

Elliptic Ruijsenaars-Schneider model via the Poisson reduction of the Affine Heisenberg Double.

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Abstract

It is shown that the elliptic Ruijsenaars-Schneider model can be obtained from the affine Heisenberg Double by means of the Poisson reduction procedure. The dynamical r -matrix naturally appears in the construction.

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1 Introduction

The recent development [1]-[7] in the theory of integrable many-body systems is mainly related with the discovery [9] of dynamical r -matrices, i.e. r -matrices depending on phase variables. One natural way to understand the origin of dynamical r -matrices is to consider the reduction procedure [10]-[14],[2, 6]. In this approach one starts with an initial phase space $\mathcal{P}$ supplied with a symplectic action of some symmetry group. Considering a relatively simple invariant Hamiltonian and factorizing the corresponding dynamics by the symmetry group one gets a smaller phase space $\mathcal{P}_{red}$ with a nontrivial dynamics. Then the L -operator coming in the Lax representation $\frac{dL}{dt} = [M, L]$ appears as a specific coordinate on $\mathcal{P}_{red}$ while the dynamical r -matrix describes the Poisson (Dirac) bracket on the reduced space.

At present the reduction procedure is elaborated for the majority of integrable many-body systems and the corresponding r -matrices are derived. One of the most interesting exception is the elliptic Ruijsenaars-Schneider model [15]. Recently two different dynamical r -matrices for this model were found in [7] and [8]. Both of these r -matrices were obtained by a direct calculation and the question of their equivalence still remains open.

In this letter we apply the Poisson reduction procedure to the affine Heisenberg Double (HD) [16] and derive the elliptic Ruijsenaars-Schneider model with the dynamical r -matrix. The reason to use the affine HD becomes apparent due to its relation with integrable many-body systems of Calogero type. As was shown in [17, 18] the Calogero-Moser and the rational and trigonometric Ruijsenaars-Schneider hierarchies can be obtained by means of the reduction procedure from the cotangent bundle of an affine Lie group $T^*G(z)$ and from a finite dimensional Heisenberg Double. The affine Heisenberg Double may be regarded as a deformation of $T^*G(z)$ and therefore one can suggest that the affine HD is a natural candidate for the phase space standing behind the elliptic Ruijsenaars-Schneider system.

The plan of the paper is as follows. In the second section we briefly describe the affine HD in terms of variables which are suitable for the reduction procedure. Then we fix the momentum map, corresponding to the natural action of the affine Poisson-Lie group on HD. The solution of the momentum map equation is shown to be equivalent to the L -operator of the elliptic Ruijsenaars-Schneider model. In the third section we study the Poisson structure of the reduced phase space and prove that it coincides with the one of the elliptic Ruijsenaars-Schneider model. The dynamical r -matrix naturally appears in our consideration and is equivalent to the one obtained in [7]. In the last section we show that the problem of solving the equations of motion is equivalent to the specific factorization problem.

In our presentation we omit the detailed description of the HD and the proof of some statements. The complete discussion will be given in a forthcoming publication.

2 Affine Heisenberg Double

The general construction of a Poisson manifold known as the Heisenberg double was elaborated in [16]. We shall discuss the HD for the affine $\widehat{GL(N)}$. It is convenient to

describe the Poisson structure of the affine HD in the following form. Let $A(x)$ and $C(x)$ be formal Fourier series in a variable x with values in $GL(N)$. The matrix elements of the harmonics of $A(x)$ and $C(x)$ can be regarded as generators of the algebra of functions on HD. The Poisson structure on HD looks as follows:

$$\begin{aligned}
\frac{1}{\gamma}\{A_1(x), A_2(y)\} &= -r_{\mp}(x-y)A_1(x)A_2(y) - A_1(x)A_2(y)r_{\pm}(x-y) \\
&\quad + A_2(y)r_{+}(x-y-2\Delta)A_1(x) + A_1(x)r_{-}(x-y+2\Delta)A_2(y), \\
\frac{1}{\gamma}\{C_1(x), C_2(y)\} &= -r_{\mp}(x-y)C_1(x)C_2(y) - C_1(x)C_2(y)r_{\pm}(x-y) \\
&\quad + C_2(y)r_{+}(x-y)C_1(x) + C_1(x)r_{-}(x-y)C_2(y), \\
\frac{1}{\gamma}\{A_1(x), C_2(y)\} &= -r_{-}(x-y)A_1(x)C_2(y) - A_1(x)C_2(y)r_{-}(x-y+2\Delta) \\
&\quad + C_2(y)r_{+}(x-y)A_1(x) + A_1(x)r_{-}(x-y+2\Delta)C_2(y), \\
\frac{1}{\gamma}\{C_1(x), A_2(y)\} &= -r_{+}(x-y)C_1(x)A_2(y) - C_1(x)A_2(y)r_{+}(x-y-2\Delta) \\
&\quad + A_2(y)r_{+}(x-y-2\Delta)C_1(x) + C_1(x)r_{-}(x-y)A_2(y),
\end{aligned}$$

where γ and Δ are complex numbers with $\text{Im } \Delta > 0$. Here we use a standard tensor notation. The matrices $r_{\pm}(x)$ are defined by their Fourier series as

$$r_{+}(x) = r_{+} + P \sum_{n>0} e^{-inx}, \quad r_{-}(x) = r_{-} - P \sum_{n>0} e^{inx},$$

where

$$r_{+} = \frac{1}{2} \sum_i E_{ii} \otimes E_{ii} + \sum_{i<j} E_{ij} \otimes E_{ji},$$

$r_{-} = -Pr_{+}P$ and P is the permutation operator. It can be easily checked that

$$r_{+}(x) - r_{-}(x) = 2\pi P\delta(x) \quad \text{and} \quad Pr_{+}(x)P = -r_{-}(-x).$$

In the region of convergence $r_{\pm}(x)$ coincide with the standard trigonometric r -matrix for the affine Lie algebra. The Poisson subalgebra generated by $A(x)$ was introduced in [19] to describe the Poisson structure of $\widehat{GL(N)}^*$.

Assuming the expansions

$$A(x) = I + \gamma J(x) + \dots, \quad C(x) = g(x) + \dots, \quad 2\Delta = \gamma k,$$

where k is a (fixed) central charge, in the deformation limit $\gamma \rightarrow 0$ we recover the standard Poisson structure on the cotangent bundle $T^*\widehat{GL(N)}$ over the level k centrally extended current group $\widehat{GL(N)}$.

The action of the current group $\widehat{GL(N)}$ on HD:

$$\begin{aligned}
A(x) &\rightarrow T^{-1}(x-\Delta)A(x)T(x+\Delta), \\
C(x) &\rightarrow T^{-1}(x-\Delta)C(x)T(x-\Delta)
\end{aligned}$$

is Poissonian. Thereby, we can consider the Poisson reduction of HD over the action of $\widehat{GL(N)}$.

The momentum map taking value in $\widetilde{GL(N)}^*$ reads as follows:

$$M(x) = A^{-1}(x - \Delta)C(x - \Delta)A(x - \Delta)C^{-1}(x + \Delta).$$

It is easy to check that $M(x)$ does generate the action of the current group. We fix the value of $M(x)$ as:

$$M(x) = e^{ih}(1 - 2\pi i\delta_\varepsilon(x)\frac{1 - e^{-ix}}{i}K). \quad (2.1)$$

Here h and ε are arbitrary complex numbers,

$$\delta_\varepsilon(x) = \frac{1}{\varepsilon} \left(\theta(x + \frac{\varepsilon}{2}) - \theta(x - \frac{\varepsilon}{2}) \right) = \frac{1}{2\pi i\varepsilon} \sum_{n=-\infty}^{n=+\infty} \frac{1}{n} (e^{in\frac{\varepsilon}{2}} - e^{-in\frac{\varepsilon}{2}}) e^{inx},$$

and K is a constant matrix $K = e \otimes e^t$, where e is the N -dimensional vector with entries $e_i = 1/\sqrt{N}$ ¹

Although eq.(2.1) can be solved for any value of ε , the reduced phase space remains to be infinite dimensional after performing the factorization procedure. To extract a finite dimensional phase space let us carry out the following trick. By multiplying the both sides of (2.1) on $C(x + \Delta)$, one gets

$$C(x + \Delta) - e^{-ih}A^{-1}(x - \Delta)C(x - \Delta)A(x - \Delta) = 2\pi i\delta_\varepsilon(x)K\frac{1 - e^{-ix}}{i}C(x + \Delta). \quad (2.2)$$

The l.h.s. of this equation does not have any explicit dependence on ε . As to the r.h.s., when ε tends to zero, $\delta_\varepsilon(x)$ tends to $\delta(x)$ and the r.h.s. is well defined only if the function $\frac{1 - e^{-ix}}{i}C(x + \Delta)$ is well defined at $x = 0$. Hence, this equation can be solved only for meromorphic functions $C(x + \Delta)$ with poles of the first order. In this case $\lim_{\varepsilon \rightarrow 0} \delta_\varepsilon(x)\frac{1 - e^{-ix}}{i}C(x + \Delta) = \delta(x)\text{Res}_{x=0}C(x + \Delta)$.

So we define the constraint surface as being the solution of the equation

$$C(x + \Delta) - e^{-ih}A^{-1}(x - \Delta)C(x - \Delta)A(x - \Delta) = 2\pi i\delta(x)K\text{Res}_{x=0}C(x + \Delta)$$

and in the following we shall explore solutions of this equation.

We start with the following difference equation

$$C(x + \Delta) - e^{-ih}D^{-1}C(x - \Delta)D = 2\pi i\delta(x)Y, \quad (2.3)$$

where D is a constant diagonal matrix and Y is an arbitrary constant matrix. Performing the Fourier expansion we get a solution of (2.3) in the form

$$C(x) = i \sum_{ij} \sum_{n=-\infty}^{n=\infty} \frac{e^{inx}}{e^{in\Delta} - e^{-is_{ij}}e^{-in\Delta}} Y_{ij} E_{ij},$$

where we use the notation $s_{ij} = h + q_{ij}$, $D = e^{iq}$, $q_{ij} = q_i - q_j$.

¹It is worthwhile to note that in the deformation limit $\gamma \rightarrow 0$, $\frac{h}{\gamma} \rightarrow \text{const}$, $\frac{\varepsilon}{\gamma} \rightarrow \text{const}$ the constraint (2.1) reduces to the one used in [17] to get the trigonometric Ruijsenaars model.

It is useful to introduce the function of two complex variables

$$w(x, s) = i \sum_{n=-\infty}^{n=\infty} \frac{e^{inx}}{e^{in\Delta} - e^{-is}e^{-in\Delta}}.$$

It is clear that $w(x, s)$ is a meromorphic function of s for any $x : |\operatorname{Im} x| < \operatorname{Im} \Delta$ and has two obvious properties

1. $w(x, s + 2\pi) = w(x, s)$,
2. $w(x, s + 2\Delta) = e^{i\Delta - ix}w(x, s)$.

Moreover, as a function of s it has simple poles at $0, \pm 2\pi, \pm 4\pi, \dots$ and $\pm 2\Delta, \pm 4\Delta, \dots$, $\operatorname{Res}_{s=0} w = 1$. By these data w is uniquely defined as:

$$w(x, s) = \frac{\sigma(s + x - \Delta)}{\sigma(x - \Delta)\sigma(s)} e^{-\frac{\zeta(\pi)}{\pi}(x - \Delta)s}. \quad (2.4)$$

Here $\sigma(x)$ and $\zeta(x)$ are the Weierstrass σ - and ζ -functions with periods equal to 2π and 2Δ . Thus, equation (2.3) has the unique solution

$$C(x) = \sum_{ij} \frac{\sigma(q_{ij} + h + x - \Delta)}{\sigma(x - \Delta)\sigma(q_{ij} + h)} e^{-\frac{\zeta(\pi)}{\pi}(x - \Delta)(h + q_{ij})} Y_{ij} E_{ij} = \sum_{ij} w(x, s_{ij}) Y_{ij} E_{ij}. \quad (2.5)$$

Now we turn to the momentum map equation

$$C(x + \Delta) - e^{-ih} A^{-1}(x - \Delta) C(x - \Delta) A(x - \Delta) = 2\pi i K Z \delta(x), \quad (2.6)$$

where $Z = \operatorname{Res}_{x=0} C(x + \Delta)$.

By using a generic gauge transformation we can diagonalize the field A . Then equation (2.6) takes the form of eq.(2.3)

$$C'(x + \Delta) - e^{-ih} D^{-1} C'(x - \Delta) D = 2\pi i K' Z' \delta(x), \quad (2.7)$$

where

$$A(x) = T(x - \Delta) D T^{-1}(x + \Delta), \quad C(x) = T(x - \Delta) C'(x) T^{-1}(x - \Delta)$$

for some T and $Z' = \operatorname{Res}_{x=\Delta} C'(x)$. We also have

$$K' = T^{-1}(0) K T(0) = T^{-1}(0) e \otimes e^t T(0) = f \otimes v^t, \quad \langle f, v \rangle = 1$$

i.e. $f = T^{-1}(0) e$ and $e^t T(0) = v^t$. According to (2.5) we find

$$C'(x) = \sum_{ij} w(x, s_{ij}) (K' Z')_{ij} E_{ij}.$$

Taking the residue of $C'(x)$ at the point $x = \Delta$ we arrive at the compatibility condition

$$Z' = K' Z' = f \otimes v^t Z', \quad \langle f, v \rangle = 1.$$

The solution of this equation is $Z' = f \otimes g^t$, where g is an arbitrary vector. Now it is easy to find Z :

$$Z = T(0)Z'T^{-1}(0) = T(0)f \otimes g^t T^{-1}(0) = e \otimes g^t T^{-1}(0) \equiv e \otimes b^t$$

Thus, we get

$$C(x + \Delta) - e^{-ih} A^{-1}(x - \Delta) C(x - \Delta) A(x - \Delta) = 2\pi i (e \otimes e^t)(e \otimes b^t) \delta(x), \quad (2.8)$$

where $e \otimes b^t$ is a residue of $C(x)$ at $x = \Delta$.

To summarize, eq.(2.6) has a solution for any field A and for any field C , having a residue at $x = \Delta$ of the form $e \otimes b^t$. For a fixed field A and a vector b this solution is unique. Note that, in general, $\langle b, e \rangle \neq 1$. The form of the r.h.s. of (2.8) shows that the isotropy group of this equation is

$$G_{isot} = \{T(x) \subset G(x) \mid T(0)e = \lambda e, \lambda \in \mathbf{C}\}.$$

This group transforms a solution of (2.8) into another one, so the reduced phase space is defined as

$$\mathcal{P}_{red} = \frac{\text{all solutions of (2.6)}}{G_{isot}}.$$

Since the group G_{isot} is large enough to diagonalize the field A , we can parametrize the reduced phase space by the section (D, L) , where L is a solution of (2.6) with $A = D$. One can easily see that $\mathcal{P}_{red}$ is finite dimensional and its dimension is equal to $2N$, i.e. N coordinates of D plus N coordinates of the vector b . Due to eq.(2.5) the corresponding L -operator has the following form:

$$L(x) = \sum_{ij} \frac{\sigma(q_{ij} + h + x - \Delta)}{\sigma(x - \Delta)\sigma(q_{ij} + h)} e^{-\frac{\zeta(\pi)}{\pi}(x-\Delta)(h+q_{ij})} e_i b_j E_{ij}. \quad (2.9)$$

Multiplying $L(x)$ by the function $\frac{\sigma(x-\Delta)\sigma(h)}{\sigma(x-\Delta+h)} e^{\frac{\zeta(\pi)}{\pi}(x-\Delta)h}$, performing the gauge transformation by means of the diagonal matrix $e^{\frac{\zeta(\pi)}{\pi}(x-\Delta)q}$, and making the shift $x \rightarrow x + \Delta$ we obtain the L -operator of the elliptic Ruijsenaars-Schneider model:

$$L^{Ruij}(x) = \frac{\sigma(x)\sigma(h)}{\sigma(x+h)} e^{\frac{\zeta(\pi)}{\pi}xh} e^{\frac{\zeta(\pi)}{\pi}xq} L(x + \Delta) e^{-\frac{\zeta(\pi)}{\pi}xq} \quad (2.10)$$

Let us briefly discuss the Hamiltonian. It is well known that the simplest nontrivial Hamiltonian invariant with respect to the action of the current group is given by:

$$H = \int dx \operatorname{tr} C(x), \quad (2.11)$$

where α is a constant. It is not difficult to show that on the reduced phase space

$$H_{red} = \frac{2\pi i}{\sqrt{N}(1 - e^{-ih})} \sum_{i=1}^N b_i \quad (2.12)$$

that is up to a constant nothing but the simplest Hamiltonian of the elliptic Ruijsenaars-Schneider model.

3 The Poisson structure on the reduced space

In this section we are going to prove that the Poisson structure on the reduced phase space does coincide with the Poisson structure of the elliptic Ruijsenaars-Schneider model. In other words we need Poisson brackets for the coordinates D -s and b -s. According to the general Dirac construction one should find a gauge invariant extension (we mean the invariance under the action of G_{isot}) of functions on the reduced phase space $\mathcal{P}_{red}$ to a vicinity of $\mathcal{P}_{red}$ and then calculate the Dirac bracket.

One can easily write down the gauge invariant extension for the matrices D and $L(x)$ while the bracket for the coordinates D_i and b_i can be extracted from the bracket for D and $L(x)$. This extension looks as follows:

$$D \rightarrow D[A] = T^{-1}[A](x - \Delta)A(x)T[A](x + \Delta) \quad (3.13)$$

$$L(x) \rightarrow \mathcal{L}[A, C](x) = T^{-1}[A](x - \Delta)C(x)T[A](x - \Delta). \quad (3.14)$$

Some comments are in order. Eq. (3.13) is a solution of the factorization problem for $A(x)$. Generally this solution is not unique but we fix the matrix $T[A]$ by the boundary condition $T[A](0)e = e$ that kills the ambiguity and makes (3.13) to be correctly defined. It is obvious that on $\mathcal{P}_{red}$: $T[A] = 1$ and $\mathcal{L}[A, C](x) = L(x)$.

We start with the calculation of the Poisson bracket for $\mathcal{L}(x)$ and $\mathcal{L}(y)$. We omit the discussion of the contribution from the second class constraints to the Dirac bracket till the end of the section. By definition, one has

$$\begin{aligned} \{\mathcal{L}_1, \mathcal{L}_2\}_{\mathcal{P}_{red}} &= (\{T_1, T_2\}L_1L_2 - L_2\{T_1, T_2\}L_1 - L_1\{T_1, T_2\}L_2 \\ &+ L_1L_2\{T_1, T_2\} + \{C_1, C_2\} - \{T_1, C_2\}L_1 - \{C_1, T_2\}L_2 \\ &+ L_2\{C_1, T_2\} + L_1\{T_1, C_2\})|_{\mathcal{P}_{red}} \end{aligned} \quad (3.15)$$

Here we took into account that $T[A]|_{\mathcal{P}_{red}} = 1$.

Let us first calculate

$$\{C_{ij}(x), T_{kl}(y)\} = \sum_{m,n} \int dz \{C_{ij}(x), A_{mn}(z)\} \frac{\delta T_{kl}(y)}{\delta A_{mn}(z)}.$$

Performing the variation of the both sides of (3.13), we get

$$X(x) = t(x - \Delta)D - Dt(x + \Delta) + d, \quad (3.16)$$

where $X(x) = \delta A(x)$, $t(x) = \delta T(x)$ and $d = \delta D$.

The general solution of (3.16) is

$$t(x) = Q - \frac{1}{2\pi i} \sum_{i,j} \int dz \frac{1}{D_i} w(x - z, q_{ij}) X_{ij}(z) E_{ij}. \quad (3.17)$$

Here Q is some constant diagonal matrix and the function $w(x, 0)$ should be understood as

$$w(x, 0) = \lim_{\varepsilon \rightarrow 0} (w(x, \varepsilon) - \frac{i}{1 - e^{-i\varepsilon}}) = \zeta(x - \Delta) - \frac{\zeta(\pi)}{\pi}(x - \Delta) - \frac{i}{2}.$$

Note that these functions solve the equations

$$\frac{1}{2\pi i}(w(x + \Delta, q_{ij}) - e^{-iq_{ij}}w(x - \Delta, q_{ij})) = \delta(x) - \frac{1}{2\pi}\delta_{ij}.$$

The solution $t(x)$ obeying the condition $t(0)e = 0$ has the following form

$$t(x) = \frac{1}{2\pi i} \sum_{i,j} \int dz \left(\frac{1}{D_i} w(-z, q_{ij}) X_{ij}(z) E_{ii} - \frac{1}{D_i} w(x - z, q_{ij}) X_{ij}(z) E_{ij} \right) \quad (3.18)$$

Performing the variation of eq.(3.18) with respect to $X_{mn}(z)$ one gets

$$\frac{\delta T_{kl}(x)}{\delta A_{mn}(z)}|_{\mathcal{P}_{red}} \equiv Q_{mn}^{kl}(x + \Delta, z) = \frac{1}{2\pi i} \frac{1}{D_k} (w(-z, q_{kn}) \delta_{kl} \delta_{km} - w(x - z, q_{kl}) \delta_{km} \delta_{ln})$$

Thus on the reduced space we obtain

$$\frac{1}{\gamma} \{C_1(x), T_2(y - \Delta)\}|_{red} = \kappa_{12}(x, y) L_1(x) - L_1(x) \omega_{12}(x, y),$$

where

$$\begin{aligned} \kappa_{12}(x, y) &= \text{tr}_3 \int dz (D_3 r_+^{13}(x - z - 2\Delta) - r_+^{13}(x - z) D_3) Q_{23}(y, z), \\ \omega_{12}(x, y) &= \text{tr}_3 \int dz (D_3 r_+^{13}(x - z - 2\Delta) - r_-^{13}(x - z) D_3) Q_{23}(y, z). \end{aligned}$$

We also get

$$\frac{1}{\gamma} \{T_1(x - \Delta), C_2(y)\} = -P \kappa_{12}(y, x) P L_2(y) + L_2(y) P \omega_{12}(y, x) P.$$

By using the relation

$$D_j Q_{ij}^{kl}(x, z) - D_i Q_{ij}^{kl}(x, z - 2\Delta) = \delta(x - z) \delta_{ik} \delta_{jl} - \delta(z - \Delta) \delta_{ik} \delta_{kl} \equiv S_{ij}^{kl},$$

we find

$$\begin{aligned} \kappa_{ij \ kl}(x, y) &= -r_+(x - y)_{ij \ kl} + \sum_m r_+(x - \Delta)_{ij \ km} \delta_{kl} \\ \omega_{ij \ kl}(x, y) &= k_{ij \ kl}(x, y) + 2\pi D_i Q_{ji}^{kl}(y, x). \end{aligned}$$

Recall that

$$\begin{aligned} \frac{1}{\gamma} \{C_1(x), C_2(y)\} &= -r_{\mp}(x - y) C_1(x) C_2(y) - C_1(x) C_2(y) r_{\pm}(x - y) \\ &+ C_2(y) r_+(x - y) C_1(x) + C_1(x) r_-(x - y) C_2(y), \end{aligned}$$

Substituting $\{C, T\}$, $\{T, C\}$ and $\{C, C\}$ brackets into (3.15) we can rewrite the $\{\mathcal{L}, \mathcal{L}\}$ bracket in the following form:

$$\begin{aligned} \frac{1}{\gamma} \{\mathcal{L}_1(x), \mathcal{L}_2(y)\}|_{red} &= -L_1(x) L_2(y) k^+(x, y) - k^-(x, y) L_1(x) L_2(y) \\ &+ L_1(x) s^-(x, y) L_2(y) + L_2(y) s^+(x, y) L_1(x), \end{aligned} \quad (3.19)$$

where

$$\begin{aligned}
k^-(x, y) &= r_-(x - y) + \kappa_{12}(x, y) - P\kappa_{12}(y, x)P - \{T_1(x - \Delta), T_2(y - \Delta)\}, \\
k^+(x, y) &= r_+(x - y) + \omega_{12}(x, y) - P\omega_{12}(y, x)P - \{T_1(x - \Delta), T_2(y - \Delta)\}, \\
s^-(x, y) &= r_-(x - y) + \omega_{12}(x, y) - P\kappa_{12}(y, x)P - \{T_1(x - \Delta), T_2(y - \Delta)\}, \\
s^+(x, y) &= r_+(x - y) + \kappa_{12}(x, y) - P\omega_{12}(y, x)P - \{T_1(x - \Delta), T_2(y - \Delta)\}.
\end{aligned}$$

It is easy to find $Pk^\pm(x, y)P = -P\delta(x - y) - k^\pm(y, x)$ and $Ps^\pm(x, y)P = \pm s^\mp(y, x)$. We also have one more important identity

$$k^+(x, y) + k^-(x, y) = s^+(x, y) + s^-(x, y).$$

To complete the calculation we should find the bracket $\{T_{ij}(x - \Delta), T_{kl}(y - \Delta)\}$ on the reduced space. The straightforward manipulations lead to a divergent result. By this reason we define this bracket as follows:

$$\{T_{ij}(x - \Delta), T_{kl}(y - \Delta)\} = \frac{1}{2} \lim_{\varepsilon \rightarrow 0} \left(\{T_{ij}(x - \Delta), T_{kl}^\varepsilon(y - \Delta)\} + \{T_{ij}^\varepsilon(x - \Delta), T_{kl}(y - \Delta)\} \right)$$

where $T_{kl}^\varepsilon(x)$ is defined as a solution of the factorization problem with the boundary condition $T(\varepsilon)e = e$. We have

$$\begin{aligned}
\{T_{ij}(x - \Delta), T_{kl}^\varepsilon(y - \Delta)\} &= \int dz dz' Q_{mn}^{ij}(x, z) Q_{sp}^{kl \varepsilon}(y, z') \{A_{mn}(z), A_{sp}(z')\} \\
&= \gamma \int dz dz' \left(-r_+(z - z')_{mn \ sp} (D_n Q_{mn}^{ij}(x, z) - D_m Q_{mn}^{ij}(x, z - 2\Delta)) S_{sp}^{kl \varepsilon}(y, z') \right. \\
&\quad \left. - 2\pi P_{mn \ sp} \delta(z - z' + 2\Delta) D_m Q_{mn}^{ij}(x, z) S_{sp}^{kl \varepsilon}(y, z') \right)
\end{aligned}$$

One can prove the cancellation of the singularities as $\varepsilon \rightarrow 0$. The result for the bracket $\{T, T\}$ is

$$\begin{aligned}
&\frac{1}{\gamma} \{T_{ij}(x - \Delta), T_{kl}(y - \Delta)\} \\
&= -r_+(x - y)_{ij \ kl} + \sum_m r_+(x - \Delta)_{ij \ km} \delta_{kl} + \sum_m r_-(\Delta - y)_{im \ kl} \delta_{ij} \\
&\quad + \frac{1}{i} w(x - y + \Delta, q_{ik}) \delta_{jk} \delta_{il} - \frac{1}{i} w(x, q_{ik}) \delta_{jk} \delta_{kl} + \frac{1}{i} w(y, q_{ki}) \delta_{ij} \delta_{il} \\
&\quad + \frac{1}{2} \delta_{ij} \delta_{ik} \delta_{il} + \frac{1}{i} \left(\zeta(q_{ik}) - \frac{\zeta(\pi)}{\pi} q_{ik} \right) \delta_{ij} \delta_{kl} (1 - \delta_{ik}) - \frac{1}{2} \sum_{a < b} (E_{ab} - E_{ba})_{ik} \delta_{ij} \delta_{kl}
\end{aligned}$$

Combining all the pieces together and taking into account the identity $e^{-is}w(x, s) = -w(-x, -s)$ we get the following expression for the coefficients:

$$\begin{aligned}
k_{ij \ kl}^-(x, y) &= -\frac{1}{i} \zeta(q_{ik}) \delta_{ij} \delta_{kl} (1 - \delta_{ik}) - \frac{1}{i} (\zeta(x - y) + \zeta(y - \Delta) - \zeta(x - \Delta)) \delta_{ij} \delta_{ik} \delta_{il} \\
&\quad - \frac{1}{i} (w(x - y + \Delta, q_{ik}) \delta_{il} \delta_{jk} + w(y, q_{ki}) \delta_{il} \delta_{ij} - w(x, q_{ik}) \delta_{jk} \delta_{kl}) (1 - \delta_{ik})
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{i} \frac{\zeta(\pi)}{\pi} q_{ik} \delta_{ij} \delta_{kl} + \frac{1}{2} \sum_{a < b} (E_{ab} - E_{ba})_{ik} \delta_{ij} \delta_{kl} \\
k_{ij}^+{}_{kl}(x, y) & = \frac{1}{i} \left(\zeta(x - y) - \frac{\zeta(\pi)}{\pi} (x - y) \right) \delta_{ij} \delta_{ik} \delta_{il} \\
& + \frac{1}{i} (w(x - y + \Delta, q_{ik}) \delta_{jk} \delta_{il} - \zeta(q_{ik}) \delta_{ij} \delta_{kl}) (1 - \delta_{ik}) \\
& + \frac{1}{i} \frac{\zeta(\pi)}{\pi} q_{ik} \delta_{ij} \delta_{kl} + \frac{1}{2} \sum_{a < b} (E_{ab} - E_{ba})_{ik} \delta_{ij} \delta_{kl} \\
s_{ij}^-{}_{kl}(x, y) & = -\frac{1}{i} \left(\zeta(y - \Delta) - \frac{\zeta(\pi)}{\pi} (y - \Delta) \right) \delta_{ij} \delta_{ik} \delta_{il} \\
& - \frac{1}{i} (w(y, q_{ki}) \delta_{ij} \delta_{il} + \zeta(q_{ik}) \delta_{ij} \delta_{kl}) (1 - \delta_{ik}) \\
& + \frac{1}{i} \frac{\zeta(\pi)}{\pi} q_{ik} \delta_{ij} \delta_{kl} + \frac{1}{2} \sum_{a < b} (E_{ab} - E_{ba})_{ik} \delta_{ij} \delta_{kl} \\
s_{ij}^+{}_{kl}(x, y) & = \frac{1}{i} \left(\zeta(x - \Delta) - \frac{\zeta(\pi)}{\pi} (x - \Delta) \right) \delta_{ij} \delta_{ik} \delta_{il} \\
& + \frac{1}{i} (w(x, q_{ik}) \delta_{jk} \delta_{kl} - \zeta(q_{ik}) \delta_{ij} \delta_{kl}) (1 - \delta_{ik}) \\
& + \frac{1}{i} \frac{\zeta(\pi)}{\pi} q_{ik} \delta_{ij} \delta_{kl} + \frac{1}{2} \sum_{a < b} (E_{ab} - E_{ba})_{ik} \delta_{ij} \delta_{kl}
\end{aligned}$$

It is instructive to note that one can check by direct calculation that the term $\frac{\zeta(\pi)}{i\pi} q_{ik} \delta_{ij} \delta_{kl}$ in the expressions obtained for k -s and s -s does not contribute to the bracket $\{\mathcal{L}, \mathcal{L}\}$.

Remind that (see eq.(2.9)):

$$L_{ii}(x) = \frac{1}{\sqrt{N}} w(x, h) b_i, \quad (3.20)$$

so to obtain the bracket $\{b_i, b_j\}$ it is sufficient to examine the $\{L_{ii}, L_{jj}\}$ bracket only. The crucial point which can be checked by the direct calculation is that the bracket of $\mathcal{L}_{ii}$ with the constraint (2.2) vanishes on $\mathcal{P}_{red}$ in the limit $\varepsilon \rightarrow 0$. Thus, there is no contribution from the Dirac term to the $\{L_{ii}, L_{jj}\}$ bracket.

By substituting the expressions obtained above for k and s into eq.(3.19), one gets for $i \neq j$

$$\frac{1}{\gamma} \{L_{ii}(x), L_{jj}(y)\} = \frac{1}{i} L_{ji}(x) L_{ij}(y) w(x - y + \Delta, q_{ij}) - \frac{1}{i} L_{ij}(x) L_{ji}(y) w(x - y + \Delta, q_{ji}). \quad (3.21)$$

It follows from this equation that

$$\frac{i}{\gamma} \{b_i, b_j\} = b_i b_j \frac{w(x, s_{ji}) w(y, s_{ij}) w(x - y + \Delta, q_{ij}) - w(x, s_{ij}) w(y, s_{ji}) w(x - y + \Delta, q_{ji})}{w(x, h) w(y, h)}. \quad (3.22)$$

By using one of the known elliptic identities [8]², we get

$$\frac{i}{\gamma}\{b_i, b_j\} = b_i b_j (2\zeta(q_{ij}) - \zeta(q_{ij} + h) - \zeta(q_{ij} - h)). \quad (3.23)$$

To complete the examination of the Poisson structure on the reduced phase space one should find the bracket $\{\mathcal{L}, D\}$ and $\{D, D\}$. Performing the straightforward but rather tedious calculations following the same line as above, we find

$$\{D[A]_1, D[A]_2\}|_{red} = 0, \quad (3.24)$$

$$\frac{1}{\gamma}\{\mathcal{L}(x)_1, D[A]_2\}|_{red} = -\sum_{i,j} L_{ij}(x) D_j E_{ij} \otimes E_{jj}. \quad (3.25)$$

It is worthwhile to point out that there are no Dirac terms in these brackets because $D[A]$ is invariant with respect to the action of the whole affine group $\widetilde{GL(N)}$.

Now for the reader's convenience we list the Poisson brackets obtained in terms of the coordinates on $\mathcal{P}_{red}$

$$\begin{aligned} \{q_i, q_j\} &= 0 \\ \frac{i}{\gamma}\{q_i, b_j\} &= b_j \delta_{ij} \\ \frac{i}{\gamma}\{b_i, b_j\} &= b_i b_j (2\zeta(q_{ij}) - \zeta(q_{ij} + h) - \zeta(q_{ij} - h)). \end{aligned} \quad (3.26)$$

One can see that the dynamical system defined by (3.26) and (2.12) is nothing but the elliptic Ruijsenaars-Schneider model.

In [7] the dynamical r -matrix for L^{Ruij} was obtained by direct calculation with the help of the Poisson structure (3.26). Comparing the r -matrix coefficients k and s with the ones in [7] we see that, in fact, they differ by the tensor $\frac{1}{2} \sum_{a < b} (E_{ab} - E_{ba})_{ik} \delta_{ij} \delta_{kl}$. However, in our calculations of the bracket $\{\mathcal{L}, \mathcal{L}\}$ we ignored the contribution from the Dirac term. We conjecture that just the Dirac term is responsible for cancelling this tensor.

4 Equations of motion

The equations of motion for the Hamiltonian (2.12) are given by

$$\dot{D} = \{\text{tr} L(x), D\} = -\gamma L(x)_{diag} D \quad (4.27)$$

and

$$\dot{L}(y) = \{\text{tr} L(x), L(y)\} = [L(y), M(x, y)], \quad (4.28)$$

where

$$M(x, y) = -\gamma i \sum_{kl} (w(x, -q_{kl}) L(x)_{kl} E_{kk} - w(x - y + \Delta, -q_{kl}) L(x)_{kl} E_{kl}). \quad (4.29)$$

²It is interesting to note that this identity can be easily obtained from the x, y -independence condition for the r.h.s. of eq.(3.22).

Here we use eq.(3.19) and the explicit form of k and s . Since $\text{tr}L(x)$ is invariant function the contribution from the Dirac term vanishes. For the reader's convenience we note that by using the elliptic function identities [8] one can rewrite $M \equiv M(x + \Delta, y + \Delta)$ in the following form

$$M = \frac{\gamma}{i} l(x, h) \left(\frac{\zeta(x + h) - \zeta(x - y)}{l(y, h)} L(y + \Delta) - (\zeta(x + h) - \zeta(x)) \left(\sum_i b_i \right) I \right) \quad (4.30)$$

$$+ \sum_k E_{kk} \sum_{i \neq k} (\zeta(q_{ik}) - \zeta(q_{ik} - h)) b_i - \frac{\zeta(\pi)}{\pi} \sum_k b_k E_{kk} \quad (4.31)$$

$$+ \sum_{k \neq l} \frac{\zeta(q_{kl}) - \zeta(q_{kl} + y + h)}{l(y, h)} L_{kl}(y + \Delta) E_{kl} \Big), \quad (4.32)$$

where we have introduced $l(x, h) = w(x + \Delta, h)$. The first two terms in (4.30) are irrelevant, so M coincides with the standard M -matrix of the elliptic RS system.

We show that the general solution of the equations of motion for the elliptic Ruijsenaars-Shneider model is given by

$$D(t) = D[e^{-2\pi\gamma L_0(x)t} D_0], \quad (4.33)$$

where $D \equiv D[A]$ denotes the solution of the factorization problem (3.13):

$$A(x) = T(x - \Delta) D[A] T(x + \Delta)^{-1} \quad (4.34)$$

and D_0 , and $L_0(x)$ are the coordinates and the L -operator at $t = 0$ respectively.

To prove (4.33) we start with calculating the derivative $\dot{D}(t)$:

$$\dot{D}(t) = \int dz \frac{\delta D[A]}{\delta A_{ij}(z)} \Big|_{A=A_t} \frac{d(A_t)_{ij}(z)}{dt}, \quad (4.35)$$

where $A_t(x) = e^{-2\pi\gamma L_0(x)t} D_0$. One can find the derivative $\frac{\delta D[A]}{\delta A_{ij}(z)}$ by performing the variation of eq.(4.34):

$$(T^{-1}(x - \Delta) \delta T(x - \Delta)) D[A] - D[A] (T^{-1}(x + \Delta) \delta T(x + \Delta)) + \delta D = X(x), \quad (4.36)$$

where the notation $X(x) = T^{-1}(x - \Delta) \delta A(x) T(x + \Delta)$ was introduced. In contrast to (3.16) in eq.(4.36) we do not impose the constraint $T = 1$.

Now we solve (4.36) for δD :

$$\delta D = \int \frac{dx}{2\pi} X(x)_{kk} E_{kk}. \quad (4.37)$$

Eq.(4.36) also allows one to find the matrix

$$T^{-1}(x) \delta T(x) = \sum_{k,l} \int \frac{dz}{2\pi i} \left(\frac{1}{D_k} w(-z, q_{kl}) X(z)_{kl} E_{kk} - \frac{1}{D_k} w(x - z, q_{kl}) X(z)_{kl} E_{kl} \right) \quad (4.38)$$

that will be used in the sequel. From (4.37), (4.38) we find

$$\frac{\delta D[A]_{kk}}{\delta A_{ij}(z)} = \frac{1}{2\pi} T_{ki}^{-1}(z - \Delta) T_{jk}(z + \Delta) \quad (4.39)$$

and

$$\begin{aligned} \left(T^{-1}(x) \frac{\delta T(x)}{\delta A_{ij}(z)} \right)_{kl} &= \frac{\delta_{kl}}{2\pi i} \sum_s \frac{w(-z, q_{ks})}{D_k} T_{ki}^{-1}(z - \Delta) T_{js}(z + \Delta) \\ &- \frac{1}{2\pi i} \frac{w(x - z, q_{kl})}{D_k} T_{ki}^{-1}(z - \Delta) T_{jl}(z + \Delta). \end{aligned} \quad (4.40)$$

Substituting (4.40) in eq.(4.35) and taking into account $\dot{A}_t(x) = -2\pi\gamma L_0(x)A_t(x)$, we get

$$\dot{D}(t)_{kk} = -\gamma \int dz T_{ki}^{-1}(z - \Delta) L_0(z)_{im} T_{mn}(z - \Delta) T_{ns}^{-1}(z - \Delta) A_t(z)_{sj} T_{jk}(z + \Delta) \quad (4.41)$$

that with the help of (4.34) reads as follows

$$\dot{D}(t) = -\gamma \int dz (T^{-1}(z - \Delta) L_0(z) T(z - \Delta))_{\text{diag}} D(t). \quad (4.42)$$

The last formula implies the notation

$$\hat{L}_t(x) = T^{-1}(x - \Delta)(t) L_0(x) T(x - \Delta)(t) \quad (4.43)$$

that provides the Lax representation $\frac{d}{dt} \hat{L}_t(x) = [\hat{L}_t(x), \hat{M}(x)]$ with $\hat{M}(x) = T^{-1}(x - \Delta) \dot{T}(x - \Delta)$.

Let us show that the Lax operator $\hat{L}_t(x)$ coincides with the L -operator of the elliptic Ruijsenaars-Shneider model. To this end we calculate explicitly $\hat{M}(x)$. We have

$$\hat{M}_{kl}(x) = \int \left(T^{-1}(x - \Delta) \frac{\delta T(x - \Delta)}{\delta A_{ij}(z)} \right)_{kl} \Big|_{A=A_t} \frac{dA_t(z)_{ij}}{dt}$$

Substituting (4.40) and using the relation $e^{-is}w(x, s) = -w(-x, -s)$ we get

$$\hat{M}(x) = -\gamma i \int dz \sum_{kl} \left(w(z, -q_{kl}) \hat{L}_t(z)_{kl} E_{kk} - w(z - x + \Delta, -q_{kl}) \hat{L}_t(z)_{kl} E_{kl} \right). \quad (4.44)$$

Note that this expression literally coincides with (4.29) if we change $\hat{L}_t$ for L . Since at $t = 0$ the operators $\hat{L}$ and L are equal to L_0 , they coincide for any t .

5 Conclusion

We have proved that the elliptic Ruijsenaars-Schneider model can be obtained by means of the reduction procedure. It is worthwhile to point out that we have used not the Hamiltonian but the Poissonian reduction technique. Our construction is specified by the choice of the trigonometric r -matrix for the Poisson structure on HD and by fixing the special value of the momentum map. By varying the r.h.s. of the momentum map equation one can derive some other systems. For instance, it is not difficult to specify the momentum map equation in a way that leads to the elliptic Calogero-Moser model. It clarifies the coincidence of the dynamical r -matrices for these two models pointed out in [7].

We have considered the simplest example of HD for $GL(N)$. It seems to be interesting to examine the Poissonian reductions of HD that correspond to some other choices of Lie groups or r -matrices.

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