Double Value-Ranges

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1 Introduction

The biggest change in Frege's logic between the 1879 *Begriffsschrift* (*Bs*) and the 1893 *Grundgesetze der Arithmetik* (*Gg*) was the introduction of value-ranges (*Wertverläufe*). There were many other differences of course, including the reinterpretation of the content stroke as the horizontal, the reinterpretation of identity as an object-language function, the new definite article operator '\', and, although it did not impinge on the proofs themselves, the distinction between sense (*Sinn*) and reference (*Bedeutung*). But nothing made such a material difference to the proofs themselves as value-ranges. Totally absent from *Bs*, which is in effect a second-order functional calculus, they are utterly pervasive in *Gg*, and figure in 25 out of the 27 definitions of *Gg.* Even the two definition where the value-range function $\dot{\varepsilon}\Phi(\varepsilon)$ is not expressly used, namely Definition Γ of the many-oneness of a relation and the Definition AB of limit, presuppose value-ranges, since their *definientia* contain symbols defined using value-ranges, and are primarily intended to apply to objects whose *raison d'être* is to be value-ranges of concepts and relations (*Beziehungen*).

The adoption of value-ranges together with Frege's principle of identity for them, the infamous Basic Law V, was, as we now know, the source of the inconsistency in *Gg.* The details of this flaw have been anatomized many times, and I do not intend to recapitulate that discussion. Rather, leaving the inconsistency aside, I shall examine Frege's use of value-ranges to provide extensions for binary functions, which include binary relations. The special value-ranges that perform this job Frege calls 'double value-ranges' (*Doppelwertverläufe*).

2 Functions in General

Value-ranges are objects that are correlated with functions. Functions come in different logical categories, according to the nature of their arguments and values. Functions that take objects as arguments and have objects as values are *first-level* functions. They are subcategorized in two further dimensions. The first simply classifies them according to the number of arguments they take. For the purposes of establishing his logicism, Frege never considers functions with more than two arguments, but in logical principle there is no upper bound to the number of arguments a function may have. The second dimension concerns whether the values of a function are or are not confined to the truth-values T (the true) and F (the false). Unary functions to only truth-values are concepts (*Begriffe*), binary functions to only truth-values are relations (*Beziehungen*). Call these in general *truth-value-functions*. Frege employs the horizontal function to trade functions taking "third" objects, ones which are neither T nor F, for truth-value-functions, since $—\Delta = T$ if $\Delta = T$ and $—\Delta = F$ if $\Delta \neq T$.

Functions which have functions as arguments are *higher-level*. The most prominent among them in *Gg* are the universal quantifier $\forall \mathfrak{a} \Phi(\mathfrak{a})$ and the value-range function $\dot{\varepsilon}\Phi(\varepsilon)$ itself. These take first-level functions as arguments and have objects as values. Of these two the former is a truth-value-function, but the latter is not. Frege also introduces third-level functions which take second-level functions as arguments, namely universal quantifiers binding unary and binary first-level functions, respectively $\forall \mathfrak{f} \mu_\beta \mathfrak{f}(\beta)$ and $\forall \mathfrak{f} \mu_{\beta\gamma} \mathfrak{f}(\beta,\gamma)$, though only the former is used "in anger" in *Gg.* Once again, there is no logical upper limit to the level of functions. Frege also gives examples of functions which are *unequal-leveled*, which is to say, have arguments from different categories. An example he gives is the differential quotient,[1] which takes a unary function and an object (a number) as arguments and returns a number as value: for example for the square function ξ^2 at the argument 3 the value of the differential $D_\beta(\beta^2,3) = 6$, or, as mathematicians might write it, $d/dx(x^2)|_3 = 6$.

According to Frege's strict separation of objects from functions, since no function is an object, all categories of functions are ontologically disjoint one from another.

In *Bs*, Frege used subtle and elegantly defined variable-binding operators to define higher-order functions such as the ancestral for relations.[2] These take functions as arguments and return functions as values. In *Gg* these subtleties are replaced by a more or less blanket use of value-ranges, so that function-valued functions are no longer required: all functions employed in *Gg* are *object-valued*.

3 Grammar

Frege was extraordinarily sensitive to matters of grammar, whether in natural language, in the hybrid semi-formal argot of mathematicians, or in his own fully formalized logical systems. It is part of what renders his work technically so superior to nearly everything else going on

[1] *Gg* § 22.
[2] Simons 1988.

around him at the time. His logical grammar fits the paradigm, developed only subsequently by Ajdukiewicz and others, of categorial grammar.[3] It possesses two *basic* categories, Judgement (J) and Name (N), and a potential infinity of derived or *functor* categories, depending on the way in which certain expressions or expression-patterns require saturation or completion to give a name. We use lower-case Greek letters as category variables. If α, β, γ … are categories, then the functor category of expressions yielding an expression of category α when saturated suitably by expressions of category β, γ … we denote by an expression of the form α(βγ…), where the dots are filled by as many category-expressions as the case requires. So for example the categories for one-place, two-place and three-place functions from objects to objects are respectively N(N), N(NN) and N(NNN). The judgement stroke works as an expression of category J(N), turning a name into a judgement, and that is all it does. So the category of judgement plays no further part in the ramification of the derived categories, and while it is interesting for illustrating various important aspects of Frege's view of logic, we leave it out of further account here.

In view of the importance of truth-value-names for Frege, it is useful to have a symbol for the subcategory of names that name truth-values, and we use 'W' for this purpose. Any expression whose subcategory is given using 'W' is of the *category* obtained by replacing the instances of 'W' in its subcateory name throughout by instances of 'N': all functor category names we shall consider are built exclusively from 'N's and parentheses, and any two functor expressions with the same category are intersubstitutible *salva congruitate*. The horizontal function for example combines with any name to yield a truth-value-name, so it has subcategory W(N), but it is still a one-place function of category N(N). The standard logical connectives of negation and material implication are of subcategories W(N) and W(NN) respectively, though their actual presence alongside horizontals in all formulas Frege writes means they work effectively as if of subcategories W(W) and W(WW) respectively. The identity function on the other hand is one designed to be saturated by any names whatsoever, so it has subcategory W(NN).

When Frege allows functions to be arguments of other functions, of higher-order, he insists that their argument-places always be marked and filled by variables that are bound by the higher-order function name. This is obvious in the case of the universal quantifier $\forall\mathfrak{a}[\ldots\mathfrak{a}\ldots]$. The presence of the letter next to the operator symbol is simply to indicate the places into which the operator reaches. In the *Begriffsschrift*, before value-ranges were introduced, and when higher-order functions played a much greater part in his logic, Frege in

3 Cf. Potts 1978.

fact separated two roles that in his later *Grundgesetze* system were run together: the role of *marking* a place for an operator to reach into, and the role of *filling* the place with a suitable expression. This led to his use of variable place-marker subscripts for constant fillers, so that for example the expression for *n* being a natural number could be telescoped into a remarkably short space invoking the ancestral operator *, amounting to $*\beta\gamma[0_\beta + 1 = n_\gamma]$.[4] That subtle complication disappears in *Gg*, which simplifies the grammatical description considerably. The quantifier has subcategory W(N(N)).

The value-range operator in Frege has category N(N(N)): it returns a name of an object in operating upon the name of a one-place function. To complete the list of categories of Frege's primitive functions, the second-order quantifier for one-place functions has subcategory W(N(N(N))), that for two-place functions is W(N(N(NN))), and the definite descriptor has category N(N). Notice that all Frege's primitives are functions: though he explains the signification of most of them through the truth-values T and F (for which any expression has the category N, subcategory W), these are actually specified stipulatively in terms of functions and value-ranges: T as $\dot{\varepsilon}(\text{—}\varepsilon)$, F as $\dot{\varepsilon}(\varepsilon = {\sim}\forall\mathfrak{a}[\mathfrak{a} = \mathfrak{a}])$.

Given the categories and subcategories of the primitives, it is possible to calculate the categories and subcategories of any well-formed expressions occurring in Frege's notation. The only complication is Frege's conventional use of Italic letters in place of German bound variables (which are reserved for use with quantifiers, presumably for the reason that they can sometimes be replaced by Italics), enabling him to give quantificational formulas containing universal quantifiers having maximal scope in effect as schemata. As an example consider the Basic Law III for identity, which (omitting the judgement stroke) is:

$$g(a = b) \to g(\forall\mathfrak{f}[\mathfrak{f}(b) \to \mathfrak{f}(a)])$$

There are four variables in this, one bound with scope of part of the consequent only, the others apparently free but in effect bound with maximal scope: two are object variables, category N, two are function variables, category N(N). The complex (i.e. non-simple) logical subcomponents of the formula (ignoring the tacit quantifiers) are as follows:

Subcomponent	Subcategory
$a = b$	W
$g(a = b)$	W
$\forall\mathfrak{f}[\mathfrak{f}(b) \to \mathfrak{f}(a)]$	W

[4] Cf. Simons 1988.

$g(\forall\mathfrak{f}[\mathfrak{f}(b) \rightarrow \mathfrak{f}(a)])$	W

This may seem surprising: after all the argument of the consequent appears to be made up of a quantifier binding into a formula which is a conditional. Here we come up against one of the subtleties of Frege's account of functions which give beginners the greatest difficulty. It is that he conceives of function names from a *logical* point of view not as being built up out of parts like a Lego toy, but as arising through designating certain places in a complex name as being such as to be replaced by one or more tokens of a single type bound variable. We must therefore distinguish the *logical* formation history of a formula from its *physical* formation history, which is the usual one for written languages of putting written sign tokens, some equiform with others, in certain spatial configurations. Looked at this way, the formation history of the formula looks very different. To designate it we use dummy place-holder variables which are then eliminated.

(Virtual) Subcomponent	Sub-category	Mode of Formation
A = B	W	Saturation of identity function expression, N(NN), by two names
G(A = B)	N	Saturation of monadic function expression, N(N), by above complex name
F(B)	N	Saturation of monadic function expression, N(N), by a name
F(A)	N	Saturation of monadic function expression, N(N), by a name
F(B) → F(A)	W	Saturation of binary conditional function, N(NN), by two above complex names
φ(B) → φ(A)	W(N(N))	"Emptying" of two occurrences of functional expression, in situ, ready for saturation
$\forall\mathfrak{f}[\mathfrak{f}(\mathrm{B}) \rightarrow \mathfrak{f}(\mathrm{A})]$	W	Saturation of resulting "gaps" by universal quantifier, W(N(N(N))
$\mathrm{G}(\forall\mathfrak{f}[\mathfrak{f}(\mathrm{B}) \rightarrow \mathfrak{f}(\mathrm{A})])$	W	Saturation of monadic function expression, N(N), by above complex name
$\mathrm{G}(\mathrm{A} = \mathrm{B}) \rightarrow \mathrm{G}(\forall\mathfrak{f}[\mathfrak{f}(\mathrm{B}) \rightarrow \mathfrak{f}(\mathrm{A})])$	W	Saturation of binary conditional function by two complex names
$g(a = b) \rightarrow g(\forall\mathfrak{f}[\mathfrak{f}(b) \rightarrow \mathfrak{f}(a)])$	W	Uniform replacement of three distinct dummy names by three distinct Italic variables

Were we to expressly replace the Italic variables by German variables bound by three universal quantifiers, we would require additional emptying/binding–operating steps, and would have to choose one of the six possible (but logically equivalent) orders in which to do this.

With this grammar in place, we are better able to investigate the way in which Frege deals with value-ranges in cases of varying complexity.

4 Why Value-Ranges?

There are no value-ranges in *Bs*, so why did Frege feel compelled to introduce them? It turns on the execution of his logicist programme. Frege was out to show that the laws of arithmetic follow from logical laws and definitions alone, and the things dealt with by those laws included numbers, finite and infinite, natural and real, ultimately probably also complex. Leaving the real and complex numbers aside, as being more complicated than we need to consider here, and also because Frege did not complete publishing his accounts,[5] we concentrate just on the simplest numbers, the finite natural numbers. It is these that are dealt with most fully both in *Gg* and in Frege's 1884 prose masterpiece, *Die Grundlagen der Arithmetik* (*Grl*).

Natural numbers occur in two different kinds of context. One is when making statements of number, as when we say there are twelve months in the year, eight planets and sixteen German *Bundesländer*. The other is in arithmetic, as when we say that $(23 + (7 \times 5)) - 8 = 50$, that $(12^3 + 1^3) = (10^3 + 9^3)$, and so on. We call the first kind *enumerative* statements and the second kind *arithmetical* statements. In enumerative statements, according to Frege, we make a statement about a concept, so that 'there are eight planets' can be treated as 'there are eight *x* such that *x* is a planet'. Here the first-level concept is indicated by 'ξ is a planet'. It is the logical subject, while the remainder, the logical predicate, saying something about this concept, is indicated by the second-level concept 'there are eight β such that $\Phi(\beta)$'. In arithmetical statements on the other hand the number-words or other number-expressions function as self-sufficient saturated designative units, singular terms or proper names (*Eigennamen*). It is Frege's mature opinion that the opposition between saturated expressions, singular terms, on the one hand, and unsaturated expressions, function expressions, on the other, is precisely correlated with the absolute ontological distinction between objects (*Gegenstände*) and functions, and that the grammatical category of an expression is a sure guide to the ontological category of its referent.

So Frege needs a way to link the intuitively related but as yet logically unconnected expressions 'there are eight β such that $\Phi(\beta)$' and '8'. The obvious way to do this is by a transitional principle

5 Cf. Simons 1987.

there are eight β such that $\Phi(\beta) \leftrightarrow$ the number of β such that $\Phi(\beta) = 8$.

and this is indeed how Frege proceeds. So he needs a way to both maintain this biconditional connection and yet allow expressions of the form 'the number of β such that Φ(β)' to designate objects. His first attempt to do this is via what Frege scholars generally call 'Hume's Principle'; for reasons of accuracy[6] and historical priority I prefer to follow Tarski, who followed Cantor, and call it the Equicardinality Principle.[7] We write '$\#\beta\Phi(\beta)$' for 'the number of β such that Φ(β)' and '$\equiv_{\beta\gamma}(\Phi(\beta),\Psi(\gamma))$' for what can easily be defined to mean 'there are as many β such that Φ(β) as there are γ such that Ψ(γ)', or more colloquially 'there are as many Φ as Ψ'. We then require

EQC $\quad \#\beta\Phi(\beta) = \#\gamma\Psi(\gamma) \leftrightarrow \equiv_{\beta\gamma}(\Phi(\beta),\Psi(\gamma))$

While this gives both a transition and an identity condition for numbers, it is prey to the "Caesar Problem", that it leaves identities of the general form $\#\beta\Phi(\beta) = a$ undecided for any case where the right-hand side is not of the requisite form, and so fails to nail the numbers as objects.[8] It is for this reason that Frege switches to giving a non-contextual definition for 'the number of β such that Φ(β)' as, roughly

the extension of the concept 'concept equinumerous with the concept Φ'

or with somewhat greater Fregean precision

$$U_{\mathfrak{f}}(\equiv_{\beta\gamma}(\mathfrak{f}(\beta),\Phi(\gamma)))$$

where 'U' is an as-yet-to-be-determined extension (*Umfang*) function, in this case a third-level function taking as argument the second-level function 'is a concept equinumerous with the concept **Φ**' and returning an object. The extension "consists of" (speaking loosely) all and only those concepts that are equinumerous with **Φ**, including of course **Φ** itself. Frege shows in *Grl* that the objects determined by such a function U satisfy the Equicardinality Principle, and so all is hunky dory.

Or is it? Simply introducing something called the extension and legislating that it does the trick looks suspiciously like theft rather than honest toil. In *Grl* itself, Frege was uncharacteristically ambivalent about extensions. One notorious footnote[9] claims the same trick can be pulled off without extensions at all, simply using the concepts themselves. But

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6 It is not close to any principle actually formulated by Hume, and the name follows a somewhat misleading historical footnote to Hume's *Treatise* in *Grl* § 63.

7 Cf. Cantor 1932, 283.

8 *Grl* § 66.

9 *Grl* § 68.

Frege soon backed away from this thought,[10] and from then on remained convinced that he needed extensions. Frege's logical *apartheid* of functions and objects was beginning to solidify around the same time, as was his assimilation of concepts, as truth-value-functions, to functions in general. So it was natural for him to look for a general operator, subsuming extensions of *Begriffe* and *Beziehungen* as special cases, which could take any function whatever as an argument, and return an object as a value, one which was, speaking with pardonable logical infelicity, uniquely correlated with the function, so that functions are distinct (insofar as we can use that talk about functions) if and only if their value-ranges are distinct, or contrapositively, that functions are the same if and only if their value-ranges are the same. Accepting that the logical analogue of identity for functions, what I shall here call 'equivalence', is having the same values for the same arguments throughout, this results in the requirement that

the value-range of $\mathfrak{f}$ = the value range of $\mathfrak{g} \leftrightarrow$ function $\mathfrak{f}$ is equivalent to function $\mathfrak{g}$

and when spelt out exactly, noting that the biconditional is identity for truth-values, this becomes (adding the judgement stroke to make clear that this is an assertion, not just a truth-value name):

V $\quad \vdash (\dot{\varepsilon}f(\varepsilon) = \dot{\alpha}g(\alpha)) = \forall\mathfrak{a}[f(\mathfrak{a}) = g(\mathfrak{a})]$

and there we have it: Basic Law V in all its grisly glory.

It did not escape Frege's attention that V was itself susceptible to the Caesar Problem: the permutation argument in *Gg* § 10 marks this recognition, and leads Frege to give conventionalist stipulations of which value-ranges are to be T and F. The effects of these stipulations are in theory stunningly far-reaching and pervasive, but since the system was inconsistent, they remain largely academic for our purpose here.

5 The Application Function

Let $f(\xi)$ be a unary object-argument object-valued function and a an argument for it. The value of $f(\xi)$ for this argument is simply $f(a)$. For example if $f(\xi)$ is the mathematical square function ξ^2 and a is the complex unit i then $f(a)$ is i^2, that is, –1. Now consider this function's value-range $\dot{\varepsilon}f(\varepsilon)$. Frege wants a way to connect this with the arguments of the function such that when the argument of the function is the first argument and the function's value-range is the second, the value of the function for the argument is the resulting value. To this end he

[10] For a masterly exposition of the development of Frege's thinking on extensions, see Burge 1984.

introduces a binary function $\xi \frown \zeta$ such that when the second argument is a value-range, the value of the binary function is the value of the function for the first argument, or in symbols

C $$a \frown \dot{\varepsilon}f(\varepsilon) = f(a)$$

This formula is in fact Theorem 1 of Frege's logical system, since he introduces the function $\xi \frown \zeta$ via the definite description operator, in a way which ensures that this result works whenever the second argument is a value-range, and gives a "don't care" value when it isn't. That is indeed the one and only use of the description operator. Having defined this function – generally known as the *application* function – he works thereafter solely with it.

The application function performs, in the case where $f(\xi)$ is a concept, i.e. a unary truth-value-function, a role strictly analogous to the relation of set-membership in naïve set theory, and C plays the role of the Comprehension Principle in that theory. As this suggests, C plays a key role in the derivation of Russell's Paradox. But C extends much more broadly than Comprehension, as the example $i \frown \dot{\varepsilon}(\varepsilon^2) = -1$ indicates. Extensions of concepts are just one kind of value-range, important though they of course are.

6 Value-Ranges for Binary Functions

The value-range operator yields objects for unary functions, but what are the value-ranges of functions with more than one argument? Consider the simple subtraction function $\xi - \zeta$ in arithmetic. It is doubly unsaturated. To obtain its value-range as an object we need to saturate both argument-places. Let us provisionally introduce a notation for a function which saturates both argument-places at once and returns a certain object as value: we write

$$\hat{}\varepsilon\alpha[\Omega(\varepsilon,\alpha)]$$

where the upper-case omega indicates a functional context where there are two available places for variables, and the circumflex is our nonce-notation for this new two-place value-range operator. The object of this sort for the subtraction function is then

$$\hat{}\varepsilon\alpha[\varepsilon - \alpha]$$

Notice that the two variables are bound at one go, together. This threatens to undercut the necessary distinction between the two places when a function is non-commutative, which in the case of a relation is when the relation is non-symetric. There are two ways see off this threat. One is to treat the written order of the variables in the binding part '^εα' as significant. That is the way adopted by Whitehead and Russell for the case of relations in *Principia*

Mathematica.[11] The significant kind of application of the value-range must then be to an *ordered* pair, so e.g. $\hat{}\varepsilon\alpha[\varepsilon - \alpha]$ applied to $\langle 5,3 \rangle$ is $5 - 3 = 2 = \hat{}\alpha\varepsilon[\varepsilon - \alpha]$ applied to $\langle 3,5 \rangle$. The other possibility is to regard the written order of the variables in '^εα' as not significant. In that case, in order to obtain the correct results for non-symmetric relations and non-commutative functions, the application of the value-range to an *unordered* pair needs to show which applicand goes with which variable. To this end we can adapt the previously mentioned device used by Frege in *Begriffsschrift*,[12] of tagging the argument places bound into using the variable, so $\hat{}\varepsilon\alpha[\varepsilon - \alpha]\{5_\varepsilon,3_\alpha\} = 2 \neq -2 = \hat{}\varepsilon\alpha[\varepsilon - \alpha]\{5_\alpha,3_\varepsilon\}$. For simplicity and definiteness, we here adopt the Whitehead–Russell way of working; it is in any case closer to the spirit of *Gg* than the tagging device. Either way, the effective category of the binary operator ^ is N(N(NN)), whereas that of the monadic smooth-breathing value-range operator ' is N(N(N)).

Frege however notably does not go down the route of binding two variables at once. He uses the unary-function value-range operator to do the work of our binary-function operator, by getting it to work twice in succession. So corresponding to $\hat{}\varepsilon\alpha[\Omega(\varepsilon,\alpha)]$ (on either understanding) we have instead the object

$$\dot{\varepsilon}\dot{\alpha}\Omega(\varepsilon,\alpha)$$

and it is this that Frege calls a double value-range (*Doppelwertverlauf*) of the binary function. To distinguish our alternative value-range with simultaneous saturation from Frege's with successive saturations we may call $\hat{}\varepsilon\alpha[\Omega(\varepsilon,\alpha)]$ the *tandem* value-range of the binary function.[13]

What is going on here? Frege explains it in *Gg* § 36. Take the subtraction function $\xi - \zeta$ and saturate just the second position with a constant, thus: $\xi - 3$. The resulting expression indicates a unary function, whose values for the arguments 0, 1, 2, 3,… are respectively –3, –2, –1, 0, … etc. This unary function has a standard value-range $\dot{\varepsilon}(\varepsilon - 3)$, so that for example $5 \frown \dot{\varepsilon}(\varepsilon - 3) = 2$. Now consider the second argument-place, hitherto occupied by 3, as variable, and we obtain again a unary function $\dot{\varepsilon}(\varepsilon - \zeta)$. Applying the unary value-range operator now to this results in an object

$$\dot{\alpha}\dot{\varepsilon}(\varepsilon - \alpha)$$

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[11] Whitehead and Russell 1910, ∗21.

[12] See Simons 1988. In Frege's case he completely separates the binding variable from the place-marking variable; here we do not need to do so. It is essentially to recover the same kind of flexibility as his function forming and applying has in *Begriffsschrift* that Frege makes so much use of value-ranges in *Grundgesetze*.

[13] As in tandem bicycles or tandem pilot seating, there is still a difference in role between the two places for variables.

and it consorts with the application function in such a fashion that, for example,

$$4 \frown (3 \frown \dot{\alpha}\dot{\varepsilon}(\varepsilon - \alpha)) = 4 \frown \dot{\varepsilon}(\varepsilon - 3) = (4 - 3) = 1$$

so we get the desired results. In the interests of notational and conceptual economy, Frege's decision is quite understandable. The strategy can be adapted for functions of more than two places, simply by repeating the move. For example take the three-place truth-value-function $\xi - \zeta = \tau$. The triple value-range for this would be

$$\dot{\eta}\dot{\varepsilon}\dot{\alpha}(\varepsilon - \alpha = \eta)$$

and a threefold use of the application function would yield for example

$$3 \frown (4 \frown (1 \frown \dot{\eta}\dot{\varepsilon}\dot{\alpha}(\varepsilon - \alpha = \eta))) = 3 \frown (4 \frown \dot{\varepsilon}\dot{\alpha}(\varepsilon - \alpha = 1)) = 3 \frown \dot{\alpha}(4 - \alpha = 1))) = (4 - 3 = 1) = \mathrm{T}$$

In this way Frege avoids any need to define a series of value-range operators, one for each arity of function.

7 Features of Double Value-Ranges

It is typical of Frege that his exposition is so smooth and elegant that it hardly invites consideration of the alternatives. But despite this elegance, we should be clear that Frege did in fact have alternatives, and that his procedure represents a presumably fully conscious decision to do things this way and not another. The obvious move of defining a special and separate value-range operator for each arity of function was considered above. The analogous move, as pointed out above, was taken some years later by Whitehead and Russell in *Principia Mathematica* for binary relations. Frege's way of doing things, that of saturating one argument-place at a time by successive uses of the unary value-range operator, in fact anticipates by some thirty years a device first invented by Moses Schönfinkel,[14] namely the representation of a binary function by a unary function from objects to a unary function. This device, subsequently rather unfairly known as *currying*, after its later adoption by Haskell B. Curry, has become a commonplace of combinatory logic, lambda calculus, and computing. Grammatically and logically speaking, a curried function is a different thing from a binary function: its category is N(N)(N), whereas a binary function's is N(NN). In Frege's terms, it is a unary function with unary functions as values, a second saturation by an object being the extra move required to get from two object arguments to an object.

One of the consequences of this is that whereas the single simultaneous saturation as in $\hat{}\varepsilon\alpha[\Omega(\varepsilon,\alpha)]$ gives one object for the function, albeit one requiring either ordered-pair applicands or tagged applicands, with the sequential saturation we have two clearly distinct

14 Schönfinkel 1924.

objects for a binary function, namely $\dot{\varepsilon}\dot{\alpha}\Omega(\varepsilon,\alpha)$ and $\dot{\alpha}\dot{\varepsilon}\Omega(\varepsilon,\alpha)$. They are not in general the same, for by Basic Law V,

$$(\dot{\varepsilon}\dot{\alpha}\Omega(\varepsilon,\alpha) = \dot{\alpha}\dot{\varepsilon}\Omega(\varepsilon,\alpha)) = \forall\mathfrak{a}[\dot{\alpha}\Omega(\mathfrak{a},\alpha) = \dot{\varepsilon}\Omega(\varepsilon,\mathfrak{a})]$$
$$= \forall\mathfrak{a}[\forall\mathfrak{e}[\Omega(\mathfrak{a},\mathfrak{e}) = \Omega(\mathfrak{e},\mathfrak{a})]]$$

and this is true only for commutative functions. It fails for the subtraction function $\xi - \zeta$ for instance, and for any non-symmetric relation. Indeed Frege uses the reversal of order of binding in the double value-range to define the value-range of the converse of a relation: if p is the value-range of a relation, then $\text{И}p$, the value-range of its converse,[15] is defined as

$$\Vdash\ \text{И}p = \dot{\alpha}\dot{\varepsilon}(\alpha \frown (\varepsilon \frown p))$$

so that it is a theorem[16] that pairs of objects applied successively to double value-ranges for a relation and its converse (more generally, a function and its commutate) yield the same value if they are applied in opposite order

$$r \frown (a \frown q) = a \frown (r \frown \text{И}q)$$

or, making it explicit when q is a double value-range

$$r \frown (a \frown \dot{\alpha}\dot{\varepsilon}\Omega(\varepsilon,\alpha)) = a \frown (r \frown \dot{\varepsilon}\dot{\alpha}\Omega(\varepsilon,\alpha))$$

This makes it clear that Frege is using the nested sequence of applications to a double value-range as a way to get at the order of the relation. In practice he always applies the second argument place first, and then working outwards applies the first argument place second. Thus procedure is of course generalisable to relations of more than two places, so that only one value-range operator is ever needed.

In fact this nested or stepwise procedure is foreshadowed at the very place where Frege introduces many-place functions, as early as *Gg* § 4. There he says that functions with two arguments

> stand *in need of double completion* insofar as a function with one argument is obtained after their completion by one argument has been effected. Only after yet another completion do we arrive at an object, and this object is then called the *value* of the function for the two arguments.[17]

As this makes clear, and as all of his examples likewise show, Frege simply does not envisage what I have called simultaneous saturation. Hence he is not tempted to define anything like our tandem value-range operator ^εα[Ω(ε,α)]. Could he have done so?

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[15] Definition E, *Gg* § 39.

[16] Theorem 21, *Gg* §61.

[17] *Gg* § 4.

8 Simultaneous Saturation

Obviously for order to be respected in the case of non-commutative functions, it cannot be the case that the objects saturating the function $\Omega(\xi,\zeta)$ are simply presented anyhow. We want in general to distinguish $\Omega(\Gamma,\Delta)$ from $\Omega(\Delta,\Gamma)$. So we need a way to indicate which of the objects Δ, Γ saturates the first (ξ) place and which the second (ζ) place. As we saw, Whitehead and Russell indicate this by saying that the members of the extension of a relation are ordered pairs. Will this work in Frege's case? Frege does himself define the ordered pair, using currying, as follows:[18]

> "We now define the pair in this way:
>
> $\Vdash o; a = \dot{\varepsilon}(o \frown (a \frown \varepsilon))$
>
> The semicolon is here a two-sided function-sign. The expression '$\Pi \frown (\Gamma; \Delta)$' is thus co-referential with '$\Gamma \frown (\Delta \frown \Pi)$' provided '$\Gamma$', '$\Delta$' and '$\Pi$' refer to objects."

The obvious thought would be to mimic the application function's definition and define a tandem application function using the ordered pair, so that the application of the ordered pair $o;a$ to $\hat{}\varepsilon\alpha[\Omega(\varepsilon,\alpha)]$ yields $\Omega(o,a)$ while $a;o$ yields $\Omega(a,o)$. Taking this analogously to Frege's definition of $a \frown u$ the tandem application function will be a function $\xi \Cap \zeta$ so that when e is an ordered pair and s is a tandem value-range, the object $e \Cap s$ is the value of the function for the two elements of the ordered pair in order

$$o;a \Cap \hat{}\varepsilon\alpha[\Omega(\varepsilon,\alpha)] = \Omega(o,a)$$

Recalling Frege's definition

$$\Vdash a \frown u = \backslash\dot{\alpha}(\exists\mathfrak{g}[u = \dot{\varepsilon}\mathfrak{g}(\varepsilon) \wedge \mathfrak{g}(a) = \alpha])$$

we can defining tandem application as

$$\Vdash e \Cap s = \backslash\dot{\eta}(\exists\mathfrak{s}\exists\mathfrak{o}\exists\mathfrak{a}[s = \hat{}\varepsilon\alpha[\mathfrak{s}(\varepsilon,\alpha)] \wedge e = \mathfrak{o};\mathfrak{a} \wedge \mathfrak{s}(\mathfrak{o},\mathfrak{a}) = \eta])$$

At this point Frege's use of currying begins to look much the more streamlined approach, as indeed approaches using currying are in general in type-free logics. He has just one value-range operator, one application function, and one Basic Law V, rather than a sheaf of them

[18] Definition Ξ, *Gg* § 144.

for each arity of function, and the ordered pairs and higher tuples are in any case defined via the unary value-range operator.

There are however reasons for looking askance at Frege's choice, elegance aside, leaving aside the paradoxes. The first is that in any logic of finite order, saying nth order, currying will not deliver suitable one-place functions in place of m-ary functions for $m > n$. It is of course a matter of debate whether one's logic *should* be truncated at some finite type or level, but that aside, the fact remains that currying only works for arbitrary arities of relation in the context of a simple type theory. The second objection is a gut-level ontological one. Four-place relations at level one are simply not the same entities as fourth-level curried properties, mappable though they may be one to another. So we should not represent the one as the other. The artificiality also shows up in the fact that there is more than one way to represent a given many-valued function via a value-range: we could represent $\Omega(\xi,\zeta)$ via either of the two available value-ranges, namely $\dot{\varepsilon}\dot{\alpha}\Omega(\varepsilon,\alpha)$ and $\dot{\alpha}\dot{\varepsilon}\Omega(\varepsilon,\alpha)$, and it is a matter of convention, not fact, as to which one we take as representing $\Omega(\xi,\zeta)$ and which we take as representing its converse.

Frege values the simplification attendant on lowering functions by a level, letting value-ranges do the work of first-level functions, so that second-level functions can be represented by first-level functions having the required output for value-ranges of first-level functions. Obviously the move can be iterated to any level, so that in effect Frege at any point can reduce $(n+1)$th level functions to first-level functions on nth degree value-ranges, plus arrangements for "don't care" cases. In view of the fact that there are k^m functions from m items to k items, one has to pause and wonder why the cardinal impossibility of the level-collapsing exercise, which shows up even in the first case, the representation of first-level functions by their value-ranges, never made it through to Frege's consciousness, especially as he was well aware of Cantor's results on the non-collapsing sequence of transfinite cardinals. Presumably we will never know why it never occurred to him, but that notwithstanding, it is difficult to think ourselves retrospectively into the mindset which could blend out the threat of numerical paradox.

9 Concluding Remarks

In a way the discussion of Frege's detailed treatment of double value-ranges is academic, in that the inconsistency of his system drove a coach and horses through the theory of value-ranges in general, and of double value-ranges in particular. But it falls short of historical futility for two reasons. The first is simply that an exploration of the choice Frege made in dealing with the extensions of relations, and more generally the value-ranges of many-placed functions, reveals not only his instinct for formal and logical elegance, but also the deeply anchored nature of his conviction that functions are what they are because of what they output for what they input. This finds expression not only in his much-overhyped context principle, but in his treatment of functions as extensional, a choice which can be disputed on philosophical grounds, but on which we have not touched here.[19]

Secondly however, and lurking behind the distinction between successive and simultaneous saturation, there are rather deep metaphysical questions about the nature of relations, whether a relation among individuals is the same kind of thing as a property of properties (of properties …). If, as I think, and against mathematical convenience, they are not, then logic ought to respect the distinction, distinguishing first-level functions N(NN) from higher-level functions N(N)(N), and so on. For the record, there was once a logician who did recognize the distinction, and built it into his logic and logical grammar, inconvenient though it may appear to be. That logician was Stanisław Leśniewski. But that is a matter for another time.

19 Cf. Tichy 1988.

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