14th International Conference on Applications of Statistics and Probability in Civil Engineering, ICASP14
Dublin, Ireland, July 9-13, 2023

Approximate WPI based Survival Probability Determination of Nonlinear Oscillator under Combined Excitation

Yuanjin Zhang
China Research Center for Emergency Management,Wuhan University of Technology,Wuhan,China

Fan Kong
School of Civil Engineering, Hefei University of Technology, Hefei, China

ABSTRACT: A Wiener path integral (WPI) technique for determining the survival probability of nonlinear oscillator subject to nonstationary stochastic and periodic combined excitation is developed in this paper. Specifically, the response of nonlinear oscillator subject to nonstationary stochastic and periodic excitation can be approximated by the combination of a zero-mean nonstationary component and a periodic component. Specifically, the concept of Wiener path integral in conjunction with stochastic averaging/linearization is utilized to obtain the nonstationary response probability density function (PDF) of the stochastic component. Next, relying on the Wiener path integral method with the concepth of most probable path yields a closed-from approximate analytical expression of joint transition PDF for small intervals corresponding to the equivalent linear SDE. Finally, relying on the analytical expression of the transition PDF of the equivalent linear system in conjunction of a discrete version of Chapman-Kolmogorov (C-K) equation, the survival probability of the original oscillator under combined excitation can be determined step by step. A Duffing oscillator is considered to demonstrate the accuracy and efficiency of the proposed method, compared with pertinent Monte Carlo simulations.

1. INTRODUCTION

Monte Carlo simulation (MCS) (Rubinstein (2008)) can be applied to assess the reliability of the stochastic systems, for example the determination of the survival probability. To improve the computational efficiency of MCS, especially in some scenarios where the occurrence probability of the event is small or the whole nonlinear system is rather complex, many numerical or analytical approximate methods have been developed, including stochastic averaging/linearization (Roberts and Spanos (1986); Spanos and Kougioumtzoglou (2014)), probability density evolution method (Li and Chen (2009); Xu et al. (2022)), method based on response statistics (Vanmarcke (1975); Barbato and Conte (2011)), Galerkin scheme (Spanos et al. (2016); Di Matteo et al. (2018)), Wiener path integral (Kougioumtzoglou and Spanos (2012, 2014); Zhang et al. (2021); Petromichelakis and Kougioumtzoglou (2020)) and so on. To extend the response determination and reliability assessment to the combined excitation scenarios such as the rotary machinery under earthquake (Iwatsubo et al. (1979); Soni and Srinivasan (1984)), and the dams excited by the water waves and earthquake (Spencer et al. (1990)), several approximate methods (Zhu and Guo (2015)) have been developed to obtain the response of the system under combined loads.

In this paper, an approximate analytical Wiener path integral (WPI) technique for determining the survival probability of nonlinear oscillator subject to nonstationary stochastic and periodic combined excitation is developed. This is done by resorting to the decomposition of system response into two parts, namely a zero-mean nonstationary stochastic component and a deterministic component. Relying on the concept of Wiener path integral in conjunction with stochastic averaging/linearization, the analytical expression of joint transition probability density function (PDF) for small intervals corresponding to the stochastic component can be obtained. In this regard, the response and survival probability of the original oscillator under combined excitation can be determined step by step.

2. STOCHASTIC AVERAGING TREATMENT FOR COMBINED EXCITATION

Consider a nonlinear single-degree-of freedom (SDOF) oscillator subject to combined periodic and evolutionary nonlinear excitation whose governing equation is given by (Han et al. (2022))

$$\ddot{x}(t)+\beta_0\dot{x}(t)+g(x,\dot{x})=w(t)+F_1\sin(\omega_1 t) \quad (1)$$

where x represents the nonlinear system response displacement, and a dot over a variable denotes differentiation with respect to time t; $g(x,\dot{x})$ denotes an arbitrary nonlinear function; $\beta_0=2\zeta_0\omega_0$ is a linear damping coefficient in which ζ_0 and ω_0 represent the corresponding linear system; $w(t)$ is a Gaussian zero-mean evolutionary nonlinear process with the power spectral density (PSD) $S_w(\omega,t)$; F_1 and ω_1 represent the amplitude and frequency of the harmonic excitation component, respectively.

Next, following Spanos et al. (2019), the response of Eq.(1) is approximately decomposed into a combination of a deterministic and stochastic component, given by

$$x(t)=\mu_x(t)+\hat{x}(t) \quad (2)$$

where $\mu_x(t)$ is the deterministic component; and $\hat{x}(t)$ is the Gaussian zero-mean nonstationary stochastic component. Employing Eq.(2), and taking expectation, yield a deterministic sub-equation as

$$\ddot{\mu}_x(t)+\beta_0\dot{\mu}_x(t)+\mathrm{E}[g(\mu_x+\hat{x},\dot{\mu}_x+\dot{\hat{x}})]=F_1\sin(\omega_1 t) \quad (3)$$

where $\mathrm{E}[\cdot]$ is the mathematical expectation operator. While, the corresponding stochastic sub-equation can be derived as following

$$\ddot{\hat{x}}(t)+\beta_0\dot{\hat{x}}(t)+h(\mu_x,\dot{\mu}_x,\hat{x}(t),\dot{\hat{x}}(t))=w(t) \quad (4)$$

where

$$\begin{aligned} h(\mu_x,\dot{\mu}_x,\hat{x}(t),\dot{\hat{x}}(t)) =& g(\mu_x+\hat{x},\dot{\mu}_x+\dot{\hat{x}}) \\ &-\mathrm{E}[g(\mu_x+\hat{x},\dot{\mu}_x+\dot{\hat{x}})] \end{aligned} \quad (5)$$

Relying next on the assumption of lightly damping (i.e., $\zeta_0 << 1$) in the system, and slowly time-varying property of the evolutionary power spectrum $S_w(\omega,t)$, it can be argued (e.g.Spanos and Lutes (1980)) that the response of stochastic component $\hat{x}$ in Eq.(4) exhibits a pseudo-harmonic behavior, which can be described by the following equations

$$\hat{x}=A(t)\cos[\omega(A)t+\phi(t)] \quad (6)$$

and

$$\dot{\hat{x}}=-\omega(A)A(t)\sin[\omega(A)t+\phi(t)] \quad (7)$$

where A and ϕ denotes the slowly varying time dependent response amplitude and phase of stochastic component $\hat{x}$, respectively.

Further, following the stochastic averaging/linearization scheme leads to a linearized version of Eq.(4) with amplitude dependent coefficients, given by

$$\ddot{\hat{x}}(t)+\beta(A)\dot{\hat{x}}(t)+\omega^2(A)\hat{x}(t)=w(t) \quad (8)$$

where

$$\begin{aligned} \beta(A)=\beta_0-\frac{1}{\pi A\omega(A)}\int_0^{2\pi}\sin\psi \\ \cdot h(\mu_x,\dot{\mu}_x,A\cos\psi,-A\omega(A)\sin\psi)\mathrm{d}\psi \end{aligned} \quad (9)$$

and

$$\begin{aligned} \omega^2(A)=\frac{1}{\pi A}\int_0^{2\pi}\cos\psi \\ \cdot h(\mu_x,\dot{\mu}_x,A\cos\psi,-A\omega(A)\sin\psi)\mathrm{d}\psi \end{aligned} \quad (10)$$

It is noted that the amplitude dependent coefficients $\beta(A)$ and $\omega(A)$ in Eqs.(9-10) can be derived by minimizing the error between Eqs.(4) and (8) in mean square sense (eg.Kougioumtzoglou and Spanos (2009)). Next, the corresponding equivalent time-dependent damping and stiffness coefficients can be derived by taking expectation on both sides of Eqs.(9-10), as

$$\beta_{\mathrm{eq}}(t) = \mathrm{E}[\beta(A)] = \int_0^{+\infty} \beta(A) p(A,t) \mathrm{d}A \quad (11)$$

and

$$\omega_{\mathrm{eq}}^2(t) = \mathrm{E}[\omega^2(A)] = \int_0^{+\infty} \omega^2(A) p(A,t) \mathrm{d}A \quad (12)$$

where $p(A,t)$ represents the nonstationary oscillator response amplitude PDF of stochastic component $\hat{x}$. In this regard, replacing $\beta(A)$ and $\omega^2(A)$ in Eq.(8) by Eqs.(11-12) respectively, yields the equivalent linear SDE with time dependent coefficients as following

$$\ddot{\hat{x}}(t) + \beta_{\mathrm{eq}}(t)\dot{\hat{x}}(t) + \omega_{\mathrm{eq}}^2(t)\hat{x}(t) = w(t) \quad (13)$$

Next, adopting a standard stochastic averaging treatment (e.g.Kougioumtzoglou and Spanos (2012); Roberts and Spanos (2003)) and manipulating Eq.(13) yields a first-order Ito SDE

$$\begin{aligned}\dot{A}(t) = &-\frac{1}{2}\beta_{\mathrm{eq}}(t)A(t) + \frac{\pi S_w(\omega_{\mathrm{eq}}(t),t)}{2A(t)\omega_{\mathrm{eq}}^2(t)} \\ &+ \frac{\sqrt{\pi S_w(\omega_{\mathrm{eq}}(t),t)}}{\omega_{\mathrm{eq}}(t)}\eta(t)\end{aligned} \quad (14)$$

where $\eta(t)$ denotes a Gaussian zero-mean stationary white noise process with intensity one with $\mathrm{E}[\eta(t)] = 0$ and $\mathrm{E}[\eta(t)\eta(t+\tau)] = \delta(\tau)$, in which $\delta(\cdot)$ is the Dirac function. Thus, the corresponding Fokker-Planck (F-P) partial differential equation (e.g.Grigoriu (2013)) becomes

$$\begin{aligned}\frac{\partial}{\partial t}p(A,t|A_1,t_1) = &-\frac{\partial}{\partial A}\left[\left(-\frac{1}{2}\beta_{\mathrm{eq}}(t)A(t)\right.\right. \\ &\left.\left.+\frac{\pi S_w(\omega_{\mathrm{eq}}(t),t)}{2A(t)\omega^2(A)}\right)p(A,t|A_1,t_1)\right] \\ &+\frac{1}{2}\frac{\partial^2}{\partial A^2}\left[\frac{\pi S_w(\omega_{\mathrm{eq}}(t),t)}{\omega_{\mathrm{eq}}^2(t)}p(A,t|A_1,t_1)\right]\end{aligned} \quad (15)$$

where $p(A,t|A_1,t_1)$ represents the transition PDF from state (A_1,t_1) to a new state (A,t). It is noted that Eq.(15) is satisfied by a solution of the Rayleigh distribution form (see Kougioumtzoglou and Spanos (2009, 2013))

$$p(A,t) = \frac{A}{c(t)}\exp\left[-\frac{A^2}{2c(t)}\right] \quad (16)$$

for $p(A,t|A_1=0,t_1=0) = p(A,t)$, where $c(t)$ denotes the variance of the stochastic component $\hat{x}$.

Further, substituting Eq.(16) into the corresponding F-P Eq.(15) in conjunction with the initial condition (i.e. $p(A,t=0) = \delta(A)$), yields

$$\dot{c}(t) = -\beta_{\mathrm{eq}}(c(t))c(t) + \frac{\pi S_w(\omega_{\mathrm{eq}}(t),t)}{\omega_{\mathrm{eq}}^2(c(t))} \quad (17)$$

In this regard, the above Eqs.(3)and (17) becomes the coupled set of ordinary differential equations (ODEs) in terms of the mean $\mu_x(t), \mu_{\dot{x}}(t)$ and the variance $c(t)$.

3. Wiener path integration for determining the transition PDF

In this section, a Wiener path integration method (see Wiener (1921); Chaichian and Demichev (2001)) is utilized for determining the transition PDF for stochastic component $\hat{x}$ under nonstationary excitation (e.g.Zhang et al. (2021)). Assuming that the nonstationary excitation $S_w(\omega,t)$ is slowly varying with time t, the nonlinear response of stochastic component can be approximated as the Gaussian same nonlinear system excited by the white noise $v_m(t)$ with the power spectrum $S_{\mathrm{eq},m} = S_w(\omega_{\mathrm{eq}}(t),t), t \in [t_{m-1},t_m]$ for this short time interval $\Delta t = \Delta t_m = (t_m - t_{m-1}) \to 0$, given by

$$\begin{aligned}&\ddot{\hat{x}}(t) + \beta_{\mathrm{eq}}(t)\dot{\hat{x}}(t) + \omega_{\mathrm{eq}}^2(t)\hat{x}(t) = v_m(t), \\ &t \in [t_{m-1},t_m], m = 1,2,\cdots,M\end{aligned} \quad (18)$$

where M is the total number of short time intervals for the considered time duration.

Next, employing the standard WPI technique, the joint transition PDF $p(\hat{x}_m,\dot{\hat{x}}_m,t_m|\hat{x}_{m-1},\dot{\hat{x}}_{m-1},t_{m-1})$ from a state $\hat{x}_{m-1},\dot{\hat{x}}_{m-1},t_{m-1}$ to a new state $\hat{x}_m,\dot{\hat{x}}_m,t_m$ within a short time interval $\Delta t = \Delta t_m = (t_m - t_{m-1}) \to 0$ can be represented as a functional over the space of all possible paths

$C\{\hat{x}_m, \dot{\hat{x}}_m, t_m; \hat{x}_{m-1}, \dot{\hat{x}}_{m-1}, t_{m-1}\}$, given by

$$p(\hat{x}_m, \dot{\hat{x}}_m, t_m | \hat{x}_{m-1}, \dot{\hat{x}}_{m-1}, t_{m-1}) = \int_{\{\hat{x}_{m-1}, \dot{\hat{x}}_{m-1}, t_{m-1}\}}^{\{\hat{x}_m, \dot{\hat{x}}_m, t_m\}} \exp\left(-\int_{t_{m-1}}^{t_m} L_m(\hat{x}, \dot{\hat{x}}, \ddot{\hat{x}}) dt \right) [\mathrm{d}\hat{x}(t)] \tag{19}$$

where $\mathrm{d}\hat{x}(t)$ represents a functional measure; and $L_m(\hat{x}, \dot{\hat{x}}, \ddot{\hat{x}})$ denotes the Lagrangian functional (e.g., Taniguchi and Cohen (2007); Kougioumtzoglou and Spanos (2014)) as

$$L_m(\hat{x}, \dot{\hat{x}}, \ddot{\hat{x}}) = \frac{v_m^2(t)}{4\pi S_w(\omega_{\text{eq}}(t_m), t_m)} \quad \text{for } t \in [t_{m-1}, t_m] \tag{20}$$

Further, following the scheme of variation calculus Ewing (1985), the functional integral in Eq.(19) can be obtained by introducing the "most probable path" $\hat{x}_c$, and the corresponding Euler-Lagrange (E-L) equation becomes

$$\frac{\partial L_m}{\partial \hat{x}_c} - \frac{\partial}{\partial t}\frac{\partial L_m}{\partial \dot{\hat{x}}_c} + \frac{\partial^2}{\partial t^2}\frac{\partial L_m}{\partial \ddot{\hat{x}}_c} = 0, t \in [t_{m-1}, t_m] \tag{21}$$

with the boundary conditions

$$\hat{x}_c(t_{m-1}) = \hat{x}_{m-1}, \qquad \dot{\hat{x}}_c(t_{m-1}) = \dot{\hat{x}}_{m-1}, \tag{22a}$$

$$\hat{x}_c(t_m) = \hat{x}_m, \qquad \dot{\hat{x}}_c(t_m) = \dot{\hat{x}}_m, \tag{22b}$$

Substituting the solution of partial differential Eq.(21) in conjunction with the boundary conditions Eqs.(22a-22b), into the Eq.(19) leads to an approximate closed-form expression for the transition PDF of stochastic component $\hat{x}$

$$p(\hat{x}_m, \dot{\hat{x}}_m, t_m | \hat{x}_{m-1}, \dot{\hat{x}}_{m-1}, t_{m-1}) = D\exp\left(-\int_{t_{m-1}}^{t_m} L_m(\hat{x}_c, \dot{\hat{x}}_c, \ddot{\hat{x}}_c) dt \right) \tag{23}$$

where D is the normalization coefficient.

Next, considering the form Eq.(20), the E-L Eq.(21) becomes a fourth-order linear ODE in the form

$$\frac{\mathrm{d}^4 \hat{x}_c}{\mathrm{d}t^4} + 2(1 - 2\zeta_{\text{eq},m}^2)\omega_{\text{eq},m}^2 \frac{\mathrm{d}^2 \hat{x}_c}{\mathrm{d}t^2} + \omega_{\text{eq},m}^4 \hat{x}_c = 0 \tag{24}$$

where for $t \in [t_{m-1}, t_m]$, denote $\beta_{\text{eq}}(t) = \beta_{\text{eq}}(t_m) = \beta_{\text{eq},m} = 2\zeta_{\text{eq},m}\omega_{\text{eq},m}$, $\omega_{\text{eq}}(t) = \omega_{\text{eq}}(t_m) = \omega_{\text{eq},m}$, and $S_{\text{eq},m} = S_w(\omega_{\text{eq}}(t_m), t_m)$ for simplicity. The most probable path $\hat{x}_c$ can be obtained by solving Eq.(24) in conjunction with the boundary conditions Eqs.(22a-22b). Furthermore, the transient transition PDF of stochastic component $\hat{x}$ can be determined by substituting the most probable path $\hat{x}_c$ into Eq.(23) (see Psaros et al. (2020) for details), given by

$$p(\hat{x}_m, \dot{\hat{x}}_m, t_m | \hat{x}_{m-1}, \dot{\hat{x}}_{m-1}, t_{m-1}) = \left[(2\pi)^2 \det(\Sigma_m)\right]^{-\frac{1}{2}} \times \exp\left(-\frac{1}{2}(\mathbf{X}_m - \mu_m)^T \Sigma_m^{-1} (\mathbf{X}_m - \mu_m) \right) \tag{25}$$

where $\mathbf{X}_m = [\hat{x}_m, \dot{\hat{x}}_m]^T$; $\mu_m = [\mu_{1,m}, \dot{\mu}_{2,m}]^T$; $\Sigma_m = [V_{11,m}, V_{12,m}; V_{12,m}, V_{22,m}]$ with

$$\mu_{1,m} = e^{-\zeta_{\text{eq},m}\omega_{\text{eq},m}\Delta t_m}\left[\left(\cos\omega_{D,m}\Delta t_m + \frac{\zeta_{\text{eq},m}\omega_{\text{eq},m}}{\omega_{D,m}} \sin\omega_{D,m}\Delta t_m \right)\hat{x}_{m-1} + \left(\frac{1}{\omega_{D,m}} \sin\omega_{D,m}\Delta t_m \right)\dot{\hat{x}}_{m-1} \right] \tag{26a}$$

$$\mu_{2,m} = e^{-\zeta_{\text{eq},m}\omega_{\text{eq},m}\Delta t_m}\left[\left(\cos\omega_{D,m}\Delta t_m - \frac{\zeta_{\text{eq},m}\omega_{\text{eq},m}}{\omega_{D,m}} \sin\omega_{D,m}\Delta t_m \right)\dot{\hat{x}}_{m-1} - \left(\frac{\omega_{\text{eq},m}^2}{\omega_{D,m}} \sin\omega_{D,m}\Delta t_m \right)\hat{x}_{m-1} \right] \tag{26b}$$

and

$$V_{11,m} = \frac{\pi S_{\text{eq},m}}{2\zeta_{\text{eq},m}\omega_{\text{eq},m}^3}\left[1 - \frac{e^{-2\zeta_{\text{eq},m}\omega_{\text{eq},m}\Delta t_m}}{\omega_{D,m}^2} \left(\omega_{D,m}^2 + \zeta_{\text{eq},m}\omega_{\text{eq},m}\omega_{D,m}\sin 2\omega_{D,m}\Delta t_m + \frac{(2\zeta_{\text{eq},m}\omega_{\text{eq},m})^2}{2}\sin^2\omega_{D,m}\Delta t_m \right) \right] \tag{27a}$$

$$V_{22,m} = \frac{\pi S_{\text{eq},m}}{2\zeta_{\text{eq},m}\omega_{\text{eq},m}}\left[1 - \frac{e^{-2\zeta_{\text{eq},m}\omega_{\text{eq},m}\Delta t_m}}{\omega_{D,m}^2} \left(\omega_{D,m}^2 - \zeta_{\text{eq},m}\omega_{\text{eq},m}\omega_{D,m}\sin 2\omega_{D,m}\Delta t_m + \frac{(2\zeta_{\text{eq},m}\omega_{\text{eq},m})^2}{2}\sin^2\omega_{D,m}\Delta t_m \right) \right] \tag{27b}$$

$$V_{12,m} = \pi S_{\text{eq},m}\frac{e^{-2\zeta_{\text{eq},m}\omega_{\text{eq},m}\Delta t_m}}{\omega_{D,m}^2}\sin^2\omega_{D,m}\Delta t_m \tag{27c}$$

where $\omega_{D,m} = \omega_{\text{eq},m}\sqrt{1-\zeta_{\text{eq},m}^2}$, and $\Delta t_m = t_m - t_{m-1}$. In this regard, the transition PDF in the form of Eq.(25) can be cast to the following form

$$p(\hat{x}_m,\dot{\hat{x}}_m,t_m|\hat{x}_{m-1},\dot{\hat{x}}_{m-1},t_{m-1}) = \frac{n_{4,m}n_{7,m}}{\pi} \exp\left(-(\mathbf{X}_{m-1,m})^T\mathbf{U}_{m-1,m}^T\mathbf{U}_{m-1,m}(\mathbf{X}_{m-1,m})\right) \tag{28}$$

where $\mathbf{X}_{m-1,m} = [\hat{x}_{m-1},\dot{\hat{x}}_{m-1},\hat{x}_m,\dot{\hat{x}}_m]^{\mathrm{T}}$; and

$$\mathbf{U}_{m-1,m} = \begin{pmatrix} n_{1,m} & n_{2,m} & n_{3,m} & n_{4,m} \\ n_{5,m} & n_{6,m} & n_{7,m} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{29}$$

The coefficients $n_{i,m}, i = 1,2,\cdots 7$ in Eq.(29) can be obtained by a Cholesky-like decomposition. Finally, following the C-K equation, the propagation of probability density $p(\hat{x}_m,\dot{\hat{x}}_m,t_m|\hat{x}_0,\dot{\hat{x}}_0,t_0)$ from an initial time instant t_0 to an arbitrary time instant t_m can be determined by recursively empolying the transition PDF Wehner and Wolfer (1983)

$$\begin{aligned} p(\hat{x}_m,\dot{\hat{x}}_m,t_m|\hat{x}_0,\dot{\hat{x}}_0,t_0) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} p(\hat{x}_m,\dot{\hat{x}}_m,t_m|\hat{x}_{m-1},\dot{\hat{x}}_{m-1},t_{m-1}) \\ & p(\hat{x}_{m-1},\dot{\hat{x}}_{m-1},t_{m-1}|x_0,\dot{\hat{x}}_0,t_0)\mathrm{d}\hat{x}_{m-1}\mathrm{d}\dot{\hat{x}}_{m-1} \\ &= \frac{k_{4,m}k_{7,m}}{\pi} \times \exp\left(-(\mathbf{X}_{0,m})^T\mathbf{U}_{0,m}^T\mathbf{U}_{0,m}(\mathbf{X}_{0,m})\right) \end{aligned} \tag{30}$$

for $m = 2,3,\cdots,M$, where $\mathbf{X}_{0,m} = [\hat{x}_0,\dot{\hat{x}}_0,\hat{x}_m,\dot{\hat{x}}_m]^T$, and

$$\mathbf{U}_{0,m} = \begin{pmatrix} k_{1,m} & k_{2,m} & k_{3,m} & k_{4,m} \\ k_{5,m} & k_{6,m} & k_{7,m} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{31}$$

In Eq.(31), the corresponding coefficients $k_{i,m}, i = 1,2,\cdots,7$ are presented in Zhang et al. (2021).

4. SURVIVAL PROBABILITY DETERMINATION

The survival probability is defined as the probability that the system stays within the prescribed thresholds for a given time interval, i.e.,

$$P_B(T) = P\Big\{B_{\mathrm{dl}} < x(t) < B_{\mathrm{du}}, t_0 < t < T| \; x(t_0) = x_0, \dot{x}(t_0) = \dot{x}_0\Big\} \tag{32}$$

where $P_B(T)$ is the survival probability; the range of $(B_{\mathrm{dl}}, B_{\mathrm{du}})$ denotes the fixed upper and lower bounds of the displacement response, respectively; and $[t_0, T]$ represents the time interval considered. In this regard, the survival problem with fixed barriers shown in Eq.(32) is equivalently cast to one with time-varying barriers, given by

$$P_B(T) = P\big\{B_{\mathrm{dl}} - \mu_x(t) < \hat{x}(t) < B_{\mathrm{du}} - \mu_x(t), t_0 < t < T|\hat{x}(t_0) = x_0 - \mu_x(t_0), \dot{\hat{x}}(t_0) = \dot{x}_0 - \mu_{\dot{x}}(t_0)\big\} \tag{33}$$

In this way, the survival problem of the equivalent system with time-varying barriers (e.g.Eq.(33)) can be addressed by recursively employing the transition PDF obtained. Further, a time-discrete version of Eq.(33) is introduced to approach the exact result of survival probability in Eq.(32), which is given by

$$\begin{aligned} P_B(T = t_M) = \mathrm{Prob}\Big\{& B_{\mathrm{dl}} - \mu_x(t_m) < \hat{x}(t_m) < B_{\mathrm{du}} - \mu_x(t_m), t_m \in [t_0, T], m = 1,2,...,M \\ & |\hat{x}(t_0) = x_0 - \mu_x(t_0), \dot{\hat{x}}(t_0) = \dot{x}_0 - \mu_{\dot{x}}(t_0)\Big\} \end{aligned} \tag{34}$$

where Prob$(\cdot)$ denotes the probability measure of intersection of simultaneously occurring M random events; $t_m = t_0 + m\Delta t, m = 1,...,M$, and $\Delta t = (T - t_0)/M$. Clearly, an small time interval Δt is appropriately selected to trade off the accuracy level and computational cost involved.

Next, manipulating Eq.(33) yields

$$\begin{aligned} P_B(T) = & \int_{B_{dl}-\mu_x(t_1)}^{B_{du}-\mu_x(t_1)}\int_{-\infty}^{+\infty}\cdots\int_{B_{dl}-\mu_x(t_M)}^{B_{du}-\mu_x(t_M)}\int_{-\infty}^{+\infty} \\ & p(\hat{x}_M,\dot{\hat{x}}_M,t_M;\hat{x}_{M-1},\dot{\hat{x}}_{M-1},t_{M-1};\cdots;\hat{x}_1,\dot{\hat{x}}_1,t_1 \\ & |\hat{x}_0,\dot{\hat{x}}_0,t_0)\mathrm{d}\hat{x}_1\mathrm{d}\dot{\hat{x}}_1\cdots\mathrm{d}\hat{x}_M\mathrm{d}\dot{\hat{x}}_M \end{aligned} \tag{35}$$

Employing the Markovian property in conjunction with C-K equation, the survival probability $P_B(T)$ can be obtained by numerical recursive manner (e.g.Iourtchenko et al. (2008))

$$P_B(T) = \int_{B_{dl}-\mu_x(t_M)}^{B_{du}-\mu_x(t_M)}\int_{-\infty}^{+\infty} q(\hat{x}_M,\dot{\hat{x}}_M,t_M|\hat{x}_0,\dot{\hat{x}}_0,t_0)\mathrm{d}\hat{x}_M\mathrm{d}\dot{\hat{x}}_M \tag{36}$$

where

$$q(\hat{x}_m,\dot{\hat{x}}_m,t_m|\hat{x}_0,\dot{\hat{x}}_0,t_0)=\int_{B_{dl}-\mu_x(t_{m-1})}^{B_{du}-\mu_x(t_{m-1})}\int_{-\infty}^{+\infty} p(\hat{x}_m,\dot{\hat{x}}_m,t_m|\hat{x}_{m-1},\dot{\hat{x}}_{m-1},t_{m-1})q(\hat{x}_{m-1},\dot{\hat{x}}_{m-1},t_{m-1}|\hat{x}_0,\dot{\hat{x}}_0,t_0)\mathrm{d}\hat{x}_{m-1}\mathrm{d}\dot{\hat{x}}_{m-1},\quad m=3,4,\ldots,M \tag{37}$$

with

$$q(\hat{x}_2,\dot{\hat{x}}_2,t_2|\hat{x}_0,\dot{\hat{x}}_0,t_0)=\int_{B_{dl}-\mu_x(t_1)}^{B_{du}-\mu_x(t_1)}\int_{-\infty}^{+\infty} p(x_2,\dot{x}_2,t_2|x_1,\dot{x}_1,t_1)p(\hat{x}_1,\dot{\hat{x}}_1,t_1|\hat{x}_0,\dot{\hat{x}}_0,t_0)\mathrm{d}\hat{x}_1\mathrm{d}\dot{\hat{x}}_1 \tag{38}$$

5. NUMERICAL EXAMPLE

In this section, a nonlinear hardening Duffing is considered for demonstrating the reliability of the developed technique. In the ensuring analysis, a small time interval $\Delta t=0.1s$ is selected. The nonlinear oscillator is assumed to be initially at rest, with the initial conditions satisfying $p(x(t_0),t_0=0)=\delta(x_0)$, and $p(\dot{x}(t_0),t_0=0)=\delta(\dot{x}_0)$ where $\delta(\cdot)$ is the Dirac delta function. A nonseparable nonstationary excitation is considered in the numerical example with its power spectrum density function given by

$$S_w(\omega,t)=S_0\left(\frac{\omega}{5}\right)^2\exp(-0.2t)t^2\exp\left[-\left(\frac{\omega}{10}\right)^2t\right] \tag{39}$$

where S_0 denotes the amplitude coefficient of the power spectrum.

Consider a Duffing nonlinear oscillator under combined excitation, whose motion of equation is given by Eq.(1) with

$$g(x,\dot{x})=\omega_0^2x+\varepsilon\omega_0^2x^3 \tag{40}$$

where the parameter ε denotes the magnitude of the nonlinearity. Substituting Eq.(40) into Eqs.(3) yields the deterministic sub-equation given by

$$\ddot{\mu}_x(t)+\beta_0\dot{\mu}_x(t)+\omega_0^2\mu_x(t)+\varepsilon\omega_0^2[3\mu_x(t)c(t)+\mu_x^3(t)]=F_1\sin(\omega_1 t) \tag{41}$$

Manipulating Eqs.(9-12) yields the equivalent linear damping and stiffness coefficients in the form

$$\beta_{\text{eq}}(c(t))=\beta_0 \tag{42}$$

and

$$\omega_{\text{eq}}^2(c(t))=\omega_0^2+3\omega_0^2\varepsilon\mu_x^2(t)+\frac{3}{2}\omega_0^2\varepsilon c(t) \tag{43}$$

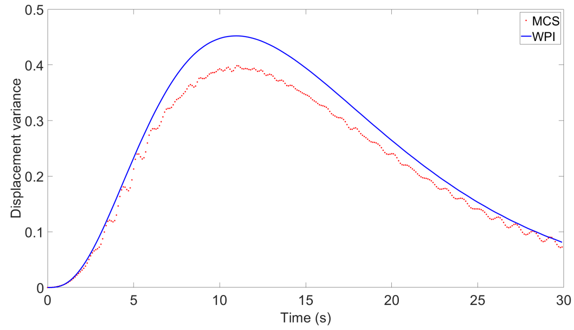

Figure 1: Variance of displacement of the Duffing oscillator; comparisons with MCS (10,000 realizations).

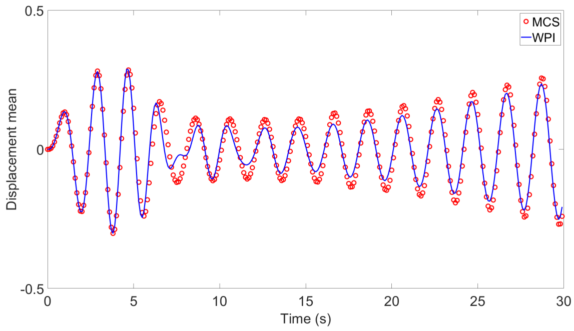

Figure 2: Mean of displacement of the Duffing oscillator; comparisons with MCS (10,000 realizations).

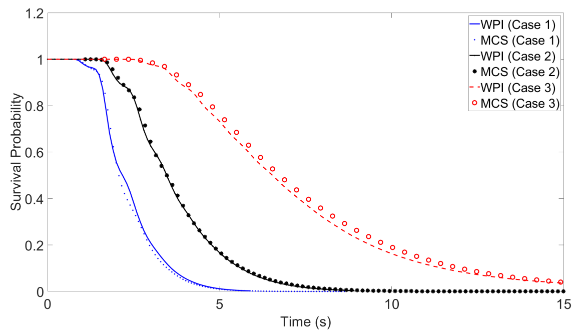

Figure 3: Survival probability for the Duffing oscillator under nonstationary excitation for various barrier levels (Case 1: $x(t)\in[-0.3,0.3]$; Case 2: $x(t)\in[-0.5,0.5]$; Case 3: $x(t)\in[-0.9,0.9]$) ; comparisons with MCS data (10,000 realizations).

In the following analysis, the Duffing oscillator under combined excitation with the parameters $(\omega_0 = \pi, \zeta_0 = 0.1, \varepsilon = 1, F_1 = 1, \omega_1 = \pi, S_0 = 2)$ is considered. In Figs.1 and 2, the mean value $\mu_x(t)$ and variance $c(t)$ of the displacement response $x(t)$ are plotted by the proposed WPI method, compared with pertinent MCS. Further, the survival probability for various barrier levels is shown in Figs.3. Clearly, the comparisons between the developed WPI method and pertinent MCS demonstrate a quite satisfactory agreement, especially the fluctuations in Figs.3 due to the influence of the harmonic excitation.

6. CONCLUDING REMARKS

In this paper, a WPI technique in conjunction with stochastic averaging/lin-earization has been developed for determining the survival probability of nonlinear oscillators under combined periodic and nonstationary stochastic excitation. Specifically, relying on a decomposition of the response into deterministic component and stochastic component, the original SDOF nonlinear system has been cast into a coupled combination of a ODE of the deterministic component and a SDE of the stochastic component. In particular, following the stochastic averaging/linearization treatment, the equivalent linear system with respect to stochastic component is obtained. Employing the analytical WPI method, a closed-form expression of the short-time joint transition PDF of stochastic component can be obtained. Finally, the survival probability and corresponding first-passage PDF have been derived. The Duffing oscillator as numerical examples has demonstrated the accuracy and efficiency of the developed method.

7. REFERENCES

Barbato, M. and Conte, J. P. (2011). “Structural reliability applications of nonstationary spectral characteristics.” *Journal of Engineering Mechanics*, 137(5), 371–382.

Chaichian, M. and Demichev, A. (2001). *Path Integrals in Physics: Volume I Stochastic Processes and Quantum Mechanics*. CRC Press.

Di Matteo, A., Spanos, P., and Pirrotta, A. (2018). “Approximate survival probability determination of hysteretic systems with fractional derivative elements.” *Probabilistic Engineering Mechanics*, 54, 138–146.

Ewing, G. M. (1985). *Calculus of Variations with Applications*. Dover Publications, Mineola, NY.

Grigoriu, M. (2013). *Stochastic Calculus: Applications in Science and Engineering*. Springer Science & Business Media.

Han, R., Fragkoulis, V. C., Kong, F., Beer, M., and Peng, Y. (2022). “Non-stationary response determination of nonlinear systems subjected to combined deterministic and evolutionary stochastic excitations.” *International Journal of Non-Linear Mechanics*, 147, 104192.

Iourtchenko, D., Mo, E., and Naess, A. (2008). “Reliability of strongly nonlinear single degree of freedom dynamic systems by the path integration method.” *Journal of Applied Mechanics*, 75(6), 061016.

Iwatsubo, T., Kawahara, I., Nakagawa, N., and Kawai, R. (1979). “Reliability design of rotating machine against earthquake excitation.” *Bulletin of JSME*, 22(173), 1632–1639.

Kougioumtzoglou, I. A. and Spanos, P. D. (2009). “An approximate approach for nonlinear system response determination under evolutionary stochastic excitation.” *Current science*, 1203–1211.

Kougioumtzoglou, I. A. and Spanos, P. D. (2012). “An analytical Wiener path integral technique for nonstationary response determination of nonlinear oscillators.” *Probabilistic Engineering Mechanics*, 28, 125–131.

Kougioumtzoglou, I. A. and Spanos, P. D. (2013). “Response and first-passage statistics of nonlinear oscillators via a numerical path integral approach.” *Journal of Engineering Mechanics*, 139(9), 1207–1217.

Kougioumtzoglou, I. A. and Spanos, P. D. (2014). “Nonstationary stochastic response determination of nonlinear systems: A Wiener path integral formalism.” *Journal of Engineering Mechanics*, 140(9), 04014064.

Li, J. and Chen, J. (2009). *Stochastic Dynamics of Structures*. John Wiley & Sons.

Petromichelakis, I. and Kougioumtzoglou, I. A. (2020). "Addressing the curse of dimensionality in stochastic dynamics: a Wiener path integral variational formulation with free boundaries." *Proceedings of The Royal Society A*, 476, 20200385.

Psaros, A. F., Zhao, Y., and Kougioumtzoglou, I. A. (2020). "An exact closed-form solution for linear multi-degree-of-freedom systems under Gaussian white noise via the Wiener path integral technique." *Probabilistic Engineering Mechanics*, 60, 103040.

Roberts, J. B. and Spanos, P. D. (1986). "Stochastic averaging: An approximate method of solving random vibration problems." *International Journal of Non-Linear Mechanics*, 21(2), 111–134.

Roberts, J. B. and Spanos, P. D. (2003). *Random Vibration and Statistical Linearization*. Dover, New York.

Rubinstein, R. Y. (2008). *Simulation and the Monte Carlo Method*. Wiley.

Soni, A. H. and Srinivasan, V. (1984). *Seismic Analysis of Rotating Mechanical Systems*. School of Mechanical and Aerospace Engineering, Oklahoma State University.

Spanos, P. D., Di Matteo, A., Cheng, Y., Pirrotta, A., and Li, J. (2016). "Galerkin scheme-based determination of survival probability of oscillators with fractional derivative elements." *Journal of Applied Mechanics*, 83(12), 121003.

Spanos, P. D. and Kougioumtzoglou, I. A. (2014). "Survival probability determination of nonlinear oscillators subject to evolutionary stochastic excitation." *Journal of Applied Mechanics*, 81(5), 051016.

Spanos, P. D. and Lutes, L. (1980). "Probability of response to evolutionary process." *Journal of Engineering Mechanics*, 106(2), 213–224.

Spanos, P. D., Zhang, Y., and Kong, F. (2019). "Formulation of statistical linearization for mdof systems subject to combined periodic and stochastic excitations." *Journal of Applied Mechanics*, 86(10), 101003.

Spencer, B., Tang, J., and Hilal, C. (1990). "Reliability of non-linear oscillators subjected to combined periodic and random loading." *Journal of Sound Vibration*, 140(1), 163–169.

Taniguchi, T. and Cohen, E. G. D. (2007). "Inertial effects in nonequilibrium work fluctuations by a path integral approach." *Journal of Statistical Physics*, 130(1), 1–26.

Vanmarcke, E. H. (1975). "On the distribution of the first-passage time for normal stationary random processes." *Journal of Applied Mechanics*, 42(1), 215–220.

Wehner, M. F. and Wolfer, W. G. (1983). "Numerical evaluation of path-integral solutions to Fokker-Planck equations." *Physical Review A*, 27(5), 2663–2670.

Wiener, N. (1921). "The average of an analytic functional." *Proceedings of the National Academy of Sciences of the United States of America*, 7(9), 253.

Xu, J., Li, Y., Mao, J., Yu, Z., and Tan, S. (2022). "Dynamic response and reliability analyses of non-linear structures driven by non-stationary non-gaussian stochastic ground motions." *Engineering Structures*, 268, 114689.

Zhang, Y., Li, S., and Kong, F. (2021). "Survival probability of nonlinear oscillators endowed with fractional derivative element and subjected to evolutionary excitation: A stochastic averaging treatment with path integral concepts." *Probabilistic Engineering Mechanics*, 66, 103156.

Zhu, H. and Guo, S. (2015). "Periodic response of a Duffing oscillator under combined harmonic and random excitations." *Journal of Vibration and Acoustics*, 137(4), 041015.