain $\mathbf{p}_i$ as well as a point $\mathbf{p}_u$, which determines the extent of the misalignment of the beam. This essentially projects the transverse wave vector component of the incident beam through the fixed point onto a plane beyond the crystal. However, we must make a correction since upon exiting the crystal the beam refracts and no longer spreads out. To achieve this, we simply project the plane at ζ onto the plane at $\zeta = 0$ to get s , as seen in Fig. 4.12.

Setting the Fresnel length of the beam and the length of the crystal to

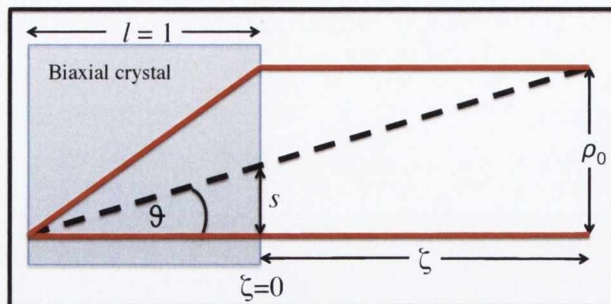


Figure 4.12: Projection of the radius at ζ onto the focal image plane. Since w , n_2 , and l have been set to unity, the FIP occurs at the exit face of the crystal. Furthermore, k_0 has been set to unity, meaning ζ spans 1 from the entrance face of the crystal to the exit face, where $\zeta = 0$.

unity, we have

$$\begin{aligned} \tan \vartheta &= s, & \tan \vartheta &= \frac{\rho_0}{\zeta + 1}, \\ \Rightarrow s &= \frac{\rho_0}{\zeta + 1}. \end{aligned} \quad (4.39)$$

The idea of ray tracing in biaxial media is not without precedent [79–82], although no simple model has yet been published for misaligned beams. A convenient method to describe this process is through the use of a conchoid of a circle, a curve defined in the following way. Given a unity circle, a fixed point $\mathbf{p}_u = \{0, u\}$, and a conchoid constant s , the conchoid curve is traced by $\{\xi, \eta\}$ as seen in Fig. 4.13 with

$$\xi = \sin(t) \pm s \frac{\sin(t) - u}{h(t)}, \quad \eta = \cos(t) \pm s \frac{\cos(t)}{h(t)}, \quad (4.40)$$

$$h(t) = \sqrt{(\sin(t) - s)^2 + \cos^2(t)}. \quad (4.41)$$

Such a curve will trace out the refracted rays which form closed regions containing $1/e$ of the intensity, since the unity circle defines the $1/e$ intensity of the incident beam. By choosing a value for the radius of the circle r other than unity, the curve generated by Eq. (4.41) will encompass a different percentage of the intensity of the beam. In this way, we may build up many isointensity contour plots, starting with $r = 0$ and increasing in equal steps

to a sufficiently large value of r such that almost all of the beam intensity is accounted for. If these steps are sufficiently small, the contour plots may be combined onto a single plot resulting in a pseudo-intensity plot. This method was performed for various values of ζ and u and compared to the exact intensity profiles as seen in Figs. 4.14 and 4.15. It is now clear that there are substantial differences between this method and that given in Berry [21], through our use of misaligned beams, the conchoid curve, and discrete energy steps.

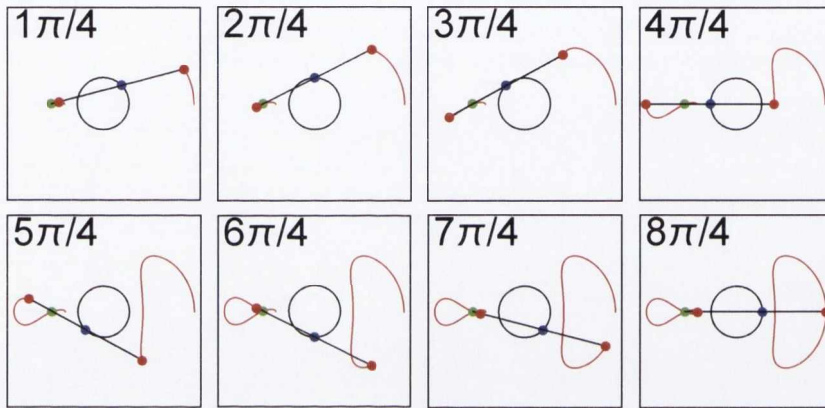


Figure 4.13: Schematic of how the conchoid of a circle is formed. The fixed distance between the red dots, s_1 and s_2 , and the blue dot p_i is s , while the green dot is a fixed point p_u . As the blue dot rotates around the circumference of the circle, the line passing through the green dot rotates to ensure the blue point remains on it. Hence, the red dots trace out paths which describe the conchoid of the circle, given by Eq. (4.40).

The conchoid of a circle method clearly reproduces the exact intensity profile with a reasonably high degree of accuracy. One notable difference is the ‘pinch point’, the region between the two branch points, which has a finite width in the exact intensity case but zero width in the conchoid case. This is simply an artefact of the ray tracing method whereby all rays pass through the fixed point p_u . The interference fringes previously discussed will of course not be present in the conchoid method, since there are no rays to interfere. The method also breaks down for small u , where we are close to full beam alignment.

The benefit of this novel conchoid method is a vastly reduced computation

time, typically a matter of seconds compared to hours for the exact intensity. This allows the parameters of a misaligned beam to be quickly inferred by comparison, since direct measurements of u are difficult due to the small angles required.

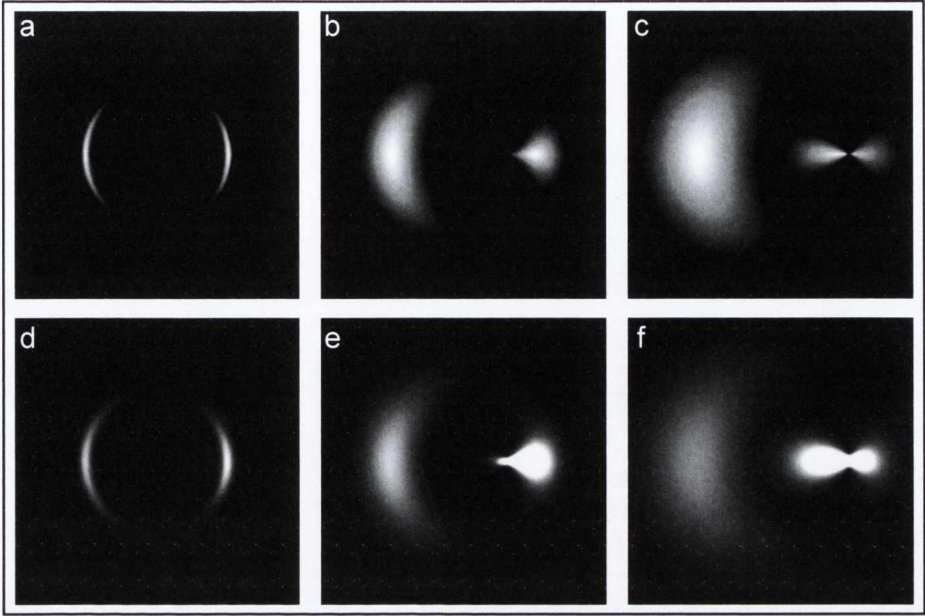


Figure 4.14: Comparison of the conchoid method (a–c) with the exact intensity (d–f) for $\rho_0 = 20$, $u = 2$, and (a,d) $\zeta = 1$, (b,e) $\zeta = 6$, (c,f) $\zeta = 11$.

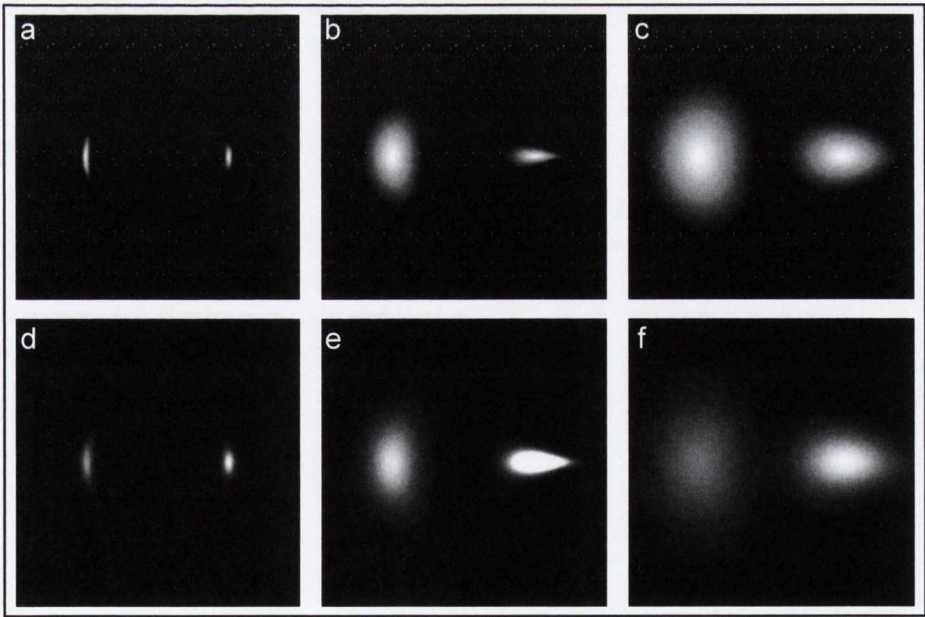


Figure 4.15: Comparison of the conchoid method (a–c) with the exact intensity (d–f) for $\rho_0 = 20$, $u = 5$, and (a,d) $\zeta = 1$, (b,e) $\zeta = 6$, (c,f) $\zeta = 11$.

4.8 Conclusion

Although it has been well over a century since conical diffraction was first observed, new features relating to the phenomenon are still being discovered. We have experimentally confirmed the presence of low-intensity interference patterns present when a Gaussian light beam enters a biaxial crystal in a direction close to, but not along an optic axis. Although these patterns exhibited high sensitivity on the orientation of the crystal, it was possible to obtain a stable image with a sufficiently high dynamic range to allow the rich structures to be seen clearly. This is the first observation of this qualitatively new optical phenomenon in biaxial materials. We anticipate further work on these low intensity features, for example using different incident beam profiles such as top-hat beams.

The effects of linearly polarised incident beams misaligned with the optic axis had not been observed before, although the theoretical intensity equation had been derived by Jeffrey [55]. The interesting profiles observed may be worth investigating further, for example over a large propagation distance to build up a beam evolution profile.

The method of intensity modelling using the conchoid of a circle proves to be useful for estimating the beam parameters u and ζ . It provides a reasonably accurate profile in a matter of seconds, compared to over an hour for the exact intensity, at the expense of some of the finer details of the beam, such as the geometrical interference spots. The method works for beams suitably far from the FIP ($\zeta > 0$) and with suitably large values of u .

Chapter 5

Further work and conclusions

5.1 Nd-doped biaxial crystal

In this section, we present some preliminary experimental data for a rare earth doped biaxial crystal, concentrating in particular on the absorption spectrum and ring radius spectrum. As this work is still in progress, we will not go into great detail but simply outline our results so far and consider potential future work and applications.

When a dielectric material is doped with a rare earth metal, energy level transitions in the rare earth atoms typically result in strong and narrow absorption peaks in the material [83]. Close to these regions, the refractive index tends to exhibit non-linear behaviour, an effect known as anomalous dispersion [84]. Furthermore, luminescence is often observed in these doped materials [85, 86]. If a biaxial material is doped with rare earth ions, the resultant luminescence and non-linear behaviour close to the absorption peaks may be of interest in non-linear conical diffraction.

There have been publications on non-linearity in biaxial materials [87, 88], as well as on the topic of second-harmonic generation in biaxial materials [64–66, 89–91]. Optically active biaxial materials have also been considered [92], although we will assume no chirality here. Neodymium (Nd) doped KGd(WO₄)₂ has been optically characterised [93], and even investigated for its potential applications as a lasing medium [34]. However, a characterisation of the ring radius spectrum has not yet been published, and owing to the non-linear behaviour of the refractive indices, we may expect to observe a distinct spectrum from the case of an undoped crystal. In this section, we present the absorption spectrum of a crystal of Nd:KGd(WO₄)₂, and make measurements of the ring radius R_0 close to the absorption peaks.

5.1.1 Absorption spectrum

Using the Beer-Lambert law [45, 94], the absorbance Abs of a material is

$$Abs = \log_{10} \frac{I_0}{I}, \quad (5.1)$$

where I is the intensity transmitted through the material, and I_0 is the intensity incident on the material. By measuring the transmitted power as a function of wavelength, full absorbance spectra were constructed for both an undoped crystal of $\text{KGd}(\text{WO}_4)_2$, and a crystal of $\text{KGd}(\text{WO}_4)_2$ doped with neodymium. These data are shown in Fig. 5.1. Both crystals have strong absorption in the ultraviolet range ($\lambda < 300$ nm), with the undoped crystal becoming almost transparent for much of the visible spectrum. The doped crystal however exhibits many strong absorption peaks, the strongest being at 582 nm and 598 nm. Other absorption peaks are labelled in Fig. 5.1.

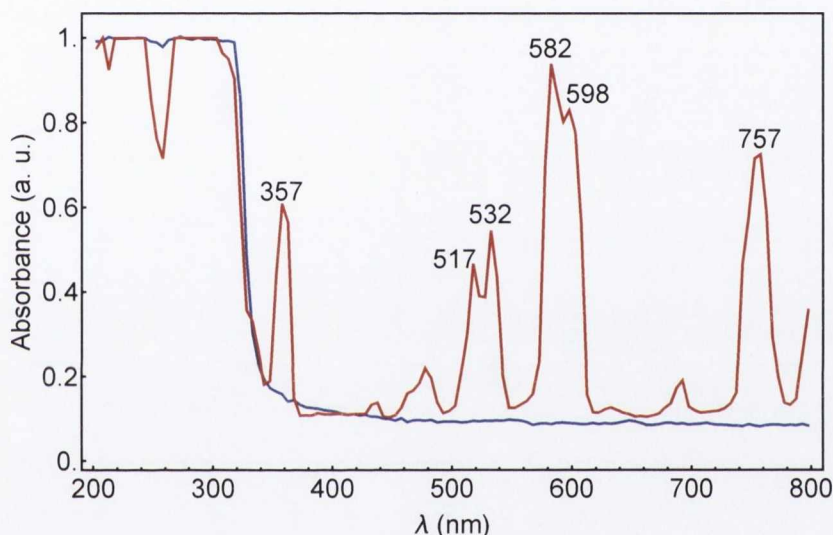


Figure 5.1: The normalised absorbance spectra of $\text{KGd}(\text{WO}_4)_2$ (blue) and $\text{Nd:KGd}(\text{WO}_4)_2$ (red) showing strong absorption peaks for the doped crystal. The peak wavelengths are indicated for some of these absorption peaks. Both crystals exhibit strong absorption in ultraviolet range, below 300 nm.

5.1.2 Variation of the radius

A change in the refractive index is expected close to the absorption peaks. This may be observed as a change in the radius of the ring R_0 at the exit face of the crystal, since as shown in Section 3.4, R_0 has a sensitive dependence on the principal refractive index values. We may simplify matters by assuming any change in the refractive index in the region of an absorption peak will be

approximately equal for each of the three principal refractive index values, *i.e.* $\delta n_1 \approx \delta n_2 \approx \delta n_3 = \delta n$. With this approximation, we greatly simplify the relationship between the ring radius in the doped crystal $R_{0\text{abs}}$, and the ring radius in the undoped crystal R_0 .

Firstly, we use Eqs. (1.20) and (1.23) to find an approximation for the ring radius R_0 in terms of the principal refractive indices n_i :

$$\begin{aligned} R_0 &\approx \frac{l}{2} \sqrt{\frac{(\epsilon_3 - \epsilon_2)(\epsilon_2 - \epsilon_1)}{\epsilon_1 \epsilon_3}} = \frac{l}{2} \sqrt{\frac{(n_3^2 - n_2^2)(n_2^2 - n_1^2)}{n_1^2 n_3^2}} \\ &= \frac{l}{2} \sqrt{\frac{(n_3 - n_2)(n_2 - n_1)(n_3 + n_2)(n_2 + n_1)}{n_1^2 n_3^2}}. \end{aligned} \quad (5.2)$$

Using the approximations $n_1 n_3 \approx n_2^2$ and $(n_1 + n_2)(n_2 + n_3) \approx 4n_2^2$, which hold for most biaxial materials, we have

$$\begin{aligned} R_0 &\approx \frac{l}{2} \sqrt{\frac{(n_3 - n_2)(n_2 - n_1)4n_2^2}{n_2^4}} \\ &= \frac{l}{n_2} \sqrt{(n_3 - n_2)(n_2 - n_1)}. \end{aligned} \quad (5.3)$$

Using this approximation, we may now find the ring radius observed in a doped crystal $R_{0\text{abs}}$ in terms of the principal refractive indices n_i and the refractive index change around the absorption peaks δn :

$$\begin{aligned} R_{0\text{abs}} &\approx \frac{l}{n_2 + \delta n} \sqrt{[(n_3 + \delta n) - (n_2 + \delta n)][(n_2 + \delta n) - (n_1 + \delta n)]} \\ &= \frac{n_2}{n_2 + \delta n} R_0, \end{aligned} \quad (5.4)$$

where R_0 is given by Eq. (5.3). Thus, by examining the variation of the ring radius around the absorption peaks, we may infer the variation of n_2 .

The spectrum of ring radius values was obtained using a monochromator and the same method outlined in Section 3.4. The output wavelength of the monochromator was varied in 10 nm increments, except close to the absorption peaks where this interval was reduced. Images of the ring at the FIP were taken for each wavelength and analysed using Mathematica to

extract the radius values. These data are presented in Fig. 5.2 as red dots, where the red lines represent the error range in the measurements. These errors primarily arise due to the resolution of the imaging device.

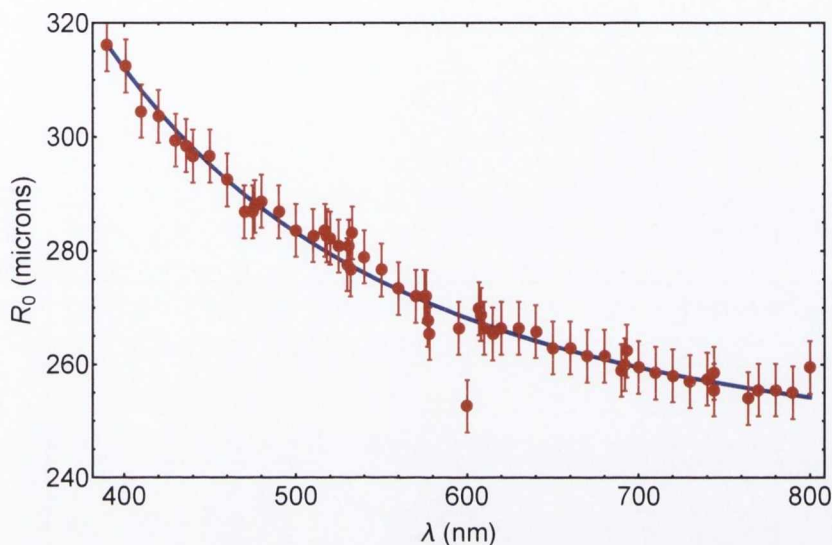


Figure 5.2: Experimental data for the radius values (red dots) at discrete wavelengths in a crystal of $\text{Nd:KGd(WO}_4)_2$. Most measurements were taken at 10 nm intervals, although close to absorption peaks this interval was reduced. The blue line represents the theoretical radius values for an undoped crystal.

Almost every data point is observed to sit on the theoretical radius curve (solid blue line), calculated using the data provided by Pujol [70]. The radius value at 598 nm is below the predicted value by an amount outside of the experimental error. This point corresponds to one of the strongest absorption peaks seen in Fig. 5.1. From the recorded ring radius values, the spectrum n_2 may be inferred using Eq. (5.4). This is shown in Fig. 5.3 where it is compared to the theoretical n_2 spectrum for an undoped crystal.

The variation of n_2 with wavelength in the doped crystal was then inferred using Eq. (5.4). As seen in Fig. 5.3, n_2 varies rapidly close to some of the absorption peaks shown in Fig. 5.1, in particular around 582 nm and 598 nm. The variation of n_2 close to 517 nm and 757 nm is indistinguishable from experimental error, although n_2 does vary rapidly between 517 nm and 532 nm.

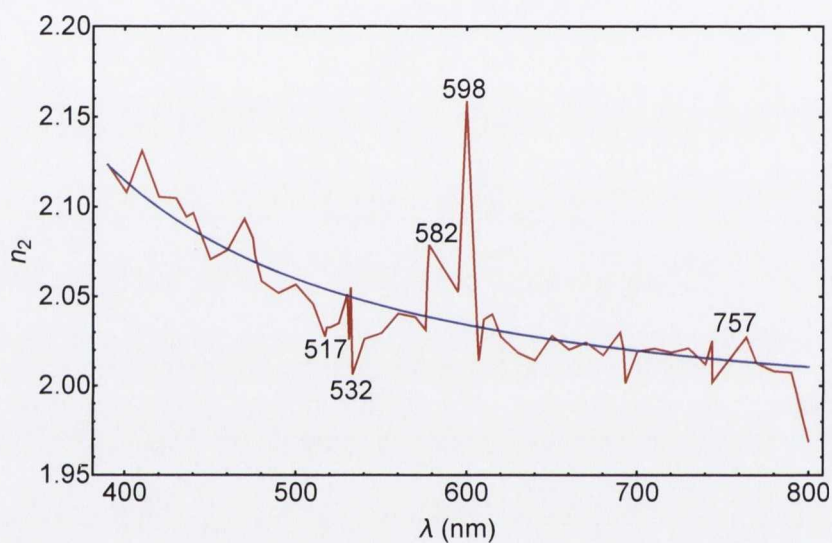


Figure 5.3: A plot of the theoretical n_2 spectrum (blue) of an undoped crystal, calculated from the data given by Pujol [70], and the n_2 spectrum for a doped crystal inferred from the radius values recorded, using Eq. (5.4). The wavelengths labelled are those which correspond to absorption peaks, as seen in Fig. 5.1.

5.2 Conclusions

In Chapter 1, we presented an historical overview of conical *refraction*, and the development of the theoretical models over the decades which resulted in conical *diffraction*. A recapitulation of the theory developed by Berry, which itself was a reformulation of the Bel'skii-Khapalyuk theory, was presented and the consequences of the theory were considered and discussed. Theoretical profiles of the beam were generated and compared.

In Chapter 2, we discussed the profile of a conically diffracted top-hat beam. The interesting intensity distribution at the focal image plane, and perhaps the even more interesting evolution of the beam were simulated. An experiment was designed to observe the features of the beam in high quality. It was found that, as expected, the intensity profile at the FIP was somewhat spread out, although it still exhibited a high-intensity, narrow-width interior ring and a wedge profile with a steep drop-off in intensity. The high frequency oscillations in the intensity at the centre of the beam were also observed, although there was a limit to how many appeared, which depended on the experimental apparatus.

In Chapter 3, we considered the theory presented by Berry which incorporated misalignment of the incident beam with the optic axis direction and derived an explicit wavelength dependence. This allowed us to consider dispersion in conical diffraction for a polychromatic light source, and we performed the first experiments using such a source since Lloyd's original observation using sunlight through a pinhole. Utilising the explicit wavelength dependent theory, it was possible to devise an experiment in which dispersion in the crystal was compensated.

In Chapter 4, the consequences of the misalignment theory were fully investigated. Berry predicted that interference effects should be observed in beams which have been misaligned with the optic axis, arising through re-expressing the beam as the sum of two geometric rays. We investigated these effects using an experimental apparatus which allowed high magnification of the beam. A high-dynamic range imaging method was utilised in order to obtain an image with a very large range of intensity values. Generating

a logarithmic intensity profile from these images confirmed the presence of these low intensity regions for the first time, and their locations matched very well with the theoretical predictions. This qualitatively new phenomenon represents the latest addition in the understanding of conical diffraction, building on those observations by Poggendorff, Raman, and Warnick and Arnold.

5.3 Future work

Over the course of my research we envisaged many experiments which, due to time constraints, were never realised. Some of these ideas are briefly outlined in this section.

Can a white light Bessel beam be generated using the method outlined in Section 3.8? A number of experimental issues would have to be overcome, for example, how could dispersion be overcome at all locations in the beam, not just a single plane at a particular ζ value? Even if the beam spatially overlapped for all ζ and λ , the beam will evolve at a difference rate depending on λ , since ζ has a dependence on n_2 .

Non-linear effects in doped biaxial crystals may give rise to interesting beam evolution patterns. It is worth studying the refractive index variation in more detail; either through the use of Kramers-Kronig relations which relate absorption to refractive index value [95, 96], or by measuring n_1 , n_2 , and n_3 directly. Incorporation of fluorescence into the theoretical model may then be more accurately achieved.

A topic which we have not discussed in this thesis, but which merits further investigation, is that of *external* conical diffraction. It occurs when an incident cone of rays with the correct semi-angle, and aligned along the correct direction, propagates as a single ray inside the crystal, refracting into an expanding cone upon exiting the crystal. In a certain sense, perhaps it cannot be distinguished as a separate phenomenon, since an incident cone of rays may be described by a superposition of single rays inclined at varying angles to the optic axis, for which we have a complete theory in Chapter 4.

Although mentioned in several publications [45, 55], no complete theoret-

ical model describing the intensity distribution either inside or outside the crystal has been developed. Furthermore, very few experimental images of the effect have been published, although some were reported by Fève *et al.* [97]. This related but mostly overlooked phenomenon may prove useful for examining non-linear effects in biaxial crystals, for example when using very high intensity lasers.

Appendices

A Method of stationary phase

Consider a function

$$f = \int_{-\infty}^{\infty} d\kappa F(\kappa) \exp[-i\Phi(\kappa)] \quad (\text{A.1})$$

where $\Phi(\kappa)$ is a rapidly varying function of κ , and $F(\kappa)$ is slowly varying by comparison. The only significant contribution from the rapidly varying exponent to the integral occurs in the region of stationary phase κ_0 , where

$$\frac{d\Phi}{d\kappa}(\kappa_0) = 0. \quad (\text{A.2})$$

Since $F(\kappa)$ is slowly varying, we can approximate $F(\kappa) \approx F(\kappa_0)$ and take it outside of the integral. The phase term may be expanded using Taylor series about κ_0 , and keeping terms to second order gives

$$\Phi(\kappa) \approx \Phi(\kappa_0) + \frac{1}{2}\Phi''(\kappa_0)(\kappa - \kappa_0)^2. \quad (\text{A.3})$$

Substituting this into Eq. (A.1) gives

$$\begin{aligned} f &\approx F(\kappa_0) \exp[-i\Phi(\kappa_0)] \int_{-\infty}^{\infty} d\kappa \exp[-i\Phi''(\kappa_0)\frac{1}{2}(\kappa - \kappa_0)^2] \\ &\approx |F(\kappa_0)| \sqrt{\frac{2\pi}{i\Phi''(\kappa_0)}} \exp[-i\Phi(\kappa_0)]. \end{aligned} \quad (\text{A.4})$$

B Misaligned beam intensity

The intensity of a misaligned beam is given by

$$I = \mathbf{D}^* \cdot \mathbf{D}, \quad (\text{B.1})$$

with $\mathbf{D}$ given by Eq. (4.7). Evaluating this equation, and after some rearrangement, we get

$$\begin{aligned} I = & |B_0|^2 + |B_1|^2 \left(|\cos \tilde{\phi}|^2 + |\sin \tilde{\phi}|^2 \right) \\ & + 2(|d_{0x}|^2 - |d_{0y}|^2) \text{Re} \left[B_0^* B_1 \cos \tilde{\phi} \right] \\ & + 4 \text{Re} [d_{0x}^* d_{0y}] \text{Re} \left[B_0^* B_1 \sin \tilde{\phi} \right] \\ & - 4|B_1|^2 \text{Im} \left[\cos \tilde{\phi}^* \sin \tilde{\phi} \right] \text{Im} [d_{0x}^* d_{0y}], \end{aligned} \quad (\text{B.2})$$

assuming a normalised polarisation vector, *i.e.* $|d_{0x}|^2 + |d_{0y}|^2 = 1$. For a circularly polarised incident beam, we can express the polarisation vectors as

$$\mathbf{d}_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \end{pmatrix}, \quad (\text{B.3})$$

and so the intensity becomes

$$\begin{aligned} I_{\text{circ}} = & |B_0|^2 + |B_1|^2 \left(|\cos \tilde{\phi}|^2 + |\sin \tilde{\phi}|^2 \right) \\ & \mp 2|B_1|^2 \text{Im} \left[\cos \tilde{\phi}^* \sin \tilde{\phi} \right]. \end{aligned} \quad (\text{B.4})$$

For a linearly polarised beam whose polarisation is inclined at an angle χ to the ξ -axis,

$$\mathbf{d}_0 = \begin{pmatrix} \cos \chi \\ \sin \chi \end{pmatrix}, \quad (\text{B.5})$$

and so the intensity beyond the crystal is

$$\begin{aligned} I_{\text{lin}} = & |B_0|^2 + |B_1|^2 \left(|\cos \tilde{\phi}|^2 + |\sin \tilde{\phi}|^2 \right) \\ & + 2 \text{Re} \left[B_0^* B_1 \cos \left(\tilde{\phi} - 2\chi \right) \right]. \end{aligned} \quad (\text{B.6})$$

To find an expression for an unpolarised incident beam, we simply take the average of two orthogonally polarised incident beams, for example using Eq. (B.6),

$$\begin{aligned} I_{\text{unpol}} &= \frac{1}{2} (I_{\text{pol}}(\chi = 0) + I_{\text{pol}}(\chi = \pi/2)) \\ &= |B_0|^2 + |B_1|^2 \left(|\cos \tilde{\phi}|^2 + |\sin \tilde{\phi}|^2 \right). \end{aligned} \quad (\text{B.7})$$

This allows us to express the intensity beyond the crystal for a circularly polarised incident beam as

$$I_{\text{circ}} = I_{\text{unpol}} \mp 2|B_1|^2 \text{Im} \left[\cos \tilde{\phi}^* \sin \tilde{\phi} \right], \quad (\text{B.8})$$

and for a linearly polarised incident beam as

$$I_{\text{lin}} = I_{\text{unpol}} + 2\text{Re} \left[B_0^* B_1 \cos \left(\tilde{\phi} - 2\chi \right) \right]. \quad (\text{B.9})$$

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