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Tightening Durbin-Watson Bounds

DENIS CONNIFFE*
The Economic and Social Research Institute

Abstract: The null distribution of the Durbin-Watson statistic, used in testing for serial correlation in regression disturbances, depends on the explanatory variable values. Although modern computing power makes it feasible to obtain exact critical values in any specific case, bounds to critical values used to be widely employed and sometimes still are. Much research was motivated by the ambiguity resulting from a test statistic falling inside the bounds, but one very simple approach to tightening bounds was overlooked. It is worth recording, if only for its pedagogic interest and possible applicability to other problems.

I INTRODUCTION

The linear regression model with possibly autocorrelated disturbances is

$$y_i = x_i'B + e_i \quad i = 1, 2, \ldots, n,$$

where x_i and B are k vectors of exogenous variables and coefficients and $e_i = \rho\, e_{i-1} + u_i$. The u_i are assumed independently normally distributed with means zero and variances σ^2. For testing the null hypothesis $\rho = 0$ against the alternative $\rho > 0$, Durbin and Watson (1950) considered general test statistics of the form

$$\hat{e}'A\hat{e} \,/\, \hat{e}'\hat{e} = e'MAMe \,/\, e'Me, \tag{1}$$

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where A is any real, symmetric matrix, $\hat{e}_i = y_i - x_i'\hat{B}$ and $\hat{B}$ is the ordinary least squares estimator of B with an intercept included, so that $\hat{e}_i = \{I - X(X'X)^{-1}X'\}e_i = Me_i$, where $X' = (x_1, x_2, \ldots, x_n)$. Noting that M is symmetric and idempotent with n–k eigenvalues equal to unity and k eigenvalues equal to zero, it followed that (1) equals

$$\sum_1^{n-k} v_i z_i^2 \Big/ \sum_1^{n-k} z_i^2, \tag{2}$$

where the z_i are independently normally distributed with means zero and variances σ^2 and the v_i are the n–k non-zero eigenvalues of the matrix MA. In continuing to the actual test that bears their names, Durbin and Watson (1951) took A to be the first-differencing matrix

$$\begin{matrix} 1 & -1 & 0 & 0 & . & . & 0 \\ -1 & 2 & -1 & 0 & . & . & 0 \\ 0 & -1 & 2 & -1 & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & . & . & 0 & -1 & 2 & -1 \\ 0 & . & . & . & 0 & -1 & 1 \end{matrix} \tag{3}$$

so that the test statistic (1) can be written in the familiar form

$$\sum_2^n (\hat{e}_i - \hat{e}_{i-1})^2 \Big/ \sum_1^n \hat{e}_i^2. \tag{4}$$

Durbin and Watson (1950) also showed that if the eigenvalues of A, other than those associated with a possible $s \leq k$ eigenvectors consisting of linear combinations of the columns of X, are $\lambda_1, \lambda_2, \ldots, \lambda_{n-s}$, then

$$\lambda_i \leq v_i \leq \lambda_{i+k-s}, \tag{5}$$

where both sets of eigenvalues are numbered in ascending order of magnitude. In the familiar case of A = (3), the vector of units corresponding to the intercept is an eigenvector of (6), so s=1 and (5) becomes

$$\lambda_i \leq v_i \leq \lambda_{i+k-1}.$$

The α level critical value of (2) could be found by solving for d_α in

$$\Pr\left\{\sum_1^{n-k} (v_i - d_\alpha) z_i^2 \leq 0\right\} = \alpha, \tag{6}$$

but as the ν_i depend on X, the value would change with the data set, so generally applicable tables of critical values cannot be obtained. Instead Durbin and Watson utilised the inequalities (5) to obtain the bounds, independent of X,

$$L = \sum_1^{n-k} \lambda_i z_i^2 \Big/ \sum_1^{n-k} z_i^2 \quad \text{and} \quad U = \sum_1^{n-k} \lambda_{i+k-s} z_i^2 \Big/ \sum_1^{n-k} z_i^2. \tag{7}$$

When A is given by (3), Durbin and Watson (1951) provided tables of L and U for ranges of n and α. Since $L \le d_\alpha \le U$, values of (4) below L imply rejection of $\rho = 0$ and values above U imply inability to reject, and these bounds do not depend on X. However, a problem remained when values of (4) fell in the inconclusive region between the bounds and much work has been devoted to seeking tighter bounds, or approximations to the exact distribution of (4), or even alternative test criteria. The research has been summarised in the reviews by Durbin and Watson (1971), Harrison (1972) and King (1987). This paper describes a simple, but sometimes quite effective, approach to adjusting bounds that seems to have been missed in the literature. Nowadays, computing power permits methods that would once have been prohibitively time consuming and exact solution of (6) by the Imhof (1961) algorithm is feasible. However, applied economists sometimes still use bounds and the approach is of pedagogic interest and may perhaps be applicable to other problems.

II MODIFIED BOUNDS

The bounds (7) take no account of X, while solving (6) for each X is a major computation. But taking some account of X need not be computationally demanding and, as Durbin and Watson (1950) remarked, the trace of MA is relatively simply obtained given that standard regression calculations have been performed. Let c = tr A − tr MA so that $\Sigma\nu + c = \Sigma\lambda$.

Then

$$\nu_i = \lambda_{i+k-s} - (c - \lambda_i - \lambda_2 - \ \dots \ - \lambda_{k-s}) + \sum_1^{n-k} (1 - \delta_j^i)(\lambda_{j+k-s} - \nu_j),$$

where δ denotes Kroneker delta. Since $\lambda_{j+k-s} \ge \nu_j$

$$\nu_i \ge \lambda_{i+k-s} - (c - \lambda_1 - \lambda_2 - \ \dots \ - \lambda_{k-s}).$$

So

$$\sum_1^{n-k} \nu_i z_i^2 \Big/ \sum_1^{n-k} z_i^2 \ge \sum_1^{n-k} \lambda_{i+k-s} z_i^2 \Big/ \sum_1^{n-k} z_i^2 - (c - \lambda_1 - \lambda_2 - \ \dots \ - \lambda_{k-s}).$$

From (7) then,

$$U-(c-\lambda_1-\lambda_2-\ \dots\ -\lambda_{k-s})\leq d_\alpha \leq U. \tag{8}$$

A precisely similar argument leads to

$$L\leq d_\alpha \leq L+(\lambda_{n-s}+\lambda_{n-s-1}+\ \dots\ +\lambda_{n-k+1}-c). \tag{9}$$

So there are three sets of bounds with (8) and (9) depending on X only through c, where

$$c=\text{tr } A-\text{tr } MA=\text{tr } X(X'X)^{-1}X'A=\text{tr } X'AX(X'X)^{-1}. \tag{10}$$

For the A matrix (3) this becomes

$$c=2\left\{k-\sum_2^n x_i'(X'X)^{-1}x_{i-1}-\tfrac{1}{2}x_1'(X'X)^{-1}x_1-\tfrac{1}{2}x'_n(X'X)^{-1}x_n\right\}. \tag{11}$$

For the case of k = 2, a single explanatory variable and an intercept, (11) becomes

$$c=2-2S_2^n x_i x_{i-1}/S_1^n x_i^2-(x_1-\bar{x})^2/S_1^n x_i^2-(x_n-\bar{x})^2/S_1^n x_i^2.$$

If the explanatory variable is highly positively serially correlated, $c-\lambda_1$ will be small, which suggests that (8) will give the tightest bound. With high negative serial correlation $c-\lambda_1$ will be large, so that if any improvement on (7) is possible, it will be because $\lambda_{n-s}-c$ is small giving advantage to (9). For example, for n = 15, λ_1 = .0437 and the (α = .05) values of L and U are 1.077 and 1.361 respectively. If the x values are taken to be the highly trended set

3.5, 3.5, 3.8, 4.1, 4.5, 4.7, 4.9, 5.3, 5.6, 5.9, 6.3, 6.6, 6.8, 7.0, 7.1,

with the serial correlation approximately .99, c = .0481, so that the bounds (8), remembering that s = 1, are the very tight 1.355 to 1.359. However, if the x values are rearranged to exhibit a high negative serial correlation of about –.9 by the ordering

3.5, 7.1, 3.5, 7.0, 3.8, 6.8, 4.1, 6.6, 4.5, 6.3, 4.7, 5.9, 4.9, 5.6, 5.3,

c = 3.6481 so that bounds based on (8) are useless. But λ_{14} = 3.9563 so that $\lambda_{14}-c$ = .2722. Then (9) gives bounds of 1.077 and 1.349, which do improve somewhat on the table bounds. Of course, if serial correlation is not high neither (8) or (9) improve on the Durbin-Watson bounds.

For several explanatory variables, the possibility of improvements seems to depend on the uniformity of positive or negative serial correlation. Taking squares of the strongly trended set as a second explanatory variable in the n = 15 example, gives c = .2423. The eigenvalue $\lambda_2 = .1729$, so $c - \lambda_1 - \lambda_2 = .0257$. The Durbin-Watson table bounds ($\alpha = .05$) are L = .946 and U = 1.543, while (8) now gives the far tighter 1.517 to 1.543. On the other hand, if the trended variable and the negatively serially correlated variable generated by reordering it are taken as the two explanatory variables, the value of c = 3.763. Then $c - \lambda_1 - \lambda_2$ is too large for (8) to be useful and, since $\lambda_{13} = 3.8271$, $\lambda_{14} + \lambda_{13} - c$ is also too large for (9) to be useful.

III CONCLUDING REMARKS

There may be scope for applying the approach to seek tighter bounds for other test statistics. Equations (1), (2), (5) and (6) hold for any real symmetric matrix A, as do the arguments that led to (8) and (9). The reduction of (10) to (11) would not hold, of course, but the computation of (10) is not onerous. Other A matrices besides (3) have been used in the literature. King (1981a) proposed an alternative test statistic to (4) for the standard regression case and also considered bounds for the classes of regression models that always contain an exact time trend or seasonal dummy variables (King, 1981b, 1983). There may also be other potential applications, as similar expressions to (2) arise with other tests, for example when testing for autocorrelation using ordinary least squares residuals and the cumulated periodogram.

Returning to the Durbin-Watson bounds, modifications could probably be further improved upon by using more information about X than is contained in tr A – tr MA. For example, there is a considerable mathematical literature on bounds for eigenvalues deducible from largest and smallest elements, row sums etc., of a matrix. But, as already mentioned, modern computing power implies tables of bounds for Durbin-Watson critical values are not indispensable, but convenient at most. So, unless the table modifications are straightforward, one might as well solve (6) directly.

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