

One-loop correction to the energy of spinning strings in S^5

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We revisit the computation of the 1-loop $\text{AdS}_5 \times S^5$ superstring sigma model correction to the energy of a closed circular string rotating in S^5 . The string is spinning around its center of mass with two equal angular momenta $J_2 = J_3$ and its center of mass angular momentum is J_1 . We revise the argument that the 1-loop correction is suppressed by $\frac{1}{J}$ factor [$J = J_1 + 2J_2$ is the total $\text{SO}(6)$ spin] relative to the classical $\frac{\lambda J_2}{J^2}$ term in the energy and use numerical methods to compute the leading 1-loop coefficient. The corresponding gauge-theory result is known only in the $J_1 = 0$ limit when the string solution becomes unstable and thus the 1-loop shift of the energy formally contains an imaginary part. While the comparison with gauge-theory may not be well defined in this case, our numerical string-theory value of the 1-loop coefficient seems to disagree with the gauge-theory one. A plausible explanation should be in the different order of limits taken on the gauge-theory and the string-theory sides of the anti-de Sitter/conformal field theory duality.

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I. INTRODUCTION

Recently, there was an interesting progress in understanding anti-de Sitter (AdS)/conformal field theories (CFT) duality by extending the Berenstein-Maldacena-Nastase (BMN) approach [1] to other sectors of semiclassical [2] string states (see [3] for reviews and references).

In general, for a classical rotating closed string solution in S^5 its energy has a regular expansion [4–8] $E_0 = J + c_1 \frac{\lambda}{J} + c_2 \frac{\lambda^2}{J^2} + \dots = J(1 + c_1 \tilde{\lambda} + c_2 \tilde{\lambda}^2 + \dots)$, where J is the total $\text{SO}(6)$ spin $J = \sum_{i=1}^3 J_i$ and $\tilde{\lambda} \equiv \frac{\lambda}{J^2}$ is an effective semiclassical expansion parameter. $c_n = c_n(\frac{J_i}{J})$ are functions of ratios of the spins which are finite in the semiclassical string-theory limit $J_i \gg 1$, $\tilde{\lambda}$ is fixed. Generic 3-spin solutions are described by an integrable Neumann model [7,8] and the coefficients c_n are expressed in terms of genus-two hyperelliptic functions.

Formally, string α' corrections are suppressed in the limit $J \rightarrow \infty$, $\tilde{\lambda}$ is fixed since $\alpha' = \frac{R^2}{\sqrt{\lambda}} \sim \frac{1}{J\sqrt{\lambda}}$. However, to expect [4] to be able to compare these classical energies to the super Yang-Mills (SYM) anomalous dimensions [9–11] one should check that the $\frac{1}{J}$ corrections are again analytic in $\tilde{\lambda}$ (as they are in the BMN case [12–14]), i.e., the expansion in large J and small $\tilde{\lambda}$ is well defined on the string side,

$$E = J \left[1 + \tilde{\lambda} \left(c_1 + \frac{d_1}{J} + \dots \right) + \tilde{\lambda}^2 \left(c_2 + \frac{d_2}{J} + \dots \right) + \dots \right],$$

$$\tilde{\lambda} \equiv \frac{\lambda}{J^2}, \quad (1)$$

with the classical energy being the $J \rightarrow \infty$ limit of the exact expression.

This question was first addressed in [5] on the example of the simplest stable 3-spin solution of [4]: a circular string orbiting in S^5 with center of mass angular momentum J_1 and two equal $\text{SO}(6)$ angular momenta $J_2 = J_3$ in the two other orthogonal planes. In addition to the values of J_1, J_2 this solution is parameterized by an integer k (winding number).¹ At the classical (but not, in general, at the quantum) level the dependence on k can be absorbed into the string tension $\sqrt{\lambda}$. Expanding E one finds the explicit form of c_1 in (1) [4,5]

$$E_0 = J + \frac{\lambda k^2 J_2}{J^2} + \dots = J \left(1 + \tilde{\lambda} k^2 \frac{J_2}{J} + \dots \right). \quad (2)$$

The main reason to consider this solution (which is a special case of a more general class of similar “rational” circular 3-spin solutions found in [8]) is its simple analytic

¹Here we change the notation compared to [5]: there we had $J_1 = J$, $J_2 = J_3 = J'$. Below J will stand for the total angular momentum $J = J_1 + 2J_2$. Written in terms of the AdS_5 time coordinate t and the angles of S^5 (with the metric $(ds^2)_{S^5} = d\gamma^2 + \cos^2 \gamma d\varphi_1^2 + \sin^2 \gamma (d\psi^2 + \cos^2 \psi d\varphi_2^2 + \sin^2 \psi d\varphi_3^2)$) the solution is [4,5] (see also Appendix A): $t = \kappa \tau$, $\gamma = \gamma_0$, $\varphi_1 = \nu \tau$, $\varphi_2 = \varphi_3 = w \tau$, $\psi = k \sigma$, where κ, γ_0, ν, w are constants, k is an integer and $w^2 = \nu^2 + k^2$, $\nu^2 = \kappa^2 - 2k^2 q$, $\nu^2 = \kappa^2 - 2k^2 q$, $q \equiv \sin^2 \gamma_0$. The three independent parameters are κ, q , and k . The nonzero $\text{SO}(6)$ spin components are $J_1 = \sqrt{\lambda} \nu (1 - q)$, $J_2 = J_3 = \frac{1}{2} \sqrt{\lambda} \sqrt{\nu^2 + k^2} q$. The classical energy $E = \sqrt{\lambda} \kappa$ can then be represented as a function of the spins $E = E(J_2, J, k; \lambda)$.

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form implying that the corresponding quadratic fluctuation action has *constant* coefficients (as in the BMN case corresponding to the limit $J_2 = J_3 = 0$). As a result, the fluctuation frequencies which determine the 1-loop correction to the energy (conjugate to the AdS_5 time $t = \kappa\tau$)

$$E_1 = \frac{1}{2\kappa} \left(\sum_{n \in \mathbb{Z}} \omega_n^B - \sum_{r \in \mathbb{Z} + 1/2} \omega_r^F \right) \quad (3)$$

can be readily found. Still, ω 's are given [5] by the roots of *4th-order polynomials* (see Appendix A) and thus are rather involved functions of J_2, J and k , making it difficult to compute the infinite sums in (3). Attempting to evaluate (3) analytically, in [5] the sums were converted into integrals, but it turns out that this direct procedure fails due to a singularity of the functions involved.

In this paper we shall first improve the general argument in [5] about the form (1) of the expansion of E_1 at large J and small $\tilde{\lambda}$ and then use numerical methods to evaluate the first subleading coefficient d_1 .

A motivation behind this work is to compare the 1-loop string correction to the corresponding $\frac{1}{J}$ correction in the anomalous dimensions of the corresponding SYM operators $\text{tr}(\Phi_1^{J_1} \Phi_2^{J_2} \Phi_3^{J_3}) + \dots$. On the gauge-theory side, one first expands in λ and then expands in $\frac{1}{J}$, so that the anomalous dimensions should have the structure

$$\Delta = J + \lambda \left(\frac{a_1}{J} + \frac{b_1}{J^2} + \dots \right) + \lambda^2 \left(\frac{a_2}{J^3} + \frac{b_2}{J^4} + \dots \right) + \dots \quad (4)$$

The form of this expansion in the 2-spin [$\text{SU}(2)$] sector was indeed verified to first few leading orders in [10,15]. Moreover, it was checked on specific examples [10,16–18] and also in general [19–21] that the expressions for a_1 and a_2 match the coefficients c_1, c_2 in E (1). Similar conclusion ($a_1 = c_1$) was reached in the $\text{SU}(3)$ sector [11,22–24] (fluctuations near the circular 3-spin solution of [4] also match [25]).

However, it was observed in [18,26] that disagreements start at λ^3 order, $a_3 \neq c_3$, with a plausible (“order-of-limits”) explanation suggested in [18,27,28]. For that reason, it would be interesting to see if the $\frac{1}{J}$ subleading coefficient b_1 in (4) agrees with the 1-loop coefficient d_1 in (1). So far, b_1 was computed [15] only for a specific 2-spin Bethe ansatz state corresponding to an unstable state on the string-theory side for which d_1 formally has an imaginary part. In that case, *a priori* the comparison does not seem to be well defined. Apart from clarifying this issue, it remains to compute b_1 for the 3-spin state with $J_1 \neq 0$, extending the Bethe ansatz analysis of [11] where a_1 was found. Once this is done, one will be in position to compare to the results for d_1 on the string side presented below.

An attempt of comparison of our numerical result for d_1 in (1) for the 2-spin (unstable) case with the gauge-theory result of [15] for b_1 indicates a disagreement (see Sec. III). We suspect that the disagreement may remain also in the

stable 3-spin case. This seems also to suggest that a similar “1-loop” (order λ) disagreement may be present for the $\frac{1}{J}$ correction to scaling dimensions of BMN operators. An explanation of these disagreements may be again related to the noncommutativity [18,27] of the “string-theory” (large J , then small $\tilde{\lambda}$) and the “gauge-theory” (small λ , then large J) limits.

II. STRUCTURE OF ONE-LOOP CORRECTION

In [5] it was attempted to find the one-loop correction E_1 in (3) as an expansion in $\frac{1}{\kappa^2} = \tilde{\lambda} + \dots$, i.e.,

$$\begin{aligned} E_1 &= \frac{1}{\kappa^2} e_1(q, k) + \frac{1}{\kappa^4} e_2(q, k) + \dots \\ &= \tilde{\lambda} d_1\left(\frac{J_2}{J}, k\right) + \tilde{\lambda}^2 d_2\left(\frac{J_2}{J}, k\right) + \dots, \end{aligned} \quad (5)$$

and the expression for the leading-order coefficient e_1 was presented. We used that [5]

$$\frac{1}{\kappa^2} = \tilde{\lambda} - \tilde{\lambda}^2 \frac{2k^2 J_2}{J} + \dots, \quad q \equiv \sin^2 \gamma_0 = \frac{2J_2}{J} + \dots, \quad (6)$$

implying

$$d_1 = e_1, \quad d_2 = e_2 - k^2 q e_1. \quad (7)$$

However, later analysis revealed that the functions that appear at higher orders have unexpected irregularities, so that the method of [5] needs a modification. Here we shall briefly discuss the nature of this modification (which turns out to be rather involved, prohibiting a direct analytic computation) and then turn to numerical methods to evaluate E_1 .

It was noticed in [5] that the bosonic and fermionic frequencies (see Appendix A below) admit the following large κ expansion (with $\frac{n}{\kappa}$ and $\frac{r}{\kappa}$ kept fixed)

$$\omega_n^B = \kappa \alpha_n^B \left(\frac{n}{\kappa} \right) + \frac{1}{\kappa} \alpha_1^B \left(\frac{n}{\kappa} \right) + \frac{1}{\kappa^3} \alpha_3^B \left(\frac{n}{\kappa} \right) + \dots, \quad (8)$$

$$\omega_r^F = \kappa \alpha_r^F \left(\frac{r}{\kappa} \right) + \frac{1}{\kappa} \alpha_1^F \left(\frac{r}{\kappa} \right) + \frac{1}{\kappa^3} \alpha_3^F \left(\frac{r}{\kappa} \right) + \dots. \quad (9)$$

One can think of $\alpha_a(\frac{m}{\kappa})$ as the values of functions $\alpha_a(x)$ at points $x_m = \frac{m}{\kappa}$. It was implicitly assumed in [5] that all $\alpha_a(x)$'s are regular. In that case one could replace the bosonic and fermionic series in (3) by integrals, and then $\alpha_{2a+1}(\frac{m}{\kappa})$ with $a \geq 1$ would not contribute to the order $\frac{1}{\kappa^2}$ in the large κ expansion. However, it turns out that $\alpha_a^B(x)$ with $a \geq 3$ in general have singularities (see (B6) and comments below it in Appendix B). A more careful analysis of α_{2a+1}^B shows that at small values of x they behave as $\frac{1}{x^{2a}} + \mathcal{O}(\frac{1}{x^{2a-2}})$. For this reason the analysis of the large κ -expansion in [5] needs to be modified. One must first subtract from $\alpha_a(x)$ the singular contributions and after that convert the series into integrals. The singular contri-

butions cannot be represented by integrals and have to be computed directly. However, then all the terms of the order $\frac{1}{x^{2a}}$ in α_{2a+1}^B with $a \geq 1$ contribute to $e_1(q, k)$ in (5). For this reason, obtaining the complete answer for the coefficient $e_1(q, k)$ (and, in general, for higher order coefficients e_p) along these lines would be hard in practice.

In the $\kappa \rightarrow \infty$ limit the one-loop energy correction must go to zero because this strict limit is essentially like a BPS limit—the bosonic and fermionic contributions should then cancel against each other due to supersymmetry.² This implies that only negative powers of κ can appear in the large κ -expansion of E_1 . Indeed, examining the functions $\alpha_a(x)$ with $a > 1$ one can show that the one-loop correction does have the large κ expansion as given in (5).

To prepare the ground for a numerical evaluation of e_1 and e_2 in (5) let us first discuss the convergence of the 1-loop correction (3) (which is expected due to the conformal invariance of the underlying $\text{AdS}_5 \times S^5$ string sigma model [30] and can be demonstrated for a generic string solution, see [5]). Each of the two sums—over the bosonic and the fermionic frequencies—is divergent, and so one should regularize them first. Let us use the standard “supersymmetry preserving” regularization ($\epsilon \rightarrow 0$)

$$E_1 = \frac{1}{2\kappa} \left(\sum_{n \in \mathbb{Z}} e^{-\epsilon|n|} \omega_n^B - \sum_{r \in \mathbb{Z}+1/2} e^{-\epsilon|r|} \omega_r^F \right). \quad (10)$$

Here w_n^B (w_r^F) is the sum of eight bosonic (fermionic) frequencies at each level and as in (3) ω 's stand for their moduli, i.e., $+\sqrt{\omega^2}$. One can then rearrange these sums as

$$E_1 = \frac{1}{2\kappa} \sum_{n \in \mathbb{Z}} \left(e^{-\epsilon|n|} \omega_n^B - \frac{1}{2} e^{-\epsilon|n-1/2|} \omega_{n-1/2}^F - \frac{1}{2} e^{-\epsilon|n+1/2|} \omega_{n+1/2}^F \right). \quad (11)$$

This can be further rewritten in a form which is more suitable for taking the $\epsilon \rightarrow 0$ limit

$$E_1 = \frac{1}{2\kappa} \left[\sum_{n \in \mathbb{Z}} e^{-\epsilon|n|} \left(\omega_n^B - \frac{1}{2} \omega_{n-1/2}^F - \frac{1}{2} \omega_{n+1/2}^F \right) + \frac{1}{2} \sum_{n \in \mathbb{Z}} (e^{-\epsilon|n|} - e^{-\epsilon|n-1/2|}) \omega_{n-1/2}^F + \frac{1}{2} \sum_{n \in \mathbb{Z}} (e^{-\epsilon|n|} - e^{-\epsilon|n+1/2|}) \omega_{n+1/2}^F \right]. \quad (12)$$

A nice feature of (12) is that the series on the first line is convergent even for $\epsilon = 0$ because at large $|n|$ (see Appendices A and B of [5])

²A heuristic reason is that in the strict limit $\kappa \rightarrow \infty$ the world surface of the string becomes a collection of BMN geodesics [29] with contribution of tension between different string bits effectively suppressed (see also [12] for a related argument).

$$\omega_n^B - \frac{1}{2} \omega_{n-1/2}^F - \frac{1}{2} \omega_{n+1/2}^F \sim \frac{1}{|n|^3}. \quad (13)$$

On the other hand, the series in the second line of (12) can be easily computed in the limit $\epsilon \rightarrow 0$. First, we note that the fermionic frequencies ω_r^F are even under $r \rightarrow -r$, and rewrite the second line of (12) as

$$\frac{1}{2\kappa} (e^{-\epsilon/2} + e^{+\epsilon/2} - 2) \sum_{r>0} e^{-\epsilon r} \omega_r^F. \quad (14)$$

Using the large r expansion in [5]

$$\omega_r^F = 8r + 4 \frac{\kappa^2 - k^2 q}{r} + \mathcal{O}\left(\frac{1}{r^3}\right),$$

we find that only the first term, $8r$, contributes in the limit $\epsilon \rightarrow 0$:

$$\lim_{\epsilon \rightarrow 0} (e^{-\epsilon/2} + e^{+\epsilon/2} - 2) \sum_{r>0} e^{-\epsilon r} \omega_r^F = 2. \quad (15)$$

Thus, the one-loop sigma model correction to the classical energy can be represented by the following convergent sum³ of the combination $\omega_n^B - \frac{1}{2} \omega_{n-1/2}^F - \frac{1}{2} \omega_{n+1/2}^F$

$$E_1 = \frac{1}{2\kappa} \left[2 + \sum_{n \in \mathbb{Z}} \left(\omega_n^B - \frac{1}{2} \omega_{n-1/2}^F - \frac{1}{2} \omega_{n+1/2}^F \right) \right]. \quad (16)$$

It is useful also to single out the contribution of the $n = 0$ term. Then

$$E_1 = \frac{1}{\kappa} \left[1 + \frac{1}{2} (\omega_0^B - \omega_{1/2}^F) + \sum_{n=1}^N \left(\omega_n^B - \frac{1}{2} \omega_{n-1/2}^F - \frac{1}{2} \omega_{n+1/2}^F \right) \right], \quad (17)$$

where we have used the symmetry of the summand under $n \rightarrow -n$. Having in mind a numerical computation of E_1 we have also introduced the upper limit $N \rightarrow \infty$. The results discussed below were obtained for $N = 40000$.

III. RESULTS OF NUMERICAL EVALUATION

Fixing the values of the parameters (κ, q, k) one can, using Mathematica or Maple, numerically solve the characteristic Eqs. (A7) and (A8) and (17) for ω_n^B and ω_r^F . The solutions are then substituted into (17) to yield numerical values of

$$E_1 = E_1(\kappa, q, k). \quad (18)$$

Since, in general, E_1 has a large κ -expansion as given in (5), it is more convenient to compute not E_1 but $\kappa^2 E_1$. For large enough κ , the value of $\kappa^2 E_1$ is very close to $d_1 = e_1$, assuming e_2/κ^2 is much smaller than d_1 .

We consider $\kappa = 50, 100, 200$ for fixed values of $k = 1, 2, 4, 8$ and $q = \frac{h}{12}$, $h = 0, 1, 2, \dots, 12$ and for $k = 1$ we set $q = \frac{h}{24}$. Evaluating $\kappa^2 E_1$ numerically it is possible then

³Let us stress again that one cannot formally rearrange the sum without loosing the convergence.

TABLE I. Values of $d_1(q, k)$.

	k = 1	k = 2	k = 4	k = 8
$q = 0$	-0.50	-1.00	-2.00	-4.00
$q = \frac{1}{12}$	-0.34	0.26	7.9	74
$q = \frac{2}{12}$	-0.20	1.33	16.1	133 + 5 i
$q = \frac{3}{12}$	-0.07	2.17	21.1 + 1.0 i	172 + 13 i
$q = \frac{4}{12}$	0.03	2.77	24.0 + 2.7 i	201 + 22 i
$q = \frac{5}{12}$	0.12	2.96	24.1 + 3.8 i	206 + 41 i
$q = \frac{6}{12}$	0.17	2.47 + 0.77 i	22.3 + 8.0 i	192 + 67 i
$q = \frac{7}{12}$	0.18	1.93 + 1.27 i	18.6 + 11.0 i	152 + 101 i
$q = \frac{8}{12}$	0.13	0.86 + 1.73 i	8.7 + 18.1 i	78 + 148 i
$q = \frac{9}{12}$	-0.23	-1.77 + 3.21 i	-7.7 + 31.0 i	-42 + 275 i
$q = \frac{10}{12}$	-0.28 + 0.50 i	-1.49 + 6.47 i	-8.6 + 57.5 i	-52 + 470 i
$q = \frac{11}{12}$	-0.35 + 0.71 i	-1.31 + 8.06 i	-5.1 + 70.0 i	-24 + 575 i
$q = \frac{12}{12}$	-0.45 + 0.87 i	-1.22 + 9.37 i	-3.4 + 81.7 i	-9 + 673 i

to estimate d_2 to be sure it is small enough, and plot the functions d_1 . For example, we find: $50^2 E_1(50, \frac{1}{12}, 2) = 0.2626$, $100^2 E_1(100, \frac{1}{12}, 2) = 0.2627$, $200^2 E_1(200, \frac{1}{12}, 2) = 0.2648$. Using the Mathematica Fit function yields the following κ -dependence,

$$\kappa^2 E_1\left(\kappa, \frac{1}{12}, 2\right) = 0.2641 - 4.1752 \frac{1}{\kappa^2}. \quad (19)$$

We see that d_2 is of order 1, and, therefore, d_2/κ^2 is much smaller than the value of $d_1 = e_1$ which for this case is $d_1 \approx 0.26$. The values of N and κ that we have used for the computation do not allow us to find d_2 reliably because we neglected the “tail” contribution ($\sum_{n=N+1}^{\infty}$) in (16). It is shown in Appendix B that the tail contribution to $\kappa^2 E_1$ is of order κ^3/N^2 . Because of that one cannot make κ too large. For $N = 400\,00$ and $\kappa = 200$ one has $\kappa^3/N^2 = 0.005$, and, therefore, our computation is accurate at least to 0.01. This procedure can be repeated for the other values of $q \approx \frac{2j}{j}$ and k . The resulting data is shown in Table I and is used to plot the q -dependence of d_1 in Fig. 1–8.

It is important to note that the circular string solution in question is stable, i.e., the frequencies ω_n^B and thus E_1 are real, only in the following range of values of q (for fixed k): $q \leq q_*$, where $q_* = 1 - (1 - \frac{1}{2k})^2$ (see [4,5] and Appendix A).⁴ While the plots are valid only in the “stable” regions of q , we have interpolated them to all values of $q \leq 1$ by simply dropping the imaginary parts.

Let us first discuss the q -dependence of d_1 for $k = 1$. The plot of d_1 is shown in Fig. 1. As discussed above the solution is stable for $q \leq 0.75$. One can see from the plot that the curve has a corner at $q = 0.75$. This is a general property for all values of k —the energy is not a differ-

entiable function of q at the edge of the stability region. An interesting feature of the graph is that it crosses the q -axis twice, at $q \approx 0.31$ and $q \approx 0.72$, i.e., for these values of q the coefficient d_1 vanishes.⁵

The plot of d_1 for $k = 2$ is shown in Fig. 2. In this case the solution is stable for $q \leq 0.4375$. One can see that the curve crosses the q -axis only once in the stability region. To see that the energy is not differentiable in this case either, it is useful to plot d_2 , see Fig. 3. Even though, as discussed above, the values of d_2 are not reliable, one can clearly see that d_2 is not smooth at the edge of the stability region.

The plots of d_1 for $k = 4$ and $k = 8$ are shown in Fig. 4 and 5. The solution is stable for $q \leq 0.2344$ and $q \leq 0.1211$, respectively. The plots of d_1 for all values of k have similar shapes. In particular, d_1 vanishes for at least one value $q = q_*$ in the stability region. This value q_* depends on k , suggesting that one should not expect to find a simple dependence of d_1 on k for fixed q .

Indeed, while the leading correction in the classical energy E_0 (2) scales with k as k^2 , there is no *a priori* reason why the leading coefficient in E_1 should also have a simple dependence on k . The dependence of frequencies ω_n^B and ω_r^F on k is such that it can be eliminated by rescaling ω 's, κ , and n, r by k [5] (see also Appendix A), but since this transformation involves a rescaling of summation indices, the resulting E_1 (3) should, in general, be a nontrivial function of k . The dependence of d_1 on k becomes nontrivial already for $q = \frac{1}{12}$ (see Fig. 6). (In the formal case of $q = 0$ it is linear ($d_1(0, k) = -\frac{1}{2}k$) as

⁴From numerical data in Table I we find, in agreement with the analytic expression $q_* = 1 - (1 - \frac{1}{2k})^2$, that E_1 becomes complex for $q > q_*$ where $18/24 < q_* < 19/24$ for $k = 1$, $5/12 < q_* < 6/12$ for $k = 2$, $2/12 < q_* < 3/12$ for $k = 4$, and $1/12 < q_* < 2/12$ for $k = 8$.

⁵Even though the coefficient d_2 does not vanish at these points, there exists a curve $F(\lambda, q) = 0$ on which the first (one-loop) sigma model correction, E_1 , vanishes. Thus, for the corresponding values of λ and q the energy of the classical circular string coincides with the exact energy of the quantum string. This may be considered as a kind of nonrenormalization theorem valid only for special λ and q (for which we do not have any obvious explanation).

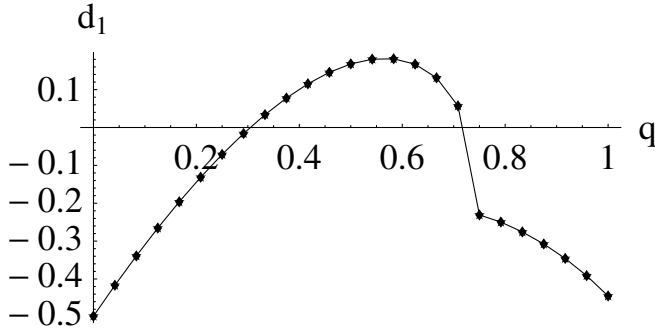
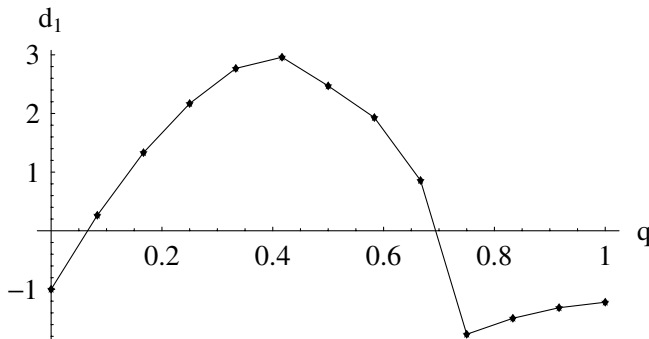
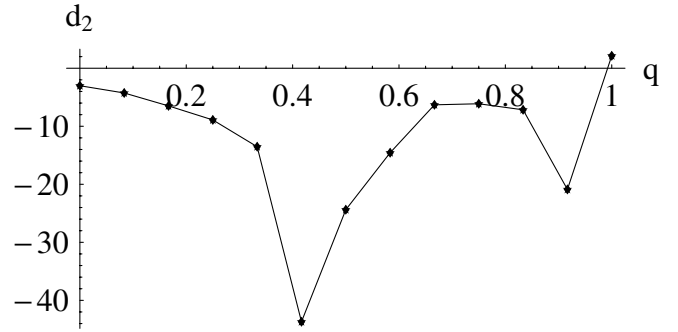
FIG. 1. q -dependence of d_1 for $k = 1$.

Table I shows.) Since $d_1(\frac{1}{12}, 1) = -0.34$ is negative and all other values of $d_1(\frac{1}{12}, k)$ are positive, the k -dependence cannot be given by a power function k^p . Rather the curve should be approximated by a polynomial of k . This example shows that in general d_1 has a complicated dependence on k . As stated in the introduction, the one-loop gauge-theory computation of the corresponding coefficient b_1 in (4) was carried out [15] only in the (unstable) $q = 1$ case with the result $b_1 = \frac{1}{2}k^2$. If that gauge-theory prediction $b_1 \sim k^2$ applies also to the stable $q < 1$ cases, then our results would indicate a disagreement between the string-theory (d_1) and the perturbative (1-loop) gauge-theory (b_1) coefficients.

Let us now consider the special case of $q = 1$, i.e., $J_1 = 0$, $J_2 = J_3 = \frac{1}{2}J$ in more detail. Here the frequencies can be found in a simple analytic form [5] and the computation of E_1 becomes more explicit (see Appendix B). The corresponding gauge-theory (4) result [15] written in the form (1) reads

$$\Delta = J \left[1 + \frac{1}{2} \tilde{\lambda} k^2 \left(1 + \frac{1}{J} + \dots \right) + \dots \right]. \quad (20)$$

Here the leading-order term agrees [10] with the classical string energy (2); in order for the $\frac{1}{J}$ term in (20) to be in agreement with the one-loop string correction in (5) one should find that $d_1(q = 1) = \frac{1}{2}k^2$. As already mentioned, an apparent problem for checking this is that the $q = 1$ solution is unstable for any $k \geq 1$ [4,5]. In the simplest

FIG. 2. q -dependence of d_1 for $k = 2$.FIG. 3. q -dependence of d_2 for $k = 2$.

case of $k = 1$ there is one imaginary bosonic frequency (for larger k there are several unstable modes). As a result, the definition and interpretation of the 1-loop correction to the energy becomes nontrivial (formally, the 1-loop correction then contains an imaginary part determining the rate of decay of the unstable state, see, e.g., [31]). In order to see if string-theory result may be put into an agreement with the gauge-theory result (20) we may try to use one of the following definitions of E_1 (for definiteness, we shall consider the case of $k = 1$):⁶

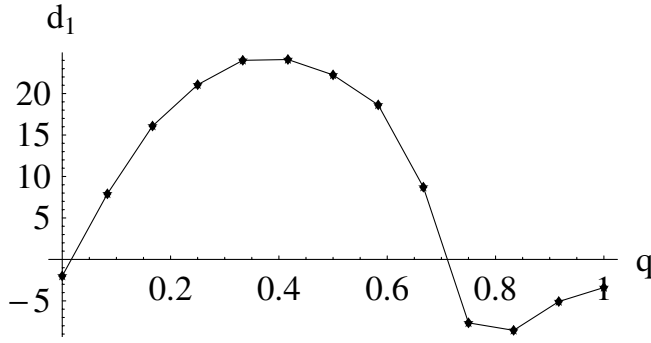
(i) compute E_1 as a sum over all frequencies as in (3) and (17), and omit the imaginary part; this amounts to ignoring the contribution of the one unstable bosonic mode with $n = 1$.

(ii) analytically continue the value of the mass of the “tachyonic” $n = 1$ mode, i.e., include the contribution of its frequency to E_1 with the modulus sign.⁷

In the case (i) we find by the numerical evaluation of the sum that $d_1 \approx -0.446$. In the case (ii) we get instead $d_1 \approx 0.42$ (the additional contribution of the modulus of the frequency of the unstable $n = 1$ mode is

⁶Formally, the unstable mode is not “seen” on the gauge-theory side [10]. More precisely, an unstable mode of the 3-spin circular solution corresponds to the configuration when a Bethe root moves off the real axis [25,26]; this case does not correspond to a true eigenstate of the Hermitian Hamiltonian of the Heisenberg ferromagnet [9]. Still, this suggests that some analytic continuation may apply. Related aspect of this problem is that the spin chain states found using the Bethe ansatz [10] are exact quantum states, while on the string-theory side we are considering semiclassical states dual to coherent states of the spin chain [19,21,32]. One can show (see also [23]) that the corresponding unstable mode is present also in the “Landau-Lifshits” sigma model which is the coherent state effective action following in the low-energy approximation from the Heisenberg spin chain Hamiltonian and which agrees [19] with a large spin limit of the string sigma model action.

⁷A possible way to support the second prescription is to view the $J_2 = J_3, J_1 = 0$ case as an analytic continuation of the stable solution with $J_2 = J_3, J_1 > \frac{2}{3}J_2$. For this stable solution the spectrum of fluctuations (and thus E_1) is real and matches with Bethe ansatz $J = \infty$ spectrum [11]. Then we may analytically continue all relations in the angular momentum plane and try to define the $J_1 \rightarrow 0$ limit.

FIG. 4. q -dependence of d_1 for $k = 4$.

$\delta d_1 = \frac{\sqrt{3}}{2} \approx 0.866$). This may look close to the 0.5 value in (20), but our estimate of the numerical error is much smaller than 0.08, so we are inclined to conclude that there is a disagreement between the gauge-theory and string-theory values for d_1 , with a plausible explanation being the same order-of-limits problem as in [27] (see also Sec. IV below).

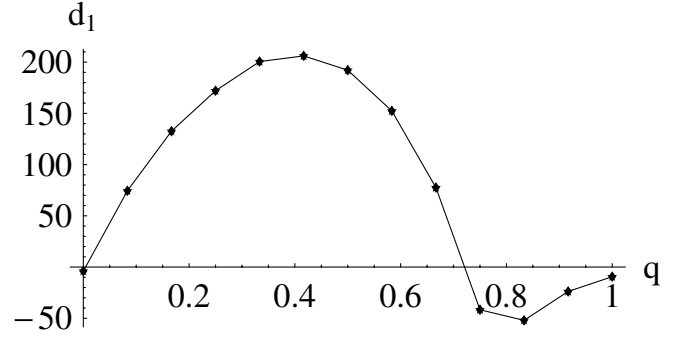
Finally, let us comment on the k -dependence of d_1 in this $q = 1$ case. Following the first prescription (i), i.e., keeping only the real part of $d_1(k)$ we get the plot in Fig. 7. It is interesting to note that the curve can be well-approximated by the power function $-0.446k^p$ with $p = 1.46$.

With the second prescription (ii), i.e., taking the sum of the real part and the imaginary part⁸ of d_1 we get the plot in Fig. 7. In this case it cannot be approximated by a power function. Comparing to (20) suggests again that there is a disagreement between the $1/J$ string-theory and gauge-theory results.

IV. CONCLUDING REMARKS

In this paper we have used numerical methods to analyze the leading 1-loop sigma model correction to the energy of the classical circular spinning string [4]. We have confirmed the expected large J expansion of the energy, and studied the dependence of the first subleading coefficient d_1 in (1) on the two parameters— $q \approx \frac{2J_2}{J}$ and “winding number” k . Comparing our results with the known gauge-theory result [15] for the corresponding spin chain state (dual to unstable 2-spin circular string with $q = 1$), we have found a discrepancy not only in the numerical value but also in the k -dependence of the leading 1-loop coefficient d_1 . Even though our computation is unambiguous and reliable only for the stable 3-spin string states with $q \leq q_* < 1$, the different k -dependence of d_1 may be viewed as an indication that there is a string/gauge-theory disagreement for the $1/J$ correction starting already at one-loop order on the gauge-theory side. We should add a reserva-

⁸All unstable modes happen to have purely-imaginary frequencies, so their modulus is equal to the absolute value of their imaginary part.

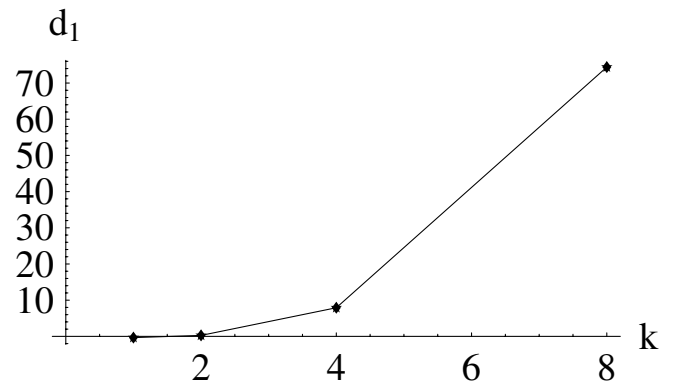
FIG. 5. q -dependence of d_1 for $k = 8$.

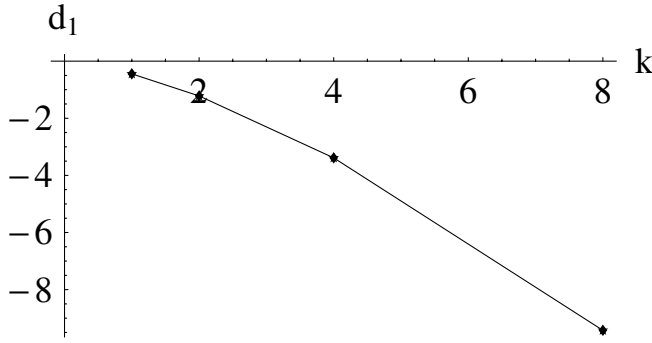
tion that it is still possible that the disagreement we find is due to the fact that the semiclassical quantization and its comparison to the gauge-theory side are not directly applicable in the case of the unstable 2-spin solution, and there is still a chance that one may find a one-loop agreement for the stable 3-spin string states once one computes the corresponding $1/J$ gauge-theory corrections using the SU(3) Bethe ansatz of [11].

Assuming this $1/J$ disagreement persists, it should have the same origin as the previously found mismatch between the string and gauge-theory results at 3-loop order in λ and the leading-order in J [18]. As was suggested in [18,27], the latter can be explained by adding “wrapping” contributions to the dilatation operator (and thus to the Bethe ansatz relations) on the gauge-theory side. For example, in the $q = 1$ case one may use the function like $\lambda^J/(1+\lambda)^J$ which is one in the string-theory limit ($J \rightarrow \infty$ with fixed $\lambda/J^2 \equiv \tilde{\lambda}$) but zero in the perturbative gauge-theory limit to interpolate between the different $\frac{\lambda^3}{J^5}$ results as follows:

$$\Delta = J + \frac{\lambda}{2J} - \frac{\lambda^2}{8J^3} + \frac{\lambda^3}{16J^5} \frac{\lambda^{J-3}}{(1+\lambda)^{J-3}} + \dots$$

This expression agrees with both the string ($E = \sqrt{J^2 + \lambda}$ [4]) and the perturbative gauge-theory ($\Delta_{\text{pert}} = J + \frac{\lambda}{2J} - \frac{\lambda^2}{8J^3} + 0 \times \lambda^3 + \dots$ [18]) results. The same idea may be applied to explain the discrepancy at order $\frac{\lambda^3}{J^5}$; for ex-

FIG. 6. k -dependence of d_1 for $q = \frac{1}{12}$.

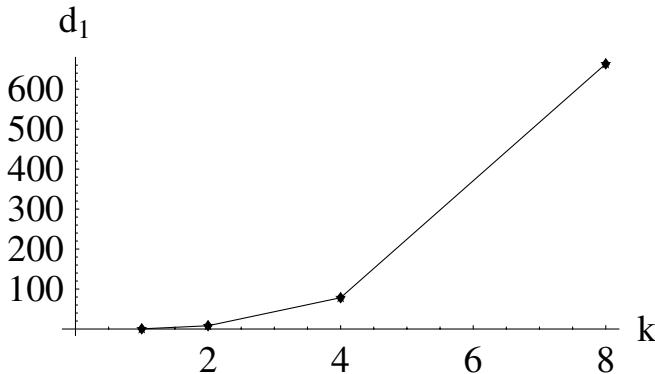
FIG. 7. k -dependence of the real part of d_1 for $q = 1$.

ample, if we assume that the interpolation formula contains also a term

$$\Delta = \dots + \frac{\lambda}{2J^2} \left[1 + a \frac{\lambda^{J-1}}{(1+\lambda)^{J-1}} \right] + \dots,$$

then the gauge-theory limit result for the coefficient d_1 will be $\frac{1}{2}$ as in [15] while the string-theory limit will give $d_1 = \frac{1}{2}(1+a)$, explaining the apparent disagreement of our result with that of [15].

A related observation is that this apparent $1/J$ disagreement can be easily accommodated and thus explained within the generalized Bethe ansatz for quantum string spectrum recently proposed in [28]. To this end all one should do is to assume a definite large $L \equiv J$ expansion of the functions $c_r(g, L)$ ($g \equiv \sqrt{8\pi^2\lambda}$) appearing in the Bethe ansatz of [28]. In particular, one can see that if c_r have an expansion of the form $c_r(g, L) \sim \tilde{\lambda}^{r+2} L^{2r+4} + \tilde{\lambda}^r L^{2r+3}$, then at order $1/L \equiv 1/J$ there is a string/gauge-theory mismatch already for the coefficient of the one-loop ($\sim \tilde{\lambda}$) term. The Bethe ansatz of [28] also implies that if there is a string/gauge-theory (dis)agreement for spinning string states at order $1/J^n$ then a similar (dis)agreement should exist also for the BMN states at order $1/J^{n+1}$. In view of the above discussion, this suggests that for the BMN states the disagreement at order $1/J^2$ should start not

FIG. 8. k -dependence of d_1 for $q = 1$ in the absolute value prescription.

for 3-loop (λ^3) terms as at $1/J$ order but already for the 1-loop (λ) terms.

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APPENDIX A: CLASSICAL SOLUTION AND QUADRATIC FLUCTUATIONS

The solution we discussed above was found in [4], and the characteristic equations for the quadratic fluctuations near it were obtained in [5]. Here we briefly review the derivation of the bosonic characteristic equation while in the fermionic case we only quote the final result referring to [5] for more details. The bosonic part of the string action in the conformal gauge is $I = \sqrt{\lambda} \int d\tau \int_0^{2\pi} \frac{d\sigma}{2\pi} \times (L_{\text{AdS}} + L_S)$, where

$$L_S = -\frac{1}{2} \partial_a X_M \partial^a X_M - \frac{1}{2} \Lambda (X_M X_M - 1), \quad (\text{A1})$$

$$L_{\text{AdS}} = -\frac{1}{2} \eta^{PQ} \partial_a Y_P \partial^a Y_Q + \frac{1}{2} \tilde{\Lambda} (\eta^{PQ} Y_P Y_Q + 1). \quad (\text{A2})$$

Here X^M , $M = 1, \dots, 6$ and Y^P , $P = 0, \dots, 5$ are the embedding coordinates with a flat Euclidean metric for S^5 and with $\eta_{MN} = (-1, +1, +1, +1, +1, -1)$ for the AdS_5 respectively. We consider the configuration where the string is located in the center of AdS_5 while rotating in S^5 . The AdS_5 part of the solution is trivial ($Y_5 + iY_0 = e^{it}$, $Y_1, \dots, Y_4 = 0$), with the global AdS_5 time being set to $t = \kappa\tau$, while the S^5 part is

$$\begin{aligned} X_1 + iX_2 &= \sqrt{q} \cos k\sigma e^{i\omega\tau}, \\ X_3 + iX_4 &= \sqrt{q} \sin k\sigma e^{i\omega\tau}, \\ X_5 + iX_6 &= \sqrt{1-q} e^{i\nu\tau} \end{aligned} \quad (\text{A3})$$

with

$$\begin{aligned} \omega^2 &= \nu^2 + k^2, & \Lambda &= \nu^2, \\ \nu^2 &= \kappa^2 - 2k^2q, & q &\equiv \sin^2 \gamma_0. \end{aligned} \quad (\text{A4})$$

It was shown in [5] that the quadratic fluctuation

Lagrangian around this solution can be written as

$$L_2 = (\partial_\tau \bar{X}_s)^2 - (\partial_\sigma \bar{X}_s)^2 + 4\nu\sqrt{q}\bar{X}_5\partial_\tau\bar{X}_6 - 4w\left(\sqrt{1-q}\bar{X}_5\partial_\tau\bar{X}_2 - \bar{X}_3\partial_\tau\bar{X}_4\right) + 4k\left(\sqrt{1-q}\bar{X}_5\partial_\sigma\bar{X}_3 - \bar{X}_2\partial_\sigma\bar{X}_4\right). \quad (A5)$$

The corresponding fluctuation spectrum is found by using

$$B_8(\Omega) = \Omega^4 + \Omega^3(-8k^2 - 4n^2 + 20k^2q - 8\kappa^2) + \Omega^2(16k^4 + 32k^2\kappa^2 + 16\kappa^4 + 8k^2n^2 + 16\kappa^2n^2 + 6n^4 - 80k^2\kappa^2q - 80k^4q - 36k^2n^2q + 96k^4q^2) + \Omega(-32k^4n^2 - 32k^2\kappa^2n^2 + 8k^2n^4 - 8\kappa^2n^4 - 4n^6 + 96k^4n^2q + 48k^2\kappa^2n^2q + 12k^2n^4q - 96k^4n^2q^2) + 16k^4n^4 - 8k^2n^6 + n^8 - 16k^4n^4q + 4k^2n^6q. \quad (A7)$$

The $4 + 4 S^5$ -frequencies are obtained as $\omega_{n,S^5}^B = \pm\sqrt{\Omega}$ where Ω is one of the four roots of $B_8 = 0$. In addition, there are $4 + 4$ AdS₅ frequencies $\omega_n = \pm\sqrt{n^2 + \kappa^2}$. By following the analogous steps one can show [5] that the fermionic characteristic frequencies ω_n^F are determined by $F_8(\Omega) = 0$ where

$$F_8 = 2\Omega^4 + \Omega^3(-8k^2 - 12\kappa^2 - 8r^2 + 20k^2q) + \Omega^2(12k^4 + 28k^2\kappa^2 + 18\kappa^4 + 8k^2r^2 + 28\kappa^2r^2 + 12r^4 - 52k^4q - 64k^2\kappa^2q - 36k^2r^2q + 59k^4q^2) + \Omega(-8k^6 - 20k^4\kappa^2 - 20k^2\kappa^4 - 8\kappa^6 + 8k^4r^2 + 8k^2\kappa^2r^2 - 20\kappa^4r^2 + 8k^2r^4 - 20\kappa^2r^4 - 8r^6 + 44k^6q + 80k^4\kappa^2q + 44k^2\kappa^4q - 24k^4r^2q + 32k^2\kappa^2r^2q + 12k^2r^4q - 78k^6q^2 - 79k^4\kappa^2q^2 + 2k^4r^2q^2 + 45k^6q^3) + 2k^8 + 4k^6\kappa^2 + 2k^4\kappa^4 - 8k^6r^2 - 4k^4\kappa^2r^2 - 4k^2\kappa^4r^2 + 12k^4r^4 - 4k^2\kappa^2r^4 + 2\kappa^4r^4 - 8k^2r^6 + 4\kappa^2r^6 + 2r^8 - 12k^8q - 16k^6\kappa^2q - 4k^4\kappa^4q + 28k^6r^2q + 16k^4\kappa^2r^2q + 4k^2\kappa^4r^2q - 20k^4r^4q + 4k^2r^6q + 27k^8q^2 + 21k^6\kappa^2q^2 + 2k^4\kappa^4q^2 - 30k^6r^2q^2 - 11k^4\kappa^2r^2q^2 + 11k^4r^4q^2 - 27k^8q^3 - 9k^6\kappa^2q^3 + 9k^6r^2q^3 + \frac{81k^8q^4}{8}. \quad (A8)$$

Unlike the bosonic case, the AdS₅ and S^5 parts are not decoupled in the fermionic case. The eight fermionic frequencies are obtained by solving $F_8 = 0$ and taking $\omega^F = \sqrt{\Omega}$ with double degeneracy.

When solving $B_8 = 0$ one may set $k = 1$; then the k -dependence can be restored by the following rescaling,

$$\omega_n \rightarrow \frac{\omega_n}{k}, \quad n \rightarrow \frac{n}{k}, \quad \kappa \rightarrow \frac{\kappa}{k}. \quad (A9)$$

Similar rescaling can be done in the fermionic case [5].

Let us now consider the large κ -expansion of the bosonic frequencies to analyze the stability condition in that limit

$$\omega_n^2 \rightarrow \frac{h_0}{4\kappa^2} + \frac{h_1}{\kappa^4} + \dots \quad (A10)$$

Here

$$h_0 = n^2 \left[2k^2(2 - 3q) + n^2 \pm 2k\sqrt{4n^2(1 - q) + k^2q(9q - 8)} \right]. \quad (A11)$$

the following mode expansion

$$\bar{X}_s = \sum_{n=-\infty}^{\infty} \sum_{h=1}^8 A_{sn}^{(h)} e^{i(\omega_{n,h}\tau + n\sigma)}, \quad (A6)$$

where h labels the different frequencies for a fixed value of n . The determinant of the characteristic matrix is proportional to (here we set $\Omega \equiv \omega_n^2$)

The stability condition that follows from positivity of h_0 is [5]

$$q \leq q_*, \quad q_* = 1 - \left(1 - \frac{1}{2k}\right)^2. \quad (A12)$$

APPENDIX B: $q = 1$ CASE: $J_1 = 0, J_2 = J_3$

In the case of $q = 1$ the string is stretched around the big circle of S^5 and rotates about its center of mass with two equal angular momenta. Here the characteristic Eqs. (A7) and (A8) can be solved explicitly [5] and one finds that the bosonic S^5 frequencies are (up to an overall sign change)

$$\omega_n^B = \left[n^2 + 2\kappa^2 - 2k^2 \pm 2\sqrt{(\kappa^2 - k^2)^2 + n^2\kappa^2} \right]^{1/2} \omega_n^B = \sqrt{n^2 + \kappa^2 - 2k^2} \pm \sqrt{\kappa^2 - 2k^2}. \quad (B1)$$

This may be compared to the AdS₅ fluctuation frequencies $\omega_n^B = \sqrt{n^2 + \kappa^2}$. The fermionic frequencies are (with double degeneracy)

$$\omega_r^F = \frac{1}{2} \left(2\sqrt{r^2 + \kappa^2 - k^2} \pm \sqrt{\kappa^2 - k^2} \pm \sqrt{\kappa^2 - 2k^2} \right). \quad (\text{B2})$$

Using (B1) and (B2) we get the explicit form of (17) is

$$E_1 = \frac{1}{\kappa} \left[1 + \left(\sqrt{\kappa^2 - 2k^2} + \sqrt{\kappa^2 - k^2} + 2\kappa - 4\sqrt{\frac{1}{4} + \kappa^2 - k^2} \right) + \sum_{n=1}^N S(n, \kappa, k) \right], \quad (\text{B3})$$

where

$$S = \sqrt{\left(n + \sqrt{n^2 - 4k^2} \right)^2 + 4\kappa^2 + 2\sqrt{n^2 - 2k^2 + \kappa^2} + 4\sqrt{n^2 + \kappa^2} - 4\sqrt{(n - 1/2)^2 - k^2 + \kappa^2} - 4\sqrt{(n + 1/2)^2 - k^2 + \kappa^2}} \quad (\text{B4})$$

We have used that

$$\begin{aligned} & \left[n^2 + 2\kappa^2 - 2k^2 + 2\sqrt{(\kappa^2 - k^2)^2 + n^2\kappa^2} \right]^{1/2} \\ & + \left[n^2 + 2\kappa^2 - 2k^2 - 2\sqrt{(\kappa^2 - k^2)^2 + n^2\kappa^2} \right]^{1/2} \\ & = \left[2(n^2 + 2\kappa^2 - 2k^2) + 2n\sqrt{n^2 - 4k^2} \right]^{1/2}. \end{aligned} \quad (\text{B5})$$

Note also that $S_{\kappa \rightarrow \infty} \rightarrow \frac{1}{2\kappa} [2k^2 - 2 - n^2 + n\sqrt{n^2 - 4k^2}] + O(\frac{1}{\kappa^3})$ and $S_{n \rightarrow \infty} \rightarrow \frac{k^2 - 2k^4 - \kappa^2}{n^3} + O(\frac{1}{n^5})$, in agreement with (13).

One may be tempted to evaluate the sum in (B3) by converting it into an integral. This conversion is not possible, however, due to singularities of the summand. To see this let us set $k = 1$ and follow the procedure of [5] to perform the large κ -expansion of S for fixed $\frac{n}{\kappa} \equiv x$,

$$S(x) \equiv S(\kappa x, \kappa, 1) \xrightarrow{\kappa \rightarrow \infty} -\frac{1}{(1 + x^2)^{3/2}} \frac{1}{\kappa} - \frac{16 + 71x^2 + 76x^4 + 16x^6}{16x^2(1 + x^2)^{7/2}} \frac{1}{\kappa^3} + \dots \quad (\text{B6})$$

Note that the first term in the coefficient of $\frac{1}{\kappa^3}$ has a singularity at $x = 0$. This is an example of the singularities

that we discussed below (9). Even though we singled out the zero mode contribution in (17), it is this divergent behavior near $x = 0$ that is responsible for the failure of the conversion of the sum to an integral.

To estimate the accuracy of the numerical method that we used in Sec. III we approximately evaluate the tail of the sum in (B3). It is possible to convert the tail part of the sum in (B3), i.e., $\frac{1}{\kappa} \sum_{n=N+1}^{\infty} S$, to an integral over $x = \frac{n}{\kappa}$ since it does not contain $x = 0$ or its neighborhood. Let us follow [5] and use that (we will not distinguish between N and $N + 1$ since $N \gg 1$)

$$\frac{1}{\kappa} \sum_{n=N}^{\infty} S \approx \int_{\frac{N}{\kappa}}^{\infty} dx g(x) + O\left(\frac{1}{\kappa^6}\right), \quad (\text{B7})$$

where

$$g(x) = -\frac{2}{15} S\left(x - \frac{1}{\kappa}\right) + \frac{6}{5} S\left(x - \frac{1}{2\kappa}\right) + \frac{1}{30} S(x) - \frac{2}{15} S\left(x + \frac{1}{2\kappa}\right) + \frac{1}{30} S\left(x + \frac{1}{\kappa}\right), \quad (\text{B8})$$

and $S(x) \equiv S(\kappa x, \kappa, 1)$. To evaluate the integral in (B7) consider the following large- x expansion of g ,

$$\begin{aligned} g = & -\frac{1 + \kappa^2}{\kappa^3} \frac{1}{x^3} - \frac{3(1 + \kappa^2)}{2\kappa^4} \frac{1}{x^4} + \frac{3(-5 - 3\kappa^2 + 2\kappa^4)}{4\kappa^5} \\ & \times \frac{1}{x^5} + \frac{5(-11 - 5\kappa^2 + 6\kappa^4)}{8\kappa^6} \frac{1}{x^6} \\ & - \frac{3(77 + 21\kappa^2 - 43\kappa^4 + 10\kappa^6)}{16\kappa^7} \frac{1}{x^7} + \dots \end{aligned} \quad (\text{B9})$$

Up to the order given in (B9) the integral yields

$$\begin{aligned} \int_{\frac{N}{\kappa}}^{\infty} dx g(x) = & -\frac{1}{2} \left(1 + \frac{1}{\kappa} \right) \frac{\kappa}{N^2} - \frac{1}{2} \left(1 + \frac{1}{\kappa} \right) \frac{\kappa}{N^3} \\ & + \frac{3}{16} \left(2 - \frac{3}{\kappa^2} - \frac{5}{\kappa^4} \right) \frac{\kappa^3}{N^4} + \frac{1}{8} \left(6 - \frac{3}{\kappa^2} - \frac{11}{\kappa^4} \right) \\ & \times \frac{\kappa^3}{N^5} + O\left(\frac{\kappa^5}{N^6}\right). \end{aligned} \quad (\text{B10})$$

For $N = 40000$ and $\kappa = 50, 100, 200$ we find that the correction is small compared to the numerically found value of E_1 (which is of order 10^{-5}). However, the correction grows if we increase κ for fixed N (e.g., consider $\kappa = 1000$).

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