



Terms and Conditions of Use of Digitised Theses from Trinity College Library Dublin

Copyright statement

All material supplied by Trinity College Library is protected by copyright (under the Copyright and Related Rights Act, 2000 as amended) and other relevant Intellectual Property Rights. By accessing and using a Digitised Thesis from Trinity College Library you acknowledge that all Intellectual Property Rights in any Works supplied are the sole and exclusive property of the copyright and/or other IPR holder. Specific copyright holders may not be explicitly identified. Use of materials from other sources within a thesis should not be construed as a claim over them.

A non-exclusive, non-transferable licence is hereby granted to those using or reproducing, in whole or in part, the material for valid purposes, providing the copyright owners are acknowledged using the normal conventions. Where specific permission to use material is required, this is identified and such permission must be sought from the copyright holder or agency cited.

Liability statement

By using a Digitised Thesis, I accept that Trinity College Dublin bears no legal responsibility for the accuracy, legality or comprehensiveness of materials contained within the thesis, and that Trinity College Dublin accepts no liability for indirect, consequential, or incidental, damages or losses arising from use of the thesis for whatever reason. Information located in a thesis may be subject to specific use constraints, details of which may not be explicitly described. It is the responsibility of potential and actual users to be aware of such constraints and to abide by them. By making use of material from a digitised thesis, you accept these copyright and disclaimer provisions. Where it is brought to the attention of Trinity College Library that there may be a breach of copyright or other restraint, it is the policy to withdraw or take down access to a thesis while the issue is being resolved.

Access Agreement

By using a Digitised Thesis from Trinity College Library you are bound by the following Terms & Conditions. Please read them carefully.

I have read and I understand the following statement: All material supplied via a Digitised Thesis from Trinity College Library is protected by copyright and other intellectual property rights, and duplication or sale of all or part of any of a thesis is not permitted, except that material may be duplicated by you for your research use or for educational purposes in electronic or print form providing the copyright owners are acknowledged using the normal conventions. You must obtain permission for any other use. Electronic or print copies may not be offered, whether for sale or otherwise to anyone. This copy has been supplied on the understanding that it is copyright material and that no quotation from the thesis may be published without proper acknowledgement.

Enhancing
the
Teaching and Learning
of
Mathematics

in the
Senior Cycle

Two Volumes

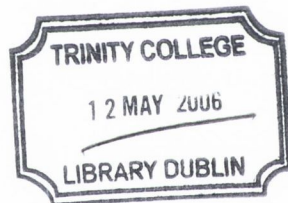
Volume I

David Flannery

A Thesis Submitted for the degree of
Doctor in Philosophy

University of Dublin

2006



THESIS
7774.1

Declaration

I declare that this thesis has not been submitted as an exercise for a degree at this or any other University.

I declare that the work submitted is entirely my own.

I agree that the Library may lend or copy this thesis upon request. This permission to cover only single copies made for study purposes, subject to normal conditions of acknowledgement.

David Flannery
18 April, 2006,

Summary

This thesis asserts that the current provision of mathematics education at Senior Cycle suffers from a number of shortcomings. These are described as a failure to show the relevance of the material being taught and an almost total neglect of the history and culture of the subject along with an ignoring of its recreational element. It is contended that the reason the current programme is not having its ambitions realised effectively is due to an overly narrow interpretation of the course material by way of texts and examinations. These points are substantiated at a number of junctures throughout the dissertation but in particular, in relation to examinations, in the chapter “Establishing the Case”. It is also maintained that there is a lack of substance/rationale in the current presentation of syllabus segments and a scant use of technology; claims which are subsequently established.

Collectively these circumstances determine the rationale and basis for the constructive element of this work which, with the exception of a minor contribution from Part I, constitutes all of Part II of this dissertation. The chapter structure of Part II is designed in such a way as to address each of the issues alluded to above in a particular order which (it is believed) lends the discussion maximum cohesion.

In order for this work’s ambitions to be realised effectively, it is argued that there needs to be a change in how subject material is presented both in textbooks and in the classroom. A central proposal as to how this might be

achieved pertains to the concept of a 'Topic'. The meaning, elaboration and practical illustration of this notion constitutes the opening chapter of Part II while also dictating much of the mathematical content and discussion appearing in the main body of the work and in its 'core' section. This core section resides on an accompanying cd and houses all of the concrete material written to support the various assertions made. It contains explicit illustrations of Topics, a great number of PowerPoint and Flash presentations, numerous written pieces as well as different sets of sample examination questions.

While the vast majority of the recommendations made in this work are fully justified by way of rationale and extensively illustrated so as to assist with their implementation in the classroom, this effort of its own is not enough to see all of them readily incorporated into an educational programme of the near future. Part III discusses various strategies by which the proposals set forth might be implemented in the framework envisaged by the National Council for Curriculum and Assessment for the 'horizon' year 2010.

A basic tenet which has driven much of the present undertaking is that one cannot hope to effect change, however gradual it may need to be in coming, in a significant and essential manner *without* spelling out in explicit and substantial detail for educationalists and teachers that which one has in mind. The result of delivering on this promise in a realistic and tangible manner accounts for the size of this work as embodied both in its written exposition and its technical demonstrations.

Acknowledgments

I wish to thank the Cork Institute of Technology for the financial assistance which allowed me to pursue this research; my supervisor Dr. John Evans for his valuable contributions throughout its gestation and evolution; the internal examiner, Dr. Aidan Seery, and external examiner, Professor John O'Donoghue, for their insightful and constructive suggestions; my friend Dr. Michel Vandyck for many thought-provoking conversations but above all I thank my family for their constant support.

Contents

1	Introduction	1
1.1	The Call for Reform of LC Mathematics	4
1.1.1	Purpose of this Thesis	9
1.1.2	Other Features Relating To Content	15
1.1.3	Methodological Issues	19
1.1.4	Consequences	24
1.1.5	What is good in the present HLC examination?	27
1.1.6	Summary	28
1.2	The Nature of the Proposed Reform	29
1.2.1	The Notion of a Topic	30
1.2.2	Use of Technology	35
1.2.3	The Three Context/Content Areas	39
1.3	Thesis Structure	51
I		55
2	Historical Aspects of Curriculum Development	57

2.1	Pre-Twentieth Century	59
2.1.1	Early Curriculum Reform in the U.S.	64
2.2	The Twentieth Century	69
2.2.1	The New Mathematics Movement	71
2.2.2	Curriculum Projects	81
2.2.3	Realistic Mathematics Education	87
2.2.4	Influence of Examinations	90
2.3	The Curriculum Development Enterprise	94
3	History of Curriculum Development in Ireland	97
3.0.1	Paucity of Documentation on Policy-Making	108
3.0.2	The Leaving Certificate Examination	111
3.0.3	Evaluation of Irish Curriculum Development	114
3.1	History of Ireland's Mathematics Provision	121
3.1.1	Recent Reviews/Revisions	138
3.1.2	The IMTA	140
4	Curriculum Matters	145
4.1	Approaches to Curriculum Development	145
4.2	Management of Curriculum Development	153
4.3	Evaluation within Curriculum Development	155
4.4	Innovation	160
4.5	The Teacher	162
4.6	Student Motivation	168

4.7	Student Activities	170
4.8	About Mathematics	172
4.8.1	What is Mathematics?	172
4.8.2	Why Mathematics?	174
4.8.3	How Much Mathematical Rigour?	177
4.9	Two Recent Endeavours	179
4.9.1	PSSM	181
4.9.2	PISA03	186
5	Establishing the Case	197
5.1	Formal Survey of 1992	199
5.2	Informal CIT Survey of 2000	205
5.2.1	HLC Mathematics Syllabus	229
5.3	The 2003 Mathematics Papers	230
5.3.1	General Remarks about Paper I	230
5.3.2	Analysis Question by Question	232
5.3.3	Analysis of Paper II	270
5.4	Current Textbooks	294
II		299
6	Topics	301
6.1	On Quadratics	308
6.2	Difference Equations	331

6.2.1	First Order Recurrence Relations	333
6.2.2	Second Order Difference Equations	352
6.3	The Newton-Raphson Algorithm	359
7	Recreational Mathematics	385
7.1	Magic Rectangles	393
7.2	Three by Three Magic Squares	399
7.2.1	Other Approaches	404
7.3	Sequential Analysis	412
7.4	The Chessboard	416
7.4.1	The Chessboard and Information Theory	424
7.5	Games	425
7.5.1	The Game of Mastermind	426
7.6	RM and Modular Arithmetic	431
7.7	RM Revealing the Power of Algebra	434
7.8	RM and Geometry	441
7.8.1	The Third Dimension	444
7.9	Algorithmic Problems	447
7.9.1	Card Tricks	450
7.10	Logic Problems	456
7.11	Summary	462
8	History and Culture of Mathematics	465
8.1	History of Mathematics	468

8.1.1	Illustrations from History	469
8.1.2	An Itemised History Time-Line	480
8.1.3	History Projects	491
8.2	The Culture of Mathematics	495
8.2.1	About Proof	498
8.2.2	Mathematical Literature	511
8.2.3	Mathematical Humour	514
8.2.4	Interesting and Intriguing Results	523
8.2.5	The Pleasures of Probability	527
8.2.6	The Nature of the Infinite	543
8.2.7	Complexity	546
8.2.8	Topological Ideas	551
8.2.9	The Platonic Solids	554
9	Applications	557
9.1	Geometry/Trigonometry	561
9.2	Counting	575
9.3	Sequences and Series	576
9.4	Applied Probability	580
9.5	Calculus	587
9.6	New Topic Areas	598
9.7	Error Analysis	604

10 Use of Technology	607
10.1 The Internet as a Resource	610
10.2 Presentations/Slide shows	611
10.3 Mathematical Packages	614
10.4 Mathematical Programming	617
10.4.1 Concrete Illustration	618
10.4.2 Overview of Mathematical Programming	624
 III	 629
 11 Contribution/Implementation	 631
11.1 Conclusion	659
 Illustration of a Topic	 663
 Mathematical Vignettes	 723
 A Socratic Dialogue	 733
 Bibliography/References	 791

Core Material on the accompanying CD.

Topics

1. On Quadratics

- (a) On Quadratics [pdf]
- (b) Threshold Schemes: How to Share a Secret [Slide show]

2. Difference Equations

- (a) Difference Equations [pdf]
- (b) The Regions Problem: A Recurrence Relation – Formulation and Solution [Slide show]
- (c) The Tower Of Hanoi [Slide show]

3. The Newton-Raphson Algorithm

- (a) The Newton-Raphson Algorithm [pdf]

Recreational Mathematics

1. Magic Rectangles

- (a) 2 x 2 Magic Rectangles [pdf]

2. 3 x 3 Magic Squares

- (a) The 3 x 3 Magic Square [Slide show]
- (b) Non Commutativity using the 3x3 Magic Square [Slide Show]
- (c) The Abstract 3 x 3 Magic Square [Slide Show]
- (d) Four by Four Magic Squares [Slide Show]
- (e) Beyond 3 x 3 Magic Squares [Slide Show]
- (f) The Adjacency Puzzle: Eight Numbers in Eight Cells [Slide Show]
- (g) Eight Numbers On The Cross [pdf]
- (h) Magic Triangle [Slide show]
- (i) The 3 x 3 Magic Square [pdf]

3. Sequential Analysis

- (a) Archibald's House Number [pdf]
- (b) The 100 Metre Dash [Slide show]
- (c) The Prison Wardens Problem [pdf]
- (d) What Are Their Ages? [pdf]
- (e) What Are Their Ages? [Slide show]

4. The Chess Board

- (a) An Information Theory Game [pdf]
- (b) Using the Chessboard to Introduce Basic Ideas from Information Theory [Slide show]

- (c) Chess Notation And Terminology [Slide show]
- (d) Weighing with a Balance Pan ($\log_2 3$) [Slide show]
- (e) The Chessboard A Source of Many Questions [Slide show]

5. Games

- (a) A Mastermind Problem [pdf]
- (b) Mastermind Mathematics Problems [pdf]
- (c) Mastermind [Flash animation]
- (d) The Game Of Fifteen [Slide show]
- (e) Tic-tac-toe [Slide show]

6. Modular Arithmetic

- (a) Examples Of Modular Arithmetic [Slide show]
- (b) The Mod Notation [pdf]

7. Revealing The Power Of Algebra

- (a) Always 1089 [pdf]
- (b) One Before And After [pdf]
- (c) Sum To One Hundred [pdf]
- (d) What Are Our Ages? [pdf]

8. Geometry

- (a) Algebra through Geometry [Slide show]
- (b) The 13 Convex Tangrams [Slide show]
- (c) The Symmetries of a Square [Slide show]

9. Algorithmic Problems

- (a) Algorithmic Problems [pdf]
- (b) The Come Forth Card Trick [pdf]
- (c) The Last Card Trick [pdf]
- (d) The Mind Reader Card Trick [pdf]
- (e) The Selected Card [pdf]
- (f) The Seven To Eleven Card Trick [pdf]
- (g) The Tree Of Clubs Card Trick [pdf]
- (h) The Twenty One Card Trick [pdf]
- (i) The Best Card Trick [pdf]
- (j) The Best Card Trick Ever [Slide show]
- (k) Hailstones [pdf]
- (l) Hailstones [Slide show]
- (m) The Twelve Balls Problem [pdf]
- (n) Twelve Balls [Flash Animation]

10. **Logic Problems**

- (a) A Cryptographic Puzzle [Slide show]
- (b) Buddhist Monk [Slide show]
- (c) Logic Problems [pdf]

History & Culture Of Mathematics

1. **History**

- (a) Illustrations From History [pdf]
- (b) Egyptian Multiplication [Slide show]
- (c) Peasant Multiplication [pdf]
- (d) The Truncated Pyramid [Slide show]
- (e) Two Tales About Thales [pdf]
- (f) Carpenter's Squares [Slide show]
- (g) An Itemised History Time Line [pdf]
- (h) Evolution of Notation [pdf]

2. **Culture**

- (a) About Proof [pdf]
- (b) Chessboard Induction [Slide show]
- (c) Mathematical Vignettes [pdf]

(d) Matheamtical Literature

i. The Square Root of Two [pdf]

ii. The Square Root of Two [Slide show]

(e) Poker Tutorial [Flash Animation]

(f) Prime Producing Polynomials [pdf]

(g) Recurrent Problems [Slide show]

(h) Repeated Doubling [Slide show]

(i) The Infinite [pdf]

(j) Cellular Automata [Slide show]

(k) Fractals [Slide show]

(l) Topological Ideas [pdf]

(m) The Platonic Solids [pdf]

Applications

1. Applied Counting Problems [pdf]

2. Applied Sequences And Series Problems [pdf]

3. Applied Trigonometric Problems [pdf]

4. Bezier Curves [pdf]

5. Derivative From First Principles [pdf]

6. Error Analysis [pdf]
7. Hamming [Flash Animation]
8. International System of Book Numbering [pdf]
9. Surveying Problems [pdf]
10. The I.S.B.N. Scheme [pdf]
11. Two Max-Min Problems Without Calculus [pdf]
12. Zero Knowledge Protocol [Slide show]

Use Of Technology

1. Oliver Byrne's 1847 Proof of Euclid's Proposition 32, Book I [Slide show]
2. Six Programming Tasks [pdf]

Implementation

1. A Fun Quiz [pdf]
2. IMTA Mathematics Problems [pdf]
3. Questions & Answers connected with the Root-Two Story [pdf]
4. Root Two Exercises [pdf]
5. Sample History Questions [pdf]

- 6. Senior Cycle Project Areas [pdf]
- 7. Teachers' Projects [pdf]

List of Tables

3.1	Cognitive categories and activities.	116
4.1	The PISA 2003 competency clusters.	193
5.1	Question type.	231
10.1	Ten by ten array of the first 100 natural numbers.	620
10.2	Triangular array of the first 55 natural numbers.	621
10.3	Second triangular array of the first 55 natural numbers.	622

List of Figures

1.1	Instructional sequence guiding the design of a Topic.	31
4.1	The mathematisation cycle.	194
4.2	The NCTM – PISA parellelism.	195
5.1	Parabolic arch.	236
5.2	Parabolic arch and bounding rectangle.	239
6.1	Instructional sequence guiding the design of a Topic.	301
7.1	Shortest routes.	422
8.1	Past examination question.	516
8.2	Effron’s nontransitive dice.	542
9.1	Sample examination question.	565
9.2	Cosine waveform.	570
9.3	Sine waveform.	570
9.4	The two waveforms.	571
9.5	The resultant waveform.	571

9.6 Component waveforms and their resultant. 572

9.7 Cosine waveform and the resultant waveform. 573

11.1 Instructional sequence guiding the design of a Topic. 635

Chapter 1

Introduction

In December 2002 the National Council for Curriculum and Assessment (NCCA) set in motion a new cycle of educational reform in the Irish education system with the publication of *Developing Senior Cycle Education – Consultative Paper on Issues and Options*¹. One of the main purposes of this consultative paper is the review and reform of the Leaving Certificate Established (LCE) along with the planning of a programme of action in this context. The paper alludes to the fact that while considerable change and extensive development in senior cycle education has taken place during the 1990s, the LCE did not receive the attention it deserves and that the time has come for it to be examined in detail.²

Shortly after the circulation of the consultative paper to the various partners in education and other concerned organisations, the NCCA entered on

¹National Council for Curriculum and Assessment (NCCA), *Developing Senior Cycle Education – Consultative Paper on Issues and Options* (Dublin: NCCA, December, 2002)

²The status of the proposed reforms as of August 2005 is described briefly in the concluding chapter.

an extensive consultative process with all these groups by means of seminars and bilateral meetings. In September 2003 the council reported in *Developing Senior Cycle Education – Report on the Consultative Process*³ on the various ideas and concerns expressed while also presenting the individual submissions prepared by many of these bodies in response to the consultative paper. Building on this report the NCCA set forth in 2004 its *Developing Senior Cycle Education - Directions for Development*⁴ a broad but clear outline of a *vision* for senior cycle education that it hopes will be attained by the ‘horizon year’ 2010. Features of this future scenario are:⁵

1. A different school culture
2. A re-structured learning experience
3. A re-balanced curriculum
4. Different assessment arrangements and a new certificate

Section 4 of the *Report on the Consultative Process* relates that already much work has been done of a specific nature concerning curriculum development in the LCE which will contribute to the envisaged new framework:⁶

³National Council for Curriculum and Assessment (NCCA), *Developing Senior Cycle Education – Report on the Consultative Process* (Dublin: NCCA, September, 2003)

⁴National Council for Curriculum and Assessment (NCCA), *Developing Senior Cycle Education - Directions for Development* (Dublin: NCCA, 2004)

⁵*Ibid.*, 3.

⁶*Report on the Consultative Process* (2003), 6.

New syllabuses have been introduced in biology, physics, chemistry, English and home economics in recent years, and plans are underway to introduce a new syllabus in religious education in 2003 and revised syllabuses in history and geography in September 2004. The four revised technology subjects, as well as physical education, are expected to be introduced shortly, and syllabus revisions in art and economics are nearing completion.

Elsewhere, in the consultative document under the heading NCCA Draft Plan of Action bullet point six reads⁷

Review syllabuses in Applied Mathematics, Classical Studies, Greek, Jewish Studies and Latin

Taken together, one sees from these extracts that, as of 2004, almost every subject on the LCE curriculum had been revisited in the recent past with a view to possible updating or was due for review; the notable exception being the 'core' subject Mathematics. The syllabi for this subject at Higher and Ordinary levels were introduced in 1992 and first examined in 1994. Between then and 2004 none had undergone any modification nor had there been any change in how each is examined. In May 2004 management authorities of second level schools were informed by the Department of Education and Science (DES) that⁸

... there will be no amendments to the syllabuses for Leaving Certificate Mathematics at higher, ordinary or foundation levels. Neither will there be any changes in the assessment of these syllabuses.

⁷*Developing Senior Cycle Education – Consultative Paper on Issues and Options* (2002), 65.

⁸DES circular M38/04, 1.

The same situation prevailed for the year 2005. However, in October of that year the NCCA circulated a discussion paper entitled *Review of Mathematics in Post-Primary Education* which sought “to address a range of issues surrounding mathematics at post-primary level in Ireland.”⁹ These issues are described in the next section.

1.1 The Call for Reform of LC Mathematics

The Task Force on the Physical Sciences: In 2002 the question of reform in mathematics education received considerable attention from the Task Force on the Physical Sciences, a committee entrusted with identifying the causes for decline in third-level uptake of the science, engineering and technology (SET) subjects and charged with making recommendations as to how matters in this regard might be remedied. In their *Report and Recommendations to the Minister for Education and Science*,¹⁰ the chairman Daniel O’Hare wrote:¹¹

Ireland’s economic future depends critically on the supply of an increasing number of people qualified in science and engineering. But at the very time this demand is increasing, there has been a sharp fall-off in interest in the sciences throughout our education system.

One reason for this sharp fall-off is identified:¹²

⁹National Council for Curriculum and Assessment (NCCA), *Review of Mathematics in Post-Primary Education: Discussion Paper* (Dublin: NCCA, October, 2005), 2.

¹⁰Task Force on the Physical Sciences, *Report and Recommendations to the Minister for Education and Science* (Dublin: 2002)

¹¹*Ibid.*, i.

¹²*Ibid.*, Summary, vii.

Serious concern about the mathematical competence of students in schools and in higher education permeates the debate on the declining uptake of the sciences. The decline in performance in mathematics at second level was highlighted by the highest-ever failure rate (17%) in ordinary level Leaving Certificate Mathematics in 2001. This high failure rate rendered one-sixth of school leavers ineligible for many science, engineering and technology courses in higher education. Because failure rates among males was higher (19%), this reduced the pool of potential candidates for engineering technology courses in institutes of technology, whose applicants are predominantly male.

The Task Force cites the Chief Examiner's report on the ordinary level mathematics for Leaving Certificate 2001: "Causes suggested for low grades included the knock-on effects of difficulty with the old junior cycle syllabus (which was revised in 1999), and the increase in part-time work by students."¹³

Parenthetically it may be added that by 2004 the picture regarding failure rates with its effect on input into third-level had changed little.¹⁴ The Education editor of the Irish Times, Seán Flynn wrote:¹⁵

Almost one in five Leaving Certificate students are ineligible for most third-level courses. Some 5,000 failed ordinary-level or higher-level maths, while a further 5,800 took foundation-level, which is not accepted by most third level colleges.

As for the reasons for this state of affairs, a very forthright Chief Examiner for Mathematics is quoted by Flynn as being of the opinion that "students

¹³Ibid., Summary vii.

¹⁴Reportedly much to the chagrin of the chairman of the Task Force: see the article *Science task force criticises lack of action*, Irish Times, 19 August, 2004, 4.

¹⁵Seán Flynn, "Leaving results show high failure rates in maths, science", *The Irish Times*, August 18, 2004.

are too preoccupied with part-time work, and their hectic social lives to give maths the concentrated attention it deserves.”¹⁶ [The 2005 results had the same education editor reporting that “... some 12 per cent (almost 4,500 students) failed ordinary-level maths. A further 5,000 took foundation-level maths, which is not recognised by many third-level colleges.”¹⁷]

The Task Force reports that, “Many science, engineering and technology departments in higher education report a lack of adeptness in basic mathematical skills on the part of students with ordinary level Mathematics.” It continues:

Students’ perception of the difficulty of mathematics and their poor performance in the subject both act as barriers in the participation and success in the sciences at second and third level. The risk in not addressing the problem with mathematics is that of undermining reform in science education.

In relation to undertaking a review of mathematics, the Task Force called on the DES to “institute an urgent investigation into the problem of decline. In addition to examining academic factors they should also examine the social and behavioral factors identified.”¹⁸ It recommended that the NCCA should undertake a review of Leaving Certificate (LC) mathematics and that a Higher Education SET group should be appointed to consult with the NCCA course committee on LC Mathematics.¹⁹ The Task Force recom-

¹⁶Cited in Ibid.

¹⁷Seán Flynn, “High rates of failure in Leaving Cert maths and science”, *The Irish Times*, August 17, 2005.

¹⁸Task Force on the Physical Sciences, *Report and Recommendations to the Minister for Education and Science*, **Section 4.3** xvii.

¹⁹Ibid., **Section 4.3** xvii.

mended four strands of action:²⁰

- Investigation in the decline in mathematics performance
- Increased input into mathematics curriculum by higher education
- Review LC Mathematics
- Credit for Foundation Level Mathematics

Thus it may be said that the Task Force made its call for reform of LCE mathematics based essentially on two features of students' attempts at the current programme: (i) the high failure rates at LC level (in particular, Ordinary level) and (ii) the lack of basic mathematical skills that many students (with Ordinary level LC mathematics) evince when at third level. One consequence of these deficiencies is that this country is not producing enough SET graduates. However, as to the nature any reform should take, the Task Force contented itself with general recommendations while entrusting the details of such reform to an NCCA committee advised by a higher education SET group. While the Task Force did not itself address the fundamental question as to why these two features prevail, significantly, it did, as the first of the four strands of action testifies, acknowledge that the question should be investigated.

The Task Force recommended reform in mathematics education in part on utilitarian grounds; because the economic well-being of the nation requires

²⁰Ibid., 130.

it. The argument is that if there are more students failing in mathematics than before, then the pool of potential students who are eligible to do SET courses at third level is thereby being reduced and so the number of SET graduates this country can hope to produce must inevitably fall off with undesirable consequences.

Review of Mathematics in Post-Primary Education: As already mentioned, the NCCA published in October of 2005 a discussion paper entitled *Review of Mathematics in Post-Primary Education* which “... set out the background and context for a review of mathematics in the post-primary curriculum.” In its concluding section it provides the following list of “a number of distinct areas that need to be considered”:²¹

- the purposes of mathematics education, including societal expectations
- the mathematics curriculum/syllabuses and changes that may be needed
- the assessment and certification of mathematics
- the requirements of third level institutions and how these may (or should) be accommodated

²¹National Council for Curriculum and Assessment (NCCA), *Review of Mathematics in Post-Primary Education: Discussion Paper* (Dublin: NCCA, October, 2005), 25.

- the perceived problems with regard to mathematics education in schools
- the role of information and communications technology in mathematics education
- the teaching and learning ‘culture’ of mathematics
- the inservice education and training of mathematics teachers

While this discussion document identifies key issues urgently in need of attention, it does not outline a programme of action as to how these issues might best be addressed in a concrete manner. Thus, as a follow-on document responding to the NCCA’s 2002 paper *Developing Senior Cycle Education – Consultative Paper on Issues and Options* it laid a greater stress on Issues than it did on Options.

1.1.1 Purpose of this Thesis

In this thesis, too, there is a call for reform. The intention here is to *help* in the specifics of the reform championed by the NCCA and to be implemented in whatever framework emerges. What is offered here are discussions and illustrations of curriculum development in a number of areas relating in particular to content and methods of teaching/learning which might, after scrutiny and modification, contribute to the provision of mathematics as a core subject in the near future.

The question as to what is ‘wrong’ with the current LCE mathematics programme, (in particular the Higher Course), is now touched upon lightly. This matter will be treated in depth in a later section.

In a word, the problem is one of *interpretation*. Central to the approach taken in this thesis is the view that the mathematical techniques taught and examined in some depth are not being justified adequately by way of application to the solving of problems external to mathematics itself. Consequently, the subject can appear bewildering and irrelevant to many students, making it difficult for them to take a genuine interest in it. The current programme which has a clearly expressed and laudatory set of aims and objectives allied to a syllabus with a rich content, is not having its ambitions realised effectively because of an overly narrow interpretation of the course material by way of texts and examinations. Broadly speaking, the single biggest manifestation of this narrowness of interpretation pertains to relevance and self-justification.

The issue of relevance: In a small survey carried out in 2000, one first-year engineering student, who gained 495 points overall and a B3 in higher course mathematics, wrote in reply to the question “Is mathematics useful?”.²²

²²Questionnaire given to first year Electronic Engineering Students at Cork Institute of Technology, September, 2000. (see Appendix A.)

To be honest, I have no idea what a lot of maths is useful for. In secondary school we were never told what the maths we were doing could be applied to. The subject was never made relevant to daily life. It was just dished out to me and I was left the monotonous task of figuring most of it out for myself.

The student continued:

I think maths would be a lot more easier to learn if it had more relevance to the receiver. Then at least there would be a reason to learn it and not just for the sake of obtaining an "A" in the exams.

One aspect of this work seeks to show that changes of emphasis and approach to existing material could be made to achieve major improvements in appreciation and understanding within a structure whose aims and objectives are essentially those of the existing programme but whose emphasis differs radically from current implementation.

The final examination: The primary source of the problem is believed by many to be the final examination. It should be remembered that at the moment this is the only assessment students are given for a two-year course.

Mc Namara writes:²³

The assessment is solely by written examination; no school based work is included. The papers contain closed questions to which there is usually a definite answer, and standard methods of solution; there are few opportunities for students to show any creativity or originality.

²³A. Mc Namara, "Investigational Mathematics in Transition Year – A Pilot Project" (M. Ed. diss., Dublin University, 1994), 77.

Under the heading of **Examination papers** the NCCA's 2005 discussion paper *Review of Mathematics in Post-Primary Education* notes that:²⁴

Examination papers still reflect the formal language and rigorous specification of questions that typify the 'modern mathematics' era. Most questions are presented as mathematical tasks (for example, "solve the equation" ...) without being set in a context.

As this work attempts to demonstrate, the syllabus material is being examined in a manner which stresses the technical almost to the exclusion of all other aspects of mathematics. The questions appear to test little more than memory recall, reproduction and technical ability. In addition, many of the questions asked seem to lack coherency by being no more than a collection of isolated and context-free fragments. It is not unusual to see in the same question sections relating to completely different areas of mathematics.²⁵ It is very rarely that one sees a coherent question of substance, all of whose sections relate to the one subject area and stepped in such a manner where, at the very least, the final portion of the question has a transparent relevance.

Despite the stated intentions of the syllabus regarding the relevance of mathematics, *viz.* "material is chosen for its intrinsic interest and general applicability"²⁶, the Higher Leaving Certificate (HLC) mathematics papers appear to fail to convey a proper sense of the power of mathematics to solve problems external to itself. An analysis of HLC Mathematics Paper I, 2003

²⁴National Council for Curriculum and Assessment (NCCA), *Review of Mathematics in Post-Primary Education: Discussion Paper* (Dublin: NCCA, October, 2005), 11.

²⁵*e. g.* Q.3, Paper I, 2003.

²⁶An Roinn Oideachais, *The Leaving Certificate, Mathematics Syllabus, (Higher and Ordinary Level)* (Dublin: The Stationary Office, n.d.), 5.

(described later) reveals that no more than one-fortieth of the material examined concerned itself with an application – and this was of the ‘intra-mathematical’ type, that is, internal to mathematics itself.²⁷

The manner in which a syllabus is taught is often influenced by how it is examined, particularly when those who teach it do not have the privilege of examining it. In fact, as O’ Donoghue and Ward note:²⁸

Kulm refers to two significant features of the impact of testing by saying “Whatever gets tested gets taught” (Kulm, 1990) and then even more depressing “only that which is tested will be taught.”

They add that, “Recent commentary on the second level system in Ireland has raised similar concerns. All those involved in the Irish education system, students, parents, teachers and employers have made similar remarks in relation to the Leaving Certificate (*Commission on Points, 1998*).²⁹

Whatever the merits or otherwise of a single national examination taken by one and all at the end of a course of study, it is generally the case that such examinations tend to settle down a short while after their inception to a predictable pattern as is certainly the case with the present HLC examination. In fact Mc Namara asserts that this predictability is intentional:³⁰

²⁷See page 189 for a fuller definition of the term ‘intra-mathematical’.

²⁸Patricia Ward, “Mathematics Assessment: A Process for Students with Special Needs” in *Proceedings of First National Conference on Research Mathematics Education* 1, no.1 (September 2005): 297.

²⁹G. Kulm (ed.) “New Directions for Mathematical Assessment.” *Assessing Higher Order Thinking in Mathematics*. American Association for the Advancement of Science (1990): 72, quoted in *ibid.*, 297.

³⁰A. Mc Namara, “Investigational Mathematics in Transition Year – A Pilot Project” (M. Ed. diss., Dublin University, 1994), 78.

A major aim of this new syllabus is to make the Higher level course more accessible to students. It was hoped to achieve this by some reduction in content, but largely by ensuring a predictable format, and level of difficulty to all examination questions, and by strict adherence to the written syllabus.

Current textbooks: A second major element to be considered in this examination of the current shortcomings in the HLC programme is the body of existing school textbooks. These books, as will be discussed later, appear to be dominated by the same technical emphasis that pervades the LC examination. The discussion document *Review of Mathematics in Post-Primary Education* comments that, “The main classroom ‘aid’ is the textbook. Again, Irish textbooks are somewhat functional by comparison to those in other countries [...]. As evidenced by inspection visits, teaching is highly dependent on the class textbook (which tends to reinforce the ‘drill and practice’ style) and the examinations, and there is frequently a very close relationship between the two.”³¹ The Review adds that Lyons *et al* (2003) found that students were generally not given insights into the applications of mathematics in everyday life; learning was a matter of memorising mathematical procedures and facts:³²

Mathematics was presented to students generally as a subject a) that had a fixed body of knowledge; b) that was abstract in character; c) that required demonstration of procedures rather than explanations; and d) that comprised discrete elements.

³¹National Council for Curriculum and Assessment (NCCA), *Review of Mathematics in Post-Primary Education: Discussion Paper* (Dublin: NCCA, October, 2005), 21.

³²Lyons *et al.*, (2003): 143 as cited in *Ibid.*, 21.

When it comes to the presentation of material Christiansen *et al.* report that, “the textbook provides for most teachers the indispensable framework for the organisation of the teaching process, both with respect to the sequencing of the mathematical content and to the time spent on the different major sub-themes.”³³

This thesis concludes that these textbooks constrain the teacher and fail both subject and student by being guilty of the same feature bedeviling the examination – a failure to show relevance.

1.1.2 Other Features Relating To Content

A number of other features of the present programme are worthy of examination. It appears that by almost exclusively concentrating on technical mathematics, facets of the subject such as its history and culture along with its recreational element are being ignored entirely.

The history/culture of mathematics: The Department of Education and Science has, over the years, made many injunctions to incorporate something of the history of mathematics into the teaching of the subject of mathematics. For example, the last paragraph of the 1988 Preamble on Mathematics related to the programme preceeding the present one, reads:³⁴

It is recommended that reference be made where appropriate to the history of Mathematics and to the lives of the great mathematicians.

³³B. Christiansen, A. G. Howson & M. Otte, *Perspectives in Mathematics Education* (Dordrecht: D. Reidel Publishing Company, 1985), 295.

³⁴An Roinn Oideachais, *Rialacha agus Clár* (Dublin: Stationary Office, 1988), 217.

The current programme requires that students:³⁵

... should be aware of the history of mathematics and hence of its past, present and future role as part of our culture.

Yet nowhere, in either programme, is there a question on the final examinations addressing this recommendation.

This work seeks to demonstrate that knowledge of mathematical history can be made to encourage students. Learning and practicing some of the rules of calculus might appear like mindless drudgery unless one is assured that the skills acquired allow more complicated problems to be tackled and solved. But even the apparent tedium of such endeavours can be alleviated by an enthusiastic teacher who can relate from history some of the blunders made by the most eminent pioneers. For example, Leibniz did not discover the product rule for differentiation at his first attempt but had a short struggle to uncover it. It might do wonders for a student's confidence to be told that one morning in the library at Hanover while developing the calculus, this towering mathematician unthinkingly assumed that $(uv)' = u'v'$, a 'false rule' which so many others since then have also unwittingly surmised to be the case. We are told that it was evening before Leibniz realised the error of his too-casual supposition and found the correct formula $(uv)' = u'v + uv'$ – a rule which to this day is known on the continent as the Leibniz rule.

An awareness that even the great are fallible and sometimes make unwarranted assumptions not unlike our own surely encourages us all. It may

³⁵An Roinn Oideachais, The Leaving Certificate, Mathematics Syllabus, (Higher and Ordinary Level), Item I of Other Objectives, 3.

also help young people realise that science or mathematics is not revealed but evolves through a spirit of enquiry which is often inept, even blundering and which rarely asks the right questions straight off or finds their answers readily. Hunter writes that:³⁶

Alvarez makes the case for enhancing all mathematics courses with readings on the evolution of the subject. She notes that the development of mathematical ideas is seldom as tidy as the concise, elegant packages we present to our students. She reminds us that mathematics appreciation can be part of every course we teach for our students' sake and for our own.

This work has much to say about using the history/culture of mathematics as an aid to the teaching/learning of the subject.

Recreational mathematics: Another feature considered in this work is recreational mathematics; that playful branch of the mathematics known to have engaged and charmed the minds of many over the ages, attracted so many to its further study and inspired much that is 'useful' within the mainstream of the subject.

Recreational mathematics (RM) has long been recognised as an ideal method for the student to 'learn by discovery' and is used as early as kindergarten to help young children develop mathematical skills. The *Review of Mathematics in Post-Primary Education* reports that the NCCA's Primary Curriculum Review 2003-2004 found in case studies conducted as part of that

³⁶David Hunter, Review of *Loving Math Infinitely* by Josephina Alvarez, *College Mathematics Journal* 32, no. 5 (November 2001): 398.

review that, “Children found mathematical games, puzzles and interesting problems a good motivational influence in their mathematics learning.”³⁷

While this aspect of RM shall be discussed and considered, it has to be admitted that the logistics of implementing this form of active learning in a genuine manner can be formidable. As a result, this work suggests a different exploitation of RM, in a way more readily realisable in the senior cycle classroom.

It seems reasonable to claim that the history/culture of mathematics and recreational mathematics, when presented in an appealing manner, can prove ideal vehicles for generating the interest so necessary to engage the learner. Interest is a key ingredient in any educational endeavour – “... above all one should seek to cultivate interest: ‘to be wearisome is the cardinal sin of instruction’.”³⁸ Besides informing students of their cultural heritage as it relates to mathematics and activating the spirit of enquiry, these two areas of study can be made to touch, in different ways, on the issue of motivation as it relates to relevance; a factor which for many is the single biggest reason for undertaking any study.

The following aspects of mathematics:

³⁷National Council for Curriculum and Assessment (NCCA), *Review of Mathematics in Post-Primary Education: Discussion Paper* (Dublin: NCCA, October, 2005), 3.

³⁸Due to the Prussian educationalist J.F. Herbart (1776-1841) as cited in A. G. Howson, C. Keitel & J. Kilpatrick, *Curriculum developments in mathematics* (Cambridge: CUP, 1981), 25.

- Applications
- History and Culture
- Recreational

are identified in this work as three significant ones, relating to content, which are either not being addressed adequately, or are being neglected entirely, by the current programme.

1.1.3 Methodological Issues

Turning to the issue of methodologies this section discusses very briefly the lack of substance/rationale in the current presentation of syllabus segments and the scant use of technology.

Rationale/substance: The preoccupation of the current programme can be described as being almost entirely with the ‘how’ of mathematics where methods are taught and learned in context-free environments rather than context-bound ones where the ‘why’ of the subject is also to the fore. Oldham wrote in 1996 that, “The ‘realistic’ approach is not familiar in Ireland. Our emphasis may be less on *concepts* and more on *procedures*.”³⁹ This situation had not changed by 2005 as evidenced by the discussion paper *Review of Mathematics in Post-Primary Education* which notes that:⁴⁰

³⁹Elizabeth Oldham, “Trends in Mathematics in Mathematics Education: Ireland in International Perspective”, *Studies in Education* 12, no. 2 (Autumn 1996): 46.

⁴⁰National Council for Curriculum and Assessment (NCCA), *Review of Mathematics in Post-Primary Education: Discussion Paper* (Dublin: NCCA, October, 2005), 6.

Post-primary mathematics syllabuses in Ireland do not currently make reference to the modelling or RME approaches.

The questions asked in the final examination tend to be short snap-shots rather than more substantial ones building towards a ‘finale’ which reveal an overall purpose. Similarly, (as will be shown) the theory and related problems to be found in the texts are largely of this form also. Consequently, there is a lack of substance and rationale to many of the topics covered in the textbooks. To indicate very briefly but concretely what is being said, the topic area of ‘cubic equations’ is used to illustrate the point being made. Many others examples, some much more serious, will be given at appropriate junctures.

What does a student know after being taught about cubic equations (cubics) in the fashion treated by the textbooks? Cubics are treated in two stages. On the first pass the student is told how to obtain the *integer* root of carefully chosen cubics (which appear unmotivated) using the factor theorem – an exercise which is straightforward but essentially technical. At the second stage he or she learns that if a cubic doesn’t have a rational root, then the Newton-Raphson algorithm (taken as a given) must be used. Both in the texts and in the examination an initial approximation is always supplied for cubics of this type (which are also served up without a hint as to their origin) and the student is expected to beaver away assiduously to get the desired root – a second straightforward exercise in applying mathematical technique. And there matters rest. Having been instructed on ‘how’ to get the roots of a cubic equation, the enquiring student cries, “But why, oh why?” His

teacher may be up to telling him but, alas, the texts don't. So at the end of a course on cubics the student may well know how to get the root of a cubic equation but will be unable to say why one might want to find such a root.

This specific illustration of failure to provide motivation and rationale is representative of a general situation. This thesis argues that it is imperative to the educational enterprise to give more substantial treatment of syllabus segments which clearly reveal purpose.

Use of Technology: “New technology has changed, not only mathematics, but also the *ways in which it can be taught and learnt*.” – so wrote Oldham shortly after the first examinations of the new mathematics courses in the senior cycle.⁴¹ She illustrated this remark by referring to the ability of both computers and calculators to perform routine calculations, render the graphs of functions and draw geometrical figures. Rather disappointingly, regarding the *use* of this new technology, she pointed out that “in this respect, the new courses can be seen as rather conservative.”⁴² Later she added that their utilisation “is surely a ‘must’ as soon as economic and other circumstances permit.”⁴³ Although economic conditions have improved, the 2005 discussion paper *Review of Mathematics in Post-Primary Education* comments that, “The image of mathematics as linked to “real life” may not have been enhanced by the late and slight adoption of ICT in mathematics

⁴¹Elizabeth Oldham, “Mathematics in the Senior Cycle: The New Leaving Certificate Courses”, *Studies in Education* 8, no. 2 (Autumn 1992): 41.

⁴²Ibid., 43.

⁴³Ibid., 45.

teaching. [...] Computers are not regularly used in mathematics classrooms, although the Schools IT2000 initiative has resulted in more teachers making use of ICT in a range of subjects.”⁴⁴ Thus it is fair to say that conservative use, if any at all, is still the norm in the senior cycle classroom.

However, there is a clear recognition that there should be an increasing use of technology. In relation to additional subjects at senior cycle, the report on the consultative process notes that there was a call for such provision,⁴⁵

... notably in civic and political education, social and personal education, physical education and information and communication technology (ICT). In the case of the latter, the emphasis was on ICT as a teaching and learning tool and on the acquisition of ICT literacy, although there was some support for a stand-alone subject in this area.

In respect of ICT this view was confirmed by respondents to the Online Survey, the results of which report that, “overwhelming support exists for the ... increased use of technology as a tool for learning (90%).”⁴⁶

In December 2004 the NCCA published a discussion paper on *Curriculum Assessment and ICT in the Irish Context*.⁴⁷ It states that “the NCCAs current ICT work aims to further elaborate the potential of ICT to support teaching and learning across the curriculum and in specific subjects in primary and post-primary schools. This support will include identifying

⁴⁴National Council for Curriculum and Assessment (NCCA), *Review of Mathematics in Post-Primary Education: Discussion Paper* (Dublin: NCCA, October, 2005), 21. The acronym ICT stands for Information and Communication Technology.

⁴⁵National Council for Curriculum and Assessment (NCCA), *Developing Senior Cycle Education – Report on the Consultative Process*, 17.

⁴⁶NCCA, *Results of Online Survey*, (Dublin: August 2003), 8.

⁴⁷National Council for Curriculum and Assessment (NCCA), *Curriculum Assessment and ICT in the Irish Context: A Discussion Paper* (Dublin: NCCA, December, 2004)

the kinds of knowledge, skills and attitudes that students can be enabled to develop through ICT key points in their primary and post-primary education.”⁴⁸ In relation to mathematics the *Review of Mathematics in Post-Primary Education* quotes a UK report by Hoyles *et al.* as stating that “the use of information (and communications) technology has ‘changed the nature of the mathematical skills required, while not reducing the need for mathematics’ itself.”⁴⁹

It is hoped that ICT will make its appearance as a stand alone subject in the horizon framework of 2010. If so, it would ensure that senior cycle students become familiar both with operating PCs and the software that can run on them. Such familiarity would be a bonus to the mathematics teacher hoping to use software and materials which assist both the teaching and learning of mathematics. However, whether this happens or not, it is strongly recommended here that such technology be still used in the mathematics classroom; firstly to assist the dynamic presentation of mathematical content by means of software programs such as PowerPoint and secondly to cultivate clear thinking via (elementary) mathematical programming implemented with powerful mathematical packages such as *Mathematica*.⁵⁰

⁴⁸Ibid., 4.

⁴⁹National Council for Curriculum and Assessment (NCCA), *Review of Mathematics in Post-Primary Education: Discussion Paper* (Dublin: NCCA, March, 2005), 2.

⁵⁰*Mathematica* is the registered trademark of Wolfram Research Inc.

1.1.4 Consequences

Since the final examinations are currently the only way in which students gain credit for their two-years' study of the LCE, and since success on this examination is, more often than not, the sole determiner of a student's post-secondary options, it is generally the case that most teachers, aided by their textbooks, gear their instruction to the nature of this examination. "IMTA meetings devoted to 'post-mortems' on the examination papers [...] especially with regard to the Higher level Leaving Certificate papers – are at times obsessively devoted to 'what will get marks' rather than 'what may improve students' learning' or what might be good mathematics education".⁵¹ Thus, in general, teachers pragmatically accept the examination for what it is and focus on optimising their students' performance on these tests by adhering to a regime which concentrates on the teaching of (i) the content most likely to be examined and (ii) the form in which it likely to be asked within the time-frame provided by school managers.

False impression: Since the mathematics examination, in turn, is almost entirely technical and context-free, treating very little practical mathematics, one serious consequence of this practice is likely to be that students acquire a false impression of the nature of mathematics.

Much as they might wish to present their subject otherwise, teachers

⁵¹National Council for Curriculum and Assessment (NCCA), *Review of Mathematics in Post-Primary Education: Discussion Paper* (Dublin: NCCA, October, 2005), 17. The acronym IMTA stands for Irish Mathematics Teachers Association.

may feel compelled by the obligation to have their students perform well on this examination and rightly regard themselves as working within “the highly constrained field of teaching in school.”⁵² Speaking in 1959 at the Royamaunt conference on *New Thinking in School Mathematics*, the great French mathematician Dieudonné remarked:⁵³

What is really at stake is the kind of mental picture of mathematics which will emerge in the mind of an *average* intelligent student after he has been subjected to that treatment for several years.

Unfortunately, this same question is as valid (with regard to current LCE mathematics) as it was almost a half-century ago and, regrettably, the answer to it is probably as it was then; the emergence of a mind-set where students view mathematics as a mindless game devoid of any sense.

What is being lost? What should our students know about mathematics upon leaving secondary school? The perspective offered by this thesis is that

- mathematics is relevant; many lack an awareness of its power to solve problems relating to the external world.

At the moment at one end of the spectrum of mathematical appreciation very able mathematicians/scientists are perplexed by, what the Hungarian/American and Nobel Prize winning physicist Eugene Wigner termed, ‘the unreasonable effectiveness of mathematics’ while

⁵²John Evans, “Education as a Preparation for Work” (Ph. D. diss., Dublin University, 2000)

⁵³Jean Dieudonné, *New Thinking in School Mathematics*, (OECD: 1960), 38.

and at the other end of this same spectrum many students reel under the bewilderment of attempting to decipher what it is all about.

- doing mathematics can be enjoyable; there is ample evidence that a great many students rarely feel any sense of excitement or pleasure from a study of the subject. They have not experienced the adventure of discovery nor the liberating influence of finding things out for themselves.⁵⁴

- mathematics has a long and rich history; that as a subject of enquiry it has always been ranked among the most daring and imaginative of human endeavours and has drawn, throughout the ages, people of every rank and station to its study and that often these ingenious and passionate individuals endowed with greatness were also possessed of the same frailties common to most.

- mathematics has its own culture: replete with ‘big ideas’ and concepts both ingenious and intriguing which lend it an elegance and grandeur both as it relates to its own methods and results but also through its intimate connections with such disciplines as art, music and architecture.

⁵⁴The OECD report *Learning for Tomorrow's World – First Results from PISA 2003* (Paris: OECD, 2004) states (115):

Student interest in mathematics is far lower, across countries, than in reading. Among OECD countries, about half of the students report being interested in things they learn in mathematics, but only 38 per cent report that they do mathematics because they enjoy it.

Each of these aspects of the mathematical task is addressed throughout this work in a manner which demands more than the mere recall of facts but the demonstration of related mathematics.

1.1.5 What is good in the present HLC examination?

While the focus of the present discussion has been one of making the case for reform by pointing to shortcomings in regard to interpretation and substance, it has to be said that there is (in the judgment of this writer) much that is good in the current programme. The potential of the present syllabus to be a much more forceful instrument has already been mentioned.

It can fairly be said that many students acquire a high level of technical ability from a study of the current HLC in that they have little difficulty with different fundamental mathematical techniques. [Those that do have difficulty, invariably have it with regard to numeracy in such matter as units of measurement, indices, logarithms and approximations, and not, perhaps surprisingly, with algebraic manipulation or techniques of differentiation and integration. All of which would seem to indicate that on-going problems in relation to skills are inherited from previous educational experiences, either at Junior Certificate level or further back at primary-school level.] While one of the assertions of this work is that the HLC is overly technical, there is no suggestion that the technical aspects of mathematics be removed in their entirety or made to virtually disappear by swamping such material by excessive context setting. The view here is simply that while technical

mathematics is necessary it is not sufficient.

The Data Analysis and Probability section is a good one in that it makes contact with applications and makes a greater call on higher order cognitive skills than many of the other sections. As a topic it has the advantage of requiring little more than the ability to count, so that one can progress quite far without invoking further technical mathematics. The statistics questions dealing with the organisation data and the drawing inferences are self-evidently relevant while many of the probability questions could be classed as applied although, in the opinion of this writer, there is a lack of emphasis on decision making. This attribute of probability to inform decision making in the presence of chance is taken up in a subsequent section along with many other ‘missed opportunities’ which could be harnessed to expand this syllabus segment into a more substantial and invigorating element.

1.1.6 Summary

Thus by way of summary, while there is much that is good with the current programme, it is asserted here that, because of a lack of emphasis on genuine applications as well as a neglect of both recreational mathematics and the history/culture of mathematics, allied to the manner in which subject matter is presented and examined, the current LCE course is failing to meet *many* of its expressed aims⁵⁵ relating to

- the engendering of interest in mathematics

⁵⁵Department of Education, *The Leaving Certificate Mathematics Syllabus (Higher Level and Ordinary Level)* (Dublin: Government Publications Sales Office, n.d.), 1.

- showing the relevance and power of mathematics
- revealing the rich heritage and special nature of mathematics as a human endeavour

Hence the need for reform.

1.2 The Nature of the Proposed Reform

Having indicated, if not yet substantiated, the need for reform, attention is now turned to the nature of this reform. This section outlines what the present work hopes to do in relation to the issues already identified.

While retaining many elements of the existing syllabus (interpreted more liberally and substantially) it is proposed here that there be a much greater emphasis on

- Recreational Mathematics
- Mathematical History/Culture
- Applications of Mathematics

and that there be a

- change in how subject material is presented both in textbooks and in the classroom.
- greater use of technology

- change in the overall assessment system, in particular in relation to the final examination

A discussion of this last point, so necessary to accommodate the other recommended changes is best left to a section dealing with implementation. However, that it is necessary to have, at the very least, a different final examination, should be evident considering the suggested new content areas and methodologies. It has already been mentioned that the fourth of the ‘directions for development’ indicated by the NCCA is one where “different assessment arrangements and a new certificate” are envisaged in the 2010 framework.⁵⁶

1.2.1 The Notion of a Topic

Turning to the proposal that there be a ‘change in how subject material is presented both in textbooks and in the classroom’, it has been pointed out, by way of the illustrative discussion on ‘cubics’, that lack of motivation combined with a failure to reveal purpose are two features which mar the current textbook treatment of many syllabus segments. Because of this, many textbook presentations of topics, in addition to lacking rationale, are short on development and scant on substance.⁵⁷ Thus it may be said in relation to the instructional sequence:

⁵⁶National Council for Curriculum and Assessment (NCCA), *Developing Senior Cycle Education - Directions for Development* (Dublin: NCCA, 2004), 3.

⁵⁷It is proposed to substantiate this point in specific detail in the section devoted to analysing the approach adopted by current textbooks.

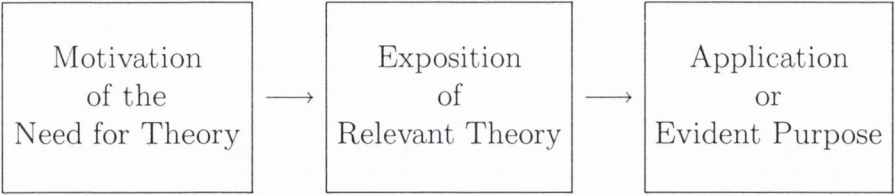


Figure 1.1: Instructional sequence guiding the design of a Topic.

that the predominant practice at the moment may be an almost total (technical) concentration on the middle element, with little or no attention being given to the other two vital components.

It is possible that there are good reasons why this is the case, one of which may be that many teachers do not know enough about the applications of their subject since they themselves also have been the victims, both at secondary school and at third-level, of this form of mathematical education. It is not the intention here to dwell on this possibility, nor indeed other contributory factors, but rather to advocate a shift from current teaching practice to one based on adherence to the above sequence even if this entails the writing of new textbooks and/or the re-educating of teachers. Implementation issues such as these are taken up again in an appropriate chapter.

In this work, any syllabus segment which is interpreted and presented according to the above sequence is referred to as a Topic. Capitalisation is used to distinguish this newly minted term from the ordinary word ‘topic’ which in the educational sense is taken to mean ‘theme’ or ‘subject’. **A Topic may be seen as the expansion of an existing syllabus entry, or a newly proposed one, to such an extent that the reason for teach-**

ing it is indisputably evident. The meaning, elaboration and practical illustration of this notion constitutes an essential element of this thesis, and dictates much of the mathematical content and discussion appearing in its core section.

To make clear the nature of a Topic, the current treatment of one segment of the present syllabus, that of *quadratic polynomials*, by both textbooks and examination, is compared with a suggested more substantial treatment which devotes more time to initial motivation, requires no more technical knowledge than the present syllabus and which justifies itself by both solving the practical problems raised initially and the showing of fresh applications. Other comprehensive illustrations are also given in relation to current syllabus entries relating to *difference equations* and *the Newton-Raphson method*. Concrete material connected to the new content areas of recreational mathematics and the history/culture of mathematics is also presented in detail. These pieces are written at the level of a teacher rather than that of the student, and as such, are to be clearly understood as being designed with mathematics teacher education in mind. They should not to be seen as constituting material which might readily be presented in the classroom, even if in certain cases this might be deemed possible.

The ultimate criterion for judging the success of a Topic hinges on whether it makes manifestly clear why one has studied the related mathematical material. The ambition is that a Topic related to a syllabus segment capable of accommodating applications be underpinned by at least one genuine applica-

tion. Material relating to history, mathematical culture and general interest may, or may not, contain a real-world application but should reveal purpose.

In relation to a purpose which would not be strictly classified as applied in the sense of solving a problem external to mathematics itself, there is a piece from recreational mathematics based on three-by-three magic squares and designed to illustrate the ‘logic of discovery’. Motivation is *exordium ex abrupto* where students are simply challenged to determine if it is indeed possible to solve this ancient puzzle and, if so, to provide a solution.⁵⁸ After being given time to make the problem their own, and possibly find (in various ways) that there are many solutions, an exposition is presented which reasons through the problem in a systematic and sequential manner to reveal a solution. But the matter does not rest here for it is only now that the teacher begins to reveal the true purpose of an exercise which began with a puzzle. The exposition is re-traced to identify key decisions, which when re-examined culminate in establishing quite simply but conclusively that there are only eight such squares. Not happy with such a significant and unsuspected outcome the teacher reveals that these apparently distinct solutions may be viewed as different manifestations of each another. The final result is substantial and fully justified by achieving, in this case, complete closure. Besides being a show case of genuine mathematics of a high order, its overall purpose is to reveal to the student the power of logical thinking.

This work presents three examples of Topics fully worked out in a manner

⁵⁸*Exordium ex abrupto* – ‘an abrupt introduction’.

which makes their pedagogical content quite evident. Should the envisaged new framework be flexible enough to accommodate this body of tangible work then it may be drawn on by teachers and spare them having to develop material from scratch. One possible direction for implementation is using a Topic as a module or part thereof. In this case one would need to ensure a significant store of Topics from which a certain number would be assigned for a given examination cycle just as a fixed number of poems and a play are prescribed for the course in English. And, as is done for this subject, by giving a suite of classes on these Topics, a teacher hopefully will be able to give students a tangible sense of learning something self-contained, purposeful and complete. The study of a number of carefully thought-out and well-chosen mathematical Topics would convey the relevance and power of mathematical thinking.

In addition, Topics with evident educational goals empower teachers to teach mathematics in a more meaningful manner and make it less isolated and divorced from its applications. By shifting to such a paradigm it may immeasurably improve the quality of the education provided to our young people while increasing the satisfaction of those who teach them. It is essential that teachers be given the freedom and time to treat some syllabi entries, in addition to proposed new items, in a manner which allows them to convey the usefulness of mathematics and which engenders the exhilaration to be had from observing mathematics ‘in action’. Equally, they should not feel restrained from considering problems which may have no immedi-

ate or apparent application but have an easily recognisable sense of purpose. The issue is not so much one of relevance but of substance combined with purpose. While pride of place may be accorded to Topics with non-trivial applications, the role of the investigation undertaken for no other reason than its intrinsic interest must never be undervalued. The criterion should simply be to consider ‘sensible’ problems.

1.2.2 Use of Technology

In relation to the use of technology it has already been indicated that there is a positive attitude on behalf of the NCCA and the partners in education towards its greater use in the classroom as an aid to teaching. While nothing specific was indicated by these bodies as to how it might aid in the teaching of mathematics, it has already been said here that technology can be used to *present* mathematics in an attractive manner and to aid precise thinking via *mathematical programming*.

The Internet as a resource: Before elaborating on these two aspects, it should also be pointed out that the Internet is a vast repository of freely available material which can be of tremendous assistance to the teacher and pupil used in the classroom. Over time the Internet is almost certain to become one of the most exciting new learning environments of all. Some Internet resources which are already available and used, particularly, at third-level, in relation to teaching and learning (of mathematics) are

Accessing class notes on web-pages and through Open Course Ware (OCW)

Obtaining exercises on-line

Problem sessions with correcting

Distance learning

Sites relating to all aspects of mathematics

The first three of these resources all help to lessen the workload of the teacher in re-enforcing and testing technical mathematics while facilitating independent learning by the student. It is envisaged that because of their sheer usefulness and convenience it will soon be the common practice of all secondary schools to actively avail of these facilities.

Presentations: Software programs such as Microsoft's PowerPoint open up a world of possibilities for those involved in education. However, to judge by what is currently available, little or nothing is being done in this area for the purposes of teaching mathematics. One current focus is on using well known mathematical packages such as *Mathematica* to aid teaching in the (third level) classroom while a second is on the creation of Java Applets for illustrating and testing some very specific mathematical skill. While these are excellent in their own way, PowerPoint-like presentations can be fashioned into a tool superior to both these modes for presenting and teaching

mathematics. If a picture can speak a thousand words then a well-thought-out presentation can be made to speak volumes. This theme and others are elaborated upon in some detail in a later chapter of this work.

Mathematical Packages: Although the PowerPoint program can be used to present mathematics very attractively it is not a mathematical package in that it lacks a mathematical engine that packages such as *Mathematica* possess. This latter program is perhaps the most powerful of the many currently available and can also be used to great effect by a teacher to present mathematics.

The power of a computer to draw the graphs of functions has already been remarked upon. Because of the tremendous speed of modern computers a mathematical package offers much greater scope for exploration than traditional pencil and paper methods. Many of the problems either a teacher or a pupil can hope to attempt in the classroom are too short to have a ring of authenticity about them. A mathematical package gives the teacher the power to display mathematics in action by presenting realistic tasks implemented efficiently and quickly. When young students see a computer encrypt a block of plaintext in the blink of an eye, they begin to see the relevance of mathematical theory which underpins such implementation.

Many of the mathematical software houses have a keen interest in mathematical education as witnessed by the articles and journals they produce dealing specifically with this area. Wolfram Research, the originators of

Mathematica and one of the pioneers in the field of computer assisted learning (CAS) have a package named **The Mathematical Explorer** which they claim, “delves into the historical and cultural details behind the most important mathematical concepts.”⁵⁹ It offers an “interactive journey through intriguing problems that have challenged mathematicians since the ancient Greeks.” The reader is invited to “walk in their computational footsteps, and experience the wonder of discovery that has fascinated amateurs and professionals throughout the ages.”⁶⁰ Clearly this resource alone is a powerful aid to a teacher wishing to engage students’ interest. Explorations such as these could be tied to the teaching of the history/culture of mathematics and be examined in that context.

Mathematical Programming: “It might appear contradictory, but the effective use of computers demands a deepened knowledge of theoretical knowledge, than the traditional way of executing by hand loosely described procedures.”⁶¹ A later section illustrates this remark in relation to the recommendation of using (mathematical) programming to cultivate the skill of precise thinking. It has been wisely said by Perlis that:⁶²

You think you know when you learn, are more sure when
you can write, even more when you can teach, but certain
when you can program.

⁵⁹Wolfram Research, *The Mathematical Explorer*TM (USA: 2001), backcover

⁶⁰Ibid., backcover

⁶¹B. Christiansen, A. G. Howson & M. Otte, *Perspectives in Mathematics Education*, 79.

⁶²Alan J. Perlis as cited by John dePillis, *777 Mathematical Conversation Starters*, (USA: Mathematical Association of America), 69.

1.2.3 The Three Context/Content Areas

Having outlined a different general methodological approach to the presentation of material via Topics and having described the power of technology to aid in this endeavour, it is time to briefly say how this work hopes to remedy the shortcomings of the current programme in relation to the three ‘context/content areas’: recreational mathematics, the history/culture of mathematics and applied/relevant mathematics. In so doing, the opportunity will be availed of to indicate briefly what little expertise this writer has attained in each of these areas.

Recreational Mathematics is discussed first because it is proposed that it should begin at the start of the first year (if not before) of the envisaged three-year programme. Although applications should be touched upon at every available opportunity, and some can be during a study of RM, others require technical mathematics which may not be developed until a little later in the course. Likewise with the history/culture of mathematics; while much can and should, be alluded to at any time, a more structured approach to both its aspects should, perhaps, wait until students have acquired a familiarity with those mathematical ideas to be first presented at senior cycle level.

Recreational Mathematics: Much of what makes mathematics attractive and exciting may not be immediately applicable but is connected with one of mankind’s most fundamental impulses – the desire to know.

Some puzzles are ideal for teaching mathematics because finding their

solutions often requires the use of numerical or geometric reasoning. By allowing the subject to be disguised in an imaginative manner puzzles can help the teacher avoid the drill-like exercises which so many students find dry and intimidating. Instead they let young people learn a great deal of mathematics in an agreeable fashion without their even realising that they are so doing. Furthermore, certain puzzles when solved suggest general questions of a genuinely mathematical nature. Thus they prove invaluable in providing the justification and impetus for particular mathematical investigations which otherwise might appear to have no relevance. The theme of providing the teacher with a framework where he or she can use puzzles, games and tricks to make contact with more conventional mathematics is a central one in this work.

Advocacy of the use of recreational mathematics in the teaching of secondary school mathematics is not new. Martin Gardner has written, "For 40 years I have done my best to convince educators that recreational math should be incorporated into the standard curriculum. It should be regularly introduced as a way to interest young students in the wonders of mathematics. So far, movement in this direction has been glacial."⁶³ Whether movement with regard to the use of RM has been glacial in countries other than the US, before or after these words saw print, will be addressed later. For the moment it is important to point out that a recreational problem

⁶³Martin Gardner, "A Quarter Century of Recreational Mathematics," *Scientific American* (August 1998): 68-75.

has the virtue of being an end in itself and so is ideally suited to serve all members of a classroom.

A number of presentations and written pieces set out for the teacher ways of making mathematics appeal to students through the study of puzzles, games and tricks. As will be seen, these impart much significant mathematical knowledge along with a number of important skills relating to habits of thought and problem-solving approaches. “Instead of a particular subject matter, however, we can teach children a more precious thing, namely how to master subject matter.”⁶⁴ Items relating to the chessboard, magic squares, the game of mastermind, simple parlour tricks and more sophisticated card tricks have been chosen for their mathematical underpinnings and the purely mathematical questions they often prompt.

This writer first became aware of the beneficial uses of RM at the start of a lecturing career in mathematics when he simply used RM as a way of rewarding a class for its attention to a lecture. The practice was to devote the last minutes of a class period to the discussion of a previously given puzzle and the description of a fresh one.

It was a welcome discovery, no doubt already experienced by countless teachers, to realise that young people love to be challenged mentally on what they perceive as the level playing field provided by a puzzle. Those puzzles in particular which require no special mathematical ability beyond a facility to

⁶⁴Hans Freudenthal, *Mathematics as an Educational Task* (Dordrecht: D. Reidel Publishing Company, 1973), 59.

reason logically and an understanding of elementary arithmetic furnish such an arena. Under circumstances of apparent equality learners are stimulated to outshine each other in finding a solution. One of the immediate benefits of this ritual was the establishment of a healthy rapport between lecturer and students.

However, it gradually dawned on this fledgling educator that the puzzles in themselves were assisting in the teaching of mathematical ideas and modes of thought. This emerged when, in the discussion of such puzzles, students became enthralled by how closely connected the solution path can be to mainstream mathematics. Thus began this writer's preoccupation with acquiring a stock of challenges that need more than a passing scrutiny to overcome them and which are of tangible assistance to the endeavour of transmitting mathematics in a stimulating and invigorating manner. It is a task which has taken time, continued reflection and careful selection.

The heartening experience of witnessing how enthusiastically puzzles were greeted and awaited by adult students prompted this author to experiment with the giving of such puzzles to his children as soon as each of them came of an age to understand the simpler specimens and have a reasonable prospect of solving them. To his delight they enjoyed them thoroughly and continued to ask for fresh challenges throughout their youth and well into their teens. Time and chance were to reveal the perceived benefits of this RM experiment in particular in the case of the eldest.

Just before her sixteenth birthday in January 2000, she became ESAT

Young Scientist for the year 1999 with a project in Cryptography which the judges remarked required a grasp of mathematics well beyond what one would expect from a sixteen-year-old. When repeatedly questioned by journalists as to the reason for this early (apparent) mathematical sophistication she often made allusion to the fact that the continual solving and discussing of puzzles at home had made her self-confident in relation to mathematics. Some time later in a book co-authored with this writer and entitled *In Code – A Mathematical Journey*, she wrote with reference to getting extra help with mathematics in the home:⁶⁵

Strictly speaking, it is not true to say that I or my brothers don't get any help with mathematics. We are given no extra classes, nor do we have to endure intense sessions at the kitchen table from over-zealous parents. But almost without knowing we have been receiving help since we were very young, out-of-the-ordinary help of a playful and subtle kind which has made us self-confident in problem-solving. [...] These puzzles were challenging and encouraged curiosity, and many of them made maths interesting and tangible. More fundamentally, they taught us to reason and think for ourselves. In this way puzzles have been far more beneficial to me than years of learning formulæ and 'proofs'.

An early chapter of *In Code – A Mathematical Journey* devotes a considerable number of pages to the discussion of many of the puzzles discussed in depth later in this work. The publishers Profile reported that this section held particular appeal for readers because of the description of the many lessons this young person claimed she learned from her encounters with these puzzles.

⁶⁵Sarah Flannery with David Flannery, *In Code – A Mathematical Journey*, (England: Profile Books, 2000), 6.

This popularity was evidenced by the number of letters received from readers worldwide, in particular from the parents of young children and teenagers. On becoming aware of the public interest in RM, Profile publishers requested that this author write a book devoted entirely to this aspect of mathematics.

Another result of this public interest was a commission to write a series of weekly articles based on extracts from *In Code* for the Weekend section of the English newspaper *The Daily Telegraph* which this writer duly did over a time span of twelve weeks under the caption of “Mindstretchers” between March and June, 2000.

The interest generated by the espousal of the benefits of RM also led to invitations to give talks at different venues both in Ireland and abroad. This writer has given a number of talks centred on a PowerPoint presentation entitled ‘Making Mathematics Fun Through Puzzles’: one such was to a meeting of Junior Certificate Teachers held in Galway in September 2000, a second to a meeting of the Irish Mathematical Society held in September 2002 at University College Cork while a third was at a plenary session at the annual general meeting of the US National Council of Teachers of Mathematics (NCTM) in San Antonio, Texas, in April 2003.

The particular interest of teachers in the use of RM was further confirmed for this author through the youngest family member who entered an individual project in the 2006 BT Young Scientist competition in the Junior section of the Physics, Chemistry and Mathematics category. The investigation: ‘Games and the mathematics they teach’ won first place for

its thirteen-year-old presenter who reported that the keenest attention and most searching questions came from mathematics teachers, many of whom expressed the opinion that teachers would value hugely a full-length book on the theme of his project. One such teacher, the editor of the *Newsletter* of the Irish Mathematics Teachers Association, on finding one of the games particularly appealing because of the number of mathematical ideas it embraces, invited the young contestant to submit an article for publication in the *Newsletter* relating to this captivating ‘trick’.⁶⁶

History/Culture of Mathematics: An equally taxing but enjoyable task conducted over many years was the one of becoming acquainted with snippets from the history/culture of mathematics that reveal to students the human side of a discipline widely regarded as a strictly impersonal one. To learn that quite the opposite is the case can induce a more positive attitude to a subject which for many induces nothing but anxiety.

“Mathematics as taught in school is perceived by most secondary school students as a subject devoid of history.”⁶⁷ While sadly this statement is also true today (as will be shown later) of most Irish school leavers, the purpose of this section is to outline briefly what this work attempts to do to remedy this situation. In addition a number of efforts relating to the culture of mathematics are described.

⁶⁶See page 456 for a discussion of the problem in question.

⁶⁷Shmuel Avital, “History of Mathematics Can Help Improve Instruction and Learning”, *Learn From the Masters*, edited by F. Swetz *et al.* (USA: Mathematical Association of America, 1995), p. 3.

A time-line of some of the more significant discoveries in mathematics throughout the ages up as far as its current frontiers is given along with mention of the more famous men and women involved. In connection with this enterprise a number of specific historical problems are discussed and related examination questions given. Furthermore, it is suggested that a number of different themes, such as the ‘solving of polynomial equations’, ‘the need for logarithms’, ‘the invention of the calculus’ be used as history projects which would become part of a student’s assessment portfolio. This exercise would require students to read books on the history of mathematics of which, fortunately, many fine examples have appeared in the last number of years treating both the past and the present.

Alternatively these themes might serve as possible Topics by which the teacher might treat certain material from the syllabus. A more comprehensive list is given in the section describing the details of what is being proposed. The virtue of this exercise will be discussed later. In addition to reading books on the history of mathematics, it is also recommended that students read some of the literature on mathematics in general by way of books or plays. There are a number of books available, sections of which can be read with no more than a knowledge of school mathematics while there are plays such as the Pulitzer Prize-winning *Proof* which convey the nature of mathematical creation without recourse to explicit details.⁶⁸

Although culture is inextricably linked with history it is proposed here

⁶⁸David Auburn, *Proof*, (USA: Faber & Faber, 2001)

to isolate a number of mathematical activities, ideas and tastes which reveal its cultural aspects. Illustrative examples might relate to the process of conjecturing, the dangers of induction and the need and nature of proof. Some of the more famous examples of indirect reasoning via *reductio ad absurdum* arguments bequeathed to posterity by the Greeks are easily understood once explained properly. Yet these cultural gems are never displayed in the classroom as part of the heritage of mathematics in the same way that many students are acquainted with the rich legacy of art and music.

Art, music and architecture are also aspects of our general culture which have been touched by mathematics, yet a complaining, “Why weren’t we told about these things at school?” is an often heard refrain from adults who later come to know that such is the case. Greater efforts could and should be made to pre-empt justified questions such as this one. One or two ways of how this might be done are indicated. Allied to this task is the one of exposing our students to the many ingenious and intriguing ideas which pervade mathematics and help characterise it as a cultural activity.

Applications of Mathematics: It might be simply said that the philosophy underpinning the entire endeavour to be described throughout the remainder of this work is the provision of sense-making contexts for revealing school mathematics along with the fullsome description of the mathematical content relating to these contexts. Traditionally, the predominant means of providing context for technical mathematics is through applications; taken to

mean the use of mathematical theory to solve ‘real-world’ problems. Sadly, as has been indicated, it is often the case at the level of Irish senior cycle provision that application-contexts by which mathematics reveals itself as a potent problem-solving discipline are at best rare.

Those teachers who may like to know more about the applications of mathematics face two problems. One is the problem of availability; where does one go to find authentic applications? Often those to be found in the recommended textbooks are spurious or contrived with the result that they can do more harm than good by making mathematics appear silly. Even if there are some that pass muster, there are not enough of them. The second problem is one of incentive; if a teacher does take the trouble to find an application, can he afford the time to teach it and will he find a receptive audience? If preparation for the current examination does not require it, might not spending time and effort on demonstrations of usefulness be construed as irresponsible.

This latter problem can be overcome by ensuring that questions relating to applications (possibly specified in advance) do appear on the examinations. The former problem is more challenging. The task of finding blue-ribbon (grade I) applications above the level of common utility is not an easy one. A call for a greater supply of meaningful everyday-life applications was made by Hersh and Davis in 1981 when they wrote:⁶⁹

⁶⁹Philip J. Davis & Reuben Hersh, *The Mathematical Experience* (USA, Penguin Books, 1981), 84.

When one moves to higher mathematics, such applications are harder to observe and to verify. It would be of enormous importance to the profession if some lively and knowledgeable investigator would devote several years to this task, and by visiting a number of businesses, laboratories, plants, etc. document just where this occurs.

Even at present, this provision remains a problem; while scientific applications of mathematics abound, particularly at third level, the supply of applications from outside this domain is disappointingly small although it must be said that the recent efforts of OECD/PISA and others seem to be addressing this shortage quite significantly.

Thus one theme of this work is the description and elaboration of many applications of mathematics accessible at the level of school mathematics which do not appear in the current textbooks and, as a consequence, are unlikely to be known to the vast majority of Irish secondary school mathematics teachers.

Many of the suggestions to be found here in relation to applications come from one who has been teaching mathematics to would-be scientists and engineers for thirty years while being continually preoccupied by the responsibility of the task. These recommendations are prompted by the views of an educator who as a young inexperienced lecturer began teaching mathematics in isolation; so few were the number of genuine applications known to him, and this, despite having been trained as a mathematician. The firm conviction that it was irresponsible to teach mathematics without knowing why, where and how it is used, accompanied by the discomfort of living in

constant fear of being thought a fraud proved the spur to begin searching immediately for applications wherever they could be found. Even now, though considerably less ignorant of the uses of mathematics, this quest continues, for no matter how large one's current supply of examples may be, there is always a welcome for a fresh application, however simple, which illustrates the power of mathematics in some way new to oneself and one's students.

One might say that the thirty years spent accumulating the experience described above formed part of the research period for a dissertation which otherwise couldn't be what it is.

Summary: One of the over-arching themes of this work is the concrete provision of rich contexts (drawn from all areas of mathematics) through which genuine mathematical content is revealed and fully elaborated upon for educators to exploit as they see fit. The second over-arching theme is that context should act as the motivation for content, and so, in terms of precedence of presentation, be revealed before the exposition of theory. This pedagogical theme is the one underlying the notion of a Topic. It receives further discussion in a later chapter.

While much of what is to be found throughout this dissertation is actively concerned with making it easier and more engaging for students to learn and understand mathematics, the general thrust of the work is of assisting and informing the teacher in this endeavour. Thus much of the exposition is written at the level of the teacher rather than of the student. As a result

this thesis might rightly be described as belonging to the domain of teacher education.

1.3 Thesis Structure

Now that the need for reform has been documented and the aspects of such reform that this work hopes to address have been outlined, it remains only to indicate the general structure of this work.

Part I, which follows this introduction begins with a survey of the history of curriculum development in mathematics in general and concludes by presenting one modern interpretation of what is entailed by curriculum development in its many different aspects. The historical focus is then narrowed to the particular case of Ireland. Then a number of the general educational issues and perceptions which emerge from these histories are discussed. A survey of two recent and comprehensive educational endeavours is then given; one benefit of which will be the acquisition of a working vocabulary which is in line with modern usage and which will be used in subsequent discussions throughout this work.

This done the criticisms made in this introduction are substantiated. Collectively this body of writing constitutes Part I of this dissertation whose structure may be summarised as shown:⁷⁰

⁷⁰While the phrase curriculum development is used for brevity it should be interpreted as curriculum development in mathematics.

	Historical Aspects of Curriculum Development
Part I	History of Curriculum Development in Ireland
:	Curriculum Matters
	Establishing the Case

Part II details the rationale and nature of the specific recommendations outlined in the introduction. Many of the proposals discussed are augmented and substantiated by extensive treatment in what is termed the ‘core’ section of the thesis. Specifically, this core consists of the concrete material placed on the CD accompanying this work. It contains the explicit expositions of the three Topics on quadratics, difference equations and the Newton-Raphson algorithm; numerous written pieces on recreational mathematics, history/culture, applications of mathematics along with a number of miscellaneous items. In addition all of the presentations prepared with PowerPoint and Flash are to be found there.

Because of the sheer extent of this expository material it was thought better to consign it to this external medium in order to circumscribe the length of a work which on its own runs to several hundred pages. As a further aid to maintaining the hardcopy portion of this treatise at a manageable size the cd also houses the 160 pages or so which make up the seven appendices. The material on this cd has been arranged so that is easily navigable by means of hyper-links. The contents of this core section are listed in the Table of Contents given at the beginning of this work.⁷¹

⁷¹See pages xiii-xx.

A broad outline of the structure of part II is summarised under the following headings:

	Topics
	Recreational Mathematics
Part II :	History and Culture of Mathematics
	Applications
	Use of Technology

Part III begins with a chapter entitled Implementation/Conclusion which synthesises the particular contribution made by this work and discusses in broad terms how the suggestions put forward in Parts I and expanded upon in Part II might be implemented. Among other things, it considers how assessment procedures being currently advocated might provide better vehicles for accommodating new schemes and initiatives. Its brief concluding section is one of reflection and appraisal.

The unnumbered chapter which follows gives the full written account of the Topic on Difference Equations – the shortest of the three Topics treated in depth. The chapter which follows presents a number of ‘mathematical vignettes’ not explicitly described elsewhere in this work while the final chapter contains the Prologue and first chapter of the Socratic piece: *The Square Root of 2: A Dialogue Concerning a Number and a Sequence*. The concluding section is that of Bibliography and References. The headings given to each of the chapters in part III are as shown overleaf:

	Implementation/Conclusion
	Illustration of a Topic
Part III :	Mathematical Vignettes
	A Socratic Dialogue
	Bibliography/References

Part I

Chapter 2

Historical Aspects of Curriculum Development

It is useful to understand that the impetus for curriculum development in mathematics can have different origins.¹ Certainly, with regard to the last four hundred years or so, one spur for curriculum improvement has been due to the almost continual change occurring in social and economic structures. The general realisation of the rewards to be had from embracing such change allied to a fear of being left behind technologically or culturally, ensured that curricula (in whatever form) were the objects of much reflection. Such change was also aided by educationalists gradually coming to know more about teaching/learning and using this knowledge to improve on existing educational provision.

Besides being initiated by pressure from society, curriculum reform and innovation comes also from within mathematics itself as it continues to grow

¹As mentioned in the Introduction a modern definition of the term curriculum development is given at the end of this chapter. See 82.

and find fresh applicability. In so far as “curriculum development in mathematics is linked to progression in mathematics”, the very broad overview of how mathematics has evolved through the ages set out in the PISA 2003 Assessment Framework document (PISA03) proves most useful.² The different epochs discernible within that synopsis are presented here as stages so as to achieve greater visibility.³

Stage I – The emergence of Arithmetic:

For centuries mathematics was predominantly the science of numbers, together with relatively concrete geometry. The period up to 500 BC in Mesopotamia, Egypt and China saw the origin of the concept of number. Operations with numbers and quantities, including quantities resulting from geometrical measurements, were developed.

Stage II – The axiomatisation of Geometry:

From 500 BC to 300 AD was the era of Greek mathematics, which focuses primarily on the study of geometry as an axiomatic theory. The Greeks were responsible for redefining mathematics as a unified science of number and shape.

Stage III – The origins of Algebra:

The next major change took place between 500 and 1300 AD in the Islamic world, India and China, which established algebra as a branch of mathematics. This founded the study of relationships.

²Directorate for Education of the Organisation for Economic Cooperation and Development, *The PISA 2003 Assessment Framework*, (OECD: 2003), 33.

³These stages are supplemented somewhat by the time-line to be found in the section relating to the teaching of history. There the approximate dates of various mathematical milestone-results are given with commentary.

Stage IV – The invention of the Calculus:

With the independent invention of calculus (the study of change, growth, limit) by Newton and Leibniz in the 17th century, mathematics became an integrated study of number, shape, change and relationships.

Stage V – The era of Modern Mathematics:

The 19th and 20th centuries saw explosions of mathematical knowledge and of the range of phenomena and problems that could be approached by means of mathematics. These include aspects of randomness and indeterminacy.

As the PISA03 document remarks in relation to this last stage, “these developments made it increasingly difficult to give simple answers to the question “what is mathematics?” At the time of the new millennium, many see mathematics as the science of patterns (in a general sense).”⁴

2.1 Pre-Twentieth Century

The time-line of educational progression in mathematics which is now set against this historical backdrop begins around the eighth century A.D. when, according to Howson, Keitel and Kilpatrick (HKK), “... two agencies for change are identified: personal contact through travel and the book.”⁵ At this time education was in the hands of the church and interest in mathematics was rudimentary, being confined to the nature of numbers, the various geometrical shapes and enough astronomy to calculate the dates of movable

⁴Ibid., 33.

⁵A. G. Howson, C. Keitel & J. Kilpatrick, *Curriculum developments in mathematics*, 16.

feasts such as Easter. “The idea that the curriculum would have to change because of the progression of knowledge was still many centuries away” – in fact, almost a thousand years in the future.⁶

By the seventeenth century technology was progressing through its many different inventions. In 1660 the Royal Society was set up in England for the purposes of improving natural knowledge which was forever expanding. The period between 1660 and 1700 was to see the most significant flowering in mathematical knowledge since the time of the Greeks with the independent discovery by Isaac Newton in England and Gottfried Leibniz in Germany of the differential and integral calculus. This newly invented branch of mathematics led to the solution of numerous, hitherto unsolved, mathematical problems connected to a host of applications from physics, astronomy and mechanics, thereby greatly broadening the then understanding of the physical world.

However, according to the great German mathematician Felix Klein (writing in 1893) it was the eighteenth century that “laid the firm foundation for the development of mathematics in all directions.”⁷ The decisive event in Klein’s opinion was the founding of the École polytechnique at Paris in 1794 for reasons he explained as follows:⁸

⁶A. G. Howson, C. Keitel & J. Kilpatrick, *Curriculum developments in mathematics*, 17.

⁷Felix Klein, *Lectures on Mathematics* (USA: AMS Chelsea Publishing, 2000), 99.

⁸*Ibid.*, 100.

That scientific research and active instruction can be directly combined, that lectures alone are not sufficient, and must be supplanted by direct personal intercourse between the lecturer and his students, that above all it is of prime importance to arouse the student's own activity, — these are the great principles that owe to this source their recognition and acceptance.

Christiansen *et al.*, referring to the eventual influence of the École polytechnique, remark “that the success of its graduates in the Napoleonic wars added weight to the argument that mathematical knowledge contributed to military and industrial power.” This was “the power of ‘real’ knowledge to improve society, *i.e.* a knowledge of the facts of the universe as distinct from a knowledge of what men had thought and said.”⁹

Even earlier in the eighteenth century the need to provide an education more suited to merchants, industrialists, surveyors and navigators had already been realised by educationalists. In the U.S. this led to the introduction of private academies where mathematics was recognised both as a mental discipline and as having practical value. By “the middle of the nineteenth century America began establishing its first technical colleges”, but by contrast with France, “the idea of a ‘research university’ did not emerge in the U.S. until the end of the nineteenth century.”¹⁰

While posterity has already judged Newton to be one of the greatest mathematicians of all time, the notation he chose for the calculus was much less suited to the nature of the subject than that fashioned for it by his

⁹B. Christiansen, A. G. Howson & M. Otte, *Perspectives in Mathematics Education*, 3.

¹⁰W. Mueller, “Reform Now, Before It’s Too Late!”, (USA: MAA Monthly, February, 2001)

great rival Leibniz. The more amenable symbolism of Leibniz when adopted by continental mathematicians was to lead from the 1800s onwards to a great flourishing of mathematical research in the European universities, in particular those of France and Germany. The obscurity of Newton's notation, on the other hand, proved to be a genuine obstacle in attracting new disciples to mathematics in England with the result that English mathematics was to languish in virtual sterility for almost two centuries. As a consequence the provision of mathematics at the English universities was considerably less in tune with the corpus of existing mathematical knowledge than was the case on the continent, a circumstance which in turn had a bearing on the mathematics being taught in its primary and secondary schools. What was taught in English schools was, in fact, to remain tied much more closely to the classical model with its emphasis on Latin and Greek and study of the ancient classics than was the case elsewhere.

While education had become institutionalised in both France and the U.S. as early as 1830, it was not until the twentieth century that the State was officially involved in secondary education in England.¹¹ Thus, according to HKK, "although governmental commissions had, in the mid-nineteenth century, pressed the claims for a modern secondary-school curriculum in which mathematics would occupy a central position, there was no way in which changes could be enforced upon schools."¹² They relate that the ed-

¹¹A. G. Howson, C. Keitel & J. Kilpatrick, *Curriculum developments in mathematics*, 32.

¹²*Ibid.*, 32.

educationalist “Godfrey did in fact strive hard and with considerable effect to change the mathematics curriculum in English schools” but “one man alone could not battle successfully against those whose decisions so constrained the school curriculum.”¹³ Further help in this enterprise was provided in 1894 when the Mathematical Association of England published the first number of the *Mathematical Gazette*, a journal containing articles aimed at improving mathematical teaching. However, it may be safely said that by the end of the nineteenth century, the provision of both primary and secondary education for the general population had been achieved in the more affluent countries of Europe and in the U.S. Moreover, in most of these schools, some form of mathematics was being taught as a subject even if, as late as 1900, some were of the opinion that “it is difficult to assign any other reason [than as a mental discipline] for teaching mathematics to the ordinary boy.”¹⁴

By the dawn of the twentieth century educational provision in the more advanced countries had reached the stage where educational programmes (with mathematics as a subject) were being administered at primary, secondary and tertiary level. Thus one might view this point in time as marking the culmination of the educational progression in the sense that henceforth it was an accepted principle that such provision was part and parcel of a government’s responsibilities to its citizens. The much broader question as to the nature and efficacy of the programmes of instruction being offered was

¹³Ibid., 25.

¹⁴Ibid., 10.

gradually to become a central one by the latter half of the twentieth century.

Before describing some of the movements relating to mathematical education that arose in the twentieth century and which, over time, were to lead to the emergence of what is entailed by a mathematical curriculum as it is understood to-day, it is worthwhile to indicate briefly some of the reforms being called for at the end of the nineteenth century as a result of dissatisfaction with the mathematics programmes of that time. This is done in the next section based on the article, “Reform Now, Before It’s Too Late!” by W. Mueller which reports in some detail on the issues pre-occupying teachers and educators in the U.S. in the 1890s.¹⁵

2.1.1 Early Curriculum Reform in the U.S.

Mueller writes that “in the late 1880s there was tremendous concern about the way in which American students were learning mathematics.” He relates that textbooks abounded with ‘proofs’ that were not proofs and which contained extremely misleading statements. Teachers who were interested in improving the quality and relevance of instruction in their classrooms – and there were many of them – organized themselves into a ‘movement’ that came to be called The New Education. The New Education stressed proceeding from the concrete to the abstract for the sake of clear realisable advantages. Specifically, they urged a de-emphasis of mindless symbolic manipulation and that students be no longer expected to work out examples which illustrate no

¹⁵W. Mueller, “Reform Now, Before It’s Too Late!”, (USA: MAA Monthly, February, 2001)

general theory and are unlikely to be of any use later on. It was urged that the so-called text-book proofs be rejected, or given only with clear indications of their limitations.

Geometric models were a required component of mathematics instruction in many states. However, there was also a warning to be on one's guard against overly simplistic 'application-oriented' curricula. Developing the intuition through 'interaction' was much in favour. E. H. Moore advocated an intuitive development of the fundamental ideas that would culminate in a more rigorous presentation, writing:

Intuition is by no means to be excluded from mathematical work. On the contrary, in every way let intuition be kept alive and active, but when a new idea is introduced as an intuition let it be sharply defined with the understanding that in so far as subsequent work depends on that intuition a new branch of mathematics is being originated.

The New Education mathematicians and mathematics educators found a forum for their ideas in the newly emerging mathematical publications such as *The American Mathematical Monthly*, the first issue of which appeared in January 1894, virtually contemporaneous with the birth of its counterpart the *Mathematical Gazette*.

In 1890 the U.S. Bureau of Education commissioned a report on the state of mathematics instruction across the country. Its author was the mathematical historian Florian Cajori and it provided a broad portrait of the general practice of mathematics teaching current at that time. In compiling his report Cajori issued the schools and colleges he surveyed with a questionnaire.

Some of the questions along with a samples of responses are given here.

In response to

Does the interest in mathematics increase as students advance to higher subjects ?

teachers wrote that

“... so long as the student sees the bearing of his work upon practical scientific investigation or can be assured that it has such bearing.”

“All who understand the principles show a growing interest.”

“With the best students only.”

“... the more evolved or abtruse the matter, the greater the interest to those who succeed.”

The question

Are topics assigned to students for special investigation?

elicited the following responses

“They ... form a very essential part of the work.”

“Independent problems given in all classes for solution, reported on paper.”

“Not to undergraduates.”

“The man who pursues original investigation with the average student will make a failure.”

In relation to

Is any attention given to the history of mathematics? Does it make the subject more interesting?

the report found that very little attention was given to the history of mathematics at the majority of schools.

One reply to

Do scientific or classical students show greater aptitude for mathematics? Which sex?

was

“Males in quantity, females in *quality*.”

Some answers to

*Do you favour memorising rules in Algebra?
What reforms are needed in teaching same?*

were

“... not to ... young boys, who contract the incurable habit of learning it by rote”

“Rules and principles are to be deducted from examples”

“Algebra should be seen”

“... less toughing, disgusting gymnastics”

“The teacher should assist the pupil to make his own rules”

“Drill on ... *raison d'être*”

“More of an inductive method; and the abolition of much that may be of use theoretically, but of little practical use”

“Students should be taught to think! think!! think!!!”

“The pupils learn to do by doing”

“Less formality and more ‘realism’ ”

“Rattle the bones of the algebraic skeleton ... and show in it its living, breathing continuity”

There were many responses to the kind of teaching that best complemented the non-mechanical approach:

“Teaching needs to be less mechanical”

“Less memorizing and more thinking, both on the part of the teacher and the student”

“ ... less blackboard work”

“ ... putting into words the ideas conveyed by symbols, equations, and operations”

“Let us have live, enthusiastic, and competent teachers – such as will teach the subject rather than the text-book.”

Responses that *did* advocate the essential benefits of memorization and drill noted that:

“The chief cause of failure in many cases is not doing enough miscellaneous examples for practice.”

“I recommend the memorization of rules, unless pupils furnish a good, working rule of their own (a rare case).”

“It is only practice that makes perfect.”

2.2 The Twentieth Century

HKK remark that “... before the rapid expansions in educational opportunities the school mathematics curriculum changed slowly and usually only in response to the innovative ideas of individuals who managed to win acceptance for new topics or approaches.”¹⁶ Curriculum development usually occurred as the expression of one person’s desire to make a planned change in the instruction.¹⁷ Reference in this regard has already been made to the single-minded efforts of the educator Godfrey in England at the end of the nineteenth century. Another individual who was to have a widespread influence on general educational policy was Dewey. He advocated that learning in schools should be learning by doing and learning by experience. He rejected the division of content into separate school subjects and proposed that teaching be based on real objects. This pragmatic philosophy was to determine

¹⁶Ibid., 67.

¹⁷Ibid., 68.

the reorientation of American curricula at the beginning of the twentieth century.

This period of the twentieth century was also to see a greater use of commissions comprising acknowledged experts in the field of education charged with scrutinising existing mathematics provision and recommending appropriate reforms. One such commission, the CIEM commission (of which Felix Klein was a member), was convened in Milan in 1911 and directed its attention to the three questions:

What Mathematics Should be Taught to Those Students Studying the Physical and Natural Sciences?

What is the Place of Rigour in Mathematics Teaching?

How Can the Teaching of the Different Branches of Mathematics Best be Integrated?

However, whatever reforms, if any, were implemented as a result of this conference, or indeed any others before the end of the 1930s, Freudenthal relates that, “the earlier-mentioned reports which led to the 1937 reform in the mathematical curriculum of academic high schools in the Netherlands provide a striking example; this reform only took place on paper, because its underlying ideas were quite impossible.”¹⁸ This disparity between what is theoretically desirable in principle and what is actually achievable in practice

¹⁸Hans Freudenthal, *Mathematics as an Educational Task*, (Dordrecht, D. Reidel Publishing Company, 1973), 158.

received little serious attention by the early advocates of reform. Nevertheless the philosophy underpinning a reform and the management of such reform were issues which were to come to the forefront over the next fifty years or so. These matters, along with others, are treated after the history of curriculum development in Ireland has been sketched in broad outline.

The second half of the twentieth century was to see the emergence of quite a number of different educational approaches to mathematics which it would be impossible to do justice to in a survey which must perforce be reasonably concise. Consequently two of the more significant movements of the twentieth century – The New Mathematics Movement and Realistic Mathematics Education – are discussed in some detail while a number of lesser projects are described briefly.

2.2.1 The New Mathematics Movement

Of all the conferences to be convened in the twentieth century, it is the Royamaunt seminar of 1959 on *New Thinking in School Mathematics* that was to prove a major landmark in the history of mathematics education.¹⁹ The official report published in 1961 was the progenitor of what came to be called the The New Mathematics Movement or New-Math for short. Written in a refreshingly straightforward manner, it began by explaining that nations worldwide were at that time facing a demand for scientists and engineers – all of whom must have a sound knowledge and understanding of mathematics.

¹⁹O.E.E.C, “New Thinking in School Mathematics” (Paris: OECD, 1961)

New applications of mathematics in industry and other branches of economic activity required more mathematicians with new kinds of skills. Hence there was an urgent need for a re-appraisal of the content and methods of school mathematics.

The mathematician Marshall Stone stated "... we cannot put off much longer a fundamental study of introduction of some modern mathematics of a suitable kind into the secondary school curriculum. Such a modification is indispensable if we are to prepare prospective university students at all adequately for the kind of mathematical thinking which will be expected of them ..."²⁰ From a purely technical point of view, educators were under a continuing obligation to all students to eliminate dead, useless, outmoded or unimportant parts of mathematics and to offer them the liveliest and most stimulating aspects of the subject in order to develop their understanding and their creative talents.

By way of counterbalancing his focus on third-level provision, Stone was careful to assert that "we need to give quite as much attention in our schools to the technical preparation of skilled workers as to the training of future university students, engineers and scientists."²¹ He noted in relation to the problems connected with mass education at secondary level that "we would have a much better grasp of what needs to be done for those secondary school students who are not planning to enter a university if there could be made

²⁰Ibid., 16.

²¹Ibid., 17.

available a thorough up-to-date analysis of the simple uses of the more elementary kinds of mathematics – arithmetic, algebra, geometry, statistics – in the skilled occupations of modern industry and in the daily life of the average citizen.”²² One might gather from this that there was a dearth of genuine applications known to educators at that time relating to the ordinary affairs of people’s lives or fields of endeavour outside the sciences; an observation which will be reinforced presently when discussing the seminar’s thoughts on applications. A later section of the report informed that “the International Commission on Mathematics Instruction, ..., has as one of its main projects new mathematics and its applications that can be taught in secondary schools.”²³

Stone asserted that it was all too evident that primary schools were failing to develop at all efficiently, or at all adequately, the latent mathematical talents and interests of the child, but worse were, in far too many cases, having the negative effect of turning the pupil permanently away from the study of mathematics. Optimistically, he declared that, “fortunately, there are clear indications that we can develop methods far superior to those now generally employed in primary teaching. Illuminating psychological investigations, particularly those of Piaget, point the way to hitherto unrecognized pedagogical possibilities.”²⁴ In passing it may be noted that the OEEC document was careful to specify the term *pedagogy*, saying that it is concerned

²²Ibid., 19.

²³Ibid., 172.

²⁴Ibid., 22.

with²⁵

- learning the psychology of human development and learning
- understanding methods of teaching
- having a philosophy of education
- observation and practice of teaching

In relation to improving teaching methods and introducing a new curriculum, Stone said it would require finding competent persons and giving them time, incentive and encouragement to collaborate in the writing of texts and in the presentation of other kinds of teaching aids. As to the use of the technology of his day – film and television – he noted that²⁶

It appears to me that when we have discovered the right ways to use these media they may very well take their places beside the printed book as indispensable standard aids for the teacher.

The vision of what form the reform should take was outlined by Jean Dieudonné. He noted that while the university curriculum has undergone deep changes, these had not been matched (at least not at all to the same extent or along the same principles) in the secondary schools. Because of this university professors found themselves having to play catch up and the net result was that their teaching was becoming every day more superficial. He referred to the existing secondary school curriculum as encyclopedic and

²⁵Ibid., 142.

²⁶Ibid., 27.

overcrowded and said that, *most students simply emerge with the haziest notions of what it is all about*. He believed that teaching had suffered long enough under the deadweight of ‘pure geometry’; that the current process was a fantastically laborious one which left one needing “to separate what is fundamental from a chaotic heap of results with no significance except as scattered relics of clumsy methods or an obsolete approach.”²⁷ He decried the parade of ‘definitions’ which do not define anything and of pseudo ‘proofs’ which cannot undergo logical analysis, and criticised those who thought it preferable to practice an intellectual swindle rather than frankly admit the situation. For his part, he saw nothing wrong or dishonorable in starting with a premise which is not an axiom, but possibly some quite complicated statement, as soon as it is possible to show without any logical flaw that the given statement implies another one.

Under the banner of ‘Euclid must go!’, he recommended that the curriculum of the secondary schools be reorganised “in order to eliminate any undue waste of time and *to absorb as much of the burden resting entirely on the university* as is compatible with the intellectual capacities of the children.”²⁸ He advocated a school curriculum where the material would, in the final years of school, be based on the modern axiomatics of such structures as groups rings, fields and vector spaces. Because “one cannot even contemplate renouncing the new methods and new results; this would be the very

²⁷Ibid., 35.

²⁸Ibid., 34; emphasis added.

negation of what is the essential mission of higher education”; the task was how to organize this material into a well-balanced curriculum and to devise methods of teaching it so that it would serve as a preparation for students who would need to study mathematics at university.²⁹

Dieudonné concluded by asking

Should we go on indefinitely in this blind adherence to tradition and close our eyes to the overwhelming evidence that there are much better ways of doing the same things? I am reluctant to believe that our profession should be so lacking in daring and imagination.

Dieudonné’s address provoked sharp controversy. However, after much discussion, it was considered that

1. the initial instruction in geometry should be through intuitive and experimental procedures, and here the triangle is paramount.
2. if one purpose of secondary school mathematics is to prepare future university students, then instruction should not be hindered by having pupils solve exercises which are too artificial, too complicated and often useless.
3. a thorough axiomatic treatment should be postponed until the last years of secondary-school instruction.
4. the secondary schools should acquaint students with modern symbols, concepts and treatments of mathematics.

²⁹Ibid., 32.

5. at secondary level the presentation of material in visual form is very important. Graphical representation, the study of functions by their curves and the use of visual imagery are procedures that future teachers must be encouraged to apply in their teaching.

The brief of mathematician Albert Tucker, an acclaimed author of a book on combinatorics, was applications of mathematics. Dieudonné, for his part, side-stepped the applications issue because, by his own admission, his real interest was in seeing the secondary school syllabus act as a precursor to a similar university syllabus in pure mathematics intended for scientists, engineers and mathematicians. Tucker's contribution might have been more substantial, if he could have mustered more examples of the concrete application of mathematics at secondary school level than he actually did. With the exception of one explicit example from linear programming, Tucker spoke in very broad terms either directly, or indirectly through Warren Weaver, about where mathematics was applied in the past and how, with its increased generalisation in the twentieth century, it was finding more diverse applications through probability, statistics and statistical inference. All of which is of value but could have been more valuable still to teachers had he discussed examples from different areas at a level which might show these teachers how to enrich their teaching.³⁰

Furthermore, one would have to question how true it would be now to say

³⁰As a matter of history one might ask if the annual presence of a linear programming problem on the current Ordinary Leaving Certificate mathematics paper owes its origins to its mention here.

that “certainly such types of limitation and artificiality are inevitable at the secondary-school level and have always occurred in almost all applications of mathematics hitherto taught without in the least destroying interest or motivation on the part of the pupils.”³¹ Tucker wisely warned that “even at the secondary school, a separation of mathematics into ‘pure’ and ‘applied’ is not desirable.”³²

Of the other speakers, Choquet advocated the use of the integers in teaching. Servais laid great stress on the role of sets while Botch alluding to the limits of Euclid, said that geometry instruction tended to be static. The aim should be to have a dynamic approach to instruction which would incorporate translations, rotations, reflections and proceed step by step to a more generalised application of groups of transformations.

E. A. Maxwell spoke on the analysis content of the proposed new programme, remarking that whereas studies like differentiation, integration and so on, say *how* the various processes of calculus work, the purpose of analysis, when fully developed, is to tell us *why* they work. He was in no doubt that both the differential and integral calculus should be taught but said that the method of introducing integral calculus is not easy to decide upon. His own personal preference was the method of summation, since that is the way in which integration arises naturally in many physical problems. He placed the accent on the method of teaching so as to divulge understanding of the basic

³¹Ibid., 53.

³²Ibid., 60.

concepts of limits, differentiation, integration and functions.

Bunt, after saying a little about the nature and purpose of statistics, pointed out that there was hardly any field of study in which statistics does not in one way or another function as an auxiliary science.³³ He noted that obstacles to making statistics a part of the secondary school programme at that time were

1. lack of teachers trained in the subject
2. lack of textbook materials
3. the necessity in some countries to make room for any new subject by dropping an established one

The remainder of the report discussed issues relating to implementation, examinations, teacher training and textbooks.

Despite what was said about catering for mass education at secondary level, it is not hard to discern from this account the almost single-minded focus of the seminar on mathematical education as a preparation for university.³⁴

According to HKK the New-Math approach resulted in significant changes in the selection and mathematical treatment of the existing curriculum so that the deduction of content from axioms could become central to the teaching of mathematics. Content which was not amenable to a deductive

³³Ibid., 90-91.

³⁴As a matter of history, under the heading "The Extent of Mathematics Study" this report of 1961 noted that "in Ireland [...] all secondary schools are university-preparatory."

approach, such as many of the applications of mathematics, was relegated to the background: the ability to supply a mathematical proof and to argue logically was considered more important than the acquisition of ‘trivial computing skills’. It was aimed almost exclusively at the better preparation of those who would go on to study mathematics at university. It made demands concerning methods but did not question existing teaching practice. Barrow writes that “in retrospect this educational experiment to introduce abstraction to children at such an early age seems to have been somewhat idealistic in its devotion to the presentation of mathematics as a work of art.”³⁵ Mc Namara asserts:³⁶

Its main error was in not taking into account the way students learn the subject. One criticism of Formalism is that it presents the student with “ready made” mathematics. While believing that mathematics is invented it does not give the student the chance to re-invent [Freudenthal]. One reason for this is the reliance on textbooks. Hersh and Davis point out that the reader is given a “polished version” and that there is no time to “fiddle around” with the text.

The Dutch Educationalist Hans Freudenthal was a major critic of the movement’s over-emphasis on the axiomatic approach. His writings in the 1970s on mathematics education were destined to see him regarded as the founder of what is now known as Realistic Mathematics Education (RME) – an approach to the learning/teaching of mathematics underpinning the

³⁵Barrow, *Pi in the Sky*, 129-134 as cited in Mc Namara.

³⁶A. Mc Namara, “Investigational Mathematics in Transition Year – A Pilot Project” (M. Ed. diss., Dublin University, 1994), 9.

mathematical education being provided by many countries at present. This is the second of the major twentieth century movements to be considered in this work. However, before doing so a number of different projects undertaken in the 1960s and 1970s are briefly surveyed.

2.2.2 Curriculum Projects

Much of the synopses given in this section are based on *Curriculum Developments in Mathematics* by HKK, who remark that “the *project* is the major contribution of the second half of the twentieth century to the field of curriculum development.”³⁷ The University of Illinois Committee on School Mathematics (UICSM) characterised the features of a project as

- trials
- published materials
- in-service education

For HKK the project represented one solution to the problem of how the curriculum development process could be managed and of how innovation could be effected.

SMSG: In 1959 the School Mathematics Study Group (SMSG) was set up in the USA. It implicitly employed the Research-Development-Dissemination

³⁷A. G. Howson, C. Keitel & J. Kilpatrick, *Curriculum developments in mathematics*, 79; emphasis added.

(RDD) model characteristic of the whole era and was the first project to offer a comprehensive programme based on the New Math approach. Some of its strongest advocates placed considerable emphasis on precision of language and the avoidance of ambiguity. The SMSG provided a stimulus and a model to innovators throughout the world, it served to train a generation of textbook authors, and it exerted considerable influence on subsequent commercial texts. Yet in the diffusion of its work, much was lost. The variety of new work and the rigorously deductive methods were reduced or watered down; and topics such as set theory and algebraic structure lost their role as ‘relational links’ and became mere inventories of concepts.

SSMCIS: The Secondary School Mathematics Curriculum Improvement Study (SSMCIS) project carried out in the USA in the 1960s was an experiment of Columbia University in developing a unified six-year curriculum. Teachers had no role in initiating and planning the project or in determining its mathematical content. An intense amount of work went into the design of the syllabus, writing of experimental textbooks, education of teachers, pilot class teaching, and evaluation of outcomes. It was designed for the most capable students, taught in the classroom but at a more advanced level than traditionally.

IMU: The Individualised Mathematics Instruction (IMU) project carried out in Sweden during the 1960s began as a grass roots study to compare the effects between completely individualised and ‘conventional’ mathematics in-

struction in the first two years of secondary school. The project developed a completely new mathematics programme for grades 7,8, and 9. Two hundred and eight separate instructional objectives in seventeen areas of mathematics were identified, and course material was developed for each objective. International interest in IMU material was high. HKK remark that “the Swedish myth is that the government should set the framework for curriculum development, in fact they implemented the IMU before its developers were ready.”³⁸

FMP: The Fife Mathematics Project (FMP) (Scotland, 1969), aimed at mixed ability classes, was a local proposal to individualise some of the mathematics instruction in the first two years of secondary school by means of a workcard system that would allow pupils to do experimental work alone or in groups in the classroom. Over time the FMP came to refer to the method of working rather than the materials developed and the teacher became more of a stage-manager. The absence of strong centralised control over the curriculum of the early secondary school years permitted the project to be flexible in its goals and procedures.

It was found that in general, the average and less able pupils seemed to be well satisfied and had a more favourable attitude to mathematics in general, although some of the brighter ones were more doubtful. The teachers said the pupils found the materials interesting and stimulating but that they would

³⁸Ibid., 45.

prefer a mixture of project and class teaching. Among other things, it was observed that the Computer Assisted Instruction (CAI) component of the programme seemed to encourage competition among pupils to see how fast they could work.

Madison: The Madison Project initiated classroom experiments to explore how interest and mathematical activity, creativity, and discovery were best fostered and it took a stand against the translation into schools of university content, with a corresponding loss in the content's substance. The teacher was assigned a central role in the process of instruction and accordingly in curriculum development. It believed that ready-to-use materials would not bring about desired changes: efforts had to be aimed rather at improving the professional competence of the teacher.

R.B. Davis writing for the Madison Project under the heading of "Some Questions of Pedagogy and Curriculum" asserted that "three aspects of traditional (and often 'New-Math') curricula fit together into an inter-locking relationship that stands in the way of significant improvement." These are

- one is trying to *tell* children new mathematical ideas
- inappropriate sequencing of topics
- rote learning without understanding *why* each step is performed when it is.

He advocated *understanding*, not memorisation; the 'do, then discuss'

sequence for developing new ideas and stated that a child usually does not need the special precision of mathematical definitions. For him mathematical statements describe reality; therefore mathematics consists of 'stories about reality'. Furthermore, he wrote:

we want mathematical ideas to grow out of problem situations. An important goal of good mathematics instruction is to have the child develop a deep conviction, based on his own experience, that mathematics consists of reasonable responses to reasonable questions. This means, in general, that any topic which might seem arbitrary had probably better be deferred until a point at which it can be experienced as a 'reasonable response to a reasonable question'.

The Madison Project was to have a greater impact on curriculum theory than on school practice.

SMP: The School Mathematics Project (SMP) was an English project initiated in 1962 at the beginning of the reform movement when the most urgent need was generally identified as being the modernisation of syllabuses. In response to criticism of existing curricula from industry and commerce, its main intention was not to prepare students for university, but to introduce them to modern applications of mathematics in a technological society. As its deputy director Douglas Quadling wrote in 1972:³⁹

Some central principles were clear to us and have remained so. One was that, in devising a course for pupils up to the age of 16, priority belonged to those for whom this would be the final stage of mathematical education.

³⁹S.M.P., *The First Ten Years* (England: Cambridge University Press, 1972), 217.

SMP, which was funded by industry, laid emphasis on a wide variety of applications such as

1. computer mathematics
2. statistics
3. probability
4. operations research (linear programming, critical path analysis, transport flow)

However, according to HKK, not all of these applications found a permanent place in the school course for, “at secondary level, what is taught still tends to be determined by what is examined than by what is possible.”⁴⁰ Howson, who was actively involved in this enterprise for some time refers, in conjunction with his co-authors, to the pains taken by the project with teaching material:⁴¹

there is little strikingly novel in the presentation: there is no pretence at ‘discovery’ or openendedness. What is new is the amount of explanation, the sheer number of words as opposed to symbols.

⁴⁰HKK, 173.

⁴¹HKK, 173.

2.2.3 Realistic Mathematics Education

The principles that underlie this approach were strongly influenced by Hans Freudenthal's concept of 'mathematics as a human activity'. He felt that students should not be considered as passive recipients of ready-made mathematics, but rather that education should guide the students towards using opportunities to reinvent mathematics by doing it themselves.⁴² Freudenthal had complained in his highly regarded *Mathematics as an Educational Task* (1973) that⁴³

pupils are taught a ready-made mathematics; a mathematics which is prefabricated by the adult mathematician who knows how the parts fit together and what is the use of each of them but a meaningless pile of isolated bricks for the pupil who is not being introduced to this knowledge.

He added, some pages later, that "every mathematician knows at least unconsciously that besides ready-made mathematics there exists mathematics as an activity. But this fact is almost never stressed, and non-mathematicians are not at all aware of it."⁴⁴ Later, in the same work, he wrote that "almost all courses start with a mathematically organized matter. The student is deprived of the finest opportunity that exists in mathematics to learn to mathematize a non-mathematical subject matter; an important connection between pure and applied mathematics is bluntly obstructed."⁴⁵

⁴²From a web page of the Freudenthal Institute, <http://www.fi.uu.nl> (accessed September 23, 2003)

⁴³Hans Freudenthal, *Mathematics as an Educational Task*, 106.

⁴⁴Ibid., 114.

⁴⁵Ibid., 124.

Freudenthal believed in the didactic value of emphasizing thought patterns, explaining that a thought-experiment is the attitude of a teacher or a textbook author teaching an imaginary group of students his own thoughts and anticipating their possible reactions. By way elaboration he wrote,⁴⁶

Some didactically-gifted authors reveal how they would have invented the results if they would have been as clever before inventing them as they were afterwards. They imagine a somewhat cleverer *Alter Ego* and have him invent the subject matter anew by a more convincing, more useful, and more intelligent method than the real *Ego* used. It is not the historical footprints of the inventor that should be followed but an improved and better course guided by history.

This doctrine which Freudenthal believed “should be the principle of all mathematical education” since “nothing is quite as convincing as self-reliant discovery” became known as the process of guided reinvention.⁴⁷

For Freudenthal the traditional way of teaching mathematics, in which the start is within the formal system, was anti-didactic. Instead of starting at the very end, students should be given the opportunity to re-invent and re-construct mathematical concepts. In the realistic view, the development of a concept begins with an intuitive exploration by the students, guided by the teacher and instructional materials, with enough room for the students to develop and use their own informal strategies to attack problems. From there on, the path leads via structuring-, abstracting- and generalizing activities, to formalised concepts.⁴⁸

⁴⁶Ibid., 103.

⁴⁷Ibid., 130, 141.

⁴⁸H. van der Kooij, “Useful Mathematics for (technical) vocational education”, *Freudenthal Institute, Utrecht University* (1998): 1.)

Contextual problems, both real world problems and realistic problems in the field of mathematics itself, are very important in the RME approach. “The problem should grow out of the situation, and the child should learn to recognize the problem in the situation.”⁴⁹ They serve as a starting point for the development of a concept and also as a source for applications and refinements of the mathematical concept. In RME students are given time to dwell on a concrete, informal level, when a new mathematical concept is introduced. At this stage in the process students develop and use their own strategies for solving problems that are offered within a context.⁵⁰

Although realistic is often interpreted as originating from ‘real life’, sense making and meaningful problem situations can also be embedded in contexts that are artificial or mathematical. The key is that the problems are real and meaningful to the students; that the problems are sense making and worth solving.

While Freudenthal contributed a wealth of ideas and conceptual tools to the development of mathematics education – on contexts, didactical phenomenology, guided reinvention, mathematisation, the constitution of mental objects, the development of reflective thinking, levels in learning processes, the development of a mathematical attitude and so on – he did not design very much concrete material.⁵¹

⁴⁹Hans Freudenthal, *Mathematics as an Educational Task*, 134.

⁵⁰H. van der Kooij, “Useful Mathematics for (technical) vocational education”, *Freudenthal Institute, Utrecht University*, (1998): 1.)

⁵¹Flier for Streefland, Leon, *The Legacy of Freudenthal*, <http://www.springer.com/sgw/cda/frontpage> (accessed March 22, 2004)

2.2.4 Influence of Examinations

According to Christiansen *et al.* examination systems were established because examinations were seen as a means of⁵²

1. classifying students
2. stimulating effort and interest through competition
3. replacing patronage and nepotism by advancement based upon recognised ability and merit.

The examination system serves several purposes. It⁵³

1. sets standards for pupils, teachers and schools
2. stimulates effort by providing a goal
3. acts as an administrative device, as a filter to higher stage courses.

Of all the different examinations taken by students, the most important ones were seen to be those which attested to the successful conclusion of secondary schooling.

However, almost from their inception it was claimed that “many qualities and educational objectives were not readily assessable by examinations” and, furthermore, “examinations tended to dominate syllabuses rather than reflect them.”⁵⁴ These objections, and others, have been constantly reiterated and

⁵²B. Christiansen, A. G. Howson & M. Otte, *Perspectives in Mathematics Education*, 4.

⁵³*Ibid.*, 20.

⁵⁴*Ibid.*, 4

re-inforced by educationalists since the end of the nineteenth century. As early as 1880 Pattison wrote that “examinations directed, as the paper examinations of the numerous examining boards now flourishing are directed, to finding out what the pupil *knows*, have the effect of concentrating the teacher’s effort upon the least important part of his function.”⁵⁵

In passing it may be remarked that Pattison’s assertion that examinations essentially test no more than the recall of basic knowledge was to be echoed ninety years later by Irish educationalists when, as will be related, they studied the examinations system prevailing in Ireland at senior cycle in the late 1960s.⁵⁶

Concerning the defects of centralised systems the 1961 OEEC report noted that though⁵⁷

a centralised system of examination has the advantage of maintaining a fixed common standard in a definite body of knowledge, most countries develop a certain pattern of examination questions, and teachers spend much time training their pupils to answer these examinations – time which could better be spent on teaching more mathematics. Many of these examinations become intelligence and endurance tests, rather than tests of mathematical reasoning and comprehension.

While agreeing that “... examinations do not always succeed in assessing a power to reason”, Nyerere in 1968 added “and they certainly do not assess

⁵⁵Cited in A. G. Howson, C. Keitel & J. Kilpatrick, *Curriculum developments in mathematics*, 28

⁵⁶George F. Madaus and John MacNamara, *Public Examinations – A study of the Irish Leaving Certificate* (Dublin, ERC, St. Patrick’s College, 1970)

⁵⁷O.E.E.C, *New Thinking in School Mathematics*, (Paris: OEEC, 1961), 97.

character or willingness to serve.”⁵⁸

Constraining the Teacher: “A major constraint in the teacher’s autonomy – not necessarily for ill – is the external public examination system” – so wrote Howson *et al.* in 1981 citing as evidence that “the French ‘bac’ exerted a near stranglehold on the country’s education.”⁵⁹ Writing sometime later Howson, Christiansen and Otte (CHO), expanding on this point, said that it can be argued that examination and university entry constraints allow the teacher of senior pupils little freedom of action:⁶⁰

The pressures exerted by the school system have been so effective that today one really can speak of ‘school mathematics’ as a specific subject rather than ‘mathematics’. In particular the influence of examinations in later secondary years dominates both the teaching and the appreciation of what is mathematics to such an extent that other approaches in other years are almost totally obliterated.

If this is compounded by a view of mathematics by the teacher which does not go beyond the bounds of the immediate task, a very restricted and unimaginative form of teaching is likely to result.

While granting that teachers have a duty to prepare students for external examinations, these authors remark that it is a vital that teachers’ views on what constitutes desirable mathematical knowledge should not be allowed to be circumscribed by the forms which such examinations take, but, as they

⁵⁸Cited in A. G. Howson, C. Keitel & J. Kilpatrick, *Curriculum developments in mathematics*, 60.

⁵⁹*Ibid.*, 59.

⁶⁰B. Christiansen, A. G. Howson & M. Otte, *Perspectives in Mathematics Education* (Dordrecht: D. Reidel Publishing Company, 1985), 13.

acknowledge, the need to secure good examination results for one's pupils and to be true to mathematics and to their mathematical education does prove a point of conflict.

Regarding teacher input, the reality was that "in most countries the teacher's part in the accreditation of students has been kept small".⁶¹ However, on a more optimistic note, in relation to the introduction of the CSE examinations in England HKK note that "a feature of the newer examinations has been the opportunity they have provided for schools to draw up their own syllabus and for teachers to be responsible, with some outside moderation, for designing and operating the assessment procedures."⁶²

The historical consensus seems to be that while examinations are necessary, they have in the past militated against the genuine evaluation of mathematical knowledge and tended to standardise teaching, thus reducing teachers' opportunity or desire to show originality in classroom work.

However, there are always difficulties in effecting change, one being that, as the 1961 OEEC report noted, "it is exceedingly difficult to get a central agency to adapt its examinations to experimental developments in the teaching of mathematics."⁶³ Some twenty years later CHO add that "there is also a suspicion that those who control the examination system are unwilling to recognise increased professionalism in the schools and have failed to adapt

⁶¹Ibid., 23.

⁶²A. G. Howson, C. Keitel & J. Kilpatrick, *Curriculum developments in mathematics*, 59.

⁶³O.E.E.C, *New Thinking in School Mathematics*, (Paris: OEEC, 1961), 97.

quickly enough to changing demands and practices.”⁶⁴ But as HKK realistically remark “the best plans will come to nought unless examination systems test desired objectives and encourage rather than act against the attainment of educational and mathematical goals.”⁶⁵

2.3 The Curriculum Development Enterprise

By the early 1980s a structured view of what is entailed by curriculum development in general had emerged. Howson *et al.* wrote in 1981 that a curriculum “can no longer merely signify ‘the syllabus’ ” – the word syllabus meaning the programme or outline of a course of study or teaching – but must encompass all aspects of the educational endeavour. These different aspects they list in the order⁶⁶

- aims
- content
- methods
- assessment procedures

That this definition, in its entirety, is not universally accepted can be seen from the acronym NCCA, which stands for the National Council for

⁶⁴B. Christiansen, A. G. Howson & M. Otte, *Perspectives in Mathematics Education*, 20.

⁶⁵A. G. Howson, C. Keitel & J. Kilpatrick, *Curriculum developments in mathematics*, 2.

⁶⁶A. G. Howson, C. Keitel & J. Kilpatrick, *Curriculum developments in mathematics*, 2.

Curriculum and Assessment. Judging from the title of this body, established in Ireland in 1991, it might be speculated that the civil servants who set up the Council view the first three items listed as being the components of a curriculum, with assessment being a separate activity.

Whatever distinctions one might make between various definitions of the term curriculum development, the different stages of that process are listed by Howson as⁶⁷

- identification (of a need)
- formulation (of a course of action)
- acceptance (by others)
- development (of new materials and other physical resources)
- dissemination (of the aims and objectives)
- implementation
- evaluation (either formal or 'informal feedback')

The authors comment that most innovations in education are concerned with the development of new materials and other physical resources saying that this is a key stage in such projects.⁶⁸ Chapter 4 entitled Curriculum Matters expands at length on the different aspects of the curriculum enterprise listed above as they relate to the teaching/learning of mathematics.

⁶⁷Ibid., 14.

⁶⁸Ibid., 15.

Summary: One might summarise the history of curriculum development in mathematics very broadly as consisting of two major phases. The first phase, which may be thought of as beginning in antiquity, didn't receive its vitalising impetus until after the invention of the calculus in the seventeenth century. From then onwards there was a gradual recognition by the nations of the western world of the empowering ability of mathematics. This, in turn, led to the realisation that the study of mathematics as a subject should be an essential part of the curriculum at the primary, secondary and tertiary levels. When this was firmly accepted by virtually all of the affluent countries towards the middle of the twentieth century curriculum development entered its second phase which endures to this day.

This phase is characterised by a focus on the nature and efficacy of the mathematics programmes that should be taught. Its early stages saw the emergence of a large number of movements and projects based on different philosophies of teaching and learning. As educational researchers reflected on the underlying theories and their consequences, curriculum development began to emerge as a major area of research in its own right concerned, as it is, with better understanding the educational endeavour.

Chapter 3

History of Curriculum Development in Ireland

The Whig government introduced a National School system to Ireland in 1831, even though very few other European countries were providing any comparable system at that time. By 1849 the number of national schools was 3500 with an attendance of 370,000 pupils. At this time and until the end of the nineteenth century the primary focus of education was at the primary level. Ó Buachalla¹ writes:

While the century witnessed an expansion of educational provision at all three levels and while various controversies attached to secondary, technical and university education, it was undoubtedly the issues involved in the elementary 'national' schools which dominated the educational scene throughout the century.

Some of the issues alluded to here concerned control of the schools, the nature of the instruction and the payment of teachers. While national school education was initially intended to be inter-denominational and integrated,

¹Séamus Ó Buachalla, *Education Policy in Twentieth Century Ireland* (Dublin: Wolfhound Press, 1988), 19.

“by the end of the century, education for the majority was parochially administered, denominationally segregated and clerically managed.”²

While many different efforts were made to train teachers, as late as 1870, no more than 38% of teachers were trained. The following year, 1871, saw the introduction of a ‘payment-by-results’ scheme which lasted until 1900 in the national schools and until 1924 in the secondary schools. This scheme “left an indelible character on the quality of the instruction and allowed external examinations to exert a dominant influence on Irish schools, an influence which still persists to some extent.”³ Ó Buachalla adds, “while the undue emphasis on examination success and the associated teaching methods may have offended against the canons of sound modern pedagogy, nevertheless the majority of those institutions offered educational opportunities to a growing number of able scholars from that section of the population which previously seldom continued their education beyond the elementary stage.”⁴

Beyond this elementary stage lay Irish secondary or superior education, described as ‘a record of poverty, of abused endowments and of numerous inquiries preceding tardy reform’. In 1871 there were 24,710 pupils receiving education in these secondary schools which showed a wide variability as to size, facilities, curriculum, teacher qualification, and the nature and quality of the instruction imparted.⁵

²Ibid., 19.

³Ibid., 25.

⁴Ibid., 32.

⁵Ibid., 32.

By the beginning of the twentieth century, Ó Buachalla says,⁶

the structures for the provision and management of education which had evolved in the course of the (nineteenth) century had been consolidated by 1900 in a manner which explicitly recognised the dominant role of the churches; these structures would emerge as an early controversial issue but would prove resistant to substantial modification for most of the new century.

In relation to its first eight decades the same author remarks that,⁷

the opening and closing periods were marked by lively debate and expansion policies whereas the middle decades were characterised by a distinct inertia in policy; in the period from the twenties to the early sixties, access to and participation in the educational system beyond the compulsory age were governed by conservative social values which also accorded a low political priority to education policy and provision.

Ó Buachalla divides the period from 1900-1986 into three phases. The first phase from 1900-1922 began with the introduction in 1900 of new programmes and regulations which governed the development of these schools until the change of the administration in 1922. These changes were inspired by the 1898 Bellmore Commission on primary schools and the 1899 Commission on Intermediate Education.⁸ McPherson's Irish Education Bill of November 1919 might have been the most significant contribution of this phase, offering "a veritable *magna carta* for Irish education which would have given a similar coherent educational structure to all parts of the island." However, it met with fierce opposition, as did the 1920 Government

⁶Ibid., 35.

⁷Ibid., 49.

⁸Ibid., 51.

of Ireland Act, so that its far-reaching proposals were never implemented.⁹ The main function of secondary-level education throughout this period was as a feeder to tertiary education.

Phase II from 1922-1957 saw the formation, by the Free State government, of the Department of Education in 1924 though the establishment of such a department had been proposed as early as 1907. With regard to this government's educational philosophy, Ó Buachalla writes that¹⁰

Quite apart from any economic constraints, the Free State government envisaged its role in education as one of minimal interference, ... (its) caution in education policy is clearly reflected in its timid approach to the Intermediate Education Act of 1924 and in the paucity of legislation on education in the early years ...

In 1926 the Ingram Commission on Technical Education was established and offered, not only a detailed analysis of existing technical education but made trenchant criticisms of primary, secondary and general education policy. It was followed in 1930 by the Vocational Educational Bill. Education policy throughout this period tended to be conservative rather than expansionist and to focus mainly on curricular issues.¹¹ Evans relates that the Vocational Education Committee (VEC) schools had their own system of public examination since 1936 and while before the 1930s there may have been, "a lack of clarity as to what technical education is; what constitutes an appropriate methodology; and the low social status associated", by 1947,

⁹Ibid., 53.

¹⁰Ibid., 64.

¹¹Ibid., 65.

“VEC programmes were markedly practical, employing a much more explicit rationale for education and curriculum development than any measure in secondary education.”¹²

Throughout this period, secondary education was still for the few and not the majority. As late as June 1940, the then Taoiseach, Eamonn de Valera, pointed out that for 90% of the population, the primary school is the “only educational experience”.¹³

Phase III from 1957-1986 was marked by developments in education which rendered this period totally different from what went before. It was an era of expansion and initiative in education that slackened somewhat in the economic recessions of the late seventies and early eighties. In 1962 Professor Patrick Lynch oversaw a major pioneering quantitative analysis of the Irish education system, resulting in the 1965 report *Investment in Education*, which among other things, recommended the establishment within the Department of Education of a planning and development branch.

The year 1967 saw “the single most significant advance in Irish Education in the century’ with the ‘free’ education and school transport system.”¹⁴ In this year also the *Report on Regional Technical Colleges* was published and would lead, by the late 1970s, to the establishment of nine Regional Technical colleges and two National Institutes of Higher Education (NIHE)

¹²John Evans, “Education as a Preparation for Work” (Ph. D. diss., Dublin University, 2000), 71; 93.

¹³Ibid., 90 (cited).

¹⁴Séamus Ó Buachalla, *Education Policy in the Twentieth Century Ireland*, 11.

whose remit was to place a pronounced emphasis on technology, industry and commerce. In 1968 the Higher Education Authority (HEA) was established to oversee education at university level, and a little later the National Council for Educational Awards (NCEA) came into being charged with promoting, coordinating and developing higher education in the non-university sector.

By 1984 the Department of Education had accepted the tripartite division of the post-primary system into vocational, secondary and community/comprehensive schools.¹⁵ In that same year the movement of reform and development promoted in the mid-eighties saw a general examination of the post-primary curriculum under the Curriculum and Examinations Board (CEB). Oldham writes¹⁶

Responsibility for producing courses of study shifted from informal, advisory committees established by the Inspectorate of the Department of Education for each subject, for each cycle, to formally constituted course committees reporting to an executive committee of the CEB (subsequently the NCCA), which in turn, advised the minister.

Contrasting the beginning of the century with its (then) current state in 1986, Ó Buachalla remarks,

The policy process and its consultative machinery of 1900 was simple, if not primitive, compared to the labyrinthine structures and networks operating in 1986. Teachers and parents were excluded then whereas in 1986 they are included as well as other interest groups.

¹⁵John Evans, "Education as a Preparation for Work" (Ph. D. diss., Dublin University, 2000), 103.

¹⁶Elizabeth Oldham, "Mathematics in the Senior Cycle: The New Leaving Certificate Courses", 3.

Current Phase of Curriculum Development in Ireland: The trend for inclusiveness has become even stronger and more wide-spread in what might be labelled the current fourth phase dating from 1986 and stretching to the ‘horizon’ year of 2010. A series of seminars, extensive consultations and surveys involving numerous groups were carried out between 2000 and 2004 by the NCCA as part of an on-going process to have in place by 2010 curricula appropriate to the first decade of the twenty-first century.

By the latter half of the 1980s, as the OECD *Reviews of National Policies for Education: Ireland* commented: “There was indeed a ferment of change beneath and surrounding the placid surface of schooling.”¹⁷ Earlier the report had noted under the heading *The curriculum criticised* that “there is dissatisfaction with present practice and frustration that new initiatives are being ignored or pursued with insufficient vigour.”¹⁸ In fact there had been trenchant criticism of the education system in general and the Leaving Certificate (established) in particular.

Nevertheless “Ireland fully recognised the necessity of adapting education to the present age of high technology and providing resources for essential technological innovations”¹⁹ As part of this process of adaptation, the CEB became in 1991 the National Council for Curriculum and Assessment (NCCA), which according to the same report was²⁰

¹⁷OECD, *Reviews of National Policies for Education: Ireland*, (1991), 123.

¹⁸Ibid., 66.

¹⁹Ibid., 122.

²⁰Ibid., 66.

the most important innovation in recent years, since it can provide a means whereby a national approach to the curriculum at both primary and secondary levels can be adopted, and the roles of the schools in implementing national guidelines and in developing their own approaches can be elaborated.

The NCCA was a major contributor to transforming post-primary education at senior cycle during the last decade of the twentieth century. The Leaving Certificate (established), which up to that point was the only option available to most schools, was augmented by Leaving Certificate Vocational Programme (LCVP). In addition, the option of a three-year senior cycle, comprising the Transition Year followed by one of the three Leaving Certificate programmes, was made available.²¹ The following sections elaborate a little on each of these developments.

The Leaving Certificate Vocational Programme (LCVP) is a two-year programme which comprises groupings of the LCE subjects with some additional modules. It was expanded in 1994 to broaden the choice of subjects and to strengthen the vocational content. The assessment consists of two elements: written examination (40%) and a portfolio of coursework (60%). LCVP students receive the same certificate as LCE students.²² However, currently the points allocated fall short of those awarded for a Leaving Certificate subject and the allocations for Institutes of Technology (ITs) and for

²¹National Council for Curriculum and Assessment (NCCA), *Developing Senior Cycle Education – Consultative Paper on Issues and Options*, 6.

²²Commission on the Points System, *Final Report and Recommendations*, (Dublin: Stationary Office, 1999), 47.

universities differ.²³

Transition Year: As early as 1974 some schools were offering students an extra year which did not follow examination syllabi. In 1991 the Department of Education and Science (DES) conferred official status on an optional year – just after the three-year Junior cycle and before the two-year Senior cycle – titled Transition Year (TY). The DES did not provide syllabi for TY but did offer guidelines such as “TY is for the personal development of the student - ‘to grow in maturity’”²⁴ It was not be used to extend the LC cycle from 2 to 3 years although it would appear that some teachers/students were doing just this.²⁵

It is up to schools to submit their own programmes for approval, a task which requires a high level of planning. The evaluation of the TY by the DES in 1996 noted the consensus among principals, teachers and students that the TY gave schools an opportunity to engage in genuine in-school curriculum development.²⁶ Research carried out by the NCCA has shown that participation in TY is associated with improved performance in the LC examination.²⁷

²³National Council for Curriculum and Assessment (NCCA), *Developing Senior Cycle Education – Consultative Paper on Issues and Options* (Dublin: NCCA, December 2002), 16.

²⁴An Roinn Oideachais (Dept of Ed), *Transition Year Option: 1987-'88, Notes for Schools*, 2.)

²⁵A. Mc Namara, “Investigational Mathematics in Transition Year – A Pilot Project”, (M. Ed. diss., Dublin University, 1994)

²⁶Department of Education and Science, *Transition Year Programme 1994-1995: An Evaluation by the Inspectorate of Education*, (Dublin: The Stationary Office, 1996)

²⁷NCCA, *From Junior to Leaving Certificate: A Longitudinal Study of 1994 Junior*

The three LCE programmes: In 1990 an Ordinary Alternative Course was introduced alongside the existing Higher and Ordinary level courses and was first examined in 1992. A separate programme, the Leaving Certificate Applied (LCA), has its own 'Maths Modules', first examined in 1995. This course came to be regarded as separate from the Leaving Certificate established programme.

The LCA is a self-contained two-year person-based programme involving a cross-curricular approach rather than a subject-based structure.²⁸ It is aimed mainly at those who do not wish to proceed to Higher Education. Additional elements were introduced in 2000. The public perception of the programme as one for 'weaker students' resulted in a lack of parity of esteem with other Leaving Certificate programmes.²⁹ While the LCA can lead to further education courses (*e.g.* PLC courses) and through this route to higher education, there is no direct access to higher education from the LCA.³⁰

Currently LCA students are graded on a series of projects and interviews that begin in January of fifth year and continue throughout the programme. Final exams count for 34 per cent of the overall mark. Significantly this form of assessment and evaluation may very well prove to be the model applied to the envisaged horizon curricula. As the national coordinator of the LCA,

Certificate Candidates who took the Leaving Certificate Examination in 1997, (Dublin: The Stationary Office, 1999)

²⁸Commission on the Points System, *Final Report and Recommendations*, 47.

²⁹National Council for Curriculum and Assessment (NCCA), *Developing Senior Cycle Education – Consultative Paper on Issues and Options*, 17.

³⁰Commission on the Points System, *Final Report and Recommendations*, 48.

Sheila O'Driscoll stated in June 2004:³¹

the evaluation methods, the ongoing assessment and the modular approach to delivery have provided an excellent example of how a reformed Leaving Cert might work.

The Ordinary Alternative Course gave birth to the present third course in the suite of LCE programmes. This is the Foundation Level course which offers alternative tracks to the Ordinary level course in the subjects Gaeilge and Mathematics.

The courses available at Senior cycle as of 2005 are synthesised in the following table:

Programmes at Senior Cycle (2005)

Transition Year (TY)

Leaving Certificate Vocational Programme (LCVP)

Leaving Certificate Applied (LCA)

Leaving Certificate Established (LCE) with three levels:

- Higher
- Ordinary
- Foundation (Gaeilge & Mathematics)

³¹Louise Holden, "The LCA offers students an alternative to the regular Leaving, and may even provide a model for reform", *The Irish Times*, November 16, 2004.

This is the same provision put in place ten years earlier. However, as stated in the introductory chapter, “it is felt timely, for a number of reasons, that the NCCA revitalise the debate and action on senior cycle education and the Leaving Certificate (established) in particular.”³²

3.0.1 Paucity of Documentation on Policy-Making

It seems to be an accepted fact that there is little publicly available material within the Department of Education pertaining to policy-matters, certainly before 1986. Professor Patrick Lynch in the Preface to *Education Policy in Twentieth Century Ireland* by Séamus Ó Buachalla alludes to “the lack of detailed documentation” when, in praising the reserach efforts of the author, he comments that “from personal experience I can understand his formidable task in seeking material on policy making from the files of the Deptartment of Education.”³³

Earlier in 1970, concerning the failure of the Department of Education to spell out in any concrete detail the intention of a syllabus, Madaus & MacNamara had this to say: “... the objectives of each course (*Rules and programme of secondary schools, 1966-67*) were stated almost exclusively in terms of content; they could only guess what students were expected to be able to do with the content. Presumably, students and teachers are

³²National Council for Curriculum and Assessment (NCCA), *Developing Senior Cycle Education – Consultative Paper on Issues and Options* (Dublin: NCCA, December 2002), 7.

³³Séamus Ó Buachalla, *Education Policy in Twentieth Century Ireland*, (Dublin: Wolfhound Press, 1988), 9.

confronted with the same difficulty.”³⁴

The same authors wrote of the consequences of such a policy (if indeed it was one):

When a public body fails to specify the content, and the related skills and abilities which it will examine, the tendency ... is for both examiners and teachers to settle for a lowest common denominator. This usually turns out to be Knowledge of a rather trivial kind. Herein, if our analysis has not completely misled us, lies the greatest single weakness of the LCE; as a whole, the examinations we studied laid an impossible and senseless burden on student's memories.

In reply to the following question of the author's:

Are there any publications available, besides the Aims and Objectives to be found in the Introduction to the (1995) Higher Leaving Certificate Mathematics syllabus, which discuss the rationale behind the different topics chosen and the reasons why others were omitted?

an official from the DES obligingly made the following reply³⁵

I am not aware of any such documents; I am reluctant to speculate as to the reasoning of those concerned, in the absence of documentation, but I believe that some of the new topics were introduced because there appeared to be gaps in the courses by comparison with other countries, and with emerging trends in the application of mathematics at third level.

³⁴George F. Madaus and John MacNamara, *Public Examinations – A study of the Irish Leaving Certificate*, 113.

³⁵Email correspondence with Mr. Hugh McManus, Post-Primary Inspector (Mathematics), DES/SEC, July 13, 2004.

It may be stated that the current (1995) syllabus of the DES which is, as of January 2005, the one to be found at its website, suffers from the very same faults identified by Madaus & Mac Namara in 1970. However, it must be pointed out that the NCCA have done much to address this state of affairs in relation to Junior Certificate Mathematics as evidenced by their recent guideline publications which expound at some length on the interpretation of syllabuses. Presumably, the intention is to do the same with whatever LC syllabus will be in existence when such work is undertaken.

Based on a reply of the same inspector, it might be asserted that the DES do have a policy, if, perhaps, not a documented one, relating to textbooks. In answer to the question:

Is there any system at DES level which, while not vetting (mathematics) textbooks, actually reads them to assess them for soundness?

the same correspondent wrote³⁶

No. None of the DES, the SEC, or the NCCA has any function in this regard, (and deliberately so). The onus is on schools and teachers to deliver the syllabus as laid down in the published syllabus document. If they choose to use a textbook to assist in this regard, then the responsibility for ensuring that it is a relevant text rests with them. When syllabuses are being reviewed, publishers are provided with updates on the ongoing state of play, by arrangement.

³⁶Ibid.

3.0.2 The Leaving Certificate Examination

In 1999 the Commission on the Points System noted that³⁷

Ireland is unique in the extent to which the Leaving Certificate relies on a single final examination taken at a single point of time using a single test instrument.

Nearly thirty years earlier in 1970, Madaus and MacNamara made the same point and remarked on its limitations:³⁸

The present examination is almost the sole means used by Irish society to assess a student's performance in five or six years of secondary schooling and also to assess his suitability for a variety of careers. Either function is more than a single examination can possibly bear, and besides present practice ignores all the information which schools accumulate about individual students over the years.

The same authors expressed a general concern about the possible negative impact of such examinations:³⁹

So powerful is the influence of the external examination that there are serious dangers associated with their use for purposes of curriculum revision. The most obvious danger is that the test maker might easily supplement his role as evaluator with that of policy dictator. Alternatively, educational administrators might themselves turn dictator and coerce schools into submission by means of external tests.

In 1991 the *OECD Reviews of National Policies for Education: Ireland* made similar points:⁴⁰

³⁷Commission on the Points System, *Final Report and Recommendations*, 50.

³⁸George F. Madaus and John MacNamara, *Public Examinations*, (Dublin: Educational Research Centre, St. Patrick's College), 1970, vii.

³⁹*Ibid.*, 131.

⁴⁰*Ibid.*, 67.

The domination of examinations, particularly at the upper secondary level, ensures that, on the one hand, teaching and the curriculum are largely determined by examination requirements, and, on the other, that major status differences persist throughout the secondary system, between the examination streams and the rest.

Later, the same report added⁴¹

Everyone agreed that examinations played a dominant role in Irish education. They helped determine the curriculum and profoundly influenced the behaviour of teachers and pupils for the last two years of the primary school and throughout secondary education.

Despite their concerns Madaus and MacNamara were by no means opposed to the use of external examinations, pointing out that “there is now a growing body of literature to support the view that external examinations, suitably modified, need not result in educational damage, and can exercise a beneficial influence of great value to education.”⁴²

However in 1999, the report of the Commission on the Points System remarked that a number of damaging effects were attributable to the examination/points system, two being: “a narrowing of the curriculum arising from the tendency to teach to the examination rather than to the aims of the curriculum and an undue emphasis on the attainment of examination results.”⁴³ Elaborating on the first point, the Commission stated that international research⁴⁴ shows that “experience with external examinations would seem to

⁴¹Ibid., 123.

⁴²Ibid., 123.

⁴³Commission on the Points System, *Final Report and Recommendations*, 49.

⁴⁴T. Kelleghan, G. F. Madaus and A. Raczek, *The Use of External Examinations to Improve Students Motivation* (Washington DC: American Educational Research Center,

indicate that all too many students focus their efforts on mastering strategies to help them over the examination hurdle rather than on developing mastery of the subject matter and honing lasting competencies.”⁴⁵

However, despite the ‘once off nature’ of the final examination the Commission had the following to say⁴⁶

While concerns were expressed ... about the limitations of the Leaving Certificate as it stands, it is clear that there is strong support for the use of a certificate of student achievement at the end of second-level education as the determinant of entry to third-level education for school leavers. It is also clear from studies in other countries that overall performance in secondary school is the single best predictor of success in higher education.

Writing in June, 2003 in an article titled *Time to leave the Leaving behind?* the education editor of the Irish Times, Seán Flynn, reported that a major poll conducted by the NCCA indicated that for many students and teachers “the Leaving Certificate examination is an unloved exam despite its towering presence in Irish education.”⁴⁷ Most respondents to the poll felt that the examination needs to be re-shaped, move away from an emphasis on regurgitating material, be more creative and more relevant. He indicated that

1996)

⁴⁵Commission on the Points System, *Final Report and Recommendations*, 50.

⁴⁶*Ibid.*, 48.

⁴⁷Seán Flynn, “Time to leave the Leaving behind?”, *The Irish Times*, June 3, 2003)

there is much support for the more flexible approach pioneered in Transition Year and in the Leaving Certificate Applied course. Many would like to see some features from these course – especially the onus it places on personal initiative integrated – into the ordinary Leaving Cert exam.”

The article says that the then Minister for Education had signalled his support for radical reform of the exam with a greater emphasis on continuous assessment and project work. However, Flynn remarks that “the challenge of formulating an alternative is not easy. As over 80 per cent in the NCCA poll acknowledge, the exam enjoys a high level of public support. Broadly it is trusted by parents and employers. The challenge ... is to change the exam and retain the public trust in any new system.”⁴⁸

3.0.3 Evaluation of Irish Curriculum Development

The *OECD Reviews of National Policies for Education: Ireland* asked back in 1991:

“Were the national authorities satisfied that the examination system was serving the best needs of society at large? Had any systematic attempt been made to analyse its effects on learning outcomes, on attitudes, and on school organisation and practices?”

Other than reporting that “great efforts continued to be made to ensure that the secondary curriculum served pupils across the whole range of abilities” and that “the modes of examination were currently under scrutiny”⁴⁹ the OECD document offered no further information on whether any evaluations

⁴⁸Ibid.

⁴⁹OECD, *Reviews of National Policies for Education: Ireland*, (1991), 124.

had taken place, or were due to be undertaken, in relation to the issues raised in these questions. However, one evaluation of the Leaving Certificate examinations was carried out at the end of the nineteen-sixties. Since so many of the findings and most of the observations made regarding the LC syllabus and examinations pertaining in 1970, still apply to the current programme, despite the syllabus change in 1992, a general synopsis of this study is presented here along with specific references it made in relation to mathematics.

In 1970 Madaus and Mac Namara (M&M) in their book *Public Examinations* reported on a detailed study overseen by them on the Irish Leaving Certificate examination.⁵⁰ M&M state that they were principally concerned about two major problems; the reliability of LC marks and the types of intellectual activity for which the marks were awarded. They found that LC marks were unreliable to a serious degree and suggested, by way of partial remedy, that certain sections of the examining be done by means of “multiple-choice questions which can be marked by mechanical devices without any unreliability whatsoever.”⁵¹ They also proposed that students should be encouraged and even required to exercise their intelligence at levels higher than rote learning.

In a chapter dealing exclusively with the mathematics examinations M&M report that “the mathematics papers proved to be both particularly inter-

⁵⁰George F. Madaus & John MacNamara, *Public Examinations*, (Dublin: Educational Research Centre, St. Patrick's College, 1970)

⁵¹*Ibid.*, vi.

esting and particularly difficult to evaluate. The main reason for this is that the LC mathematics courses were thoroughly revised in the early 1960s, and 1967 was only the second LC examination of the new LC courses.”⁵² Guided by Bloom’s Taxonomy they give an analysis of each of the papers based on a hierarchical classification of the intellectual skills brought into play by each examination.⁵³ The Hierarchical Structure of the Taxonomy used is shown in the following table:

Category	Cognitive Activity
Knowledge	a
Comprehension	a+b
Application	a+b+c
Analysis	a+b+c+d
Synthesis	a+b+c+d+e
Evaluation	a+b+c+d+e+f

Table 3.1: Cognitive categories and activities.

They found that “of all the examinations which were investigated, the mathematics papers has the fewest items classified under knowledge and most under analysis. Other examinations, such as English and Irish, called for a limited amount of synthesis, which admittedly was not called for at all in the mathematics examinations. However, most of the content of the

⁵²George F. Madaus & John MacNamara, *Public Examinations*, 1970, 85.

⁵³*Taxonomy of education objectives: Handbook 1, the cognitive domain* (1956), Benjamin S. Bloom *et al.* as cited in George F. Madaus & John MacNamara, *Public Examinations*.

English and Irish syllabus was examined at the Knowledge level. Taking one thing with another, and recalling our reservations as classifying English and Irish essays as synthesis, we are inclined to conclude that the mathematics examinations were the most successful in tapping a wide range of intellectual abilities.”⁵⁴

In relation to the question of “how well the LC mathematics papers sampled the universe of items constituted by the syllabus” the authors commented that “an honours student who knew nothing of calculus could still have gained full marks”, a circumstance which “must be considered a serious defect in the examination, since the meaning of a student’s exam responses derives mainly from the assumption that these responses reveal how well he would have fared on the universe of problems.” They added, “moreover, the fact that different students attempted different problems casts grave doubts on the comparability of marks from student to student” and concluded that, “in short, for several reasons, the idea of allowing a choice of questions was not a happy one.”⁵⁵ They re-emphasised this point later saying, “the practice of allowing optional questions undermines the major function of an external examination – it renders non-comparable the results of students who answered different sets of questions.”⁵⁶

Thirty-five years later, as noted earlier, the DES have opted for no choice on the Junior Certificate examinations.

⁵⁴George F. Madaus & John MacNamara, *Public Examinations*, 91.

⁵⁵*Ibid.*, 92.

⁵⁶*Ibid.*, 115.

Lack of Guidance: Elaborating on the reductionism which is a consequence of failing to specify explicitly how a syllabus is to be interpreted, M&M noted that:⁵⁷

in the past, syllabuses like those for the Irish LCE, have for the most part confined themselves to a statement of content and neglected to specify what students are expected to be able to do with it.

Examiners, teachers and students, lacking directions about anything except content, tend to feel that they had done their duty when they had in their respective capacities 'covered' this content. The inertia of human systems did the rest. The patterns of examinations once established, examiners, teachers and students knew what to expect of one another, and against all three of them the most high-minded pleading in the syllabus was of no avail.

This problem of interpretating syllabi in an overly-narrow fashion so that they fail to show relevance was discussed in the introduction where it is asserted to be a shortcoming that still bedevils the current programme.⁵⁸

Influence of the universities: M&M remarked that "education is run by educators who naturally take the 'educational' point of view, and they tend to place too high a premium on subjects required for university entrance to the neglect of other subjects which might serve the best interests of the majority of students."⁵⁹ In fact, the Commission on Higher Education had written in its 1967 report:⁶⁰

⁵⁷Ibid., 125.

⁵⁸Refer to 10.

⁵⁹George F. Madaus & John MacNamara, *Public Examinations*, 127.

⁶⁰Commission on Higher Education (1960-67), *Report, Volume II* (Dublin: Stationary Office, 1967), 572.

We are of the opinion that the universities and other institutions exclusively concerned with higher education should, from the beginning, work at a level which is appropriate to them. We do not accept that the first university year should be looked upon as a trial year. The secondary students entering the university or other institution should come prepared to work at the appropriate level.

M&M state that the LC examinations themselves deal almost exclusively with material considered pre-requisite for future university study, and in support of this view comment that “the American raters of the Leaving certificate examination papers all observed that the content of the courses was in their view university orientated.” They continued⁶¹

University requirements can be quite arbitrary and secondary courses designed as prolegomena to university ones might not suit the students for whom they were intended. Our point is that secondary school curricula should be designed and examined primarily from the point of view of those for whom they are intended, with due regard to their age, ability and attainment, rather than from the point of view of what those students may be some years later.

In 1999 The Commission on the Points System in its Final Report and Recommendations re-iterated the view that the demands of third-level entry should not dominate the curriculum at second level nor unduly influence the lives of second level students. “The Commission strongly supports the integrity and independence of the second level system and agrees that second-level education should provide a broad and balanced curriculum which

⁶¹George F. Madaus & John MacNamara, *Public Examinations*, 128.

would enable young people to benefit from a wide range of educational experiences.”⁶²

It may be said that the point made by the M&M in 1970 has been addressed somewhat in the present time by the LCVP and the LCA programmes. However, it will be argued that some criticism is still a valid one in relation to the Ordinary and Higher level courses where more needs to be done to establish the criteria referred to in this extract.

Alternative methods of evaluation: Quoting the educational researcher Tyler, the authors wrote that his “experience was that no change could be made in the schools until the power of the external examinations was broken by supplementing those examinations with alternative methods of evaluating students.” They add that time proved him right in that “broadening the base of evaluation to include teacher’s assessments and a scholastic aptitude test as well as external tests of attainment [...] has the effect that Tyler described.”⁶³

Revisiting the question of the reliability of marks in the LC examination the authors concluded that:⁶⁴

There is a high degree of unreliability associated with the marks in each subject, not due to any malice or carelessness on the part of the markers, but due rather to the nature of the task which they have been set. It follows that to the extent which LCE marks are unreliable, to that extent they are invalid selectors for any purpose whatsoever.

⁶²Commission on the Points System, *Final Report and Recommendations*, 49.

⁶³Ibid., 129-130.

⁶⁴Ibid., 134.

By way of advocating a more comprehensive form of evaluation they opined that "... a combination of all three, marks in external applications, marks in internal applications and rank in class, form a better predictor than any on its own."⁶⁵

While often alluded to in the interim period between 1970 and 2005, these recommendations to broaden the base of evaluation were never adopted.

3.1 History of Ireland's Mathematics Provision

The paucity of documentation at the DES relating to educational policy in general has already been alluded to in an earlier section, and was confirmed for this writer during his searches for material treating specifically with mathematics provision in the period 1924-1964. Outside of the actual examination papers which date back to 1924 there is little or nothing touching on the rationale shaping these papers. However, since what is examined is almost certainly close to what is taught, one may gauge to some extent from these papers the nature of the mathematics teaching at LC level in the first decades of the new state.

Phase I: 1924-1964: This writer perused the examination papers for each of the years 1924 to 1965 inclusive, before making a particular examination of those for 1933, 1943, 1953 and 1963. With the exception of some minor

⁶⁵Ibid., 134.

points of detail, one observes very little change in the nature of the material being examined over this time span of forty years.

Each student was obliged to take a common paper in Arithmetic and also a paper in Algebra and one in Geometry at 'Pass' or 'Honours' level. The time allowed for each paper never exceeded two-and-a-half hours with choice being offered in the majority of cases either across the whole of a paper or within sections.

The general Arithmetic questions are mostly at the level of common utility, covering such competencies as costing, mensuration, percentages, interest repayments, distances and speeds with a number of technical questions relating to decimals and logarithms.

Pass Algebra: The 1933 paper examines items on indices, simultaneous equations, factorisations, surds, arithmetic/geometric progressions and series, a binomial expansion with a natural index and quadratics. The years 1943, 1953 and 1963 sees similar type questions on the same material along with slight variations involving the solution of simultaneous equations of mixed type; logarithms with a mention of the natural base e ; algebraic identities; number theory/set theory; use of graphs to approximate roots and the remainder theorem for cubics. All in all the level of the questions and the format of the paper are uniform throughout, save that the 1963 paper has a slightly different layout and is typed in a new style.

Pass Geometry: The 1933 paper consists mostly of pure geometry problems for which proofs are required along with two trigonometry problems, one theoretical and the other practical in the sense of relating to bearings and distances. The 1943 paper has two sections. The first section has problems on: tangency, cyclic quadrilaterals; the division of a line; the construction of an isosceles triangle having each of the base angles double the vertical angles; similarity of triangles and a right-angled triangle problem. The second section has four trigonometric problems; two technical and two applied of the surveying type. In addition to problems of the type described, the years 1953 and 1963 have additional items on the trigonometric rules and the compound angle formulae.

Honours Algebra: The 1933 paper has questions on: the solution of polynomial equations; functions of two variables; summation of series (quite sophisticated); combinatorics; determining all three roots of a cubic using Newton's method by supplying the initial approximations oneself, either graphically or otherwise; convergency tests; a binomial expansion involving fractional indices; straightforward differentiation and a related problem; two applications of integration. The other years sampled furnish extra items on complex numbers, differentiation from first principles, max-min problems, curve sketching with the aid of calculus paying special attention to maxima and minima and infinite branches, related rate problems and the determination of areas and volumes via integration.

Honours Geometry: The 1933 paper has a geometrical problem relating to overlapping circles, another on transversals, Heron's formula, a number of trigonometrical problems both theoretical and practical and a number of coordinate geometry problems relating to straight lines, circles and parabolas. The year 1943 has questions on: the converse of Ceva's theorem; polars and the inversion of circles; Ptolemy's theorem and cyclic quadrilaterals; coordinate geometry relating to orthocentres, tangents to circles, centroids of triangles and the parametric form of a parabola. It also has trigonometric problems relating to maxima and minima, evaluation and the solution of trigonometric equations. In addition to the problems of the type described, the year 1953 sees a theoretical problem involving inverse trigonometric functions while the year 1963 conforms to the pattern of previous years.

General comments in relation to the papers sampled:

1. Outside of the general arithmetical problems which are at the level of common utility but of genuine value, the algebra and geometry questions have a pronounced theoretical bias with only the rare one touching on an application. For the most part, half of the trigonometrical questions are of the applied type.
2. There is a very sparse use of diagrams in the format of the papers.
3. The 1933 paper uses the Greek letter θ explicitly.
4. The question in 1933 on Newton's method is slightly more realistic than

the corresponding current questions, in that one is expected to find graphical approximations to the roots for oneself, rather than being supplied with initial approximation as is done at present. Of the five years sampled, 1933 is the one time Newton’s method appears.

5. There is a reference made also in the 1933 paper to functions of more than one variable in relation to symmetric and homogeneous functions although there is no explicit use of function notation. One recommendation of this work is that use be made of such notation in contexts where it is both appropriate and convenient to do so.⁶⁶
6. Some of the summations asked are quite taxing in relation to current expectations, viz. HLC’33

Sum to n terms the series:

- (a) $x^2 + (x + 1)^2 + (x + 2)^2 + \cdots$
- (b) $1.2.4 + 2.3.5 + 3.4.6 + \cdots$

7. The following question at honours level in 1928 requires quite a sophisticated use of quadratic theory:

Find between what limits x must be so that y may be real
if

$$4x^2 + 2xy + y^2 - 39x - 6y + 99 = 0$$

Find also the maximum and minimum values of y within
these limits.

⁶⁶One case in point occurs in the treatment of the trajectory of a projectile where quantities such as the range and maximum height attained are functions of the two independent variables – angle of projection and initial speed.

8. Questions on the binomial expansion often involve both negative and fractional indices.
9. Of the four years sampled, only the HLC 1943 had a question on complex numbers.
10. Some of the graph questions are above the current level. For example

Trace

$$y^2 = x(x - 3)^2$$

Missing List: Reading through these papers one notes that there is no use of set notation nor is there an explicit use of function notation, although a 1933 question refers to functions of two and three variables. Neither is there any reference to physical notions other than those of length and speed. This is the very much the same as at present where the practice is to have Applied mathematics as a separate subject as it is now and was then. There was very little examining of complex numbers and no emphasis placed on the waveform nature of the sinusoidal functions. Outside of one reference to e there is no mention of the exponential or natural logarithmic functions. Furthermore, there are not any questions relating to statistics or probability nor is there any reference to vectors, matrices or groups. The next phase, to be discussed a little later, was to address many of these omissions in line with the prevailing New-Math thinking.

Mathematical Provision for Girls: In 1932 an alternative mathematics course, “elementary mathematics” – for girls only – was introduced. This course was examined until late into the nineteen sixties. What follows is a brief overview of the papers set in 1933, 1943, 1953 and 1963. In 1933 there are three papers in Arithmetic, Algebra and Geometry:

1. Arithmetic: Two hours to answer 6 questions from a choice of 9.
2. Algebra: Two-and-a-half hours to answer 6 questions from 9.
3. Geometry: Two-and-a-half hours to answer 6 questions from 9.

Arithmetic: The questions are all at the level of common utility, and cover costing, measuring, percentages and related matters. One question tests the ability to read electricity-metres which are nicely illustrated.

Algebra: The questions cover algebraic manipulation, factorisation, the solving of linear equations and two-by-two systems of linear equations along with quadratic equations. None of the questions could be described as bearing directly on real-life problems. The paper contains no diagrams.

Geometry: Questions included the following:

1. Circumscribe a given triangle.
2. Prove that the angle in a semicircle is a right angle.
3. Prove that the sum of the angles in a triangle is two right angles.

4. Find the locus of points equidistant from two given points.
5. P is any point inside a circle and AB is any diameter: prove that the angle APB is greater than a right angle.
6. Draw a rectangle seven square inches in area and construct a square equal to it in area.
7. $ABCD$ is a quadrilateral in which $AB=CD$. Find a point P so that the triangles APB , CPD may be congruent.
8. Illustrate by a diagram the identity $(a + b)^2 = a^2 + 2ab + b^2$.
9. A person travels 20 miles West and then 30 miles North-West. He then proceeds directly to a place which is 40 miles North of his starting point. Illustrate the journey by means of a scale drawing. Find the total length of the journey and the shortest distance he was away from the starting point during the last stage of the journey.

This paper contains no diagrams.

Ten years later in 1943 there is no discernible difference in the Arithmetic paper, save that candidates are given only an hour-and-a-half to answer it. A similar time allotment is given to the algebra paper, which represents a drop of one hour in the time allowed in 1933 while the expectation is still to answer 6 questions out of 9. The questions are a shade more difficult with one calling for the sketching of the graph of x^2 over an interval. There are at least two questions testing formulation. Many of the geometry questions to

be answered in 1943 in one-and-a-half hours had been asked in 1933. Other questions include:

1. ABCD is a rhombus: prove that the diagonals AC, BD bisect each other at right angles.
2. Show geometrically how to divide a given straight line into three equal parts. Give proof.
3. Prove that the angle subtended by an arc of a circle at the centre is double the angle subtended by it at the circumference.
4. Prove that parallelograms on the same base and between the same two parallel lines are equal in area.
5. Using squared paper, protractor and a large diagram, drawn to scale, solve the following:

From a point A on level ground a man saw two rocks, R_1 , R_2 in a line due east of A. On walking 4 miles due north to a point B he found the angle $ABR_1 = 40^\circ$ and the angle $ABR_2 = 70^\circ$. Find the distance between R_1 and R_2 .

Besides the genuine practical nature of the problem, the use of subscripts in its statement is worthy of note.

In 1953 all six questions in Arithmetic have to be answered in two hours. These questions are of the same character and level of difficulty as in 1933

and 1943. Similarly all six questions in Algebra have to be answered in two hours. These questions are of the same character and level of difficulty as in 1943. The geometry paper of six questions has to be answered in full in two hours. In addition to the type of question already described for the 1933 and 1943 papers, there are ones such as:

1. Prove that in a right-angled triangle the square on the hypoteneuse is equal to the sum of the squares on the other two sides.
2. P is a point outside a circle, centre O. Show, with proof, how to draw two tangents PA and PB to the circle. Prove that the tangents are equal and that OP bisects the chord AB.
3. Show, with proof, how to construct on a given straight line
 - (a) a segment of a circle containing an angle equal to a given angle
 - (b) a segment of a circle containing an angle equal to the supplement of a given angle.

The 1963 papers in the three subjects Arithmetic, Algebra and Geometry are each of two hours duration and require that all questions be answered. There are 8 arithmetic questions similar in type to those asked in 1933, 1943 and 1953. The Algebra paper resembles that of 1953 while the geometry paper is similar to its counterpart of 1953.

The Intermediate Certificate for Girls Only was discontinued in 1967.

It is apparent from the samples given that the level of questions asked on these papers between 1932 and 1967 is similar to that of the question being set on the current Junior and Ordinary Leaving Certificate but with the radical difference that there was a much greater call for proof in the former papers.

Outside of the arithmetic questions which were at the level of colloquial mathematics and a number of geometrical applications, the remaining questions could not be described as bearing directly on real-life problems.

Phase II: 1964-1992 According to Oldham the impact of contemporary views on mathematics and mathematics education in the early sixties brought about major alterations in what had been a very stable programme.⁶⁷ The stability of the previous programme has been attested to in the survey given above.

In 1964 new Leaving Certificate courses were introduced and new Intermediate courses followed two years later. In the senior programme the terms Ordinary Level and Higher Level replace the previously used designations of Pass and Honours respectively. Furthermore, there are now only two papers at each level with six questions out of ten, carrying 50 marks each, to be answered within each paper in two-and-a-half hours. To discern the general nature of the examinations in this second phase the relevant papers from this period were perused. In particular, by simply comparing the 1973 pa-

⁶⁷Elizabeth Oldham, "Mathematics in the Senior Cycle: The New Leaving Certificate Courses", *Studies in Education* 8, no. 2 (Autumn 1992).

per with the those of the first-phase examinations, it was possible to gauge many of the changes of emphasis introduced in the new programme as well as determine which portions of the older syllabuses were dropped in favour of new material aimed at modernising the existing provision.

Specifically, changes of emphasis include: a greater use of diagrams; use of the function notation as in $f : \mathbb{R} \rightarrow \mathbb{R} : x \longrightarrow 2 + 4x - x^2 = f(x)$; use of set language in relation to the domain and range of a function, solution set of an equation as well as questions directly related to the properties of sets; complex numbers as far as De Moivre's theorem; tests of convergence relating to infinite series; graphs of the sinusoidal functions; parametrisation; treatment of the exponential and logarithmic functions; more applications of differentiation to optimisation and related rate problems; applications of integration based on understanding the process as a form of summation.

The new topics introduced were vectors (to include the scalar product and orthogonality); linear transformations and matrices; central and axial symmetries; groups with examples based on rotations and reflections as well as modular arithmetic. On the other hand, absent from the new programme were most of the arithmetic based questions familiar in the older one along with most of the geometrical questions related to constructions and proofs.

However, as Oldham relates, while these courses “offered much in terms of the stark beauty and logical rigour of mathematics”, it was felt by 1988 that there was a need to make the Higher course more attractive and accessible in terms of style and content without radically changing the topics addressed.

Thus 1988 marked the beginning of the most recent period of curriculum change in Mathematics at the Leaving Certificate level of post-primary.⁶⁸ At that time the Ordinary course was enlivened, for instance by the inclusion of work on probability and numerical methods readily applicable in the world outside academe.

Change in the Junior Cycle mathematics curriculum was already underway by 1988, with three courses, known as Syllabus A, Syllabus B, and Syllabus C, already introduced in 1987 within the *Intermediate Certificate* programme.⁶⁹ In 1989, these courses were absorbed, unchanged, into the New Junior Certificate structure as the Higher, Ordinary, and Foundation courses in mathematics.⁷⁰ Thus changes in the mathematics courses had started at least two years in advance of changes in the entire curriculum structure.

Courses of instruction in mathematics within the Leaving Certificate programme had been designed to follow on from courses of mathematics within the Junior Cycle. In 1988, mathematics could be studied at two levels, Higher and Ordinary, within the Leaving Certificate Programme. Updated versions of these courses came into effect in Autumn 1992.

⁶⁸Elizabeth Oldham, "Senior Cycle Mathematics Curricula in the Republic of Ireland: Courses for the 1990s – Gospels, Epistles and Revelations?", paper presented at the 17th Annual Conference of the Educational Studies Association of Ireland, March 27, 1992. "Mathematics in the Senior Cycle: The New Leaving Certificate Courses", *Studies in Education*, 8, no. 2 (Autumn 1992).

⁶⁹Elizabeth Oldham, "Mathematics in the Senior Cycle: The New Leaving Certificate Courses".

⁷⁰Department of Education, *Rules and Programme for Secondary Schools 1988/89*. (Dublin: The Stationary Office, n.d.), 39-64.

Phase III: 1992- The new Higher and Ordinary Leaving Certificate courses were designed to have a ‘core and options’ structure; one reason being, according to Oldham, that it helps individual courses to be tailored in the classroom, both to match students’ needs and interests and to reflect teachers’ own visions and enthusiasms. A second reason is to facilitate change over time. Eventually, as part of a more profound revision, suitable options could move into the core, while core topics of declining importance could become optional rather than undergoing sudden death. When the 2003 Higher level papers are discussed in some detail at a later juncture, this structure will be made explicit. These Higher and Ordinary courses are described by Oldham as ‘specialist’ courses, that is, ones appropriate for students proceeding to further study in mathematics, science, engineering, technology or economics. Until 1990, effectively all students followed mathematics courses of the specialist type.⁷¹

Regarding the Ordinary Alternative course in mathematics proposed in 1990, Oldham writes that it was a ‘general’ course, geared to the needs of students who would not need to use advanced mathematics in their future careers. It was intended to offer lively, relevant and mathematically meaningful education to those who did not require the specialist content of the Ordinary course. She emphasises that the Ordinary Alternative was neither merely a maintenance course for social arithmetic nor a washed-out version

⁷¹Elizabeth Oldham, “Mathematics in the Senior Cycle: The New Leaving Certificate Courses”, *Studies in Education* 8, no. 2 (Autumn 1992).

of the Ordinary course. The serious heed given to calculators, the recognition given to investigational and constructive learning, and the incorporation of strong visual and artistic elements made it genuinely complementary in scope and style. As already mentioned, this course became part of the Leaving Certificate established programme, and is now known as Foundation level. According to the DES the Foundation Level mathematics course “is designed for students who have only a very limited acquaintance with abstract mathematics”⁷² and “is intended to equip students with the knowledge and techniques required in everyday life in various kinds of employment ... and to lay the groundwork for students who proceed to further education and training in areas in which specialist mathematics is not required.”⁷³

By way of making comparisons with other educational systems and drawing attention to perceived shortcomings in the Irish system as of 1992, Oldham relates that pressure groups were advocating the extension of a ‘modelling’ approach in school courses based on the fact that many national reports emphasised the importance of problem-solving.⁷⁴ She adds that the Americans count problem-solving as a basic skill rather than an add-on extra while in England and Wales extended problem-solving activities and open-ended investigations are mandatory (and assessed) at junior cycle level if not yet at senior level. She remarks that it may appear that the world trend in

⁷²An Roinn Oideachais, “The Leaving Certificate, Mathematics Syllabus (Foundation Level)”, (Dublin, The Stationary Office) n.d., 5.

⁷³Ibid., 5.

⁷⁴Elizabeth Oldham, “Mathematics in the Senior Cycle: The New Leaving Certificate Courses”, *Studies in Education* 8, no. 2 (Autumn 1992): 41.

this area is not being followed, adding that there is still much to be learnt as regards

1. modelling
2. investigations
3. practical work
4. utilisation of modern technology

The question of using investigations in the classroom was explored in 1994 by Mc Namara, one of Oldham's students. In a study entitled "Investigational Mathematics in Transition Year"⁷⁵ Mc Namara points out that since Transition Year (TY) is the one time that students are free from examination pressure, TY offers the possibility of introducing Mathematical Investigations (MIs). Coincidentally, the '94-'95 TY guidelines say that:⁷⁶ "the approach taken to Mathematics is as important as the content itself. It should seek to stimulate the interest and enthusiasm of the pupils in identifying problems through practical activities and *investigating* appropriate ways of solving them."⁷⁷ The same document recognises the poor attitude that many students have to mathematics and suggests that TY can be used to help overcome it: "the aim is to guide the pupil towards confidence and attainment of a balanced view of her/his potential for the subject."

⁷⁵A. Mc Namara, "Investigational Mathematics in Transition Year – A Pilot Project", (M. Ed. diss., Dublin University, 1994)

⁷⁶Department of Education, *Transition Year Programmes: Guidelines 1994-'95*, 26.

⁷⁷Emphasis added.

In the final paragraph of the Abstract of her thesis Mc Namara writes that

Doing Investigational Mathematics is of benefit to students both intellectually and socially. Working in an investigational mode can create difficulties for some, but not all, teachers; these difficulties can be overcome with experience and the provision of good resource materials. Finally, because of the creative and personal nature of this work it has a definite place as part of a Mathematics programme in Transition year.

Since this passage was written it would appear that many more teachers are using the TY to carry out such investigations or to engage in mathematical activities closely allied to such pursuits. As evidence of this, the textbook writer Oliver Murphy, in an Irish Times article of November 2004, titled *What I teach*, describes how he uses the freedom afforded by the TY to enhance the teaching of mathematics.⁷⁸ He relates that he uses “a five-module maths syllabus for transition year which the students really enjoy”. He describes the first module as a practical one based on probability which gives students “a chance to see maths at work in everyday situations”. A second module on sequences and series (labelled by him as more traditional) is followed by “an innovative module on applied maths” which demonstrates “the links between physics and maths”. The fourth module is a do-it-yourself research topic connected with Fibonacci numbers and the golden section where the students are exposed to ideas from art and architecture. The final module is one on algebra, designed (in the words of the author) “to bring the students

⁷⁸Oliver Murphy, “What I teach”, *The Irish Times*, November 24, 2004.

back down to earth". Most interestingly, the final paragraph of this valuable article reads:

I used to run a bridge module on the transition-year maths course because the game uses so much maths. It became so popular that it is now a transition year course on its own.

The significance of this extract from this writer's perspective is how it bears testament to the beneficial use of recreational mathematics. The popularity of this module, connected as it is with card playing, is telling confirmation that topics of this type from recreational mathematics can be very engaging. Since many additional items relating to the game of poker and card tricks are explored in this work and suggested for inclusion in the horizon syllabus, it is to be hoped that this new material would also be seen to teach valuable mathematical lessons while exercising a comparable appeal.

3.1.1 Recent Reviews/Revisions

While the mathematics courses at Senior level inaugurated in 1992 have not yet, at any rate, undergone any specific review since being first examined in 1994, a revised Junior Certificate mathematics syllabus was introduced in 2000 at each of the three levels: foundation, ordinary and higher. The revised syllabuses were first examined in June 2003. The DES 2004 circular discussing the implications of these revisions notes that where a topic has been included, students can now be expected to be familiar with this topic when they progress to Leaving Certificate, while those topics that have been

excluded will require greater attention. It states that in any consideration of 'knock-on' effects, account must be taken of the increased emphasis on teaching for understanding, with the use of active methodologies designed to improve the student's grasp of the mathematical concepts involved. It points out that the structure of the examination papers is now closely aligned to that currently in use for Leaving Certificate. Questions typically have a three-part structure – (a), (b), (c) – that leads students from easier work to more challenging tasks. It remarks that absence of choice on the examination should contribute to complete syllabus coverage and to students having a more solid basis when embarking upon the Leaving Certificate programme.⁷⁹

Amongst the most significant changes the following may be listed: Junior Certificate students will now be familiar with calculator use but the concept of logarithms, rules for their use and the solving of equations will be new material for Leaving Certificate. There will be a different treatment of geometry with fewer proofs being examined while topics such as the composition of functions and inverse functions will require treatment 'ab initio' at Leaving Certificate. In relation to number, students will now be required to find prime decompositions and determine HCFs.⁸⁰

⁷⁹Department of Education and Science, "The Implications of Revisions to the Junior Certificate Mathematics Syllabuses for Leaving Certificate Mathematics – Guideline for Teachers", (Dublin: Qualifications, Curriculum and Assessment Policy Unit, 2004), circular M38/04, 2.

⁸⁰*Ibid.*, 2.

3.1.2 The IMTA

In most countries, associations of mathematics teachers have played a key role in the development of the mathematical provision of their national curricula. Often it is only through membership of bodies such as these that teachers, acting either individually or collectively, can hope to have an input into the educational programme they're expected to administer. Mention has already been made of the British Mathematical Association and the American Mathematical Association, both of whom came into existence before 1900. It was not until April 1964 that the Irish Mathematics Teachers' Association (IMTA) was founded to primarily "further the study and teaching of Mathematics in Ireland, and to foster co-operation among teachers at all levels."⁸¹ As of 2001 it had thirteen branches nationwide holding regular meetings "to discuss topics of interest to practising mathematics teachers." Furthermore, "through lectures, courses, *etc.* members are helped to keep up to date in their own subject areas." In relation to bringing influence to bear on curriculum decisions, "the association is represented on all NCCA mathematics course committees" so that, "members, through their branch meetings and syllabus committee representatives, have a direct input into syllabus revisions and curriculum changes."⁸²(accessed January 4, 2004)

The OEEC report of 1961 had noted that "there are four countries that publish periodicals which appear four to ten times each year and which are

⁸¹IMTA Newsletter, *How we began* 1 (June, 1964)

⁸²Official website, "Irish Mathematics Teachers' Association (IMTA) Home Pages", <http://www.imta.ie>

concerned exclusively with problems of teaching mathematics, interpreting mathematics, producing new proofs of theorems but not concerned with research in mathematics *per se*.”⁸³ It added that “journals could do much to inform teachers of necessary changes and innovations ... if only all teachers could be persuaded to subscribe to them and to read them”. It continued “the writing of mathematical treatises and interpretations by teams of university mathematicians and educators working together is of urgent concern to all countries.”⁸⁴ While the British and American mathematics associations have, since their inception, been publishing through their respective journals articles aimed at improving mathematical teaching in all its aspects, it was not until 1978 that the IMTA began the publication of its Newsletter.

This quarterly has proved an invaluable resource for primary and post-primary teachers in Irish education from that day to this. Each of the IMTA Newsletters from 1980 to 2001 were surveyed by this writer who made a synopsis of each of the mathematical articles followed by brief comments, some of which are negative in tone and should be viewed as no more than simply the opinion of this reviewer. This commentary is given in Appendix B.

A number of both the general and mathematical articles lend historical support to the many of the suggestions being put forward in this present work. However, rather than present résumés of these at this juncture, it is

⁸³O.E.E.C, *New Thinking in School Mathematics*, (Paris: OEEC, 1961), 153.

⁸⁴Ibid., 154.

thought more appropriate to do so in a later section which deals specifically with making the case for reform. Nevertheless, it is worthwhile to indicate here the broad range of items and topics covered by the general Newsletter articles since its inception. These are categorised as follows:

Reports from the local branches of the IMTA

News from the course committees

Reports on submissions by syllabus committees of the IMTA to the CEB, NCCA or DES

Reports on International Conferences

Draft guidelines for the construction and assessment of courses

Draft syllabii and sample/specimen papers issued by the Department of Education

Publication of examination papers and model solution sets at the various levels in Mathematics and Applied Mathematics

Problem sets with solutions, quizzes, puzzles

Results of the Irish National Mathematics Contests and International Mathematical Olympiads

Information Technology articles include: The Computer in the Teaching of Mathematics; Calculators Myth or Magic?; The Use of Spreadsheets in Teaching

Book reviews

History articles on items such as Euclidean geometry; sites of mathematical interest; the development of mathematics; the Calculus

Mathematics education articles include:

- Teaching Mathematics Creatively
- Some Thoughts on the relevance of second-level Maths to Third level needs
- The Changing Role of the Teacher in the Mathematics Class
- Mathematics Education as Cultural Induction
- Context as a Factor in Determining School Mathematics
- Challenges for Enriching the Curriculum: Algebra
- What Irish Pupils say about Mathematics
- Whither Mathematics in Education in Ireland?
- Mathematical Projects for the Young Scientist Exhibition

Advanced articles include “Random Thoughts on Relativity” and a two-part article on using the Cayley-Hamilton theorem in the evaluation of exponential matrices.

Chapter 4

Curriculum Matters

In this chapter a number of themes touched upon in the general history of curriculum development given heretofore are brought into sharper focus. In addition a number of key questions relating to mathematics itself are discussed.

It is evident from the previous historical discussion that there were many different approaches to teaching in general and to teaching mathematics in particular. A number of the approaches to the learning/teaching of mathematics, which were the basis of many of the educational projects carried out in the 1960s, are briefly synthesised in the next section. As will become evident, the boundaries between the various approaches are very fluid.

4.1 Approaches to Curriculum Development

With the exception of the last two summaries, the synopses in this section are based upon material in chapters five and six of *Curriculum developments*

in mathematics by HKK.¹

The Behaviourist approach: This approach arose from a desire to achieve a more effective mathematical education through a reform of the methods of learning and teaching being practiced at the end of the 1930s. The underlying theory proceeds on the assumption that any learning process can be described in terms of a stimulus-response pattern, that the learning process can be ‘objectivised’ as observable changes in behaviour. The learning objective (goal) determines the desired change of behaviour, which can be checked.

Formulating and grouping objectives bulks large in the behaviourist-oriented development of curricula. The most consistent and obvious applications of this stimulus-response design are programmed learning and computer-assisted instruction where pupils are provided with (in theory) lucid and unambiguous directions and with a clear control system.

The Structuralist approach: Here the view is that at the lower levels of cognition mathematical objects are discovered empirically in the pupil’s environment. Later on, analytic thinking is developed step by step, and the axiomatic method is used for explaining and analysing mathematical structures. The mathematical objects themselves only function as examples and rank lower than processes, methods and working procedures – a case of

¹A. G. Howson, C. Keitel & J. Kilpatrick, *Curriculum developments in mathematics* (Cambridge: CUP, 1981)

process above product.

With this approach the major task for the curriculum developer is to devise appealing and meaningful teaching models for these processes of discovery as embodiments of underlying structures. The discovery approach, in which a student is asked to explore a situation in his own way, is held to be invaluable in developing creative and independent thinking in an individual.²

It is obvious, however, that the discovery method is slow. Some believe that with this approach the teacher should be prepared to introduce required ideas when they are not forthcoming from the class; that he should bring attention to misleading statements in the way of the discussion, and summarise results clearly as they come forward.³ These reasons and others account for the approach taken in this work regarding the use of recreational mathematics.

The Formative approach: This approach is formulated without reference to particular school subjects. It proceeds on the assumption that any school education should aim at endowing the pupil with an optimal basic body of cognitive abilities and affective and motivational attitudes. These may be described in terms of personality traits as creative ability, 'intelligence' and performance motivation.

According to The International Conference on Public Education, 1956, it

²Cambridge Conference on the Correlation of Elementary Science and Mathematics, 1963 as cited in *Ibid.*, 109.

³*Ibid.*, 110.

is important

1. to guide the student into forming his own ideas and discovering the mathematical relations and properties himself, rather than imposing ready-made adult thought upon him
2. to make sure that he acquires operational processes and ideas before introducing him to formalism
3. not to entrust to automatism any operations that are not already assimilated

HKK say that Piaget's research has demonstrated that it is fundamental to the processes of concept formation that they should be initiated at the level of concrete operations by the child's self-reliant manipulation of real objects. The emphasis is placed on reality and the rejection of models. The teacher's key function is to create situations in which the child can enjoy 'real activity' and translate this activity into learning processes. So mathematical applications, discarded by the followers of the New-Math movement because they did not fit neatly into the system, assume a greater significance in the formative approach, being the manifestation of the mathematics of reality.

The Integrated-Teaching approach: Here one seeks to go beyond mere statements on methods and to consider also problems of content. The selection of the content of learning must be geared to the interests and needs of the pupils. This means that real need and problems concerning their private and professional life, both present and future, have to be considered. Problem areas from reality should determine the content of the teaching.

This approach does away with the division into subjects of instruction; instead it integrates them according to the requirements of the problem at hand. Progress in problem solving, in the sense of discovery learning, is enhanced by the application-oriented means of cognition and by the procedures of the various disciplines. In this approach the motivation of pupils is of great importance.

HKK writing in 1981 suggested that “a determined effort should now be made to integrate science and mathematics teaching” because “the advantages of emphasizing the organic interconnections between mathematics and the various branches of science would far outweigh any disadvantages.”⁴ They argued that a great deal can be accomplished in the time span of nine years given to primary and junior high if one starts slowly and moves ahead at a natural pace. They added⁵

To the argument that mathematics has stood for years on its own pedestal in schools, we reply that it has hardly been a screaming success. For every child who feels comfortable with mathematical abstractions we can surely find two who are bewildered and repelled.

⁴Ibid., 124

⁵Ibid., 124.

The New-Math approach: Because the New-Math movement has already been discussed at some length, this synopsis here is kept to a minimum. This approach offers a systematic description of mathematics, organised so as to emphasise structural conditions. It advocates that material be presented in a uniform language with great precision and regards the basic principle of deduction of content from axioms to be central to its teaching.

The RME approach: Since this movement has also been discussed in detail the synopsis given here is also a brief one borrowed from Oldham who characterises the RME approach as follows:⁶

1. The starting points of instructional sequences should be experimentally real to students so that they immediately engage in personally meaningful mathematical activity.
2. The initial informal mathematical activity should constitute a basis from which students can abstract and construct increasingly sophisticated mathematical concepts.
3. The instructional sequence should involve activities in which the students create and elaborate symbolic models like drawings, diagrams, tables or notations.
4. The above can only be effective if they are realised in interactive instruction: explaining and justifying solutions, understanding other student's

⁶Elizabeth Oldham, *Trends in Mathematical Education*, 44.

solutions, to agree and disagree, questioning alternatives, to reflect afterwards.

5. Real phenomena in which mathematical structures and concepts manifest themselves lead to intertwining of learning strands.

The Constructivist approach: Constructivists believe in giving learners the freedom of their own construction or reconstruction. A person must construct, reconstruct, create or recreate all knowledge for themselves. The paramount requirement of constructivism is that the teacher realises and accepts the personal nature of knowledge. Mc Namara notes that “the constructivist views communication as an integral part of the learning” and adds that “the importance of communication can never be underestimated. Employers, when surveyed on skills they require in employees, named it as one of the most important.”⁷

The main difference between RME and constructivism in general is that RME is applied only to mathematics education while constructivism is used in many subjects.⁸

Three different shades of constructivism are identified by some writers:

Radical constructivism: knowledge cannot simply be transferred ready-made from parent to child or from teacher to student but has to be

⁷A. Mc Namara, “Investigational Mathematics in Transition Year – A Pilot Project” (M. Ed. diss., Dublin university, 1994)

⁸J. de Lange, “Using and Applying Mathematics in Education”, *International Handbook of mathematics education, Part One* (Holland: Kluwer Academic Publisher, 1996)

actively built by each learner in his or her mind.⁹ From this description it is clear that this approach has close links with RME. “The RME pillar of guided reinvention becomes radical constructivism in the limit as the guidance goes to zero.”¹⁰

Social constructivism: students can better construct their knowledge when it is embedded in a social process.¹¹

Socio-Constructivist: this type of social constructivism is developed only in mathematics education. The characteristics of this approach are similar to those of RME in that mathematics should be taught through problem solving, students should interact with teachers and other students and should solve problems based on their own strategies.

This approach and RME both view mathematics as a creative human activity and consider that mathematical learning occurs as students develop effective ways to solve problems. “The difference between the socio-constructivist approach and the realistic approach is that the former does not offer heuristics for developing constructional activities for students.”¹²

⁹Ernest von Glasersfeld, “Some Like it Radical” in *Constructivist View on the Teaching and Learning of Mathematics*, 1992.

¹⁰B. Braams, “Report on a Visit to the Freudenthal Institute” (2001), <http://www.nyu.edu/mfdd/braams/links/fi-visit.html>, 2. (accessed March 22, 2004)

¹¹Paul Ernest, “The Psychology of Teaching Mathematics: A Conceptual Framework”, *Mathematics Education for Teaching*, (1989)

¹²K. P. E. Gravemeijer, *Developing realistic mathematics education*, (Utrecht:CD β press, 1994), 81.

Koehler and Grouws¹³ quoting Cobb remark that “although constructivist theory is attractive when the issue of learning is concerned, deep-rooted problems arise when attempts are made to apply it to instruction”, explaining that this is especially true if one subscribes to the ‘transmission of knowledge’ view of teaching, rather than the view that “students construct knowledge for themselves by restructuring their internal cognitive structures.”¹⁴

In the next sections some of the modern issues associated with curriculum development are expanded upon.

4.2 Management of Curriculum Development

It has already been remarked that in the nineteenth century and the earlier decades of the twentieth century it was an accepted practice for educational reformers to urge changes both in content and methods of instruction within existing programmes without feeling the least need to test the proposed recommendations in any way beforehand. It was not until late in the twentieth century that curriculum development was to be managed on an on-going basis.¹⁵

HKK remark that even as late as the 1970s most countries attempted radical reform of their mathematics curricula but with little use of classical

¹³M. S. Koehler & D. A. Grouws, “Mathematics Teaching Practices and Their Effects” in the *Handbook of Research on Mathematics Teaching and Learning*, (US: Macmillan Publishing, 1992), 118-119

¹⁴P. Cobb, “The tension between theories of learning and instruction in mathematics education.”, *Educational Psychologist* 23(2), (1988): 87.

¹⁵A.G. Howson, C. Keitel & J. Kilpatrick, *Curriculum developments in mathematics*, 68.

research lines which use¹⁶

- control groups
- test explicit hypotheses
- evaluation

Even when curricula (in whatever form) were believed to be in place they were, as Freudenthal wrote:¹⁷

... mostly vague and usually admit of a great variety of interpretation.

However, the OEEC report of 1961 clearly understood and appreciated the need for such management. It stated that there was a need for a large-scale, systematic, scientific research in actual classroom situations where theories relating to (i) subject matter and (ii) the organisation and methods of instruction, could be tested against controlled conditions over a long period of time. The research should aim to give knowledge on the quality and quantity of output in an educational system, as well as a means for improving the quality of teaching. There was also a need for fundamental research into problems of intellectual development and concept formation.

The Dutch were amongst the first to establish a body charged with managing curriculum development. "In 1970 the Dutch Ministry of Education decided to raise development of subject matter and teacher retraining to the status of a permanent institution."¹⁸

¹⁶Ibid., Preface.

¹⁷Hans Freudenthal, *Mathematics as an Educational Task*, 158.

¹⁸Ibid., 169. Some fourteen years later the general examination of the post-primary

4.3 Evaluation within Curriculum Development

Just as the curriculum had not been widely perceived as something to be ‘developed’ until the middle of the twentieth century with the advent of curriculum development projects, so the need for an explicit, formal evaluation of the curriculum did not arise until people began to ask whether the projects had been worthwhile.¹⁹

Evaluation is the process of judging the worth of something. It is necessary in education because a curriculum is an abstraction that can only be glimpsed through such means as the analysis of statements of aims, observation of content actually taught and assessment of what the pupils have learned. According to Jacobsen evaluation consists of three activities:²⁰

asking significant questions

gathering information to answer the questions

interpreting the results

It is a psychological process because it affects people’s beliefs about their work and themselves and it is also a socio-political process because it affects decisions as to what shall be taught to whom.

curriculum in Ireland was entrusted to the Curriculum and Examinations Board (CEB). Séamus Ó Buachalla, *Education Policy in Twentieth Century Ireland*, 76.

¹⁹A. G. Howson, C. Keitel & J. Kilpatrick, *Curriculum developments in mathematics*, 183.

²⁰As cited in *Ibid.*, 184.

HKK relate that since the 'New-Math' curriculum development projects were primarily concerned with updating the content of the mathematics curriculum, evaluations of these projects tended to be directed at the question of content: could pupils learn the new content and still learn the old content too?²¹ In fact, the New-Math people saw their project's work as virtually self-validating. The materials they developed were intrinsically worthwhile and seemed to be usable. Why was a 'full-dress' evaluation necessary except to satisfy the bureaucrats? They worried that the subtle kinds of 'higher-order' learning sought by their projects would not be easily detected by evaluation studies.²²

However, as these authors point out, not all studies of New-Math curriculum development projects were cast in the 'let's compare pupils' achievement mould. These evaluations tended to stress the justifications for undertaking a project more than the effects of its work. They were aimed at making a curriculum acceptable rather than improving its quality. Thus they were of the *formative* type where one attempts to provide direction to a project's work in contrast to *summative* evaluation which looks at the final products of a curriculum development project.²³

HKK report that a large number of the evaluation studies conducted in the past had a negligible impact on decisions regarding the curricula being evaluated. They give an account of the evaluations, or lack thereof, carried

²¹Ibid., 187.

²²Ibid., 206.

²³Ibid., 196.

out in relation to a number of the different curriculum approaches already described and a brief synopsis is given here.

With regard to the evaluation of projects based on the behaviourist approach, Bloom's taxonomy was an attempt to develop a scheme for classifying both the outcomes of instruction as measured by tests and the objectives of instruction. However, Eisenberg (1975) argued vehemently that behaviourism, by equating education with training, missed the essence of the discipline of mathematics. Standardised achievement tests (SATs) are poor devices for such diagnosis as distinguishing between effective and ineffective parts of new programmes because they tend to emphasise the recall of factual information and performance of routine calculations. They rely heavily on the pupil's memory for facts and procedures and so little on such things as understanding of why the procedures worked and his ability to apply his knowledge of mathematics to new situations.

The difficulty of comparing something as amorphous as 'the discovery method of teaching' with any other teaching method seemed to convince curriculum developers who advocated a discovery approach that they should not permit their projects to stand or fall on the basis of an experimental comparison of teaching methods. Projects based on the structuralist approach have, therefore, seldom received extensive formal evaluation. An exception is the Comprehensive School Mathematics Project (CSMP). The CSMP Evaluation Program was a behaviourist attack on humanistic mathematics. It dismissed 'team teaching' as having being clearly shown to have 'no advan-

tages over conventional teaching’.

The formative approach to curriculum development requires that teachers play a strong role in helping to devise situations in which children can learn mathematics. Projects based on the formative approach tended to provide materials which were illustrative and suggestive rather than comprehensive and definite. They left much of the developmental work to the teacher, who was often expected to treat the materials as a supplement to the regular mathematics programme. Such projects were difficult to evaluate and usually did not set forth a list of goals, but when they did, the goals tended to deal with vague, albeit important, notions about such things as learning to pursue mathematics on one’s own and developing an educated intuition.

To the extent that a curriculum project based on the integrated-teaching approach attempts to do away with boundaries between mathematics and other disciplines, it will encounter problems in implementing its programme with teachers whose own education has been conducted largely within these boundaries. Programmes based on the integrated-teaching approach, therefore, have faced greater problems of dissemination and teacher education (and re-education) than those based on other approaches.

In relation to the evaluation of RME, Freudenthal had written “it is believed that knowledge and ability acquired by re-invention are better understood and more easily preserved than if acquired in a less active way. I do not know whether this thesis has been formally tested but there are many

indications which make it probable.”²⁴ As of now several positive results of the RME theory can be found in the literature. For instance, in the USA, RME is adopted in the “Mathematics in context” textbooks for grades 5-8. After the books were used by students in several school districts from different states, a preliminary research showed that the students achievement on the national test highly increased.²⁵ Furthermore, in the country where RME originally has been developed, the Netherlands, there are also positive results that can be used as indicators for the success of RME in the reform of mathematics education. The results of the Third International Mathematics and Science Study (TIMSS) show that students in the Netherlands gained high achievements in mathematics education.²⁶

In 1994 Mc Namara wrote that there was “no research findings on the efficacy of constructivism as of yet at secondary school level.”²⁷

In summing up their discussion of evaluation HKK say that “the art and science of education evaluation is still in a primitive state, and it is not surprising that its practice does not follow the well-worn paths. Nevertheless, evaluation of sorts is constantly taking place, and decisions are being made

²⁴Hans Freudenthal, *Mathematics as an Educational Task*, 118.

²⁵A. Romberg & D. de Lange, *Mathematics in Context: Teacher Resource and Implementation Guide* (Britannica Mathematics system, USA, 1998)

²⁶I.V. Mullis, M. O. Martin, A. E. Beaton, E. J. Gonzalez, D. Kelly, & T. A. Smith, *Mathematics Achievement in the primary school years: IEA's Third International Mathematics and Science Study* (USA: Chestnut Hill, MA: TIMSS International Study Centre, 1997)

²⁷A. Mc Namara, “Investigational Mathematics in Transition Year – A Pilot Project” (M. Ed. diss., (1994)

on the basis of so-called evidence. The need to develop improved methods and to inculcate new attitudes is, therefore, obvious. For, if we do not do so, we run the risk that those programmes and initiatives which are best fitted to evaluative procedures as they currently exist will thrive at the expense of others. The challenge, then, to evaluators of mathematics curricula of the future is to build on the best of present practices without forgetting the lessons to be learned from the past.”²⁸

4.4 Innovation

The known provides a degree of security while innovation often means risk-taking. Tradition, the past training of teachers, the existence of long-standing popular textbooks, and examinations based on fixed syllabi all tend to prevent or seriously delay changes, especially innovations in school programmes.

HKK at the very beginning of *Curriculum developments in mathematics* point out that “it is important that proposed innovations, not only reach classrooms, but do so in an ungarbled form reflecting the originators’ aims and objectives.”²⁹ Later they indicate optimistically that “progress can be made in a climate in which educational innovation is expected and encouraged. That this might in fact lead, without adequate safeguards, to over-ambitious and ill-advised changes is no excuse for inertia.”³⁰ Harking back to the dangers of inertia they ask, “How can one attempt to ensure that

²⁸Ibid., 236.

²⁹A. G. Howson, C. Keitel & J. Kilpatrick, *Curriculum developments in mathematics*, 1.

³⁰Ibid., 52.

teacher-innovators do not, in the eyes of less enthusiastic colleagues, ‘disappear over the horizon’?”³¹

Whereas innovation had in the past been university driven, HKK relate that “the emphasis had to switch to the school and the teacher became the focus of the process of innovation. ... No longer could he be regarded as the mere performer of a ready-to-use curriculum. ... Curriculum development could only be done by teachers, with the assistance where necessary from other mathematicians and educators, for teachers.” They warned that “if innovation is to proceed more satisfactorily in the future then it is essential that we ensure better understanding and acceptance by teachers.”³²

Attempts to change the subject matter, its organisation, or its mode of presentation have to be accommodated to teachers’ beliefs as to what good teaching is and what it is reasonable to expect of pupils. The effectiveness of any such changes depends very much on the teacher’s willingness to accept and adapt them, or more particularly to *absorb* them especially if the change embodies new mathematics as far as the teacher is concerned.

In order that teachers can embrace proposed changes, it is important that, in addition to their pre-service training, that there be provision for in-service training, for constant up-dating of their professional knowledge. Furthermore it is necessary to take decisive steps in producing material and setting new examinations. It has already been pointed out that “the best

³¹Ibid., 243.

³²Ibid., 128.

plans will come to nought unless examination systems test desired objectives and encourage (rather than act against) the attainment of educational and mathematical goals.”³³ As to the provision of adequate guidance through classroom material, HKK observe that “innovators who believe that teachers are the key curriculum developers tend to elevate process above product; the materials these developers produce are often no more than samples or source materials that require the teacher to do his own developing.”³⁴ Nevertheless many projects in the 1960s were keenly aware that in order to facilitate the innovation process **it is essential to provide concrete materials for both teacher and student.** It is hoped that the present work will be judged to have this characteristic.

4.5 The Teacher

The 1961 OEEC report “New Thinking in School Mathematics” asserts that a reform in the instruction of secondary-school mathematics is not possible unless competent teachers are attracted and retained in the profession. It stresses that if the teacher of mathematics is to be an active, appreciated and respected member of society, then the prestige of the teacher must be raised; it is necessary to provide to this end:³⁵

- an adequate salary

³³Ibid., 2.

³⁴Ibid., 65.

³⁵O.E.E.C. *New Thinking in School Mathematics*, 123.

- favourable conditions for personal development
- possibilities of promotion to higher rank
- satisfactory conditions of work (hours and class size).

Remarking that the mathematical education of the teacher is of prime importance, it maintains that not only should the teacher's knowledge of each subject he or she teaches extend beyond what he or she actually teaches, but it should also be in harmony with contemporary thinking at the advanced and research levels of mathematics. Consequently to teach in the upper cycle of the secondary school the requirement is, without exception, graduation from a university (or the equivalent state examination). Furthermore, to teach mathematics it is desirable that the candidate place major study in mathematics and minor study in physics.

After asking "once a teacher is permanently certified, what are his professional obligations to continue studying?" the report says that (at the time of writing) no country requires a teacher who is permanently certified to³⁶

do further study

become a member of a professional mathematical association

attend conferences.

It suggests that the shocking lack of responsibility in keeping abreast of new developments in mathematics and education is a serious flaw in maintain-

³⁶Ibid., 151.

ing an adequate educational programme. All countries should seek ways of helping teachers to continue studying and to constantly improve their teaching of mathematics. As a result of a lack of concentrated and co-operative effort to keep secondary school teachers informed of modern developments in mathematics, most programmes in mathematics education have remained static.

Referring to the real need for educators to keep abreast of new developments, it says that it is as necessary to do this in the teaching of mathematics as it is in medicine and other professions. The retraining of teachers should be the product of close collaboration between university mathematicians, educators and secondary-school teachers. In this way, it would be possible for teachers to take courses to³⁷

- see how the mathematics developed in recent years should be closely related to what goes on in the classroom
- take part in experimental work in teaching secondary-school mathematics from a modern point of view
- sense the genuine need for further study in subjects of “modern mathematics”.

Twenty years later Howson *et al.* concur on the importance of re-training: “It must, however, be accepted that no matter how well planned that (pre-service training) may be, it cannot provide the background mathematical

³⁷Ibid., 96.

knowledge sufficient to span a teaching career of forty years. Further study will be needed.”³⁸ Speaking as educators they add that “... it is our professional duty to acquaint ourselves with new ideas and to make arrangements to see new methods in action, where necessary, to make small-scale experiments.”³⁹

The OEEC report believes that it can be brought home to teachers that they render a service to their pupils by keeping up with developments. Teachers who keep in contact with new knowledge and trends in mathematics and educational practice will certainly exert a favourable influence on teaching and have open minds towards the introduction of new subject matter. Those keen to acquire more knowledge should be given satisfaction in moderation and care should be taken not to discourage them or to oversimplify the problems.⁴⁰

CHO writing in 1985 assert that teachers are often with good reason criticised for their lack of experience of the world outside education. The school (pupil) – college (student) – school (teacher) route is cramping and a determined effort is required to overcome the shortcomings it induces.⁴¹ They see it as vital that the teacher learn more about how mathematics is used in other disciplines and be ready to explain to

³⁸A. G. Howson, C. Keitel & J. Kilpatrick, *Curriculum Development in Mathematics*, 261.

³⁹Ibid., 263.

⁴⁰O.E.E.C. *New Thinking in School Mathematics*, 96.

⁴¹B. Christiansen, A. G. Howson & M. Otte, *Perspectives in Mathematics Education*, 44.

others the effects and benefits of any changes planned in the mathematics curriculum. For the mathematics teacher to be part of the community and to maintain a credibility in the world at large he must at least show some recognition of the use the world makes of mathematics, and preferably to be involved with it. Only in this way will he be able to make a proper expression to his pupils (and to their parents) of the role of mathematics in society. Quoting Burkhardt they observe that, while there will always be a few for whom mathematics has a beauty within itself, the majority will find their involvement with the subject enhanced by the application of it to what are seen to be worthwhile interesting practical problems.⁴²

For CHO it is essential that the mathematics teacher have his professional competence and status based firmly on the ability to relate mathematics to its use in the 'real-world'. This requires a willingness to be involved with a large number of other school subjects which in turn involves communication. Intending and practising teachers should be given time to find out how professional mathematicians work, and preferably to gain experience for themselves. They say it will mean a deeper understanding of the way mathematical *ideas* and *skills* (and not just particular techniques) are used, thus enhancing the special ability of the mathematics teacher – to be able to translate and interpret what other mathematicians have done in a form which their pupils can understand.⁴³

⁴²Ibid., 94.

⁴³Ibid., 94.

It is important, according to the same authors, that teacher-students learn to view mathematics positively. Pupils of teachers who view mathematics as interesting, enjoyable, and useful are likely to develop much more positive attitudes towards the subject than pupils of teachers who see mathematics as dull, difficult, or likely to be rewarding for certain people who possess special aptitudes for it. They say that in addition to attitudes, beliefs about mathematics as a subject matter will affect teachers' presentation of the content to their pupils. Teachers must be well versed in mathematics to teach the subject effectively. Without such breadth and depth of mathematical knowledge, teachers are likely to rely too heavily on the textbook, to present the content in a fragmented way without sufficient explanation of key concepts or problem-solving strategies, and to be ineffective at individualising instruction, diagnosing error patterns, or responding to unanticipated difficulties or opportunities that arise during instruction.

It is not enough that teachers become confident about their knowledge of mathematics; they must also become confident that they can teach the subject effectively to their pupils. This involves not only the ability to present material with clarity, but also the ability to diagnose the reasons for pupil confusion and to draw on an easily accessible repertoire of alternative and remedial methods of instruction. CHO report that research on teacher effectiveness has shown that one important factor common to teachers who are highly successful in producing pupil learning gains in basic skills is the expectation of success. These teachers believe that their pupils are capable

of mastering the prescribed curriculum, and that they (the teachers) are capable of teaching it to them effectively. Teachers who set higher goals and work systematically toward them will be more successful than teachers who set lower goals or who give up easily when their efforts do not yield quick success. The development of teaching competence is a process which requires training and practical experience, but practical professional knowledge is also necessary to understand the skills to be trained, as well as to understand the complexity of the teacher's situation itself.⁴⁴

These authors provocatively enquire "are we trying to produce teachers who will 'fit in' with existing conditions and practices, or are we trying instead to develop creative and autonomous teachers prepared to try anything?"⁴⁵

4.6 Student Motivation

Christiansen *et al.* assert that a student who does not regard mathematics as significant relative to his particular norms will make no efforts to learn it.⁴⁶ The major problems of mathematics education in school would seem to stem from those many pupils who, according to research findings, *decide* to stop learning at school.⁴⁷ This research indicates that the low-attainer is unlikely to take any great interest in the wider world outside him. The 'relevance' of mathematics must be demonstrated then in a stronger and more individual

⁴⁴Ibid., 137.

⁴⁵Ibid., 310.

⁴⁶B. Christiansen, A. G. Howson & M. Otte, *Perspectives in Mathematics Education*, 29.

⁴⁷Ibid., 31.

way to such students; they must be helped to see how mathematics can contribute to their knowledge of survival, indeed to their whole process of socialisation. Mathematics must be shown to help the inarticulate to express their wishes and demands.

CHO are very critical of the nature of the mathematical provision and constraints on teachers prevailing in 1985. They complain that too often it is the case that the motivation to teach or to learn does not come as a result of one's own mathematical or other activities or of one's own experiences. Instead it is the abstract content of the subject itself that is expected to carry the incentive for studying it. They point out that the development of school mathematics does not depend on a factual and objective relation between the pupils and/or teachers and mathematics. It depends to a great extent on external influences exerted by various professional groups with specific interests in mathematics or education in general which impose certain expectations on the outcome of mathematical education. The result is that the pupil cannot find the motivation for learning in his present activity; it is hidden in the future after his leaving school.

In their view almost every topic is associated with certain types of exercises; these exercises are mostly of a routine character which aim to mechanise the presented path of calculations. Only rarely will there be genuine problems which demand the application of previously learned concepts and methods to new or essentially changed contexts and situations. For the learner this results in separation of school mathematics from other fields of mathematical

activity; it serves its own purposes which are neither mathematical nor oriented towards some applications but are prescribed by the type of schooling now experienced.

For CHO the applications of mathematics as motivational activities must become central to its teaching. They write that to be involved in real applications means the pupils finding their own problems to solve – such experience is important to mathematical development. However, the problems must appear *real* (and, equally important worth the mathematical effort) to the pupil. Problems – or the need to solve them mathematically – must make ‘common sense’. They observe that the use of motivated activity on tasks as the major organising principle in the teaching/learning process means in fact to let a totally different pattern become dominant in the mathematics classroom.

4.7 Student Activities

The 1961 OEEC report says that it would seem wise to provide interested and capable mathematics students with opportunity – in addition to and beyond classroom study – for the development of their talents. They recommend activities such as student clubs or circles, voluntary competitive examinations, publishing of mathematical journals suited to the ability and interest level of the secondary school student. They remark that, “as a whole secondary school teachers have not participated to any great extent in this

phase of mathematical education.”⁴⁸

The criticism in this statement is not as valid to-day as it may have been then. For one thing the IMTA Newsletters (referred to in the chapter on the history of Irish curriculum development) carry a number of articles on mathematics which are accessible to upper-level secondary school students and on one occasion at least have contained an article written by such a student. It is also the case, as can be gleaned from the survey given of the IMTA Newsletters, that many Irish teachers and third level educators do indeed devote considerable time and effort to preparing mathematically talented students for special National and International Mathematics competitions. Furthermore, some schools have mathematics clubs while some universities and institutes of technology have at various times offered extra-curricular classes to Higher LC students with a particular interest in and aptitude for mathematics.⁴⁹ Provision for specially gifted children in mathematics has been “a long-standing Russian tradition” where the best university mathematicians “inspire specially-selected school pupils, gathered together in mathematical ‘circles’ ...”⁵⁰

⁴⁸O.E.E.C, *New Thinking in School Mathematics*, 175.

⁴⁹UCC offers “Enrichment classes in Mathematics” on Saturday mornings throughout the academic year while CIT ran a “Mathematical Excursions” class on Tuesday nights two years in succession during term.

⁵⁰SMP: *The First Ten Years*, 260.

4.8 About Mathematics

The history of curriculum development (as well as current endeavours) are for the greater part concerned with the two central questions “What mathematics should be taught?” and “How should mathematics be taught?” As the discussion to-date shows, the many ways in which various individual thinkers and movements have attempted to answer these questions give rise to the different approaches described. Though important, the questions “What is mathematics” and “Why should one teach mathematics?” have, perhaps, unconsciously been considered of lesser significance or, in fact, not in need of answering because the answers were assumed to be almost self-evident. Although these questions have in part been already answered implicitly or explicitly in some form it is appropriate to dwell a little further on them and the answers they have prompted.

4.8.1 What is Mathematics?

The writers Christiansen B., Howson A. G. & Otte M. (CHO) present two views of the nature of mathematics which they describe as extreme:⁵¹

⁵¹B. Christiansen, A. G. Howson & M. Otte, *Perspectives in Mathematics Education*, 143.

In the formalistic view, mathematics is a set of concepts, rules, theorems and structures. ... To acquire the knowledge you need certain skills, such as algorithmical skills and deductive reasoning ability. This latter is to be considered the only proper way of reasoning in mathematics. Plausible and intuitive reasoning do not belong among genuine mathematical activities, and serve only as preliminary stages towards the real thing, as phases between not-knowing and knowing. A teacher is, however, permitted to use them, but solely as a means to motivating his students, to interest them in deductive or formal reasoning. Thus, despite all the demands of the unadulterated method, a teacher is encouraged to use the visual power of geometrical illustrations to make algebraic truths plausible.

On the other hand:⁵²

Opposing the formalistic view, in which mathematics is considered as a body of knowledge, stands the activist's view that mathematics consists of being engaged in activities like generalising, classifying, formalising, ordering, quantifying, abstracting, exploring, patterns etc. Intuitive reasoning is to be considered as a useful mathematical activity and therefore, in this view, as genuine mathematics. Students are supposed to discover patterns, rules, concepts, structures, etc. It is, in this extreme view, of no importance what they discover as long as the activities are mathematical ones.

They add that "in the first view, the main goal of education is knowledge, and activities are a means to an end. In the second view it is just the reverse. The first view leads to education by receptive learning; education according to the second view mainly consists of discovery learning."

Whatever the answer to the question posed in the section title might be, if indeed there is one, the 1961 OEEC report, "New Thinking in School

⁵²Ibid., 144.

Mathematics” was in no doubt as to who were qualified to answer it. It declares that “the nature of mathematics – and the designation of the types of mathematics that are important – are rightfully the decisions of mathematicians.”⁵³ The report appears to broaden this base of entitlement when it adds, “What portion of this mathematics

- can be taught below university level
- to whom it can be taught
- the way it can be taught

are the decisions of educators, teachers and writers of textbooks.”⁵⁴

This question will be taken up again in a later section dealing with a recent curriculum initiative connected with the RME approach to mathematics.

4.8.2 Why Mathematics?

Tate, writing in 1857, says that⁵⁵

Mental arithmetic cultivates the memory and the powers of conception and reasoning. It also especially fosters the habit of promptitude, presence of mind and mental activity.

Arithmetic cultivates the reasoning powers and induces habits of exactness and order.

⁵³O.E.E.C, *New Thinking in School Mathematics*, 61.

⁵⁴*Ibid.*, 61.

⁵⁵T. Tate, *The Philosophy of Education* (London: Longmans, 1857) as cited in A. G. Howson, C. Keitel & J. Kilpatrick, *Curriculum developments in mathematics*, 24.

Mathematics and Natural Philosophy cultivate the reasoning powers chiefly in relation to the acquisition of necessary truths; they also cultivate habits of abstraction.

Over a hundred years later, writing in 1969 of the value of teaching both science and mathematics, the Cambridge Conference on the Correlation of Elementary Science and Mathematics notes eloquently that “because of the comparative simplicity with which they can go beyond the superficial, science and mathematics lend themselves to the development of attitudes of lifelong and general value.” Among these it lists:

A healthy skepticism regarding accepted knowledge and a willingness to abandon ideas which are demonstrably erroneous.

The humility inherent in the realization that our understanding can never be complete, coupled with the optimism that, nevertheless, our understanding can always be increased.

The realization that understanding, while indeed a means to power, is a joy and an end in itself.

However, as CHO remark, “since the 1960s mathematics educators have tended to stress the utility of their subject and to underplay its possible contributions to all-round educational development.” They add “for employers mathematics is seen first, second and last in vocational, utilitarian terms; it is almost never referred to as part of human culture and employ-

ers remain unconvinced by arguments as to ‘the beauty of the subject’.”⁵⁶ However, “employers do not necessarily criticise the mathematical content of the curriculum, rather the lack of enough process-oriented work – problem formulation, problem solving, etc.”⁵⁷

CHO quoting the 1982 Cockcroft report say that “there can be no doubt that there is general agreement that every child should study mathematics at school; indeed the study of mathematics is regarded by most people as being essential” but they add “why that study should be essential is hard to specify.”⁵⁸ However the same authors give a comprehensive list of convincing reasons as to why mathematics claims a place on the curriculum when they write that it can be seen as⁵⁹

1. contributing to the basic knowledge of any citizen
2. contributing to the study and advancement of numerous disciplines, professions and trades
3. contributing to a student's general education through the inculcation of particular attitudes and approaches
4. possessing an inherent interest and appeal.

The important issue of how much mathematical rigour should or can hope to achieve at second level is now discussed briefly. The role of proof as part of the discipline of mathematics along with a little of its history is dealt with later.

⁵⁶B. Christiansen, A. G. Howson & M. Otte, *Perspectives in Mathematics Education*, 52.

⁵⁷*Ibid.*, 53.

⁵⁸*Ibid.*, 49.

⁵⁹*Ibid.*, 9.

4.8.3 How Much Mathematical Rigour?

As early as 1893 the German mathematician Felix Klein remarked:⁶⁰

It is my opinion that in teaching it is not only admissible, but absolutely necessary, to be less abstract at the start, to have constant regard to applications, and to refer to the refinements only gradually as the student becomes able to understand them. This is, of course, nothing but a pedagogical principle to be observed in all mathematical instruction.

Earlier he had observed in relation to possible scruples about mathematical rigour:⁶¹

Now, just here a practical difficulty presents itself in the teaching of mathematics, let us say of the elements of the differential and integral calculus. The teacher is confronted with the problem of harmonising two opposite and almost contradictory requirements. On the one hand, he has to consider the limited and as yet undeveloped intellectual grasp of his students and the fact that most of them study mathematics mainly with a view to practical applications; on the other, his conscientiousness as a teacher and a man of science would seem to compel him to detract in nowise from perfect mathematical rigour and therefore to introduce from the beginning all the refinements and niceties of modern abstract mathematics.

Speaking in 1959 at the Royaumont conference on “New Thinking in School Mathematics”, the French mathematician Jean Dieudonné, while allowing that complete rigour was unobtainable and ill-advised at the level of secondary school, was anxious that “... when logical inference is introduced in some mathematical question, it should always be presented with absolute

⁶⁰Felix Klein, *Lectures on Mathematics*, Lecture VI: Mathematics and the Applied Sciences, (USA: AMS Chelsea Publishing, 2000), 50.

⁶¹*Ibid.*, 49.

honesty – that is, without trying to hide gaps or flaws in the argument; any other way, in my opinion, is worse than giving no proof at all.”⁶² The 1961 report on this seminar was to augment this call for transparent honesty by writing:⁶³

At the secondary-school level it is not always possible to be rigorous, but the results of instruction should be correct. If gaps exist in a proof, the student must be aware of them. At this period of study, students should have some mathematical maturity and readily recognise the difference between intuitive illustrations and proofs of theorems.

Freudenthal wrote in 1971 that “some teachers can hardly understand that there are viewpoints besides the logical system.”⁶⁴ Those teachers were the product of a system where “mathematics has been synonymous with geometry ... a perfect conceptual system, where things rigorously followed from each other, and finally everything from definitions and axioms.”⁶⁵ Yet, as Freudenthal observes, “... there have always existed other branches too, algebra, trigonometry, calculus, which, however, were not more than collections of haphazard, badly founded rules ...”⁶⁶ Even in relation to the teaching of geometry Freudenthal stressed that “I would not claim that geometric axiomatics is a “must” at school. [...] I am convinced that more and better geometry can be taught by a teacher who in mathematical rigour follows his own conscience rather than having his scruples prescribed by the frowning

⁶²O.E.E.C., *New Thinking in School Mathematics*, 39.

⁶³Ibid., 116.

⁶⁴Hans Freudenthal, *Mathematics as an Educational Task*, 101.

⁶⁵Ibid., 401.

⁶⁶Ibid., 401.

axiomatization.”⁶⁷

The German mathematician Cantor, a contemporary of Klein’s and ‘tamer of the infinite’, famously claimed that “the essence of mathematics is freedom”. The following quote from Freudenthal, used to close this discussion on the place and role of mathematical rigour, is very much in accord with Cantor’s dictum:⁶⁸

Experimental mathematics, that is, the mathematics of free discovery, is much more important than that which is confined to the axioms imposed by the teacher or textbook author, and there is no reason to claim that it is any less rigorous. There are levels of rigour and for each subject matter there is a level of rigour adapted to it; the learner should pass through the levels and acquire their rigour. The adult mathematician, too, operates according to various levels of rigour; it is blindness to assert that there is only one rigour (which, of course, happens to be that exercised by the person who is asserting it) and any others below are fake and all above are hair-splitting.

The final section of this chapter considers two recent (2000 and 2003) large-scale endeavours in the realm of mathematics education.

4.9 Two Recent Endeavours

The *Principles and Standards in School Mathematics* (PSSM), published in 2000 by the US National Council of Teachers of Mathematics (NCTM), is a mission statement covering the whole school curriculum from pre-kindergarten

⁶⁷Ibid., 456.

⁶⁸Ibid., 149.

up to 12th grade.⁶⁹ The second publication, in 2003, comes from the Directorate for Education of the Organisation for Economic Cooperation and Development (OECD) and is entitled *The PISA 2003 Assessment Framework* (PISA03) where PISA stands for Programme for International Student Assessment.⁷⁰ This programme, conceived in 1997, is charged with generating comparative international data on the achievement of students up to the age of 15 in three domains – reading literacy, mathematical literacy and scientific literacy – as well as informing policy in participating countries on issues associated with achievement.⁷¹ In 2000 this programme reported on an international assessment which assessed reading literacy more comprehensively than it did mathematics and scientific literacy. Thus in PISA2000 as it is now known, reading literacy was the ‘major’ domain while mathematics and scientific literacy were ‘minor’ domains.

PISA03 is a *framework* document discussing how it was intended to assess mathematical literacy as a major domain, and reading and scientific literacy as minor domains in the year 2003. The first report on its findings appeared in December 2004. Though PISA03 is concerned with assessment and describes in considerable detail its focus and methods in relation to each of the domains, this document also sets out clearly and extensively what it means by the mathematical literacy domain, its theoretical basis and its organisation. In

⁶⁹National Council of Teachers of Mathematics, *Principles and Standards for School Mathematics* (USA: April, 2000)

⁷⁰OECD, *The PISA 2003 Assessment Framework*

⁷¹G. Shiel, J. Cosgrove, N. Sofroniou, A. Kelly, *Ready for Life? The Literacy Achievements of Irish 15-Year-Olds*, (Dublin: ERC, 2001), xiii.

so doing, it describes a particular overview of the nature and structure of school mathematics in a vocabulary which has many terms not seen in earlier descriptions of mathematical curriculum development projects. It is PISA03 as it relates to mathematics and PSSM that are used here to provide many of the terms used subsequently. By furnishing modern mathematical pedagogy with a rich terminology and a structural clarity these studies provide a frame of reference which facilitates the presentation of educational ideas in a concise manner.

4.9.1 PSSM

The NCTM has been heavily involved in mathematical curriculum development. Principles and Standards for School Mathematics is a major and comprehensive “resource and guide for all who make decisions that affect the mathematics education of students in prekindergarten through grade 12.”⁷² Given the broader nature of its brief, the Irish NCCA consultative papers and reports referred to in the Introduction are concerned with a general vision of education, and so make little or no explicit mention of mathematics, nor any other individual subject. However, just as the NCCA document *Directions for Development*⁷³ sets out a vision for education in general, the PSSM begins by setting out its particular vision for school mathematics. It then discusses in detail

⁷²National Council of Teachers of Mathematics, *Principles and Standards for School Mathematics*, ix.

⁷³The word mathematics does not appear in this document.

The Need for Mathematics in a Changing World⁷⁴

The Need for Continued Improvement of Mathematics Education.⁷⁵

PSSM continues by elaborating on the role and purpose of standards before proceeding to give an overview of these principles and standards. It says that

the principles are statements reflecting basic precepts that are fundamental to a high-quality mathematics education

while

the standards are descriptions of what mathematics instruction should enable students to know and do - statements of what is valued for school mathematics education.

PSSM describes six Principles:⁷⁶

Equity: Excellence in mathematics education requires equity - high expectations and strong support for all students.

Curriculum: A curriculum is more than a collection of activities: it must be coherent, focused on important mathematics, and well articulated across the grades.

Teaching: Effective mathematics teaching requires understanding what students know and need to learn and then challenging and supporting them to learn it well.

⁷⁴National Council of Teachers of Mathematics, *Principles and Standards for School Mathematics*, 4.

⁷⁵Ibid., 5.

⁷⁶Ibid., 16.

Learning: Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge.

Assessment: Assessment should support the learning of important mathematics and furnish useful information to both teachers and students.⁷⁷

Technology: Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning.

Considerable space is devoted to the elaboration of each of these principles. For the moment, it will simply be noted that the elevation of Technology to that of a Principle indicates the importance American educators attach to technology as an educational tool. That the Irish NCCA are also beginning to attach a similar importance to the role of technology in teaching/learning has already been mentioned.

PSSM identifies ten Standards, five of which are classified as the Content Standards.⁷⁸

Numbers and Operations

Algebra

Geometry

⁷⁷Again it is to be noted the assessment appears to be something different than curriculum for this body of educators.

⁷⁸Ibid., 29.

Measurement

Data Analysis and Probability

with the remaining five being described as the Process Standards:⁷⁹

Problem Solving

Reasoning and Proof

Connections

Communication

Representation

Upon listing these standards, it is emphasised that the Content and Process Standards are inextricably linked:⁸⁰

One cannot solve problems without understanding and using mathematical content. Establishing geometric knowledge calls for reasoning. The concepts of algebra can be examined and communicated through representations.

PSSM describes how each of the ten standards should be addressed across each of the grade-bands, while supplying many concrete illustrative examples at each of the levels. All of this is supplemented by what are termed e-examples, many of which are interactive applets. It is, perhaps, worth remarking that in the US the subject of calculus is not treated at the level of secondary school.

⁷⁹Ibid., 29.

⁸⁰Ibid., 30.

Content is dictated by context. So what are the mathematical contexts recognised by the NCTM? They are to be found in the chapter treating *The Need for Mathematics in a Changing World* listed as:⁸¹

- Mathematics for life
- Mathematics as a part of cultural heritage
- Mathematics for the workplace
- Mathematics for the scientific and technical community

It is heartening to see explicit mention of mathematics as part of cultural heritage, a theme already considered here in the introduction and treated in some detail later in this work with, perhaps, a different emphasis than that of PSSM.

If one thinks of the process standards acting on the content standards and context as determining what these latter standards should be, then the sequence

$$\text{Context} \longrightarrow \text{Content} \longrightarrow \text{Processes}$$

imposes an order on these aspects of curriculum design which, it must be remarked, mirror the sequence: aims, content and methods, that form part of the definition of curriculum given by Howson *et al.*⁸²

⁸¹Ibid., 4.
⁸²See 83.

4.9.2 PISA03

The OECD/PISA (2003) is an international undertaking, involving in the region of 42 countries. Its evolution and character probably owe most to the Dutch Realistic Mathematics Education (RME) movement of the 1970s. However, one must add to this European influence that of the Australian educational system which embraced and actively implemented many of the ideas espoused by the RME movement.

In setting out the rationale for the form of assessment used, the PISA03 framework document parallels much of what is said in the first part of the NCTM’s *Principles and Standards*. The differing language used by the NCTM and OECD/PISA to describe mathematics education reveals slightly different perspectives. By contrast with the sequence

$$\text{Context} \longrightarrow \text{Content} \longrightarrow \text{Processes}$$

that of PISA03 might be described by

$$\text{Situation} \longrightarrow \text{Overarching Ideas} \longrightarrow \text{Mathematical Competencies}$$

Situations: For PISA03 ‘the situation is the part of the student’s world in which the tasks are placed. It is located at a certain distance from the student.’⁸³ Four situation-types are defined:

- 1. personal

⁸³OECD, *The PISA 2003 Assessment Framework*, 32.

2. educational/occupational
3. public
4. scientific

While there is not an exact isomorphism between what might be termed the ‘context list’ of the PSSM and the PISA03 ‘situations list’, one can see that the overall correspondence is very close. However, PISA03 views ‘the context of an item in its specific setting within a situation’. It illustrates this notion by an example relating to a savings account where there are two options regarding the investment of a 1000 units of currency, saying⁸⁴

The situation of this term is ‘finance and banking’, which is a situation from the local community and society that OECD/PISA would classify as ‘public’. The context of this item concerns money and interest rates for a bank account.

Intra-mathematical and extra-mathematical tasks: In discussing the situation and context of a problem, PISA03 draws a distinction between two types of task. It explains an ‘intra-mathematical task’ as follows:⁸⁵

If a task refers only to mathematical objects, symbols or structures, and makes no reference to matters outside the mathematical world, the context of that task is considered as intra-mathematical, ...

It adds that such tasks are to be classified as belonging to the ‘scientific’ situation-type.

The term ‘extra-mathematical task’ is explained as follows:⁸⁶

⁸⁴Ibid., 32.

⁸⁵Ibid., 33.

⁸⁶Ibid., 33.

More typically, problems encountered in the day-to-day experience of the student are not stated in explicit mathematical terms. They refer to real-world objects. These tasks are called “extra-mathematical”, ...

where “the student must translate the problem contexts into a mathematical form.”⁸⁷

These categorisations will be found useful later in the discussion of applications and in the analysis of examination problems. The second term of the PISA03 sequence labelled Overarching Ideas is now examined in relation to its Contents counterpart in the NCTM sequence.

Overarching ideas: Recall that the five *Content Standards* of PSSM are: Numbers and Operations; Algebra; Geometry; Measurement; Data Analysis and Probability. In the PISA framework mathematical content is “illustrated by four categories that encompass the kind of problems that arise through interaction with day-to-day phenomena, and are based on a conception of the ways in which mathematical content presents itself to people.”⁸⁸ These are called the *Overarching Ideas*:

- Quantity
- Space and Shape
- Change and Relationships
- Uncertainty

⁸⁷Ibid., 33.

⁸⁸Ibid., 31.

Here mathematical content is “organized into a sufficient number of areas to ensure a spread of items across the curriculum, but at the same time a number small enough to avoid too fine a division that would work against a focus on problems based in real situations.”⁸⁹ PISA03 relates, by means of the historical sketch given earlier,⁹⁰ how the order of these overarching ideas corresponds to the chronological order in which they evolved as areas of human study.

The PISA descriptors avoid the use of ‘mathematical’ terms such as algebra and geometry, preferring terms closer to ordinary language. PISA03 observes that its approach to content is somewhat different from “perspectives of mathematics instruction and the curricular strands typically taught at schools”, adding, however, that “the overarching ideas taken together broadly encompass the range of mathematical topics that students are expected to have learned.”⁹¹

Mathematical competencies: The PISA03 counterparts of the five PSSM process standards described previously as: Problem Solving; Reasoning and Proof; Connections; Communication; Representation are now considered.

Rather than use the term “processes” PISA03 prefers that of “competencies”, which it says, “have to be activated to connect the real world, in which the problems are generated, with mathematics, and thus solve the

⁸⁹Ibid., 35.

⁹⁰Refer to 46-47.

⁹¹Ibid., 31.

problems.”⁹²

Based on the work of M. Niss (1999)⁹³ and his Danish colleagues it singles out eight mathematical competencies “which, taken together, can be seen as constituting comprehensive mathematical competence.”⁹⁴ They are

1. Thinking and Reasoning
2. Argumentation
3. Communication
4. Modelling
5. Problem posing and solving
6. Representation
7. Symbolic, formal and technological language and operations
8. Aids and tools

It is to be noted that with the exception of Connections all of the PSSM process standards appear somewhere on the PISA list of competencies. PISA defines and elaborates on these competencies and this will be considered in more depth later in the work. For now one example is considered, the definition of the communication competency.⁹⁵

⁹²Ibid., 30.

⁹³“Kompetencer og Uddannelsebeskrivelse” (Competencies and subject description), Uddanneise, 9 21-29 as cited in PISA03, 40.

⁹⁴Ibid., 40.

⁹⁵Ibid., 40.

This involves understanding and expressing oneself, in a variety of ways, on matters with a mathematical content, in oral as well as written form, and understanding others' written or oral statements about such matters.

By contrast, PSSM, strictly speaking, does not define its use of the process term Communications; rather it gives reasons why it is important and describes different modes of communication including that of technology.

Competency clusters: PISA03 describes the cognitive activities that the eight competencies encompass according to three competency clusters:⁹⁶

the *reproduction* cluster

the *connections* cluster

the *reflection* cluster

Use of the term “connections”, explicitly identified as a process standard by the PSSM, should be noted. PISA03 characterises with considerable detail what is meant by each of these clusters, and “the ways in which the individual competencies are played out in each cluster.”⁹⁷ Before attempting a very brief synopsis of what is involved in each case, the interest here is the machinery they provide for classifying problems in a reasonably tight fashion.

The competencies in the reproduction cluster ‘essentially involve reproduction of practiced knowledge’.⁹⁸ For a problem of this type, the student

⁹⁶Ibid., 41.

⁹⁷Ibid., 41.

⁹⁸Ibid., 42.

needs to *know* basic facts, be able to *recognise* equivalent scenarios, *recall* objects and propositions and *perform* routine procedures. After discussing the depth to which each of the eight competencies (listed earlier) are invoked at the level of this cluster, PISA03 presents a number of examples of reproduction cluster problems, four of which are shown.

Heading the hierarchy of competency clusters is the *reflection* cluster. “The competencies in this cluster include an element of reflectiveness ... about the processes needed or used to solve a problem.”⁹⁹ The solver has to be able to plan and implement solution strategies in settings that contain more elements and may be more original or unfamiliar than those in the connections cluster. The PISA03 discussion of how each of the eight individual competencies are called upon by problems classified in this category is longer in each case than the corresponding treatments for either of the other two competency clusters, reflecting the greater demands such problems make on their would-be solver.

Two concrete problems from this competency cluster are presented and discussed in detail by PISA03.

The following table is a diagrammatic representation of the competency clusters:

⁹⁹Ibid., 46.

The Representation Cluster	The Connections Cluster	The Reflection Cluster
• Standard representations and definitions • Routine computations • Routine procedures • Routine problem solving	• Modelling • Standard problem-solving translation and interpretation • Multiple well-defined methods	• Complex problem-solving and posing • Reflection and insight • Original mathematical approach • Multiple complex methods • Generalisation

Table 4.1: The PISA 2003 competency clusters.

The PISA03 document devotes considerable discussion to the fundamental process by which real-life problems are solved, referring to it as ‘mathematisation’ and characterising it as consisting of the following sequence of five steps:¹⁰⁰

1. Starts with a problem situated in reality.
2. Organises it according to mathematical concepts and identifying the relevant mathematics.
3. Gradually trims away the reality through processes such as making

¹⁰⁰Organisation for Economic Cooperation and Development, *The PISA 2003 Assessment Framework* (OECD: Paris, 2003), 38.

assumptions, generalising and formalising, which promote the mathematical features of the situation and transform the real-world problem into a mathematical problem that faithfully represents the situation.

4. Solves the mathematical problem.
5. Makes sense of the mathematical solution in terms of the real situation.

Undertaken in the order listed these five steps constitute *the mathematisation cycle* which may be depicted as shown:

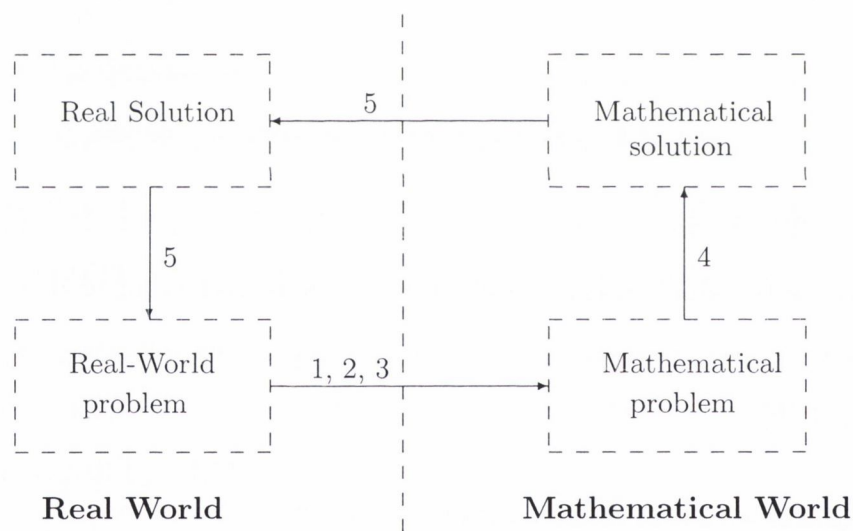


Figure 4.1: The mathematisation cycle.

Latterly writers have found it useful to refer to the combination of steps 1, 2 and 3, whereby the real-world problem gets translted into a mathematical one, as ‘horizontal mathematisation’ and to employ the term ‘vertical

mathematisation’ in relation to step 4, which involves technical mathematical theory being applied to solve the mathematical formulation of the original problem.¹⁰¹

Summary: The discussion of this section has set before the reader a number of concepts and terms in current usage which will be employed throughout this work.

By considering both recent publications of the NCTM and OECD relating to mathematical education, a parallelism between the former’s PSSM and the latter’s PISA03 has been identified which might be represented diagrammatically as shown:

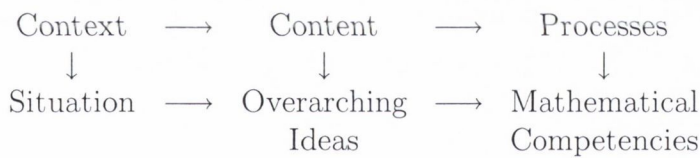


Figure 4.2: The NCTM – PISA parellelism.

The direction of the three arrows from each of the terms in the upper ‘NCTM sequence’ to its counterpart in the lower ‘PISA sequence’ is downward rather than two-way since, as discussed, from the PISA perspective each element in the NCTM sequence is seen as being under the umbrella of the corresponding term in the PISA sequence.

¹⁰¹Paul F. Conway, & Finbarr C. Sloane, *International Trends in Post-Primary Education*, (Dublin: NCCA, 2005), 125.

The PISA03 document has much more to say in relation to many of the themes considered in this work and so will be called upon again.¹⁰²

¹⁰²A virtue of both the *Principles and Standards of School Mathematics* and the *PISA 2003 Assessment Framework* documents is that they are comprehensive and explanatory, if, perhaps, overly-long to the point of intimidation. The PISA problems alone are worth reading. The DES Aims and Objectives, to be discussed later, are by no means as comprehensive, being more classically frugal but serve a purpose. One would expect future editions to follow the trend towards greater elaboration judging by recent DES/NCCA publications relating to other courses. See, for example, *Junior Certificate: Guidelines for Teachers*.

Chapter 5

Establishing the Case

The purpose of this chapter is to substantiate a case for reform of the current programme.

In order to tackle this task it is useful to recall the general criticisms made, which is, perhaps, best done by simply listing them in the order in which they were (first) raised. These are that

1. the mathematical techniques taught and examined in some depth are not being justified adequately by way of application to the solving of problems external to mathematics itself.¹
2. syllabus material is being examined in a manner which stresses the technical almost to the exclusion of all other aspects of mathematics.²
3. existing school textbooks are permeated by the same technical emphasis that pervades the LC examination.³

¹See page 10.

²See page 12.

³See page 14.

4. by almost exclusively concentrating on technical mathematics, facets of the subject such as its history and culture along with its recreational element are being ignored entirely.⁴
5. there is at best a scant use of technology.⁵

At the moment this last criticism is not contested since, as has already mentioned, there is an acceptance that there should be a greater use of technology.⁶

Two surveys are considered now; the first was a formal research survey carried out with Junior Certificate students in 1992 while the second was an informal one conducted in September 2000 with students who had just completed the Higher Leaving Certificate.

Because of the moderate sizes of each of the groups surveyed, it is not statistically valid to conclude that the results of either survey are indicative of a general situation prevailing in the nation's classrooms. However, when one bears in mind the fact that all 'examination' classes both at Junior and Senior cycle level have the passing of a common national examination as their ultimate objective, and given that these examinations vary little in style from year to year, the findings of these surveys may reflect much of what is 'typical' in many, if not all, of the classrooms nationwide.

It will be seen that the determinations of the first survey are very much in

⁴See page 15.

⁵See page 19.

⁶See pages 22-23.

keeping with those reported earlier in this work in relation to the more recent study of Lyons *et al.*⁷ One of the values of the second survey for this writer is in what it reveals, in however limited a manner, of students' exposure to and knowledge of both recreational mathematics and the history of mathematics.

5.1 Formal Survey of 1992

What Irish pupils say about Mathematics is a 1992 report of a study conducted in Ireland to investigate pupils' attitudes to school mathematics, which its authors, Bajpai, English and O'Donoghue, "feel is a neglected area of research which could produce useful insights for practitioners in mathematics education."⁸ These researchers maintain that in the Irish context insufficient attention has been given to students' attitudes to mathematics and schooling in general, their expectations and their motivating drives.

They found that for many students, mathematics is associated with a strong sense of failure, and their memories of school mathematics are of tests and exams, of courses, of the fear of 'getting it wrong'. An over-emphasis on procedures and techniques in the Irish context has led to a neglect of the 'pleasure' and aesthetic side of mathematics. This imbalance does not help to promote pupil interest and positive attitudes to the subject. In the words of one upper-secondary school student "I believe it is necessary to change

⁷See page 14.

⁸J. English, J. O'Donoghue and A. C. Bajpai, "What Irish pupils say about Mathematics", *International Journal of Mathematical Education in Science and Technology* 23, no. 5 (September-October 1992): 749.

students' attitudes to the subject, that it becomes a subject to be enjoyed, not endured."⁹ They point out that despite the onslaught of the information revolution the potential of the microcomputer for enhancing pupils' understanding of mathematics (and in the process promoting positive attitudes) has not been satisfactorily exploited.

The sample population was confined to 25 of 140 pupils taking mathematics in the third year of the Junior cycle programme. The size of the sample was in accordance with the methodological approach employed which was an evaluation strategy of the "*illuminative and qualitative*" type.¹⁰ Data was collected by means of individual tape-recorded interviews which were of a semi-structured nature. The framework for each interview was based on eight subscales:¹¹

1. mathematics as an interesting subject (5 questions)
2. satisfaction and success from mathematics (3 questions)
3. attitude of pupils to school as a whole (4 questions)
4. home environment/study habits (2 questions)
5. pupil confidence (6 questions)
6. parental attitudes (3 questions)

⁹Ibid., 750.

¹⁰Ibid., 753.

¹¹Ibid., 754.

7. relation to other subjects (3 questions)
8. miscellaneous (6 questions)

The report's findings are briefly synthesised here (along with a number of student replies) under the relevant sub-scales' headings.

Mathematics as a useful and interesting subject: Those pupils (the majority, 21 out of 25) who perceived mathematics as having a utility value related it to its usefulness to future life and practical situations:

"Mathematics is useful to find jobs and solve problems at home."

"Mathematics is not useful except for bills and measuring."

On the issue of mathematics as an interesting subject the feelings expressed in almost all cases were strong and negative in nature and directed in the main towards the dull content of the mathematics syllabus:

"More than 50% of the course is boring"

A number of topics were heavily criticized, most notably much of the geometry (especially theorems), logarithms and equations involving more than one variable.

Parenthetically it must be noted that the OECD/PISA report of December 2004 on PISA03 shows that these rather negative findings, concerning the

perceived utility of mathematics and the interest that the subject generates, persist and are ubiquitous:¹²

Student interest in mathematics is far lower, across countries, than in reading. Among OECD countries, about half of the students report being interested in things they learn in mathematics, but only 38 percent report that they do mathematics because they enjoy it. On the other hand, the great majority of students believe that studying mathematics will help them in their future.

Pupil anxiety and confidence: A slight majority (56%) expressed a lack of confidence in handling mathematics problems:

"I have absolutely no confidence in solving mathematics problems. I don't like difficult problems and it annoys me very much as mathematics is an essential subject."

In general, those pupils who lacked confidence related their plight to 'very difficult' problems and to particular topics. The authors noted that confidence in solving mathematics problems does not necessarily imply confidence in asserting oneself in class when difficulties arise:

"I feel confident in solving problems because I am good at maths. Yet, if I get stuck, I would not raise my hand because I would be afraid of the teacher and what she might say."

Enjoyment and satisfaction from mathematics: Eighty four percent of the students perceived satisfaction in mathematics as being linked to success in solving mathematics problems especially in examinations. However,

¹²"Top Performer Finland Improves Further in PISA Survey as Gap Between Countries Widens", <http://www.oecd.org/document/28/, 2>; See also OECD, *Learning for Tomorrow's World – First Results from PISA 2003* (OECD: Paris, 2004), 115. (accessed January 6, 2005)

even this type of satisfaction was all too often a rarity for the majority. Significantly, when pupils were asked: "Have you ever experienced a feeling of joy from doing mathematics over three years?" not one pupil replied in the affirmative. The authors assert that the pupils' responses clearly indicate that the utility value of mathematics overshadows the debate on the enjoyment/satisfaction issue.

Transfer of mathematical knowledge: Only three pupils were able to perceive a link between algebra and geometry and sets and functions. "It appears that pupils have pockets of knowledge which are specific to the time they 'did it in class'.¹³

Pupil passivity: In response to: "Are mathematics classes generally active with pupils participating or does the teacher do most of the work at the board?" two replies were:

"The teacher generally asks a question, he calls people up to do it on the board. At the start of each topic, the teacher does all the work and we take down notes but then we tend to become more active."

"The teacher usually does most of the work at the board and always decides what topic is to be done next. There have been occasions when the pupils asked the teacher a question and the teacher didn't have time to answer."

In general the scenario is one of students sitting in their seats following an inactive and regular routine of 'exposition', examples and exercises.

¹³Ibid., 757.

Teachers, teaching and the pace of lessons: One student remarked

"I think the pace is too fast for most students and they miss out on the basics as the teacher is concerned about getting the course finished."

Another said:

"What I like about our mathematics teacher is that he doesn't force us to do the work, but will help pupils if asked to do so. Yes, the teacher affects the way you work in school. I like the way he gets us to tease out the answer."

Parental attitudes: It is evident from the comments that mathematics is held in very high esteem by parents who recognize that, in reality, mathematics is used as the primary screening device or 'filter' for entry to numerous professions.

"Yes, parents think that maths is very important. They feel that if you're good at maths then you must be good at everything else."

Summary: These researchers found that the aesthetic and 'pleasure' aspect of mathematics is neglected at Junior Cycle where the 'profit' motive predominates in an over-crowded syllabus. In top stream classes, individualism, rivalry and competition is widespread at the expense of co-operation and collaboration. In general, pupils are dissatisfied and expressed emotive feelings with the present mathematics courses from the point of view of no applications value to the world outside. Pupils equate the utility value of mathematics almost exclusively to its usefulness for employment purposes.

The authors assert that mathematics teachers need to spend more time talking about mathematics and to help students appreciate the power and

role of mathematics in its applications at the level which pupils have reached. Pupils are unable to transfer knowledge gained in one topic to other areas, or to other subjects or into usable forms in the world outside the school. For the most part, the amount of active participation by pupils in the learning of mathematics is very limited. They remark that ‘an acquiescent’ participation aptly describes the reality.

All in all, this survey presents a rather bleak picture of what may have pertained in general in the Junior cycle classrooms of 1992. This picture is one of a joyless mathematics, lacking in application, being passively imbibed by students who believed in the subject’s utility because of its perceived value in the market place but who had never had this utility visibly demonstrated within their classroom.

5.2 Informal CIT Survey of 2000

At the start of each academic year, it has been part of this writer’s practice to discuss with incoming engineering students their overall perception of mathematics in its various aspects. Each year the students in question are those hoping to pursue the degree in Electronic Engineering at the Cork Institute of Technology (CIT), a course for which the minimum mathematical requirement is a grade of C3 on the Higher level paper. These informal talks normally follow a simple format whereby the lecturer asks questions followed by various students responding in their own way. It has been this author’s experience that, over the course of at least twenty years, the answers to the

questions asked differ little from year to year and are largely independent of school background.

In an effort to have a concrete record of the general nature of these ‘standard’ answers, the written views of the first-year students of September of 2000 were sought. The twenty-two students, most of whom had just completed the HLC course, obliged by writing replies, each in their own fashion over a period of fifty minutes, to the question

What do you think about the subject of mathematics?

By way of assistance it was added that:

Answering some or all of the following questions may help you. Feel free to pose and answer any others that spring to mind or to write about any aspect of mathematics that occurs to you.

1. What do you think mathematics is about?
2. Is mathematics useful?
3. List a number of ‘areas’ where a knowledge of mathematics is essential?
4. Is mathematics needed for this course?
5. Did you like maths at national school?
6. Did you like studying mathematics at secondary school?
7. Would you prefer if you didn’t have to study mathematics now at tertiary level?

8. What is your favourite part of mathematics?
9. What do you not like about mathematics?
10. Do you like puzzles? If so give some of your favourites.
11. How well do you think mathematics is taught in general?
12. What would you teach if you were given the opportunity?
13. What other mathematics, if any, would you have liked to learn?
14. What in your opinion is the most important piece of mathematics that you have learned?
15. What do you know of the history of mathematics?
16. How did mathematics evolve?
17. Who in your opinion is a mathematician?
18. Do you know of any mathematicians, alive or dead?
19. Would you say, in general, that secondary school teachers of mathematics are mathematicians?
20. Would you say, in general, that third level lecturers of mathematics are mathematicians?

This questionnaire was given as a form of guide for those who might find it difficult to write concerning something as broad as 'about mathematics'. It

was re-emphasised verbally to the students that they could approach the task in any way they wished. They were also told to be completely honest and not to overly worry about style and presentation since the primary objective of the exercise was to learn their opinions.

The twenty-two essays are given in appendix A in the section *The Student Replies*. These show that most used the questionnaire to structure their responses but, as can be seen, some gave free rein to their thoughts and powers of expression. After typing up all twenty-two essays, the replies were extracted on a question by question basis. Other than using a spell checker to correct words misspelt in the original essay material, no other changes whatsoever were made. In each case, the source of the opinion being quoted is given using the individual's initials along with a short profile of the respondent in terms of the overall number of points scored in the Leaving Certificate (LC) examination, the maths grade obtained and the number of questions out of 20 answered correctly on a Basic Mathematics Skills (BMS) test.

The BMS test is a one-hour examination given to each student in the first days of this course and is used to gauge the student's level of competence at answering very basic but essential mathematical questions of a technical nature. Further details of the test and the rationale behind it can also be found appendix A.

The different answers to the first question

What do you think mathematics is about ?

arranged in descending order according to mathematics grade are as follows:

1. Mathematics is the universal language. In its simplest form it is a tool for every day life. In its most complex form, it is quite simply everything. (CD, 550-A1 (BMS: 20/20))

2. I believe the universe we live in is written in the language of mathematics. It is used by everybody every day of their lives in one form or the other. One of main reasons for enjoying maths is that I believe that mathematics is the prime key in answering the puzzles of the universe. Why are we here? Where did we come from and most importantly where are we going.

Mathematics is fundamental in the study of any engineering course. It will play a large part in every subject be it computer science or physics. My attraction to maths is based on its factuality and conciseness. Unlike English or Philosophy, maths in general can have only one answer. I like the idea that with maths we are dealing with truths and not opinions. I believe the human race has compromised its ability to think for itself and solve the mysteries of the universe with suspicious ignorance i.e. religion. I don't believe in any thing except the absolute truth of mathematics.

If the laws and rules of this universe are written down mathematics then is mathematics as old as the universe itself. We didn't simply invent maths, we discovered its existence therefore I believe we have only scratched the surface of what we know with mathematics. There

is a vast amount of learning to be done and that is what I wish to be part of. Mathematics is like an undetectable energy that envelopes absolutely everything; the ink on this page, the pen that writes this. I believe I would maths if I had the opportunity because I want people to understand that without maths nothing would exist. Maybe mathematics is our true God !

(MJoS, 480-A1 (BMS: 16/20))

3. Mathematics is about logically and methodically understanding the way things work and why.

(TH, 445-A2 (BMS: 17/20))

4. In my opinion mathematics is about numbers and problems and using numbers to solve problems, real or imaginary.

(KS, 440-A2 (BMS: 19/20))

5. I don't think it is possible to accurately define what maths is all about. Obviously there is a lot of problem solving, numbers & even shapes involved but there is also much more. For example it has a very strong influence on the theory of music also. I think that one either understands or doesn't understand what maths is about, but it is very difficult to teach or explain it. It's almost like a picture to me, I can visualise & understand what it is about in my own head but find it very difficult to "explain" it.

(ÉF, 500-A2 (BMS: 16/20))

6. Maths is a means of calculating equations so as to get solutions to problems.

(JS, 470-B1 (BMS: 18/20))

- 7. The calculation of amounts and quantities of all events and actions to decipher a link between situations eg. logarithmic progressions.
.
(DM, 455-B1 (BMS: 8/20))

- 8. Mathematics is about learning and understanding equations and theorems which help you throughout your life and career.
.
(CAO'D, 410-B1 (BMS: 16/20))

- 9. Maths is about solving problems and situations through the use of equations. By defining certain things, we can put these "things" into an equation and therefore solve the problems. (CoK, 485-B2 (BMS: 16/20))

- 10. Maths is like a different language. It helps us in every aspect of life. It is more than just a way of counting. (PoC, 405-C1 (BMS: 11/20))

- 11. Mathematics is about the counting of things. (PoS, 410-C2 (BMS: 13/20))

- 12. Maths not only involves numbers and letters, adding, subtracting etc. but also involves important aspects of our human lives like for example when we walk we are using maths to get from one place to another.
.
(JD, 325-C2 (BMS: 8/20))

- 13. It is the creating of equations which help us to understand the world around us. As many people believe this world is made up of numbers and equations. (JS, 315-C2 (BMS: 10/20))

- 14. I think maths is about calculating results for every day things "If a train leaves" etc (TR, 305-C2 (BMS: 18/20))

15. Maths is about solving problems and using equations to do so.

. (SD, 370–C3 (BMS: 10/20))

16. Mathematics is about the solving of numerical equations. It is about seeing a problem and finding a solution for it. (CM, 345–C3 (BMS: 8/20))

17. Mathematics is about the study and understanding of numbers. It is also about the way in which numbers effect our daily lives.

. (DoC, 330–C3 (BMS: 13/20))

18. I think Mathematics is about using numerical figures to explain things differently to a verbal description. For example, if you were given a quadratic equation in x and were asked to solve this equation you would get two values for x . This means that x can be two possible numbers which can be substituted for x .

. (DH, 320–D1 (BMS: 12/20) – (65% EE))

As can be seen the answers to this question vary from the quite sophisticated to the almost naïve. Perspectives on the nature and meaning of mathematics by those practised in the art are many and varied. Here are the students' answers to the more specific question

Is mathematics useful?

1. There are some areas of life in which maths is a more widely used and essential tool – all forms of engineering for example.

. (CD, 550–A1 (BMS: 20/20))

2. It is used by everybody every day of their lives in one form or the other.

(MJoS, 480-A1 (BMS: 16/20))

3. Mathematics is undoubtedly a vital aspect of life. From simple counting right through to the applications of complex binary functions, maths is ever present in our world. Everybody needs some sort of understanding of maths however basic.

(ÉF, 500-A2 (BMS: 16/20))

4. It is very useful in that it develops the logical part of the brain, and provides us with a systematic understanding of the fundamentals of physics, electronics, indeed anything with quantitative elements. It develops the ability to think through and return with probable conclusion and so “makes the world go round.”

(TH, 445-A2 (BMS: 17/20))

5. It is a very useful subject which can be applied to many aspects of life, not least engineering.

(KS, 440-A2 (BMS: 19/20))

6. It is very useful to calculate the positions of satellites, in calculating bank balance and for finding solutions to various problems which occur in the business world.

(JS, 470-B1 (BMS: 18/20))

7. Yes because without it certain quantities would have to be accounted for manually which is time wasted when you consider a formula can be used taking the first few values and solutions can be worked out at hypothetical values which otherwise couldn't be counted.

(DM, 455-B1 (BMS: 8/20))

8. Its useful in many areas such as science, computers, architecture and engineering. Maths teaches you how to be precise about your calculations and solve problems, which is useful in these areas.

. (CAO'D, 410-B1 (BMS: 16/20))

9. Yes. maths is definitely useful as it enables people to solve complex problems and get a grasp of the world and how it works. It also enables people to develop new technology to benefit humankind.

. (CoK, 485-B2 (BMS: 16/20))

10. To be honest, I have no idea what a lot of maths is useful for. In secondary school we were never told what the maths we were doing could be applied to. The subject was never made relevant to daily life. It was just dished out to me and I was left the monotonous task of figuring most of it out for myself.

(SF, 495-B3 (BMS: 16/20))

11. Maths is useful in a variety of ways. Computers use maths. Nearly every one uses maths.

(PoC, 405-C1 (BMS: 11/20))

12. It is used almost everywhere; construction, computing and travelling.

. (PoS, 410-C2 (BMS: 13/20))

13. Mathematics is very useful to us but we don't always think that way. Everything on this earth is in some way connected to maths.

. (JD, 325-C2 (BMS: 8/20))

14. It is a basic necessity for everyday Life.

(JS, 315-C2 (BMS: 10/20))

15. In my opinion it is extremely useful, for instance, calculating the volume of an object when there is no other way to find out i.e. a solid. It is useful for calculating the approx. distance of the stars when we know it is impossible to travel there. (TR, 305-C2 (BMS: 18/20))
16. Maths has to be used everywhere and it is therefore vital to have a basic understanding of the subject. (BoC, 400-C3 (BMS: 13/20))
17. Mathematics is very useful in everyday life. It is used in every aspect of life especially engineering. It is important for the day to day running of the country *e.g.* budgets and accounts. Maths is interlinked with many subjects like physics and chemistry. (JM, 395-C3 (BMS: 10/20))
18. Maths is essential in areas such as electronics, Physics and business. Maths is a very important part of electronic engineering. (SD, 370-C3 (BMS: 10/20))
19. Mathematics is very useful in Life because it gives you greater understanding of problems and how to solve them. (CM, 345-C3 (BMS: 8/20))
20. Without mathematics modern civilisation could not take place or exist We would not be able to tell the time or build a suspension bridge. Early farmer would not have been able to see if lost any of animals and Neil Armstrong would not have stepped on the moon.

For all the reasons listed above mathematics is useful. It is a key-stone of our way of life. Mathematics is useful for science, accounting, engineering and day to day living.

(DoC, 330-C3 (BMS: 13/20))

21. For example, if you were given a quadratic equation in x and were asked to solve this equation you would get two values for x . This means that x can be two possible numbers which can be substituted for x .

In English this would take much longer, therefore Maths is quite useful in my opinion.

(DH, 320-D1 (BMS: 12/20) – (65% EE))

These answers show that some are content with naming a number of disciplines for which they believe maths is useful without going into further detail. The list of all the disciplines mentioned explicitly along with sub-disciplines is as follows:

Engineering/Electronics/Build Bridges

Science/Physics/Chemistry/Space Travel

Computing

Architecture/Construction

Business & Accounting

Other answers are more encouraging, showing, as they do, a general appreciation of the importance of mathematics to the needs and advancement

of society. One example is, ‘to get a grasp of the world and how it works’, though how this is done is not mentioned. One can also draw hope from “Mathematics is very useful to us but we don’t always think that way. Everything on this earth is in some way connected to maths” even if it be untainted by any mention of a specific use. It is to be hoped that the almost parliamentary acknowledgment of the importance of mathematics is a sincerely expressed opinion rather than one designed to please the questioner. In any case the utterance is valid even if the motive be false. The concrete uses of mathematics cited in these replies are those of

Simple counting.

Aiding logical thinking and a systematic understanding of the fundamentals of quantitative disciplines.

Teaching how to be precise and concise.

Calculating the positions of satellites.

Calculating bank balances and budgets.

Using formulae to save on manual labour.

Calculating volumes of solids.

The answers to the more specific question

List a number of ‘areas’ where a knowledge of mathematics is essential

are contained in the appendix *The Students' Replies*. They add Civil and Mechanical Engineering and Computer Software Design to the list of disciplines. One answer states that maths is also used by NASA and nuclear plants while another states that it is used in Formula One racing and the Airplane industry to calculate optimal fuel loads. The jobs of electrician, plumber, carpenter, shopkeeper, teacher and mason are mentioned as ones where a knowledge of maths is essential while explicit mention of further concrete uses of maths appends the following items to the previous list:

Algebra for the plotting of projectile courses, moving objects, population curves.

Calculus for measuring area and plotting graphs.

Trigonometry for technical and construction drawing.

... to calculate distance, speed, time, height etc.

... for architectural drawings

... to calculate statistics and percentages.

The remainder of this section is concerned with the replies to those questions which touched on recreational mathematics and the history of mathematics.

Recreational Mathematics: The question

Do you like puzzles? If so give some of your favourites

received the following replies:

1. I enjoy cross word puzzles, mainly crosswords on the topic of sport.
(JS, 470-B1 (BMS: 18/20))
2. I like puzzles to a certain extent. (CAO'D, 410-B1 (BMS: 16/20))
3. Not really. (CO'K, 485-B2 (BMS: 16/20))
4. I love puzzles mainly riddles for instance. You have a row of ducks one duck has two ducks behind him. Another duck has two ducks in front of him and there is one in the middle. How many duck are there. It is just a simple riddle but I love to sort things like that out. I could spend days at them sometimes. (PO'C, 405-C1 (BMS: 11/20))
5. I like getting a few correct answers on a crossword or else getting out a picture puzzle. (PO'S, 410-C2 (BMS: 13/20))
6. Yes. Rubix cube (JS, 315-C2 (BMS: 10/20))
7. Mathematic Puzzles ! None spring to mind at the moment but I could sit down contented for hours trying to figure them out. I remember in first year we had a free maths class. A sub teacher came in and gave us a Mathematical Illusion (not sure if I have the right terminology).

he told us to keep count through the whole story and we should end up with the same number we started with (27 I think). No multiplication, no division just passing three objects from person to person. Everyone ended up with one short (26). Sounds quite boring but it was enjoyable at the time.

(TR, 305–C2 (BMS: 18/20))

8. I enjoy puzzles but find them quite difficult.

(BO'C, 400–C3 (BMS: 13/20))

9. I never really practised puzzles before and never did it in second level.

(JM, 395–C3 (BMS: 10/20))

10. I like to do Brain teasers from newspapers which involve use of Maths.

(SD, 370–C3 (BMS: 10/20))

11. I do like certain puzzles. My favourite is:

A shopkeeper brought a balance for his shop. If the shopkeep must weight items between 1kg and 40kg what weights will he buy to keep the number of weights to a minimum.

(DO'C, 330–C3 (BMS: 13/20))

12. I enjoy some puzzles. I like crossword puzzles and wordsearchers.

(DH, 320–D1 (BMS: 12/20) – (65% EE))

It would appear that most students have not had a great deal of exposure to recreational matheamtics either at home or at school. However, the remarks:

It is just a simple riddle but I love to sort things like that out. I could spend days at them sometimes.

and

Sounds quite boring but it was enjoyable at the time.

are very positive and so significantly encouraging.

History of Mathematics: The following are the responses

What do you know of the history of mathematics?

of the twenty-two students surveyed:

1. With regards to the history of mathematics, there is the famous abacus system which everyone knows but then came the more complex systems of decimal, binary and hexadecimal. Each of these have an advantage in their own use but of the major ones binary (computing) and decimal (mathematics) are most widely used. (TH, 445-A2 (BMS: 17/20))
2. Very little other than a few mathematicians names. (DM, 455-B1 (BMS: 8/20))
3. I don't know much about the history of mathematics only that most of the ideas behind maths came from mathematicians such as Maclaurin, Pythagoras and others. (CAO'D, 410-B1 (BMS: 16/20))
4. I know that the Greeks were very intelligent and liked maths as a few theorems and equations are called after them. (CO'K, 485-B2 (BMS: 16/20))

5. I know absolutely nothing about the history of maths only that numbers evolved from Egyptian or Rome. (PO'C, 405-C1 (BMS: 11/20))
6. I know about Pythagoras, but that is it. (PO'S, 410-C2 (BMS: 13/20))
7. I don't know much about the history of maths. I used to know a good bit because our old teacher used example to us where it all started but I have forgotten most of it. (JD, 325-C2 (BMS: 8/20))
8. Mathematics has a long and distinguished history involving the greatest minds ever known. (JS, 315-C2 (BMS: 10/20))
9. I think maths dates back to Greek times when Archimedes jumped out of the bath and shouted Eureka when he developed Archimedes Principle..... (TR, 305-C2 (BMS: 18/20))
10. I don't know much about the history of Mathematics only that Pythagoras was Greek! (BO'C, 400-C3 (BMS: 13/20))
11. I know very little about the history of Maths apart from the way abacus was used to help solve Mathematical problems. (SD, 370-C3 (BMS: 10/20))
12. I know nothing at all about the History of maths or where it originated from. (CM, 345-C3 (BMS: 8/20))
13. Our Maths teacher often told us the history of each topic were started.

Mostly who specialised in it, and who developed new theories and techniques.

(DH, 320-D1 (BMS: 12/20) – (65% EE))

The responses to

How did mathematics evolve ?

read

1. Astrology, science and mathematics all sort of developed together, as people discovered more they needed something to explain to them and others why things happened the way they did. These systems originated from this curiosity of how things worked, and easiest way to represent and teach this new found knowledge. (TH, 445-A2 (BMS: 17/20))

2. I don't know much about the history of maths or how it evolved.
.
(JS, 470-B1 (BMS: 18/20))

3. I think maths was probably the first subject to evolve as it comes into play in everyday life such as counting objects or people etc. It also comes into play in physics and many other areas. It evolved through various mathematicians who first came up with ideas and then proved them. Mathematics will keep evolving as mathematicians' progress continues and they continue to perfect the information they already have before them. The approximation of π or e .
.

(CAO'D, 410-B1 (BMS: 16/20))

4. I know absolutely nothing about the history of maths only that numbers evolved from Egyptian or Rome. (PO'C, 405-C1 (BMS: 11/20))
5. I believe it evolved quite slowly to begin with but I believe it was the first language used by man. In the last 2 centuries I believe it has evolved very rapidly. (JS, 315-C2 (BMS: 10/20))
6. Theories were formulated. (DH, 320-D1 (BMS: 12/20) - (65% EE))

The responses to

Who in your opinion is a mathematician ?

read

1. Mathematicians, people who solve problems with maths or presents it with a new formula are invaluable to society. They help us all to keep learning new ways of understanding things. $E = mc^2$ being a very famous example, by none other than Albert Einstein. Not everyone who uses or teaches maths are mathematicians. They only use or pass on what was taught to them or learned through others. A real mathematician creates new mathematics. (TH, 445-A2 (BMS: 17/20))
2. A mathematician to me is somebody who has a deep understanding of maths and is capable of deriving equations for use in industry and various calculations. (JS, 470-B1 (BMS: 18/20))

3. In my opinion my secondary school teacher was a mathematician. She was involved in many areas and made the learning of mathematics considerably easier. She knew everything there was for us to know but I'm sure there will be more. (CAO'D, 410-B1 (BMS: 16/20))
4. Pythagoras as his theorem is very useful and effective. (CO'K, 485-B2 (BMS: 16/20))
5. A mathematician to me is someone who knows every object in maths and thinks up of new ways to do maths. (PO'C, 405-C1 (BMS: 11/20))
6. In my opinion a mathematician is a person who finds out new ways of using maths and invents some form of new mathematics. (JD, 325-C2 (BMS: 8/20))
7. A mathematician is someone with an extremely high understanding of mathematics and is someone who can develop equations and formulas to help him understand the world around him. (JS, 315-C2 (BMS: 10/20))
8. I think that a mathematician is someone who has a good general understanding of all aspects of Maths. (BO'C, 400-C3 (BMS: 13/20))
9. A Mathematician is a person who has a very good knowledge of Maths and enjoys doing problems for pleasure. (SD, 370-C3 (BMS: 10/20))
10. I suppose a mathematician is someone who has the ability to solve any maths problem. (CM, 345-C3 (BMS: 8/20))

11. Someone who takes a mathematical topic and masters it until he/she can make no further progress. (DH, 320-D1 (BMS: 12/20) – (65% EE))

The responses to

Do you know of any mathematicians, alive or dead?

read

1. I can't say I know of any mathematicians alive but is said Carol Vordermann is. Dead mathematicians would include Pythagoras, Archimedes.
 . (CAO'D, 410-B1 (BMS: 16/20))
2. A few old Greeks, e.g. Archimedes, Isaac Newton, Einstein.
 . (CO'K, 485-B2 (BMS: 16/20))
3. Einstein was a mathematician.. I think. A lot of Renaissance inventors like De Vinci were mathematicians. I know a man who lives near me he was supposed to be a mathematician but he is now gone mad and he shouts at everyone. Nobody talks to him anymore.
 . (PO'C, 405-C1 (BMS: 11/20))
4. I know about Pythagoras, but that is it. (PO'S, 410-C2 (BMS: 13/20))
5. Isaac Newton and Albert Einstein. (JS, 315-C2 (BMS: 10/20))
6. Newton, De Moivre, Pythagoras. (DH, 320-D1 (BMS: 12/20) – (65% EE))

While the Greek geometer Pythagoras is cited a number of times as is the polymath Archimedes there is no mention of Euclid, whose books on geometry were, after the bible, the second most translated and copied of all time. Neither is there any mention of Leibniz, the co-founder, along with Isaac Newton, of the calculus. It has to be speculated that De Moivre appears because the theorem bearing his name is on the LC syllabus. Thus, when one discounts the inclusion of the great physicist Einstein and the TV presenter Carol Vordermann as mathematicians, the respondents allude to no other mathematicians of the eighteenth, nineteenth or twentieth centuries.

There is no mention of the Arab mathematician al Khwarazimi from whom the words algorithm and algebra originate and whose work on quadratics allowed later mathematicians to express the solution of all quadratic equations in a general algebraic form. Neither is there reference to those Italian mathematicians involved with finding the general solutions of both cubic and quartic equations. However, the mention of Leonardo da Vinci by one student is heartening.

It is rather disappointing that there is not one reference to Carl Friederich Gauss. One might have expected that some student would have heard the celebrated story of how, as an eight-year-old, Gauss summed the first hundred natural numbers. This particular story is a most striking instance of where history can prove a wonderful aid to pedagogy.

Summary: Based on the replies given in relation to recreational mathematics, it can be said that the students involved appear to have received no formal exposure to this aspect of mathematics in the classroom. Furthermore, the responses fail to reveal any awareness on the students' part that puzzles/riddles can be regarded as valid mathematics which can be made to connect with 'school mathematics'. The answers given by this group of students relating to the history of mathematics and mathematicians would have to be regarded collectively as a rather discouraging response to item I of *Other Objectives* of the HLC syllabus which, as already cited, reads:

They (the students) should be aware of the history of mathematics and hence of its past, present and future role as part of our culture.

However, this objective is not acted upon by way of examining it. There has not been a single historical allusion of any kind on the examination papers of the past five years (nor any number of years for that matter). Is it the case that as a result few teachers take the time to mention the great historical figures or is that students do hear about them but quickly forget?

With regard to evincing knowledge of the applications of mathematics, it must be said that while the students' replies indicate that most can point to areas in which mathematics is used, it would have been more encouraging to have seen more detail in the answers given.

On reflection, it would have been useful to have framed a question asking for specific applications of HLC mathematics to real-world responses. The answers elicited to such a question may have proved more revealing. It

would also have been a valuable exercise to have designed questions seeking opinions and perspectives regarding the LC examination and the nature of the textbooks used.

5.2.1 HLC Mathematics Syllabus

The current HLC Mathematics syllabus as set out in the department of education publication *The Leaving Certificate Mathematics Syllabus (Higher Level and Ordinary Level)* is given in Appendix C. The format of this document appears to consist of a comprehensive general Introductory section, a short second section relating to the Higher Level Course in general followed by a third section detailing the syllabus content.

The first section begins by describing the context of mathematics, then sets out the general Aims associated with the teaching of mathematics and then expounds on general and fundamental objectives in great detail. The final subsection is devoted to the principles that guide the course design. The second section begins by elaborating on the rationale, style and aims of the higher course before outlining the prior knowledge assumed on behalf of the student. It then describes the assessment objectives and the structure and content of the higher course. The third section gives a complete breakdown of the different syllabus elements under the headings

1. Algebra
2. Geometry

3. Trigonometry
4. Sequences and Series
5. Functions and Calculus
6. Discrete Mathematics and Statistics

It also indicates the topics available in the **Options** section as

1. Further Calculus and Series
2. Further Probability and Statistics
3. Groups
4. Further Geometry

and describes briefly the syllabus entries for each.

5.3 The 2003 Mathematics Papers

Appendix D contains Paper I and Paper II of the 2003 Mathematics Examination for the Higher Leaving Certificate.

5.3.1 General Remarks about Paper I

The overriding impression this paper makes on this writer is that its predominant focus is on examining mathematical techniques in context-free fragments. As such it is principally an exercise in drill which does not address the ‘why’ of mathematics.

The following table gives the general type of each of the questions asked and the number of each type:

Purely technical	37
Intra-mathematical applications	1
Extra-mathematical applications	0

Table 5.1: Question type.

While question 8(c)(ii) is about explaining why the volume of a sphere of radius r has volume $4/3\pi r^3$, there are very few other question segments, if any, which could be judged as mathematics explaining or revealing anything either internal or external to itself.

While it is subjective to ask if there are any questions that ‘vibrate’ by engaging one’s interest or appealing to curiosity, many of the questions have a bland seriousness to them which fail to convey the interest or even sense of wonder/enjoyment of which mathematics is capable. This writer’s major criticism is that while the questions examine essential skills, the relevance of these skills to ‘problems in context’ is not visible in the structure of the questions. It can be said that, between them, the questions test adequately the skills of

- 1. algebraic manipulation

2. differentiation
3. anti-differentiation

but do not test the important skills of

1. formulation
2. inference
3. proof
4. description

It is proposed now to look at each of the questions individually, segment by segment. Since the contention is that the questions being asked are unduly technical and context-free, the opportunity is taken whenever it presents itself to suggest how a minor change might connect the question to a context or give it a more applied appearance. Brief pointers to classroom activities are provided.

5.3.2 Analysis Question by Question

- 1 (a) Express the following as a single fraction in its simplest form:

$$\frac{6y}{x(x+4y)} - \frac{3}{2x}$$

A purely technical question requiring the addition of two rational expressions in x and y followed by simplifying as far as possible. This is an isolated fragment examining algebraic manipulation.

The question is not set in any context nor is the student being asked to draw any conclusion from the simplification which might justify it. Such a question would be much more powerful if set in some framework where a sought-for conclusion would be obvious only after the requested simplification. An example of this type is related to the standard deviation σ of a random variable uniformly distributed on the interval $[a, b]$. Here, by definition,

$$\sigma^2 = \frac{1}{b-a} \frac{b^3 - a^3}{3} - \left(\frac{b+a}{2} \right)^2$$

When this expression is simplified one gets

$$\sigma = \frac{b-a}{\sqrt{12}}$$

The simplified result shows that σ – the measure of spread of the variable – is proportional to $b-a$ which is the length of the interval $[a, b]$. This is a geometrically plausible result – the larger the interval on which values can be assumed at random, the larger one would expect the measure of dispersion σ to be. Without simplifying the expression fully, the geometrically important interpretation is not visible. So the significant lesson is that the simplification leads to greater understanding;

in this case the ability to see that the expression defining the spread σ is indeed geometrically related to a spread – the spread of the interval in question. Hence, one simplifies, not just for convenience but often for further insight.

One might frame a question relating to this example along the lines:

When numbers are being generated at random from the interval $[a, b]$, the standard deviation σ of these numbers is known to satisfy

$$\sigma^2 = \frac{1}{b-a} \frac{b^3 - a^3}{3} - \left(\frac{b+a}{2} \right)^2$$

Find σ in simplest form.

Hence interpret σ geometrically in terms of the interval $[a, b]$.

Another example, less applied but closer to HLC level might be:

If

$$f(a, b) = \frac{1}{2} [(a+b)^2 + 3a + b]$$

show that

$$f(a, b) = \frac{(a+b)(a+b+1)}{2} + a$$

Hence, or otherwise, deduce that $f(a, b)$ is always a natural number whenever a and b are.

Deriving the second expression from the first tests algebraic skill. Alternatively, of course, one can expand both versions of $f(a, b)$ to see that they are equivalent.

However, the purpose of the simplification is that it is easy to establish that $f(a, b)$ is a natural number whenever a and b are from the second expression but much more difficult to do so using the first expression.

For The Teacher (FTT): The real significance of the algebraic simplification and the subsequent question is that it can be used to show that the set of (positive) rational numbers can be injectively embedded as a subset of the natural numbers – a startling and intriguing result which should find a place in a topic such as: ‘Ideas of Mathematics’. If this were done, the question posed above could be expanded to a full-blown rich question whose purpose is obvious and which students would have heard discussed in class.

1(b) (i) $f(x) = ax^2 + bx + c$ where $a, b, c \in \mathbb{R}$.

Given that k is a real number such that $f(k) = 0$ prove that $x - k$ is a factor of $f(x)$.

This examines the theorem about roots/factors of a quadratic expression. A technical question which finds a ‘technical application’ in

1(b) (ii) Show that $2x - \sqrt{3}$ is a factor of $4x^2 - 2(1 + \sqrt{3})x + \sqrt{3}$ and find the other factor.

i.e. show that a certain linear expression is a factor of a given quadratic and hence find the other factor.

However, one is given no reason why one might want to do this. Consequently parts b(i) and b(ii) solely examine mathematical technique.

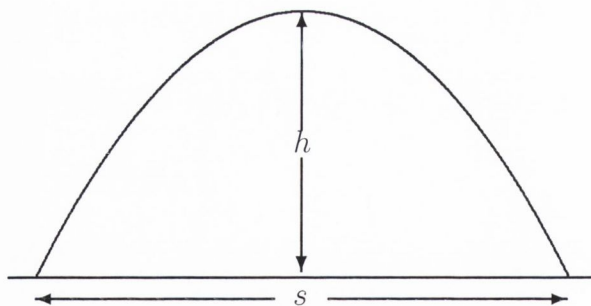
A question such as this can be given a real-world context. By way of establishing some needed preliminary theory one might ask:

Show that if α and β are roots of the general quadratic $p(x) = ax^2 + bx + c$ then

$$p(x) = a(x - \alpha)(x - \beta)$$

Then one might state:

It is desired to design an entrance in the form of a parabolic arch shown:



with prescribed span s and a prescribed height h .
Determine the area of the entrance in terms of s and h .

Figure 5.1: Parabolic arch.

Most students will allow that there is some sense to this type of question. It is not entirely contrived, is visual and solving it draws on the theoretical knowledge of the first part of the question. Furthermore, the question connects with integration in order to determine area. Thus it is far from being an isolated context-free fragment but blends nicely two different aspects of the syllabus.

Specifically, placing the origin at the left-end of the arch gives

$$p(x) = a(x - 0)(x - s) = ax(x - s)$$

as the equation of a family of parabolas passing through $(0, 0)$ and $(s, 0)$. Since this parabola also passes through the point $(s/2, h)$ it follows that

$$h = a \frac{s}{2} \left(\frac{s}{2} - s \right) = -a \left(\frac{s}{2} \right)^2$$

Hence

$$p(x) = \frac{4h}{s^2} x(s - x)$$

is the equation of *the* given parabola.

Because this parabolic arch stands on the interval $[0, s]$ the area enclosed by it and the horizontal axis is given by

$$\int_0^s p(x) \, dx = \frac{4h}{s^2} \int_0^s (sx - x^2) \, dx$$

$$\begin{aligned} &= \frac{4h}{s^2} \left[s \frac{x^2}{2} - \frac{x^3}{3} \right]_0^s \\ &= \frac{4h}{s^2} \left[\frac{s^3}{2} - \frac{s^3}{3} \right] \end{aligned}$$

$$\Rightarrow \text{Area under arch} = \frac{2}{3}hs$$

With this form of question a considerable amount of technical theory is examined in a context. Specifically,

1. A mathematical theorem relating to the roots of a quadratic equation.
2. Knowing that a parabola is described by a quadratic polynomial.
3. Introduction of a coordinate system and direct use of the theorem to formulate the appropriate quadratic polynomial ‘interpolating’ at the three ‘given’ points.
4. Formulation and evaluation of a definite integral over an appropriate interval.

Something substantial has been achieved; if nothing else one sees an amount of mathematical technique brought to bear on an applied problem. The problem is also visual which makes it livelier and is not as open to questions of the type:

“Why do I want to know this?” or “Where is this used?”

Observations for extending the question:

- 1. This answer is dimensionally correct. This aspect is not evident from a specific question where the dimensions s and h are assigned numerical values at the outset.
- 2. This is an ancient result which Archimedes might have stated as: “The area of the parabolic arch is two-thirds that of its bounding rectangle.”

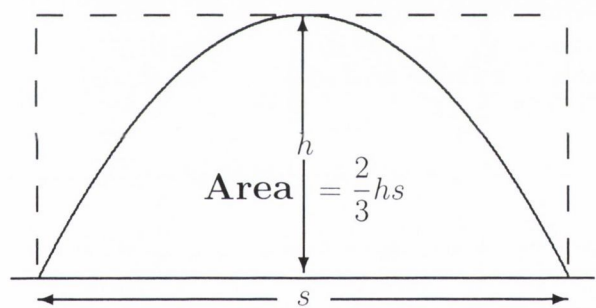


Figure 5.2: Parabolic arch and bounding rectangle.

- 3. The area is linear in both h and s .

In fact, to explain and emphasise for the students what is meant by the phrase “the area is linear in both h and s ”, the teacher might record the result obtained about the area, A , of the arch as

$$A(s, h) = \frac{2}{3} s h$$

– that is, display A as the simple function of the two independent variables s and h that it is. With this in place he can then expand on the terminology used by having the students show that

$$A(ks, h) = k A(s, h) \quad ; \quad A(s, lh) = l A(s, h)$$

for arbitrary constants k and l . Then by choosing specific values of k and l along with discussing what is being said in each case, he can further embed the important notion of linearity.

This exercise is also valuable because it provides the teacher with an opportunity to impress on his students how efficient function notation can be by the manner in which it can express briefly what it would take paragraphs to say in words. To reinforce this point the teacher might set the students the task of explaining, via the notation, why doubling h and halving s leaves the area unchanged, even if the effect is to radically alter the shape of the arch in question.

4. What is the ‘best’ shape aesthetically? engineering-wise?
5. How does the enclosed area compare with that enclosed by a semi-circular arch on the same base?

In the terminology of PISA an integrated problem of this type is from the *reflections* cluster.

1(c) The real roots of $x^2 + 10x + c$ differ by $2p$ where $c, p \in \mathbb{R}$ and $p > 0$.

(i) Show that $p^2 = 25 - c$.

(ii) Given that one root is greater than zero and the other is less than zero, find the range of possible values of p .

To answer these questions requires theoretical knowledge and an amount of algebraic confidence. Part (ii) examines ability at handling algebraic inequalities. The question is context-free.

In a different framework one might see all three parts of this question, relating entirely to quadratic polynomials as parts (b) and (c) do here. In this way the question would be underpinned by a single theme and might be cross-fertilised with other aspects of the course as is done in the recommendation which followed part (b).

2 (a) Solve the simultaneous equations:

$$3x - y = 8$$

$$x^2 + y^2 = 10$$

A 2×2 system of simultaneous equations; the first representing a line, the other a circle. This question tests algebraic technique without the provision of a context.

The question could have been enlivened by asking for a geometrical interpretation of the solutions or made it more substantial still, as well as visual, by asking for a sketch of the solutions in relation to the two equations. In this way one could have also tested a geometric understanding of the system of equations.

The question might have asked afterwards:

Why are there two solutions of the form (x, y) ?

An alternative formulation might be:

Where does the line $3x - y = 8$ intersect the circle $x^2 + y^2 = 10$?

Here the student needs to understand that this question is equivalent to solving the given system of simultaneous equations. This formulation goes a very short way towards the first part of the aim which reads:¹⁴

developing their modelling abilities, problem-solving skills,
creative talents, and powers of communication;

2 (b)(i) Solve for x :

$$|4x + 7| < 1$$

¹⁴No. 2 aim.

This question appears not to be related to part (a). It is concerned with the solving of an inequality which tests understanding of the very useful absolute value, $|\cdot|$, notation.

The problem is not given in any context and tests algebra alone. It might have been enhanced by requiring a sketch of the open interval of the real line which it defines.

An easier (but perhaps more important) question of this type might be:

Sketch the interval of the real line defined by $|x| < 1$ or the intervals of the real line described by $|x| > 1$.

A better attempt at displaying the absolute value notation in a context is provided by the following question:

To determine whether a coin is fair (has an equal probability of landing tails up or heads up), an experimenter tosses it 100 times and records the number of heads h . The coin is declared unfair if

$$\left| \frac{h - 50}{5} \right| \geq 1.645$$

For what values of h will the coin be declared unfair?

The correct setting for this question is in relation to the binomial distribution where it can be interpreted in terms of deviations from the mean.

FTT: The absolute value notation recommends itself on the grounds of succinctness and is regularly employed in the definition of piecewise

continuous functions. For example, the symmetrical pulse function of amplitude 1 is defined by

$$\Pi(t) := \begin{cases} 1 & ; \quad 2|t| \leq 1 \\ 0 & ; \quad 2|t| > 1 \end{cases}$$

Here the role of the absolute value function is quite visible in its role of specifying the values the function takes over different intervals. This definition is very close to a form which could be given to a computer for graphing purposes. On the other hand, without the use of this function the pulse function Π might be defined as follows:

$$\Pi(t) := \begin{cases} 1 & ; \quad -1 \leq 2t \leq 1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

However, this form has either to be written as above or be expanded further to

$$\Pi(t) := \begin{cases} 0 & ; \quad 2t < -1 \\ 1 & ; \quad -1 \leq 2t \leq 1 \\ 0 & ; \quad 2t > 1 \end{cases}$$

so that it can be implemented by a computer package.

This extended version, eliminating the pseudo-mathematical ‘otherwise’ for the purposes of being machine intelligible shows why the first description is the preferred one.

An even more striking example illustrating the descriptive conciseness of the $|\cdot|$ notation is the definition of the symmetrical triangular pulse function $\Lambda : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\Lambda(t) := \begin{cases} 1 - |t| & ; \quad |t| \leq 1 \\ 0 & ; \quad |t| > 1 \end{cases}$$

Note that the $|\cdot|$ notation is being used in two roles

- 1. to specify the domain of the function
- 2. to describe the function

It is a useful, if substantial, exercise for a student to unpack this definition so as to be able to sketch the graph of the function. This requires some thought. As a first step one might opt for the less concise, but perhaps more accessible piecewise definition:

$$\Lambda(t) := \begin{cases} 1 + t & ; \quad -1 \leq t \leq 0 \\ 1 - t & ; \quad 0 \leq t \leq 1 \\ 0 & ; \quad |t| > 1 \end{cases}$$

Here, in this partially ‘opened out’ version, it is still useful to retain the $|\cdot|$ notation. To eliminate it would require replacing the last line of this definition with two more lines, thus lengthening the definition even more and unnecessarily.

2 (b)(ii) Given that $x^2 - ax - 3$ is a factor of $x^3 - 5x^2 + bx + 9$ where $a, b \in \mathbb{R}$, find the value of a and the value of b .

This question is not related to (b) (i). It is concerned with the determination of two parameters a and b , which are coefficients in a quadratic

and cubic respectively where (the reader is told that) the quadratic is a factor of the cubic.

It can be viewed as an algebraic exercise in long division which is followed by the solving of two simple equations, one linear in the parameter a alone and the other linear in a and b .

This question examines algebraic technique and the understanding of factors. It is not set in any context.

This question could have been extended to ask for the complete factorisation of the cubic. Doing so would have rendered the problem complete, in the sense that the cubic is ‘split’ or ‘completely reduced’ to a product of linear factors.

2 (c) (i) Solve for y :

$$2^{2y+1} - 5(2^y) + 2 = 0.$$

Another context-free question testing a knowledge of indices and the factorising of a quadratic into linear terms.

-
- 2 (c) (ii) Given that $x = \alpha$ and $x = \beta$ are the solutions of the quadratic equation $2k^2x^2 + 2ktx + t^2 - 3k^2 = 0$, where $k, t \in \mathbb{R}$ and $k \neq 0$, show that $\alpha^2 + \beta^2$ is independent of k and t .
-

This work will discuss in its core section ‘the dipstick calibration problem’ as an application of the Newton-Raphson algorithm. This particular problem shows students how a family of equations involving a parameter can arise in a practical way. By contrast, the examples encountered in the above questions seem contrived and artificial.

- 3 (a) Evaluate $\begin{pmatrix} 1 & -2 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.
-

This is a technical exercise in vector-matrix multiplication.¹⁵ This context-free segment is the only one of its kind on Paper I. There are no further references to matrices or vectors.

- 3 (b) (i) Given that $z = 2 - i$, calculate $|z^2 - z + 3|$ where $i^2 = -1$.
-

This question examines the multiplication, addition and subtraction of complex numbers along with obtaining the modulus of a complex

¹⁵Evaluating a quadratic form.

number.

-
- 3 (b) (ii) k is a real number such that $\frac{-1 + i\sqrt{3}}{-4\sqrt{3} - 4i} = ki$.
Find k .
-

This question requires solving a linear equation involving complex numbers by comparing real and imaginary parts.

- 3 (c) $1, \omega, \omega^2$ are the roots of the equation $z^3 - 1 = 0$.
- (i) Prove that $1 + \omega + \omega^2 = 0$.
- (ii) Hence, find the value of $(1 - \omega - \omega^2)^5$.
-

These questions concern the cubed roots of unity.

This is a technique question which can be slogged out by direct evaluations but which can also be answered quite cleverly using the factorisation of $z^3 - 1 = (z - 1)(z^2 + z + 1)$.

A nice question allowing a student to show some flair; as such it is an important question. However, there is no attempt to link it to a context.

The level of knowledge needed to answer these complex numbers questions amounts to knowing the arithmetic of complex numbers and the

modulus of a complex number. While it is undoubtedly important that students can carry out calculations such as these correctly and with ease, it would also be useful to test for some geometric understanding of complex numbers, perhaps a circular region defined by an equality or inequality.

- 4 (a) Express the recurring decimal $0.252525\dots$ in the form p/q where $p, q \in \mathbb{N}$ and $q \neq 0$.
-

This question asks that the repeated decimal $0.\overline{25}$ be expressed as a rational number. A technical question, it requires a simple but attractive trick if one is to avoid summing a geometric series.

FTT: Periodic decimal expansions are used to generate pseudo-random numbers, a topic requiring a little elementary number theory but which would make for a reasonably accessible project for an able sixth year student. It was this topic which began Gauss's fascination with number theory.

- 4 (b) In an arithmetic series, the sum of the second term and the fifth term is 18. The sixth term is greater than the third term by 9.
- (i) Find the first term and the common difference.

- (ii) What is the smallest value of n such that $S_n > 600$, where S_n is the sum of the first n terms of the series.
-

Part (i) of this question could be described as a word-problem concerning an arithmetic progression. In order to be able to solve it, the student must first be able to formulate a 2×2 system of simultaneous equations which must then be solved. In this case the system is simple yet the question is useful since formulation is a vital skill.

Part(ii) requires the solution of (i). It is clearly designed to ‘work out’ easily.

A ‘practical’ alternative might have been

A company invests €10000 in its first year and in each subsequent year the amount is increased by €1000.
How much is invested in the tenth year.
What is the *least* number of years required for the total invested to amount to €186,000?

- 4 (c) (i) $u_1, u_2, u_3, u_4, u_5, \dots$ is a sequence where $u_1 = 2$ and

$$u_{n+1} = (-1)^n u_n + 3.$$

Evaluate u_2, u_3, u_4, u_5 and u_{10} .

A question which gives a starting value and a first order recurrence relation and asks for the next four terms and the tenth value.

After evaluating the next four terms, one finds that the fifth term matches the initial value. Thus the sequence is periodic and so one can find the tenth term easily.

In obtaining their answers the students might have been asked to indicate the nature of the underlying sequence.

This purely technical question could have been turned into a finer one had the students been asked to provide an algorithm (step-by-step procedure) for determining *any* term in the sequence. Doing this would have rounded out the question to a complete one where a general conclusion is drawn about the sequence.

Specifically, the first part of the question leads the student to discover that the sequence is periodic with principal period 4. It then checks to see if the student has observed this periodicity by asking for the tenth term. It is possible that some students will have answered this question by recursively finding all the intermediate terms and so arrive at the correct answer for the tenth term without ever having made use of the periodic structure of the sequence. Perhaps it would have been better to ask for the 100th term, for then the student would need to discover and exploit the periodicity.

It would be closer to what occurs in any process of generalisation to ask the student to say how he or she would calculate any term in

the sequence. This is not hard to answer but does require a written explanation as to how this is to be achieved.

As already mentioned, there is very little by way of written explanations expected from students in the current HLC mathematics examinations. In this particular case, one would hope that the student would answer that the n^{th} term in the sequence can be determined from the remainder n leaves after division by 4. Specifically, if upon division by 4

$$\begin{array}{llll} n \text{ leaves a remainder of 1 then} & u_n & = & u_1 & = & 5 \\ n \text{ leaves a remainder of 2 then} & u_n & = & u_2 & = & -2 \\ n \text{ leaves a remainder of 3 then} & u_n & = & u_3 & = & 1 \\ n \text{ leaves a remainder of 4 then} & u_n & = & u_4 & = & 2 \end{array}$$

This answer is algorithmic and covers all possibilities. It is not difficult and tests if a student is able to draw a general conclusion from an examination of particular cases.

FTT: The above elaboration provides the teacher with an ideal reason for introducing the mod notation. Using it one renders the above as

$$u_n = u_{n \bmod 4}$$

for each $n \in \mathbb{N}$.

The conciseness of this reveals the usefulness (and necessity) of the notation to maximum effect.

Perhaps, this example also shows why it is preferable to begin sequences at $n = 0$.

A more advanced question might ask that the observed periodicity of the sequence be established using the recurrence relation itself. This involves some algebra and a facility with indices but is a fine exercise in showing the efficaciousness of the function notation.

Here it must be shown, for each $n \in \mathbb{N}$, that

$$u(n+1) = (-1)^n u(n) + 3 \Rightarrow u(n+4) = u(n)$$

The proof goes as follows:

$$\begin{aligned}
 u(n+4) &= (-1)^{n+3} u(n+3) + 3 \\
 &= (-1)^{n+3} [(-1)^{n+2} u(n+2) + 3] + 3 \\
 &= -u(n+2) + 3[(-1)^{n+3} + 1] \\
 &= -[(-1)^{n+1} u(n+1) + 3] + 3[(-1)^{n+3} + 1] \\
 &= (-1)^{n+2} u(n+1) + 3[-1 + (-1)^{n+3} + 1] \\
 &= (-1)^{n+2} [(-1)^n u(n) + 3] + 3[-1 + (-1)^{n+3} + 1] \\
 &= u(n) + 3[(-1)^{n+2} - 1 + (-1)^{n+3} + 1] \\
 \Rightarrow u(n+4) &= u(n)
 \end{aligned}$$

as asserted.

Note that this result is independent of the starting value. It is the nature of the recurrence relation alone which gives the sequence its periodic structure. Furthermore, the additive constant 3 has no bearing on the periodicity. It can be replaced by any other value without affecting the periodicity.

The section of the syllabus dealing with difference equations is discussed in some detail later where it is recommended that the focus should be on first order recurrence relations drawn from recreational or applied contexts. [Appendix F which contains the HLC questions set on difference equations up to 2003 reveals all of the equations to be of the second order.] One example from recreational mathematics that one might study in class is the Tower of Hanoi puzzle. Here the task is to determine, $t(n)$, the minimum number of moves needed to reconstruct a tower of size n subject to the restrictions of the game.

A typical exam question relating to this entry might require a description of the simple but insightful reasoning that leads to the recurrence relation and specified initial condition:

$$t(n) = 2t(n - 1) + 1 \quad ; \quad t(1) = 1$$

which allows $t(n)$ to be calculated recursively.

A question might ask the student to determine how many moves it will take a child, using optimal play to construct a tower of size 8.¹⁶

¹⁶The form of the puzzle sold commercially has a tower with eight discs.

It is also suggested that the syllabus include a discussion of the first order recurrence difference equation relating to mortgage repayments. This is perhaps one of the most useful applications of mathematics lying outside of the scientific situation that bears on a student's (future) personal life and which can be undertaken at the level of senior-cycle.

- 4 (c) (ii) a, b, c, d are the first, second, third and fourth terms of a geometric sequence, respectively.

Prove that $a^2 - b^2 - c^2 + d^2 \geq 0$.

A purely algebraic problem concerning a geometric sequence which requires some skill to solve.

A question of the following more applied type might have been asked:

In a drilling process it is found that each successive drill is four fifths of the previous one.

How many drillings would it require to drill to a depth of 800 metres if the initial drilling was 180 metres?

What is the maximum depth one could hope to achieve with this drilling process?

- 5 (a) Solve for x :

$$x = \sqrt{7x - 6} + 2.$$

This is a test of algebraic technique. Here squaring both sides of a slightly modified version of the original equation leads to a quadratic equation which has two distinct roots. However, when the student comes to check these putative solutions of the original equation, only one of the roots satisfies the original equation. Any example, such as this one, which highlights the importance of checking purported solutions is very valuable and may encourage a student always to check his work.

This problem could be extended by asking for an explanation, either analytical or graphical, as to why the original equation has only one solution while the associated quadratic has two solutions.

The question would be all the more appealing if it were set in some context.

-
- 5 (b) Use induction to prove that 8 is a factor of $7^{2n+1} + 1$ for any positive integer n .
-

This question on induction tests technique. To draw attention to the fact that the result only holds in general for odd exponents it might also have been asked:

Does this result hold in general if $2n + 1$ is replaced by $2n$ for each $n \in \mathbb{N}$? Explain.

This question tests if a student understands the role of a counterexample. It also requires that he communicate his negative result using a certain amount of written explanation along the lines: “The result does not hold in general as can be seen by examining the case $n = 1$, when $7^{2n} + 1 = 7^2 + 1 = 50$ – a number not exactly divisible by 8.”

- 5 (c) Consider the binomial expansion of $\left(ax + \frac{1}{bx}\right)^8$, where a and b are non-zero real numbers.
- (i) Write down the general term.
- (ii) Given that the coefficient of x^2 is equal to the coefficient of x^4 , show that $ab = 2$.
-

These two parts examine the general term of a binomial expansion and a related algebraic problem.

The emphasis here is on technique.

- 6 (a) Differentiate $\sqrt{1 + 4x}$ with respect to x .
-

A problem in differentiation emphasising technique. The question is given without context and no use is made of the result.

- 6 (b) Show that the equation $x^3 - 4x - 2 = 0$ has a root between 2 and 3.

Taking $x_1 = 2$ as the first approximation to this root, use the Newton-Raphson method to find x_3 , third approximation. Give your answer correct to two decimal places.

An elementary problem to show that a cubic equation has a root in a given interval, followed by an application of the Newton-Raphson (NR) method to approximate this root to a certain number of decimal places using the initial approximation $x_1 = 2$. It must be observed that the question does not even ask for a verification that the approximation obtained is indeed ('very close to') a root of the given equation. Since the cubic is simply given along with an estimate of the desired root, the solving of this problem is routine.

Students need to be made aware that a great deal of equations encountered in any form of quantitative investigation are not amenable to solution via formulae such as the quadratic formula, that the availability of such a formula is the exception. The important educational lesson that they should carry with them is that *most equations are hard to solve* and often require the assistance of a computer to execute the steps of the algorithm chosen to solve them.

The NR algorithm is such a useful one that it deserves realistic problems

for its proper illustration. Though by no means desirable, it would be preferable to demonstrate this algorithm on one single cubic derived from a realistic context rather than indulge in the current activity even if this meant that at examination time, answers were to be no more than the regurgitation of the teacher's classroom treatment. At least one could be assured that the students were exposed to an example which showed the need for and power of the NR method. The business of making the NR algorithm the basis of a Topic is taken up in the core section. There, in addition to discussing a number of naturally arising cubics, other equations derived from applied contexts and not readily soluble by known formulae are provided as genuine grist for the NR mill.

6 (c) The function $f(x) = \frac{1}{1-x}$ is defined for $x \in \mathbb{R} \setminus \{1\}$.

(i) Prove that the graph of f has no turning points and no points of inflection.

(ii) Write down a reason that justifies the statement " f is increasing at every value of $x \in \mathbb{R} \setminus \{1\}$ ".

(iii) Given that $y = x + k$ is a tangent to the graph of f where k is a real number, find the two possible values of k .

Part (i) is quite a sophisticated one, employing calculus reasoning to

deduce the nature of a graph.

Part (ii), by asking for a reason, tests the understanding of concepts, an essential element of mathematical knowledge.

Part (iii) requires some skill to answer and the techniques needed prove useful in other problems.

The question might have asked for a graph of the function under discussion after the material in parts (i) and (ii) are established. Had this been done, then the sense of (iii) becomes more evident.

This question would have benefitted by being made more visual.

If the topic of transformation of functions were part of the current provision then a question such as this could be treated differently. If the object of the exercise is to understand the nature of the graph of the function f so as to be able to sketch it, then this understanding can be arrived at from a knowledge of the graph of $g : \mathbb{R} \rightarrow \mathbb{R}$ defined for each $x \neq 0$ by

$$g(x) = \frac{1}{x}$$

One would expect that every student should be able to sketch a graph of this function. In this case, noting that

$$f(x) = -g(x - 1)$$

indicates that the graph of f is that of g translated horizontally to the right by 1 unit and reflected in the horizontal axis.

Thus, one can see that there are no turning points, no points of inflection and why there should be two values of k for which the graph of $y = x + k$ is tangent to the graph of f . It is also clear why one value of k will be positive with the other value negative and greater in absolute value than the positive value.

The current syllabus does not make it clear exactly which functions (along with their respective graphs) students are expected to be familiar with. For the higher course one might expect that such a list might include the

1. general constant function
2. general linear function
3. general quadratic function
4. general cubic function
5. reciprocal function $1/x$
6. square-root function $\sqrt{x}$
7. trigonometric functions $\cos x$, $\sin x$, $\tan x$
8. exponential function
9. logarithmic function
10. the function $1/(x^2 + 1)$, e^{-x^2} and $\ln(1 + x)$

7 (a) Differentiate each of the following with respect to x :

$$(i) \quad \cos^4 x \qquad (ii) \quad \sin^{-1} \frac{x}{5} .$$

This is an exercise in the techniques of differentiation, requiring the use of a table and the chain rule.

However, current students' strong base in techniques can be harnessed to teach them the lesson of how useful it is to have a 'calculus' or set of working rules which cover all contingencies. And the use is that one can teach them to a computer which can then be programmed to execute all manner of differentiations far more efficiently than any human. Inform a mathematical package (in the language of its underlying code) of the general rules of differentiation (sum, difference, product, quotient and chain) and of the specific rules relating to differentiating powers, the trigonometric functions $\cos$ and $\sin$, the exponential and logarithmic functions and it will produce the derivatives of even the most convoluted combinations of these functions.

The important lesson here for the student is that knowing the calculus rules allows humans to harness the computer to perform differentiation tasks routinely and so spare drudgery. The educational point is that

mathematics instructs the machine. Here the stress is on the idea that it is the human who identifies the correct working rules along with the relevant information pertaining to the different classes of functions which he passes to the computer to carry out the calculations, a master and slave scenario.

7 (b) (i) The parametric equations of a curve are:

$$x = \cos t + t \sin t$$

$$y = \sin t - t \cos t \quad \text{where} \quad 0 < t < \frac{\pi}{2}$$

Find $\frac{dy}{dx}$ and write your answer in simplest form.

Here the student needs to know that $dy/dx = \dot{y}/\dot{x}$ and simplifies via $\sin t/\cos t$ to $\tan t$.

The derivative expression is not specified as a function of x but in terms of the parameter t . One might ask why the point $t = 0$ is omitted from the domain of the parameter or, for that matter, the point $t = \pi/2$. It might have made a little more sense to ask for

$$\lim_{t \rightarrow \frac{\pi}{2}^+} \frac{dy}{dx}$$

and that this limit be interpreted geometrically.

Here the tangent line is parallel to the vertical axis at the ‘point’ $t = \pi/2$. Asking a question such as this imbues the question with a little more rationale.

Would the students who are being asked this question, know the shape of the curves given by the following parametrisations?

$$\begin{aligned}(\alpha): x(t) &= \cos t & ; & \quad y(t) = \sin t & ; & \quad 0 \leq t < 2\pi \\(\beta): x(t) &= \cos 2t & ; & \quad y(t) = \sin 2t & ; & \quad 0 \leq t < 2\pi \\(\gamma): x(t) &= t & ; & \quad y(t) = t^2 & ; & \quad 0 \leq t < 1\end{aligned}$$

Visualisation and Interest: There are a great many very intriguing curves described parametrically. While it is most unlikely that one could hope to derive these parametrisations in all cases, though parametrising some, such as the cycloid, can prove most instructive, generating these curves graphically using a mathematical package reveals their value by producing spectacular curves. This visual aspect breathes life into the equations and brings home to the student what can lie latent or implicit in a set of equations.

7 (b) (ii) Given that $\frac{1}{x} + \frac{1}{y} = \frac{1}{6}$, find the value of $\frac{dy}{dx}$ at the point $(2, -3)$.

A nice technical exercise (partially related to part (i)) in implicit differentiation to evaluate dy/dx at a specific point.

But could the student explain in general, why it is a very useful device

to be able to obtain an expression for dy/dx implicitly? Perhaps he or she could be asked this question and be expected to answer by illustrating in a specific case.

With regard to this example, implicit differentiation gives

$$\begin{aligned} -\frac{1}{x^2} - \frac{1}{y^2} \frac{dy}{dx} &= 0 \\ \Rightarrow \frac{dy}{dx} &= -\frac{y^2}{x^2} \\ \Rightarrow \frac{dy}{dx} \Big|_{(2,-3)} &= -\frac{9}{4} \end{aligned}$$

On the other hand, proceeding to first determine y as an explicit function of x gives

$$\begin{aligned} \frac{1}{x} + \frac{1}{y} &= \frac{1}{6} \\ \Rightarrow 6y + 6x &= xy \\ \Rightarrow (x - 6)y &= 6x \\ \Rightarrow y &= \frac{6x}{x - 6} \quad ; \quad (x \neq 6) \end{aligned}$$

Then, provided $x \neq 6$,

$$\frac{dy}{dx} = \frac{(x - 6)6 - 6x}{(x - 6)^2} = -\frac{36}{(x - 6)^2}$$

with

$$\frac{dy}{dx} \Big|_{(2,-3)} = \frac{(-36)}{(-4)^2} = -\frac{9}{4}$$

The implicit and explicit treatments just given make it very clear that the implicit approach to differentiation gets to the desired derivative more efficiently.

However, as a result of obtaining y as an explicit function of x , one could sketch the graph of the function making use of the observation that

$$y = \frac{6x}{x-6} = 6 + \frac{36}{x-6}$$

One could turn this into a *compare and contrast* question seeking an opinion from the student - a judgment call.

1. Ask to quantify the comparison by counting the number of steps and facts required by each of the two methods.
2. A good student might argue that whereas the explicit method is longer, pupils feel more comfortable and surer of their footing tackling the problem in this manner. He or she may also point out that the explicit approach yields

$$y = \frac{6x}{x-6} = 6 + \frac{36}{x-6}$$

which can be harnessed to draw the graph of y versus x and so help illustrate the result that

$$\left. \frac{dy}{dx} \right|_{(2,-3)} = -\frac{9}{4}$$

7 (c) (i) Given that $y = \log \frac{1+x^2}{1-x^2}$ for $0 < x < 1$, find $\frac{dy}{dx}$ and write your answer in the form $\frac{kx}{1-x^k}$ where $k \in \mathbb{N}$.

A technical question in differentiation requiring a knowledge of

$$\frac{d}{dx} [\ln |f(x)|] = \frac{f'(x)}{f(x)}$$

– No context is provided.

7 (c) (ii) Given that $f(\theta) = \sin(\theta + \pi) \cos(\theta - \pi)$, find the derivative of $f(\theta)$ and express it in the form $\cos n\theta$ where $n \in \mathbb{Z}$.

This question tests differentiation of the trigonometric functions to include the compound-angle formulae. No context is provided.

Sophisticated solutions:

$$\begin{aligned} f(\theta) &= \sin(\theta + \pi) \cos(\theta - \pi) \\ &= (-\sin \theta)(-\cos \theta) \\ &= \frac{1}{2} \sin 2\theta \\ \Rightarrow f'(\theta) &= \cos 2\theta \end{aligned}$$

If the simplifications are not made at line 2, then a longer process of

solution might proceed via the product rule as follows:

$$f'(\theta) = \cos(\theta + \pi) \cos(\theta - \pi) - \sin(\theta + \pi) \sin(\theta - \pi)$$

Two different approaches possible here: The quickest is to recognise that the rhs here is of the form

$$\cos \alpha \cos \beta - \sin \alpha \sin \beta$$

where $\alpha = \theta + \pi$ and $\beta = \theta - \pi$ and simplifies to $\cos(\alpha + \beta)$. Thus

$$f'(\theta) = \cos 2\theta$$

as in the slick solution.

There is a danger that students will take a longer route and expand all four expressions in $f'(\theta)$ and then simplify by eventually taking a step similar to the one just taken.

The ‘bright student’ who sees that the ‘correct approach’ to solving this problem is to first write $f(\theta)$ in a more natural form, might comment that as it stands $f(\theta)$ is in artificial form, something which detracts from the realism of the question posed.

$$8 \text{ (a)} \quad \text{(i)} \quad \int (x^3 + 2) dx \quad \text{(ii)} \quad \int e^{7x} dx.$$

A purely technical question requiring two simple anti-differentiations – at the level of reproduction.

8 (b) (i) Evaluate $\int_0^1 \frac{2x}{\sqrt{1+x^2}} dx$.

A question requiring the evaluation of a definite integral via substitution. No context is provided.

8 (b) (ii) By letting $u = \sin x$, evaluate $\int_0^{\pi/2} \cos x \sin^6 x dx$.

This is a technical question given without context.

8 (c) (i) Show that $\int_a^{2a} \sin 2x dx = \sin 3a \sin a$.

This question involves a simple trigonometric integration followed by a conversion of a trigonometric difference to the desired product form. This is another technical fragment given without context.

8 (c) (ii) Use integration methods to show that the volume of a sphere with radius r is $\frac{4}{3} \pi r^3$.

The above analysis strongly suggests that there is an overly technical emphasis in most of the questions set on Paper I of the Higher Level Mathematics examination. A perusal of the 2004, 2005 papers and those prior to 2003 reveal little variation in their format so that the criticisms made here of the year 2003 may be seen to apply also to these other years.

The situation with Paper II is essentially the same, as the discussion to follow will make clear. However, due to its more varied structure and the presence of probability related questions which touch somewhat on the personal and social lives of those taking the examination, it might be argued that this paper suffers less from the shortcomings that allegedly bedevil paper I.

5.3.3 Analysis of Paper II

- 1 (a) For all values of $t \in \mathbb{R}$, the point $\left(\frac{3-3t^2}{1+t^2}, \frac{6t}{1+t^2}\right)$ lies on the circle $x^2 + y^2 = r^2$.

Find r , the radius of the circle.

This question tests an understanding of an identity (to obtain with ease) the radius of a circle. It is a nice technical question but is not set in a context.

-
- 1 (b) $C_1 : x^2 + y^2 + 2x - 2y - 23 = 0$ and $C_2 : x^2 + y^2 - 14x - 2y + 41 = 0$ are two circles.

(i) Prove that C_1 and C_2 touch externally.

(ii) K is a third circle. Both C_1 and C_2 touch internally.

Find the equation of K .

A technical question about circles which is enlivened by an accompanying diagram.

- 1 (c) The line $ax + by = 0$ is tangent to the circle $x^2 + y^2 - 12x + 6y + 9 = 0$ where $a, b \in \mathbb{R}$ and $b \neq 0$.

(i) Show that $\frac{a}{b} = -\frac{3}{4}$.

(ii) Hence, or otherwise, find the coordinates of the point of contact.

Another technical question without a context.

The student might have been asked for a diagram illustrating the solution by way of showing the line drawn tangential to the given circle with the point of tangency clearly marked.

- 2 (a) $oabc$ is a parallelogram where o is the origin, $\vec{a} = 3\hat{i} - \hat{j}$ and $\vec{b} = 4\hat{i} + 3\hat{j}$.
Express $\vec{c}$ in terms of $\hat{i}$ and $\hat{j}$.

A question which requires drawing a diagram and realising that one vector must be subtracted from another.

A simple question relating to displacements of the type:

A man travels 50 km east, then 30 km south, then 20 km west and finally 10 km north.

Determine the resultant displacement of the traveller from his starting point.

Determine also how far the man is from his starting point and give his bearing with respect to the eastward direction.

has the virtue of setting vectors in some form of context while the question asked is a 'sensible' one.

- 2 (b) $\vec{p} = 2\hat{i} + \hat{j}$, $\vec{q} = 3\hat{i} + k\hat{j}$, $\vec{r} = 3\hat{i} + t\hat{j}$ where $k, t \in \mathbb{R}$ and o is the origin.
- (i) Given that $\vec{p} \perp \vec{q}$, calculate the value of k .
- (ii) Given that $|\angle por| = 45^\circ$, calculate the two possible values of t .

This question tests understanding that the dot product of two perpendicular vectors is zero and requires the factorisation of a quadratic. This is a technical question without a context.

2 (c) oab is a triangle where o is the origin.

(i) x is a point on $[a, b]$ such that $|ax| : |xb| = 1 : 3$.

Express $\vec{x}$ in terms of $\vec{a}$ and $\vec{b}$.

(ii) y is a point on $[oa]$ such that $|oy| : |ya| = 2 : 1$.

Express $\vec{by}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(iii) $[ox]$ and $[by]$ intersect at g . Given that $\vec{g} = m\vec{x}$ and $\vec{bg} = n\vec{by}$ where $m, n \in \mathbb{R}$, find the value of m and the value of n .

The question is quite a testing one, if once again not set in a context. It might have been replaced with a very similar question which would establish the wonderful theorem that the medians of a triangle are concurrent, divide each other in the ratio $2 : 1$ and can form the sides of a triangle – a result which should be illustrated visually.

Between all its three major parts, this question tests a significant amount of technical skill relating to vectors. Any student who successfully answers all three parts could be considered quite competent technically in this particular area.

- 3 (a) f is the transformation $(x, y) \rightarrow (x', y')$ where $x' = x + y$ and $y' = x - y$.
 L is the line $4x - 2y - 1 = 0$.

Find the equation of $f(L)$, the image of L under f .

This question requires determining the image of a line under a linear transformation.

- 3 (b) K is the line $3x - 4y + 9 = 0$.

The point $a(-3, 0)$ is on K .

The line M is parallel to K .

The point $p(2, -1)$ is midway between K and M .

(i) Find the equation of M .

(ii) Calculate the distance between K and M .

(iii) Calculate the measure of the acute angle between ap and K . Give your answer correct to the nearest degree.

(iv) $b(x, y)$ is a point on K such that $|ab| = 15$ and $x > 0$. Find the value of x and the value of y .

These four parts are a good test of (i) geometrical perception and (ii)

knowledge of the equation of a line.

This kind of skill is needed for describing the various signals that arise in electrical engineering.

The question is not in a context. However, it conveys a greater sense of integration than others of this parts-type and gives the impression of being more substantial.

4 (a) The circumference of a circle is 30π cm.

The area of a sector of the circle is 75 cm^2 .

Find, in radians, the angle in this sector.

A simple but partially practical question involving an understanding of radian measure.

4 (b) Find all the solutions of the equation

$$\sin 2x + \sin x = 0$$

in the domain $0^\circ \leq x \leq 360^\circ$.

A technical trigonometric question requiring knowledge of a trigonometric identity and basic knowledge as to when the sin and cos functions attain the values 0 and $-1/2$ respectively. This is a nice technical question but context-free.

A better effort might be:

The current flowing in a circuit at any time t (measured in seconds) is given by

$$i(t) = \sin t + \sin 2t$$

Explain why this current is periodic with principal period 2π and determine the times in this principal interval when the current is zero.

While this form of the question is still a contrived one it does attempt a certain rationale.

4 (c) C_1 is a circle with centre a and radius r .

C_2 is a circle with centre b and radius r .

C_1 and C_2 intersect at k and p .

$a \in C_2$.

$b \in C_1$.

(i) Find, in radians, the measure of the angle kap .

(ii) Calculate the area of the shaded region.

Give your answer in terms of r and π .

The first part of the question requires a little reflection but is easy once the right perspective is acquired. The second part calls for a certain amount of ingenuity by way of ‘looking at things from both sides’.

A nice exercise testing geometrical thinking but very much in the style of previous questions in that it is not tied to a context. This question was posed in an even balder form seventy years earlier in 1933.¹⁷

5 (a) Find the value of $\sin 15^\circ$ in surd form.

A question solvable by use of the half angle formulae or a clever use of geometry. In a historical context it can be seen as a theoretical use of trigonometry to determine an accurate value for the $\sin 15^\circ$ (and from which other trigonometric values can be obtained). An alternative approach might be:

How was value of $\sin 15^\circ$ originally obtained in surd form?
How is this result used to deduce in surd form corresponding values of the other trigonometric functions?

A question such as this might be classed as a practical technical question. The accurate evaluation of the trigonometric functions (as well as logarithms and roots) was a matter of great practical concern as recently as the 1800s and many nations, such as the English and French, expended much time and effort on the compiling of such tables giving values to five decimal places of accuracy.

¹⁷See 137.

Here in addition to hoping the student would furnish the values of $\cos 15^\circ$ and $\tan 15^\circ$, one would hope that the values of the trigonometric functions for the complementary angle 75° would also be supplied, since it involves little or no extra labour.

5 (b) a, f and e are points on horizontal ground.

d is a point on a vertical wall directly above a .

$$|ad| = 72 \text{ m}, |de| = 97 \text{ m}.$$

$$|af| = 35 \text{ m and } |fe| = 40 \text{ m}.$$

(i) Calculate $|ae|$.

(ii) Hence, calculate $|\angle afe|$.

A question with a practical feel to it. The first part requiring a length can be solved using Pythagoras's theorem, whereupon the second part follows by an application of the cosine rule. The figures are chosen so that everything works out quite readily.

5 (c)(i) Using the identity $\cos(A - B) = \cos A \cos B + \sin A \sin B$, or otherwise, prove:

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

(ii) Prove:

$$\sin(A + B) \sin(A - B) = (\sin A + \sin B)(\sin A - \sin B)$$

Two technical exercises in trigonometry without a context.

- 6 (a) Eight people, including Kieran and Anne, are available to form a committee.

Five people must be chosen for the committee.

- (i) In how many ways can the committee be formed if both Kieran and Anne must be chosen?
- (ii) In how many ways can the committee be formed if neither Kieran nor Anne can be chosen?
-

An exercise in the important art of counting which is easy but requires clear thinking.

- 6 (b)(i) Solve the difference equation $u_{n+2} - 4u_{n+1} + 3u_n = 0$, where $n \geq 0$, given that $u_0 = -2$ and $u_1 = 4$.

(ii) Verify that the solution you have obtained in (i) satisfies the difference equation.

The first part of this question is a drill exercise in solving an unmotivated difference equation using a ‘classical procedure.’ It requires solving a quadratic equation to acquire the solution basis of the equation and then the matching of the arbitrary constants to the initial conditions. It must be questionable if students understand why they proceed as they are instructed to do. It is a pity to use a syllabus segment on difference equations in such a futile manner when it could be harnessed in a much more engaging and vigorous manner. As already mentioned, this syllabus segment is the subject of a later discussion.

The second part, that of checking a purported solution is an essential exercise in all real-life calculations. An approach to this entire topic is discussed in some detail later.

6 (c) Ten discs, each marked with a different whole number from 0 to 10, are placed in a box.

Three of the discs are drawn at random (without replacement) from the box.

(i) What is the probability that the disc with the number 7 is drawn?

(ii) What is the probability that the three numbers on the discs drawn are odd?

(iii) What is the probability that the product of the three numbers on the discs drawn is even?

(iv) What is the probability that the smallest number on the discs drawn is 4?

(i) is a completely elementary application of Laplace's definition of a probability.

(ii) requires more thinking than (i). One way of answering is via

$$\frac{\binom{5}{3}}{\binom{10}{3}}$$

which is equivalent to

$$\frac{5}{10} \times \frac{4}{9} \times \frac{3}{8}$$

the answer one would give using the probability laws relating to conditioning.

(iii) is another testing question best answered using the result from (ii) with the complementary principle.

(iv) requires clear thinking which is almost enough justification for its inclusion.

With the exception of part (b) this could be judged to a satisfactory question which if answered well indicates a certain amount of ability on the part of the respondent.

7 (a) Five cars enter a car park. There are exactly five vacant spaces in the car park.

(i) In how many different ways can the five cars park in the vacant spaces?

(ii) Two of the cars leave the car park without parking.

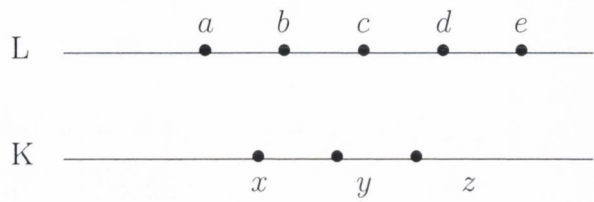
In how many different ways can the remaining three cars park in the five vacant spaces?

(i) is a simple question on permutations with the answer $5! = 120$.

(ii) A variation on (i) which is a good test of the ability to count.

Both parts of this strong question require that one think clearly.

7 (b)



L and K are distinct parallel lines.

a, b, c and d are points on L such that $|ab| = |bc| = |cd| = 1\text{ cm}$.

x, y and z are points on K such that $|xy| = |yz| = 1\text{ cm}$.

- (i) How many different triangles can be constructed using three of the named points as vertices?
- (ii) How many different quadrilaterals can be constructed using four of the named points as vertices?
- (iii) How many different parallelograms can be constructed using four of the named points as vertices?
- (iv) If one qudrilateral is constructed at random, what is the probability that it is *not* a parallelogram?

This is an excellent question in all its parts, the first three of which require the skill to subdivide a problem into mutually exclusive cases for

the purposes of making an exhaustive enumeration. Again, a student who scores highly on such a question is certainly evincing the ability to reason clearly and in some depth.

The role of probability is taken up later where it is proposed that it be widened somewhat to treat matters such as lotteries, gambling, etc. which touch on people's personal lives.

7 (c) The mean of the real numbers a and b is $\bar{x}$.

The standard deviation is σ .

(i) Express σ in terms of a, b and $\bar{x}$.

(ii) Hence, express σ in terms of a and b only.

(iii) Show that $\bar{x}^2 - \sigma^2 = ab$.

A mainly technical question examining algebraic skills but also testing the definitions of the mean and standard deviation. However, the question is not set in a context.

The four remaining questions are from the options section B of the paper.

8 (a) Use integration by parts to find $\int x e^{-5x} dx$.

A simple technical question examining the all-important integration by parts technique which is used over and over again at third level in such topics as Laplace transforms and Fourier series/transforms. No context is given.

One might ask the students for a sketch of the graph of $f(t) = t e^{-5t}$ for $t \geq 0$ and then for the area subtended by this graph and the horizontal axis.

One area where a function such as $f(t)$ arises is in probability in connection with the Poisson distribution (with parameter $\lambda = 5$).

8 (b) $f(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \dots$ is the Maclaurin series.

(i) Derive the Maclaurin series for $f(x) = \log_e(1+x)$ up to and including the term containing x^4 .

(ii) Write down the general term and use the Ratio Test to show that the series converges for $-1 < x < 1$.

Leaving aside the slight inaccuracy in the wording of this question, its

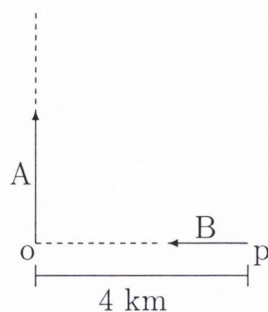
first part requires 4 separate differentiations, 4 evaluations of functions at 0 and the marshalling of these evaluations to give, after simplification, the first five terms of the desired series.

Part (ii) requires recognising a general pattern along with knowledge of the Ratio test and how to apply it.

Both parts are technically testing but neither part is set in context. When the leading terms of the series have been obtained and the interval of convergence established, would it not have been sensible to demonstrate or test a use of these results?

-
- 8 (c) The point p is 4 km due east of the point o .

At noon, A leaves o and travels north at a steady speed of 12 km/h. At the same time, B leaves p and travels towards o at a steady speed of 6 km/h.



- (i) Write down expressions in x for the distances that A and B will each

have travelled at x minutes after noon.

(ii) Find an expression in x for the distance that B will be from A at x minutes after noon.

(iii) At how many minutes after noon will B be closest to A ?

This is a very important type of problem requiring, as it does, the ability to formulate. Even though the question is stepped in order to guide the student through the formulation, the student must possess a certain amount of skill, particularly in (ii).

It might have been more natural to use t to denote time measured in minutes rather than the suggested variable x . It would appear that at the level of the HLC the variable x is forever called on to act as the unknown in all manner of problems, where other letters may be more appropriate in the context. One danger of this practice (actually experienced) is that some students come to believe that one must use x as the unknown! In fact, the Chief Examiner's report on the HLC 1995 paper contains the following comment "candidates had difficulties in calculus when the functions to be differentiated contained variables other than x and y ."¹⁸

¹⁸Department of Education, *Chief Examiner's Report on the Leaving Certificate Examination 1995, Mathematics*, (Dublin: DES 1996), 19.

- 9 (a) z is a random variable with standard normal distribution.

Calculate $P(-2.13 < z \leq 1.46)$.

No context is provided.

- 9 (b) Whenever Anne's mobile phone rings, the probability that she answers the call is $\frac{3}{4}$.

A friend phones Anne six times.

- (i) What is the probability that she misses all the calls?
 - (ii) What is the probability that she misses the first two calls and answers the others?
 - (iii) What is the probability that she answers exactly one of the calls?
 - (iv) What is the probability that she answers at least two of the calls?
-

Parts (i) and (ii) require the use of basic probabilistic reasoning (on the assumption of independence). Part (iii) is a 'single' binomial calculation while (iv) is a compound binomial calculation which can be answered using parts (i) and (iii). A contrived question which requires clear thinking to answer it correctly; yet, one would prefer that less artificial questions, requiring the same clarity of analysis, be used.

-
- 9 (c) In a newspaper advertisement, a driving school claimed that 80% of its clients passed their driving test on their first attempt.

1000 people who attended the school and who had taken the test for the first time were randomly selected.

Find, at the 5% level of significance, the interval in which the number who passed should lie in order that the claim made in the advertisement be accepted.

A return to the type of question in part (a) involving standard deviations from the mean of a normal distribution. This might have been placed after (a) with (b), a question on binomial probabilities, replacing (c)

- 10 (a) $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$
 $M = \{A, B, C, D\}$ is a cyclic group under multiplication.

Verify that A is a generator of M .

The group in question is a matrix representation of C_4 with identity D , A and C its two cyclic generators and B its single element of order 2. In essence this question tests no more than knowledge recall of

the group-theoretic terms employed and requires no more than matrix multiplication for its execution.

Students might have been asked for a demonstration (via the construction of a Cayley table) that the set M under matrix multiplication is a commutative group followed by a request to identify the nature of the group, *i.e.* as being isomorphic to the abstract group $\mathcal{C}_4$ as opposed to the (abstract) Klein-four group $\mathcal{V}_4$?

This question might also have been enriched by asking for an interpretation of the transformations induced by each of the matrices on vectors of $\mathbb{R}^2$ to ascertain if the student appreciated their geometric nature.

10 (b) abc is an equilateral triangle.

L, M and N , the perpendicular bisectors of the sides, intersect at o .

$D_3 = \{I_\pi, R_{120^\circ}, R_{240^\circ}, S_L, S_M, S_N\}$, under composition, is the symmetry group of abc .

(i) Investigate whether $S_L \circ S_M = S_M \circ S_L$.

(ii) Write down the centralizer of R_{120° .

A fine technical question relating to the group of symmetries of an equilateral triangle. It would benefit from a comment on the significance of the result obtained in part (i) with, perhaps, extra credit for

an answer which would reveal that this is the smallest finite group for which commutativity breaks down.

$$10 \text{ (c) } P = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \right\}$$

P is a group under composition of permutations.

D_3 is the symmetry group of the equilateral triangle abc , as described in (b).

$f : D_3 \rightarrow P$ is an isomorphism where $f(R_{120^\circ}) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$.

Find $f(R_{120^\circ}^{-1})$, justifying your answer.

Another substantial technical question in that it furnishes a different concrete realisation of the abstract group $\mathcal{S}_3$ in terms of permutations and where the student must establish its relation with part (b) by verifying the indicated isomorphism.

As a technical question about groups this is an excellent one in all its three parts. However, one might worry whether or not group questions such as this one are seen as something more than mere occasions for skillful drill.

- 11 (a) The polar of a point p with respect to the circle $x^2 + y^2 = 9$ is $2x - 5y = 27$.

Given that the polar of the point (x_1, y_1) with respect to the circle $x^2 + y^2 = r^2$ is $xx_1 + yy_1 = r^2$, find the coordinates of p .

A technical geometrical question.

- 11 (b) f is a similarity transformation. f maps the angle θ onto the angle ϕ .
Prove that θ and ϕ are equal in measure.
-

A technical geometrical question.

- 11 (c) $[cd]$ is a diameter of the ellipse $\frac{x^2}{100} + \frac{y^2}{25} = 1$ where c is the point $\left(5\sqrt{2}, \frac{5}{2}\sqrt{2}\right)$.

(i) Find the equation of the tangent to the ellipse at c .

(ii) The diameter $[st]$ is conjugate to the diameter $[cd]$. Find the equation of st and the coordinates of the points s and t .

A technical geometrical question.

General Observations

1. With the possible exception of the probabilistic questions there are no others addressing the communications standard where the expected answers necessitates a certain amount of written explanation.
2. Despite the syllabus's injunction that the history of mathematics be alluded to in the classroom there is not one question relating to history in the examination papers.
3. There is no question or part thereof relating to a recreational problem.
4. In neither paper is there a question touching on financial matters relating to investments, interest or the like.

The findings of the above analyses are supported by a study reported on by Elwood and Carlisle in 2003. As Conway and Sloane relate:¹⁹

Elwood and Carlisle (2003, pp. 72-74) in their examination of Leaving Certificate mathematics papers (they examined both content and question type) found that questions in the Leaving Certificate were abstract, focused on 'pure' mathematics, relied more heavily on traditional notations and symbols than comparable examination papers in other jurisdictions, and were notable also for the lack of real-life contexts in questions.

¹⁹J. Elwood & K. Carlisle, *Examining Gender: Gender Achievement in the Junior and Leaving Certificate Examinations 2000/2001*. (Dublin: NCCA Research Report no. 1, 2003) as quoted in Paul F. Conway, & Finbarr C. Sloane, *International Trends in Post-Primary Education*, (Dublin: NCCA, 2005), 188.

5.4 Current Textbooks

In this section existing textbooks are examined for the same technical emphasis that characterises the LC examination.²⁰ Four texts currently being used for the Higher mathematics course were consulted:

1. *New Concise Maths 4 for Leaving Cert Level* by George Humphrey.²¹
2. *Texts and Tests 4, Mathematics Higher Level* by O. D. Morris.²²
3. *Discovering Maths 4, Leaving Certificate Higher Level* by Oliver Murphy.²³
4. *Leaving Certificate Maths, Mathematics for Higher Level Students* by Aidan Roantree.²⁴

It would be harsh to attribute blame to the authors of these books if it is truly the case that these writers feel the structure and nature of their texts to be predetermined by the DES syllabi and interpreted and illustrated by that body via batches of sample papers at the inception of new programmes – a case of test determining text. The situation prevailing at present is that there have now been more than ten years of examination papers based on

²⁰See page 14.

²¹G. Humphrey, *New Concise Maths 4 for Leaving Cert Level* (Dublin: Gill & Macmillan, 2003)

²²O. D. Morris, *Texts and Tests 4, Mathematics Higher Level* (Dublin: The Celtic Press, 2003)

²³O. Murphy, *Discovering Maths 4, Leaving Certificate Higher Level* (Dublin: Folens Publishers, 2002)

²⁴A. Roantree, *Leaving Certificate Maths, Mathematics for Higher Level Students* (Dublin: CJ Fallon, April, 2003)

the current syllabus, a factor which has caused each of these four authors to update their respective earlier texts. In passing it should be remarked that there has been a noticeable and welcome improvement in the presentation of mathematical material in school textbooks, particularly in the past ten years, aided, no doubt, by the interim advances in typesetting and drawing. One can see this clearly by comparing *Discovering Maths 4, Leaving Certificate Higher Level* by Oliver Murphy published in 2002 with its 1992 predecessor by the same author *Leaving Certificate – MATHEMATICS – Higher Level*.²⁵ This latter publication was relieved by a number of cartoon-like drawings and quotes at the heading of each chapter.

The statements to follow, which are extracts from the prefaces of each of these new texts, indicate clearly that it is the nature of the HLC examination which is dictating their structure, mode of presentation and emphasis. Humphrey writes that “full analysis of the pattern and level of difficulty of the examination question was taken into account.” Some lines later he adds that “each revision exercise contains an extensive collection of ‘part a’, ‘part b’ and ‘part c’ type questions similar in standard to the Leaving Certificate questions.”²⁶

Likewise, Morris writes that “the book also responds to and reflects the pattern and style of question that has appeared on the Leaving Certificate

²⁵O. Murphy, *Leaving Certificate – Mathematics – Higher Level* (Dublin: Folens Publishers, 1992)

²⁶G. Humphrey, *New Concise Maths 4 for Leaving Cert Level*, Preface, iv.

papers since the course was first examined in 1994.”²⁷

Even more pragmatically, Murphy writes in the Preface that²⁸

The Leaving Certificate Higher Level Maths syllabus is over ten years old, and we can see a pattern in the *kind of question* and level of difficulty that appears in the exams and this new textbook is designed to reflect both of these. It contains examples and exercises based on what has turned up on the exam;

After listing a number of specific topic areas he continues²⁹

The worked examples are at the level of difficulty of actual exam questions. Furthermore, all of the exercises are carefully graded to lead the students to the level that past exam papers demand, including the more difficult problems that appear in part (c) of Leaving Certificate questions.

For his part, Roantree writes in relation to his two-volume text that “the many changes and improvements to all topics have been inspired by student and teacher reaction to existing textbooks and analysis of the pattern of examination questions.”³⁰

Thus it may be fairly said that it is a case of the examination dictating the text. Since the previous sections of this chapter have been devoted to supporting the assertion that the current examinations place an undue emphasis on the technical aspects of mathematics at the expense of others, it follows that the same contention is true of these texts whose authors make

²⁷O.D. Morris, *Texts and Tests 4, Mathematics Higher Level*, Preface.

²⁸O. Murphy, *Discovering Maths 4, Leaving Certificate Higher Level*, vi.

²⁹*Ibid.*, vi.

³⁰A. Roantree, *Leaving Certificate Maths, Mathematics for Higher Level Students*, preface.

no secret of the fact that both their content and manner of presentation is dictated by the nature of the current Leaving Certificate examinations.

As a test case, details are given as to how each textbook handles the relatively straightforward topic of ‘quadratics’. The discussion may be found in Appendix E.

While the over-riding concern of the latter sections of this chapter has been to substantiate the claim that at present both the examinations and textbooks place an undue emphasis on mathematical techniques, it must be re-iterated that this writer is keenly aware that mathematics students must have a firm technical mastery. It is perhaps appropriate to conclude this chapter by quoting and taking heed of what Douglas Quadling had to say on this important matter in relation to the experience of the Schools’ Mathematics project.³¹

But technique, of course, remains an important feature of mathematics. Not only is it essential if mathematical ideas are to be handled with any facility and confidence – one cannot be going back to first principles all the time – but much of the power and fascination of mathematics lies in the patterns formed by symbols and their transformation into other patterns. In the initial enthusiasm of reform some of us were inclined to play this down, and the result was an unacceptable decline in manipulative skill;

³¹Douglas Quadling, “The Mathematics of S.M.P.”, *The First Ten Years*, 218.