R-matrix Quantization of the Elliptic Ruijsenaars–Schneider model.

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Abstract

It is shown that the classical L-operator algebra of the elliptic Ruijsenaars-Schneider model can be realized as a subalgebra of the algebra of functions on the cotangent bundle over the centrally extended current group in two dimensions. It is governed by two dynamical r and $\bar{r}$-matrices satisfying a closed system of equations. The corresponding quantum R and $\overline{R}$-matrices are found as solutions to quantum analogs of these equations. We present the quantum L-operator algebra and show that the system of equations on R and $\overline{R}$ arises as the compatibility condition for this algebra. It turns out that the R-matrix is twist-equivalent to the Felder elliptic R^F-matrix with $\overline{R}$ playing the role of the twist. The simplest representation of the quantum L-operator algebra corresponding to the elliptic Ruijsenaars-Schneider model is obtained. The connection of the quantum L-operator algebra to the fundamental relation $RLL = LLR$ with Belavin's elliptic R matrix is established. As a byproduct of our construction, we find a new N-parameter elliptic solution to the classical Yang-Baxter equation.

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1 Introduction

The appearance of classical dynamical r-matrices [1, 2] in the theory of integrable many-body systems raises an interesting problem of their quantization. On this way one may hope to separate the variables explicitly.

At present, the classical dynamical r-matrices are known for the rational, trigonometric [2, 3] and elliptic [4, 5] Calogero–Moser (CM) systems, as well as for their relativistic generalizations – the rational, trigonometric [6, 7] and elliptic [8, 9] Ruijsenaars–Schneider (RS) systems [10]. It is recognized that dynamical systems of the Calogero type can be naturally understood in the framework of the Hamiltonian reduction procedure [11, 12]. Moreover, the reduction procedure provides an effective scheme to compute the corresponding dynamical r-matrices [3, 13].

Depending explicitly on the phase variables, the dynamical r-matrices do not satisfy a single closed equation of the Yang–Baxter type, that makes the problem of their quantization rather nontrivial. In [14], the spin generalization of the Calogero–Sutherland system was quantized by using the particular solution [15] of the Gervais–Neveu–Felder equation [16, 15] and in [17], it was interpreted in terms of quasi-Hopf algebras. This system is not integrable, but zero-weight representations of the quantum L-operator algebra admit a proper number of commuting integrals of motion. However, it seems to be important to find such a quantum L-operator algebra for the Calogero-type systems that possesses a sufficiently large abelian subalgebra.

Recently, an algebraic scheme for quantizing the rational RS model in the R-matrix formalism was proposed [18]. We introduced a special parameterization of the cotangent bundle over $GL(N, \mathbf{C})$. In new variables, the standard symplectic structure was described by a classical (Frobenius) r-matrix and by a new dynamical $\bar{r}$-matrix. The classical L-operator was introduced as a special matrix function on the cotangent bundle. The Poisson algebra of L inherited from the cotangent bundle coincided with the L-operator algebra of the rational RS model. It is this reason why we called L the L-operator. Quantizing the Poisson structure for L, we found the quantum L-operator algebra and constructed its particular representation corresponding to the rational RS system. This quantum algebra has a remarkable property, namely, it possesses a family of N mutually commuting operators directly on the algebra level.

It is well-known that the elliptic RS model is the most general among the systems of the CM and RS types. In this paper, we are aimed to include this model in our scheme. Recall that the classical L-operator algebra for the elliptic RS model can be obtained by means of the Poissonian [19] or the Hamiltonian reduction schemes [20]. In the first scheme, the affine Heisenberg double is used as the initial phase space and in the second one, the cotangent bundle over the centrally extended current group in two dimensions is considered. Thus, the appropriate phase space we choose to deal with the elliptic RS model is the cotangent bundle $T^{\star}\widehat{GL}(N)(z, \bar{z})$ over the centrally extended group $\widehat{GL}(N)(z, \bar{z})$ of double loops. The application of our approach [18] is not straightforward since one should work with the infinite-dimensional phase space and, therefore, the correct description of the Poisson structure on $T^{\star}\widehat{GL}(N)(z, \bar{z})$ in the desired parameterization requires an intermediate regularization.

Describe briefly the content of the paper and the results obtained. In the second section we start with describing the Poisson structure on $T^{\star}\widehat{GL}(N)(z, \bar{z})$ that depends on two complex parameters k and α. Then we parametrize $T^{\star}\widehat{GL}(N)(z, \bar{z})$ in a special way. The Poisson structure in new variables is ill defined due to the presence of singularities. To overcome this problem, we introduce an intermediate regularization. Removing the regularization we find

that only for $\alpha = 1/N$ the resulting Poisson structure is well defined. The value $\alpha = 1/N$ corresponds, in fact, to the case where only the $SL(N)(z, \bar{z})$-subgroup is centrally extended. The corresponding Poisson structure is described by two matrices $\mathbf{r}$ and $\bar{\mathbf{r}}$, which depend on N dynamical variables q_i. It follows from the Jacoby identity (see Appendix A) that $\mathbf{r}$ is an N-parameter elliptic solution to the classical Yang-Baxter equation (CYBE). It is worthwhile to mention that the main elliptic identities (see Appendix B) follow from the fulfillment of the CYBE for $\mathbf{r}$. We expect that the matrix $\mathbf{r}$ is related to a special Frobenius subgroup in $GL(N)(z, \bar{z})$ as it was in the finite-dimensional case [18]. The Jacoby identity also implies a closed system of equations on $\mathbf{r}$ and $\bar{\mathbf{r}}$.

We define a special matrix function L on $T^\star \widehat{GL}(N)(z, \bar{z})$. We call this function the "L-operator" since as we show the Poisson algebra of L inherited from $T^\star \widehat{GL}(N)(z, \bar{z})$ literally coincides with the one for the elliptic RS model [9, 20]. Thus, the classical L-operator algebra can be realized as a subalgebra of the algebra of functions on $T^\star \widehat{GL}(N)(z, \bar{z})$. It turns out that the L-operator as a function on $T^\star \widehat{GL}(N)(z, \bar{z})$ admits a factorization $L = WP$, where $p_i = \log P_i$ are the variables canonically conjugated to q_i and W belongs to some special subgroup in $GL(N)(z, \bar{z})$. The Poisson bracket for the entries of W is given by the matrix $\mathbf{r}$ and coincides with the Sklyanin bracket defining the structure of the Poisson–Lie group.

Although the quantum analogs of equations on $\mathbf{r}$ and $\bar{\mathbf{r}}$ can be easily established, it is rather difficult to find the corresponding quantum $\mathbf{R}$ and $\overline{\mathbf{R}}$-matrices. The matter is that the matrices $\mathbf{r}$ and $\bar{\mathbf{r}}$ have the complicated structure, $\mathbf{r} = r - s \otimes I + I \otimes s$ and $\bar{\mathbf{r}} = \bar{r} - s \otimes I$, due to the presence of the s-matrix. However, we observe that the classical L-operator algebra does not depend on s and, moreover, the matrices r and $\bar{r}$ also obey a closed system of equations. We show that this system arises as the compatibility condition for a new Poisson algebra. This algebra contains both the Poisson algebra of $T^\star \widehat{GL}(N)(z, \bar{z})$ and the classical L-operator algebra as its subalgebras. In the third section, using this key observation, we pass to the quantization.

We find the corresponding quantum R and $\overline{R}$-matrices as solutions to quantum analogs of the equations for r and $\bar{r}$. In particular, the R-matrix satisfies a novel triangle relation that differs from the standard quantum Yang-Baxter equation by shifting the spectral parameters in a special way. The Felder elliptic R^F-matrix naturally arises in our construction. It turns out that the R-matrix is twist-equivalent to the R^F-matrix with the $\overline{R}$-matrix playing the role of the twist.

Then we derive a new quadratic algebra satisfied by the "quantum" L-operator. This algebra is described by the quantum dynamical R-matrices, namely, R, R^F, and $\overline{R}$:

$$R_{12} L_1 \overline{R}_{21} L_2 = L_2 \overline{R}_{12} L_1 R^F_{12}.$$

We show that the system of equations on R, R^F, and $\overline{R}$-matrices arises as the compatibility condition for this algebra. We present the simplest representation of the quantum L-operator algebra corresponding to the elliptic RS model. We note that when performing a simple canonical transformation, the quantum L-operator coincides in essential with the classical L-operator found in [10].

The quantum integrals of motion for the elliptic RS model were obtained in [10]. In [21], it was shown that any operator from the Ruijsenaars commuting family can be realized as the trace of a proper transfer matrix for the special $\hat{L}$-operator that obeys the relation $R\hat{L}\hat{L} = \hat{L}\hat{L}R$ with Belavin's elliptic R-matrix [26]. We note that our L-operator is gauge-equivalent to $\hat{L}$. It follows from this observation that the determinant formula for the commuting family [21] is

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also valid for L. We show that any representation of our L-operator algebra is gauge equivalent to a representation of the relations $R\hat{L}\hat{L} = \hat{L}\hat{L}R$.

In Conclusion we discuss some problems to be solved.

2 Classical L-operator algebra

2.1 Poisson structure of $T^\star\widehat{GL}(N)(z, \bar{z})$

Let T_τ be a torus endowed with the standard complex structure and periods 1 and τ. Denote by G a group of smooth mappings from T_τ into the group $GL(N, \mathbf{C})$. Then $g \in G$ is a double-periodic matrix function $g(z, \bar{z})$. The dual space to the Lie algebra of G is spanned by double-periodic functions $A(z, \bar{z})$ with values in $\mathrm{Mat}\,(N, \mathbf{C})$. In what follows, we often use the concise notation $g(z, \bar{z}) = g(z)$ and $A(z, \bar{z}) = A(z)$. The group G admits central extensions $\widehat{G}$ [22]. The Poisson structure on $T^*\widehat{G}$ with fixed central charges reads

$$\{A_1(z), A_2(w)\} = \frac{1}{2}[C, A_1(z) - A_2(w)]\delta(z - w) - k(C - \alpha I)\frac{\partial}{\partial\bar{z}}\delta(z - w) \qquad (2.1)$$

$$\{g_1(z), g_2(w)\} = 0 \qquad (2.2)$$

$$\{A_1(z), g_2(w)\} = g_2(w)C\delta(z - w), \qquad (2.3)$$

where k, α are central charges and $\delta(z)$ is the two-dimensional δ-function. Here we use the standard tensor notation, and C is the permutation operator.

One can consider the following Hamiltonian action of G on $T^*\widehat{G}$

$$A(z) \rightarrow T^{-1}(z)A(z)T(z) + kT^{-1}(z)\bar{\partial}T(z), \qquad (2.4)$$

$$g(z) \rightarrow T^{-1}(z)g(z)T(z).$$

We restrict our consideration to the case of smooth elements $A(z)$. Then generic element $A(z)$ can be diagonalized by the transformation (2.4) [23]:

$$A(z) = T(z)DT^{-1}(z) - k\bar{\partial}T(z)T^{-1}(z). \qquad (2.5)$$

Here D is a constant diagonal matrix with entries D_i, $D_i \neq D_j$ and $T(z)$ is double-periodic. Matrix D is defined up to the action of the elliptic Weyl group. One can fix D by choosing the fundamental Weyl chamber.

Matrix $T(z)$ in eq.(2.5) is not uniquely defined. Any element $\tilde{T}(z, \bar{z}) = T(z, \bar{z})h(z)$, where a diagonal matrix $h(z)$ is an entire function of z, also satisfies (2.5). Demanding $\tilde{T}(z, \bar{z})$ to be double-periodic, we obtain that $h(z)$ is a constant matrix. We can remove this ambiguity by imposing the condition

$$T(\varepsilon)e = e, \qquad (2.6)$$

where e is a vector such that $e_i = 1\ \forall i$, and ε is an arbitrary point on T_τ. In what follows, we denote the matrix $T(z)$ that solves eq.(2.5) and satisfies eq.(2.6) by $T^\varepsilon(z)$. Such matrices evidently form a group.

Now we try to rewrite the Poisson structure (2.1) in terms of variables T and D. Since D_i are G-invariant functions, they belong to the center of (2.1) and, therefore, it is enough to

calculate the bracket $\{T^\varepsilon(z), T^\varepsilon(w)\}$. However, the straightforward calculation reveals that this bracket is ill defined. So, we begin with calculating the bracket $\{T^\varepsilon(z), T^\eta(w)\}$, where $T^\varepsilon(z)$ and $T^\eta(w)$ satisfy (2.6) at different points ε and η,

$$\{T^\varepsilon_{ij}(z), T^\eta_{kl}(w)\} = \sum_{mnps} \int d^2z' d^2w' \frac{\delta T^\varepsilon_{ij}(z)}{\delta A_{mn}(z')} \frac{\delta T^\eta_{kl}(w)}{\delta A_{ps}(w')} \{A_{mn}(z'), A_{ps}(w')\}. \tag{2.7}$$

To calculate the functional derivative $\frac{\delta T^\varepsilon_{ij}(z)}{\delta A_{mn}(z')}$, we consider the variation of (2.5):

$$X(z) = t(z)D - Dt(z) - k\bar{\partial}t(z) + d, \tag{2.8}$$

where $X(z) = T^{-1}(z)\delta A(z)T(z)$, $t(z) = T^{-1}(z)\delta T(z)$ and $d = \delta D$.

First, from (2.8), we immediately obtain

$$\frac{\delta D_i}{\delta A_{kl}(z)} = \frac{1}{(\tau - \bar{\tau})} T^{-1}_{ik}{}^\varepsilon(z) T^\varepsilon_{li}(z). \tag{2.9}$$

Let us introduce the function $\Phi(z, s)$ of two complex variables

$$\Phi(z, s) = \frac{\sigma(z+s)}{\sigma(z)\sigma(s)} e^{-2\zeta(\frac{1}{2})zs} e^{2\pi i s \frac{z-\bar{z}}{\tau - \bar{\tau}}}. \tag{2.10}$$

Here $\sigma(z)$ and $\zeta(z)$ are the Weierstrass σ- and ζ-functions with periods equal to 1 and τ. The function $\Phi(z, s)$ is the only double-periodic solution to the following equation:

$$\bar{\partial}\Phi(z, s) + \frac{2\pi i s}{\tau - \bar{\tau}}\Phi(z, s) = 2\pi i \delta(z). \tag{2.11}$$

It is also convenient to define $\Phi(z, 0)$ as follows:

$$\Phi(z, 0) = \lim_{\varepsilon \to 0}\left(\Phi(z, \varepsilon) - \frac{1}{\varepsilon}\right) = \zeta(z) - 2\zeta(\frac{1}{2})z + 2\pi i \frac{z - \bar{z}}{\tau - \bar{\tau}}.$$

This function solves the equation

$$\bar{\partial}\Phi(z, 0) = 2\pi i \delta(z) - \frac{2\pi i}{\tau - \bar{\tau}}.$$

We introduce the notation $q_{ij} \equiv q_i - q_j$, where $q_i = \frac{\tau - \bar{\tau}}{2\pi i k}D_i$.

Using these functions, one can write the solution to (2.8) obeying the condition $t(\varepsilon)e = 0$ [20]:

$$t(z) = \kappa \sum_{i,j} \int d^2w (\Phi(\varepsilon - w, q_{ij})X_{ij}(w)E_{ii} - \Phi(z - w, q_{ij})X_{ij}(w)E_{ij}). \tag{2.12}$$

Hereafter, we denote $\frac{1}{2\pi i k}$ by κ.

Performing the variation of eq.(2.12) with respect to $A_{mn}(w)$ one gets

$$\frac{\delta T^\varepsilon_{ij}(z)}{\delta A_{mn}(w)} = \kappa \left(\sum_k \Phi(\varepsilon - w, q_{jk}) T^\varepsilon_{ij}(z) T^{-1}_{jm}{}^\varepsilon(w) T^\varepsilon_{nk}(w) \right.$$

$$\left. - \sum_k \Phi(z - w, q_{kj}) T^\varepsilon_{ik}(z) T^{-1}_{km}{}^\varepsilon(w) T^\varepsilon_{nj}(w) \right).$$

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To compute the bracket (2.7), one needs the following relation between $T^\varepsilon(z)$ and $T^\eta(z)$:

$$T^\varepsilon(z) = T^\eta(z)H^{\eta\varepsilon}, \tag{2.13}$$

where $H^{\eta\varepsilon}$ is a constant diagonal matrix.

By direct computation, one finds

$$\frac{1}{\kappa}\{T_1^\varepsilon(z), T_2^\eta(w)\} = T_1^\varepsilon(z)T_2^\eta(w)(H_1^{\varepsilon\eta}H_2^{\eta\varepsilon}r_{12}^{\varepsilon\eta}(z,w) - \alpha f^{\varepsilon\eta}(z,w)), \tag{2.14}$$

where

$$
\begin{aligned}
r_{12}^{\varepsilon\eta}(z,w) &= \sum_{ij}\Phi(\varepsilon - \eta, q_{ij})E_{ii}\otimes E_{jj} + \sum_{ij}\Phi(z-w, q_{ij})E_{ij}\otimes E_{ji} \tag{2.15}\\
&- \sum_{ij}\Phi(z-\eta, q_{ij})E_{ij}\otimes E_{jj} + \sum_{ij}\Phi(w-\varepsilon, q_{ij})E_{jj}\otimes E_{ij} \tag{2.16}
\end{aligned}
$$

and

$$f^{\varepsilon\eta}(z,w) = \Phi(\varepsilon - \eta, 0) + \Phi(\omega - \varepsilon, 0) + \Phi(z - \omega, 0) - \Phi(z - \eta, 0).$$

The bracket (2.14) has the r-matrix form with the r-matrix depending not only on coordinates q_i but also on the additional variables H.

In the limit $\eta \to \varepsilon$, one encounters the singularity. This shows that the variable $T(z)$ is not a good candidate to describe the Poisson structure (2.1). However, one can use the freedom to multiply $T(z)$ by any functional of A. So, we introduce a new variable

$$\mathbf{T}^\varepsilon(z) = T^\varepsilon(z)(\det T^\varepsilon(\varepsilon))^\beta. \tag{2.17}$$

We use $\det T^\varepsilon(\varepsilon)$ in the definition of $\mathbf{T}^\varepsilon(z)$ in order to have the group structure for the new variables.

Using the Poisson bracket (2.14) one immediately finds

$$
\begin{aligned}
\frac{1}{\kappa}\{\mathbf{T}_1^\varepsilon(z), \mathbf{T}_2^\eta(w)\} = \ &\mathbf{T}_1^\varepsilon(z)\mathbf{T}_2^\eta(w)\,(H_1^{\varepsilon\eta}H_2^{\eta\varepsilon}r_{12}^{\varepsilon\eta}(z,w) - \alpha f^{\varepsilon\eta}(z,w)\\
&+ \ \beta I \otimes \text{tr}_3 H_3^{\varepsilon\eta}H_2^{\eta\varepsilon}r_{32}^{\varepsilon\eta}(\varepsilon, w) + \beta\,\text{tr}_3 H_1^{\varepsilon\eta}H_3^{\eta\varepsilon}r_{13}^{\varepsilon\eta}(z,\eta)\otimes I\\
&+ \ \beta^2\,\text{tr}_{34} H_3^{\varepsilon\eta}H_4^{\eta\varepsilon}r_{34}^{\varepsilon\eta}(\varepsilon,\eta)\,I\otimes I\Big), \tag{2.18}
\end{aligned}
$$

since $f(\varepsilon, w) = f(z, \eta) = 0$.

Now we are going to pass to the limit $\eta \to \varepsilon$.[1] For this purpose one should take into account the following behavior of $H^{\varepsilon\eta}$ when η goes to ε: $H^{\varepsilon\eta} = 1 + (\varepsilon - \eta)h + o(\varepsilon - \eta)$, where h is a constant diagonal matrix being the functional of A. It turns out that there exists a unique choice for α and β, namely, $\alpha = 1/N$, $\beta = -1/N$, for which the singularities cancel and there is no contribution from the matrix h. In the limit $\eta \to \varepsilon = 0$,[2] for these values of α and β, one gets

$$\frac{1}{\kappa}\{\mathbf{T}_1(z), \mathbf{T}_2(w)\} = \mathbf{T}_1(z)\mathbf{T}_2(w)\mathbf{r}_{12}(z,w). \tag{2.19}$$

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[1]Without loss of generality we assume that ε and η are real.

[2]The ε-dependence can be easily restored by shifting both z and w by ε.

Here the limiting **r**-matrix is given by

$$\mathbf{r}_{12}(z,w) = r_{12}(z,w) - s(z) \otimes I + I \otimes s(w) - \frac{1}{N}\Phi(z-w,0)I \otimes I, \tag{2.20}$$

where

$$\begin{aligned}
r(z,w) &= \sum_{i \neq j} \Phi(q_{ij})E_{ii} \otimes E_{jj} + \sum_{ij}\Phi(z-w,q_{ij})E_{ij} \otimes E_{ji} \\
&\quad - \sum_{ij}\Phi(z,q_{ij})E_{ij} \otimes E_{jj} + \sum_{ij}\Phi(w,q_{ij})E_{jj} \otimes E_{ij}
\end{aligned} \tag{2.21}$$

and

$$s(z) = \frac{1}{N}\sum_{ij}\left(\Phi(q_{ij})E_{ii} - \Phi(z,q_{ij})E_{ij}\right). \tag{2.22}$$

Here we denote by the function $\Phi(q_{ij})$ the regular part of $\Phi(\varepsilon, q_{ij})$ for $\varepsilon \to 0$:

$$\begin{aligned}
\Phi(q_{ij}) &= \zeta(q_{ij}) - 2\zeta(\tfrac{1}{2})q_{ij}, \quad i \neq j, \\
\Phi(q_{ii}) &= 0.
\end{aligned}$$

Note that both **r** and r are skew-symmetric: $\mathbf{r}_{12}(z,w) = -\mathbf{r}_{21}(w,z)$.

A natural conjecture is that the **r**-matrix obtained satisfies the classical Yang–Baxter equation

$$[[\mathbf{r},\mathbf{r}]] \equiv [\mathbf{r}_{12}(z_1,z_2),\mathbf{r}_{13}(z_1,z_3) + \mathbf{r}_{23}(z_2,z_3)] + [\mathbf{r}_{13}(z_1,z_3),\mathbf{r}_{23}(z_2,z_3)] = 0. \tag{2.23}$$

It can be verified either by direct calculation or by considering the limiting case of the Jacoby identity for the bracket (2.18) as is done in the Appendix A. Thereby, the **r**-matrix (2.20) is an N-parameter solution of the classical Yang–Baxter equation.

Let us note that, as one could expect, the condition $\det \mathbf{T}(0) = 1$ is compatible with the bracket (2.19), since $\det \mathbf{T}(z)$ is a central element of algebra (2.19).

Remark that the choice $\alpha = 1/N$ corresponds to the case where only the $sl(N)(z,\bar{z})$-subalgebra is centrally extended. In terms of $\mathbf{T}(z)$ ($\beta = -1/N$), the boundary condition looks like $\mathbf{T}(0)e = \lambda e$ and $\det \mathbf{T}(0) = 1$. One can also check that the field $A(z)$ defined by (2.5) with the substitution $\mathbf{T}(z)$ for $T(z)$ obeys Poisson algebra (2.1).

The next step is to consider the special parameterization for the field $g(z)$. To this end, we introduce $\tilde{A}(z)$:

$$\tilde{A} = gAg^{-1} - k\bar{\partial}gg^{-1} + \frac{k}{N}\operatorname{tr}\bar{\partial}gg^{-1}. \tag{2.24}$$

One can check that $\tilde{A}(z)$ Poisson commutes with $A(w)$ and obeys the Poisson algebra:

$$\begin{aligned}
\{\tilde{A}_1(z),\tilde{A}_2(w)\} &= -\frac{1}{2}[C,\tilde{A}_1(z) - \tilde{A}_2(w)]\delta(z-w) + k(C - \frac{1}{N}I)\frac{\partial}{\partial\bar{z}}\delta(z-w) \\
\{\tilde{A}_1(z),g_2(w)\} &= Cg_2(w)\delta(z-w).
\end{aligned} \tag{2.25}$$

Now we factorize $\tilde{A}(z)$ in the same manner as it was done for $A(z)$,

$$\tilde{A}(z) = \mathbf{U}(z)D\mathbf{U}^{-1}(z) - k\bar{\partial}\mathbf{U}(z)\mathbf{U}^{-1}(z), \tag{2.26}$$

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where $\mathbf{U}(z)$ satisfies the boundary condition $\mathbf{U}(0)e = \lambda e$ and $\det \mathbf{U}(0) = 1$. Obviously, $\mathbf{U}(z)$ Poisson commutes with $\mathbf{T}(w)$ and satisfies the Poisson algebra

$$\frac{1}{\kappa}\{\mathbf{U}_1(z), \mathbf{U}_2(w)\} = -\mathbf{U}_1(z)\mathbf{U}_2(w)\mathbf{r}_{12}(z, w). \tag{2.27}$$

One can find from (2.24) and (2.26) the representation for the field g,

$$g(z) = (\det g(z))^{\frac{1}{N}} \mathbf{U}(z)\mathbf{P}\mathbf{T}^{-1}(z), \tag{2.28}$$

where $\mathbf{P}$ is a constant diagonal matrix.

Computing the determinants of the both sides of eq.(2.28) one gets

$$\det \mathbf{P} = \det \left(\mathbf{T}(z)/\mathbf{U}(z)\right).$$

Since the l.h.s. does not depend on z and $\det \mathbf{T}(0) = \det \mathbf{U}(0) = 1$ we obtain that $\det \mathbf{P} = 1$ and $\det \mathbf{T}(z) = \det \mathbf{U}(z)$.

Calculating the Poisson brackets of $\mathbf{P}$ with $\mathbf{P}$ and $Q = \mathrm{diag}(q_1, \dots q_N)$ in the same manner as above one reveals that

$$\{\mathbf{P}_1, \mathbf{P}_2\} = 0, \tag{2.29}$$

$$\frac{1}{\kappa}\{Q_1, \mathbf{P}_2\} = \mathbf{P}_2(\sum_{ii} E_{ii} \otimes E_{ii} - \frac{1}{N}I \otimes I). \tag{2.30}$$

In fact, it means that $\log \mathbf{P}_i = p_i - \frac{1}{N}\sum_i p_i$, where p_i are canonically conjugated to q_i.

For the remaining Poisson brackets of $\mathbf{P}$ with the fields $\mathbf{T}$, $\mathbf{U}$, we have

$$\frac{1}{\kappa}\{\mathbf{T}_1(z), \mathbf{P}_2\} = \mathbf{T}_1(z)\mathbf{P}_2\bar{\mathbf{r}}_{12}(z), \tag{2.31}$$

$$\frac{1}{\kappa}\{\mathbf{U}_1(z), \mathbf{P}_2\} = \mathbf{U}_1(z)\mathbf{P}_2\bar{\mathbf{r}}_{12}(z). \tag{2.32}$$

Here

$$\bar{\mathbf{r}}_{12}(z) = \bar{r}_{12}(z) - s(z) \otimes I - \frac{1}{N}I \otimes \sum_{ij} \Phi(q_{ij})E_{jj}, \tag{2.33}$$

where we introduced the $\bar{r}$-matrix:

$$\bar{r}_{12}(z) = \sum_{ij} \Phi(q_{ij})E_{ii} \otimes E_{jj} - \sum_{ij} \Phi(z, q_{ij})E_{ij} \otimes E_{jj}. \tag{2.34}$$

To complete the description of the classical Poisson structure of the cotangent bundle we present the Poisson bracket of $\det g$ with other variables:

$$\frac{1}{\kappa}\{Q, \det g(w)\} = \det g(w), \qquad \{\mathbf{P}, \det g(w)\} = 0,$$

$$\frac{1}{\kappa}\{\mathbf{T}(z), \det g(w)\} = -\det g(w)\mathbf{T}(z)(\Phi(z - w, 0) + \Phi(w, 0)),$$

$$\frac{1}{\kappa}\{\mathbf{U}(z), \det g(w)\} = -\det g(w)\mathbf{U}(z)(\Phi(z - w, 0) + \Phi(w, 0)).$$

Recall that the Jacoby identity for bracket (2.19) reduces to the classical Yang-Baxter equation for the **r**-matrix. As to the Poisson relations (2.31) and (2.32), one finds that the Jacoby identity is equivalent to the following quadratic in $\bar{\mathbf{r}}$ equation:

$$[\bar{\mathbf{r}}_{12}(z), \bar{\mathbf{r}}_{13}(z)] - \mathbf{P}_3^{-1}\{\bar{\mathbf{r}}_{12}(z), \mathbf{P}_3\} + \mathbf{P}_2^{-1}\{\bar{\mathbf{r}}_{13}(z), \mathbf{P}_2\} = 0 \tag{2.35}$$

and the equation involving **r** and $\bar{\mathbf{r}}$,

$$[\mathbf{r}_{12}(z, w), \bar{\mathbf{r}}_{13}(z) + \bar{\mathbf{r}}_{23}(w)] + [\bar{\mathbf{r}}_{13}(z), \bar{\mathbf{r}}_{23}(w)] - \mathbf{P}_3^{-1}\{\bar{\mathbf{r}}_{12}(z, w), \mathbf{P}_3\} = 0. \tag{2.36}$$

One can check by direct calculations that the matrices **r** and $\bar{\mathbf{r}}$ given by (2.20) and (2.33) do solve these equations.

Finally, we remark that the fields $A(z)$ and $g(z)$ defined by (2.5) and (2.28) obey Poisson relations (2.1–2.3).

Let us note that the Poisson relation for the generator $\mathbf{W}(z) = \mathbf{T}^{-1}(z)\mathbf{U}(z)$ turns out to be the Sklyanin bracket:

$$\frac{1}{\kappa}\{\mathbf{W}_1(z), \mathbf{W}_2(w)\} = [\mathbf{r}_{12}(z, w), \mathbf{W}_1(z)\mathbf{W}_2(w)], \tag{2.37}$$

which, therefore, defines the structure of a Poisson-Lie group. This group is an infinite-dimensional analog of the Frobenius group appeared in [18], where the Poisson-Lie group structure was related to the existence of a non-degenerate two-cocycle on the corresponding Lie algebra. It would be interesting to find a similar interpretation in the infinite-dimensional case.

2.2 Classical L-operator

In this subsection, we define a special function on the cotangent bundle, which we call the *classical L-operator*. The motivation to treat this function as the L-operator is that the Poisson algebra of L is equivalent to the one found in [9] for the L-operator of the elliptic RS model.

Denote by L the following function

$$L(z) = \mathbf{T}^{-1}(z)g(z)\mathbf{T}(z) = (\det g(z))^{\frac{1}{N}}\mathbf{T}^{-1}(z)\mathbf{U}(z)\mathbf{P}. \tag{2.38}$$

By using the formulas of the previous subsection, one can easily derive the Poisson bracket of L and Q:

$$\frac{1}{\kappa}\{Q_1, L_2(z)\} = L_2(z)\sum_i E_{ii} \otimes E_{ii}, \tag{2.39}$$

and the Poisson algebra of the L-operator,

$$\begin{aligned}
\frac{1}{\kappa}\{L_1(z), L_2(w)\} &= \mathbf{r}_{12}(z, w)L_1(z)L_2(w) \\
&+ L_1(z)L_2(w)(\bar{\mathbf{r}}_{12}(z) - \bar{\mathbf{r}}_{21}(w) - \mathbf{r}_{12}(z, w)) \\
&+ L_1(z)\bar{\mathbf{r}}_{21}(w)L_2(w) - L_2(w)\bar{\mathbf{r}}_{12}(z)L_1(z).
\end{aligned} \tag{2.40}$$

Clearly, the generators Q and L form a Poisson subalgebra in the Poisson algebra of the cotangent bundle. An important feature of this subalgebra is that $I_n(z) = \operatorname{tr} L^n(z)$ form a set of mutually commuting variables.

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Just as in the finite-dimensional case [18], one can see from (2.39) that the L-operator admits the following factorization: $L(z) = W(z)P$, where Q and $\log P$ are canonically conjugated variables, the W-algebra coincides with (2.37), and the bracket of W and P is

$$\frac{1}{\kappa}\{W_1(z), P_2\} = -P_2[\bar{\mathbf{r}}_{12}(z), W_1(z)]. \tag{2.41}$$

In fact, everything what we need to quantize the L-operator algebra (2.40) is prepared. The problem of quantization is reduced to finding the quantum $\mathbf{R}$ and $\overline{\mathbf{R}}$-matrices satisfying the quantum analogs of eqs.(2.23), (2.35), and (2.36),

$$\begin{aligned}
\mathbf{R}_{12}(z_1, z_2)\mathbf{R}_{21}(z_2, z_1) &= 1, & (2.42) \\
\mathbf{R}_{12}(z_1, z_2)\mathbf{R}_{13}(z_1, z_3)\mathbf{R}_{23}(z_2, z_3) &= \mathbf{R}_{23}(z_2, z_3)\mathbf{R}_{13}(z_1, z_3)\mathbf{R}_{12}(z_1, z_2), & (2.43) \\
\mathbf{R}_{12}(z_1, z_2)\overline{\mathbf{R}}_{13}(z_1)\overline{\mathbf{R}}_{23}(z_2) &= \overline{\mathbf{R}}_{23}(z_2)\overline{\mathbf{R}}_{13}(z_1)P_3^{-1}\mathbf{R}_{12}(z_1, z_2)P_3, & (2.44) \\
\overline{\mathbf{R}}_{12}(z)P_2^{-1}\overline{\mathbf{R}}_{13}(z)P_2 &= \overline{\mathbf{R}}_{13}(z)P_3^{-1}\overline{\mathbf{R}}_{12}(z)P_3. & (2.45)
\end{aligned}$$

These matrices are assumed to have the standard behavior near $\hbar = 0$:

$$\mathbf{R} = 1 + \hbar\mathbf{r} + o(\hbar), \qquad \overline{\mathbf{R}} = 1 + \hbar\bar{\mathbf{r}} + o(\hbar),$$

where $\hbar$ is a quantization parameter.

The problem formulated seems to be rather complicated due to the presence of the s-matrix in the classical $\mathbf{r}$ and $\bar{\mathbf{r}}$-matrices. However, Poisson algebra (2.40) possesses an important property allowing one to avoid the problem at hand. Namely, the matrix $s(z)$ coming both in $\mathbf{r}$ and $\bar{\mathbf{r}}$ drops out from the r.h.s. of (2.40). Thereby, eq.(2.40) can be eventually rewritten as

$$\begin{aligned}
\frac{1}{\kappa}\{L_1(z), L_2(w)\} &= r_{12}(z, w)L_1(z)L_2(w) & (2.46) \\
&\quad - L_1(z)L_2(w)(r_{12}(z, w) + \bar{r}_{21}(w) - \bar{r}_{12}(z)) \\
&\quad + L_1(z)\bar{r}_{21}(w)L_2(w) - L_2(w)\bar{r}_{12}(z)L_1(z).
\end{aligned}$$

Moreover, if we denote by r_{12}^F the sum

$$r_{12}^F(z, w) = r_{12}(z, w) + \bar{r}_{21}(w) - \bar{r}_{12}(z), \tag{2.47}$$

then using (2.21) and (2.34) we obtain

$$r_{12}^F(z - w) = -\sum_{ij}\Phi(q_{ij})E_{ii}\otimes E_{jj} + \sum_{ij}\Phi(z - w, q_{ij})E_{ij}\otimes E_{ji}. \tag{2.48}$$

In this expression one can recognize the elliptic solution to the classical Gervais–Neveu–Felder equation [16, 15]:

$$\begin{aligned}
[r_{12}^F(z_1 - z_2), r_{13}^F(z_1 - z_3) + r_{23}^F(z_2 - z_3)] + [r_{13}^F(z_1 - z_3), r_{23}^F(z_2 - z_3)] & (2.49) \\
-P_3^{-1}\{r_{12}^F(z_1 - z_2), P_3\} + P_2^{-1}\{r_{13}^F(z_1 - z_3), P_2\} - P_1^{-1}\{r_{23}^F(z_2 - z_3), P_1\} = 0.
\end{aligned}$$

In fact, r^F emerges as the semiclassical limit of the quantum R-matrix found in [15].

The absence of the s-matrix in the resulting L-operator algebra and the appearance of the r^F-matrix show that there may exist a closed system of equations involving only r- and

$\bar{r}$-matrices in the classical case, and R- and $\overline{R}$-matrices in the quantum one. In the next subsection we find the desired system of equations and describe a Poisson structure for which these equations ensure the fulfillment of the Jacoby identity.

Note that the algebra (2.46) literally coincides with the one obtained in [20] by using of the Hamiltonian reduction procedure. A mere similarity transformation of L turns algebra (2.46) to the one previously found in [9]. In contrast to [9] where (2.46) was derived by direct calculation with the usage of the particular form of the L-operator for the RS model, our treatment does not appeal to the particular form of L.

2.3 Quadratic Poisson algebra with derivatives

In the first subsection, we obtained the matrices $\mathbf{r}$ and $\bar{\mathbf{r}}$ obeying system of equations (2.23), (2.35) and (2.36). Clearly, these equations are not satisfied when substituting r and $\bar{r}$ for $\mathbf{r}$ and $\bar{\mathbf{r}}$. However, computing the l.h.s. of these equations after this substitution we arrive at surprisingly simple result:

$$[r_{12}(z_1, z_2), r_{13}(z_1, z_3) + r_{23}(z_2, z_3)] + [r_{13}(z_1, z_3), r_{23}(z_2, z_3)] = \tag{2.50}$$
$$-(\partial_1 + \partial_2)r_{12}(z_1, z_2) + (\partial_1 + \partial_3)r_{13}(z_1, z_3) - (\partial_2 + \partial_3)r_{23}(z_2, z_3),$$

$$[\bar{r}_{12}(z), \bar{r}_{13}(z)] - P_3^{-1}\{\bar{r}_{12}(z), P_3\} + P_2^{-1}\{\bar{r}_{13}(z), P_2\} = -\partial(\bar{r}_{12}(z) - \bar{r}_{13}(z)), \tag{2.51}$$

and

$$[r_{12}(z_1, z_2), \bar{r}_{13}(z_1) + \bar{r}_{23}(z_2)] + [\bar{r}_{13}(z_1), \bar{r}_{23}(z_2)] - P_3^{-1}\{r_{12}(z_1, z_2), P_3\}$$
$$= -(\partial_1 + \partial_2)r_{12}(z_1, z_2) + \partial_1\bar{r}_{13}(z_1) - \partial_2\bar{r}_{23}(z_2). \tag{2.52}$$

Here $\partial = \frac{\partial}{\partial x}$, where $x = \operatorname{Re} z$. Note that eqs.(2.35) and (2.36) are formulated with the help of $\mathbf{P}$. However, since all the matrices depend only on the difference $q_{ij} = q_i - q_j$, we simply replace $\mathbf{P}$ by P.

Comparing eqs.(2.50–2.52) for r and $\bar{r}$ with (2.23), (2.35), and (2.36) for $\mathbf{r}$ and $\bar{\mathbf{r}}$, we come to the conclusion that the $s(z)$-matrix coming in $\mathbf{r}$ and $\bar{\mathbf{r}}$ effectively plays the role of the derivative with respect to the spectral parameter.

It is worth mentioning that eqs.(2.50)-(2.52) obtained for r and $\bar{r}$ can be rewritten in the same form as eqs.(2.23), (2.35), and (2.36) if we replace r and $\bar{r}$ by $r_{12} - \partial_1 + \partial_2$ and $\bar{r}_{12} - \partial_1$. In particular, for (2.50), we have

$$[r_{12}-\partial_1+\partial_2, r_{13}-\partial_1+\partial_3]+[r_{13}-\partial_1+\partial_3, r_{23}-\partial_2+\partial_3]+[r_{12}-\partial_1+\partial_2, r_{23}-\partial_2+\partial_3] = 0. \tag{2.53}$$

Thus, $r_{12} - \partial_1 + \partial_2$ is a matrix first-order differential operator satisfying the standard classical Yang-Baxter equation. Using this fact we write down the Poisson algebra generated by the fields $T(z)$, $U(z)$, Q and P, having eqs.(2.50)-(2.52) as the consistency conditions:

$$\frac{1}{\kappa}\{T_1(z), T_2(w)\} = T_1(z)T_2(w)r_{12}(z, w) + T_1'T_2 - T_1T_2', \tag{2.54}$$

$$\frac{1}{\kappa}\{U_1(z), U_2(w)\} = -U_1(z)U_2(w)r_{12}(z, w) - U_1'U_2 + U_1U_2', \tag{2.55}$$

$$\frac{1}{\kappa}\{T_1(z), P_2\} = P_2T_1(z)\bar{r}_{12}(z) + P_2T_1'(z), \tag{2.56}$$

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$$\frac{1}{\kappa}\{U_1(z), P_2\} = P_2 U_1(z)\bar{r}_{12}(z) + P_2 U_1'(z), \tag{2.57}$$

$$\{Q_1, P_2\} = P_2 \sum_i E_{ii} \otimes E_{ii}, \quad \{Q_1, T_2\} = \{Q_1, U_2\} = 0, \tag{2.58}$$

where $T' = \partial T$.

It is worth mentioning that the Poisson structure (2.54-2.58) is not compatible with the boundary condition $T(0)e = \lambda e$.

Let us note that there exists a Poisson subalgebra of Poisson algebra (2.54-2.58), formed by the generators:

$$A(z) = T(z)DT^{-1}(z) - k\bar{\partial}T(z)T^{-1}(z), \quad g(z) = U(z)PT^{-1}(z)$$

that coincides with the Poisson algebra of the cotangent bundle with the central charge $\alpha = 0$.

Defining the L-operator as $L(z) = T^{-1}(z)g(z)T(z) = T^{-1}(z)U(z)P$, we get for L algebra (2.46) obtained previously. As in the previous subsection the commutativity of $I_n(z)$ follows again from the one of $g(z)$.

The main advantage of Poisson algebra (2.54-2.58) is that it can be easily quantized.

3   Quantization

3.1   Quantum R-matrices

In this section, following the ideology of the Quantum Inverse Scattering Method [24, 25], we quantize the classical r and $\bar{r}$-matrices and derive the quantum L-operator algebra.

We start with quantization of the relations (2.50)-(2.52). Let $T(z)$, $U(z)$ be matrix generating functions being the formal Fourier series in variables x and y:

$$T(z) = \sum_{mn} T_{mn} e^{2\pi i(mx+ny)}, \quad U(z) = \sum_{mn} U_{mn} e^{2\pi i(mx+ny)},$$

where $z = x + \tau y$.

Denote by $\mathcal{A}$ a free associative unital algebra over the field $\mathbf{C}$ generated by matrix elements of the Fourier modes of $T(z)$, $U(z)$, and by the entries of the diagonal matrices P and Q modulo the relations

$$T_1(z)T_2(w - \hbar) = T_2(w)T_1(z - \hbar)R_{12}(-\hbar, z, w), \tag{3.1}$$

$$U_1(z)U_2(w + \hbar) = U_2(w)U_1(z + \hbar)R_{12}(\hbar, z, w), \tag{3.2}$$

$$T_1(z + \hbar)P_2\overline{R}_{12}(\hbar, z) = P_2 T_1(z), \tag{3.3}$$

$$U_1(z + \hbar)P_2\overline{R}_{12}(\hbar, z) = P_2 U_1(z), \tag{3.4}$$

$$[Q_1, P_2] = -\hbar P_2 \sum_i E_{ii} \otimes E_{ii}, \tag{3.5}$$

$$[T_1(z), U_2(w)] = [T_1(z), Q_2] = [U_1(z), Q_2] = [P_1, P_2] = [Q_1, Q_2] = 0.$$

Here $R(\hbar, z, w)$ and $\overline{R}_{12}(\hbar, z)$ are double-periodic matrix functions of spectral parameters. These functions also depend on the coordinates q_i and have the following semiclassical behavior at $\hbar = 0$:

$$R = 1 + \hbar r + o(\hbar), \quad \overline{R} = 1 + \hbar \bar{r} + o(\hbar). \tag{3.6}$$

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The next step is to find the conditions on R and $\overline{R}$ that ensure the consistency of the defining relations for $\mathcal{A}$. In the sequel we often use $R(z, w)$ as a shorthand notation for $R(\hbar, z, w)$.

First, we write down the compatibility condition for algebra (3.1) or (3.2), which reduces to the Quantum Yang-Baxter equation with spectral parameters shifted by $\hbar$,

$$R_{12}(z, w)R_{13}(z - \hbar, s - \hbar)R_{23}(w, s) = R_{23}(w - \hbar, s - \hbar)R_{13}(z, s)R_{12}(z - \hbar, w - \hbar). \quad (3.7)$$

Analogously to the classical case, one can introduce the following matrix differential operator $\mathcal{R}(z, w) = e^{\hbar \frac{\partial}{\partial w}} R(z, w) e^{-\hbar \frac{\partial}{\partial z}}$ in terms of which eq.(3.7) reads as the standard Quantum Yang-Baxter equation

$$\mathcal{R}_{12}(z, w)\mathcal{R}_{13}(z, s)\mathcal{R}_{23}(w, s) = \mathcal{R}_{23}(w, s)\mathcal{R}_{13}(z, s)\mathcal{R}_{12}(z, w). \quad (3.8)$$

Relation (3.1) also requires the fulfillment of the "unitarity" condition for R,

$$R_{12}(z, w)R_{21}(w, z) = 1. \quad (3.9)$$

Analogously, we find the following compatibility conditions for (3.3):

$$P_3^{-1}\overline{R}_{12}(z)P_3\overline{R}_{13}(z - \hbar) = P_2^{-1}\overline{R}_{13}(z)P_2\overline{R}_{12}(z - \hbar) \quad (3.10)$$

and

$$P_3^{-1}R_{12}(z, w)P_3\overline{R}_{13}(z - \hbar)\overline{R}_{23}(w) = \overline{R}_{23}(w - \hbar)\overline{R}_{13}(z)R_{12}(z - \hbar, w - \hbar). \quad (3.11)$$

Now taking into account (3.6) one can easily see that in the semiclassical limit

$$-\frac{1}{\kappa}\{\cdot, \cdot\} = \lim_{\hbar \to 0} \frac{1}{\hbar}[\cdot, \cdot]$$

relations (3.1-3.5) determine Poisson structure (2.54-2.58), while eqs.(3.7), (3.10), and (3.11) turn into (2.50), (2.51), and (2.52), respectively, in order $\hbar^2$. In the first order in $\hbar$, the unitarity condition (3.9) requires r to be skew-symmetric. Hence, the algebra $\mathcal{A}$ with defining relations (3.1)-(3.5), where R and $\overline{R}$ are the solutions of (3.7-3.11) obeying (3.6), is a quantization of the Poisson structure (2.54-2.58).

Now we are in a position to find the matrices R and $\overline{R}$ explicitly. We start with the R-matrix for which we assume the following natural ansatz:

$$fR(\hbar, z, w) = \sum_{ij} \Phi(\hbar_1, q_{ij} + \hbar_2)E_{ii} \otimes E_{jj} + \sum_{ij} \Phi(z - w + \hbar_3, q_{ij} + \hbar_4)E_{ij} \otimes E_{ji}$$

$$- \sum_{ij} \Phi(z + \hbar_5, q_{ij} + \hbar_6)E_{ij} \otimes E_{jj} + \sum_{ij} \Phi(w + \hbar_7, q_{ij} + \hbar_8)E_{jj} \otimes E_{ij}. \quad (3.12)$$

This form is compatible with the structure of the classical r-matrix. Here $\hbar_1, \ldots, \hbar_8$ are arbitrary parameters that should be specified by eqs.(3.7) and (3.9), and f is a scalar function that may depend only on $\hbar_i$ and spectral parameters. It turns out that the parameters h_i are almost uniquely fixed by the unitarity condition (3.9). Substituting (3.12) into (3.9) and using the elliptic function identities we obtain

$$\hbar_2 = \hbar_3 = \hbar_4 = \hbar_6 = \hbar_8 = 0, \quad \hbar_5 = \hbar_1 + \hbar_7, \quad f^2(z, w) = \mathcal{P}(\hbar_1) - \mathcal{P}(z - w),$$

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where $\mathcal{P}(z)$ is the Weierstrass $\mathcal{P}$-function. Now it is a matter of direct calculation to check that eq.(3.7) holds for $\hbar_1 = \hbar$. The remaining parameter $\hbar_7$ is inessential since it corresponds to an arbitrary common shift of the spectral parameters z and w. In the sequel, we choose $\hbar_7 = 0$. Therefore, the obtained solution to (3.7) and (3.6) reads as follows:

$$
\begin{aligned}
f(z,w)R(\hbar,z,w) \ = \ & \sum_{ij}\Phi(\hbar,q_{ij})E_{ii}\otimes E_{jj} + \sum_{ij}\Phi(z-w,q_{ij})E_{ij}\otimes E_{ji} \qquad (3.13) \\
& - \sum_{ij}\Phi(z+\hbar,q_{ij})E_{ij}\otimes E_{jj} + \sum_{ij}\Phi(w,q_{ij})E_{jj}\otimes E_{ij},
\end{aligned}
$$

where $f(z,w) = \sqrt{\mathcal{P}(\hbar) - \mathcal{P}(z-w)}$. One must be careful in the definition of $R(-\hbar,z,w)$. This matrix is defined by (3.13) with the replacement $\hbar \to -\hbar$ and $f \to -f$. Therefore, $R(\hbar,z,w)$ and $R(-\hbar,z,w)$ are related as

$$
R_{12}(-\hbar,z,w) = R_{21}(\hbar,w-\hbar,z-\hbar). \qquad (3.14)
$$

To find the $\overline{R}$-matrix, we adopt the following ansatz:

$$
\frac{1}{\sigma(\hbar)}\overline{R}_{12}(\hbar,z) = \sum_{ij}\Phi(\hbar_1,q_{ij}+\hbar_2)E_{ii}\otimes E_{jj} - \sum_{ij}\Phi(z+\hbar_3,q_{ij}+\hbar_4+\delta_{ij}\hbar_5)E_{ij}\otimes E_{jj}. \quad (3.15)
$$

It has almost the same matrix structure as the classical $\bar{r}$-matrix. Since eq.(3.10) is easier to deal with than eq.(3.11), we first substitute (3.15) into eq.(3.10) thus obtaining $\overline{R}$:

$$
\frac{1}{\sigma(\hbar)}\overline{R}_{12}(\hbar,z) = \sum_{i\neq j}\Phi(\hbar,q_{ij})E_{ii}\otimes E_{jj} - \sum_{i\neq j}\Phi(z+\hbar_3,q_{ij})E_{ij}\otimes E_{jj} - \Phi(z+\hbar_3,-\hbar)\sum_i E_{ii}\otimes E_{ii}. \qquad (3.16)
$$

where $\hbar_3$ remains unfixed.

Eq.(3.11) involves both the R- and $\overline{R}$-matrices and is independent on (3.7) and (3.10). One can verify by direct calculations that R and $\overline{R}$ given by eqs.(3.13) and (3.16) also satisfy (3.11) as soon as $\hbar_3 = \hbar$.

One can easily check that in the case of real $\hbar$, the matrices R and $\overline{R}$ have the proper semiclassical behavior (3.6).

In what follows we also need the $\overline{R}^{-1}$-matrix,

$$
\frac{1}{\sigma(\hbar)}\overline{R}_{12}^{-1}(\hbar,z) = -\sum_{ij}\Phi(-\hbar,q_{ij}+\hbar)E_{ii}\otimes E_{jj} + \sum_{ij}\Phi(z,q_{ij}+\hbar)E_{ij}\otimes E_{jj}, \qquad (3.17)
$$

It would be of interest to mention that just as in the rational case without the spectral parameter [18], one can introduce the formal variable $W(z) = T^{-1}(z)U(z)$ with permutation relations following from (3.1–3.5):

$$
\begin{aligned}
R_{12}(z,w)W_1(z)W_2(w+\hbar) \ &= \ W_2(w)W_1(z+\hbar)R_{12}(z,w), \qquad (3.18) \\
W_1(z+\hbar)P_2\overline{R}_{12}(z) \ &= \ P_2\overline{R}_{12}(z)W_1(z). \qquad (3.19)
\end{aligned}
$$

In analogy with the rational case, it is natural to treat eq.(3.18) as the defining relation of the quantum elliptic Frobenius group.

3.2 Quantum L-operator algebra

Just as in the classical case, we introduce a new variable:

$$L(z) = T^{-1}(z)U(z)P = W(z)P, \tag{3.20}$$

which we call a *quantum L-operator*. Using the relations of the algebra $\mathcal{A}$ one can formally derive the following algebraic relations satisfied by the quantum L-operator:

$$[Q_1, L_2(z)] = -\hbar L_2(z) \sum_i E_{ii} \otimes E_{ii}, \tag{3.21}$$

$$R_{12}(z, w)L_1(z)\overline{R}_{21}(w)L_2(w) = L_2(w)\overline{R}_{12}(z)L_1(z)\overline{R}_{21}(w - \hbar)R_{12}(z - \hbar, w - \hbar)\overline{R}_{12}^{-1}(z - \hbar). \tag{3.22}$$

In spite of the fact that L has the form $L(z) = W(z)P$, we can not reconstruct from eqs.(3.21) and (3.22) the relations (3.18) and (3.19) for W and P. So, in the sequel, we do not assume any relations on W and P.

Let us define

$$R_{12}^F(z, w) = \overline{R}_{21}(w)R_{12}(z, w)\overline{R}_{12}^{-1}(z). \tag{3.23}$$

Then, by using the explicit form of the R- and $\overline{R}$-matrices and elliptic function identities, we obtain

$$fR_{12}^F(z - w) = -\sum_{i \neq j} \Phi(-\hbar, q_{ij})E_{ii} \otimes E_{jj} + \sum_{i \neq j} \Phi(z - w, q_{ij})E_{ij} \otimes E_{ji} + \Phi(z - w, \hbar)\sum_i E_{ii} \otimes E_{ii}, \tag{3.24}$$

which is nothing but the R-matrix by Felder [15], i.e., an elliptic solution to the quantum Gervais–Neveu–Felder equation [16, 15]:

$$P_1^{-1}R_{23}^F(w - s)P_1 R_{13}^F(z - s)P_3^{-1}R_{12}^F(z - w)P_3 = R_{12}^F(z - w)P_2^{-1}R_{13}^F(z - s)P_2 R_{23}^F(w - s). \tag{3.25}$$

Recall that one feature of R^F is the "weight zero" condition:

$$[P_1 P_2, R_{12}^F(z - w)] = 0. \tag{3.26}$$

Developing R^F in powers of $\hbar$, we have $R^F = 1 + \hbar r^F + o(\hbar)$, where r^F is given by (2.48).

Let us stress that in our consideration R^F arises to account for the explicit form of R and $\overline{R}$, and that the Gervais–Neveu–Felder equation does not follow from system (3.7-3.11). Formula (3.23) shows that the matrix $\overline{R}$ plays the role of the twist, which transforms the matrix $R(z, w)$ – a particular solution of (3.7) – into such solution of (3.25) which depends only on the difference $z - w$.

Thus, the quantum L-operator algebra (3.22) can be presented in the following form:

$$R_{12}(z, w)L_1(z)\overline{R}_{21}(w)L_2(w) = L_2(w)\overline{R}_{12}(z)L_1(z)R_{12}^F(z - w). \tag{3.27}$$

The quantum L-operator algebra seems to be automatically compatible as $\mathcal{A}$ is compatible. However, a simple analysis shows that $\mathcal{A}$ and the algebra (3.27) admit different supplies of representations. In particular, the simplest representation for L we present below does not realize the algebra (3.18), (3.19). Therefore, we find it necessary to give a direct proof of the

compatibility of (3.27). In this way, we come across eq.(3.25) and discover a new relation involving R^F and $\overline{R}$. To this end, let us multiply both sides of (3.27) by

$$P_2^{-1}\overline{R}_{31}(s+\hbar)P_2\overline{R}_{32}(s)L_3(s)$$

and subsequently using eq.(3.27) we transform the string $L_1 \cdots L_2 \cdots L_3$ into $L_3 \cdots L_2 \cdots L_1$. For the l.h.s., we have

$$
\begin{aligned}
&R_{12}(z,w)L_1\overline{R}_{21}(w)L_2P_2^{-1}\overline{R}_{31}(s+\hbar)P_2\overline{R}_{32}(s)L_3 = \\
&R_{12}(z,w)L_1\overline{R}_{21}(w)\overline{R}_{31}(s+\hbar)L_2\overline{R}_{32}(s)L_3 = \\
&R_{12}(z,w)L_1\overline{R}_{21}(w)\overline{R}_{31}(s+\hbar)R_{32}(s,w)L_3\overline{R}_{23}(w)L_2R^F_{23}(w-s) = \\
&R_{12}(z,w)R_{32}(s+\hbar,w+\hbar)L_1\overline{R}_{31}(s)L_3P_3^{-1}\overline{R}_{21}(w+\hbar)P_3\overline{R}_{23}(w)L_2R^F_{23}(w-s) = \\
&R_{12}(z,w)R_{32}(s+\hbar,w+\hbar)R_{31}(s,z)L_3\overline{R}_{13}(z)L_1 \times \\
&\qquad R^F_{13}(z-s)P_3^{-1}\overline{R}_{21}(w+\hbar)P_3\overline{R}_{23}(w)L_2R^F_{23}(w-s).
\end{aligned}
\tag{3.28}
$$

At this point, we interrupt the chain of calculations by remarking that the next step implies the possibility to push R^F somehow through $P_3^{-1}\overline{R}_{21}(w+\hbar)P_3\overline{R}_{23}(w)$. It can be done by virtue of the following new relation involving R^F and $\overline{R}$:

$$R^F_{12}(z)P_2^{-1}\overline{R}_{31}(w)P_2\overline{R}_{32}(w-\hbar) = P_1^{-1}\overline{R}_{32}(w)P_1\overline{R}_{31}(w-\hbar)R^F_{12}(z),
\tag{3.29}$$

which can be checked directly by using the explicit forms (3.24) and (3.16) of R^F and $\overline{R}$ respectively.

Now we pursue calculation (3.28) with the relation (3.29) at hand.

$$
\begin{aligned}
&R_{12}(z,w)R_{32}(s+\hbar,w+\hbar)R_{31}(s,z)L_3\overline{R}_{13}(z)\overline{R}_{23}(w+\hbar) \times \\
&\qquad L_1\overline{R}_{21}(w)L_2P_2^{-1}R^F_{13}(z-s)P_2R^F_{23}(w-s).
\end{aligned}
$$

As to the r.h.s., the same technique yields

$$
\begin{aligned}
&L_2\overline{R}_{12}(z)L_1R^F_{12}(z-w)P_2^{-1}\overline{R}_{31}(s+\hbar)P_2\overline{R}_{32}(s)L_3 = \\
&L_2\overline{R}_{12}(z)\overline{R}_{32}(s+\hbar)L_1\overline{R}_{31}(s)L_3P_3^{-1}R^F_{12}(z-w)P_3 = \\
&L_2\overline{R}_{12}(z)\overline{R}_{32}(s+\hbar)R_{31}(s,z)L_3\overline{R}_{13}(z)L_1R^F_{13}(z-s)P_3^{-1}R^F_{12}(z-w)P_3 \\
&= R_{31}(s+\hbar,z+\hbar)L_2\overline{R}_{32}(s)L_3P_3^{-1}\overline{R}_{12}(z+\hbar)P_3 \times \\
&\qquad \overline{R}_{13}(z)L_1R^F_{13}(z-s)P_3^{-1}R^F_{12}(z-w)P_3 = \\
&R_{31}(s+\hbar,z+\hbar)R_{32}(s,w)L_3\overline{R}_{23}(w)\overline{R}_{13}(z+\hbar)L_2\overline{R}_{12}(z)L_1 \times \\
&\qquad P_1^{-1}R^F_{23}(w-s)P_1R^F_{13}(z-s)P_3^{-1}R^F_{12}(z-w)P_3 = \\
&R_{31}(s+\hbar,z+\hbar)R_{32}(s,w)R_{12}(z+\hbar,w+\hbar)L_3\overline{R}_{13}(z)\overline{R}_{23}(w+\hbar)L_1 \times \\
&\qquad \overline{R}_{21}(w)L_2R^F_{21}(w-z)P_1^{-1}R^F_{23}(w-s)P_1R^F_{13}(z-s)P_3^{-1}R^F_{12}(z-w)P_3.
\end{aligned}
\tag{3.30}
$$

Therefore, comparing the resulting expressions we conclude that the compatibility condition for the L-operator algebra (3.27) reduces to four equations (3.7), (3.11), (3.24), and (3.29).

The existence of the Poisson commuting functions $I_n(z)$ in the classical case implies that the commuting family should exist in the quantum case as well. It should be the intrinsic property

of the algebra (3.27) itself, without referring to the explicit form of its representations. Let us demonstrate the commutativity of the simplest quantities $\operatorname{tr} L(z)$ and $\operatorname{tr} L^{-1}(z)$ postponing the discussion of the general case to the next section. To this end, we need one more relation involving the matrices R^F, R and $\overline{R}$.

In analogy with the rational case, it is useful to introduce the variable $g(z) = U(z)PT^{-1}(z)$. Calculation of the commutator $[g_1(z), g_2(w)]$ with the help of the defining relations of $\mathcal{A}$ results in

$$
\begin{aligned}
[g_1(z), g_2(w)] &= U_2(w)U_1(z+\hbar)\left(R_{12}(\hbar, z, w)P_1\overline{R}_{21}(w)P_2\overline{R}_{12}(z-\hbar)R_{12}(-\hbar, z, w)\right. \\
&- \left. P_2\overline{R}_{12}(z)P_1\overline{R}_{21}(w-\hbar)\right)T_2^{-1}(w-\hbar)T_1^{-1}(z).
\end{aligned}
$$

When the spectral parameter is absent, the algebra $\mathcal{A}$ allows one to establish a connection with the quantum cotangent bundle (see [18] for details). Then, in particular, the quantity $[g_1, g_2]$ is equal to zero. In the case at hand, we can not construct a subalgebra of $\mathcal{A}$ that is isomorphic to the quantum cotangent bundle. However, one can note that in the elliptic case, the commutativity of $g(z)$ with $g(w)$ follows from the identity

$$
R_{12}(\hbar, z, w)P_1\overline{R}_{21}(w)P_2\overline{R}_{12}(z-\hbar)R_{12}(-\hbar, z, w) = P_2\overline{R}_{12}(z)P_1\overline{R}_{21}(w-\hbar).
$$

Using the definition of R^F, eq.(3.14), and the "weight zero" condition (3.26), the last formula can be written in the following elegant form

$$
R_{12}(z, w) = P_2\overline{R}_{12}(z)P_2^{-1}R^F_{12}(z-w)P_1\overline{R}_{21}^{-1}(w)P_1^{-1}. \tag{3.31}
$$

Identity (3.31) plays the primary role in proving the commutativity of the family $\operatorname{tr} L(z)$. To prove the commutativity, let us multiply both sides of

$$
L_2(w)\overline{R}_{12}(z)L_1(z) = R_{12}(z, w)L_1(z)\overline{R}_{21}(w)L_2(w)R^F_{21}(w-z).
$$

by $P_2\overline{R}_{12}^{-1}(z)P_2^{-1}$ and take the trace in the first and the second matrix spaces. We get

$$
\begin{aligned}
\operatorname{tr}_{12} P_2\overline{R}_{12}^{-1}(z)P_2^{-1}L_2(w)\overline{R}_{12}(z)L_1(z) &= \\
\operatorname{tr}_{12} P_2\overline{R}_{12}^{-1}(z)P_2^{-1}R_{12}(z, w)L_1(z)\overline{R}_{21}(w)L_2(w)R^F_{21}(w-z)&. \tag{3.32}
\end{aligned}
$$

It is useful to write $\overline{R}_{12}^{-1}$ in the factorized form

$$
\overline{R}_{12}^{-1}(z) = \sum_{ij}\overline{R}_{ij}^{-1}\otimes E_{jj}, \tag{3.33}
$$

where

$$
\overline{R}_{ij}^{-1} = -\sigma(\hbar)\Phi(-\hbar, q_{ij}+\hbar)E_{ii} + \sigma(\hbar)\Phi(z, q_{ij}+\hbar)E_{ij}.
$$

Then the l.h.s. of (3.32) reads as

$$
\sum_{ij}\operatorname{tr}_{12}(P_j\overline{R}_{ij}^{-1}P_j^{-1}\otimes E_{jj})L_2\overline{R}_{12}(z)L_1.
$$

Taking into account that $\overline{R}_{12}$ is diagonal in the second matrix space and using the cyclic property of the trace, we obtain

$$\sum_{ij} \operatorname{tr}_{12}(P_j \overline{R}_{ij}^{-1} P_j^{-1} \otimes I)(I \otimes LE_{jj})\overline{R}_{12}(z)L_1\,.$$

Since $L = WP$, where all entries of W commutes with q_i, we arrive at

$$\sum_{ij} \operatorname{tr}_{12}(P_j \overline{R}_{ij}^{-1} P_j^{-1} \otimes I)(I \otimes WP_j E_{jj})\overline{R}_{12}(z)L_1 \; =$$

$$\sum_{ij} \operatorname{tr}_{12} L_2(\overline{R}_{ij}^{-1} \otimes E_{jj})\overline{R}_{12}(z)L_1 \; = \; \operatorname{tr} L(w) \, \operatorname{tr} L(z)\,.$$

As to the r.h.s. of (3.32), we use identity (3.31) to rewrite it as

$$\operatorname{tr}_{12} R_{12}^F(z - w) P_1 \overline{R}_{21}^{-1}(w) P_1^{-1} L_1(z)\overline{R}_{21}(w) L_2(w) R_{21}^F(w - z)\,.$$

Having in mind that $\overline{R}_{21}$ is diagonal in the first matrix space and taking into account the property (3.26) one can easily see that under the trace sign, the matrix R^F can be pushed to the right where it cancels with R_{21}^F. Therefore, we get

$$\operatorname{tr}_{12} P_1 \overline{R}_{21}^{-1}(w) P_1^{-1} L_1(z)\overline{R}_{21}(w) L_2(w)\,. \tag{3.34}$$

Now applying to this expression the technique we used above for the l.h.s of (3.32) we conclude that eq.(3.34) is equal to $\operatorname{tr} L(z) \, \operatorname{tr} L(w)$. Thus, we proved that $\operatorname{tr} L(z)$ commutes with $\operatorname{tr} L(w)$. Quite analogously one can prove that $\operatorname{tr} L^{-1}(z)$ commutes with $\operatorname{tr} L^{-1}(w)$ and with $\operatorname{tr} L(w)$.

Now we give an example of the simplest representation of algebra (3.21) and (3.27) associated with the elliptic RS model. Namely, the following L-operator satisfies algebra (3.27):

$$L(z) = \sum_{ij} \Phi(z, q_{ij} + \gamma) b_j P_j E_{ij}, \tag{3.35}$$

where

$$b_j = \prod_{a \neq j} \Phi(\gamma, q_{aj}). \tag{3.36}$$

Here the parameter γ is a coupling constant of the elliptic RS model. This can be checked by straightforward calculations. Some comments are in order. The L-operator of the form (3.35) was already appeared in [20] as a result of the Hamiltonian reduction procedure applied to $T^\star \widehat{GL}(N)(z, \bar{z})$. To guess the explicit form of b_j one should note that in the rational limit algebra (3.27) tends to the one obtained in [18], where the coefficients b_j are found to be

$$b_j = \prod_{a \neq j} \frac{q_{aj} + \gamma}{q_{aj}}. \tag{3.37}$$

Therefore, it is natural to assume that the elliptic analog of (3.37) is given by (3.36).

It is worthwhile to mention that b_j are not uniquely defined since one can perform a canonical transformation of (q, p)-variables. In particular, the variables

$$\tilde{b}_j = \prod_{a \neq j} \frac{\sigma(q_{aj} + \gamma)}{\sigma(\gamma)\sigma(q_{aj})} \tag{3.38}$$

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are related to b_j by the canonical transformation $q_i \to q_i$ and

$$P_i \to e^{\alpha \sum_a q_{ai}} P_i,$$

where $\alpha = 2\zeta(1/2)\gamma - i\frac{\gamma - \bar\gamma}{\tau - \bar\tau}$.

We call this L the *quantum L-operator of the elliptic RS model*. Indeed, taking the Hamiltonian to be $H = \operatorname{tr} L(z)$ one can see that the quantum canonical transformation of the form:

$$P_i^R = \prod_{a \neq i} \left(\frac{\sigma(q_{ai} + \gamma)}{\sigma(q_{ai})} \right)^{1/2} P_i \prod_{a \neq i} \left(\frac{\sigma(q_{ai})}{\sigma(q_{ai} - \gamma)} \right)^{1/2}, \tag{3.39}$$

where P_i^R is the momentum in the Ruijsenaars Hamiltonian, turn H into the first integral S_1 from the Ruijsenaars commuting family [10]. Moreover, after the canonical transformation (3.39), the L-operator (3.35) coincides in essential with the classical L-operator of the RS model.

The generating function for the commuting family in terms of L can be written as

$$I(z, \mu) =: \det\left(L(z) - \mu\right) := \sum_k (-\mu)^{N-k} I_k(z), \tag{3.40}$$

where the normal ordering :: means that all momentum operators are pushed to the right. It follows from the results obtained in the next section.

4 Connection to the fundamental relation $RLL = LLR$

In this section we establish a connection of the quantum L-operator algebra (3.27) with the fundamental relation $RLL = LLR$.

In [21], the operators from the Ruijsenaars commuting family were obtained by using a special representation $\hat{L}$ of the algebra

$$R_{12}^B(z - w)\hat{L}_1(z)\hat{L}_2(w) = \hat{L}_2(w)\hat{L}_1(z)R_{12}^B(z - w), \tag{4.1}$$

where $R^B(z)$ is Belavin's R-matrix being an elliptic solution to the quantum Yang-Baxter equation [26]. The explicit form of R^B we use here can be found in [27]. For reader's convenience we recall a construction of $\hat{L}$ [28, 29].

Denote by $\mathbf{h}^*$ the weight space for $sl_N(\mathbf{C})$ that can be realized in $\mathbf{C}^N$ with a basis ϵ_i, $<\epsilon_i, \epsilon_j> = \delta_{ij}$, as the orthogonal complement to $\sum_{i=1}^N \epsilon_i$. Let $\bar\epsilon_k$ be the orthogonal projection of ϵ_k: $\bar\epsilon_k = \epsilon_k - \frac{1}{N}\sum_{i=1}^N \epsilon_i$.

For each $q \in \mathbf{h}^*$ one can introduce the intertwining vectors [32, 33]

$$\phi(z)_q^{q + \hbar\bar\epsilon_k}{}_j = \theta_j\left(\frac{z}{N} - <q, \bar\epsilon_k>\right)/i\eta(\tau), \tag{4.2}$$

where

$$\theta_j(z) = \sum_{n \in \frac{N}{2} - j + N\mathbf{Z}} \exp 2\pi i \left[n\left(z + \frac{1}{2}\right) + \frac{n^2}{2N}\tau \right]$$

and $\eta(\tau) = p^{1/24} \prod_{m=1}^\infty (1 - p^m)$ is the Dedekind eta function with $p = \exp 2\pi i\tau$.

Following [28] we denote by $\bar{\phi}(z)_q^{q+\hbar\bar{\epsilon}_k j}$ the entries of the matrix inverse to $\phi(z)_q^{q+\hbar\bar{\epsilon}_k}{}_j$. Then the orthogonality relations read as follows

$$\sum_{j=1}^{n} \bar{\phi}(z)_q^{q+\hbar\bar{\epsilon}_k j} \phi(z)_q^{q+\hbar\bar{\epsilon}_{k'}}{}_j = \delta_{kk'} \qquad \sum_{k=1}^{n} \phi(z)_q^{q+\hbar\bar{\epsilon}_k}{}_j \bar{\phi}(z)_q^{q+\hbar\bar{\epsilon}_k j'} = \delta_{jj'}. \tag{4.3}$$

In the sequel, the following formula [21] will be of intensive use

$$\sum_{m=1}^{n} \bar{\phi}(z)_{q'}^{q'+\hbar\bar{\epsilon}_j m} \phi(z)_q^{q+\hbar\bar{\epsilon}_i}{}_m = \frac{\theta(z+ < q', \bar{\epsilon}_j > - < q, \bar{\epsilon}_i >)}{\theta(z)} \prod_{j' \neq j} \frac{\theta(< q', \bar{\epsilon}_{j'} > - < q, \bar{\epsilon}_i >)}{\theta(< q', \bar{\epsilon}_{j'} > - < q', \bar{\epsilon}_j >)} \tag{4.4}$$

Here $\theta(z)$ denotes the Jacoby θ-function

$$\theta(z) = -\sum_{n} e^{2\pi i(z+\frac{1}{2})(n+\frac{1}{2})+i\pi\tau(n+\frac{1}{2})^2} = \theta'(0)e^{-\zeta(\frac{1}{2})z^2}\sigma(z).$$

It is shown in [29, 30] that the $\hat{L}$-operator

$$\hat{L}_{ij}(z) = \sum_{k=1}^{N} \phi(z+\gamma N)_q^{q+\hbar\bar{\epsilon}_k}{}_i \bar{\phi}(z)_q^{q+\hbar\bar{\epsilon}_k j} e^{\hbar\bar{\epsilon}_k \frac{\partial}{\partial q_k}}, \tag{4.5}$$

acting on the space of functions on $\mathbf{h}^*$ satisfies relation (4.1). This $\hat{L}$ is an $N \times N$ generalization of the 2×2 Sklyanin L-operator [31].

The intertwining vectors $\phi(z)_q^{q+\hbar\bar{\epsilon}_k}{}_j$ coming in the definition of $\hat{L}$ relate the matrix R^B with the Boltzmann weights for the $A_{n-1}^{(1)}$ face model. Recall [33] that the nonzero Boltzmann weights depending on the spectral parameter z are explicitly given by

$$\check{W}\left[\begin{array}{ccc} & q+\hbar\bar{\epsilon}_i & \\ q & z & q+2\hbar\bar{\epsilon}_i \\ & q+\hbar\bar{\epsilon}_i & \end{array} \right] = \frac{\theta(z+\hbar)}{\theta(\hbar)}, \tag{4.6}$$

$$\check{W}\left[\begin{array}{ccc} & q+\hbar\bar{\epsilon}_i & \\ q & z & q+\hbar(\bar{\epsilon}_i+\bar{\epsilon}_j) \\ & q+\hbar\bar{\epsilon}_i & \end{array} \right] = \frac{\theta(-z+q_{ij})}{\theta(q_{ij})} \quad (i \neq j),$$

$$\check{W}\left[\begin{array}{ccc} & q+\hbar\bar{\epsilon}_i & \\ q & z & q+\hbar(\bar{\epsilon}_i+\bar{\epsilon}_j) \\ & q+\hbar\bar{\epsilon}_j & \end{array} \right] = \frac{\theta(z)}{\theta(\hbar)} \frac{\theta(\hbar+q_{ij})}{\theta(q_{ij})} \quad (i \neq j),$$

where $q_{ij} = < q, \bar{\epsilon}_i - \bar{\epsilon}_j >$.

The relation between R^B and the face weights is given by

$$\sum_{i'j'} R^B(z-w)_{ij}^{i'j'} \phi(z)_q^{q+\hbar\bar{\epsilon}_k}{}_{i'} \phi(w)_{q+\hbar\bar{\epsilon}_k}^{q+\hbar(\bar{\epsilon}_k+\bar{\epsilon}_m)}{}_{j'} = \tag{4.7}$$

$$\sum_{s} \phi(w)_q^{q+\hbar\bar{\epsilon}_s}{}_j \phi(z)_{q+\hbar\bar{\epsilon}_s}^{q+\hbar(\bar{\epsilon}_k+\bar{\epsilon}_m)}{}_i \check{W}\left[\begin{array}{ccc} & q+\hbar\bar{\epsilon}_k & \\ q & z-w & q+\hbar(\bar{\epsilon}_k+\bar{\epsilon}_m) \\ & q+\hbar\bar{\epsilon}_s & \end{array} \right]$$

In what follows we use the concise notation

$$W_s^k[k+m] = \check{W} \begin{bmatrix} & q+\hbar\bar{\epsilon}_k & \\ q & z-w & q+\hbar(\bar{\epsilon}_k+\bar{\epsilon}_m) \\ & q+\hbar\bar{\epsilon}_s & \end{bmatrix}$$

Then the dual relation to (4.7) is

$$\sum_{i'j'} \bar{\phi}(w)_q^{q+\hbar\bar{\epsilon}_k j'} \bar{\phi}(z)_{q+\hbar\bar{\epsilon}_k}^{q+\hbar(\bar{\epsilon}_k+\bar{\epsilon}_m)i'} R^B(z-w)_{i'j'}^{ij} = \sum_s W_k^s[k+m]\bar{\phi}(z)_q^{q+\hbar\bar{\epsilon}_s i}\bar{\phi}(w)_{q+\hbar\bar{\epsilon}_s}^{q+\hbar(\bar{\epsilon}_k+\bar{\epsilon}_m)j} \quad (4.8)$$

In [21] another L-operator $\tilde{L}$ appeared. It is related to $\hat{L}$ in the following way:

$$\hat{L}_{ij}(z) \to \tilde{L}_{ij}(z) = \sum_{i'j'} \bar{\phi}(z)_q^{q+\hbar\bar{\epsilon}_i i'}\phi(z)_q^{q+\hbar\bar{\epsilon}_j}{}_{j'}\hat{L}_{i'j'}(z) \quad (4.9)$$

$$= \frac{\theta(z+\gamma+q_{ij})}{\theta(z)} \prod_{n\neq i} \frac{\theta(\gamma+q_{nj})}{\theta(q_{ni})} e^{\hbar\bar{\epsilon}_j \frac{\partial}{\partial q_j}}.$$

In what follows we need to remove from the quantum L-operator algebra (3.27) the non-holomorphic dependence on the spectral and quantization parameters. This can be achived by considering the following transformation of the L-operator

$$L(z) \to e^{\alpha(z)Q}e^{-\beta Q}L(z)e^{\beta Q}e^{-\alpha(z)Q}, \quad (4.10)$$

where $\alpha(z)$ is an arbitrary function of the spectral parameter and β is a complex number. Since the transformed L-operator also has the structure WP, then the following formula is valid

$$L_2(w)e^{\alpha(z)Q_1} = e^{\alpha(z)Q_1}L_2(w)e^{\hbar\alpha(z)r_0}, \quad (4.11)$$

where the notation $r_0 = \sum_i E_{ii} \otimes E_{ii}$ was used.

Recalling that the L-operator (3.35) satisfies the quantum L-operator algebra (3.27) and using eq.(4.11) one can easily establish the algebra satisfied by the transformed L:

$$\check{R}_{12}(z,w)L_1(z)\check{\overline{R}}_{21}(w)L_2(w) = L_2(w)\check{\overline{R}}_{12}(z)L_1(z)\check{R}_{12}^F(z-w),$$

where the matrices $\check{R}$, $\check{\overline{R}}$ and $\check{R}^F$ are

$$\check{R}_{12}(z,w) = e^{-\alpha(z)Q_1-\alpha(w)Q_2+\beta Q_2}R_{12}(z,w)e^{\alpha(z)Q_1+\alpha(w)Q_2-\beta Q_1}, \quad (4.12)$$

$$\check{\overline{R}}_{12}(z) = e^{\hbar\alpha(z)r_0-\alpha(z)Q_1+\beta Q_2}\overline{R}_{12}(z)e^{\alpha(z)Q_1-\beta Q_1}, \quad (4.13)$$

$$\check{R}_{12}^F(z,w) = \quad (4.14)$$

$$e^{-\alpha(z)Q_1-\alpha(w)Q_2+\frac{\hbar}{2}(\alpha(w)-\alpha(z))r_0+\beta Q_1}R_{12}^F(z,w)e^{\alpha(z)Q_1+\alpha(w)Q_2+\frac{\hbar}{2}(\alpha(w)-\alpha(z))r_0-\beta Q_2}.$$

Since the transformation in question keeps the form of the quantum L-operator algebra intact, the transformed matrices $\check{R}$, $\check{\overline{R}}$ and $\check{R}^F$ also satisfy all the compatibility conditions. In particular, the transformation (4.14) defines another solution of (3.25). For $\beta = 0$ this was observed in [14].

For the particular choice $\beta = 2\pi i \frac{h-\bar{h}}{\tau-\bar{\tau}}$ and $\alpha(z) = 2\pi i \frac{z-\bar{z}}{\tau-\bar{\tau}} + \beta$ we find that the matrices $\check{R}$, $\check{\bar{R}}$ and $\check{R}^F$ are given by the same formulae (3.13), (3.16), (3.24) with the total change of $\Phi(z,s) = \frac{\theta(z+s)}{\theta(z)\theta(s)}\theta'(0)e^{2\pi i \frac{z-\bar{z}}{\tau-\bar{\tau}}}$ by the meromorphic function $\frac{\theta(z+s)}{\theta(z)\theta(s)}$. The transformation with such a choice of α and β transforms (up to an unessential multiplier) the L-operator (3.35) into

$$L_{ij}(z) = \frac{\theta(z+q_{ij}+\gamma)}{\theta(z)\theta(q_{ij}+\gamma)} \prod_{n \neq j} \frac{\theta(q_{nj}+\gamma)}{\theta(q_{nj})} \mathbf{P}_j \qquad (4.15)$$

which is a quasi-periodic meromorphic matrix function of the spectral parameter:

$$L(z+1) = L(z), \quad L(z+\tau) = e^{-2\pi i(\gamma+\hbar)}e^{-2\pi i Q}L(z)e^{2\pi i Q}.$$

We assume that the L-operator is of the form $W\mathbf{P}$, where $\mathbf{P}_i = e^{\hbar\bar{\epsilon}_i \frac{\partial}{\partial q_i}}$. The replacement of P by $\mathbf{P}$ preserves all the consistency conditions because the R-matrices depend only on the difference $q_i - q_j$. Thus, eq.(3.27) with R-matrices defined via $\Phi(z,s) = \frac{\theta(z+s)}{\theta(z)\theta(s)}$ refers to the meromorphic version of the quantum L-operator algebra while (4.15) provides its particular meromorphic representation. In what follows we use only this meromorphic version.

Comparing (4.15) to (4.9) we can read off that L and $\tilde{L}$ are related in the following way

$$\tilde{L}_{ij}(z) = \frac{\prod_{n \neq j} \theta(q_{nj})}{\prod_{n \neq i} \theta(q_{ni})} L_{ij}(z) \qquad (4.16)$$

Since the combined transformation (4.9), (4.16) from $\hat{L}$ to L depends only on q we can conjecture that *any* representation L of the quantum L-operator algebra (3.27) is gauge equivalent to some representation $\hat{L}$ of (4.1) with a gauge-equivalence defined as

$$\hat{L}_{ij}(z) = \sum_{i'j'} \phi(z)_q^{q+\hbar\bar{\epsilon}_{i'}}{}_i \bar{\phi}(z)_q^{q+\hbar\bar{\epsilon}_{j'}}{}_j \frac{\prod_{n \neq j'} \theta(q_{nj'})}{\prod_{n \neq i'} \theta(q_{ni'})} L_{i'j'}(z). \qquad (4.17)$$

Now we are in a position to prove this conjecture. Suppose $\hat{L}$ be an abstract L-operator satisfying algebra (4.1) and introduce $\tilde{L}$ by eq.(4.9). Assume that $\tilde{L}$ has the structure $W\mathbf{P}$, where the entries of the diagonal matrix $\mathbf{P}$ are $\mathbf{P}_i = e^{\hbar\bar{\epsilon}_i \frac{\partial}{\partial q_i}}$ and the entries of W commute with q_i. Then substituting $\hat{L}$ expressed via $\tilde{L}$ in (4.1) and performing the straightforward calculation with the use of (4.8) one finds an algebra satisfied by $\tilde{L}$:

$$\sum_{\substack{i'j' \\ abcd}} W_k^a[a+c]\delta_{a+c,i+k} \bar{\phi}(z)_q^{q+\hbar\bar{\epsilon}_a}{}^{i'}\phi(z)_q^{q+\hbar\bar{\epsilon}_b}{}_{i'} \bar{\phi}(w)_{q+\hbar\bar{\epsilon}_a}^{q+\hbar(\bar{\epsilon}_a+\bar{\epsilon}_c)}{}^{j'} \bar{\phi}(w)_{q+\hbar\bar{\epsilon}_j}^{q+\hbar(\bar{\epsilon}_j+\bar{\epsilon}_d)}{}_{j'} \tilde{L}_{bj}(z)\tilde{L}_{dl}(w) =$$

$$\sum_{\substack{i'j' \\ abcd}} W_a^j[j+l]\delta_{j+l,a+c} \bar{\phi}(w)_q^{q+\hbar\bar{\epsilon}_k}{}^{i'}\phi(w)_q^{q+\hbar\bar{\epsilon}_d}{}_{i'} \bar{\phi}(z)_{q+\hbar\bar{\epsilon}_k}^{q+\hbar(\bar{\epsilon}_k+\bar{\epsilon}_i)}{}^{j'} \bar{\phi}(z)_{q+\hbar\bar{\epsilon}_c}^{q+\hbar(\bar{\epsilon}_c+\bar{\epsilon}_b)}{}_{j'} \tilde{L}_{dc}(w)\tilde{L}_{ba}(z).$$

Performing the summation in i' and j' with the help of (4.4) we obtain

$$\sum_{abc} W_k^b[b+a]\delta_{a+b,i+k} \frac{\theta(w+q_{ac}+\hbar\delta_{ab}-\hbar\delta_{jc})}{\theta(w)} \prod_{n \neq a} \frac{\theta(q_{nc}+\hbar\delta_{nb}-\hbar\delta_{jc})}{\theta(q_{na}+\hbar\delta_{nb}-\hbar\delta_{ab})} \tilde{L}_{bj}(z)\tilde{L}_{cl}(w) =$$

$$\sum_{abc} W_a^j[j+l]\delta_{j+l,a+c} \frac{\theta(z+q_{ib}+\hbar\delta_{ik}-\hbar\delta_{bc})}{\theta(z)} \prod_{n \neq i} \frac{\theta(q_{nb}+\hbar\delta_{nk}-\hbar\delta_{bc})}{\theta(q_{ni}+\hbar\delta_{nk}-\hbar\delta_{ik})} \tilde{L}_{kc}(w)\tilde{L}_{ba}(z).$$

Let us introduce an operator L by inverting (4.16). Substituting this L in the last formula, taking into account the nonzero components of the face weights and multiplying both sides by the function

$$\frac{\prod_{n\neq k}\theta(q_{nk})}{\prod_{n\neq j}\theta(q_{nj})}\frac{\prod_{n\neq i}\theta(q_{ni}+\hbar\delta_{nk}-\hbar\delta_{ik})}{\prod_{n\neq l}\theta(q_{nl}+\hbar\delta_{nj}-\hbar\delta_{jl})}$$

we finally arrive to the algebra satisfied by L:

$$\sum_s W_k^k[2k]\delta_i^k\frac{\theta(w+q_{ks}+\hbar-\hbar\delta_{js})}{\theta(w)}\frac{\prod_{n\neq k}\theta(q_{ns}-\hbar\delta_{js})}{\prod_{n\neq s}\theta(q_{ns}+\hbar\delta_{nj}-\hbar\delta_{js})}L_{kj}(z)L_{sl}(w)+$$

$$\sum_s W_k^k[k+i]\frac{\theta(w+q_{is}-\hbar\delta_{js})}{\theta(w)}\frac{\prod_{n\neq i}\theta(q_{ns}+\hbar\delta_{nk}-\hbar\delta_{js})}{\prod_{n\neq s}\theta(q_{ns}+\hbar\delta_{nj}-\hbar\delta_{js})}L_{kj}(z)L_{sl}(w)+$$

$$\sum_s W_k^i[i+k]\frac{\theta(q_{ki}+\hbar)}{\theta(q_{ki}-\hbar)}\frac{\theta(w+q_{ks}-\hbar\delta_{js})}{\theta(w)}\frac{\prod_{n\neq k}\theta(q_{ns}+\hbar\delta_{ni}-\hbar\delta_{js})}{\prod_{n\neq s}\theta(q_{ns}+\hbar\delta_{nj}-\hbar\delta_{js})}L_{ij}(z)L_{sl}(w)=$$

$$\sum_s W_j^j[j+l]\left(\frac{\theta(q_{lj}+\hbar)}{\theta(q_{lj}-\hbar)}+2\delta_{lj}\right)\frac{\theta(z+q_{is}+\hbar\delta_{ik}-\hbar\delta_{ls})}{\theta(z)}\times$$

$$\times\frac{\prod_{n\neq i}\theta(q_{ns}+\hbar\delta_{nk}-\hbar\delta_{ls})}{\prod_{n\neq s}\theta(q_{ns}+\hbar\delta_{nl}-\hbar\delta_{ls})}L_{kl}(w)L_{sj}(z)+$$

$$\sum_s W_l^j[j+l]\frac{\theta(z+q_{is}+\hbar\delta_{ik}-\hbar\delta_{js})}{\theta(z)}\frac{\prod_{n\neq i}\theta(q_{ns}+\hbar\delta_{nk}-\hbar\delta_{js})}{\prod_{n\neq s}\theta(q_{ns}+\hbar\delta_{nj}-\hbar\delta_{js})}L_{kj}(w)L_{sl}(z). \qquad (4.18)$$

Here in the second and the third lines $i\neq k$. The ratio of products of theta-functions occurring in each term in (4.18) allows one to take off the sum over s, e.g., when $i\neq k$, we have

$$\frac{\prod_{n\neq i}\theta(q_{ns}+\hbar\delta_{nk}-\hbar\delta_{js})}{\prod_{n\neq s}\theta(q_{ns}+\hbar\delta_{nj}-\hbar\delta_{js})}=$$

$$\delta_{is}\left(\delta_{ij}\frac{\theta(q_{ki})}{\theta(q_{ki}-\hbar)}|_{j\neq k}+\delta_{kj}(1-\delta_{ij})+\frac{\theta(q_{ki}+\hbar)\theta(q_{ji})}{\theta(q_{ki})\theta(q_{ji}+\hbar)}|_{\substack{j\neq i\\j\neq k}}\right)+$$

$$\delta_{ks}(1-\delta_{kj})\left(\delta_{ij}\frac{\theta(\hbar)}{\theta(q_{ik}+\hbar)}+\frac{\theta(q_{jk})\theta(\hbar)}{\theta(q_{jk}+\hbar)\theta(q_{ik})}|_{j\neq i}\right)+$$

$$\delta_{js}\frac{\theta(q_{kj})\theta(\hbar)}{\theta(q_{kj}-\hbar)\theta(q_{ji}+\hbar)}|_{j\neq i}.$$

To compare (4.18) to (3.27) we rewrite relation (3.27) with the help of eq.(3.31) in the following form:

$$R_{12}^F(z-w)P_1\overline{R}_{21}^{-1}(w)P_1^{-1}L_1(z)\overline{R}_{21}(w)L_2(w)=P_2\overline{R}_{12}^{-1}(z)P_2^{-1}L_2(w)\overline{R}_{12}(z)L_1(z)R_{12}^F(z-w). \qquad (4.19)$$

In the component form algebra (4.19) is presented in Appendix C. Comparing the components of (4.18) to the ones of (4.19) we establish that they coincide up to the overall multiplicative factor $\theta(z-w)\theta(\hbar)^2$. Thus, we have shown that any representation of algebra (3.27) by the transformation (4.17) turns into a representation of (4.1). The connection established gives right to assert that algebra (3.27) possesses a family of N-commuting integrals and that the formula of the determinant type (3.40) for the commuting family proved in [21] is also valid for the L-operator (3.35).

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5 Conclusion

In this paper, we described the dynamical R-matrix structure of the quantum elliptic RS model. The quantum L-operator algebra possesses a family of commuting operators. It turns out that this algebra has a surprisingly simple structure and can be analysed explicitly in the component form. Furthermore, one can hope that the problem of finding new representations of the algebra obtained is simpler than the corresponding problem for the algebra $RLL = LLR$.

There are several interesting problems to be discussed.

First, we recall that in the classical case we obtained two different Poisson algebras, which lead to the same classical L-operator algebra. Only one of them was quantized. It is desirable to quantize the second one and to show that the corresponding quantum L-operator algebra is isomorphic to the algebra obtained in the paper.

The elliptic RS model we dealt with in the paper corresponds to the A_{N-1} root system. It seems to be possible to extend our approach to other root systems and to derive the corresponding L-operator algebras. To this end, one should find a proper parameterization of the corresponding cotangent bundle.

Generalizing our approach to the cotangent bundle over a centrally extended group of smooth mappings from a higher-genus Riemann surface into a Lie group, one may expect to obtain new integrable systems.

It is known that the CM systems admit spin generalizations [34, 35, 36]. Recently, the spin generalization was found for the elliptic RS model [37]. However, the Hamiltonian formulation for the model is not found yet. One may hope that in our approach the spin models can arise as higher representations of the L-operator algebra.

Probably, the most interesting and complicated problem is to separate variables for the quantum elliptic RS model. Up to now only the three-particle case for the trigonometric RS model was solved explicitly [38]. One could expect that the L-operator algebra obtained in the paper may shed light on the problem.

Appendix A

In this Appendix, we prove that the limiting **r**-matrix (2.20) satisfies the classical Yang–Baxter equation. For this, let us write the equation, which follows from the Jacoby identity for the bracket (2.18):

$$[\mathbf{r}_{12}^{\varepsilon\eta}(z,w), \mathbf{r}_{13}^{\varepsilon\rho}(z,s) + \mathbf{r}_{23}^{\eta\rho}(w,s)] + [\mathbf{r}_{13}^{\varepsilon\rho}(z,s), \mathbf{r}_{23}^{\eta\rho}(w,s)]$$
$$+\frac{1}{\kappa}T_3^{\rho\ -1}(s)\{T_3^{\rho}(s), \mathbf{r}_{12}^{\varepsilon\eta}(z,w)] + \frac{1}{\kappa}T_1^{\varepsilon\ -1}(z)\{T_1^{\varepsilon}(z), \mathbf{r}_{23}^{\eta\rho}(w,s)] \qquad (A.1)$$
$$-\frac{1}{\kappa}T_2^{\eta\ -1}(w)\{T_2^{\eta}(w), \mathbf{r}_{13}^{\varepsilon\rho}(z,s)] = 0.$$

Here

$$\mathbf{r}_{12}^{\varepsilon\eta}(z,w) \quad = \quad H_1^{\varepsilon\eta}H_2^{\eta\varepsilon}r_{12}^{\varepsilon\eta}(z,w) - \alpha f^{\varepsilon\eta}(z,w)$$

$$\begin{aligned}
+ \quad & \beta I \otimes \operatorname{tr}_3 H_3^{\varepsilon\eta} H_2^{\eta\varepsilon} r_{32}^{\varepsilon\eta}(z,w) + \beta \operatorname{tr}_3 H_1^{\varepsilon\eta} H_3^{\eta\varepsilon} r_{13}^{\varepsilon\eta}(z,w) \otimes I \\
+ \quad & \beta^2 \operatorname{tr}_{34} H_3^{\varepsilon\eta} H_4^{\eta\varepsilon} r_{34}^{\varepsilon\eta}(z,w) \, I \otimes I.
\end{aligned} \tag{A.2}$$

To study (A.1), one needs to know the Poisson bracket of $\mathbf{T}^\varepsilon(z)$ with $H^{\eta\rho}$. This bracket can be easily derived from eqs.(2.13), (2.14), and (2.17):

$$\frac{1}{\kappa} \mathbf{T}_1^{\varepsilon\,-1}(z)\{\mathbf{T}_1^\varepsilon(z), H_2^{\eta\rho}\} = \sum_{i\neq j} H_i^{\varepsilon\rho} H_j^{\eta\varepsilon} \Phi(\varepsilon-\rho, q_{ij})(E_{ii} - \frac{1}{N} I) \otimes E_{jj} \tag{A.3}$$

$$- \sum_{i\neq j} H_i^{\varepsilon\eta} H_j^{\eta\varepsilon} H_j^{\eta\rho} \Phi(\varepsilon-\eta, q_{ij})(E_{ii} - \frac{1}{N} I) \otimes E_{jj}$$

$$- \sum_{i\neq j} H_i^{\varepsilon\rho} H_j^{\eta\varepsilon} \Phi(z-\rho, q_{ij}) E_{ij} \otimes E_{jj} + \sum_{i\neq j} H_i^{\varepsilon\eta} H_j^{\eta\varepsilon} H_j^{\eta\rho} \Phi(z-\eta, q_{ij}) E_{ij} \otimes E_{jj}$$

$$+ \sum_j \frac{1}{N} H_j^{\eta\rho} \left(\Phi(\varepsilon-\rho, 0) + \Phi(\eta-\varepsilon, 0) + \Phi(z-\eta, 0) - \Phi(z-\rho, 0) \right) (E_{jj} - \frac{1}{N} I) \otimes E_{jj}.$$

Eq.(A.1) holds at arbitrary values of all the parameters. Without loss of generality one can put $\rho = 0$ and $\eta = a\varepsilon$, where a and ε are real. Let us now perform the change of variables $H_i^{\varepsilon\eta}$:

$$H_i^{\varepsilon\eta} = 1 + h_i^{\varepsilon\eta}$$

and consider the expansion of the l.h.s. of (A.1) in powers of ε and h.

Let us note that the matrix $\mathbf{r}^{\varepsilon\eta}(z,w)$ has the following expansion in powers of h at $\varepsilon-\eta \to 0$:

$$\mathbf{r}^{\varepsilon\eta}(z,w) = \mathbf{r}(z,w) + \mathbf{r}_{\mathrm{reg}}^{(1)}(z,w,\varepsilon)h + \mathbf{r}^{(2)}(z,w,\varepsilon)h^2 + o(\varepsilon-\eta), \tag{A.4}$$

where $\mathbf{r}(z,w)$ is given by (2.20). The matrix $\mathbf{r}_{\mathrm{reg}}^{(1)}(z,w,\varepsilon)$ is regular when $\varepsilon \to 0$ and the ε-dependence of $\mathbf{r}^{(2)}(z,w,\varepsilon)$ is inessential. Substituting (A.4) into the bracket $\mathbf{T}^{-1}\{\mathbf{T},\mathbf{r}\}$ one gets:

$$\mathbf{T}^{-1}\{\mathbf{T},\mathbf{r}\} = \mathbf{r}_{\mathrm{reg}}^{(1)}(\varepsilon)\mathbf{T}^{-1}\{\mathbf{T},h\} + \mathbf{r}^{(2)}(\varepsilon)h\mathbf{T}^{-1}\{\mathbf{T},h\}. \tag{A.5}$$

Clearly, eq.(A.1) should be satisfied in any order in h and ε. Since we are interested in finding an equation for $\mathbf{r}$, we consider only the terms independent of h and ε in the expansion of eq.(A.1). The terms of zero order in h and ε occurring in (A.1) come from $\mathbf{r}^{\varepsilon\eta}$ and from the first term in (A.5). However, from the explicit expression for the bracket $\{\mathbf{T}, H\}$ one can see that

$$\{\mathbf{T}, h\}|_{h=0} = o(\varepsilon)$$

Now, taking into account that $\mathbf{r}_{\mathrm{reg}}^{(1)}(\varepsilon)$ is regular at $\varepsilon \to 0$, we conclude that the last three terms in (A.1) do not contribute. Thus, in the zero order in h and ε, eq.(A.1) reduces to the CYBE for $\mathbf{r}(z,w)$.

The remarkable thing is that the main elliptic identities (see Appendix B) follow from the Jacoby identity for the bracket (2.18) or, equivalently, from the Yang-Baxter equation for $\mathbf{r}(z,w)$.

Appendix B

Here we present the basic elliptic function identities, formulated as a set of functional relations on $\Phi(z, w)$ [8]:

$$\Phi(z, x)\Phi(w, y) = \Phi(z, x - y)\Phi(z + w, y) + \Phi(z + w, x)\Phi(w, y - x), \tag{B.1}$$

$$\Phi(z, x)\Phi(z, y) = \Phi(z, x + y)\left(\Phi(z, 0) + \Phi(x, 0) + \Phi(y, 0) - \Phi(z + x + y, 0)\right), \tag{B.2}$$

$$\Phi(z - w, a - b)\Phi(z, x + b)\Phi(w, y + a) - \Phi(z - w, x - y)\Phi(z, y + a)\Phi(w, x + b) =$$
$$\Phi(z, x + a)\Phi(w, y + b)\left(\Phi(a - b, 0) + \Phi(x + b, 0) - \Phi(x - y, 0) - \Phi(a + y, 0)\right). \tag{B.3}$$

Eq.(B.2) is the limiting case of eq.(B.1) where $w \to z$, and eq.(B.3) is a consequence of (B.1) and (B.2). Note that the exponent term $e^{-2\zeta(1/2)zs + 2\pi i s \frac{z - \bar{z}}{\tau - \bar{\tau}}}$ in $\Phi(z, s)$ as well as the linear term $-2\zeta(1/2)z + 2\pi i \frac{z - \bar{z}}{\tau - \bar{\tau}}$ in $\Phi(z, 0)$ are irrelevant since they drop out from (B.1–B.2).

To establish the unitarity relation for R, one also needs the identity involving the Weierstrass $\mathcal{P}$-function

$$\Phi(z, s)\Phi(z, -s) = \mathcal{P}(z) - \mathcal{P}(s),$$

and to prove eq.(2.50), the following relation between the derivatives of Φ is of use:

$$\frac{\partial \Phi(z, q_{ij})}{\partial z} = \frac{\partial \Phi(z, q_{ij})}{\partial q_{ij}} - (\Phi(z, 0) - \Phi(q_{ij}))\Phi(z, q_{ij}).$$

Appendix C

In this Appendix we present the quantum L-operator algebra

$$R_{12}^F(z-w)P_1 \overline{R}_{21}^{-1}(w)P_1^{-1}L_1(z)\overline{R}_{21}(w)L_2(w) = P_2 \overline{R}_{12}^{-1}(z)P_2^{-1}L_2(w)\overline{R}_{12}(z)L_1(z)R_{12}^F(z-w) \tag{C.1}$$

in the component form. The l.h.s. of (C.1) has the form

$$\sum_{i \neq k; j, l} \Phi(\hbar, q_{ki})\Phi(\hbar, q_{ik})\Phi(\hbar, q_{kj} - \hbar)L_{ij}(z)L_{kl}(w)E_{ij} \otimes E_{kl}$$

$$- \sum_{i \neq k; j, l} \Phi(\hbar, q_{ki})\Phi(\hbar, q_{ij} - \hbar)\Phi(w, q_{kj} - \hbar)L_{ij}(z)L_{jl}(w)E_{ij} \otimes E_{kl}$$

$$+ \sum_{i \neq k; j, l} \Phi(\hbar, q_{ki})\Phi(w, q_{ki})\Phi(\hbar, q_{ij} - \hbar)L_{ij}(z)L_{il}(w)E_{ij} \otimes E_{kl}$$

$$+ \sum_{i \neq k; j, l} \Phi(z - w, q_{ik})\Phi(\hbar, q_{ki})\Phi(\hbar, q_{ij} - \hbar)L_{kj}(z)L_{il}(w)E_{ij} \otimes E_{kl}$$

$$- \sum_{i \neq k; j, l} \Phi(z - w, q_{ik})\Phi(\hbar, q_{kj} - \hbar)\Phi(w, q_{ij} - \hbar)L_{kj}(z)L_{jl}(w)E_{ij} \otimes E_{kl}$$

$$+ \sum_{i \neq k; j, l} \Phi(z - w, q_{ik})\Phi(w, q_{ik})\Phi(\hbar, q_{kj} - \hbar)L_{kj}(z)L_{kl}(w)E_{ij} \otimes E_{kl}$$

$$+ \sum_{j, k, l} \Phi(z - w, \hbar)\Phi(w, \hbar)\Phi(\hbar, q_{kj} - \hbar)L_{kj}(z)L_{kl}(w)E_{kj} \otimes E_{kl}$$

$$- \sum_{j, k, l} \Phi(z - w, \hbar)\Phi(w, \hbar)\Phi(w + \hbar, q_{kj} - \hbar)L_{kj}(z)L_{jl}(w)E_{kj} \otimes E_{kl}$$

The r.h.s. of (C.1) reads

$$\sum_{i\neq k;j\neq l}\Phi(\hbar,q_{ki})\Phi(\hbar,q_{il}-\hbar)\Phi(\hbar,q_{lj})L_{kl}(w)L_{ij}(z)E_{ij}\otimes E_{kl}$$

$$+\sum_{i\neq k;j,l}\Phi(\hbar,q_{ki})\Phi(\hbar,q_{ij}-\hbar)\Phi(z-w,q_{lj}+\hbar\delta_{lj})L_{kj}(w)L_{il}(z)E_{ij}\otimes E_{kl}$$

$$-\sum_{i\neq k;j\neq l}\Phi(\hbar,q_{kl}-\hbar)\Phi(z,q_{il}-\hbar)\Phi(\hbar,q_{lj})L_{kl}(w)L_{lj}(z)E_{ij}\otimes E_{kl}$$

$$-\sum_{i\neq k;j,l}\Phi(\hbar,q_{kj}-\hbar)\Phi(z,q_{ij}-\hbar)\Phi(z-w,q_{lj}+\hbar\delta_{lj})L_{kj}(w)L_{jl}(z)E_{ij}\otimes E_{kl}$$

$$+\sum_{i\neq k;j\neq l}\Phi(z,q_{ik})\Phi(\hbar,q_{kl}-\hbar)\Phi(\hbar,q_{lj})L_{kl}(w)L_{kj}(z)E_{ij}\otimes E_{kl}$$

$$+\sum_{i\neq k;j,l}\Phi(z,q_{ik})\Phi(\hbar,q_{kj}-\hbar)\Phi(z-w,q_{lj}+\hbar\delta_{lj})L_{kj}(w)L_{kl}(z)E_{ij}\otimes E_{kl}$$

$$+\sum_{j\neq l;i}\Phi(z,\hbar)\Phi(\hbar,q_{il}-\hbar)\Phi(\hbar,q_{lj})L_{il}(w)L_{ij}(z)E_{ij}\otimes E_{il}$$

$$+\sum_{i,j,l}\Phi(z,\hbar)\Phi(\hbar,q_{ij}-\hbar)\Phi(z-w,q_{lj}+\hbar\delta_{lj})L_{ij}(w)L_{il}(z)E_{ij}\otimes E_{il}$$

$$-\sum_{j\neq l;i}\Phi(z,\hbar)\Phi(z+\hbar,q_{il}-\hbar)\Phi(\hbar,q_{lj})L_{il}(w)L_{lj}(z)E_{ij}\otimes E_{il}$$

$$-\sum_{i,j,l}\Phi(z,\hbar)\Phi(z+\hbar,q_{ij}-\hbar)\Phi(z-w,q_{lj}+\hbar\delta_{jl})L_{ij}(w)L_{jl}(z)E_{ij}\otimes E_{il}$$

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