

Probabilistic Response Predictions using Few Nonlinear Finite Element Analyses

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ABSTRACT: This paper combines second-moment statistics with exact and efficiently calculated finite element response sensitivities. The objective is to gain insights from a single, or a few, nonlinear static or dynamic finite element analyses. Three contributions are included. First, the Bouc-Wen material model that is most recently implemented in OpenSees is amended with the algorithmically consistent tangent that is required for correct response sensitivity calculations. Second, the direct differentiation method is implemented for that model. Third, the response sensitivities are transformed into a standardized parameter space; thereafter, points along that vector are transformed back to the original parameter space in order to provide probabilistic insights into the response and the relative importance of correlated input parameters.

1. INTRODUCTION

The long-term vision behind this paper is to improve computer simulation models, plus to gain as much insight as possible from each analysis. One application is earthquake engineering, where static and dynamic nonlinear finite element analyses are commonplace. The finite element models are comprehensive, with hundreds or thousands of input parameters, finite elements, and material instances. Oftentimes, a large number of deterministic analyses are conducted. The post-processing of such results may provide desirable insights into the performance of the structure. However, this paper explores another approach. The idea is to learn more, ideally in a probabilistic manner, from one or just a few nonlinear finite element analyses.

In the ICASP community, there are two distinct approaches for employing numerical simulation models. One is to run the simulation model several times, at selected input values, in order to train surrogate models to the results (Blatman & Sudret 2010). That approach is non-intrusive because the computer code of the numerical simulation model remains unaltered. Conversely, the approach adopted in this paper is intrusive. Following work by Zhang & Der Kiureghian (1993) and others, the computer code is amended to produce response sensitivities in an

exact and efficient manner. Additionally, ideas related to new history variable and state determinations are discussed at the end of this paper.

2. STRUCTURAL MODEL

The finite element method discretizes structural boundary value problems by means of degrees of freedom connected by elements. As a demonstration, nonlinear displacement-based frame elements are considered in this paper. They have wide-flange cross-sections discretized into fibers containing an inelastic uniaxial material model. This distributed plasticity element is appealing for several reasons and quintessential for the software framework OpenSees (McKenna et al. 2010). OpenSees is an open-source platform for creating and analyzing numerical simulation models for nonlinear static or dynamic analysis in earthquake engineering. The framework was spearheaded by Dr. Fenves and his student Dr. McKenna in the late 1990s at the University of California at Berkeley. A limited Matlab-based and object-oriented twin, called G2 because OpenSees was then called G3, was created by Dr. Fenves around the same time for coursework purposes. The author of the present conference paper took the course from Dr. Fenves and is here developing and extending a Python version of G2.

Figure 1 shows a steel column employed to demonstrate the developments in this paper. The black dots are nodes, each with three degrees of freedom, connecting elements shown as thick black lines. Each element has five integration points, i.e., cross-section instances, shown as red dots. Each of those cross-sections are discretized into 3 fibers in each flange and 8 fibers in the web, as shown in Figure 1. As a result, the model has $5 \cdot 5 \cdot (8 + 2 \cdot 3) = 350$ material instances. The length of the cantilever is 5m, the width of the flanges is $b_f = 365\text{mm}$, the thickness of the flanges is $t_f = 18\text{mm}$, the height of the web, from outer edge to outer edge, is $h_w = 355\text{mm}$, and the thickness of the web is $t_w = 11\text{mm}$. The material is characterized by modulus of elasticity $E = 200,000\text{MPa}$, yield stress $f_y = 350\text{MPa}$, and second-slope stiffness ratio $\alpha = 2\%$. The distributed load is denoted by q .

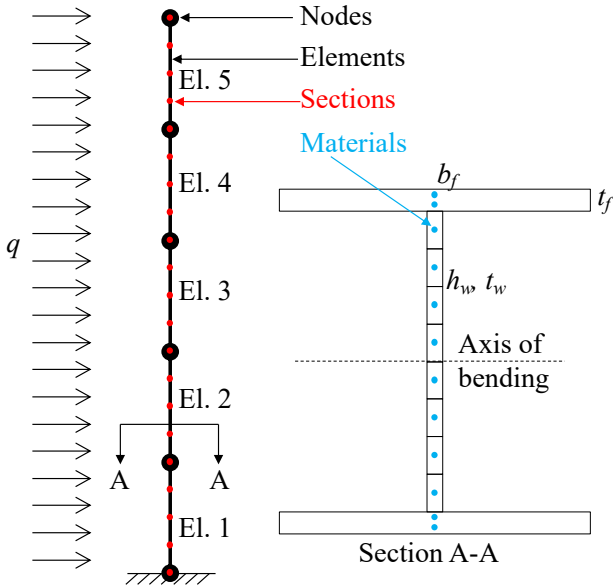


Figure 1: Finite element demonstration model.

For the probabilistic analysis, the following coefficients of variation are employed: $\delta q = 15\%$, $\delta E = 10\%$, $\delta f_y = 20\%$, $\delta \alpha = 20\%$, $\delta h_w = 5\%$, $\delta b_f = 5\%$, $\delta t_f = 5\%$, and $\delta t_w = 5\%$. The material model is presented later in this paper; it is a Bouc-Wen model that takes E , f_y , and α as input, plus a positive integer $\eta = 3$, which governs the smoothness of the transition from elastic to yielding. It also takes the two parameters β and γ

that are explained later in this paper. Figure 2 shows the lateral displacement response at the top of the cantilevered column Figure 1 for $q = 100\text{kN/m}$, applied in 20 increments plus three unloading steps for the subsequent demonstration of response sensitivities.

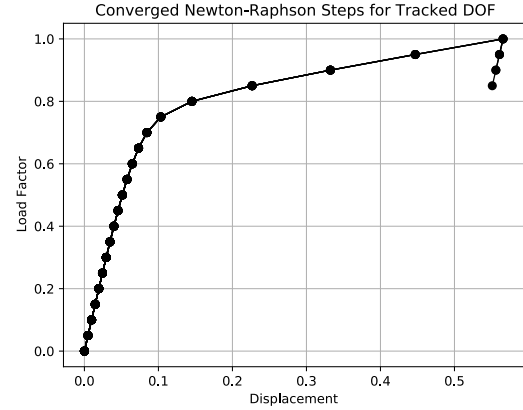


Figure 2: Tip displacement for the model in Figure 1.

3. STATISTICAL MODEL

For large simulation models it is demanding for the engineer to determine detailed probability distributions for all the input parameters. Instead, a second-moment approach is adopted in this paper. That implies that the analyst provides best estimates, i.e., mean values, plus standard deviations and correlation coefficients. The second-moment approach is appealing for two reasons. First, it lends itself to straightforward and transparent probabilistic calculations, as demonstrated shortly. Second, it carries appeal in engineering practice because the specification of that information is relatively simple and intuitive. The second-moment information for the vector, $\mathbf{x}$, of selected input parameters is expressed by the mean vector, $\mathbf{M}$, the diagonal matrix of standard deviations, $\mathbf{D}$, and the correlation matrix, $\mathbf{R}$. With that notation, the covariance matrix is $\mathbf{\Sigma} = \mathbf{DRD}$.

4. BOUC-WEN MATERIAL MODEL

The author of this paper implemented a version of the Bouc-Wen model in OpenSees two decades ago. That model, presented by Baber and Noori (1985), facilitates the modeling of deterioration. However, it is also a complex model; a neat alternative that takes E , f_y , and α as input is

implemented more recently in OpenSees by Schellenberg et al. (2015) employing the formulation by Casciati (1989). The Bouc-Wen model was first developed by Wen (1976) with reference to a 1967 conference paper by R. Bouc, who also published a research report with this model in French in 1971. In this model, the stress is written as a sum of a linear part and a hysteretic part:

$$\sigma = \alpha \cdot E \cdot \varepsilon + (1 - \alpha) \cdot f_y \cdot z \quad (1)$$

where all parameters are defined earlier, except z , which is a variable that governs the evolution of the hysteretic material behavior, defined by the differential equation

$$\dot{z} = \frac{1}{\varepsilon_y} \cdot (\dot{\varepsilon} - \gamma \cdot \dot{\varepsilon} \cdot |z|^\eta - \beta \cdot |\dot{\varepsilon}| \cdot z^\eta) \quad (2)$$

where $\varepsilon_y = f_y/E$ is the yield strain and β and γ are parameters that govern the evolution of the value of z . In this context, ε_y is essentially a scaling factor that provides a transition from elastic to yield within appropriate strain values. The specification of γ and β in a manner such that the sum of their individual absolute values equal unity implies that z takes on values between -1 and 1 . That is because Eq. (2) then suggests that the rate of change of z approaches zero as z approaches -1 or 1 . The value $\gamma = \beta = 0.5$ is adopted in this paper for a bilinear material with smooth transitions.

As will be seen shortly, it is helpful to isolate the strain rate in Eq. (2), which yields the following revised version of Eq. (2):

$$\dot{z} = \frac{1}{\varepsilon_y} \cdot (1 - (\gamma + \beta \cdot \text{sign}(\dot{\varepsilon} \cdot z)) \cdot |z|^\eta) \cdot \dot{\varepsilon} \quad (3)$$

The signum function appears in order to accommodate the switch of the absolute value operator in the last term on the right-hand side of Eq. (2). The rewrite in Eq. (3) is helpful because $\dot{\varepsilon} = d\varepsilon/dt = \Delta\varepsilon/\Delta t$, and the strain increment, $\Delta\varepsilon$, is available to all material models in a nonlinear analysis. Eq. (3), which governs the evolution of z , is now discretized using the Backward Euler scheme. This is done in order to obtain z_{n+1} from z_n at increment n . For a generic

differential equation $\dot{z} = r(z(t))$ the algorithm reads $z_{n+1} = z_n + \Delta t r(z_{n+1})$. Given the right-hand-side of Eq. (3), and the fact that $\dot{\varepsilon} = \Delta\varepsilon/\Delta t$, the Backward Euler scheme reads, after recognizing that Δt cancels:

$$z_{n+1} = z_n + \frac{1}{\varepsilon_y} \cdot (1 - (\gamma + \beta \cdot \text{sign}(\dot{\varepsilon}_{n+1} \cdot z_n)) \cdot |z_{n+1}|^\eta) \cdot \Delta\varepsilon_{n+1} \quad (4)$$

As expected for the Backward Euler scheme, z_{n+1} appears on both sides of Eq. (4). For a moment, let z_{n+1} be labelled x and consider the Newton algorithm $x_{j+1} = x_j - f(x_j)/f'(x_j)$ for the generic root-finding problem $f(x) = 0$, where f' means df/dx . The function f is, from Eq. (4)

$$f = z_{n+1} - z_n - (1 - (\gamma + \beta \cdot \text{sign}(\Delta\varepsilon_{n+1} \cdot z_n)) \cdot |z_{n+1}|^\eta) \cdot \frac{\Delta\varepsilon_{n+1}}{\varepsilon_y} \quad (5)$$

where $\Delta\varepsilon_{n+1}$ takes the place of $\dot{\varepsilon}_{n+1}$ in the signum function because the sign is determined by $\Delta\varepsilon_{n+1}$. The derivative of f is

$$\frac{df}{dz_{n+1}} = 1 + \eta \cdot \text{sign}(z_{n+1}) \cdot (\gamma + \beta \cdot \text{sign}(\Delta\varepsilon_{n+1} \cdot z_n)) \cdot |z_{n+1}|^{\eta-1} \cdot \frac{\Delta\varepsilon_{n+1}}{\varepsilon_y} \quad (6)$$

Once z_{n+1} is determined, the stress is obtained from Eq. (1).

5. ALGORITHMICALLY CONSISTENT TANGENT

In addition to the stress, the material algorithm must return the tangent stiffness. The tangent stiffness is used in the Newton-Raphson algorithm to compute the structural response. The accuracy and consistency of the tangent stiffness is vital for the convergence rate of the Newton-Raphson algorithm; however, it is also crucial for sensitivity analysis with the direct differentiation method. The starting point for the derivation of the tangent for the Bouc-Wen material model is the derivative of Eq. (1) with respect to the current strain:

$$\frac{d\sigma_{n+1}}{d\varepsilon_{n+1}} = \alpha \cdot E + (1 - \alpha) \cdot f_y \cdot \frac{dz_{n+1}}{d\varepsilon_{n+1}} \quad (7)$$

When calculating the last factor, i.e., $\partial z_{n+1}/\partial \varepsilon_{n+1}$, the concept of a continuum tangent must be carefully distinguished from the algorithmically consistent tangent. It is the latter that should be implemented on the computer. It is obtained by differentiating the previously presented equations that compute z_{n+1} , with respect to the strain, and substituting the result, $\partial z_{n+1}/\partial \varepsilon_{n+1}$, into Eq. (7). The Newton-Raphson algorithm within the material calculates z_{n+1} using the iterative algorithm

$$(z_{n+1})_{j+1} = (z_{n+1})_j - \frac{f((z_{n+1})_j)}{\frac{df((z_{n+1})_j)}{dz_{n+1}}} \quad (8)$$

Any change in z_{n+1} after convergence would emanate solely from a change in the fraction in the last term of Eq. (8). That means the sought derivative is the derivative of that fraction with respect to the strain, i.e.,

$$\frac{dz_{n+1}}{d\varepsilon_{n+1}} = -\frac{d}{d\varepsilon_{n+1}} \left(\frac{f(z_{n+1})}{\frac{df(z_{n+1})}{dz_{n+1}}} \right) \quad (9)$$

By adopting the notation $f = f(z_{n+1})$ and $f' = df(z_{n+1})/dz_{n+1}$ the chain rule of differentiation applied to Eq. (9) yields

$$\frac{dz_{n+1}}{d\varepsilon_{n+1}} = - \left(\frac{\left(\frac{df}{dz_{n+1}} \right)}{f'} - \frac{f}{(f')^2} \cdot \left(\frac{df'}{d\varepsilon_{n+1}} \right) \right) \quad (10)$$

where

$$\frac{df}{d\varepsilon_{n+1}} = -(1 - (\gamma + \beta \cdot \text{sign}(\Delta \varepsilon_{n+1} \cdot z_{n+1})) \cdot |z_{n+1}|^\eta) \cdot \frac{1}{\varepsilon_y} \quad (11)$$

and

$$\frac{df'}{d\varepsilon_{n+1}} = \eta \cdot \text{sign}(z_{n+1}) \cdot (\gamma + \beta \cdot \text{sign}(\Delta \varepsilon_{n+1} \cdot z_{n+1})) \cdot |z_{n+1}|^{\eta-1} \frac{1}{\varepsilon_y} \quad (12)$$

Eq. (10) substituted into Eq. (7) is the algorithmically consistent tangent that is implemented in G2.

6. RESPONSE SENSITIVITIES

Response sensitivity analysis aims at calculating $\partial \mathbf{u} / \partial \mathbf{x}$ or $\partial u / \partial \mathbf{x}$, where u is one component of the

displacement vector $\mathbf{u}$ and $\mathbf{x}$ contains the selected input parameters. By implementing the direct differentiation method, such derivatives are calculated alongside the response during each finite element analysis (Zhang & Der Kiureghian, 1993). Characteristics of that method are:

- *One-time effort*: All response equations and algorithms are differentiated analytically, and implemented in the finite element code.
- *Exact*: Analytical differentiation means that the response sensitivities are exact.
- *Efficient*: There is no need for additional finite element analyses for each parameter.
- *Linear system*: For any analysis type, response sensitivities are obtained from a linear system of equations, for each parameter.
- *Identical coefficient matrix*: The coefficient matrix in the linear system is independent of the parameter, prompting the efficient use of LU-decomposition, or similar solvers.
- *Consistent tangent*: The linear system must employ the current algorithmically consistent tangent stiffness.
- *Two phases*: The internal forces depend implicitly on all parameters via the displacements $\mathbf{u}$. The forces may also depend explicitly on any given parameter. This leads to the calculation of conditional internal force derivatives in “Phase 1” and storage of unconditional derivatives of history variables in “Phase 2” (Zhang & Der Kiureghian, 1993).

Turning to the Bouc-Wen model, the first objective is to calculate the stress derivative $(\partial \sigma / \partial \theta)|_{\varepsilon \text{ fixed}}$ in Phase 1, where θ is one variable from $\mathbf{x}$. Three of the input parameters to the Bouc-Wen model are considered as options for θ , namely, E , f_y , and α . Note that the derivatives $dE/d\theta$, $df_y/d\theta$, $d\alpha/d\theta$, and $d\varepsilon_y/d\theta$, which appear in subsequent equations, are easily calculated once it is determined what θ is. Differentiation of Eq. (1) yields (continued on the next page)

$$\left. \frac{d\sigma_{n+1}}{d\theta} \right|_{\varepsilon_{n+1} \text{ fixed}} = \frac{d\alpha}{d\theta} \cdot E \cdot \varepsilon_{n+1} + \alpha \cdot \frac{dE}{d\theta} \cdot \varepsilon_{n+1} \quad (13)$$

$$-\frac{d\alpha}{d\theta} \cdot f_y \cdot z_{n+1} + (1 - \alpha) \cdot \frac{df_y}{d\theta} \cdot z_{n+1} + (1 - \alpha) \cdot f_y \cdot \frac{dz_{n+1}}{d\theta} \Big|_{\varepsilon_{n+1} \text{ fixed}}$$

To assist the further derivations, it is recalled that the hysteresis evolution variable z is obtained by solving the equation $f(z_{n+1})=0$, where

$$f = z_{n+1} - z_n - \left(1 - (\gamma + \beta \cdot \text{sign}(\Delta\varepsilon_{n+1} \cdot z_{n+1})) \cdot |z_{n+1}|^\eta\right) \cdot \frac{\varepsilon_{n+1} - \varepsilon_n}{\varepsilon_y} \quad (14)$$

using the Newton algorithm. As a result, at the converged state at increment $n+1$, the hysteresis evolution variable, which is a history variable for this material model, reads

$$z_{n+1} = z_n + \left(1 - (\gamma + \beta \cdot \text{sign}(\Delta\varepsilon_{n+1} \cdot z_{n+1})) \cdot |z_{n+1}|^\eta\right) \cdot \frac{\varepsilon_{n+1} - \varepsilon_n}{\varepsilon_y} \quad (15)$$

Differentiating Eq. (15) with respect to θ yields

$$\begin{aligned} \frac{dz_{n+1}}{d\theta} &= \frac{dz_n}{d\theta} - (\gamma + \beta \cdot \text{sign}(\Delta\varepsilon_{n+1} \cdot z_{n+1})) \cdot \eta \cdot |z_{n+1}|^{\eta-1} \cdot \text{sign}(z_{n+1}) \cdot \frac{dz_{n+1}}{d\theta} \cdot \frac{\varepsilon_{n+1} - \varepsilon_n}{\varepsilon_y} \\ &\quad + \left(1 - (\gamma + \beta \cdot \text{sign}(\Delta\varepsilon_{n+1} \cdot z_{n+1})) \cdot |z_{n+1}|^\eta\right) \cdot \frac{d\varepsilon_{n+1}}{d\theta} - \frac{d\varepsilon_n}{d\theta} \cdot |z_{n+1}|^\eta \cdot \frac{d\varepsilon_y}{d\theta} \\ &\quad - \left(1 - (\gamma + \beta \cdot \text{sign}(\Delta\varepsilon_{n+1} \cdot z_{n+1})) \cdot |z_{n+1}|^\eta\right) \cdot \frac{\varepsilon_{n+1} - \varepsilon_n}{\varepsilon_y^2} \cdot \frac{d\varepsilon_y}{d\theta} \end{aligned} \quad (16)$$

where $d\varepsilon_{n+1}/d\theta$ is kept in order to facilitate Phase 2. Solving Eq. (16) for $dz_{n+1}/d\theta$ yields

$$\frac{dz_{n+1}}{d\theta} = \frac{\frac{dz_n}{d\theta} - a + b}{1 + c} \quad (17)$$

where the auxiliary constants are

$$a = \frac{\frac{d\varepsilon_y}{d\theta} (\varepsilon_{n+1} - \varepsilon_n) (1 - (\gamma + \beta \cdot \text{sign}(\Delta\varepsilon_{n+1} \cdot z_{n+1})) \cdot |z_{n+1}|^\eta)}{\varepsilon_y^2} \quad (18)$$

$$b = \frac{(\frac{d\varepsilon_{n+1}}{d\theta} - \frac{d\varepsilon_n}{d\theta}) (1 - (\gamma + \beta \cdot \text{sign}(\Delta\varepsilon_{n+1} \cdot z_{n+1})) \cdot |z_{n+1}|^\eta)}{\varepsilon_y} \quad (19)$$

$$c = \frac{(\varepsilon_{n+1} - \varepsilon_n) (\gamma + \beta \cdot \text{sign}(\Delta\varepsilon_{n+1} \cdot z_{n+1})) \cdot \eta \cdot |z_{n+1}|^{\eta-1} \cdot \text{sign}(z_{n+1})}{\varepsilon_y} \quad (20)$$

Eq. (17) is implemented in G2, with $d\varepsilon_{n+1}/d\theta=0$ in Phase 1. Not setting that derivative equal to

zero addresses the second objective in this section; the calculation of the unconditional derivative of history variables for Phase 2, once the strain derivative, $\partial\varepsilon/\partial\theta$, is available. Note that both Phase 1 and Phase 2 must be conducted for all input parameters, even if a parameter is not associated with the material. That is because $\partial\mathbf{u}/\partial\mathbf{x}$ and therefore $\partial\varepsilon/\partial\theta$ is non-zero even for, say, θ representing a load variable. This means that the unconditional derivative of a history variable in Phase 2 is a vector with dimension equal to the total number of parameters in $\mathbf{x}$.

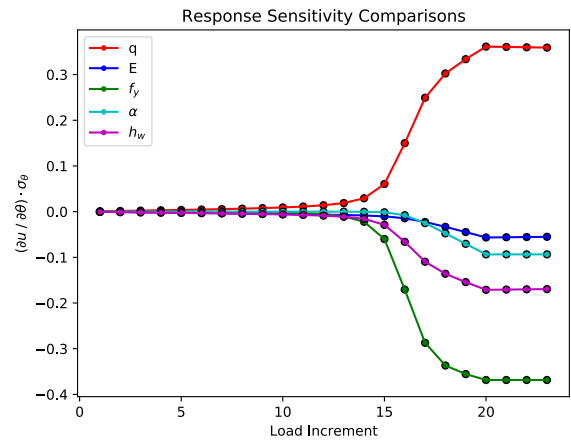


Figure 3: Response sensitivity demonstration.

Figure 3 shows response sensitivity results obtained from the same analysis that produced the load-displacement curve in Figure 2. In this analysis, but not later in the paper, all five finite elements have identical q , E , f_y , α , and cross-section dimensions. It is arbitrarily selected to plot h_w amongst those dimensions. One purpose of Figure 3 is to verify the correctness of the previously presented equations. Therefore, the black dots are included to show the result of finite difference calculations with parameter perturbations equal to 10^{-8} times the original parameter value. It is observed in the detailed numbers, not presented here, that the finite difference results match the direct differentiation results to the expected accuracy. Notice that Figure 3 displays the product $(\partial u / \partial \theta) \sigma_i$, where σ_i is the standard deviation of the respective parameter, in order to facilitate a comparison. The

figure suggests that the load, q , and the yield stress, f_y , are particularly influential on the response after yielding.

7. LIKELY INPUT PERTURBATIONS

The objective in this section is to make use of the second-moment information and the response sensitivity results that are described in the previous sections. One idea explored here and in a recent journal paper submission by the author (Haukaas 2023) is to first transform the vector of response sensitivities, $\partial u / \partial \mathbf{x}$, into a standardized parameter space. That space is denoted by $\mathbf{y}$, where $\mathbf{y}$ are uncorrelated zero-mean variables with unit variances. The transformation between the two spaces is well-known and reads (Der Kiureghian 2022)

$$\mathbf{x} = \mathbf{M} + \mathbf{DLy} \quad (21)$$

where $\mathbf{L}$ is the lower-triangular Cholesky decomposition of the correlation matrix. As a result, the gradient vector in the $\mathbf{y}$ -space is

$$\frac{\partial u}{\partial \mathbf{y}} = \frac{\partial u}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial \mathbf{y}} = \frac{\partial u}{\partial \mathbf{x}} \mathbf{DL} \quad (22)$$

Two points along that gradient vector in the $\mathbf{y}$ -space will be transformed back to the $\mathbf{x}$ -space studied. The first is the end point of the vector in Eq. (22), which in light of Eq. (21) reads

$$\mathbf{x} = \mathbf{M} + \mathbf{DL} \frac{\partial u}{\partial \mathbf{y}} \quad (23)$$

However, instead of keeping both terms in the right-hand side of Eq. (23) it is the *change* in parameter values that is highlighted in this paper. That means keeping only the last term in Eq. (23), which is denoted

$$\Delta \mathbf{x}_V = \mathbf{DL} \frac{\partial u}{\partial \mathbf{y}} = \mathbf{DL} \left(\frac{\partial u}{\partial \mathbf{x}} \mathbf{DL} \right) \quad (24)$$

where the parenthesis enforces the correct order of vector-matrix multiplications and the subscript V is explained shortly. The other point along the vector in Eq. (22) that is studied here is the point that sits a unit distance from origin in the $\mathbf{y}$ -space. In the $\mathbf{y}$ -space, that point is the end point of the vector in Eq. (22) once that vector has been scaled to unit length. Again, omitting the mean, focusing

on parameter perturbations, that point in the $\mathbf{x}$ -space is

$$\Delta \mathbf{x}_\sigma = \frac{1}{\|\frac{\partial u}{\partial \mathbf{x}} \mathbf{DL}\|} \cdot \mathbf{DL} \left(\frac{\partial u}{\partial \mathbf{x}} \mathbf{DL} \right) \quad (25)$$

where the subscript σ and the subscript V in Eq. (24) are now addressed. In this context, V means variance and σ means standard deviation. The reason for those labels is as follows. $\Delta \mathbf{x}_V$ is the parameter perturbations that cause a unit variance change in the response, u , if the relationship between u and $\mathbf{x}$ is linear. Similarly, $\Delta \mathbf{x}_\sigma$ is the parameter perturbations that cause a unit standard deviation change in the response, if the input-output relationship is linear.

To substantiate those claims, first rewrite Eq. (24) as follows:

$$\Delta \mathbf{x}_V = \frac{\partial u}{\partial \mathbf{x}}^T \boldsymbol{\Sigma} \quad (26)$$

which shows that a multiplication with the gradient vector yields the exact variance for a linear function, i.e., $\text{Var}[u] = (\partial u / \partial \mathbf{x})^T \boldsymbol{\Sigma} (\partial u / \partial \mathbf{x})$.

Next, recall that $\text{Var}[u] = \sigma_u^2$, where σ_u is the standard deviation of u , and observe that $\partial u / \partial \mathbf{x} \mathbf{DL}$ in Eq. (22) is essentially the square root of $\text{Var}[u] = (\partial u / \partial \mathbf{x})^T \boldsymbol{\Sigma} (\partial u / \partial \mathbf{x})$. The result is that the Euclidean norm $\|(\partial u / \partial \mathbf{x}) \mathbf{DL}\|$ is the standard deviation of the response u , and that $\Delta \mathbf{x}_\sigma$ in Eq. (25) contain the parameter perturbations that would cause a unit standard deviation change in the response, if the input-output relationship is linear.

Even in linear elastic finite element analysis, the relationship between the input parameters $\mathbf{x}$ and the response u is nonlinear. For instance, geometry parameters enter in a nonlinear manner in the stiffness matrix of a structural model. However, $\Delta \mathbf{x}_V$ and $\Delta \mathbf{x}_\sigma$ are nonetheless employed in this paper, including in the development of a new importance measure for correlated variables.

The finite element model in Figure 1 has 40 parameters because each of the five elements has unique q , E , f_y , α , h_w , t_w , b_f , and t_f parameters. $\Delta \mathbf{x}_V$ and $\Delta \mathbf{x}_\sigma$ are now evaluated for $q=100\text{kN/m}$, without any of the three unloading steps shown at

the end of the analysis in Figure 2. Those 40 values, for each vector, representing the most likely parameter perturbations to attain a unit variance or standard deviation in the response, respectively, are not presented here. Instead, two ideas are now pursued, one in the next section.

Table 1: Effect of likely parameter perturbations.

q [kN/m]	$u(\mathbf{x} + \Delta\mathbf{x}_\sigma) - u(\mathbf{x})$	$\sigma_{u,FOSM}$	$\frac{u(\mathbf{x} + \Delta\mathbf{x}_\sigma) - u(\mathbf{x})}{\sigma_{u,FOSM}}$
10	0.0014	0.0013	1.08
30	0.0044	0.0041	1.08
50	0.0092	0.0080	1.15
60	0.019	0.012	1.54
65	0.053	0.017	3.01
70	0.17	0.030	5.53
75	0.29	0.072	4.05
80	0.38	0.20	1.87
90	0.50	0.42	1.20
100	0.60	0.48	1.26
110	0.70	0.60	1.16
130	0.89	0.78	1.15

The first idea is to reanalyze the finite element model with perturbed parameter values $\mathbf{x} + \Delta\mathbf{x}_\sigma$. The results are shown in Table 1 together with the first-order second-moment estimate $\sigma_{u,FOSM}$, which is the square root of $(\partial u / \partial \mathbf{x})^T \Sigma (\partial u / \partial \mathbf{x})$. Keeping in mind that if u were linear in terms of $\mathbf{x}$ then $u(\mathbf{x} + \Delta\mathbf{x}_\sigma) - u(\mathbf{x})$ would equal $\sigma_{u,FOSM}$, the first two columns in Table 1 show that the finite element response is always nonlinear in terms of the input parameters, even in the linear elastic response regime. However, that nonlinearity, measured by the difference in $u(\mathbf{x} + \Delta\mathbf{x}_\sigma) - u(\mathbf{x})$ and $\sigma_{u,FOSM}$, varies significantly as the load varies. The third column in Table 1, which provides the ratio between the two estimates of the response standard deviation, shows that the difference is particularly large when the load factor in Figure 1 is around 0.7, i.e., when q is around 70kN/m. This is highlighted in gray in the table. This means that the ratio in the third column of Table 1 can be used to track how close the response is to becoming significantly nonlinear.

8. RANKING OF CORRELATED INPUT

The second idea related to the use of $\Delta\mathbf{x}_V$ and $\Delta\mathbf{x}_\sigma$ is to employ them as importance measures in order to rank the input parameters according to relative importance. To that end, first multiply the components of $\Delta\mathbf{x}_V$ by the respective response derivatives and thereby define

$$\boldsymbol{\tau}_V = \Delta\mathbf{x}_V[i] \cdot \frac{\partial u}{\partial x_i} \quad (27)$$

According to Eq. (26) and the paragraph below it, summation of the components of $\boldsymbol{\tau}_V$ produces the variance of the response, if $u(\mathbf{x})$ is linear. For that reason, the components of $\boldsymbol{\tau}_V$ are indicative of the contribution to the response variance from each input parameter. Importantly, $\boldsymbol{\tau}_V$ accounts for correlation between the parameters, and is obtained from a single finite element analysis. An alternative for comparing variance contributions in this manner is Sobol indices (Sobol 2001). However, Sobol indices require sampling, i.e., many response analyses, and are not always sensitive to correlation (Haukaas 2023).

Second, divide the components of $\Delta\mathbf{x}_\sigma$ by the respective standard deviation and thereby define

$$\boldsymbol{\tau}_\sigma = \mathbf{D}^{-1} \Delta\mathbf{x}_\sigma = \frac{\Delta\mathbf{x}_\sigma[i]}{\sigma_i} \quad (28)$$

Although $\boldsymbol{\tau}_V$ and $\boldsymbol{\tau}_\sigma$ are expected to give quite similar importance rankings of the input variables, $\boldsymbol{\tau}_\sigma$ cannot be interpreted as individual variance contributions. Instead, the reference for $\boldsymbol{\tau}_\sigma$ is the basic importance measure $(\partial u / \partial x) \sigma_i$, explained on Page 39 in the book by Der Kiureghian (2022), except $\boldsymbol{\tau}_\sigma$ includes the effect of correlation between the input parameters. Another way to view $\boldsymbol{\tau}_\sigma$ is as the most likely parameter perturbations to cause a unit standard deviation change in the response, relative to the standard deviation of each parameter.

The top-ten list for the model in Figure 1, ranked according to $\boldsymbol{\tau}_V$ scaled to unit length, at the load $q=100\text{kN/m}$, is shown in Table 2. The yield stress in the element at the bottom of the cantilever is naturally the most important parameter, followed by the web height in that element. It also makes sense that the load, q , of

Elements 4 and 5 feature in the top-ten list. An advantage of the single-analysis importance measures presented here is that they account for correlation. Suppose the value of q in all elements are correlated by 0.9. Then, the load in Elements 4 and 5 move up to become the 2nd and 3rd most important parameters and the loads in Elements 3 and 2 appear as the 5th and 6th most important parameters. Correlation “groups” the variables and give them jointly more importance.

Table 2: Top-ten most important parameters.

#	Element	Parameter	τ_v
1	1	f_y	0.96
2	1	h_w	0.20
3	5	q	0.14
4	4	q	0.080
5	1	b_f	0.080
6	1	t_f	0.066
7	1	α	0.063
8	3	q	0.037
9	1	E	0.020
10	2	q	0.010

9. CONCLUSIONS

This paper presents exact response sensitivity calculations within a finite element code, followed by the use of that information to gain additional insights from a single analysis. The approach adopted in this paper is “intrusive” in the sense that it requires amendments to the finite element code. Within the spatial and temporal boundaries of this submission, several additional ideas are omitted. The first is to present nonlinear dynamic examples. The peak response of the nonlinear static analyses presented here is related to the peak response from a nonlinear dynamic analysis. However, additional points emerge that should be addressed. The second idea is to conduct additional state determinations, especially at the peak response. A state determination is a query to the elements without iterations that may or may not converge. In dynamic analyses, additional history variables committed at the peak response may assist in executing additional state determinations or

additional loading at the peak response, after the last time step. Finally, the misunderstanding that the nonlinear finite element model will reveal severe damage and collapse if pushed far enough should be avoided. Rather, it should be an objective to learn, from one analysis, where the model is outside the expected physical behavior.

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