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Conformal bootstrap and thermalization in holographic CFTs

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Declaration and coauthorship

I declare that this thesis has not been submitted as an exercise for a degree at this or any other university and it is entirely my own work. I agree to deposit this thesis in the open access institutional repository of the University, or allow the library to do so on my behalf, subject to Irish Copyright Legislation and Trinity College Library conditions of use and acknowledgement. I consent to the examiners retaining a copy of the thesis beyond the examining period, should they so wish. This thesis contains, among others, results of four coauthored research papers, three with Robin Karlsson, Manuela Kulaxizi and Andrei Parnachev [12-14], and one with Robin Karlsson and Andrei Parnachev [16].

Petar Tadic
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This thesis is the result of three years of PhD studies. Parts of this thesis are based on the publications [12-14], which were published in the journal JHEP. Furthermore, parts of this thesis are based on the preprint [16]. The content of [12] can be found in Section 4. The content of [13] can be found in Section 5. The content of [14] can be found in Section 6. Section 7 is based on the preprint [16].

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Summary

This thesis covers a number of topics in conformal field theories that are supposed to have gravity duals according to the AdS/CFT correspondence. We use the conformal bootstrap in the Regge and lightcone limits as the technique for studying these theories. We also explore their thermal properties by studying the large-N conformal field theories at finite temperature.

In Section 2 we review the basic implications of conformal symmetry in quantum field theories in spacetime with the number of dimensions $d \geqslant 3$ and $d = 2$ separately. We precisely define the holographic CFTs and briefly describe the idea of conformal bootstrap as the consistency condition of all conformal field theories.

In Section 3 we introduce the "heavy-heavy-light-light" correlator in Regge and lightcone limit. We review the calculation of correlators of this type and we set up the notation for the rest of the thesis.

In Section 4 we study the heavy-heavy-light-light correlation function of the holographic CFTs in the Regge limit, based on [12]. The gravitational dual of this correlator in the Regge limit is the high energy scattering of the light probe with the fixed impact parameter in the asymptotically AdS black hole background. The Schwarzschild radius of the black hole in AdS units is proportional to the ratio of the conformal dimension of the heavy operator and the central charge. This ratio serves as a useful expansion parameter whose power counts the number of stress tensors in the multi-stress tensor operators which contribute to the four-point correlation function. In the cross-channel the four-point function is determined by the OPE coefficients and anomalous dimensions of the heavy-light double-trace operators. We explain how this data can be obtained and explicitly compute the first and second order terms in the expansion of the anomalous dimensions. We observe perfect agreement with known results in the lightcone limit, which were obtained by computing perturbative corrections to the energy eigenstates in AdS spacetimes.

In Section 5 we study the heavy-heavy-light-light correlation function in the lightcone limit, based on [13]. Near-lightcone correlators are dominated by the contributions of exchanged operators with the lowest twist. We consider the contributions of such leading lowest twist multi-stress tensor operators to the

correlator in a holographic CFT of any even dimensionality. An infinite number of such operators contribute, but their sum is described by a simple ansatz. We show that the coefficients in this ansatz can be determined recursively, thereby providing an operational procedure to compute them. This is achieved by bootstrapping the corresponding near lightcone correlator: conformal data for any minimal-twist determines that for the higher-order minimal-twist contributions and so on. To illustrate this procedure in four spacetime dimensions we determine the contributions of double- and triple-stress tensors. We compute the OPE coefficients; whenever results are available in the literature, we observe the complete agreement. We also compute the contributions of double-stress tensors in six spacetime dimensions and determine the corresponding OPE coefficients. In all cases the results are consistent with the exponentiation of the near lightcone correlator. This is similar to the situation in two spacetime dimensions for the Virasoro vacuum block.

In Section 6 we generalize the technique developed in Section 5 to include the contributions of multi stress tensor operators of arbitrary twist to the heavy-heavy-light-light correlator, based on [14]. We show how one can compute the unknown coefficients in the generalized version of the ansatz from Section 5 by the lightcone bootstrap, for the entire stress tensor sector of the correlator. Therefore, iteratively computing the OPE coefficients of multi-stress tensor operators with an increasing twist. Some parameters are not fixed by the bootstrap - they correspond to the OPE coefficients of multi-stress tensors with spin zero and two. We further show that in holographic CFTs one can use the phase shift computed in the dual gravitational theory to reduce the set of undetermined parameters to the OPE coefficients of multi-stress tensors with spin zero. Finally, we verify some of these results using the Lorentzian OPE inversion formula and comment on its regime of applicability.

Finally, in Section 7 we study the thermalization of the stress tensor sector in CFTs with a large number of degrees of freedom, based on [16]. We show that the thermalization of operators from this sector, or the equality between their expectation values in heavy states and at finite temperature, is equivalent to a universal behavior of their OPE coefficients with a pair of identical heavy scalar operators. We verify this behavior in a number of examples which include holographic and free CFTs and provide a bootstrap argument for the general case. In a free CFT we check the thermalization of multi stress tensor operators directly and also confirm the equality between the contributions of multi stress tensors to heavy-heavy-light-light correlators and to the corresponding thermal light-light two-point functions by disentangling the contributions of other light operators. Unlike multi stress tensors, we show that these light operators violate the Eigenstate Thermalization Hypothesis and do not thermalize.

Dedicated to my wife Tijana, my mother Daliborka, my father Lazar,

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Contents

1. Motivation

Constructing the theory of quantum gravity has been an open problem for many years. The lack of experimental data at energies so high that the quantum-gravitational effects are detectable implies the necessity for alternative ways for learning about quantum gravity. Modern approaches to this problem include using the dualities between gravitational theories and theories without gravitational degrees of freedom. The term "duality" means that theories have equivalent Hilbert spaces and dynamics. Their mathematical descriptions, on the other hand, can differ, for example, they can have different Lagrangians, degrees of freedom, and be situated in different spacetimes. The hope with this type of duality is that one can indirectly approach the regime where the quantum effects in gravity are important using the dual, non-gravitational description and therefore learn the general properties of quantum gravity. Knowing these properties will help the achievement of the ultimate goal, i.e. the construction of the fundamental microscopic theory of quantum gravity.

1.1. Holographic principle and AdS/CFT duality

The works of Bekenstein and Hawking [1-3] gave the first indirect hint of the existence of dualities that include gravitational theories. They showed that black holes are dynamic objects that emit thermal radiation. The entropy of a neutral, non-rotating black hole is shown to be proportional to the area of the horizon A:

$$S_{BH} = \frac{Ac^3}{4G\hbar}. \tag{1.1}$$

Colloquially, one can interpret this entropy as being proportional to the amount of information in the physical system. Naively, one might have expected that it scales with the volume of the space behind the horizon (or volume of the black hole). The fact that it scales with the area of the horizon was the first indication that the gravitational theory where the black hole is present has the degrees of freedom that fit in spacetime with one spatial dimension less. This was the first sign of the so-called holographic principle of gravitational theories [4-5].

Another important indication of the existence of dualities between gravitational and non-gravitational theories came from the work of 't Hooft [6], where he studied non-Abelian gauge theories in the limit of a large number of colors (large-N limit). He showed that in this limit the Feynman diagrams rearrange in a such way that the expansion looks the same as the perturbative expansion in string theory with string coupling $1/N$. Since the string theories necessarily include gravity, this was strong evidence of the relation between gravitational string theories and non-gravitational gauge theories at the deep fundamental level.

The work of Brown and Henneaux [7] gave further insight into what kind of theories one should be looking at to find the dual descriptions of the gravitational theories. Namely, by studying gravity in three-dimensional Anti de-Sitter spacetime, they found the conformal symmetry algebra of conformal field theory in two-dimensional spacetime as the algebra of asymptotic symmetries in three-dimensional gravity. This was the first evidence of the relation between gravitational theories in Anti-de Sitter spacetime and the conformal field theories in spacetime with one spatial dimension less.

The first concrete instance of the duality between gravitational and non-gravitational theories was established in the work of Maldacena [8] in 1997. By studying black branes in supersymmetric string theory he proposed the famous anti-de Sitter/conformal field theory (AdS/CFT) conjecture which states that type IIB string theory with string length $l_s = \sqrt{\alpha'}$ and coupling constant g_s on $AdS_5 \times S^5$ with radius of curvature L of both five-dimensional Anti-de Sitter AdS_5 and sphere S^5, is dual to $\mathcal{N} = 4$ super-Yang-Mills (SYM) in flat four-dimensional spacetime with gauge group $SU(N)$ and gauge coupling g_{YM}. The parameters of these theories are related by the following equations

$$g_{YM}^2 = 2\pi g_s, \qquad g_{YM}^2 N = \frac{L^4}{2\alpha'^2}. \tag{1.2}$$

The $\mathcal{N} = 4$ gauge theory is conformally invariant which is the reason why it is called the "CFT side" of the duality. The gravitational part (type IIB string theory) of the duality is usually called the "AdS side", or simply the "gravity side".

The string theory reasoning behind this duality is the equivalence between the open and closed string descriptions of the Dirichlet branes (D-branes). These are the non-perturbative higher-dimensional objects present in the superstring theory. They can be viewed in the open and closed string perspectives, and which one is right depends on the value of the string coupling g_s. In terms of open strings, D-branes might be viewed as the higher-dimensional objects where the open strings end. Since the open strings must be treated as small perturbations that do not affect the gravitational background, this description is only valid when the coupling between open and closed strings g_s is small $g_s \ll 1$. Furthermore, if we neglect the massive string excitations by focusing on the low energy regime, the dynamics of the D-branes will be described in terms of the pure supersymmetric gauge theory. If there are N coincident branes, the gauge group will be[1] $SU(N)$ and the effective 't Hooft coupling will be $g_s N$. It follows that this description works for $g_s N \ll 1$. In terms of the closed strings, the D-branes can be viewed as the solitonic solutions of the low-energy limit of the string theory, i.e. the supergravity, and they represent the source of the gravitational field, therefore, they curve the spacetime around them. For the supergravity approximation to hold, there must be a scale separation between the characteristic length L of the spacetime considered and the string length $\sqrt{\alpha'}$, or in other words, $L^4/\alpha'^2 \to \infty$. In the case of N coincident branes, $L^4/\alpha'^2 \propto g_s N \gg 1$. Therefore, this description is valid in the opposite limit compared to the open string description, $g_s N \gg 1$.

One concludes that there are two, very different theories with different degrees of freedom and that even live in different spacetimes, but still describe the dynamics of the same physical objects, the D-branes, in the different limits of the effective coupling. The conjectured part of the duality is that the respective descriptions are valid even beyond the limits specified above and one can relate the parameters of these two theories by (1.2). Concretely, the candidate for the theory of quantum gravity (type IIB string theory) can be mapped to the $\mathcal{N} = 4$ SYM gauge theory without gravitational degrees of freedom. Additionally, the information about gravitational theory in five-dimensional spacetime (obtained after Kaluza-Klein reduction of type IIB string theory on five-dimensional sphere S^5) is mapped to the gauge conformal field theory in four-dimensional spacetime, following the idea from the works of Hawking and Bekenstein and therefore satisfying the holographic principle. In the rest of the thesis, we do not

[1] Actually, the gauge group is $U(N)$, but it turns out that the $U(1) \subseteq U(N)$ degrees of freedom decouple from the $SU(N)$ degrees of freedom.

use the string theory language behind the AdS/CFT and for the more detailed discussion of the duality, we refer the reader to the original papers [8-10].

The duality conjecture as stated above is the strongest version of the conjecture as it is assumed to hold for all values of parameters in both theories. Unfortunately, studying and proving the duality at a generic value of parameters is not feasible at the moment as one would have to study string theory away from the classical limit in gravity and gauge theory at an arbitrary value of coupling constant. This is the reason why the practical exploration of duality is mostly limited to the classical limit in the string theory, intending to go beyond it.

The classical limit in gravity is the limit where the string length, in units of AdS radius $\sqrt{\alpha'}/L$, and the string coupling g_s go to zero $\sqrt{\alpha'}/L \to 0$, $g_s \to 0$. This implies that the limit of classical gravity is equivalent to the limit of infinite 't Hooft coupling $\lambda = g_{YM}N^2 \to \infty$ and infinite number of colors $N \to \infty$ in $\mathcal{N} = 4$ SYM gauge theory according to eqs. (1.2). Therefore, the classical limit in gravity is dual to the limit of infinite coupling in the dual non-gravitational theory.

The idea of holographic duality is generalized using the previous example. The general statement of the AdS/CFT correspondence is that a gravitational theory in $(d{+}1)$-dimensional asymptotically Anti-de Sitter spacetime is dual to a certain, strongly coupled conformal field theory in d-dimensional flat spacetime. This d-dimensional flat spacetime where the CFT is present is the boundary of AdS spacetime [9-10]. The generalization of the AdS/CFT correspondence beyond the type IIB string theory and $\mathcal{N} = 4$ SYM is assumed to be independent of the string realization of both of the theories in the duality. The properties of strongly coupled conformal field theories that have the gravitational duals are conjectured in [11] and theories satisfying these are called holographic CFTs. We give the precise definition of holographic CFTs in Section 2. Most of the work in this thesis is devoted to studying the properties of holographic CFTs and their relations to the dual gravitational theories on the AdS side of the duality.

1.2. Outline

To understand the generic properties of quantum gravity using the AdS/CFT correspondence, two things are needed: first, we need to fully understand the duality at the classical level in gravity and therefore we need to establish the mappings between all variables in classical gravity with their counterparts in terms of observables in the dual CFT, and second, we need a better understanding of strongly coupled, holographic CFTs. This is the reason why we are interested in understanding better the whole class of holographic CFTs, including the parametrization of the class as well as the universal aspects of the dynamics of theories within the class.

Generally, there are two different approaches for studying holographic CFTs. The first approach is to use the classical gravitational dual of a particular CFT, such as Einstein-Hilbert gravity or the modifications thereof, and to extract the implications to the strongly coupled, holographic CFT by studying the gravity side of the duality. These implications can sometimes be generalized to the whole class of the holographic CFTs. This was the dominant approach in the early days of the AdS/CFT duality and we use it in Section 4. The other approach is to directly use the field theory techniques to solve and classify holographic CFTs and it became more dominant recently. We mostly use this approach in the rest of the thesis.

In Section 2 we give a brief introduction to the conformal field theories in general, with special attention to the holographic CFTs and the conformal bootstrap that is a technique used for studying them. In Section 3 we give a detailed introduction for studying "heavy-heavy-light-light" correlator. We also review the calculation of such correlator, we establish the notation and define Regge and lightcone limits that will be used later. In Section 4 we study the CFT version of the high energy scattering of the light probe with the fixed impact parameter in the black hole background [12]. In Section 5 and Section 6 we develop an algorithm for computing the CFT analog of the contributions of multiple gravitons to the scattering of the light probe in the black hole background [13,14]. We show that some of these contributions are universal, i.e. they are the same for all holographic CFTs, in agreement with the claim in [15]. In Section 7 we study the holographic CFTs at the finite temperature [16].

2. The dictionary of the conformal field theory

In this section we review the basics of conformal field theories in two-dimensional and higher-dimensional spacetime and we precisely define the class of holographic CFTs that have a gravitational dual according to [11]. We discuss the basics of the conformal bootstrap, as the technique used in studying the holographic CFTs. We conclude the section with the discussion of holographic CFTs at finite temperature and thermalization.

2.1. Brief introduction to the CFT

In this subsection we give a brief introduction to the kinematics of conformal field theories and establish the notation that we will be using in the rest of the thesis. We consider CFTs in spacetime with number of dimensions $d \geqslant 3$ and $d = 2$ separately.

2.1.1. CFT in spacetime with the number of dimensions $d \geqslant 3$

Conformal field theory is a quantum field theory invariant under the transformations of the conformal group. These are the diffeomorphisms that transform the metric in the following way

$$ds^2 \to ds'^2 = f(x)^2 ds^2, \tag{2.1}$$

where $f(x)$ is an arbitrary function of coordinates. These transformations preserve angles (i.e. causal structure in Lorentzian signature) but not the distances. In spacetime with d dimensions and the Lorentzian signature, these transformations make conformal group $SO(d, 2)$, while in the Euclidean signature, the conformal group is $SO(d + 1, 1)$. For now, we assume that $d \geqslant 3$, as the conformal symmetry when $d = 2$ is described differently than in the case of higher-dimensional spacetime. The conformal group in spacetime with $d \geqslant 3$ consists of transformations in Poincaré group plus the scale transformations and the special conformal transformations. Poincaré group includes translations, with generators P_μ, rotations with generators[2] M_{ij}, and boosts

[2] In our notation Greek letters $(\mu, \nu, \ldots)$ denote all spacetime coordinates, while Latin letters $(i, j, \ldots)$ denote just the spatial part of the coordinates.

with generators M_{0j}. Scale transformations, whose generator is denoted by D, act on spacetime coordinates as

$$D: \qquad x^\mu \to \lambda x^\mu, \tag{2.2}$$

while special conformal transformations, with generators denoted by K_μ, act as

$$K_\mu: \qquad x^\mu \to \frac{x^\mu - b^\mu x^2}{1 - 2xb + b^2 x^2}, \tag{2.3}$$

where b^μ is an arbitrary constant vector. The full conformal algebra in spacetime with the number of dimensions $d \geqslant 3$ is given by

$$\begin{aligned}
[P_\mu, P_\nu] &= [K_\mu, K_\nu] = [M_{\mu\nu}, D] = 0, \\
[D, P_\mu] &= iP_\mu, \\
[D, K_\mu] &= -iK_\mu, \\
[K_\mu, P_\nu] &= 2i(\delta_{\mu\nu} D - M_{\mu\nu}), \\
[M_{\mu\nu}, M_{\rho\sigma}] &= i\left(\delta_{\nu\rho} M_{\mu\sigma} - \delta_{\mu\rho} M_{\nu\sigma} - \delta_{\nu\sigma} M_{\mu\rho} + \delta_{\mu\sigma} M_{\nu\rho}\right), \\
[M_{\mu\nu}, P_\rho] &= i\left(\delta_{\nu\rho} P_\mu - \delta_{\mu\rho} P_\nu\right), \\
[M_{\mu\nu}, K_\rho] &= i\left(\delta_{\nu\rho} K_\mu - \delta_{\mu\rho} K_\nu\right),
\end{aligned} \tag{2.4}$$

where $\delta_{\mu\nu}$ is the identity matrix if we are working in the Euclidean signature and if we are working in the Lorentzian signature it should be exchanged with the Minkowski metric $\eta_{\mu\nu}$. The Euclidean space $\mathbf{R}^d$ with the metric in the spherical coordinates

$$ds^2 = dr^2 + r^2 d\Omega^2_{d-1}, \tag{2.5}$$

can be conformally mapped to the cylinder $\mathbf{R} \times S^{d-1}$, with the transformation $\tau = \log(r)$, and one obtains the metric

$$ds^2 = e^{2\tau}(d\tau^2 + d\Omega^2_{d-1}). \tag{2.6}$$

The coordinate τ is the Euclidean time on the cylinder. Dilatation operator D, that in the $\mathbf{R}^d$ shifts r coordinate $r \to \lambda r$, generates the time translations on the cylinder $\tau \to \tau + \log(\lambda)$. Therefore, the dilatation operator plays the role of the Hamiltonian on the cylinder and its eigenvalues are treated as energies. This is the physical reason why we need to demand that the eigenvalues of the dilatation operator, also known as the conformal dimensions, are bounded from below. From the algebra (2.4) it is obvious that the generators of translations P_μ increase the conformal dimension of the operator by one, while the generators of special conformal transformations K_μ lower the conformal dimension by one. We define the set of operators whose conformal dimensions can not be lowered further and we call these primary operators

$$[K_\mu, \mathcal{O}_{\Delta,s}(0)] = 0. \tag{2.7}$$

These operators are also known as the highest weight vectors that define the irreducible representation of the conformal group. They are characterized by the eigenvalues of the dilatation operator (i.e. the conformal dimensions) and their spin

$$\begin{aligned} [D, \mathcal{O}_{\Delta,s}(0)] &= i\Delta \mathcal{O}_{\Delta,s}(0), \\ [M_{\mu\nu}, \mathcal{O}_{\Delta,s}(0)] &= \hat{\mathcal{M}}_{\mu\nu} \mathcal{O}_{\Delta,s}(0), \end{aligned} \tag{2.8}$$

where $\hat{\mathcal{M}}_{\mu\nu}$ are generators of representation s of the group[3] $SO(d)$. Starting from the primary operator $\mathcal{O}_{\Delta,s}$, the entire, infinite-dimensional, irreducible representation of the conformal group is obtained by acting on primary operators $\mathcal{O}_{\Delta,s}$ with generators of translations, which are represented by derivatives in coordinate representation. The Hilbert space of the conformal field theory factorizes in a sum of irreducible representations generated from highest weight vectors, i.e. primary operators with quantum numbers (Δ, s).

The main objects of study in the conformal field theory are the correlation functions of the local operators present in the spectrum of the theory. The conformal symmetry fixes the one-point, two-point and three-point correlation functions up to the position-independent constants [17,18]. For external scalar primary operators ϕ_i with conformal dimension Δ_i, these are given by

$$\begin{aligned} \langle \phi_i(x) \rangle = 0, \qquad \langle \phi_i(x_i)\phi_j(x_j)\rangle &= \frac{\delta_{ij}}{x_{ij}^{2\Delta_i}}, \\ \langle \phi_i(x_i)\phi_j(x_j)\phi_k(x_k)\rangle &= \frac{\lambda_{ijk}}{x_{ij}^{\Delta-2\Delta_k} x_{ik}^{\Delta-2\Delta_j} x_{jk}^{\Delta-2\Delta_i}}, \end{aligned} \tag{2.9}$$

where $x_{ab} = x_a - x_b$ and $\Delta = \Delta_i + \Delta_j + \Delta_k$. The two-point functions are fixed by the normalization of the operators and we are using a basis of orthogonal

[3] s denotes the spin of these representations.

operators, hence the Kronecker delta δ_{ij}. The coefficients λ_{ijk} are called operator product expansion coefficients. Once the normalization of operators is fixed, they can not be scaled away, therefore they represent physical parameters of the theory. For spinning operators, multiple independent tensor structures are allowed by conformal symmetry in the three-point functions, so these are determined by more than one position-independent constant multiplying every allowed tensor structure.

The simplest non-trivial correlation functions whose spacetime dependence is not fixed by the conformal symmetry are the four-point correlation functions of the external scalar primary operators. In this thesis, we mostly focus on them in four-dimensional spacetime.

2.1.2. CFT in spacetime with the number of dimensions $d = 2$

The conformal symmetry in two-dimensional spacetime is described differently than the conformal symmetry in higher-dimensional spacetime. Namely, the conformal algebra in two dimensions is an infinite algebra that puts much stronger constraints on the dynamics of the theory compared to the conformal algebra in higher dimensions (2.4).

In two-dimensional spacetime with Euclidean spherical coordinates (r, ϕ) and metric given by

$$ds^2 = dr^2 + r^2 d\phi^2 = dz d\bar{z}, \tag{2.10}$$

where

$$z = re^{i\phi}, \qquad \bar{z} = re^{-i\phi}, \tag{2.11}$$

the set of all conformal transformations is equal to the set of all holomorphic and anti-holomorphic functions of complex coordinates z and $\bar{z}$

For practical purposes, we can lift the condition that z and $\bar{z}$ are the complex conjugate and treat them as the independent complex variables. Then, at the end of the calculation, if one wants to evaluate the result in the Euclidean signature, one requires $z^* = \bar{z}$. The vector space of infinitesimal conformal transformations has a basis given by $\{(\ell_n, \bar{\ell}_n) | n \in \mathbf{Z}\}$, where $\ell_n = -z^{n+1}\partial_z$ and

$\bar{\ell}_n = -\bar{z}^{n+1}\partial_{\bar{z}}$. It is easy to check that generators ℓ_n and $\bar{\ell}_n$ create the two copies of the Witt algebras given by

$$\begin{aligned}
[\ell_n, \ell_m] &= (n-m)\ell_{n+m}, \\
[\bar{\ell}_n, \bar{\ell}_m] &= (n-m)\bar{\ell}_{n+m}, \\
[\ell_n, \bar{\ell}_m] &= 0.
\end{aligned} \tag{2.12}$$

The subalgebras $\{\ell_{-1}, \ell_0, \ell_1\}$ and $\{\bar{\ell}_{-1}, \bar{\ell}_0, \bar{\ell}_1\}$ generate the globally well-defined conformal transformations on the Riemann sphere $S^2 \sim \mathbf{C} \cup \{\infty\}$. These six generators are the two-dimensional version of P_μ, K_μ and $M_{\mu\nu}$ in higher dimensions. It is obvious that generators ℓ_{-1} and $\bar{\ell}_{-1}$ generate translations in the complex plane, therefore they are the two-dimensional version of P_μ, and similarly one can check that ℓ_1 and $\bar{\ell}_1$ generates the special conformal transformations, therefore being the two-dimensional version of K_μ. Generators ℓ_0 and $\bar{\ell}_0$ are related to the dilatation operator D and the generator of rotations. It is useful to write these generators in terms of (r, ϕ) coordinates

$$\ell_0 = -\frac{1}{2}r\partial_r + \frac{i}{2}\partial_\phi, \qquad \bar{\ell}_0 = -\frac{1}{2}r\partial_r - \frac{i}{2}\partial_\phi. \tag{2.13}$$

Now, it is easy to see that the dilatation and the rotation generators can be expressed as

$$D = (\ell_0 + \bar{\ell}_0) = -r\partial_r, \qquad M = (\ell_0 - \bar{\ell}_0) = i\partial_\phi. \tag{2.14}$$

By the standard quantum-mechanical reasoning, for example, by demanding the unitarity of the theory, one concludes that the conformal algebra of the charges in two-dimensional spacetime has actually to be a central extension of the Witt algebra, also known as the Virasoro algebra and given by

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}, \tag{2.15}$$

and similarly for $\bar{L}_n$, while $[L_n, \bar{L}_m] = 0$. Here, c is a constant that multiplies the center of the Virasoro algebra given by the identity operator, and it is called the central charge of a two-dimensional conformal field theory. The physical meaning of the two-dimensional central charge c is that it counts the number of degrees of freedom in the theory, as shown in [19].

One immediately observes that in contrast with the higher-dimensional case, the conformal algebra in two-dimensional spacetime is infinite-dimensional and therefore it gives stronger constraints on the dynamics of the theory. From (2.14) we conclude that the eigenvalues of the dilatation operator are related to the eigenvalues of the operators L_0 and $\bar{L}_0$. We use the same logic as in the higher-dimensional case to argue that the eigenvalue of the dilatation operator must be bounded from below, which now translates to the fact that the eigenvalues of L_0 and $\bar{L}_0$ are bounded from below as well. From the Virasoro algebra (2.15) it follows that operators L_n, for $n > 0$, decreases the eigenvalue of L_0, while the operators L_{-n}, for $n > 0$, increases this eigenvalue. Therefore, we define Virasoro primary operators as those whose eigenvalue of L_0 (and $\bar{L}_0$) can not be decreased further

$$[L_n, \mathcal{O}(0)] = [\bar{L}_n, \mathcal{O}(0)] = 0, \qquad n \geq 1. \tag{2.16}$$

These are characterized by quantum numbers $(h, \bar{h})$, which are eigenvalues of L_0 and $\bar{L}_0$

$$[L_0, \mathcal{O}(0)] = h\mathcal{O}(0), \quad [\bar{L}_0, \mathcal{O}(0)] = \bar{h}\mathcal{O}(0). \tag{2.17}$$

The quantum numbers $(h, \bar{h})$ can be related to the two-dimensional version of the quantum numbers (Δ, s) using (2.14), by the following relations:

$$\Delta = h + \bar{h}, \qquad s = h - \bar{h}. \tag{2.18}$$

Descendants of the Virasoro primary state are obtained by acting with L_{-n} and $\bar{L}_{-n}$, for $n \geqslant 1$, on the primary state. This way, the highest weight irreducible representation of the Virasoro algebra is generated. These representations are called Verma modules. The Hilbert space of two-dimensional CFT factorizes into a sum of the Verma modules.

Generators $\{L_{-1}, L_0, L_1\}$ and $\{\bar{L}_{-1}, \bar{L}_0, \bar{L}_1\}$ make the global subalgebra of the Virasoro algebra. One can notice that there is no central charge in the commutators of elements of the global subalgebra, similarly to the conformal algebra in higher-dimensional spacetime (2.4). If eqs. (2.16) hold only for $n = 1$ generators from the global subalgebra, we say that operator $\mathcal{O}$ is quasi-primary. Therefore, every Virasoro primary is a quasi-primary as well, but not the other way around. Each Verma module can be factorized into the sum of the highest weight representations of the global subalgebra, where the highest weight vectors are quasi-primary operators from the given Verma module.

The four-point correlation functions are not fixed by the conformal symmetry even in two dimensions. But here the Virasoro algebra grants a much better analytic control over them compared to the higher-dimensional cases, which is the reason we often use the two-dimensional CFT as the toy model for the calculation of the four-point correlators in CFT in higher-dimensional spacetime.

2.2. Holographic CFTs

In [11] it was conjectured that there are two necessary and sufficient conditions for CFTs to have a gravitational dual with local physics below the AdS scale. The first condition can be stated as:

The central charge C_T is large, $C_T \to \infty$, and the correlation functions in the theory factorize at large C_T.

The central charge C_T of the conformal field theory in spacetime with Euclidean signature and arbitrary number of dimensions, is defined via the two-point correlation function of the canonically normalized stress tensor

$$\langle T_{\mu\nu}(x) T_{\rho\sigma}(0)\rangle = \frac{C_T}{\Omega_{d-1}^2 x^{2d}} \mathcal{I}_{\mu\nu,\rho\sigma}(x), \tag{2.19}$$

where

$$\begin{aligned} \mathcal{I}_{\mu\nu,\rho\sigma}(x) &= \frac{1}{2}\left(I_{\mu\rho}(x) I_{\nu\sigma}(x) + I_{\mu\sigma}(x) I_{\nu\rho}(x)\right) - \frac{1}{d}\delta_{\mu\nu}\delta_{\rho\sigma}, \\ I_{\mu\nu} &= \delta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}, \qquad \Omega_{d-1} = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}. \end{aligned} \tag{2.20}$$

In the two-dimensional spacetime, the central charge defined this way is related to the central charge c defined via the Virasoro algebra (2.15) with the relation $C_T = c/2$. However, one should remember that C_T in higher-dimensional CFTs does not satisfy the c-theorem and therefore it can not be treated as the number of degrees of freedom in the theory. The coordinate dependence of the two-point correlation function (2.19) is fixed by the conformal symmetry, while the coefficient C_T is a characteristic of a particular theory. The stress tensor in the holographic CFT is a CFT dual of the single graviton state on the AdS side of duality.

In the example of the duality with the type IIB string theory and $\mathcal{N}=4$ SYM, the central charge of $\mathcal{N}=4$ SYM scales in terms of numbers of colors N as

$$C_T \propto N^2. \tag{2.21}$$

Therefore, the fact that the classical gravity limit in type IIB string theory is obtained by taking the large number of colors ($N \to \infty$) is equivalent to the statement that the central charge of the $\mathcal{N}=4$ SYM has to be large.

The gravitational interpretation of the central charge in holographic CFTs follows from [7]. Namely, in the case of three-dimensional gravitational theory in Anti-de Sitter spacetime, the infinite-dimensional Virasoro algebra (2.15) is found as the algebra of the asymptotic symmetry with the central charge $c = 3L/2G_N$, where G_N is the Newton's constant. Therefore, we generally interpret the central charge as the inverse Newton's constant G_N of the dual gravitational theory, or equivalently, the inverse graviton coupling constant. In the case when the CFT has a finite central charge, the effects of the finiteness of the central charge would, therefore, correspond to the graviton quantum loop corrections in the dual gravitational theory.

The second condition that the holographic CFTs have to satisfy can be stated as follows:

There is a parametrically large gap $\Delta_{\text{gap}} \to \infty$ in the spectrum of conformal dimensions of primary single-trace operators with spin greater than two.

When considering CFTs with the gauge symmetry, the physically relevant operators are the gauge-invariant operators. In the CFTs with the matter in the adjoint representation of the gauge group, the gauge-invariant operators are made of multiple traces of fundamental fields. The second condition which the holographic CFTs have to satisfy requires that the primary single-trace operators with spin greater than two have a large conformal dimension.

Multi-trace, gauge-invariant, primary operators are constructed as the products of single-trace primaries with a number of derivatives inserted in a such way that the product is still the primary operator (or in other words, commutes with K_μ). The large gap condition does not affect these operators in holographic CFTs.

In the language of dual gravity, this condition can be interpreted as the requirement that the fields with spin greater than two in gravity decouple from the rest of the degrees of freedom. Namely, the single-trace primary operators in CFT are dual of the single particle states in gravity. Mass of the field in gravity is proportional to the conformal dimension of the CFT operator. The large gap in the spectrum of conformal dimensions of single-trace primary operators with spin greater than two means that higher-spin fields in gravity have a large mass, and therefore, decouple from the rest of the degrees of freedom. In the example with type IIB string theory and $\mathcal{N} = 4$ SYM, the fact that in the classical limit in gravitational theory we have large 't Hooft coupling ($\lambda \to \infty$) in $\mathcal{N} = 4$ SYM is responsible for this decoupling of higher-spin primary single-trace operators. Namely, these operators receive the large anomalous dimensions at large coupling ($\lambda \to \infty$) and end up with large conformal dimensions, even though they have conformal dimensions of order one at the weak coupling ($\lambda \to 0$).

In the context of the string theory, requiring the large gap in the holographic CFTs accounts for the decoupling of the stringy degrees of freedom in the dual gravitational description. For the CFT with the finite gap, we would have a dual gravitational description with higher-spin, stringy modes in the spectrum.

Additionally, one should notice that having the gauge symmetry is not necessary for holographic CFTs. Without the gauge symmetry, we still use the notion of single and multiple-trace operators in the spectrum of the CFT, but in this case they differ by the large-C_T scaling of the corresponding three-point functions. In the conformal field theories where we do not assume the existence of the gauge symmetry, we also still use the "large-N" terminology by which we mean the large central charge, according to (2.21).

2.3. The conformal bootstrap

We review the important technique for studying strongly coupled CFTs, called conformal bootstrap. It was developed in [20-23]. For recent, more detailed reviews of the conformal bootstrap see [24-26].

The conformal symmetry allows us to write the convergent operator product expansion (OPE)

$$\phi_1(x_1)\phi_2(x_2) = \sum_{\text{primary } \mathcal{O}} \lambda_{12\mathcal{O}} C(x_{12}, \partial_y)\mathcal{O}(y)\Big|_{y=x_2}, \tag{2.22}$$

where the sum goes over all unit-normalized primary operators $\mathcal{O}$ (with arbitrary spin) in the spectrum, functions $C(x_{12}, \partial_y)$ are fixed by the conformal symmetry and they account for the contribution of the descendants of the primary operator $\mathcal{O}$. The operator $\mathcal{O}$ and the derivatives do not have to be evaluated at x_2, they can be evaluated at any point between x_1 and x_2, only the function C will change accordingly. The expansion converges within correlation functions in the Euclidean signature as long as x_1 is closer to x_2 than any other operators inserted at y_i, or in other words, as long as there is a sphere enclosing points x_1 and x_2 with no other external operator inserted within the sphere (see eg. [27-29]). The operator product expansion is always done first in the Euclidean signature and then it can be analytically continued to the Lorentzian signature if necessary.

Using the operator product expansion we reduce any n-point to $(n-1)$-point functions and so on until we arrive at two-point functions that are fixed by the conformal symmetry. Therefore, if we know quantum numbers (Δ, s) of all primary operators in the spectrum of the theory and all OPE coefficients λ_{ijk}, we can, in principle, compute any correlation function and the theory is solved. These data are colloquially called OPE data.

The conformal bootstrap relies on the fact that the operator product expansion is the convergent expansion, therefore it has to be associative. By requiring the associativity of the operator product expansion, also called the crossing symmetry, we put bounds on the OPE data, or sometimes even solve the theory. Let us consider the four-point correlation function of scalar primary operators ϕ_i, $\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle$. Using the following operator product expansions

$$\begin{aligned} \phi_1(x_1)\phi_2(x_2) &= \sum_{\mathcal{O}} \lambda_{12\mathcal{O}} C(x_{12}, \partial_y)\mathcal{O}(y)\Big|_{y=x_2}, \\ \phi_3(x_3)\phi_4(x_4) &= \sum_{\mathcal{O}} \lambda_{34\mathcal{O}} C(x_{34}, \partial_z)\mathcal{O}(z)\Big|_{z=x_4}, \end{aligned} \tag{2.23}$$

the four-point correlation function can be written as

$$\begin{aligned}&\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \\ &\sum_{\mathcal{O}}\lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}}C(x_{12},\partial_y)C(x_{34},\partial_z)\langle\mathcal{O}(y)\mathcal{O}(z)\rangle\Big|_{y=x_2,z=x_4}.\end{aligned} \tag{2.24}$$

On the other hand, we can choose to write the operator product expansions between ϕ_1 and ϕ_4, as well as ϕ_2 and ϕ_3

$$\begin{aligned}\phi_1(x_1)\phi_4(x_4) &= \sum_{\mathcal{O}'}\lambda_{14\mathcal{O}}C(x_{14},\partial_y)\mathcal{O}'(y)\Big|_{y=x_4}, \\ \phi_2(x_2)\phi_3(x_3) &= \sum_{\mathcal{O}'}\lambda_{23\mathcal{O}'}C(x_{23},\partial_z)\mathcal{O}'(z)\Big|_{z=x_3}.\end{aligned} \tag{2.25}$$

In this case, the four-point correlation function is given by

$$\begin{aligned}&\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \\ &\sum_{\mathcal{O}'}\lambda_{14\mathcal{O}'}\lambda_{23\mathcal{O}'}C(x_{14},\partial_y)C(x_{23},\partial_z)\langle\mathcal{O}'(y)\mathcal{O}'(z)\rangle\Big|_{y=x_4,z=x_3}.\end{aligned} \tag{2.26}$$

The different ways of writing the OPE are called different channels of expansion. Now, the condition of consistency of the theory (or the associativity of the OPE or the crossing symmetry) requires that these two channels give the same result

$$\begin{aligned}&\sum_{\mathcal{O}}\lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}}C(x_{12},\partial_y)C(x_{34},\partial_z)\langle\mathcal{O}(y)\mathcal{O}(z)\rangle\Big|_{y=x_2,z=x_4} \\ =&\sum_{\mathcal{O}'}\lambda_{14\mathcal{O}'}\lambda_{23\mathcal{O}'}C(x_{14},\partial_y)C(x_{23},\partial_z)\langle\mathcal{O}'(y)\mathcal{O}'(z)\rangle\Big|_{y=x_4,z=x_3}.\end{aligned} \tag{2.27}$$

This equation is a non-trivial constraint on the OPE data and it is known as the bootstrap equation. It bounds the spectrum of the theory, as well as on the OPE coefficients.

One should notice that generally we do not have the contributions of the same primary operators in both channels, that is why they are denoted differently by $\mathcal{O}$ and $\mathcal{O}'$. The parts of the contributions of these operators that are fixed by conformal symmetry, $C(x_{12},\partial_y)C(x_{34},\partial_z)\langle\mathcal{O}(y)\mathcal{O}(z)\rangle|_{y=x_2,z=x_4}$ and $C(x_{14},\partial_y)C(x_{23},\partial_z)\langle\mathcal{O}'(y)\mathcal{O}'(z)\rangle|_{y=x_4,z=x_3}$, are called conformal blocks (or conformal partial waves). They are not crossing symmetric independently, but their corresponding sums have to be, which is the reason why the condition (2.27) is the non-trivial requirement on the spectrum of the theory. The explicit analytic expressions for the blocks are known in even-dimensional spacetime since the work of Dolan and Osborn [30,31]. Their expressions play a crucial role in the conformal bootstrap technique as they help to solve (2.27). The conformal bootstrap technique has brought many new results for holographic CFTs in recent years (see e.g. [11-14,32-55]).

We solve this equation by taking a particular kinematic limit which isolates the small number of operators that contribute in one channel and their contributions have to be recovered by an infinite number of operators in the other channel. Examples of such limits are the lightcone and Regge limits. Physically, the lightcone limit is a limit where one of the external scalar operators in the four-point function approaches the lightcone of one of the other external operators, while the Regge limit corresponds to special kinematics, which on the gravity side is described by the scattering of highly energetic particles whose trajectories in the bulk are approximately null.

The usefulness of the conformal bootstrap technique for solving CFTs lies in the fact that one does not have to specify the theory, for example, with its Lagrangian. The set of reasonable assumptions about the spectrum of the theory allows one to run the bootstrap program and learn more about the whole class of theories for which the assumptions work. The particular class of theories that we are interested in are the strongly coupled, holographic CFTs. The bootstrap equation is perfectly well-defined for the strongly coupled theories as it does not rely on the perturbative expansion in the coupling constant.

In the rest of the thesis we mostly consider the holographic CFTs in four-dimensional spacetime. We study the four-point correlation function of two pair-wise identical primary single-trace scalar operators $\mathcal{O}_L$ and $\mathcal{O}_H$, with conformal dimensions Δ_L and Δ_H, that scale as $\Delta_L \sim \mathcal{O}(1)$ and $\Delta_H \sim \mathcal{O}(C_T)$. Because of this scaling, operator $\mathcal{O}_L$ is called "light", while $\mathcal{O}_H$ is called "heavy". Taking into account the dual gravitational picture, $\mathcal{O}_H$ is a CFT analog of the black hole, while $\mathcal{O}_L$ represents a light probe. Instead of the usual $1/C_T$ expansion parameter in the correlators of the holographic CFTs, for the four-point correlation function that includes two heavy states $\mathcal{O}_H$ as the external operators, one has to use $\mu \sim \Delta_H/C_T$ as the expansion parameter.

Two channels of the expansion are called “T-channel”, where two OPEs are written between $\mathcal{O}_L \times \mathcal{O}_L$ and $\mathcal{O}_H \times \mathcal{O}_H$, and “S-channel”, where we have two OPEs $\mathcal{O}_H \times \mathcal{O}_L$ and $\mathcal{O}_L \times \mathcal{O}_H$. The important set of primary operators that contribute to the T-channel is called “the stress tensor sector” and consists of the stress tensor (which is a single-trace operator) and multi-trace operators made of the stress tensor. Operators made of k stress tensors contribute at order μ^k to the correlator. As the stress tensor is a conserved current of the theory, its OPE coefficient with two external scalar operators is fixed by the Ward identity. On the other hand, the operators with $k \geqslant 2$ are not the conserved currents, therefore, their OPE coefficients are not fixed by the conformal symmetry. The gravitational analog of the stress tensor sector contributions to the correlator is the exchange of the single graviton in the Witten diagram and Witten diagrams with multi-graviton exchanges (the graviton loops).

In the kinematical limits we consider the stress tensor sector decouples from the rest of the operators contributing in the T-channel, which allows us to solve this sector completely, i.e. to write an algorithm for computing all of the OPE coefficients of operators in the stress tensor sector with external scalar operators. The contributions of these operators in the T-channel are reproduced by the double-trace operators made of one $\mathcal{O}_L$ and one $\mathcal{O}_H$ operator in the S-channel. These double-trace operators receive an anomalous dimension and correction to the zeroth order OPE coefficients due to the stress tensor sector contributions in T-channel. Solving the stress tensor sector of the holographic CFT accounts also for finding the analytic expressions for these anomalous dimensions and corrections to the MFT OPE.

2.4. Thermalization in holographic CFTs

Due to the presence of finite temperature states (the black holes and black branes with finite Hawking temperature) on the gravity side of AdS/CFT duality, at least in the classical gravity limit, studying the thermal properties of the dual CFTs represents the essential task to understand the full scope of the duality and to be able to describe these states via their CFT counterparts.

One should notice that the conformal symmetry is broken at the finite temperature, as the one-point functions of the local operators are not set to zero by symmetry anymore. This is because the temperature introduces the dimensionful parameter in the theory which allows the thermal one-point functions to be non-zero. All other assumptions on the spectrum of the CFT, including the factorization of the Hilbert space and the operator product expansion[4], are assumed to hold when the CFT is set at finite temperature.

We have already suggested that the pure heavy scalar primary state $\mathcal{O}_H$ is the CFT equivalent of the black hole in gravity. This might seem to be in contradiction with the usual AdS/CFT dictionary which says that black holes (or black branes) in gravity correspond to a (mixed) thermal state in CFT, and the Hawking temperature of the black hole is equal to the temperature of the thermal state. Additionally, there is an obvious mismatch between the finite Bekenstein-Hawking entropy of the black hole and the zero entropy of the pure state, which could seem to completely invalidate the assumption.

However, we argue that the operators in the stress tensor sector thermalize in the pure heavy states, such that their OPE coefficients with scalar heavy operators are equal to their thermal one-point functions at the temperature that is related to the conformal dimension of the heavy operator Δ_H. This justifies the identification of the black hole with a pure heavy state in the dual holographic CFT, as long as one is interested in studying the graviton contributions to the gravitational or the CFT correlators. This also explains why the OPE coefficients of stress tensor sector computed in the thermal black hole background in gravity [15] are equal to those computed in CFT by the conformal bootstrap [13,14]. Other operators that are generically present in the spectrum of the holographic CFTs do not thermalize in this sense, and their contribution would see a difference between the mixed thermal state and the pure heavy state.

We extend the analysis of thermalization from holographic CFTs to all large-C_T (or equivalently, large-N) theories, without assuming the large gap in the spectrum. We show that thermalization works even in the opposite limit, for free large-C_T theories whose gap goes to zero.

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[4] with modified convergence criteria

3. Introduction

The AdS/CFT correspondence provides a non-perturbative definition of quantum gravity in negatively curved spacetimes [8-10]. The correspondence in principle provides an opportunity to study generic properties of quantum gravity, possibly probing regimes unattainable by low-energy effective theories. Recent years have seen a development in conformal bootstrap techniques following [20-23], leading to many results for CFTs with holographic duals (see e.g. [11,32-55]). CFT methods have therefore become a powerful tool in the study of quantum gravity.

Crossing symmetry in CFTs imposes highly non-trivial constraints on the theory. The idea of conformal bootstrap is to use these constraints to put restrictions on the CFT data or, if possible, even solve the theory. One way to make use of the crossing symmetry is to isolate a small number of contributing operators in one channel, e.g. by going to a certain kinematical regime. This typically has to be reproduced by the exchange of an infinite number of operators in another channel. One such example is the lightcone limit where the separation between two operators in a four-point function is close to being null. One can then infer [56,57] the existence of double-trace operators at large spin in any CFT in dimensions $d > 2$. The Regge limit provides another opportunity to isolate the contribution of a class of operators, those of highest spin.

3.1. The conformal bootstrap in the Regge limit

In holographic CFTs the Regge limit of a four-point function, extensively studied in [58-63][5], is dominated by operators of spin two – the stress tensor and the double-trace operators (this is a consequence of the gap in the spectrum). In gravity, it reproduces a Witten diagram with graviton exchange (see e.g. [43]). The Regge limit corresponds to special kinematics, which on the gravity side is described by the scattering of highly energetic particles whose trajectories in the bulk are approximately null.

Such scattering can be described in the eikonal approximation where particles follow classical trajectories but their wavefunctions acquire a phase shift $\delta(S, L)$. The phase shift is a function of the total energy S and the impact

[5] See also [45,64-86] for other recent applications of Regge limit in CFTs.

parameter L. In the CFT language, this phase shift can be extracted from the Fourier transform of the four-point function. In [59] the phase shift extracted from the four-point function of the type $\langle \mathcal{O}_1\mathcal{O}_1\mathcal{O}_2\mathcal{O}_2\rangle$ was shown to be equal (up to a factor of $-\pi$) to the anomalous dimension of the double-trace operators $[\mathcal{O}_1\mathcal{O}_2]_{n,l} =: \mathcal{O}_1\partial^{2n}\partial_{\mu_1}\dots\partial_{\mu_l}\mathcal{O}_2 :$, at leading order in $1/N^2$. The Regge limit implies that the calculation is valid for $n, l \gg 1$. These anomalous dimensions have been subsequently verified in [49,87-95].

So far both operators $\mathcal{O}_1$ and $\mathcal{O}_2$ were assumed to have conformal dimensions of order one. In the following, we will refer to them as "light" operators and denote them by $\mathcal{O}_L$. In [55] one pair of operators (which we denote by $\mathcal{O}_H$) was taken to be "heavy", with conformal dimension Δ_H scaling as the central charge. The ratio $\mu \sim \Delta_H/C_T$ is a useful expansion parameter; its power k corresponds to the number of stress tensors in the multi-stress tensor operators exchanged in the T-channel ($\mathcal{O}_H\times\mathcal{O}_H \to (T_{\mu\nu})^k \to \mathcal{O}_L\times\mathcal{O}_L$)[6]. As explained in [55], one can define the phase shift as a Fourier transform of the $\langle\mathcal{O}_H\mathcal{O}_H\mathcal{O}_L\mathcal{O}_L\rangle$ four-point function. It is related to the time delay and angle deflection of a highly energetic particle traveling along a null geodesic in the background of an asymptotically AdS black hole. The black hole corresponds to the insertion of the heavy operator $\mathcal{O}_H$; its mass in the units of AdS radius is proportional to μ.

The phase shift $\delta(S, L)$ was computed in gravity in [55] as an infinite series expansion in μ, *i.e.*,

$$\delta(S, L) = \sum_{k=1}^{\infty} \delta^{(k)}\mu^k\,, \tag{3.1}$$

with terms subleading in $1/N^2$ suppressed. Conformal dimensions Δ and spins s of heavy-light double-trace operators $[\mathcal{O}_H\mathcal{O}_L]_{n,l}$ are given by

$$\Delta = \Delta_H + \Delta_L + 2n + l + \gamma(n, l), \qquad s = l, \tag{3.2}$$

where the anomalous dimensions $\gamma(n, l)$ admit a similar expansion in powers of μ

$$\gamma(n, l) = \sum_{k=1}^{\infty} \gamma^{(k)}_{n,l}\mu^k. \tag{3.3}$$

[6] Recently a similar limit was studied in [15].

In [55] it was also proven that

$$\gamma^{(1)}_{n,l} = -\frac{\delta^{(1)}}{\pi}, \tag{3.4}$$

where the following identifications are implied:

$$h = n + l, \qquad \bar{h} = n, \qquad S = 4h\bar{h}, \qquad e^{-2L} = \frac{\bar{h}}{h}. \tag{3.5}$$

However, it was observed that this relation does not hold for higher order terms, i.e. in general $\gamma^{(k)}_{n,l}$ is not proportional to $\delta^{(k)}$. One of the aims of Section 4 is to explain how higher order anomalous dimensions are related to higher order terms in the phase shift.

3.2. The minimal twist multi stress tensors

As reviewed before, the two-point function of the stress tensor in Conformal Field Theories is proportional to a single parameter, the central charge C_T. It generally serves as a measure of the number of degrees of freedom in the theory. In two spacetime dimensions this statement can be made precise: one can define a c-function which monotonically decreases along Renormalization Group flows and reduces to the central charge at conformal fixed points [19]. In four spacetime dimensions the situation is a bit more subtle and it is the a-coefficient in the conformal anomaly which necessarily satisfies $a_{IR} \leq a_{UV}$ [96]. Nevertheless, in any unitary conformal field theory a and C_T can only differ by a number of $\mathcal{O}(1)$ (see [97] for the original argument and [98-104] for more recent field theoretic proofs.) Hence, to consider the limit of infinite number of degrees of freedom one needs to take C_T to infinity.

In two spacetime dimensions conformal symmetry is described by the infinite-dimensional Virasoro algebra. This symmetry strongly constrains correlators, especially when combined with the $C_T \to \infty$ limit. Of particular interest is the heavy-heavy-light-light correlator, which involves two heavy operators with conformal dimension $\Delta_H \sim C_T$ and two light operators with conformal dimension $\Delta_L \sim \mathcal{O}(1)$. In this case the contribution of the identity operator and all its Virasoro descendants is known as the Virasoro vacuum block and has been calculated in several ways [40,105-110]. The Virasoro vacuum block (and finite C_T corrections to it) is instrumental in a variety of settings, such as e.g. the problem of information loss [111-116] and properties of the Renyi and entanglement entropies [117-120] (see also [121,122] for the original applications of large C_T correlators in this context).

The heavy-heavy-light-light Virasoro vacuum block exponentiates

$$\langle \mathcal{O}_H(\infty)\mathcal{O}_L(1)\mathcal{O}_L(z)\mathcal{O}_H(0)\rangle \sim e^{\Delta_L \mathcal{F}(\mu;z)}, \tag{3.6}$$

with $\mathcal{F}$ a known function which admits an expansion in powers of $\mu \sim \Delta_H/C_T$

$$\mathcal{F}(\mu;z) = \sum_k \mu^k \mathcal{F}^{(k)}(z). \tag{3.7}$$

One can consider contributions of various quasi-primaries made out of the stress tensor to $\mathcal{F}^{(k)}$. At $k=1$ the only such quasi-primary is the stress tensor itself, while for $k=2$ one needs to sum an infinite number of quasi-primaries quadratic in the stress tensor (double-stress operators) and labeled by spin. The situation is similar for all other values of k. It is possible to compute the OPE coefficients of the corresponding quasi-primaries, starting from the known result for the Virasoro vacuum block. Interestingly, at each order in μ, $\mathcal{F}^{(k)}$ can be written as a sum of particular terms [55][7]

$$\mathcal{F}^{(k)}(z) = \sum_{\{i_p\}} b_{i_1\dots i_k} f_{i_1}(z)\dots f_{i_k}(z), \qquad \sum_{p=1}^{k} i_p = 2k, \tag{3.8}$$

where $f_a(z) = (1-z)^a {}_2F_1(a,a,2a,1-z)$.

It is an interesting question whether a similar structure appears when the number of spacetime dimensions d is greater than two. Unlike in two spacetime dimensions, in addition to spin, multi-stress tensor operators are also labeled by their twist. An interesting subset of multi-stress tensor operators is comprised out of those with minimal twist. These operators dominate in the lightcone limit over those of higher twist. In [124] an expression for the OPE coefficients of two scalars and minimal-twist double-stress tensor operators in $d=4$ was obtained, and the sum was performed to obtain a remarkably simple expression for the near lightcone $\mathcal{O}(\mu^2)$ term in the heavy-heavy-light-light correlator. It was

[7] Similar expressions in a slightly different context appeared in [123].

shown to have a similar form to (3.8). One may now wonder if the minimal-twist multi-stress tensor part of the correlator in higher dimensions exponentiates

$$\langle \mathcal{O}_H(\infty)\mathcal{O}_L(1)\mathcal{O}_L(z,\bar{z})\mathcal{O}_H(0)\rangle\big|_{\text{multi-stress tensors}} \sim e^{\Delta_L \mathcal{F}(\mu;z,\bar{z})}, \tag{3.9}$$

and whether $\mathcal{F}(\mu;z,\bar{z})$ can be expressed as

$$\mathcal{F}(\mu;z,\bar{z}) = \sum_k \mu^k \mathcal{F}^{(k)}(z,\bar{z}), \tag{3.10}$$

with

$$\mathcal{F}^{(k)}(z,\bar{z}) = (1-\bar{z})^{k(\frac{d-2}{2})} \sum_{\{i_p\}} b_{i_1\ldots i_k} f_{i_1}(z)\ldots f_{i_k}(z), \qquad \sum_{p=1}^{k} i_p = k\left(\frac{d+2}{2}\right), \tag{3.11}$$

and d an even number.

In Section 5 we investigate this further. We start by assuming that the multi-stress tensor sector of the heavy-heavy-light-light correlator in the near lightcone regime $\bar{z} \to 1$ admits an expansion in μ

$$\langle \mathcal{O}_H(\infty)\mathcal{O}_L(1)\mathcal{O}_L(z,\bar{z})\mathcal{O}_H(0)\rangle\big|_{\text{multi-stress tensors}} \sim \sum_k \mu^k \mathcal{G}^{(k)}(z,\bar{z}), \tag{3.12}$$

where each coefficient function $\mathcal{G}^{(k)}(z,\bar{z})$ takes a particular form:

$$\mathcal{G}^{(k)}(z,\bar{z}) = \frac{(1-\bar{z})^{k(\frac{d-2}{2})}}{[(1-z)(1-\bar{z})]^{\Delta_L}} \sum_{\{i_p\}} a_{i_1\ldots i_k} f_{i_1}(z)\ldots f_{i_k}(z), \quad \sum_{p=1}^{k} i_p = k\left(\frac{d+2}{2}\right). \tag{3.13}$$

We subsequently use this ansatz to compute the contributions of the multi-stress tensor operators to the near lightcone correlator and extract the corresponding OPE coefficients.

For even d, the hypergeometric functions in (3.13) reduce to terms which contain at most one power of $\log(z)$ each. Their products contain multi-logs whose coefficients turn out to be rational functions of z. We use the conformal bootstrap approach initiated in [22] (for a review and references see eg. [24-26]) to relate these functions to the anomalous dimensions and OPE coefficients of the heavy-light double-trace operators in the cross channel. The ansatz (3.13) has just a few coefficients at any finite k which can be determined completely from the cross-channel data derived using the $(k-1)$th term. This is related to the fact that all the $\log^m(z)$ terms with $2 \leq m \leq k$ are completely determined by the anomalous dimensions and OPE coefficients at $\mathcal{O}(\mu^{k-1})$. At each step, we obtain an overconstrained system of equations solved by the same set of $a_{i_1 \ldots i_k}$. This provides strong support to the ansatz (3.11). We then proceed to derive the OPE coefficients of the multi-stress tensor operators with two light scalars from our result. In practice, we complete this program to $\mathcal{O}(\mu^3)$ in $d=4$ and to $\mathcal{O}(\mu^2)$ in $d=6$. However the procedure outlined can be easily generalised to arbitrary order in μ and any even d.

In [15] the authors considered holographic CFTs dual to gravitational theories defined by the Einstein-Hilbert Lagrangian plus higher derivative terms and a scalar field minimally coupled to gravity in AdS_{d+1}. Interpreting the scalar propagator in an asymptotically AdS_{d+1} black hole background as a heavy-heavy-light-light four-point function, enabled the authors of [15] to extract the OPE coefficients of a few multi-stress tensor operators from holography (see also [125-127] for related work). Ref. [15] also argued that the OPE coefficients of the leading, minimal-twist multi-stress operators are universal – they do not depend on the higher derivative terms in the Lagrangian. Their results agree with the general expressions obtained in this section, upon substitution of the relevant quantum numbers. We do not use holography in Section 5; our major assumption is (3.13).

3.3. The full stress tensor sector of conformal correlators

In Section 6 we study the contribution of the entire stress-tensor sector to the scalar CFT correlation functions, $\langle \mathcal{O}_H \mathcal{O}_L \mathcal{O}_L \mathcal{O}_H \rangle$. We go beyond the limitation of considering only the minimal twist multi stress tensor operators. We investigate the stress tensor sector further by considering contributions from multi-stress tensors with non-minimal twist. Our goal is to determine the structure of the correlator to subleading orders in the lightcone limit and extract the relevant OPE coefficients. Once more, we motivate an ansatz similar to the one successfully describing the leading lightcone behavior of $\mathcal{G}(z,\bar{z})$ and show that most of the parameters in the ansatz can be fixed using lightcone bootstrap. A few parameters are, however, left undetermined and might depend on the details of the theory. They correspond to the OPE coefficients of multi-stress tensors with spin $s = 0, 2$. Our approach can be employed to study the stress-tensor sector to arbitrary orders in μ and $(1 - \bar{z})$. We completed this program for the $\mathcal{O}(\mu^2)$ subleading, subsubleading and subsubsubleading terms as well as the $\mathcal{O}(\mu^3)$ subleading and subsubleading terms.

The OPE coefficients of the minimal twist multi stress tensors can be obtained by the Lorentzian inversion formula [73,78] as shown in [128][8]. We also investigate this complementary approach to computing the OPE data of the stress tensor sector. As noted earlier, the validity of the Lorentzian inversion formula for the HHLL correlator has not been rigorously established. It is however natural to expect that it is applicable in the large-C_T and small-μ expansion, as long as a Regge bound is observed. Here we assume that the Regge behavior of the correlator is given by σ^{-k} at $\mathcal{O}(\mu^k)$ in the large-C_T limit, which is consistent with the behaviour of the scattering phase shift from a black hole (or a massive star) computed classically in AdS. We then find that whenever the Lorentzian inversion formula is applicable, *i.e.*, for operators of spin $s > k+1$ at $\mathcal{O}(\mu^k)$, OPE data extracted with both methods are in perfect agreement. However, already at order $\mathcal{O}(\mu^3)$, our ansatz combined with the crossing symmetry or Lorentzian inversion formula is more powerful than the Lorentzian inversion formula alone. For instance, while the former procedure allows us to determine the OPE coefficient of a triple-stress tensor with spin $s = 4$ and twist $\tau = 8$, this is not possible using solely the Lorentzian inversion formula.

Finally, we explore the possibility of obtaining the unknown OPE data from the gravitational description of the CFT. We use the phase shift calculation in the dual gravitational theory. The scattering phase shift – acquired by a highly energetic particle travelling in the background of the AdS black hole – was first computed in the Regge limit in Einstein gravity in [55]. To explicitly see how the presence of higher derivative gravitational terms affects the OPE data, we work in Einstein-Hilbert + Gauss-Bonnet gravity with small Gauss-Bonnet coupling λ_{GB}. To combine the gravitational results with those of the CFT in the lightcone regime, we follow the approach first discussed in [124]

[8] One should exercise caution when using the Lorentzian inversion formula in the context of the HHLL correlator as the Regge behaviour of the correlator has not been rigorously established.

and further developed in [13], which involves an analytic continuation of the lightcone results around $z = 0$ and an expansion around $z = 1$. Matching terms in the correlator obtained from the gravitational calculation to those obtained from the CFT enables us to completely fix the stress tensor sector of the HHLL correlator up to the OPE coefficients of the spin-0 multi-stress tensors which are left undetermined. Non-universality is manifest by the presence of the Gauss-Bonnet coupling in the expressions for the OPE coefficients.

3.4. Review of heavy-heavy-light-light correlator in holographic CFTs

In this section, crossing relations for a heavy-heavy-light-light correlator of pair-wise identical scalars are reviewed. We consider large-N CFTs, with $C_T \sim N^2$, with a parametrically large gap Δ_{gap} in the spectrum of single trace operators with spin $J > 2$. The object that we study is a four-point correlation function between two light scalar operators $\mathcal{O}_L$, with scaling dimension of order one, and two heavy scalar operators $\mathcal{O}_H$, with scaling dimension Δ_H of $\mathcal{O}(C_T)$. Explicitly, we expand the CFT data in the parameter μ defined in [55] as

$$\mu = \frac{4\Gamma(d+2)}{(d-1)^2\Gamma(d/2)^2}\frac{\Delta_H}{C_T}, \tag{3.14}$$

which is kept fixed as $C_T \to \infty$.

The four-point function is fixed by conformal symmetry up to a function $\mathcal{A}(u,v)$ of the cross-ratios as

$$\langle \mathcal{O}_H(x_4)\mathcal{O}_L(x_3)\mathcal{O}_L(x_2)\mathcal{O}_H(x_1)\rangle = \frac{\mathcal{A}(u,v)}{x_{14}^{2\Delta_H}x_{23}^{2\Delta_L}}, \tag{3.15}$$

where u, v are cross-ratios[9]

$$\begin{aligned} v &= z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \\ u &= (1-z)(1-\bar{z}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \end{aligned} \tag{3.16}$$

and $x_{ij} = x_i - x_j$. Using conformal symmetry we can fix $x_1 = 0$, $x_3 = 1$ and $x_4 \to \infty$, with the last operator confined to a plane parameterized by $(z, \bar{z})$.

[9] Note that (u, v) are exchanged compared to the more common convention.

The main object of study is an appropriately rescaled version of (3.15)

$$G(z,\bar{z}) = \lim_{x_4\to\infty} x_4^{2\Delta_H} \langle \mathcal{O}_H(x_4)\mathcal{O}_L(1)\mathcal{O}_L(z,\bar{z})\mathcal{O}_H(0)\rangle. \tag{3.17}$$

This can be expanded in the S-channel $\mathcal{O}_L(z,\bar{z}) \to \mathcal{O}_H(0)$ as

$$G(z,\bar{z}) = (z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)} \sum_{\mathcal{O}'} \left(-\frac{1}{2}\right)^{J'} \lambda_{\mathcal{O}_H\mathcal{O}_L\mathcal{O}'}\lambda_{\mathcal{O}_L\mathcal{O}_H\mathcal{O}'} g_{\mathcal{O}'}^{\Delta_{HL},-\Delta_{HL}}(z,\bar{z}), \tag{3.18}$$

where $\Delta_{HL} = \Delta_H - \Delta_L$, λ_{ijk} are OPE coefficients and the sum runs over primaries $\mathcal{O}'$ with spin J' and corresponding conformal blocks $g'_{\mathcal{O}}$. The correlator can likewise be expanded in the T-channel $\mathcal{O}_L(z,\bar{z}) \to \mathcal{O}_L(1)$ as

$$G(z,\bar{z}) = \frac{1}{[(1-z)(1-\bar{z})]^{\Delta_L}} \sum_{\mathcal{O}} \left(-\frac{1}{2}\right)^{J} \lambda_{\mathcal{O}_H\mathcal{O}_H\mathcal{O}}\lambda_{\mathcal{O}_L\mathcal{O}_L\mathcal{O}} g_{\mathcal{O}}^{0,0}(1-z,1-\bar{z}), \tag{3.19}$$

where we again sum over primaries $\mathcal{O}$ with spin J. The equality of (3.18) and (3.19) constitutes an example of a crossing relation, in both channels we sum over an infinite set of conformal blocks $g_{\mathcal{O}}^{\Delta_1,\Delta_2}(z,\bar{z})$. These contain the contribution from a primary $\mathcal{O}$ and all its descendants. (For recent reviews on conformal bootstrap see [24-26].) Here we have distinguished between operators $\mathcal{O}'$ and $\mathcal{O}$, in the S- and T-channel, respectively, in order to stress that generically different operators are relevant in different channels. As an example of this, in the lightcone limit in $d > 2$ one finds [56,57] that the T-channel is dominated by the identity operator, while in the S-channel an infinite number of operators contribute. These are the so-called double-trace operators that exist at large spin in any $\mathrm{CFT}_{d>2}$.

We will assume the following scaling for a non-trivial single trace operator $\mathcal{O}$ (not including the stress tensor)

$$\langle \mathcal{O}_{H,L}\mathcal{O}_{H,L}\mathcal{O}\rangle \sim \frac{1}{\sqrt{C_T}}. \tag{3.20}$$

The conformal Ward identity fixes the following 3-pt function for the stress tensor

$$\langle \mathcal{O}_{H,L}\mathcal{O}_{H,L}T_{\mu\nu}\rangle \sim \Delta_{H,L}, \tag{3.21}$$

which implies the following scaling for the exchange of the stress tensor in the T-channel

$$\frac{\langle \mathcal{O}_H \mathcal{O}_H T_{\mu\nu}\rangle \langle T_{\mu\nu} \mathcal{O}_L \mathcal{O}_L\rangle}{\langle T_{\mu\nu} T_{\mu\nu}\rangle} \sim \frac{\Delta_H \Delta_L}{C_T} \sim \mu. \tag{3.22}$$

Keeping μ small, it follows that the leading contribution in the T-channel is given by the disconnected correlator $\langle \mathcal{O}_H \mathcal{O}_H\rangle\langle \mathcal{O}_L \mathcal{O}_L\rangle$, i.e. the exchange of the identity operator. Decomposing the disconnected correlator in the S-channel, we will infer the existence of the "double-trace operators" $[\mathcal{O}_H \mathcal{O}_L]_{n,l}$ for all integers n, l, with scaling dimension $\Delta = \Delta_H + \Delta_L + 2n + l + \gamma(n,l)$ and spin l. Moreover, the OPE coefficients scale as

$$\langle \mathcal{O}_H \mathcal{O}_L [\mathcal{O}_H \mathcal{O}_L]_{n,l}\rangle \sim 1. \tag{3.23}$$

Keeping $\mu \sim \Delta_H / C_T$ fixed as $C_T \to \infty$, (3.22) implies that the CFT data of double-trace operators $[\mathcal{O}_H \mathcal{O}_L]_{n,l}$ receives perturbative corrections in μ. We therefore expand the anomalous dimensions of these double-trace operators, as well as the OPE coefficients

$$P^{(HL,HL)}_{n,l} \equiv \left(-\frac{1}{2}\right)^l \lambda_{\mathcal{O}_H \mathcal{O}_L [\mathcal{O}_H \mathcal{O}_L]_{n,l}} \lambda_{\mathcal{O}_L \mathcal{O}_H [\mathcal{O}_H \mathcal{O}_L]_{n,l}}\,, \tag{3.24}$$

in μ as

$$\begin{aligned} &\gamma(n,l) = \mu \gamma^{(1)}_{n,l} + \mu^2 \gamma^{(2)}_{n,l} + \dots \\ &P^{(HL,HL)}_{n,l} = P^{(HL,HL);\text{MFT}}_{n,l}(1 + \mu P^{(HL,HL);(1)}_{n,l} + \mu^2 P^{(HL,HL);(2)}_{n,l} + \dots), \end{aligned} \tag{3.25}$$

with ... denoting higher order terms. The OPE coefficients $P^{(HL,HL);\text{MFT}}_{n,l}$ are fixed and can be found in [38]:

$$\begin{aligned} P^{(HL,HL);\text{MFT}}_{\bar{h},h-\bar{h}} =& \frac{(\Delta_H + 1 - d/2)_{\bar{h}} (\Delta_L + 1 - d/2)_{\bar{h}} (\Delta_H)_h (\Delta_L)_h}{\bar{h}!(h-\bar{h})!(\Delta_H + \Delta_L + \bar{h} + 1 - d)_{\bar{h}} (\Delta_H + \Delta_L + h + \bar{h} - 1)_{h-\bar{h}}} \\ &\times \frac{1}{(h - \bar{h} + d/2)_{\bar{h}} (\Delta_H + \Delta_L + h - d/2)_{\bar{h}}}, \end{aligned} \tag{3.26}$$

where $(a)_b$ is the Pochhammer symbol and the relation between (n, l) and $(h, \bar{h})$ variables is given by

$$h = n + l, \qquad \bar{h} = n. \tag{3.27}$$

In the limit $\Delta_H \gg \Delta_L, h, \bar{h}$, (3.26) simplifies

$$P^{(HL,HL);\text{MFT}}_{\bar{h},h-\bar{h}} \approx C_{\Delta_L} \frac{\Gamma(\Delta_L + \bar{h} - d/2 + 1)\Gamma(\Delta_L + h)}{\bar{h}!(h-\bar{h})!(h - \bar{h} + d/2)_{\bar{h}}}, \tag{3.28}$$

where $C_{\Delta_L} = (\Gamma(\Delta_L - d/2 + 1)\Gamma(\Delta_L))^{-1}$.

3.4.1. Regge limit of HHLL correlator

After the analytic continuation $z \to z e^{-2i\pi}$ we use (σ, ρ) coordinates defined by:

$$\begin{aligned} 1 - z &= \sigma e^{\rho} \\ 1 - \bar{z} &= \sigma e^{-\rho}. \end{aligned} \tag{3.29}$$

The Regge limit corresponds to $\sigma \to 0$ with ρ kept fixed. The easiest way to understand the kinematics of the Regge limit is to consider the four-point correlation function on Euclidean cylinder with coordinates $(\tau, \hat{n})$, as in [55]:

$$\langle \mathcal{O}_H(\tau_4, \hat{n}_4) \mathcal{O}_L(\tau_3, \hat{n}_3) \mathcal{O}_L(\tau_2, \hat{n}_2) \mathcal{O}_H(\tau_1, \hat{n}_1) \rangle, \tag{3.30}$$

where the relation between the operators on the cylinder and plane (with coordinates x) is given by

$$\mathcal{O}_\Delta(x) = e^{-\tau\Delta} \mathcal{O}_\Delta(\tau, \hat{n}), \qquad x^2 = e^{2\tau}. \tag{3.31}$$

The heavy operators are evaluated at $\tau_4 \to \infty$ and $\tau_1 \to -\infty$ and via the operator-state correspondence, they correspond to heavy states. The light operators are evaluated near reference points P_2 and P_3, that are null separated after Wick rotation to Lorentzian signature $\tau = -it$. Basically, their time separation is $t_{3P} - t_{2P} = \pi$, and they are placed on the opposite sides of S^{d-1} sphere $\hat{n}_{3P} = -\hat{n}_{2P}$. Using the translational symmetry along τ direction and rotational symmetry on S^{d-1} we set $\mathcal{O}_L(\tau_3, \hat{n}_3) = \mathcal{O}_L(P_3)$. Now, time component of the other light operator τ_2 is measured with respect to P_3, and it is given by

$$\tau_2 = \tau_3 + i x^0. \tag{3.32}$$

We use $(x^0, \hat{n}_2)$ as the coordinates of the other light operator that will be Fourier transformed to get the gravitational phase shift as in [55]. It is easy to check

$$\begin{aligned} z\bar{z} &= \frac{x_2^2}{x_3^2} = e^{2(\tau_2 - \tau_3)} = e^{2ix^0}, \\ (1-z)(1-\bar{z}) &= \frac{x_{23}^2}{x_3^2} = 1 + e^{2ix^0} - 2e^{ix^0} \cos\phi, \end{aligned} \tag{3.33}$$

where ϕ is the angle between vectors $\hat{n}_{3P} = \hat{n}_3$ and $\hat{n}_2$. These conditions are satisfied by the following relations:

$$z = e^{ix^+}, \qquad \bar{z} = e^{ix^-}, \qquad x^\pm = x^0 \pm \phi. \tag{3.34}$$

In the correlation function (3.30) we start from the configuration where light operators are inserted close to each other, $x^0 \approx \phi \approx 0$, and we shift one of them $\mathcal{O}_L(x^0, \hat{n}_2)$ near the reference point P_2. In this case, $x^0 \approx \phi \approx -\pi$. Therefore, we need to shift $x^+ \to x^+ - 2\pi$, or in terms of z variable $z \to z e^{-2\pi i}$. Correlation function on the cylinder (3.30) is related to the correlation function on the plane (3.17) by the simple conformal transformation.

After the analytic continuation, in $\sigma \to 0$ limit the dominant contribution to the correlator comes from high-spin operators, which follows from the fact that the conformal blocks that scale as σ^{-J+1}. These contributions can be calculated via the means of conformal Regge theory, as explained in [63].

The conformal blocks in the S-channel transform as (see e.g. [44, 75])

$$g_{\Delta,J}(z,\bar{z}) \to \mathrm{e}^{-i\pi(\Delta-J)} g_{\Delta,J}(z,\bar{z}), \tag{3.35}$$

after the analytic continuation. In particular, for double-trace operators $[\mathcal{O}_H\mathcal{O}_L]_{n,l}$ with scaling dimension $\Delta = \Delta_H + \Delta_L + 2n + l + \gamma(n,l)$, the blocks transform as

$$g^{\Delta_{HL},-\Delta_{HL}}_{[\mathcal{O}_H\mathcal{O}_L]_{n,l}}(z,\bar{z}) \to e^{-i\pi(\Delta_H+\Delta_L)} e^{-i\pi\gamma(n,l)} g^{\Delta_{HL},-\Delta_{HL}}_{[\mathcal{O}_H\mathcal{O}_L]_{n,l}}(z,\bar{z}). \tag{3.36}$$

In what follows it will be convienent to do a change of variables to $h = n + l$ and $\bar{h} = n$ and to denote the block due to a heavy-light double-trace operator $[\mathcal{O}_H\mathcal{O}_L]_{\bar{h},h-\bar{h}}$ as $g^{\Delta_{HL},-\Delta_{HL}}_{h,\bar{h}}$. Substituting the μ expansion (3.25) in the S-channel (3.18) and performing the usual analytic continuation to $\mathcal{O}(\mu)$ leads to

$$\begin{aligned} G(z,\bar{z})|_{\mu^0} &= (z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)} \sum_{h\geq\bar{h}\geq 0}^{\infty} P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}} g^{\Delta_{HL},-\Delta_{HL}}_{h,\bar{h}}(z,\bar{z}) \\ G(z,\bar{z})|_{\mu^1} &= (z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)} \sum_{h\geq\bar{h}\geq 0}^{\infty} P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}} \Bigg(P^{(HL,HL);(1)}_{\bar{h},h-\bar{h}} \\ &\qquad + \gamma^{(1)}_{\bar{h},h-\bar{h}} \left(\frac{1}{2}\left(\partial_h + \partial_{\bar{h}}\right) - i\pi \right) \Bigg) g^{\Delta_{HL},-\Delta_{HL}}_{h,\bar{h}}(z,\bar{z}). \end{aligned} \tag{3.37}$$

The new single trace operators that can possibly appear here would be subleading in $1/N^2$. Continuing to $\mathcal{O}(\mu^2)$, the imaginary part of the S-channel is given by

$$\mathrm{Im}(G(z,\bar{z}))|_{\mu^2} = -i\pi(z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)} \sum_{h\geq\bar{h}\geq 0}^{\infty} P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}} \times$$
$$\times \Big(\gamma^{(2)}_{\bar{h},h-\bar{h}} + \gamma^{(1)}_{\bar{h},h-\bar{h}} P^{(HL,HL);(1)}_{\bar{h},h-\bar{h}} + \frac{(\gamma^{(1)}_{\bar{h},h-\bar{h}})^2}{2}(\partial_h + \partial_{\bar{h}})\Big) g^{\Delta_{HL},-\Delta_{HL}}_{h,\bar{h}}(z,\bar{z}). \tag{3.38}$$

Moreover, the real part of the correlator at the same order is given by

$$\begin{aligned}\mathrm{Re}(G(z,\bar{z}))|_{\mu^2} = (z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)} \sum_{h\geq\bar{h}\geq 0}^{\infty} & P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}} \Big(P^{(HL,HL);(2)}_{\bar{h},h-\bar{h}} \\ & - \frac{1}{2}\pi^2(\gamma^{(1)}_{\bar{h},h-\bar{h}})^2 + \frac{1}{2}(\gamma^{(2)}_{h,\bar{h}} + P^{(HL,HL);(1)}_{\bar{h},h-\bar{h}}\gamma^{(1)}_{\bar{h},h-\bar{h}})(\partial_h + \partial_{\bar{h}}) \\ & + \frac{1}{8}(\gamma^{(1)}_{\bar{h},h-\bar{h}})^2(\partial_h + \partial_{\bar{h}})^2\Big) g^{\Delta_{HL},-\Delta_{HL}}_{h,\bar{h}}(z,\bar{z})\,.\end{aligned} \tag{3.39}$$

As we will see below, in the Regge limit the dominant contribution in the S-channel comes from double-trace operators with $h, \bar{h} \gg 1$. In this limit the OPE coefficients are given by

$$P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}} \approx C_{\Delta_L}(h\bar{h})^{\Delta_L-\frac{d}{2}}(h-\bar{h})^{\frac{d}{2}-1}. \tag{3.40}$$

We will further need $\lambda_{\mathcal{O}_L\mathcal{O}_L T}\lambda_{\mathcal{O}_H\mathcal{O}_H T}$ in (3.19), these are fixed by Ward Identities to be

$$P^{(HH,LL)}_{T_{\mu\nu}} = \left(-\frac{1}{2}\right)^2 \lambda_{\mathcal{O}_L\mathcal{O}_L T}\lambda_{\mathcal{O}_H\mathcal{O}_H T} = \frac{\Delta_L}{4}\left(\frac{d}{d-1}\right)^2 \frac{\Delta_H}{C_T} = \mu\frac{\Delta_L}{4}\frac{\Gamma\left(\frac{d}{2}+1\right)^2}{\Gamma\left(d+2\right)}. \tag{3.41}$$

Note that as explained in [55], an expansion in μ corresponds in the bulk to an expansion in the black hole Schwarzschild radius in AdS units. The bulk description of Regge limit follows almost intuitively from the CFT consideration. Namely, the heavy operators generate the black hole background where the light (but not massless) operators propagate on almost null geodesics. This implies that the energy of the light operators has to be large compared to the inverse impact parameter. On the more formal level, one can analyze the CFT Regge limit in the momentum space, for Mellin amplitude, where it is given by $s \to \infty$ and fixed t. This is the limit where we have high energy scattering with fixed momentum transfer which justifies the usage of the eikonal approximation in the bulk calculation.

3.4.2. Lightcone limit of the HHLL correlator

Below we review the setup of a heavy-heavy-light-light correlator with focus on its behaviour in the lightcone limit. Using conformal transformations we define the stress tensor sector of the correlator by

$$\mathcal{G}(z,\bar{z}) = \lim_{x_4\to\infty} x_4^{2\Delta_H} \langle \mathcal{O}_H(x_4)\mathcal{O}_L(1)\mathcal{O}_L(z,\bar{z})\mathcal{O}_H(0)\rangle\Big|_{\text{multi}-\text{stress tensors}}, \tag{3.42}$$

where z and $\bar{z}$ are the cross-ratios given by (3.16). In (3.42) the "multi-stress tensor" subscript stands to indicate the contribution of the identity and all multi-stress tensor operators.

The correlator $\mathcal{G}(z,\bar{z})$ can be expanded in the "T-channel" $\mathcal{O}_L(1) \times \mathcal{O}_L(z,\bar{z}) \to \mathcal{O}_{\tau,s}$ as[10]

$$\mathcal{G}(z,\bar{z}) = [(1-z)(1-\bar{z})]^{-\Delta_L} \sum_{\mathcal{O}_{\tau,s}} P^{(HH,LL)}_{\mathcal{O}_{\tau,s}} g^{(0,0)}_{\tau,s}(1-z,1-\bar{z}), \tag{3.43}$$

where $\tau = \Delta - s$ and s denote the twist and spin of the exchanged operator, respectively, and $g^{(0,0)}_{\tau,s}(z,\bar{z})$ the conformal block of the primary operator $\mathcal{O}_{\tau,s}$. Moreover, $P^{(HH,LL)}_{\mathcal{O}_{\tau,s}}$ are defined as

$$P^{(HH,LL)}_{\mathcal{O}_{\tau,s}} = \left(-\frac{1}{2}\right)^s \lambda_{\mathcal{O}_H\mathcal{O}_H\mathcal{O}_{\tau,s}} \lambda_{\mathcal{O}_L\mathcal{O}_L\mathcal{O}_{\tau,s}}, \tag{3.44}$$

where $\lambda_{\mathcal{O}_L\mathcal{O}_L\mathcal{O}}$ and $\lambda_{\mathcal{O}_H\mathcal{O}_H\mathcal{O}}$ denote the respective OPE coefficients.

We will mainly be interested in the lightcone limit defined by $u \ll 1$ or equivalently $\bar{z} \to 1$. In this limit the T-channel expansion (3.43) is dominated by minimal-twist operators as follows from the behaviour of the conformal blocks

$$\mathcal{G}(u,v) \underset{u\to 0}{\approx} u^{-\Delta_L} \sum_{\mathcal{O}_{\tau,s}} P^{(HH,LL)}_{\mathcal{O}_{\tau,s}} u^{\frac{\tau}{2}} (1-v)^{-\frac{\tau}{2}} f_{\frac{\tau}{2}+s}(v), \tag{3.45}$$

where twist τ is given just above and

$$f_{\frac{\tau}{2}+s}(v) = (1-v)^{\frac{\tau}{2}+s} {}_2F_1\Big(\frac{\tau}{2}+s, \frac{\tau}{2}+s, \tau+2s, 1-v\Big) \tag{3.46}$$

[10] For reasons of convenience, here and in the rest of the thesis we refer to $\mathcal{G}(z,\bar{z})$ as the correlator; the reader should keep in mind that $\mathcal{G}(z,\bar{z})$ is not the full correlator $G(z,\bar{z})$ but only its stress tensor sector, as defined in (3.42).

is a $SL(2;R)$ conformal block.

For any CFT in $d>2$ the leading contribution in the lightcone limit comes from the exchange of the identity operator with twist $\tau=0$. Another operator present in any unitary CFT is the stress tensor with twist $\tau=d-2$. Its contribution to the correlator is completely fixed by a Ward identity and given by (3.41).

As explained in [55], the correlator admits a natural perturbative expansion in μ,

$$\mathcal{G}(z,\bar{z})=\sum_k \mu^k \mathcal{G}^{(k)}(z,\bar{z})\,. \tag{3.47}$$

Using (3.45) and (3.41), we find the following contribution due to the stress tensor at $\mathcal{O}(\mu)$

$$\mathcal{G}^{(1)}(z,\bar{z})\underset{\bar{z}\to 1}{\approx}\frac{(1-\bar{z})^{\frac{d-2}{2}}(1-z)^{\frac{d+2}{2}}}{[(1-z)(1-\bar{z})]^{\Delta_L}}\frac{\Delta_L\Gamma(\frac{d}{2}+1)^2}{4\Gamma(d+2)}\,{}_2F_1\Big(\frac{d+2}{2},\frac{d+2}{2};d+2;1-z\Big). \tag{3.48}$$

Let us study the correlator perturbatively in μ in the lightcone limit. At k-th order in that expansion we expect contributions from minimal-twist multi-stress tensor operators of the schematic form

$$[T^k]_{\tau_{k,\min},s} =: T_{\mu_1\nu_1}\dots\partial_{\lambda_1}\dots\partial_{\lambda_\ell}T_{\mu_k\nu_k}:, \tag{3.49}$$

where the minimal-twist $\tau_{k,\min}$ and spin s of these operators are given by

$$\begin{aligned}\tau_{k,\min}&=k(d-2),\\ s&=2k+\ell\end{aligned} \tag{3.50}$$

and ℓ an even integer denoting the number of uncontracted derivatives. We moreover define the product of OPE coefficients for minimal-twist operators at order k as

$$P^{(HH,LL)}_{[T^k]_{\tau_{k,\min},s}}=\mu^k P^{(HH,LL);(k)}_{\tau_{k,\min},s}. \tag{3.51}$$

Compared to the $k=1$ case, there exists an infinite number of minimal-twist multi-stress tensor operators for each value of $k>1$. To obtain their contribution to the correlator in the lightcone limit, we thus have to sum over all these operators.

The correlator can likewise be expanded in the "S-channel" $\mathcal{O}_L(z,\bar{z}) \times \mathcal{O}_H(0) \to \mathcal{O}_{\tau',s'}$ as

$$\mathcal{G}(z,\bar{z}) = (z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)} \sum_{\mathcal{O}_{\tau',s'}} P^{(HL,HL)}_{\mathcal{O}_{\tau',s'}} g^{(\Delta_{HL},-\Delta_{HL})}_{\tau',s'}(z,\bar{z}). \tag{3.52}$$

where $P^{(HL,HL)}_{\mathcal{O}_{\tau',s'}}$ are the products of the corresponding OPE coefficients and $\Delta_{HL} = \Delta_H - \Delta_L$. Operators contributing in the S-channel are heavy-light double-trace operators [55,12][11] $[\mathcal{O}_H\mathcal{O}_L]_{n,l}$. In the $\Delta_H \to \infty$ limit the $d = 4$ blocks are given by

$$g^{(\Delta_{HL},-\Delta_{HL})}_{\Delta_H+\Delta_L+2n+\gamma,l}(z,\bar{z}) \approx \frac{(z\bar{z})^{\frac{1}{2}(\Delta_H+\Delta_L+2n+\gamma)}}{\bar{z}-z}\left(\bar{z}^{l+1} - z^{l+1}\right), \tag{3.53}$$

and similarly in $d = 6$

$$\begin{aligned} g^{(\Delta_{HL},-\Delta_{HL})}_{\Delta_H+\Delta_L+2n+\gamma,l}(z,\bar{z}) \approx & \frac{(z\bar{z})^{\frac{1}{2}(\Delta_H+\Delta_L+2n+\gamma(n,l))}}{(\bar{z}-z)^3} \\ & \times \left(\bar{z}^{l+3} - \frac{l+3}{l+1}\bar{z}^{l+2}z^1 - (z \leftrightarrow \bar{z})\right). \end{aligned} \tag{3.54}$$

The anomalous dimensions $\gamma(n,l)$ and the product of the OPE coefficients of the double-trace operators $P^{(HL,HL)}_{n,l}$ admit an expansion in μ given by (3.25). The zeroth order OPE coefficients in large Δ_H limit, given by (3.28), can be subsequently expanded in large l limit ($l \gg n \sim 1$). We obtain

$$P^{(HL,HL);\mathrm{MFT}}_{n,l} \approx \frac{l^{\Delta_L-1}(\Delta_L - \frac{d}{2}+1)_n}{n!\,\Gamma(\Delta_L)}. \tag{3.55}$$

To reproduce the correct singularities manifest in the T-channel one has to sum over infinitely many heavy-light double-trace operators with $l \gg 1$. For such operators the dependence of the OPE data on the spin l for $l \gg 1$ is[12]:

$$\begin{aligned} P^{(HL,HL);(k)}_{n,l} &= \frac{P^{(k)}_n}{l^{\frac{k(d-2)}{2}}}, \\ \gamma^{(k)}_{n,l} &= \frac{\gamma^{(k)}_n}{l^{\frac{k(d-2)}{2}}}. \end{aligned} \tag{3.56}$$

3.4.3. Higher-twist multi stress tensors contributions to HHLL correlator

Here, we review the contributions of the higher-twist multi stress tensor operators to the heavy-heavy-light-light correlator in four-dimensional spacetime ($d = 4$).

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[11] This the analogue of light-light double-trace operators that are present in the cross channel of $\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_2\mathcal{O}_1\rangle$, with $\mathcal{O}_1$ and $\mathcal{O}_2$ both light, in any CFT [56,57].

[12] This behavior in the large l limit is different from that of the OPE data of light-light double-trace operators [56,57].

3.4.3.1. T-channel expansion

Consider the T-channel expansion (3.43) in $d = 4$. Conformal blocks in $d = 4$ are given by [30,31]

$$g^{(0,0)}_{\tau,s}(1-z, 1-\bar{z}) = \frac{(1-z)(1-\bar{z})}{\bar{z}-z} \left(f_{\frac{\beta}{2}}(z) f_{\frac{\tau-2}{2}}(\bar{z}) - f_{\frac{\beta}{2}}(\bar{z}) f_{\frac{\tau-2}{2}}(z) \right), \quad (3.57)$$

with conformal spin, $\beta = \Delta + s$, and

$$f_a(z) = (1-z)^a {}_2F_1(a, a, 2a, 1-z). \quad (3.58)$$

In the lightcone limit, defined by $\bar{z} \to 1$ and z fixed, the leading contribution to the conformal blocks (3.57) comes from the first term in parenthesis in (3.57)

$$g^{(0,0)}_{\tau,s}(1-z, 1-\bar{z}) = (1-\bar{z})^{\frac{\tau}{2}} \left(f_{\frac{\beta}{2}}(z) + \mathcal{O}((1-\bar{z})) \right). \quad (3.59)$$

From (3.59) it is clear that the operators with the lowest twist in the T-channel dominate the correlator in the lightcone limit.

Note that the only single-trace primaries with twist equal to or lower than that of the stress tensor are scalars $\mathcal{O}$ with dimension $1 \leq \Delta_{\mathcal{O}} \leq 2$, or conserved currents with twist $\tau = 2$. In a theory without supersymmetry there is no *a priori* reason for the contributions of these operators, even if they exist, to be enhanced by a factor of Δ_H, so generically we expect them to be subleading in $C_T \to \infty$ limit.[13]

The stress tensor sector of the correlator (3.42) admits a perturbative expansion in μ given by (3.47), where the cases $k = 0$ and $k = 1$ correspond to the exchange of the identity and the stress tensor, respectively. For higher k we expect "multi-stress tensors" to contribute to $\mathcal{G}(z, \bar{z})$. Since we are interested in the four-point function of pairwise identical scalar operators, only multi-stress tensor operators with even spin give a nonvanishing contribution. At $\mathcal{O}(\mu^2)$, the contribution of these operators was explicitly calculated in [124]. Following that, it was shown in [13] how one can write the contributions of these operators

[13] Interestingly, in [129] it is conjectured that OPE coefficients $\lambda_{\phi\psi\psi}$ of operators ϕ with conformal dimension $\Delta_\phi \ll \Delta_{\text{gap}}$ and ψ with conformal dimension Δ_ψ, such that $\Delta_\phi \ll \Delta_\psi \ll C_T^{\#>0}$, scale as $\lambda_{\phi\psi\psi} \propto \frac{\Delta_\psi}{\sqrt{C_T}}$. Note however that here we are working in different regime, as $\Delta_H \propto \mathcal{O}(C_T)$.

at arbitrary order in the μ-expansion, in the lightcone limit $(1-\bar{z}) \ll 1$, using an appropriate ansatz and lightcone bootstrap. We briefly review this procedure here since the contribution from non-minimal-twist operators is obtained in a similar manner.

At $\mathcal{O}(\mu^k)$, there are infinitely many minimal-twist multi-stress tensors with twist $2k$ according to (3.50) which are distinguished by their conformal spin $\beta = \Delta + s$ given by $\beta = 6k + 4\ell$ with $\ell = 0, 1, 2, \ldots$. Inserting the leading behavior of the blocks (3.59) in (3.43) one finds

$$\mathcal{G}^{(k)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^k}{[(1-z)(1-\bar{z})]^{\Delta_L}} \sum_{\ell=0} P^{(k)}_{\Delta(\ell),s(\ell)} f_{\frac{\beta(\ell)}{2}}(z), \tag{3.60}$$

with

$$\mu^k P^{(k)}_{\Delta(\ell),s(\ell)} = P^{(HH,LL)}_{[T^k]_{\tau,s(\ell)}}, \tag{3.61}$$

where $\Delta(\ell) = \frac{\tau_{k,\min}+\beta}{2}$, $\tau_{k,\min} = 2k$, $s(\ell) = 2k + 2\ell$ and conformal spin $\beta = 6k + 4\ell$. Here $\underset{\bar{z}\to 1}{\approx}$ means that only the leading contribution as $\bar{z} \to 1$ is kept. It was shown in [13] that the infinite sum in (3.60) takes a particular form

$$\mathcal{G}^{(k)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^k}{[(1-z)(1-\bar{z})]^{\Delta_L}} \sum_{\{i_p\}} a_{i_1\ldots i_k} f_{i_1}(z)\ldots f_{i_k}(z), \qquad \sum_{p=1}^{k} i_p = 3k, \tag{3.62}$$

with i_p being integers and $a_{i_1\ldots i_k}$ are coefficients that can be determined via lightcone bootstrap. Furthermore, using an identity for the product of two f_a functions (Eq. (A.1) in [124]) one can express the $\mathcal{G}^{(k)}(z,\bar{z})$ in the form of (3.60) to read off the OPE coefficients for the exchange of minimal-twist multi-stress tensors of arbitrary conformal spin.

In this section, we want to consider multi-stress tensors with non-minimal twist. These operators are obtained by contracting indices in (3.49) either between the derivatives or between the operators. At $\mathcal{O}\left(\mu^k\right)$ there exist operators $[T^k]_{\tau_{k,m},s}$ with twist

$$\tau_{k,m} = \tau_{k,\min} + 2m, \tag{3.63}$$

for any non-negative integer m. For $m \neq 0$, these operators provide subleading contributions to the correlator in the lightcone limit. To consider these subleading contributions it is convenient to expand $\mathcal{G}^{(k)}(z,\bar{z})$ from (3.47) as

$$\mathcal{G}^{(k)}(z,\bar{z}) = \sum_{m=0}^{\infty} (1-\bar{z})^{-\Delta_L+k+m} \mathcal{G}^{(k,m)}(z), \tag{3.64}$$

where $\mathcal{G}^{(k,m)}(z)$ comes from operators of twists $\tau_{k,m}$ and less.

For illustration, let us consider the case $k=2$ with $m=1$. There exist two infinite families of operators with twist $\tau_{2,1}=6$ of the schematic form

$$\begin{aligned} \mathcal{O}_{6,2\ell_1+2} &\sim :T_{\mu\kappa}\partial_{\lambda_1}\dots\partial_{\lambda_{2\ell_1}}T^{\kappa}{}_{\nu}:, \\ \mathcal{O}'_{6,2\ell_2+4} &\sim :T_{\mu\nu}\partial_{\lambda_1}\dots\partial_{\lambda_{2\ell_2}}\partial^2 T_{\rho\sigma}:. \end{aligned} \tag{3.65}$$

These two families share the same twist and spin for $\ell_1=\ell_2+1$. Hence, they are indistinguishable for $\ell_1\geq 1$ at order $1/C_T$ in the large C_T expansion. A single operator stands out; it corresponds to $\ell_1=0$ and is of the schematic form $:T_{\mu\alpha}T^{\alpha}{}_{\nu}:$. Note that $:T_{\mu\alpha}T^{\alpha}{}_{\nu}:$ has minimal conformal spin $\beta=10$, among the ones in (3.65), since $\beta_{\ell_1}=\beta_{\ell_2+1}=10+4\ell_1$, for $\ell_1\geq 1$.

Let us now move on to the case $k=2$ and $m=2$. Here, there are three infinite families $\mathcal{O}_{8,s}$, $\mathcal{O}'_{8,s}$ and $\mathcal{O}''_{8,s}$ with conformal spin $8+4\ell_1$, $12+4\ell_2$ and $16+4\ell_3$, respectively. Schematically, these families can be represented as

$$\begin{aligned} \mathcal{O}_{8,2\ell_1} &\sim :T_{\alpha\beta}\partial_{\lambda_1}\dots\partial_{\lambda_{2\ell_1}}T^{\alpha\beta}:, \\ \mathcal{O}'_{8,2\ell_2+2} &\sim :T_{\mu\alpha}\partial_{\lambda_1}\dots\partial_{\lambda_{2\ell_2}}\partial^2 T^{\alpha}{}_{\nu}:, \\ \mathcal{O}''_{8,2\ell_3+4} &\sim :T_{\mu\nu}\partial_{\lambda_1}\dots\partial_{\lambda_{2\ell_3}}(\partial^2)^2 T_{\rho\sigma}:. \end{aligned} \tag{3.66}$$

Notice once more that the infinite families are indistinguishable for conformal spin $\beta\geq 16$. Here, operators with $\beta=8,12$ stand out. The operator with $\beta=8$ is of the schematic form $:T_{\alpha\beta}T^{\alpha\beta}:$. For $\beta=12$, there are two indistinguishable operators of the schematic form $:T_{\mu\alpha}\partial^2 T^{\alpha}{}_{\nu}:$ and $:T_{\alpha\beta}\partial_\mu\partial_\nu T^{\alpha\beta}:$.

The same holds for $m\geq 3$ (and $\tau\geq 10$) since there is no other independent way to contract stress tensor indices. The discussion above generalizes straightforwardly to $\mathcal{O}(\mu^k)$ with $k+1$ number of infinite families at high enough twist.

3.4.3.2. S-channel expansion

The anomalous dimensions and the product of OPE coefficients for heavy-light double-trace operators that contribute to the S-channel admit an expansion in powers of μ given by (3.25).

We begin by briefly reviewing the calculation in the lightcone expansion, i.e. due to the multi-stress tensors in the T-channel. Inserting the blocks (3.53) in the S-channel expansion (3.52) one finds that

$$\mathcal{G}(z,\bar{z}) = \sum_{n=0}^{\infty} \frac{(z\bar{z})^n}{\bar{z}-z} \int_0^{\infty} dl P_{n,l}^{(HL,HL)} (z\bar{z})^{\frac{1}{2}\gamma_{n,l}} (\bar{z}^{l+1} - z^{l+1}), \tag{3.67}$$

where the sum was approximated by an integral over l, with $(1-\bar{z})$ being the small parameter. Namely, the difference between sum and integral is $\mathcal{O}(1-\bar{z})$ and since we are always interested in the singular terms when $1-\bar{z} \ll 1$, this difference does not affect the calculation. Expanding the OPE data in (3.67) according to (3.25) and noting that

$$(z\bar{z})^{\frac{1}{2}\gamma_{n,l}} = \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{\gamma_{n,l} \log(z\bar{z})}{2} \right)^j, \tag{3.68}$$

it follows that terms proportional to $\log^i z$ at $\mathcal{O}(\mu^k)$, with $i = 2, 3, \ldots k$, in (3.67) are determined by OPE data at $\mathcal{O}(\mu^{k-1})$. These terms can therefore be matched with the T-channel in order to fix the coefficients in the ansatz.

In [13], the leading contribution of the OPE data of heavy-light double-trace operators as $l \to \infty$, together with the leading contribution of the conformal blocks as $\bar{z} \to 1$, was used to determine the minimal-twist contributions in the stress tensor sector of the T-channel. This section extends that analysis by considering subleading corrections in the lightcone expansion and therefore probing non-minimal-twist contributions in the T-channel. In particular, the S-channel OPE data have the following dependence on the spin l as $l \to \infty$:

$$\begin{aligned} \gamma_{n,l}^{(k)} &= \frac{1}{l^k} \sum_{p=0}^{\infty} \frac{\gamma_n^{(k,p)}}{l^p}, \\ P_{n,l}^{(HL,HL);(k)} &= \frac{1}{l^k} \sum_{p=0}^{\infty} \frac{P_n^{(HL,HL);(k,p)}}{l^p}, \end{aligned} \tag{3.69}$$

which is necessary in order to reproduce the correct power of $(1-\bar{z})$ as $\bar{z} \to 1$. This can be seen by substituting the expansion of (3.28) in the large-l limit

$$P_{n,l}^{(HL,LH);MFT} = l^{\Delta_L} \left(\frac{(\Delta_L - 1)_n}{n!\Gamma(\Delta_L) l} + \frac{(2n(\Delta_L-2) + \Delta_L(\Delta_L-1))(\Delta_L-1)_n}{2(n!)\Gamma(\Delta_L) l^2} + \mathcal{O}\left(\frac{1}{l^3}\right) \right), \tag{3.70}$$

and (3.69) in (3.67) which result in integrals of the form

$$\int_0^\infty dl \bar{z}^l l^{\Delta_L - m - 1} = \frac{\Gamma(\Delta_L - m)}{(-\log \bar{z})^{\Delta_L - m}}, \tag{3.71}$$

where m is a positive integer. Expanding (3.71) for $\bar{z} \to 1$, the correct $\bar{z}$-behavior of the stress tensor sector in the T-channel is reproduced from the S-channel.

3.5. The rest of the thesis

Our first goal is to get insight into the heavy-heavy-light-light correlator at $\mathcal{O}(\mu^2)$, where there are the contributions of double-trace operators, made of two stress tensors, whose OPE coefficients, in general, are unknown, although some of these OPE coefficients are found in [15]. In Section 4 we study the heavy-heavy-light-light correlator in the Regge limit. We show how to compute the anomalous dimensions and the corrections to the MFT OPE coefficients of the double-trace operators in the S-channel at $\mathcal{O}(\mu^2)$ using the gravitational calculation of the wave function phase shift of the light probe in the black hole background.

In Section 5 we establish the complete algorithm for computing the contributions of the minimal-twist[14] subset of the stress tensor sector at arbitrary order μ^k in even-dimensional spacetime. The algorithm relies on the appropriate functional ansatz for these contributions and the lightcone conformal bootstrap that fixes the coefficients in the ansatz. When these coefficients in the ansatz are fixed, one can read off the OPE coefficients of arbitrary multi stress tensor operator with a minimal twist at the given order μ^k. We show that these operators have universal OPE coefficients in agreement with the statement in [15], which means that they are the same in the whole class of holographic CFTs.

In Section 6 we extend the algorithm to include the contributions of multi stress tensor operators with an arbitrary twist at each order in μ. We show that the only OPE coefficients that can not be determined by the conformal bootstrap technique are those of operators of spin $s = 0, 2$. These OPE coefficients can be thought of as the parameters of the theory, that are not fixed by the consistency condition and they parametrize the class of holographic CFTs.

[14] Twist τ of the operator is defined as the difference between the conformal dimension and spin, $\tau = \Delta - s$.

We show how the OPE coefficients of operators of spin two can be extracted directly from the gravitational computation of the wave function phase shift of the light probe in the black hole background.

Finally, in Section 7 we study the thermalization properties of the stress tensor sector in general large-N theories, without the large gap assumptions. We show that the stress tensor sector of the theory thermalizes in the pure heavy scalar state $\mathcal{O}_H$ with the large conformal dimension $\Delta_H \sim \mathcal{O}(C_T)$. The thermalization is manifested by the equality of the OPE coefficients of the multi stress tensor operators with two heavy operators and the thermal expectation value of stress multi stress tensor. Section 8 sums up the conclusions.

4. Black holes and conformal Regge bootstrap

4.1. Summary of the results

In this section we explain how to compute the anomalous dimensions of heavy-light double-trace operators $[\mathcal{O}_H\mathcal{O}_L]_{n,l}$ order by order in μ, using the phase shift result of [55]. In particular, we show that the $\mathcal{O}(\mu^2)$ anomalous dimensions in any d are given by

$$\gamma^{(2)}_{\bar{h},h-\bar{h}} = -\frac{\delta^{(2)}}{\pi} + \frac{\gamma^{(1)}_{\bar{h},h-\bar{h}}}{2}(\partial_h + \partial_{\bar{h}})\gamma^{(1)}_{\bar{h},h-\bar{h}}, \qquad \Delta_H \gg h, \bar{h} \gg 1. \tag{4.1}$$

Using known results for $\delta^{(1)}$ and $\delta^{(2)}$ from [55], we find an explicit expression for $\gamma^{(2)}_{\bar{h},h-\bar{h}}$ and compare it with the known results in the lightcone limit ($\Delta_H \gg h \gg \bar{h} \gg 1$). We find perfect agreement.

The rest of this section is organized as follows. In Section 4.2 we focus on four-dimensional holographic CFTs. At $\mathcal{O}(\mu)$, we use the crossing equation between the S- and T-channel to solve for the anomalous dimensions of heavy-light double-trace operators $[\mathcal{O}_H\mathcal{O}_L]_{n,l}$. The result is eq. (3.4), valid for $l, n \gg 1$. We then introduce the impact parameter representation which allows us to rewrite the S-channel expansion as a Fourier transform. We use this to relate the phase shift to the anomalous dimensions of $[\mathcal{O}_H\mathcal{O}_L]_{n,l}$ at $\mathcal{O}(\mu^2)$, thereby deriving (4.1). Using a known result for the phase shift $\delta^{(2)}$, we write down an explicit expression for $\gamma^{(2)}_{n,l}$. In the subsequent $l \gg n$ limit it reduces to the result which has been obtained in [55] in a completely different way (by computing corrections to the energies of excited states in the AdS-Schwarzschild background).

In Section 4.3 we generalize these results to any d ($d = 2$ is treated separately in Appendix A.4.). By solving the Casimir equation in the limit $\Delta_H \gg \Delta_L, l, n$, we obtain the conformal blocks for heavy-light double-trace operators in the S-channel. Using the explicit expression for the blocks together with the zeroth order OPE coefficients, we derive an impact parameter representation valid in general dimensions. Just as in the $d = 4$ case, this allows us to write the S-channel sum as a Fourier transform. Hence, we show that (4.1) holds for any d. We compute $\gamma^{(2)}_{n,l}$ in the lightcone limit and find perfect agreement with the results quoted in [55]. In addition, we find an expression for the $\mathcal{O}(\mu^2)$ corrections to the OPE coefficients.

Section 4.4 discusses various observations and mentions some open problems. Appendices contain additional technical details. The conformal bootstrap calculations are summarized in Appendix A.1, the proof of the impact parameter representation in $d = 4$ in Appendix A.2 and the proof in general dimension d in Appendix A.3. The special case of $d = 2$ is treated in Appendix A.4. Appendix A.5 discusses the fate of some boundary terms. Appendices A.6 and A.7 contain some identities which are used in Section 4.4.

4.2. Anomalous dimensions of heavy-light double-trace operators in $d = 4$

In this section we investigate the anomalous dimensions of heavy-light double-trace operators $[\mathcal{O}_H\mathcal{O}_L]_{\bar{h},h-\bar{h}}$ in $d = 4$ using conformal bootstrap. Moreover, using a four-dimensional impact parameter representation we relate the anomalous dimensions to the bulk phase shift to $\mathcal{O}(\mu^2)$. This procedure can be repeated order by order in μ to obtain the OPE data (anomalous dimensions and OPE coefficients – see also Section 4.3) to the desired order.

4.2.1. Anomalous dimensions in the Regge limit using bootstrap

The conformal blocks in $d = 4$ are given by [30]

$$g^{\Delta_{12},\Delta_{34}}_{\Delta,J}(z,\bar{z}) = \frac{z\bar{z}}{z-\bar{z}}\left(k_{\Delta+J}(z)k_{\Delta-J-2}(\bar{z}) - (z \leftrightarrow \bar{z})\right) \tag{4.2}$$

where

$$k_\beta(z) = z^{\beta/2}{}_2F_1\left(\frac{\beta-\Delta_{12}}{2}, \frac{\beta+\Delta_{34}}{2}, \beta, z\right). \tag{4.3}$$

In the limit $\Delta_H \sim C_T \gg 1$ the hypergeometric functions in (4.2) can be substituted by the identity up to $1/\Delta_H$ corrections. Explicitly, the conformal blocks of $[\mathcal{O}_H\mathcal{O}_L]_{\bar{h},h-\bar{h}}$ in the heavy limit are given by

$$g^{\Delta_{HL},-\Delta_{HL}}_{h,\bar{h}}(z,\bar{z}) = \frac{(z\bar{z})^{\frac{1}{2}(\Delta_H+\Delta_L)}\left(z^{h+1}\bar{z}^{\bar{h}} - z^{\bar{h}}\bar{z}^{h+1}\right)}{z-\bar{z}}. \tag{4.4}$$

Inserting this form of the conformal blocks in (3.37) together with the OPE coefficients in the Regge limit (3.40), we approximate the sums by integrals and find the following expression at $\mathcal{O}(\mu^0)$ in the S-channel

$$G(z,\bar{z})|_{\mu^0} = \frac{C_{\Delta_L}}{z-\bar{z}}\int_0^\infty dh\int_0^h d\bar{h}(h\bar{h})^{\Delta_L-2}(h-\bar{h})\left(z^{h+1}\bar{z}^{\bar{h}} - z^{\bar{h}}\bar{z}^{h+1}\right). \tag{4.5}$$

The integrals are explicitly computed in Appendix A.1; the result is the disconnected correlator in the T-channel $[(1-z)(1-\bar{z})]^{-\Delta_L}$ in the Regge limit $\sigma \to 0$.

At $\mathcal{O}(\mu)$ in holographic CFTs the leading corrections in the T-channel come from the exchanges of the stress tensor and double-trace operators $[\mathcal{O}_L\mathcal{O}_L]_{n,l=2}$ ($[\mathcal{O}_H\mathcal{O}_H]_{n,l=2}$ are heavy and therefore decouple). The conformal block for the T-channel exchange of the stress tensor is found after $z \to e^{-2\pi i}z$ to be given by

$$g_{T_{\mu\nu}} = \frac{360 i\pi e^{-\rho}}{\sigma(e^{2\rho}-1)} + \dots, \tag{4.6}$$

where ... denotes non-singular terms. The contribution from the stress tensor exchange in the T-channel is thus imaginary for real values of σ and ρ. The only imaginary term at order μ in the S-channel expansion (3.37) comes from the term proportional to $-i\pi\gamma$; it must reproduce (4.6).

In the Regge limit, we approximate the sum in the S-channel by an integral and insert the OPE coefficients from (3.40); the imaginary part at $\mathcal{O}(\mu)$ in the S-channel is thus given by

$$\begin{aligned}\operatorname{Im}(G(z,\bar{z}))|_{\mu^1} = \frac{-i\pi C_{\Delta_L}}{z-\bar{z}} \int_0^\infty dh \int_0^h d\bar{h} (h\bar{h})^{\Delta_L-2}(h-\bar{h})\gamma^{(1)}_{\bar{h},h-\bar{h}} \\ \left(z^{h+1}\bar{z}^{\bar{h}} - z^{\bar{h}}\bar{z}^{h+1}\right).\end{aligned} \tag{4.7}$$

With the ansatz $\gamma^{(1)}_{\bar{h},h-\bar{h}} = c_1 h^a \bar{h}^b/(h-\bar{h})$ the integrals in (4.7) can be computed (for more details see Appendix A.1). In order to reproduce the exchange of the stress tensor, the anomalous dimensions at $\mathcal{O}(\mu)$ must be equal to

$$\begin{aligned}\gamma^{(1)}_{\bar{h},h-\bar{h}} &= -\frac{90\lambda_{\mathcal{O}_H\mathcal{O}_H T_{\mu\nu}}\lambda_{\mathcal{O}_L\mathcal{O}_L T_{\mu\nu}}}{\mu\Delta_L}\frac{\bar{h}^2}{h-\bar{h}} \\ &= -\frac{3\bar{h}^2}{h-\bar{h}},\end{aligned} \tag{4.8}$$

where in the second line we inserted the OPE coefficients from (3.41). With the form (4.8) not only the stress tensor exchange is reproduced, but also an infinite sum of spin-2 double-trace operators $[\mathcal{O}_L\mathcal{O}_L]_{n,l=2}$ with scaling dimension $\Delta_n = 2\Delta_L + 2 + 2n$. This is similar to what happens in the light-light case [75].

To determine the second order corrections to the anomalous dimensions we use the derivative relationship:

$$P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}} P^{(HL,HL);(1)}_{\bar{h},h-\bar{h}} = \frac{1}{2}(\partial_h + \partial_{\bar{h}}) \left(P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}} \gamma^{(1)}_{\bar{h},h-\bar{h}} \right). \tag{4.9}$$

We will prove below (see Section 4.3.3) that this relationship is true in the limit $h, \bar{h} \gg 1$. The imaginary part at $\mathcal{O}(\mu^2)$ in the S-channel from (3.37) is then given by

$$\begin{aligned} \mathrm{Im}(G(z,\bar{z}))|_{\mu^2} = -i\pi \int_0^\infty dh \int_0^h d\bar{h} P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}} \Bigg(& \gamma^{(2)}_{\bar{h},h-\bar{h}} \\ & + \gamma^{(1)}_{\bar{h},h-\bar{h}} P^{(HL,HL);(1)}_{\bar{h},h-\bar{h}} + \frac{(\gamma^{(1)}_{\bar{h},h-\bar{h}})^2}{2}(\partial_h + \partial_{\bar{h}}) \Bigg) g_{h,\bar{h}}. \end{aligned} \tag{4.10}$$

With the help of (4.9), one can write (4.10) as

$$\begin{aligned} \mathrm{Im}(G(z,\bar{z}))|_{\mu^2} =& - i\pi \int_0^\infty dh \int_0^h d\bar{h} P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}} g_{h,\bar{h}} \\ & \times \left(\gamma^{(2)}_{\bar{h},h-\bar{h}} - \frac{\gamma^{(1)}_{\bar{h},h-\bar{h}}}{2}(\partial_h + \partial_{\bar{h}}) \gamma^{(1)}_{\bar{h},h-\bar{h}} \right) + \text{total derivative}, \end{aligned} \tag{4.11}$$

where the total derivate term does not contribute (see Appendix A.5 for details). In order to fix $\gamma^{(2)}_{\bar{h},h-\bar{h}}$ completely from crossing symmetry, we would need to consider the exchange of infinitely many double-trace operators made out of the stress tensor in the T-channel. Instead, we will use an impact parameter representation to relate $\gamma^{(2)}_{\bar{h},h-\bar{h}}$ to the bulk phase shift calculated from the gravity dual in [55].

4.2.2. 4d impact parameter representation and relation to bulk phase shift

In [59] the anomalous dimensions of light-light double-trace operators in the limit $h, \bar{h} \gg 1$ were shown to be related to the bulk phase shift. An impact parameter representation for the case when one of the operators is heavy was introduced in [55], where it was also shown that the bulk phase shift and the anomalous dimensions are equal at $\mathcal{O}(\mu)$. The goal of this section is to see explicitly how the bulk phase shift and the anomalous dimensions are related to $\mathcal{O}(\mu^2)$.

The correlator (3.17) can be written in an impact parameter representation as

$$G(z,\bar{z}) = \int_0^\infty dh \int_0^h d\bar{h}\, \mathcal{I}_{h,\bar{h}} f(h,\bar{h}), \tag{4.12}$$

with $\mathcal{I}_{h,\bar{h}}$ given by

$$\mathcal{I}_{h,\bar{h}} = (z\bar{z})^{-\frac{(\Delta_H+\Delta_L)}{2}} P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}} g^{\Delta_{HL},-\Delta_{HL}}_{h,\bar{h}}(z,\bar{z}) \tag{4.13}$$

and $f(h,\bar{h})$ some function that generically depends on the anomalous dimension and corrections to the OPE coefficients. In particular, for $f(h,\bar{h})=1$, (4.12) is equal to the disconnected correlator. In Appendix A.2 it is shown that $\mathcal{I}_{h,\bar{h}}$ can be equivalently written as

$$\mathcal{I}_{h,\bar{h}} \equiv C(\Delta_L) \int_{M^+} \frac{d^4p}{(2\pi)^4} (-p^2)^{\Delta_L-2} e^{-ipx} (h-\bar{h}) \delta(p\cdot\bar{e}+h+\bar{h})\, \delta\left(\frac{p^2}{4}+h\bar{h}\right) \tag{4.14}$$

where M^+ is the upper Milne wedge with $\{p^2 \leq 0,\ p^0 \geq 0\}$, $C(\Delta_L)$ given by (with $d=4$)

$$C(\Delta) \equiv \frac{2^{d+1-2\Delta}\pi^{1+\frac{d}{2}}}{\Gamma(\Delta)\Gamma(\Delta-\frac{d}{2}+1)} \tag{4.15}$$

and $\bar{e}=(1,0,0,0)$. Moreover, following [55], we will set $z=e^{ix^+}$ and $\bar{z}=e^{ix^-}$, with $x^+=t+r$ and $x^-=t-r$ in spherical coordinates.

Using the identity

$$\delta(p\cdot\bar{e}+h+\bar{h})\, \delta\left(\frac{p^2}{4}+h\bar{h}\right) = \frac{1}{|h-\bar{h}|}\left(\delta\left(\frac{p^+}{2}-h\right)\delta\left(\frac{p^-}{2}-\bar{h}\right)+(h\leftrightarrow\bar{h})\right), \tag{4.16}$$

with $p^+=p^t+p^r$, $p^-=p^t-p^r$, the integrals over $h,\bar{h}$ in (4.12) are easily computed. With the identification $h=\frac{p^+}{2}$ and $\bar{h}=\frac{p^-}{2}$ it follows that a generic term like (4.12) can be written as a Fourier transform

$$\int_0^\infty dh \int_0^h d\bar{h}\, \mathcal{I}_{h,\bar{h}} f(h,\bar{h}) = C(\Delta_L)\int_{M^+} \frac{d^4p}{(2\pi)^4}(-p^2)^{\Delta_L-2} e^{-ipx} f\left(\frac{p^+}{2},\frac{p^-}{2}\right). \tag{4.17}$$

We thus see that the impact parameter representation allows us to rewrite the S-channel expression as a Fourier transform.

The phase shift $\delta(p)$ for a pair of operators $\mathcal{O}_H$ and $\mathcal{O}_L$, with scaling dimensions $\Delta_H/C_T \propto \mu$ and $\Delta_L/C_T \ll 1$, respectively, was defined in [55] by

$$\mathcal{B}(p) \equiv \int d^4x e^{ipx} G(x) = \mathcal{B}_0(p) e^{i\delta(p)}, \tag{4.18}$$

where $G(x)$ is given in (3.17) and $\mathcal{B}_0(p)$ denotes the Fourier transform of the disconnected correlator. As the OPE data, the phase shift admits an expansion in μ:

$$\delta(p) = \mu\delta^{(1)}(p) + \mu^2\delta^{(2)}(p) + \dots, \tag{4.19}$$

where $\dots$ denotes higher order terms in the expansion. Expanding the exponential in (4.18) in μ we get

$$\mathcal{B}(p) = \mathcal{B}_0(p)\Big(1 + i\mu\delta^{(1)} + \mu^2(-\frac{(\delta^{(1)})^2}{2} + i\delta^{(2)}) + \dots\Big). \tag{4.20}$$

With (4.20) the relationship between the anomalous dimensions and the bulk phase shift to $\mathcal{O}(\mu^2)$ can be established using (3.37), (3.38) and (4.17):

$$\begin{aligned} \gamma^{(1)}_{\bar{h},h-\bar{h}} &= -\frac{\delta^{(1)}}{\pi} \\ \gamma^{(2)}_{\bar{h},h-\bar{h}} &= -\frac{\delta^{(2)}}{\pi} + \frac{\gamma^{(1)}_{\bar{h},h-\bar{h}}}{2}(\partial_h + \partial_{\bar{h}})\gamma^{(1)}_{\bar{h},h-\bar{h}}. \end{aligned} \tag{4.21}$$

The phase shift was calculated in closed form to all orders in μ for the four-dimensional case [55], with the first and second order terms given by

$$\begin{aligned} \delta^{(1)} &= \frac{3\pi}{2}\sqrt{-p^2}\frac{e^{-L}}{e^{2L}-1} \\ \delta^{(2)} &= \frac{35\pi}{8}\sqrt{-p^2}\frac{2e^{L}-e^{-L}}{(e^{2L}-1)^3}, \end{aligned} \tag{4.22}$$

where

$$-p^2 = p^+p^-, \qquad \cosh L = \frac{p^+ + p^-}{2\sqrt{-p^2}}. \tag{4.23}$$

Using (4.22) and (4.23), the $\mathcal{O}(\mu)$ corrections to the anomalous dimensions are given by $\gamma^{(1)}_{n,l} = -3n^2/l$, which agrees with (4.8). From (4.22) and (4.21), we deduce the anomalous dimensions at $\mathcal{O}(\mu^2)$:

$$\gamma^{(2)}_{n,l} = -\frac{35}{4}\frac{(2l+n)n^3}{l^3} + 9\frac{n^3}{l^2}. \tag{4.24}$$

Taking the lightcone limit ($l \gg n \gg 1$) in (4.24) we find

$$\gamma^{(2)}_{n,l} \approx -\frac{17}{2}\frac{n^3}{l^2}. \tag{4.25}$$

The anomalous dimensions in the lightcone limit (4.25) agree with eq. (6.40) in [55], which was obtained independently by considering corrections to the energy levels in the AdS-Schwarzschild background.

4.3. OPE data of heavy-light double-trace operators in generic d

In this section we will write the general form of conformal blocks for heavy-light double-trace operators in the limit $\Delta_H \sim C_T \gg 1$ and general $d > 2$. These blocks will be used to confirm the validity of the impact parameter representation in Appendix A.3. Using the impact parameter representation the OPE data will be related to the bulk phase shift. In particular, we show that (4.21) remains valid in any number of dimensions and find explicit expressions for the corrections to the OPE coefficients up to $\mathcal{O}(\mu^2)$.

4.3.1. Conformal blocks in the heavy limit

In order to find conformal blocks in general spacetime dimension d in the limit $\Delta_H \gg \Delta_L, h, \bar{h}$, we write them in the following form:

$$g_{h,\bar{h}}^{\Delta_{HL},-\Delta_{HL}}(z,\bar{z}) = (z\bar{z})^{\frac{\Delta_H+\Delta_L}{2}} F(z,\bar{z}), \tag{4.26}$$

where the function $F(z,\bar{z})$ does not depend on Δ_H and is symmetric with respect to the exchange $z \leftrightarrow \bar{z}$. Let us now insert the expression (4.26) into the Casimir equation and consider the leading $\mathcal{O}(\Delta_H)$ term:

$$z\frac{\partial}{\partial z}F(z,\bar{z}) + \bar{z}\frac{\partial}{\partial \bar{z}}F(z,\bar{z}) - (h+\bar{h})F(z,\bar{z}) = 0. \tag{4.27}$$

The most general solution to eq. (4.27) is:

$$F(z,\bar{z}) = z^{h+\bar{h}} f\Big(\frac{\bar{z}}{z}\Big), \tag{4.28}$$

where f is an arbitrary function that satisfies $f(\frac{1}{x}) = x^{-h-\bar{h}} f(x)$, since conformal blocks must be symmetric with respect to the exchange $z \leftrightarrow \bar{z}$.

The behavior of the conformal blocks as $z, \bar{z} \to 0$ and $z/\bar{z}$ fixed is given by [30,130]

$$g_{\Delta,l}^{\Delta_{12},\Delta_{34}}(z,\bar{z}) \to \frac{l!}{(\frac{d}{2}-1)_l}(z\bar{z})^{\frac{\Delta}{2}} C_l^{(\frac{d}{2}-1)}\Big(\frac{z+\bar{z}}{2\sqrt{z\bar{z}}}\Big), \tag{4.29}$$

where $\Delta = \Delta_1 + \Delta_2 + 2n + l$ and $C_q^{(p)}(x)$ are the Gegenbauer polynomials. Using (4.29), we can completely determine the function f:

$$f\Big(\frac{\bar{z}}{z}\Big) = \frac{(h-\bar{h})!}{(\frac{d}{2}-1)_{h-\bar{h}}}\Big(\frac{\bar{z}}{z}\Big)^{\frac{h+\bar{h}}{2}} C_{h-\bar{h}}^{(\frac{d}{2}-1)}\Big(\frac{z+\bar{z}}{2\sqrt{z\bar{z}}}\Big). \tag{4.30}$$

That is, the conformal blocks in the limit of large Δ_H are given by

$$g^{\Delta_{HL},-\Delta_{HL}}_{h,\bar{h}}(z,\bar{z}) = \frac{(h-\bar{h})!}{(\frac{d}{2}-1)_{h-\bar{h}}}(z\bar{z})^{\frac{\Delta_H+\Delta_L+h+\bar{h}}{2}} C^{(\frac{d}{2}-1)}_{h-\bar{h}}\Big(\frac{z+\bar{z}}{2\sqrt{z\bar{z}}}\Big). \tag{4.31}$$

It is easy to explicitly check that this form of the conformal blocks agrees with the one we used in $d=4$ in the previous section.

4.3.2. Anomalous dimensions

In Appendix A.3 we prove the validity of the impact parameter representation in any d. This means that the derivation of (4.21) goes through for arbitrary d. Using known results for the bulk phase shift from [55], we thus find

$$\gamma^{(1)}_{\bar{h},h-\bar{h}} = -\frac{\bar{h}^{\frac{d}{2}}}{h^{\frac{d}{2}-1}} \frac{\Gamma(d)}{\Gamma(\frac{d}{2})\Gamma(\frac{d}{2}+1)} {}_2F_1(\frac{d}{2}-1, d-1, \frac{d}{2}+1, \frac{\bar{h}}{h}). \tag{4.32}$$

In the lightcone limit ($h = l \gg \bar{h} = n$) this reduces to

$$\gamma^{(1)}_{\bar{h},h-\bar{h}} \approx -\frac{\bar{h}^{\frac{d}{2}}}{h^{\frac{d}{2}-1}} \frac{\Gamma(d)}{\Gamma(\frac{d}{2})\Gamma(\frac{d}{2}+1)}. \tag{4.33}$$

Similarly, using (4.21) together with eq. (2.29) and eq. (A.5) from [55], we find the $\mathcal{O}(\mu^2)$ corrections to the anomalous dimensions in the limit $h, \bar{h} \gg 1$:

$$\begin{aligned}
\gamma^{(2)}_{\bar{h},h-\bar{h}} &= -\frac{\delta^{(2)}}{\pi} + \frac{1}{2}\gamma^{(1)}_{\bar{h},h-\bar{h}} \left\{ \frac{2}{h+\bar{h}}\gamma^{(1)}_{\bar{h},h-\bar{h}} - \frac{\Gamma(d)}{\Gamma\left(\frac{d}{2}\right)^2} \bar{h}^{\frac{d}{2}-1} h^{\frac{d}{2}-1} \frac{(h-\bar{h})^{3-d}}{h+\bar{h}} \right\} = \\
&= -\left(\frac{\bar{h}^{d-1}}{h^{d-2}}\right) \frac{2^{2d-4}\Gamma\left(d+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(d)} \; {}_2F_1[2d-3, d-2, d, \frac{\bar{h}}{h}] + \\
&+ \frac{\bar{h}^d h^{2-d}}{(h+\bar{h})} \frac{4\Gamma^2(d)}{d^2\,\Gamma^4\left(\frac{d}{2}\right)} \left({}_2F_1[\frac{d}{2}-1, d-1, \frac{d}{2}+1, \frac{\bar{h}}{h}] \right)^2 + \\
&+ \frac{\bar{h}^{d-1}(h-\bar{h})^{3-d}}{h+\bar{h}} \frac{\Gamma^2(d)}{d\,\Gamma^4\left(\frac{d}{2}\right)} \; {}_2F_1[\frac{d}{2}-1, d-1, \frac{d}{2}+1, \frac{\bar{h}}{h}]
\end{aligned} \tag{4.34}$$

Taking further the lightcone limit ($h \gg \bar{h}$) we find that

$$\gamma^{(2)}_{\bar{h},h-\bar{h}} \approx \frac{\bar{h}^{d-1}}{h^{d-2}} \frac{2^{2d-4}}{\pi} \left(\frac{d\Gamma\left(\frac{d+1}{2}\right)^2}{\Gamma\left(\frac{d+2}{2}\right)^2} - \frac{\sqrt{\pi}\Gamma\left(d+\frac{1}{2}\right)}{\Gamma\left(d\right)} \right). \tag{4.35}$$

The result (4.35) agrees with eq. (6.42) in [55] which was obtained independently using perturbation theory in the bulk. In order to see this explicitly, one should notice the following expression for the hypergeometric function:

$$ {}_3F_2(1,-\frac{d}{2},-\frac{d}{2};1+\frac{d}{2},1+\frac{d}{2};1)=\frac{1}{2}\Big(1+\frac{\Gamma^4(1+\frac{d}{2})\Gamma(2d+1)}{\Gamma^4(d+1)}\Big). \tag{4.36} $$

4.3.3. Corrections to the OPE coefficients

So far, we have only considered the imaginary part of the S-channel. The real part at $\mathcal{O}(\mu)$ is given by the following expression:

$$ \begin{aligned} \mathrm{Re}(G(z,\bar{z}))|_\mu =&(z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)}\int_0^{+\infty}dh\int_0^h d\bar{h}P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}}\\ &\times\Big(P^{(HL,HL);(1)}_{\bar{h},h-\bar{h}}+\frac{1}{2}\gamma^{(1)}_{\bar{h},h-\bar{h}}(\partial_h+\partial_{\bar{h}})\Big)g^{\Delta_{HL},-\Delta_{HL}}_{h,\bar{h}}(z,\bar{z}), \end{aligned} \tag{4.37} $$

which can be rewritten as:

$$ \begin{aligned} &\mathrm{Re}(G(z,\bar{z}))|_\mu=(z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)}\int_0^{+\infty}dh\int_0^h d\bar{h}g^{\Delta_{HL},-\Delta_{HL}}_{h,\bar{h}}\times\\ &\times\Big(P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}}P^{(HL,HL);(1)}_{\bar{h},h-\bar{h}}-\frac{1}{2}(\partial_h+\partial_{\bar{h}})(P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}}\gamma^{(1)}_{\bar{h},h-\bar{h}})\Big)\\ &+\text{total derivative}. \end{aligned} \tag{4.38} $$

The total derivative term in (4.38) can be shown to vanish as explained in Appendix A.5.

To derive a relation between the corrections to the OPE coefficients and the anomalous dimensions at $\mathcal{O}(\mu)$, let us consider the limit $h,\bar{h}\gg 1$ and substitute $\bar{h}$ by h everywhere. Using (4.32), one can deduce $\gamma^{(1)}_{\bar{h},h-\bar{h}}\propto h$. Then, it follows that $(\partial_h+\partial_{\bar{h}})(P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}}\gamma^{(1)}_{\bar{h},h-\bar{h}})\propto P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}}$ and hence the second term on the right hand side of (4.38) behaves as:

$$ \begin{aligned} \frac{(z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)}}{2}\int_0^{+\infty}dh\int_0^h d\bar{h}g^{\Delta_{HL},-\Delta_{HL}}_{h,\bar{h}}(\partial_h+\partial_{\bar{h}})(P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}}\gamma^{(1)}_{\bar{h},h-\bar{h}})\\ \propto\frac{1}{\sigma^{2\Delta_L}}. \end{aligned} \tag{4.39} $$

On the other hand, we know that in the Regge limit the leading contribution in the T-channel at $\mathcal{O}(\mu)$ comes from the exchange of the stress tensor. The real part of its conformal block is proportional to σ^d, so the T-channel result behaves as $\frac{1}{\sigma^{2\Delta_L - d}}$. This is way less singular than (4.39). Hence (4.39) must be canceled by the first term on the right hand side of (4.38), at least in the limit $h, \bar{h} \gg 1$. That is:

$$P^{(HL,HL);\text{MFT}}_{\bar{h},h-\bar{h}} P^{(HL,HL);(1)}_{\bar{h},h-\bar{h}} = \frac{1}{2}(\partial_h + \partial_{\bar{h}})(P^{(HL,HL);\text{MFT}}_{\bar{h},h-\bar{h}} \gamma^{(1)}_{\bar{h},h-\bar{h}}). \tag{4.40}$$

A similar relation holds for the OPE coefficients of light-light double-trace operators, e.g. see [11,38,53]. In that case it was observed in [49] that the relation is not exact in $(h, \bar{h})$. We expect the same to be true here. Furthermore, the real part at $\mathcal{O}(\mu^2)$ was given in (3.39) as:

$$\begin{aligned}\text{Re}(G(z,\bar{z}))|_{\mu^2} = (z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)} \sum_{h \geq \bar{h} \geq 0}^{\infty} P^{(HL,HL);\text{MFT}}_{\bar{h},h-\bar{h}} \Big(& P^{(HL,HL);(2)}_{\bar{h},h-\bar{h}} \\ - \frac{1}{2}(\pi\gamma^{(1)}_{\bar{h},h-\bar{h}})^2 + \frac{1}{2}(\gamma^{(2)}_{\bar{h},h-\bar{h}} & + P^{(HL,HL);(1)}_{\bar{h},h-\bar{h}} \gamma^{(1)}_{\bar{h},h-\bar{h}})(\partial_h + \partial_{\bar{h}}) \\ + \frac{1}{8}(\gamma^{(1)}_{\bar{h},h-\bar{h}})^2(\partial_h + \partial_{\bar{h}})^2 \Big) & g^{\Delta_{HL},-\Delta_{HL}}_{h,\bar{h}}.\end{aligned} \tag{4.41}$$

Using the impact parameter representation this can be expressed as:

$$\begin{aligned}\text{Re}(G(z,\bar{z}))|_{\mu^2} = \int_0^\infty dh \int_0^h d\bar{h} \mathcal{I}_{h,\bar{h}} \Big(& P^{(HL,HL);(2)}_{\bar{h},h-\bar{h}} - \frac{\pi^2}{2}(\gamma^{(1)}_{\bar{h},h-\bar{h}})^2 \\ - \frac{1}{2P^{(HL,HL);\text{MFT}}_{\bar{h},h-\bar{h}}}(\partial_h + \partial_{\bar{h}})(P^{(HL,HL);\text{MFT}}_{\bar{h},h-\bar{h}} & (\gamma^{(2)}_{\bar{h},h-\bar{h}} + P^{(HL,HL);(1)}_{\bar{h},h-\bar{h}} \gamma^{(1)}_{\bar{h},h-\bar{h}})) \\ + \frac{1}{8P^{(HL,HL);\text{MFT}}_{\bar{h},h-\bar{h}}}(\partial_h + \partial_{\bar{h}})^2 (P^{(HL,HL);\text{MFT}}_{\bar{h},h-\bar{h}} & (\gamma^{(1)}_{\bar{h},h-\bar{h}})^2)\Big),\end{aligned} \tag{4.42}$$

where we repeatedly integrated by parts. It follows from (4.20) and (4.17), together with $\pi\gamma^{(1)}_{\bar{h},h-\bar{h}} = -\delta^{(1)}$, that the corrections to the OPE coefficients at $\mathcal{O}(\mu^2)$ satisfy the following relationship:

$$\begin{aligned}P^{(HL,HL);\text{MFT}}_{\bar{h},h-\bar{h}} P^{(2)}_{\bar{h},h-\bar{h}} = & -\frac{1}{8}(\partial_h + \partial_{\bar{h}})^2 (P^{(HL,HL);\text{MFT}}_{\bar{h},h-\bar{h}} (\gamma^{(1)}_{\bar{h},h-\bar{h}})^2) \\ + \frac{1}{2}(\partial_h + \partial_{\bar{h}})(P^{(HL,HL);\text{MFT}}_{\bar{h},h-\bar{h}} & (\gamma^{(2)}_{\bar{h},h-\bar{h}} + P^{(HL,HL);(1)}_{\bar{h},h-\bar{h}} \gamma^{(1)}_{\bar{h},h-\bar{h}})).\end{aligned} \tag{4.43}$$

The arguments above are similar to the ones used in [75,59].

4.3.4. Flat space limit

In the flat space limit the relation between the scattering phase shift and the anomalous dimensions has been previously discussed in [131]. Hence, it is interesting to consider the flat space limit of eq. (4.1). This limit is achieved by taking the apparent impact parameter to be much smaller than the AdS radius. This corresponds to the small L regime or, equivalently, using $e^{-2L} = \bar{h}/h$ to the $1 \ll l \ll n \ll \Delta_H$ limit.

In this limit, according to (4.32), the behavior of $\gamma^{(1)}_{n,l}$ is given by

$$\gamma^{(1)}_{n,l} \propto n \left(\frac{n}{l}\right)^{d-3} . \tag{4.44}$$

Hence, the $\gamma^{(1)}_{\bar{h},h-\bar{h}}(\partial_h + \partial_{\bar{h}})\gamma^{(1)}_{\bar{h},h-\bar{h}}$ term in eq. (4.1) behaves as

$$\gamma^{(1)}_{n,l}\partial_n \gamma^{(1)}_{n,l} \propto n \left(\frac{n}{l}\right)^{2d-6} . \tag{4.45}$$

Similarly, using equation (A.5) from [55], one finds that $\delta^{(2)}$ behaves as

$$\delta^{(2)} \propto n \left(\frac{n}{l}\right)^{2d-5} . \tag{4.46}$$

Since (4.45) is subleading to (4.46), in the flat space limit the anomalous dimensions are proportional to the phase shift,

$$\gamma^{(2)}_{n,l} \approx -\frac{\delta^{(2)}}{\pi} . \tag{4.47}$$

4.4. Discussion

In this section we studied a four-point function of pairwise identical scalar operators, $\mathcal{O}_H$ and $\mathcal{O}_L$, in holographic CFTs of any dimensionality. Scaling Δ_H with the central charge, the CFT data admits an expansion in the ratio $\mu \sim \Delta_H/C_T$ which we keep fixed. Using crossing symmetry and the bulk phase shift calculated in [55], we studied $\mathcal{O}(\mu^2)$ corrections to the OPE data of heavy-light double-trace operators $[\mathcal{O}_H\mathcal{O}_L]_{n,l}$ for large l and n. In particular, the relationship between the bulk phase shift and the OPE data of heavy-light double-trace operators is found using an impact parameter representation. Furthermore, this allows us in principle to determine the OPE data of $[\mathcal{O}_H\mathcal{O}_L]_{n,l}$, for $l, n \gg 1$

to all orders in μ, i.e., to all orders in an expansion in the dual black hole Schwarzschild radius.

Scaling Δ_H with the central charge enhances the effect of stress tensor exchanges compared to the $1/C_T$ corrections due to the exchange of generic operators. At $\mathcal{O}(\mu^2)$ and higher, we therefore expect multi-stress tensor operators to contribute. The OPE coefficients for such exchanges are not known in general. They would be needed to determine corrections to the OPE data of heavy-light double-trace operators using purely CFT methods. In a recent paper [15] some of these OPE coefficients have been computed. In particular, the OPE coefficients with the multi-stress tensor operators of lowest twist have been argued to be universal (independent of the higher derivative couplings in the bulk gravitational lagrangian). It would be interesting to connect these results to the ones discussed in this section.

It is a curious fact that each term in the μ-expansion of the bulk phase shift as computed in gravity in [55] can be expressed as an infinite sum of "Regge conformal blocks" corresponding to operators of dimension $\Delta = k(d-2)+2n+2$ and spin $J = 2$. Explicitly,

$$i\,\delta^{(k)}(S,L) = \; f(k) \sum_{n=0}^{\infty} \lambda_k(n)\;\, g^R_{k(d-2)+2n+2,\,2}(S,L)\,, \tag{4.48}$$

where the coefficients $(f(k), \lambda_k(n))$ are listed in Appendix A.6 and we set $S \equiv \sqrt{-p^2}$ compared to [55]. Here $g^R_{\Delta,J}(S,L)$ denotes a "Regge conformal block", and is equal to the leading behaviour of the analytically continued T-channel conformal block in the Regge limit [132,74]

$$g^R_{\Delta,J}(S,L) = i\,c_{\Delta,J}\; S^{J-1}\; \Pi_{\Delta-1,d-1}(L) \tag{4.49}$$

defined in terms of

$$1-z = \frac{e^L}{S}, \quad 1-\bar{z} = \frac{e^{-L}}{S} \tag{4.50}$$

as $S \to \infty$ and L fixed. Here $c_{\Delta,J}$ are known coefficients which can be found in Appendix A.6 and $\Pi_{\Delta-1,d-1}(L)$ denotes the $(d-1)$-dimensional hyperbolic space propagator for a massive scalar of mass square $m^2 = (\Delta-1)$.

To understand the implications of (4.48) let us focus on $k = 2$ and consider large impact parameters, a.k.a. the lightcone limit. In this case, one expects that the dominant contribution to the bulk phase shift comes from the infinite sum of the minimal twist double-trace operators built from the stress tensor, schematically denoted by $T_{\mu\nu}\partial_{\mu_1}\cdots\partial_{\mu_\ell}T_{\rho\sigma}$. (4.48) implies that this infinite sum gives rise to a contribution which can be interpreted as coming from a *single* conformal block of an "effective" operator of the same twist $\tau = 2(d-2)$, but spin $J = 2$. At finite impact parameter, one would then need to add the contributions of an infinite tower of such effective operators of twist $\tau = 2(d-2)+2n$ and spin $J = 2$, as expressed by the infinite sum in (4.48). From this point of view, the coefficients λ_n in (4.48) can be interpreted as ratios of sums of OPE coefficients of double-trace operators. It is clear that this picture appears to hold to all orders in $\left(\frac{\Delta_H}{C_T}\right)$ or equivalently, the Schwarzschild radius of the black hole.

It would be interesting to investigate whether Rindler positivity constrains the Regge behaviour of the bulk phase shift to grow at most linearly with the energy S, similarly to Section 5.2 in [74]. If this were true, one would perhaps only need to understand the origin of the λ_n to compute the bulk phase shift to arbitrary order in $\left(\frac{\Delta_H}{C_T}\right)$ purely from CFT techniques.

5. The minimal twist multi-stress tensors and conformal bootstrap

5.1. Summary of the results

In this section we argue that for a large class of CFTs (including holographic CFTs) in even d, the contribution of minimal-twist multi-stress tensors to the correlator in the lightcone limit can be written as a sum of products of certain hypergeometric functions. To be explicit, let us define functions $f_a(z)$ as

$$f_a(z) = (1-z)^a {}_2F_1(a, a, 2a; 1-z). \tag{5.1}$$

The stress tensor contribution to the correlator in the lightcone limit is given in any dimension d by

$$\mathcal{G}^{(1)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^{\frac{d-2}{2}}}{[(1-z)(1-\bar{z})]^{\Delta_L}} \frac{\Delta_L \Gamma(\frac{d}{2}+1)^2}{4\Gamma(d+2)} f_{\frac{d+2}{2}}(z). \tag{5.2}$$

At $\mathcal{O}(\mu^2)$ the contribution from twist-four double-stress tensor operators in $d=4$ is

$$\begin{aligned}\mathcal{G}^{(2)}(z,\bar{z}) &\underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^2}{[(1-z)(1-\bar{z})]^{\Delta_L}} \left(\frac{\Delta_L}{28800(\Delta_L-2)}\right) \times \\ &\left((\Delta_L-4)(\Delta_L-3) f_3^2(z) + \frac{15}{7}(\Delta_L-8) f_2(z) f_4(z) + \frac{40}{7}(\Delta_L+1) f_1(z) f_5(z)\right).\end{aligned} \tag{5.3}$$

This result agrees with the expression obtained by different methods in [124].

The contribution from twist-six triple-stress tensors in the lightcone limit in $d=4$ at order $\mathcal{O}(\mu^3)$ is

$$\begin{aligned}\mathcal{G}^{(3)}(z,\bar{z}) &\underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^3}{[(1-z)(1-\bar{z})]^{\Delta_L}} \Big(a_{117} f_1(z)^2 f_7(z) + a_{126} f_1(z) f_2(z) f_6(z) \\ &+ a_{135} f_1(z) f_3(z) f_5(z) + a_{225} f_2(z)^2 f_5(z) + a_{234} f_2(z) f_3(z) f_4(z) + a_{333} f_3(z)^3\Big),\end{aligned} \tag{5.4}$$

where coefficients a_{ijk} are given by (5.26).

Furthermore, from (5.4) and (5.26), we find the OPE coefficients of twist-six triple-stress tensor operators as a finite sum (for details see Section 5.2.5). Two such OPE coefficients for twist-6 triple-stress tensors were calculated holographically in [15] and agree with our results.

The contribution from twist-eight double-stress tensors to the correlator in the lightcone limit in $d=6$ at order $\mathcal{O}(\mu^2)$ is

$$\begin{aligned} \mathcal{G}^{(2)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} & \frac{(1-\bar{z})^4}{[(1-z)(1-\bar{z})]^{\Delta_L}} \times \\ & \Big(a_{13} f_1(z) f_7(z) + a_{26} f_2(z) f_6(z) + a_{35} f_3(z) f_5(z) + a_{44} f_4(z)^2\Big), \end{aligned} \tag{5.5}$$

where a_{mn} are given by (5.49). Using (5.5) and (5.49) we find the OPE coefficients for operators of type $: T_{\mu\nu} \partial_{\lambda_1} \dots \partial_{\lambda_{2\ell}} T_{\alpha\beta} :$ in $d=6$ to be equal to:

$$P_{8,s}^{(HH,LL)} = \mu^2 \frac{c\Delta_L}{(\Delta_L-3)(\Delta_L-4)} (a_3 \Delta_L^3 + a_2 \Delta_L^2 + a_1 \Delta_L + a_0), \tag{5.6}$$

where c and a_m, given by (5.57), are functions of the total spin $s = 4 + 2\ell$.

In general we propose that the contribution from minimal-twist multi-stress tensor operators to the correlator in even d at $\mathcal{O}(\mu^k)$ in the lightcone limit takes the form

$$\mathcal{G}^{(k)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^{k(\frac{d}{2}-1)}}{[(1-z)(1-\bar{z})]^{\Delta_L}} \sum_{\{i_p\}} a_{i_1 \dots i_k} f_{i_1}(z) \dots f_{i_k}(z), \quad \sum_{p=1}^{k} i_p = k\Big(\frac{d+2}{2}\Big), \tag{5.7}$$

where the sum goes over all sets of $\{i_p\}$ with $i_p \le i_{p+1}$ and $a_{i_1 \dots i_k}$ coefficients that need to be fixed.[15]

We also check that the stress tensor sector of the near lightcone correlator exponentiates

$$\langle \mathcal{O}_H(x_4) \mathcal{O}_L(1) \mathcal{O}_L(z,\bar{z}) \mathcal{O}_H(0) \rangle|_{\text{multi-stress tensors}} \underset{\bar{z}\to 1}{\approx} \frac{e^{\Delta_L \mathcal{F}(\mu;z,\bar{z})}}{[(1-z)(1-\bar{z})]^{\Delta_L}}, \tag{5.8}$$

where $\mathcal{F}(\mu;z,\bar{z})$ is a rational function of Δ_L that remains $\mathcal{O}(1)$ as $\Delta_L \to \infty$. We explicitly verify this up to $\mathcal{O}(\mu^3)$ in $d=4$ and $\mathcal{O}(\mu^2)$ in $d=6$.

5.1.1. Outline

The rest of Section 5 is organized as follows. In Section 5.2 we find the contribution of minimal-twist double- and triple-stress tensor operators in $d=4$ in the lightcone limit. We show that this contribution exponentiates and we write

[15] One only needs to sum the linearly independent products of functions f_a.

an expression for the OPE coefficients of minimal-twist triple-stress tensors of spin s with scalar operators, in the form of a finite sum. In Section 5.3, we repeat this program up to $\mathcal{O}(\mu^2)$ in $d = 6$. Again we confirm exponentiation and we find a closed form expression for the OPE coefficients of minimal-twist double-stress tensors of arbitrary spin with scalar operators. We discuss our results in Section 5.4.

5.2. Multi-stress tensors in four dimensions

In this section we describe how to use crossing symmetry to fix the contribution of minimal-twist multi-stress tensors to the heavy-heavy-light-light correlator in $d = 4$ to $\mathcal{O}(\mu^3)$. The methods described generalize to other even spacetime dimensions, with the six-dimensional case to $\mathcal{O}(\mu^2)$ described in Section 5.3. In principle the same technology can also be used to determine the correlator at higher orders. Moreover, the resulting expression can be decomposed into multi-stress tensor blocks of minimal-twist, allowing us at each order to read off the OPE coefficients of minimal-twist multi-stress tensors.

The idea is to study the S-channel expansion in (3.52) in the limit $1-\bar{z} \ll z \ll 1$. In this limit operators with $l \gg 1$ and low values of n dominate. Namely, the OPE coefficients and the anomalous dimensions of the S-channel operators can be expanded in powers of $1/l$. The leading contribution in $1-\bar{z} \ll 1$ limit is due to the leading term in $1/l$ expansion and same is true for subleading contributions. Subsequently, the leading contribution in $z \ll 1$ limit is due to the $n = 0$ term, the subleading is due to the $n = 1$, and so on. Expanding the conformal blocks in (3.53) for small $\gamma(n,l)$ and $\bar{z} \to 1$, the blocks in $d = 4$ reduce to

$$(z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)} g^{(\Delta_{HL},-\Delta_{HL})}_{\Delta_H+\Delta_L+2n+\gamma,l}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \bar{z}^l p(\log z, \gamma(n,l)) \frac{z^n}{1-z}, \qquad (5.9)$$

where $p(\log z, \gamma(n,l))$ is given by

$$p(\log z, \gamma(n,l)) = z^{\frac{1}{2}\gamma(n,l)} = \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{\gamma(n,l)\log z}{2} \right)^j . \qquad (5.10)$$

Inserting (5.9) into (3.52) and converting the sum into an integral, we have the following expression for the correlator in the limit $\bar{z} \to 1$

$$\mathcal{G}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \sum_{n=0}^{\infty} \frac{z^n}{1-z} \int_0^{\infty} dl P^{(HL,HL)}_{n,l} \bar{z}^l p(\log z, \gamma(n,l)). \qquad (5.11)$$

In the following we consider an expansion of (5.11) around $z = 0$. The key point is to note that by expanding the anomalous dimensions and OPE coefficients, as in (3.25), terms proportional to $z^p \log^i z$ with $i = 2, 3, \ldots, k$ and any p at $\mathcal{O}(\mu^k)$, in (5.11) are completely determined in terms of OPE data at $\mathcal{O}(\mu^{k-1})$. Moreover, using (3.56) one sees that the integral over the spin l yields[16]

$$\int_0^\infty dl l^{\Delta_L - 1 - k} \bar{z}^l = \frac{\Gamma(\Delta_L - k)}{(-\log \bar{z})^{\Delta_L - k}} \underset{\bar{z}\to 1}{\approx} \frac{\Gamma(\Delta_L - k)}{(1-\bar{z})^{\Delta_L - k}}, \tag{5.12}$$

at $\mathcal{O}(\mu^k)$ in the limit $\bar{z} \to 1$. This correctly reproduces the expected $\bar{z}$ behavior of minimal-twist multi-stress tensors in the T-channel, thus verifying (3.56). Additionally, it is easy to check that the difference between the integral and the sum of expression $\bar{z}^l l^{\Delta_L - k - 1}$ is $\mathcal{O}(1-\bar{z})$ in $1 - \bar{z} \ll 1$ limit, while both integral and sum scale as $(1-\bar{z})^{-\Delta_L + k}$ when $\bar{z} \to 1$. As we always (implicitly) assume that $\Delta_L > k$, at each order in μ we are only interested in singular terms of sum or integral, therefore, the $\mathcal{O}(1-\bar{z})$ difference will never affect the calculation.

We now make the following ansatz for the correlator

$$\mathcal{G}^{(k)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^k}{[(1-z)(1-\bar{z})]^{\Delta_L}} \sum_{\{i_p\}} a_{i_1 \ldots i_k} f_{i_1}(z) \ldots f_{i_k}(z), \tag{5.13}$$

where the sum goes over all sets of $\{i_p\}$ with i_p integers and $i_p \le i_{p+1}$ such that $\sum_{p=1}^k i_p = 3k$ and $a_{i_1 \ldots i_k}$ coefficients that need to be fixed. Generally $f_a(z)$ are given by

$$f_a(z) = q_{1,a}(z) + q_{2,a}(z) \log z, \tag{5.14}$$

where $q_{(1,2),a}(z)$ are rational functions and the ansatz (5.13) at $\mathcal{O}(\mu^k)$ is therefore a polynomial in $\log z$ of degree k. By crossing symmetry terms with $\log^a z$, with $2 \le a \le k$, are determined by OPE data at $\mathcal{O}(\mu^{k-1})$. This is what we will use to determine the coefficients $a_{i_1 \ldots i_p}$.

[16] At each order in μ we implicitly assume that integrals of this type converge, i.e. that $\Delta_L > k$.

5.2.1. Stress tensor

We start by determining the OPE data at $\mathcal{O}(\mu)$. This is easily obtained by matching (5.11) at $\mathcal{O}(\mu)$ with the stress tensor contribution (3.48). Explicitly, multiplying both channels by $(1-z)$ we have at $\mathcal{O}(\mu)$

$$\frac{\Delta_L f_3(z)}{120[(1-z)(1-\bar{z})]^{\Delta_L-1}} = \frac{1}{(1-\bar{z})^{\Delta_L-1}} \sum_{n=0}^{\infty} \frac{\Gamma(\Delta_L+n-1)z^n}{\Gamma(\Delta_L)n!} \times \left(P_n^{(1)} + \frac{\gamma_n^{(1)}}{2}\log z\right). \tag{5.15}$$

Expanding the LHS in (5.15) for $z \ll 1$ we find

$$\begin{aligned}\frac{\Delta_L/120}{[(1-z)(1-\bar{z})]^{\Delta_L-1}} f_3(z) = \frac{1}{(1-\bar{z})^{\Delta_L-1}}\Big(&-\frac{\Delta_L}{4}(3+\log z) \\ &- z\frac{\Delta_L}{4}(3(\Delta_L+1)+(\Delta_L+5)\log z) \\ &- z^2\frac{\Delta_L}{8}\left(3\Delta_L(\Delta_L+3)+(12+\Delta_L(\Delta_L+11))\right) \\ &+ \mathcal{O}(z^3, z^3\log z)\Big),\end{aligned} \tag{5.16}$$

while the RHS is given by

$$\begin{aligned}\frac{\sum_{n=0}^{\infty}\frac{\Gamma(\Delta_L+n-1)z^n}{\Gamma(\Delta_L)n!}(P_n^{(1)}+\frac{\gamma_n^{(1)}}{2}\log z)}{(1-\bar{z})^{\Delta_L-1}} = \frac{1}{(1-\bar{z})^{\Delta_L-1}}\Big(&\frac{P_0^{(1)}+\frac{\gamma_0^{(1)}}{2}\log z}{\Delta_L-1} \\ &+ z(P_1^{(1)}+\frac{\gamma_1^{(1)}}{2}\log z) \\ &+ z^2\frac{\Delta_L}{2}(P_2^{(1)}+\frac{\gamma_2^{(1)}}{2}\log z) \\ &+ \mathcal{O}(z^3, z^3\log z)\Big).\end{aligned} \tag{5.17}$$

Comparing (5.16) and (5.17) order-by-order in z one finds the following OPE data

$$\begin{aligned}\gamma_0^{(1)} &= -\frac{\Delta_L(\Delta_L-1)}{2}, \\ \gamma_1^{(1)} &= -\frac{\Delta_L(\Delta_L+5)}{2}, \\ \gamma_2^{(1)} &= -\frac{12+\Delta_L(\Delta_L+11)}{2},\end{aligned} \tag{5.18}$$

which agrees with eq. (6.10) in [55], and the OPE coefficients

$$\begin{aligned} P_0^{(1)} &= -\frac{3\Delta_L(\Delta_L - 1)}{4}, \\ P_1^{(1)} &= -\frac{3\Delta_L(\Delta_L + 1)}{4}, \\ P_2^{(1)} &= -\frac{3\Delta_L(\Delta_L + 3)}{4}. \end{aligned} \tag{5.19}$$

It is straightforward to continue and compute the $\mathcal{O}(\mu)$ OPE data in the S-channel for any value of n.

5.2.2. *Twist-four double-stress tensors*

From (5.13) we infer the following expression for the contribution due to twist-four double-stress tensors to the heavy-heavy-light-light correlator in the limit $\bar{z} \to 1$:

$$\mathcal{G}^{(2)}(z, \bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^2}{[(1-z)(1-\bar{z})]^{\Delta_L}} \Big(a_{15} f_1(z) f_5(z) + a_{24} f_2(z) f_4(z) + a_{33} f_3^2(z) \Big). \tag{5.20}$$

By expanding (5.20) further in the limit $z \ll 1$ and collecting terms that goes as $z^p \log^2 z$, we will match with known contributions obtained from (5.11).

Inserting (5.18) and (5.19) in the S-channel (5.11) fixes terms proportional to $z^p \log^2 z$ up to $\mathcal{O}(z^2 \log^2 z)$. Expanding the ansatz (5.20) and matching with the S-channel reproduces the result obtained in [124]:

$$\begin{aligned} \mathcal{G}^{(2)}(z, \bar{z}) \underset{\bar{z}\to 1}{\approx} & \frac{(1-\bar{z})^2}{[(1-z)(1-\bar{z})]^{\Delta_L}} \left(\frac{\Delta_L}{28800(\Delta_L - 2)} \right) \times \\ & \Big((\Delta_L - 4)(\Delta_L - 3) f_3^2(z) + \frac{15}{7}(\Delta_L - 8) f_2(z) f_4(z) \\ & + \frac{40}{7}(\Delta_L + 1) f_1(z) f_5(z) \Big). \end{aligned} \tag{5.21}$$

Using the $\mathcal{O}(\mu)$ OPE data in the S-channel for $n > 2$ in (5.16) and (5.17) one gets an overconstrained system which is still solved by (5.21). This is a strong argument in favor of the validity of our ansatz (5.13).

We can now use (5.21) to derive the $\mathcal{O}(\mu^2)$ OPE data in the S-channel by matching terms proportional to $z^p \log^i z$ as $z \to 0$, with $i = 0, 1$, by comparing with (5.11). This is done in the same way it was done for $\mathcal{O}(\mu)$ OPE data in the S-channel. For example, one finds the following data for $n = 0, 1, 2, 3$:

$$\begin{aligned}
\gamma_0^{(2)} &= -\frac{(\Delta_L - 1)\Delta_L(4\Delta_L + 1)}{8}, \\
\gamma_1^{(2)} &= -\frac{\Delta_L(\Delta_L + 1)(4\Delta_L + 35)}{8}, \\
\gamma_2^{(2)} &= -\frac{(3 + \Delta_L)(68 + \Delta_L(69 + 4\Delta_L))}{8}, \\
\gamma_3^{(2)} &= -\frac{(5 + \Delta_L)(204 + \Delta_L(4\Delta_L + 103))}{8},
\end{aligned} \tag{5.22}$$

which agrees with Eq. (6.39) in [55], and for the OPE coefficients

$$\begin{aligned}
P_0^{(2)} &= \frac{(\Delta_L - 1)\Delta_L(-28 + \Delta_L(-145 + 27\Delta_L))}{96}, \\
P_1^{(2)} &= \frac{\Delta_L(-596 + \Delta_L(-399 + \Delta_L(-64 + 27\Delta_L)))}{96}, \\
P_2^{(2)} &= \frac{-1248 + \Delta_L(-2252 + \Delta_L(-699 + \Delta_L(44 + 27\Delta_L)))}{96}, \\
P_3^{(2)} &= \frac{-3744 + \Delta_L(-4940 + \Delta_L(-783 + \Delta_L(152 + 27\Delta_L)))}{96}.
\end{aligned} \tag{5.23}$$

It is again straightforward to extract the OPE data for any value of n.

5.2.3. Twist-six triple-stress tensors

We now consider the multi-stress tensor sector of the correlator at $\mathcal{O}(\mu^3)$ and proceed similarly to the previous section. From (5.13) we infer the following expression for the contribution due to twist-six triple-stress tensors:

$$\begin{aligned}
\mathcal{G}^{(3)}(z, \bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^3}{[(1-z)(1-\bar{z})]^{\Delta_L}} \Big(& a_{117} f_1^2 f_7 + a_{126} f_1 f_2 f_6 + a_{135} f_1 f_3 f_5 \\
& + a_{225} f_2^2 f_5 + a_{234} f_2 f_3 f_4 + a_{333} f_3^3 \Big),
\end{aligned} \tag{5.24}$$

where $f_i = f_i(z)$ is given by (3.46).[17] Taking the limit $1 - \bar{z} \ll z \ll 1$ of (5.24), we fix the coefficients by matching with terms proportional to $z^p \log^2 z$

[17] Note that we omitted a potential term of the form $f_1 f_4^2$. This can be written in terms of f_3^3, $f_1 f_3 f_5$, $f_2^2 f_5$ and $f_2 f_3 f_4$, as follows from:

$$f_3^3(z) = \frac{20}{21} f_1(z) f_3(z) f_5(z) - \frac{27}{28} f_1(z) f_4^2(z) - \frac{20}{21} f_2^2(z) f_5(z) + \frac{55}{28} f_2(z) f_3(z) f_4(z). \tag{5.25}$$

and $z^p \log^3 z$, with $p = 0, 1, 2$ from (5.11). This requires using the OPE data of the heavy-light double-trace operators $[\mathcal{O}_H \mathcal{O}_L]_{n,l}$ for $n = 0, 1, 2$ and $l \gg 1$ to $\mathcal{O}(\mu^2)$, given in (5.18), (5.19), (5.22) and (5.23).

We find the following solution:

$$
\begin{aligned}
a_{117} &= \frac{5\Delta_L(\Delta_L+1)(\Delta_L+2)}{768768(\Delta_L-2)(\Delta_L-3)}, \\
a_{126} &= \frac{5\Delta_L(5\Delta_L^2-57\Delta_L-50)}{6386688(\Delta_L-2)(\Delta_L-3)}, \\
a_{135} &= \frac{\Delta_L(2\Delta_L^2-11\Delta_L-9)}{1209600(\Delta_L-3)}, \\
a_{225} &= -\frac{\Delta_L(7\Delta_L^2-51\Delta_L-70)}{2903040(\Delta_L-2)(\Delta_L-3)}, \\
a_{234} &= \frac{\Delta_L(\Delta_L-4)(3\Delta_L^2-17\Delta_L+4)}{4838400(\Delta_L-2)(\Delta_L-3)}, \\
a_{333} &= \frac{\Delta_L(\Delta_L-4)(\Delta_L^3-16\Delta_L^2+51\Delta_L+24)}{10368000(\Delta_L-2)(\Delta_L-3)}.
\end{aligned}
\tag{5.26}
$$

We can also consider higher values of p and obtain an overconstrained system of equations, whose solution is still (5.26). Inserting (5.26) into (5.24), we obtain the contribution from minimal-twist triple-stress tensor operators to the heavy-heavy-light-light correlator in the lightcone limit.

Note that for $\Delta_L \to \infty$, the correlator is determined by the exponentiation of the stress tensor discussed e.g. in [124], i.e.

$$
\mathcal{G}^{(3)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^3}{[(1-z)(1-\bar{z})]^{\Delta_L}} \frac{1}{3!} \left(\frac{\Delta_L}{120}(1-z)^3 {}_2F_1(3,3;6;1-z) \right)^3 + \cdots , \tag{5.27}
$$

which one indeed obtains by taking $\Delta_L \to \infty$ of (5.24) with (5.26). Here ellipses denote terms subleading in Δ_L.

By analytically continuing $z \to e^{-2\pi i} z$ and sending $z \to 1$, one can access the large impact parameter regime of the Regge limit. To do this we use the following property of the hypergeometric function (see e.g. [98]):

$$
{}_2F_1(a,a,2a,1-ze^{-2\pi i}) = {}_2F_1(a,a,2a,1-z) + 2\pi i \frac{\Gamma(2a)}{\Gamma(a)^2} {}_2F_1(a,a,1,z). \tag{5.28}
$$

Using (5.28) the leading term from (5.24) with the coefficients (5.26) in the limit $1-\bar{z} \ll 1-z \ll 1$ is given by

$$\mathcal{G}^{(3)}(z,\bar{z}) \underset{\bar{z}\to 1, z\to 1}{\approx} \frac{1}{[(1-z)(1-\bar{z})]^{\Delta_L}} \times \left(-\frac{9i\pi^3 \Delta_L(\Delta_L+1)(\Delta_L+2)(\Delta_L+3)(\Delta_L+4)}{2(\Delta_L-2)(\Delta_L-3)} \left(\frac{1-\bar{z}}{(1-z)^2}\right)^3 \right). \tag{5.29}$$

This agrees with the holographic calculation in a shockwave background at $\mathcal{O}(\mu^3)$ given by Eq. (45) in [126] based on techniques developed in [58-60,63,133].

5.2.4. Exponentiation of leading-twist multi-stress tensors

In $d = 2$ the heavy-heavy-light-light correlator is determined by the heavy-heavy-light-light Virasoro vacuum block. This block contains the exchange of any number of stress tensors and derivatives thereof in the T-channel [40,105,110], and therefore all multi-stress tensor contributions. This block, together with the disconnected part, exponentiates as

$$\langle \mathcal{O}_H(\infty)\mathcal{O}_L(1)\mathcal{O}_L(z)\mathcal{O}_H(0)\rangle = e^{\Delta_L \mathcal{F}(z)}, \tag{5.30}$$

for a known function $\mathcal{F}(z)$ independent of Δ_L. It is interesting to ask if something similar happens for the contribution of the minimal-twist multi-stress tensors in the lightcone limit of the correlator in higher dimensions. By this we mean whether the stress tensor sector of the correlator can be written as

$$\mathcal{G}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{1}{[(1-z)(1-\bar{z})]^{\Delta_L}} e^{\Delta_L \mathcal{F}(\mu; z,\bar{z})}, \tag{5.31}$$

for some function $\mathcal{F}(\mu; z, \bar{z})$ which is a rational function of Δ_L and remains $\mathcal{O}(1)$ as $\Delta_L \to \infty$.

The $\bar{z}$ dependence implies the following form of $\mathcal{F}(\mu; z, \bar{z})$:

$$\mathcal{F}(\mu; z,\bar{z}) = \mu(1-\bar{z})\mathcal{F}^{(1)}(z) + \mu^2(1-\bar{z})^2\mathcal{F}^{(2)}(z) + \mu^3(1-\bar{z})^3\mathcal{F}^{(3)}(z) + \mathcal{O}(\mu^4). \tag{5.32}$$

At leading order we observe $\mathcal{F}^{(1)}(z) = \frac{1}{120} f_3(z)$, which is just the stress tensor contribution. At second order we find:

$$\mathcal{F}^{(2)}(z) = \frac{(12-5\Delta_L)f_3(z)^2 + \frac{15}{7}(\Delta_L-8)f_2(z)f_4(z) + \frac{40}{7}(\Delta_L+1)f_1(z)f_5(z)}{28800(\Delta_L-2)}. \tag{5.33}$$

Note that $\mathcal{F}^{(2)}(z)$ is independent of Δ_L in the limit $\Delta_L \to \infty$.

To find $\mathcal{F}^{(3)}(z)$ we parametrise it as

$$\begin{aligned}\mathcal{F}^{(3)}(z) = \Big(& b_{117} f_1^2(z) f_7(z) + b_{126} f_1(z) f_2(z) f_6(z) + b_{135} f_1(z) f_3(z) f_5(z) \\ &+ b_{225} f_2^2(z) f_5(z) + b_{234} f_2(z) f_3(z) f_4(z) + b_{333} f_3^3(z)\Big).\end{aligned} \tag{5.34}$$

It is clear that for terms which do not contain a factor of $f_3(z)$, the coefficients b_{ijk} should satisfy $b_{ijk} = a_{ijk}/\Delta_L$. This is not true for terms which contain a factor of f_3. Inserting $\mathcal{F}^{(1)}$, $\mathcal{F}^{(2)}$ and Eq. (5.34) in (5.31), expanding in μ and matching with (5.24) yields

$$\begin{aligned}
b_{117} &= \frac{a_{117}}{\Delta_L}, \\
b_{126} &= \frac{a_{126}}{\Delta_L}, \\
b_{225} &= \frac{a_{225}}{\Delta_L}, \\
b_{135} &= -\frac{11\Delta_L^2 - 19\Delta_L - 18}{1209600(\Delta_L - 2)(\Delta_L - 3)}, \\
b_{234} &= \frac{(\Delta_L - 2)(\Delta_L + 2)}{1209600(\Delta_L - 2)(\Delta_L - 3)}, \\
b_{333} &= \frac{7\Delta_L^2 - 18\Delta_L - 24}{2592000(\Delta_L - 2)(\Delta_L - 3)}.
\end{aligned} \tag{5.35}$$

From (5.33) and (5.35), one finds that the correlator exponentiates to $\mathcal{O}(\mu^3)$ in the sense described above, i.e. $\mathcal{F}(\mu; z, \bar{z})$ is a rational function of Δ_L of $\mathcal{O}(1)$ as $\Delta_L \to \infty$.

To leading order in Δ_L, exponentiation for large Δ_L is a prediction of the AdS/CFT correspondence. The two-point function of the operator $\mathcal{O}_L$ in the state created by the heavy operator $\mathcal{O}_H$ is given in terms of the exponential of the (regularized) geodesic distance between the boundary points in the dual bulk geometry. For details on this, see e.g. [124].

5.2.5. OPE coefficients of triple-stress tensors

In this section we describe how to decompose the correlator (5.24) into an infinite sum of minimal-twist triple-stress tensor operators. In order to do this we use the following multiplication formula for hypergeometric functions [124]:

$$
\begin{aligned}
{}_2F_1(a,a;2a;w){}_2F_1(b,b;2b;w) = \sum_{m=0}^{\infty} p[a,b,m]w^{2m}\times \\
{}_2F_1[a+b+2m, a+b+2m, 2a+2b+4m, w],
\end{aligned}
\tag{5.36}
$$

where

$$
\begin{aligned}
p[a,b,m] = \frac{\Gamma(a+b+2m)}{\Gamma(a+b+2m-\frac{1}{2})}\times \\
\frac{2^{-4m}\Gamma(a+\frac{1}{2})\Gamma(b+\frac{1}{2})\Gamma(m+\frac{1}{2})\Gamma(a+m)\Gamma(b+m)\Gamma(a+b+m-\frac{1}{2})}{\sqrt{\pi}\Gamma(a)\Gamma(b)\Gamma(m+1)\Gamma(a+m+\frac{1}{2})\Gamma(b+m+\frac{1}{2})\Gamma(a+b+m)}.
\end{aligned}
\tag{5.37}
$$

It is useful to note that by using (5.36) we can write a similar formula for the functions f_a defined in (3.46):

$$
f_a(z)f_b(z) = \sum_{m=0}^{\infty} p[a,b,m] f_{a+b+2m}(z), \tag{5.38}
$$

where $p[a,b,m]$ is defined in (5.37). It is now clear that the correlator (5.24) can be written as a double sum over functions $f_{9+2(n+m)}$. We can thus write the stress tensor sector of the correlator in the lightcone limit at $\mathcal{O}(\mu^3)$ as

$$
\mathcal{G}^{(3)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^3}{[(1-z)(1-\bar{z})]^{\Delta_L}} \sum_{n,m=0}^{\infty} c[m,n] f_{9+2(n+m)}(z), \tag{5.39}
$$

with

$$
\begin{aligned}
c[m,n] =\big(& a_{333}p[3,3,m]p[3,6+2m,n] + a_{117}p[1,7,m]p[1,8+2m,n] \\
&+ a_{126}p[2,6,m]p[1,8+2m,n] + a_{135}p[3,5,m]p[1,8+2m,n] \\
&+ a_{225}p[2,5,m]p[2,7+2m,n] + a_{234}p[3,4,m]p[2,7+2m,n]\big),
\end{aligned}
\tag{5.40}
$$

where coefficients a_{ijk} are fixed in (5.26).

Comparing (5.39) with (3.45) we see that the contribution at $\mathcal{O}(\mu^3)$ comes from operators of the schematic form $:T_{\alpha\beta}T_{\gamma\delta}\partial_{\rho_1}\ldots\partial_{\rho_{2\ell}}T_{\mu\nu}:$. These operators have $\frac{\tau_{3,\min}}{2} + s = 9 + 2\ell$, where s is total spin $s = 6 + 2\ell$. The corresponding OPE coefficients of such operators will be a sum of all contributions in (5.39) for which $n + m = \ell$.

Now, one can write OPE coefficients of operators of type : $T_{\alpha\beta}T_{\gamma\delta}\partial_{\rho_1}\dots\partial_{\rho_{2\ell}}T_{\mu\nu}$: as

$$P^{(HH,LL);(3)}_{6,6+2\ell} = \sum_{n=0}^{\ell} c[\ell-n,n]. \tag{5.41}$$

Let us write a few of the coefficients explicitly here:

$$\begin{aligned}
\mu^3 P^{(HH,LL);(3)}_{6,6} &= \mu^3 \frac{\Delta_L(3024+\Delta_L(7500+\Delta_L(7310+143\Delta_L(25+7\Delta_L))))}{10378368000(\Delta_L-2)(\Delta_L-3)}, \\
\mu^3 P^{(HH,LL);(3)}_{6,8} &= \mu^3 \frac{\Delta_L(2688+\Delta_L(7148+\Delta_L(9029+13\Delta_L(464+231\Delta_L))))}{613476864000(\Delta_L-3)(\Delta_L-2)}, \\
\mu^3 P^{(HH,LL);(3)}_{6,10} &= \mu^3 \frac{\Delta_L(888+\Delta_L(2216+\Delta_L(3742+17\Delta_L(181+143\Delta_L))))}{9468531072000(\Delta_L-3)(\Delta_L-2)}.
\end{aligned} \tag{5.42}$$

We further find that $P^{(HH,LL);(3)}_{6,6}$ and $P^{(HH,LL);(3)}_{6,8}$ agree with the expression obtained holographically in [15].

5.3. *Minimal-twist double-stress tensors in six dimensions*

In this section we derive the contribution of minimal-twist double-stress tensors to the heavy-heavy-light-light correlator in the lightcone limit in $d = 6$. The method is analogous to the four-dimensional case described in Section 5.2.

From (5.7) we make the following ansatz for the stress tensor sector in the lightcone limit:

$$\begin{aligned}
\mathcal{G}^{(2)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} & \frac{(1-\bar{z})^4}{[(1-z)(1-\bar{z})]^{\Delta_L}} \times \\
& \Big(a_{17}f_1(z)f_7(z) + a_{26}f_2(z)f_6(z) + a_{35}f_3(z)f_5(z) + a_{44}f_4^2(z)\Big).
\end{aligned} \tag{5.43}$$

The S-channel conformal blocks in six dimensions in the limit $\Delta_H \to \infty$ are given by (3.54). In the lightcone limit $\bar{z} \to 1$ operators with $l \gg 1$ dominate and the blocks can be approximated by

$$(z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)} g^{(\Delta_{HL},-\Delta_{HL})}_{\Delta_H+\Delta_L+2n+\gamma,l}(z,\bar{z}) \simeq \frac{\bar{z}^l z^n p(\log z,\gamma)}{(1-z)^2}, \tag{5.44}$$

with p given by (5.10). Replacing the sum in (3.52) with an integral and inserting (5.44) we have

$$\mathcal{G}^{(2)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \sum_{n=0}^{\infty} \frac{z^n}{(1-z)^2} \int_0^{\infty} dl P^{(HL,HL)}_{n,l} \bar{z}^l p(\log z,\gamma). \tag{5.45}$$

As in $d=4$ one finds that terms proportional to $\log^i z$ with $i=2,3,\ldots,k$ at $\mathcal{O}(\mu^k)$, are determined by the OPE data at $\mathcal{O}(\mu^{k-1})$.

At $\mathcal{O}(\mu)$ we can use the known contribution from the stress tensor exchange (3.48) to derive the anomalous dimensions $\gamma_n^{(1)}$ and the OPE coefficients $P_n^{(1)}$ just as it was done in four dimensions. This is done by matching (5.45) order by order in the small z expansion. Using (3.56) one can integrate over spin. E.g. for $n=0,1,2,3$:

$$\begin{aligned}
\gamma_0^{(1)} &= -\frac{(\Delta_L-2)(\Delta_L-1)\Delta_L}{2}, \\
\gamma_1^{(1)} &= -\frac{(\Delta_L-1)\Delta_L(\Delta_L+10)}{2}, \\
\gamma_2^{(1)} &= -\frac{\Delta_L(\Delta_L+2)(\Delta_L+19)}{2}, \\
\gamma_3^{(1)} &= -\frac{(\Delta_L+4)(\Delta_L(\Delta_L+29)+30)}{2},
\end{aligned} \tag{5.46}$$

These anomalous dimensions agree with eq. (6.10) in [55]. Similarly, we obtain the following OPE coefficients:

$$\begin{aligned}
P_0^{(1)} &= -\frac{11(\Delta_L-2)(\Delta_L-1)\Delta_L}{12}, \\
P_1^{(1)} &= -\frac{(\Delta_L-1)\Delta_L(11\Delta_L+38)}{12}, \\
P_2^{(1)} &= -\frac{\Delta_L(22+\Delta_L(87+11\Delta_L))}{12}, \\
P_3^{(1)} &= -\frac{\Delta_L(202+\Delta_L(147+11\Delta_L))}{12}.
\end{aligned} \tag{5.47}$$

It is straightforward to continue to higher values of n.

Plugging (5.46) into (5.45) in the limit $1-\bar{z} \ll z \ll 1$ one finds the following contribution to the terms proportional to $\frac{z^p \log^2 z}{(1-\bar{z})^{\Delta_L-4}}$ at $\mathcal{O}(\mu^2)$:

$$\begin{aligned}
p=0 :& \frac{\Delta_L^2(\Delta_L-1)(\Delta_L-2)}{32(\Delta_L-3)(\Delta_L-4)}, \\
p=1 :& \frac{\Delta_L^2(\Delta_L-1)(\Delta_L+6)(\Delta_L+16)}{32(\Delta_L-3)(\Delta_L-4)} \\
p=2 :& \frac{\Delta_L^2(\Delta_L^4+46\Delta_L^3+599\Delta_L^2+1898\Delta_L+1056)}{64(\Delta_L-3)(\Delta_L-4)}, \\
p=3 :& \frac{\Delta_L^7+72\Delta_L^6+1651\Delta_L^5+13344\Delta_L^4+40180\Delta_L^3+41952{\Delta_L}^2+14400\Delta_L}{192(\Delta_L-3)(\Delta_L-4)}.
\end{aligned} \tag{5.48}$$

It is now straightforward to expand the ansatz (5.43) in the limit $1-\bar{z} \ll z \ll 1$, collect terms that behave as $z^p \log^2 z$ and compare them to the S-channel (5.48). This determines the coefficients:

$$
\begin{aligned}
a_{17} &= \frac{\Delta_L(\Delta_L+1)(\Delta_L+2)}{64064(\Delta_L-3)(\Delta_L-4)}, \\
a_{26} &= \frac{\Delta_L(-18+(-12+\Delta_L)\Delta_L)}{133056(\Delta_L-3)(\Delta_L-4)}, \\
a_{35} &= \frac{\Delta_L(\Delta_L-6)(\Delta_L-15)}{302400(\Delta_L-3)(\Delta_L-4)}, \\
a_{44} &= \frac{\Delta_L(\Delta_L-5)(\Delta_L-6)}{627200(\Delta_L-3)}.
\end{aligned}
\tag{5.49}
$$

One can consider higher values of p; eq. (5.49) is still the solution of the corresponding overconstrained system.

The double-stress tensor contribution to the correlator in the lightcone limit $\bar{z} \to 1$ is therefore given by

$$
\begin{aligned}
\mathcal{G}^{(2)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} & \frac{(1-\bar{z})^4}{[(1-z)(1-\bar{z})]^{\Delta_L}} \frac{\Delta_L}{(\Delta_L-3)(\Delta_L-4)} \left(\frac{1}{627200}\right) \\
\times \Big(& (\Delta_L-4)(\Delta_L-5)(\Delta_L-6) f_4^2(z) + \frac{56(\Delta_L-6)(\Delta_L-15)}{27} f_3(z) f_5(z) \\
& + \frac{1400(\Delta_L(\Delta_L-12)-18)}{297} f_2(z) f_6(z) \\
& + \frac{1400(\Delta_L+1)(\Delta_L+2)}{143} f_1(z) f_7(z) \Big).
\end{aligned}
\tag{5.50}
$$

Using (5.50) one can deduce the second order OPE data in the S-channel. The anomalous dimensions at this order can then be compared to the holographic calculations in [55] to reveal perfect agreement.

5.3.1. Exponentiation of minimal-twist multi-stress tensors in six dimensions

It is interesting to study whether the stress tensor sector of the correlator exponentiates in the lightcone limit

$$
\mathcal{G}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{1}{[(1-z)(1-\bar{z})]^{\Delta_L}} e^{\Delta_L \mathcal{F}(\mu;z,\bar{z})}, \tag{5.51}
$$

with $\mathcal{F}(\mu; z, \bar{z})$ a rational function of Δ_L that is of $\mathcal{O}(1)$ as $\Delta_L \to \infty$. In the lightcone limit $\mathcal{F}(\mu; z, \bar{z})$ admits an expansion

$$\mathcal{F}(\mu; z, \bar{z}) = \mu(1-\bar{z})^2 \mathcal{F}^{(1)}(z) + \mu^2 (1-\bar{z})^4 \mathcal{F}^{(2)}(z) + \mathcal{O}(\mu^3). \tag{5.52}$$

At $\mathcal{O}(\mu)$ one finds $\mathcal{F}^{(1)}(z) = \frac{\Gamma(\frac{6}{2}+1)^2}{4\Gamma(6+2)} f_4(z)$ from the stress tensor contribution. Using (5.50) we find

$$\mathcal{F}^{(2)}(z) = b_{17} f_1(z) f_7(z) + b_{26} f_2(z) f_6(z) + b_{35} f_3(z) f_5(z) + b_{44} f_4^2(z) \tag{5.53}$$

with

$$\begin{aligned} b_{17} &= \frac{a_{17}}{\Delta_L}, \\ b_{26} &= \frac{a_{26}}{\Delta_L}, \\ b_{35} &= \frac{a_{35}}{\Delta_L}, \\ b_{44} &= -\frac{4\Delta_L^2 - 31\Delta_L + 60}{313600(\Delta_L - 3)(\Delta_L - 4)}. \end{aligned} \tag{5.54}$$

From (5.54) we indeed see that the stress tensor sector of the correlator exponentiates at least to $\mathcal{O}(\mu^2)$ in $d = 6$.

5.3.2. OPE coefficients of minimal-twist double-stress tensors

In this section we decompose the stress tensor sector of the correlator (5.43) into a sum over minimal-twist double-stress tensors. The discussion follows that of Section 5.2.5.

Applying (5.38) to (5.50), we find that $a+b+2\ell = 8+2\ell$ which is $\frac{\tau_{2,\min}}{2}+s+2\ell$, with $\tau_{2,\min} = 8$ and $s = 4$ being the twist and spin of the simplest minimal-twist double-stress tensor operator $:T_{\mu\nu}T_{\rho\lambda}:$. Non-zero value of ℓ thus gives the contribution from operators of higher spin of the form $:T_{\mu\nu}\partial_{\rho_1}\ldots\partial_{\rho_{2\ell}}T_{\delta\lambda}:$, where no indices are contracted and only even spin operators contribute to the OPE between identical scalars.

It is now straightforward to write down the OPE coefficients for minimal-twist double-stress tensors in six dimensions. E.g. one finds for the lowest-spin operators the following OPE coefficients:

$$\begin{aligned} \mu^2 P_{8,4}^{(HH,LL);(2)} &= \mu^2 \frac{\Delta_L(600 + \Delta_L(1394 + \Delta_L(677 + 429\Delta_L)))}{269068800(\Delta_L - 3)(\Delta_L - 4)}, \\ \mu^2 P_{8,6}^{(HH,LL);(2)} &= \mu^2 \frac{\Delta_L(30 + \Delta_L(187 + \Delta_L(-120 + 143\Delta_L)))}{3430627200(\Delta_L - 3)(\Delta_L - 4)}, \\ \mu^2 P_{8,8}^{(HH,LL);(2)} &= \mu^2 \frac{\Delta_L(60 + \Delta_L(1382 + \Delta_L(-1857 + 1105\Delta_L)))}{657033721344(\Delta_L - 3)(\Delta_L - 4)}. \end{aligned} \tag{5.55}$$

For general spin we have ($s = 4 + 2\ell$)

$$P_{8,s}^{(HH,LL)} = \mu^2 \frac{c\Delta_L}{(\Delta_L - 3)(\Delta_L - 4)} (a_3 \Delta_L^3 + a_2 \Delta_L^2 + a_1 \Delta_L + a_0) \tag{5.56}$$

where

$$\begin{aligned}
c &= \frac{2^{-9-2s}\sqrt{\pi} s(s+2)\Gamma(s-1)}{(s-3)(s+4)(s+6)(s+8)(s+10)\Gamma(s+\frac{7}{2})}, \\
a_3 &= (s-2)s(s+2)(s+5)(s+7)(s+9), \\
a_2 &= -3(2880 + s(s+7)(-276 + s(s+7)(-56 + s(s+7)))), \\
a_1 &= 2(25920 + s(s+7)(3276 + s(s+7)(-80 + s(s+7)))), \\
a_0 &= 675 \times 2^7.
\end{aligned} \tag{5.57}$$

5.4. Discussion

In this section we consider the minimal-twist multi-stress tensor contributions to the heavy-heavy-light-light correlator of scalars in large C_T CFTs in even spacetime dimensions. We provide strong evidence for the conjecture that all such contributions are described by the ansatz (5.7) and determine the coefficients by performing a bootstrap procedure. In practice this is completed for twist-four double-stress tensors and twist-six triple-stress tensors in four dimensions as well as twist-eight double-stress tensors in six dimensions. In principle it is straightforward to use our technology to determine the coefficients $a_{i_1 \ldots i_k}$ to arbitrarily high order in μ; this must be related to the universality of the minimal-twist OPE coefficients.

In two dimensions the heavy-heavy-light-light Virasoro vacuum block exponentiates [see eq. (3.6)], with $\mathcal{F}(\mu; z)$ independent of Δ_L. In higher dimensions we observe a similar exponentiation with $\mathcal{F}(\mu; z, \bar{z})$ a rational function of Δ_L that remains $\mathcal{O}(1)$ as $\Delta_L \to \infty$. It would be interesting to see whether it is possible to write down a closed-form recursion formula for $\mathcal{F}(\mu; z, \bar{z})$. Solving such a recursion formula would give a higher-dimensional analogue of the two-dimensional Virasoro vacuum block.

An immediate technical question concerns CFTs in odd spacetime dimensions. We could not immediately generalize our results in this context – the ansatz in eq. (5.7) fails in odd dimensions. However, the heavy-light conformal blocks are known [12], so a similar approach should be feasible.

It would be interesting to study the regime of applicability of our results. We have not used holography; our main assumption is the ansatz (3.13), known to be true for holographic CFTs to $\mathcal{O}(\mu^2)$ in $d = 4$ [124]. Yet, our general expressions for the OPE coefficients agree with the OPE coefficients computed in some holographic examples [15]. What happens once one goes beyond holographic CFTs - will our ansatz need to be modified by the inclusion of terms suppressed by the gap or the central charge? We leave these questions for subsequent investigations.

Another interesting direction concerns the study of the bulk scattering phase-shift in the presence of a black hole background. In the context of higher dimensional CFTs, this problem was first considered in [55] where the gravitational expression was given to all orders in μ and the CFT computation was performed to $\mathcal{O}(\mu)$. Subsequently, $\mathcal{O}(\mu^2)$ was discussed in [12]. In [126] the $\mathcal{O}(\mu)$ contribution was exponentiated to yield the scattering phase shift in the presence of a shock-wave geometry. A CFT computation of the phase shift to all orders in μ is still lacking. This would in principle involve understanding Regge theory beyond the leading order. It will be interesting to see whether the results of this article could be helpful in this regard.

6. Stress tensor sector of conformal correlators

6.1. Summary of the results

In this section, we show that the stress tensor sector of the HHLL correlator in $d = 4$ can be written in terms of products of $f_a(z)$ functions defined as

$$f_a(z) = (1-z)^a {}_2F_1(a, a, 2a, 1-z). \tag{6.1}$$

The stress tensor sector of the HHLL correlator can be expanded in powers of μ and then in powers of $(1-\bar{z})$ as

$$\begin{aligned}\mathcal{G}(z,\bar{z}) = \sum_{k=0}^{\infty} \mu^k \mathcal{G}^{(k)}(z,\bar{z}) &= \frac{1}{((1-z)(1-\bar{z}))^{\Delta_L}} \\ &+ \sum_{k=1}^{\infty}\sum_{m=0}^{\infty} \mu^k (1-\bar{z})^{-\Delta_L+k+m} \mathcal{G}^{(k,m)}(z),\end{aligned} \tag{6.2}$$

where we have explicitly separated the contribution of the identity operator.[18] We explain how one can write $\mathcal{G}^{(k,m)}(z)$ for arbitrary k and m.

We write an ansatz for each $\mathcal{G}^{(k,m)}(z)$ with a few unknown coefficients and fix all, but a handful of them, via lightcone bootstrap. The undetermined coefficients correspond to the OPE coefficients of spin-0 and spin-2 exchanged operators. We further show that in holographic CFTs one can use the phase shift computed in the dual gravitational theory to reduce the set of undetermined parameters to the OPE coefficients of multi-stress tensors with spin zero.

Operators of non-minimal twist give a subleading contribution in the light-cone limit, $1-\bar{z} \ll 1$, which can be expressed as a sum of products of the functions $f_a(z)$ (times an appropriate power of $(1-\bar{z})$). This form is similar to the contribution of minimal-twist multi-stress tensor operators considered in [13]. While our method can be used to address the contribution of operators of arbitrary twist, here we focus on determining the specific contributions of operators with twist $\tau = 6, 8, 10$, at $\mathcal{O}(\mu^2)$ and $\tau = 8, 10$, at $\mathcal{O}(\mu^3)$.

At $\mathcal{O}(\mu)$, the only operator that contributes to the stress tensor sector of the correlator is the stress tensor and its contribution is completely fixed by

[18] The contribution of the identity operator is denoted with $k = 0$, schematically $(T_{\mu\nu})^0 = \mathbf{1}$.

conformal symmetry. In $d = 4$ its exact (to all orders in $\bar{z}$) contribution is given by

$$\mathcal{G}^{(1)}(z, \bar{z}) = \frac{1}{[(1-z)(1-\bar{z})]^{\Delta_L - 1}} \frac{\Delta_L}{120(\bar{z} - z)} \Big(f_3(z) - f_3(\bar{z}) \Big). \tag{6.3}$$

At $\mathcal{O}(\mu^2)$, the leading contribution in the lightcone limit, due to twist-four double-stress tensors, was evaluated in [124]

$$\begin{aligned} \mathcal{G}^{(2,0)}(z) =& \frac{1}{(1-z)^{\Delta_L}} \left(\frac{\Delta_L}{28800(\Delta_L - 2)} \right) \times \\ & \Big((\Delta_L - 4)(\Delta_L - 3) f_3^2(z) + \frac{15}{7}(\Delta_L - 8) f_2(z) f_4(z) \\ & + \frac{40}{7}(\Delta_L + 1) f_1(z) f_5(z) \Big). \end{aligned} \tag{6.4}$$

We show that the subleading contribution in the lightcone limit, due to twist-four and twist-six double-stress tensors, is given by

$$\begin{aligned} \mathcal{G}^{(2,1)}(z) = \frac{1}{(1-z)^{\Delta_L}} \Bigg(& \left(\frac{3-z}{2(1-z)} \right) \left(a_{33} f_3(z)^2 + a_{24} f_2(z) f_4(z) + a_{15} f_1(z) f_5(z) \right) \\ & + \left(b_{14} f_1(z) f_4(z) + c_{16} f_1(z) f_6(z) + c_{25} f_2(z) f_5(z) + c_{34} f_3(z) f_4(z) \right) \Bigg), \end{aligned} \tag{6.5}$$

with coefficients a_{mn} and c_{mn} given in (6.30). The coefficient b_{14} is non-universal and generically depends on the details of the theory. It corresponds to the OPE coefficient of twist-six double-stress tensor with spin $s = 2$

$$b_{14} = P^{(2)}_{8,2}, \tag{6.6}$$

obtained holographically in [15] and here, via the gravitational phase-shift calculation in (6.107).

The subsubleading contribution in the lightcone limit, due to twist-four, six and eight double-stress tensor operators, is

$$\begin{aligned} \mathcal{G}^{(2,2)}(z) = \frac{1}{(1-z)^{\Delta_L}} \Bigg(& \left(\frac{z(2z-7)+11}{6(z-1)^2} \right) (a_{33} f_3^2 + a_{24} f_2 f_4 + a_{15} f_1 f_5) \\ & + \left(\frac{2-z}{1-z} \right) (b_{14} f_1 f_4 + c_{16} f_1 f_6 + c_{25} f_2 f_6 + c_{34} f_3 f_4) + (d_{17} f_1 f_7 + d_{26} f_2 f_6 \\ & + d_{35} f_3 f_5 + d_{44} f_4^2 + e_{15} f_1 f_5 + g_{13} f_1 f_3) \Bigg), \end{aligned} \tag{6.7}$$

with coefficients d_{mn} given in (6.35). By f_a we mean $f_a(z)$ which we will use for brevity. The coefficients g_{13} and e_{15} are theory dependent and are related to the OPE coefficients of twist-eight double-stress tensors with spin $s = 0, 2$ by

$$\begin{aligned} g_{13} &= P^{(2)}_{8,0}, \\ e_{15} &= P^{(2)}_{10,2} - \frac{5}{252} P^{(2)}_{8,0}. \end{aligned} \tag{6.8}$$

These coefficients were also obtained by a gravitational computation in [15]. Here we have used the calculation of the phase shift in the dual gravitational theory to determine the OPE coefficient of the spin-2 operator, $P^{(2)}_{10,2}$, in (6.110).

The subsubsubleading contribution in the lightcone limit, due to double-stress tensors with twists $\tau = 4, 6, 8, 10$, is given by

$$\begin{aligned} \mathcal{G}^{(2,3)}(z) = \frac{1}{(1-z)^{\Delta_L}} \Bigg(& \left(\frac{z((13-3z)z-23)+25}{12(1-z)^3} \right) (a_{33} f_3^2 + a_{24} f_2 f_4 + a_{15} f_1 f_5) \\ & + \left(\frac{1}{(1-z)^2} + \frac{1}{1-z} + \frac{9}{10} \right) (b_{14} f_1 f_4 + c_{16} f_1 f_6 + c_{25} f_2 f_5 + c_{34} f_3 f_4) \\ & + \left(\frac{1}{1-z} + \frac{3}{2} \right) (d_{17} f_1 f_7 + d_{26} f_2 f_6 + d_{35} f_3 f_5 + d_{44} f_4^2 + e_{15} f_1 f_5 + g_{13} f_1 f_3 +) \\ & + g_{13} f_3 + (h_{18} f_1 f_8 + h_{27} f_2 f_7 + h_{36} f_3 f_6 + h_{45} f_4 f_5 + j_{16} f_1 f_6 + i_{14} f_1 f_4) \Bigg), \end{aligned} \tag{6.9}$$

with h_{mn} given in (6.41). The non-universal coefficients here are i_{14} and j_{16} which are related to the OPE coefficients of twist-ten double-stress tensor operators with spin $s = 0, 2$

$$\begin{aligned} i_{14} &= P^{(2)}_{10,0}, \\ j_{16} &= P^{(2)}_{12,2} - \frac{2}{99} P^{(2)}_{10,0}. \end{aligned} \tag{6.10}$$

The OPE coefficient $P^{(2)}_{12,2}$ is determined in (6.111) using the phase shift calculation in the dual gravitational theory. Non-universality is manifest through dependence on the Gauss-Bonnet coupling.

Using the results above, we also extract the OPE coefficients $P^{(2)}_{\Delta,s}$ of double-stress tensors of given twist. For $\tau = 6$:

$$\begin{aligned} P^{(2)}_{10+2\ell,4+2\ell} = & \frac{\sqrt{\pi} 2^{-4\ell-17} \Gamma(2n+7)}{(\ell+4)(\ell+5)(\ell+6)(2\ell+1)(2\ell+3)(2\ell+5)\Gamma\left(2\ell+\frac{13}{2}\right)} \\ & \times \frac{\Delta_L}{(\Delta_L-3)(\Delta_L-2)} (a_{1,\ell}\Delta_L^3 + b_{1,\ell}\Delta_L^2 + c_{1,\ell}\Delta_L + d_{1,\ell}), \end{aligned} \tag{6.11}$$

where $a_{1,\ell}$, $b_{1,\ell}$, $c_{1,\ell}$, $d_{1,\ell}$ can be found in (6.33). For $\tau = 8$:

$$P^{(2)}_{12+2\ell,4+2\ell} = \frac{\sqrt{\pi}\Delta_L 2^{-4\ell-19}\Gamma(2\ell+7)}{3(\Delta_L-4)(\Delta_L-3)(\Delta_L-2)(\ell+4)(\ell+5)} \times \frac{a_{2,\ell}\Delta_L^4 + b_{2,\ell}\Delta_L^3 + c_{2,\ell}\Delta_L^2 + d_{2,\ell}\Delta_L + e_{2,\ell}}{(\ell+6)(\ell+7)(2\ell+1)(2\ell+3)(2\ell+5)\Gamma\left(2\ell+\frac{15}{2}\right)}, \tag{6.12}$$

with $a_{2,\ell}$, $b_{2,\ell}$, $c_{2,\ell}$, $d_{2,\ell}$ and $e_{2,\ell}$ given in (6.38). Similarly for $\tau = 10$:

$$P^{(2)}_{14+2\ell,4+2\ell} = \frac{\sqrt{\pi}2^{-4\ell-22}\Gamma(2\ell+9)}{5(2\ell+1)(2\ell+3)(2\ell+5)(2\ell+7)\Gamma\left(2\ell+\frac{17}{2}\right)} \times \frac{\Delta_L(\Delta_L+1)(a_{3,\ell}\Delta_L^4 + b_{3,\ell}\Delta_L^3 + c_{3,\ell}\Delta_L^2 + d_{3,\ell}\Delta_L + e_{3,\ell})}{(\ell+5)(\ell+6)(\ell+7)(\ell+8)(\Delta_L-5)(\Delta_L-4)(\Delta_L-3)(\Delta_L-2)}, \tag{6.13}$$

with $a_{3,\ell}$, $b_{3,\ell}$, $c_{3,\ell}$, $d_{3,\ell}$ and $e_{3,\ell}$ expressed in terms of Δ_L in (6.44). Note that in all of these formulas $\ell \geq 0$ and, therefore, the OPE coefficients of operators with spin $s = 0, 2$ are not included here. It appears that at $\mathcal{O}(\mu^2)$, the OPE coefficients of all operators with spin $s \geq 4$ are universal in the sense that they only depend on Δ_L and C_T. On the other hand, the OPE coefficients of double-stress tensors with $s = 0, 2$ are non-universal.

At $\mathcal{O}(\mu^3)$, the leading contribution of twist-six triple-stress tensors in the lightcone limit, was computed in [13]

$$\mathcal{G}^{(3,0)}(z) = \frac{1}{(1-z)^{\Delta_L}}\Big(a_{117}f_1(z)^2 f_7(z) + a_{126}f_1(z)f_2(z)f_6(z) + a_{135}f_1(z)f_3(z)f_5(z) + a_{225}f_2(z)^2 f_5(z) + a_{234}f_2(z)f_3(z)f_4(z) + a_{333}f_3(z)^3\Big), \tag{6.14}$$

where the coefficients a_{ijk} can be found in (6.46).

The subleading contribution to the correlator is due to twist-eight and twist-six triple-stress tensors

$$\mathcal{G}^{(3,1)}(z) = \frac{1}{(1-z)^{\Delta_L}}\left(\left(\frac{2-z}{1-z}\right)(a_{117}f_1^2 f_7 + a_{126}f_1 f_2 f_6 + a_{135}f_1 f_3 f_5 + a_{225}f_2^2 f_5 + a_{234}f_2 f_3 f_4 + a_{333}f_3^3) + (b_{116}f_6 f_1^2 + c_{118}f_8 f_1^2 + c_{145}f_4 f_5 f_1 + c_{127}f_2 f_7 f_1 + c_{244}f_2 f_4^2 + c_{334}f_3^2 f_4 + c_{235}f_2 f_3 f_5 + c_{226}f_2^2 f_6)\right), \tag{6.15}$$

with b_{ijk} and c_{ijk} given in (B.2.1). Terms proportional to a_{ijk} come from the subleading contribution due to the minimal-twist triple-stress tensors in (6.14). Note that all of these coefficients are non-universal, since they depend on b_{14} from the $\mathcal{O}(\mu^2)$ result. Accordingly, no OPE coefficients of non-minimal-twist triple-stress tensors are universal.

A similar story holds for the subsubleading contribution to the correlator at $\mathcal{O}(\mu^3)$. This is due to multi-stress tensors with twist six, eight and ten and takes the following form

$$\begin{aligned}\mathcal{G}^{(3,2)}(z) = \frac{1}{(1-z)^{\Delta_L}}\Bigg(&\left(\frac{144z^2-448z+464}{160(z-1)^2}\right)(a_{117}f_1^2f_7 + a_{126}f_1f_2f_6 \\ &+ a_{135}f_1f_3f_5 + a_{225}f_2^2f_5 + a_{234}f_2f_3f_4 + a_{333}f_3^3) + \left(\frac{1}{1-z}+\frac{3}{2}\right)(b_{116}f_6f_1^2 \\ &+ c_{118}f_8f_1^2 + c_{145}f_4f_5f_1 + c_{127}f_2f_7f_1 + c_{244}f_2f_4^2 + c_{334}f_3^2f_4 + c_{235}f_2f_3f_5 \\ &+ c_{226}f_2^2f_6) + (d_{117}f_1^2f_7 + e_{115}f_1^2f_5 + g_{119}f_1^2f_9 + g_{128}f_1f_2f_8 + g_{155}f_1f_5^2 \\ &+ g_{227}f_2^2f_7 + g_{236}f_2f_3f_6 + g_{245}f_2f_4f_5 + g_{335}f_3^2f_5 + g_{344}f_3f_4^2)\Bigg),\end{aligned} \tag{6.16}$$

with d_{117} and g_{ijk} in (B.3.1)-(B.3.3) and e_{115} in (6.115).

We further explain how one can write an ansatz for the correlator at arbitrary order in μ and the lightcone expansion. All unknown coefficients in the ansatz, except those that correspond to OPE coefficients of spin-0 and spin-2 operators, can be fixed by means of the lightcone bootstrap. We further show that in holographic CFTs one can use the phase shift computed in the dual gravitational theory to reduce the set of undetermined parameters to the OPE coefficients of multi-stress tensors with spin zero. Our results for these OPE coefficients precisely match those in [15] whenever available in the latter.

The OPE coefficients of multi-stress tensors can also be calculated using the Lorentzian inversion formula as in [128]. In order to determine for which operators the formula can be applied, one should consider the behavior of the correlation function in the Regge limit. The Regge behavior of the correlator at $\mathcal{O}(\mu^k)$ is $1/\sigma^k$, implying that the Lorentzian inversion formula can be used to extract the OPE coefficients of the operators with spin $s > k+1$. Accordingly, already at $\mathcal{O}(\mu^3)$, fixing the relevant OPE coefficients by combining an ansatz with the lightcone bootstrap allows one to determine more OPE data compared to those obtained with the sole use of the Lorentzian inversion formula. We explicitly check that it is not possible to extract the OPE coefficient of a triple-stress tensor with spin $s = 4$ and twist $\tau = 8$ using the Lorentzian inversion formula. Note, however, that this coefficient is completely determined in this article (where an ansatz is additionally employed).

6.1.1. Outline

This section is organized as follows. In Section 6.2 we analyze the stress tensor sector of the correlator at $\mathcal{O}(\mu^2)$, where we compute the subleading, subsubleading and subsubsubleading contributions in the lightcone expansion. We also compute the OPE coefficients of double-stress tensors with twist $\tau = 6, 8, 10$ and spin $s > 2$. In Section 6.3, we analyze the stress tensor sector of the correlator at $\mathcal{O}(\mu^3)$, where we explicitly calculate the subleading and subsubleading contributions in the lightcone expansion. In Section 6.4, we investigate the Gauss-Bonnet dual gravitational theory and give additional evidence for the universality of the OPE coefficients of minimal-twist multi-stress tensors using the phase shift calculation. Furthermore, we calculate the OPE coefficients of double- and triple-stress tensors with spin $s = 2$ (up to undetermined spin zero data). In Section 6.5, we show how one can use the Lorentzian inversion formula in order to extract the OPE coefficients of double-stress tensors with twist $\tau = 4, 6$. We discuss our results in Section 6.6. Appendix B.1 contains certain relations that products of f_a functions satisfy, while Appendices B.2 and B.3 contain explicit expressions for the coefficients which determine the correlator in subleading and subsubleading lightcone order at $\mathcal{O}(\mu^3)$. Several OPE coefficients of twist-eight triple-stress tensors are listed in Appendix B.4. In Appendix B.5 we clarify the relationship between the scattering phase shift as defined in [55] and the deflection angle and finally, in Appendix B.6 we explicitly write some of the S-channel anomalous dimensions at $\mathcal{O}(\mu^2)$ and we investigate their relation with the phase shift.

6.2. Double-stress tensors in four dimensions

In this section, we analyze the stress tensor sector of the HHLL correlator at $\mathcal{O}(\mu^2)$ in $d = 4$. The operators that contribute at this order in the T-channel are the double-stress tensors. Here, we investigate the subleading contributions that are coming from families of operators with nonminimal twist, specifically, $\tau_{2,1} = 6$, $\tau_{2,2} = 8$ and $\tau_{2,3} = 10$, according to (3.63).

The dominant contribution in the lightcone limit at $\mathcal{O}(\mu^2)$ was calculated in [124]. It comes from the operators with minimal twist $\tau_{2,\min} = 4$ and they are of the schematic form $: T_{\mu\nu}\partial_{\alpha_1} \dots \partial_{\alpha_{2\ell}} T_{\rho\sigma} :$. These operators have conformal dimension $\Delta = 8 + 2\ell$ and spin $s = 4 + 2\ell$. The result is [124]

$$\mathcal{G}^{(2,0)}(z) = \frac{1}{(1-z)^{\Delta_L}} \left(\frac{\Delta_L}{28800(\Delta_L - 2)} \right) \times \\ \Big((\Delta_L - 4)(\Delta_L - 3) f_3^2(z) + \frac{15}{7}(\Delta_L - 8) f_2(z) f_4(z) + \frac{40}{7}(\Delta_L + 1) f_1(z) f_5(z) \Big), \tag{6.17}$$

where $f_a(z) = (1-z)^a {}_2F_1(a, a, 2a, 1-z)$.

6.2.1. Twist-six double-stress tensors

Twist-six double-stress tensors contribute at $\mathcal{O}(\mu^2)$ and at subleading order in the lightcone expansion $\sim (1-\bar{z})^{-\Delta_L+3}$ as $\bar{z} \to 1$. As shown in this section, this contribution again takes a particular form with a few undetermined coefficients which, except for a single one, can be fixed using lightcone bootstrap. The undetermined data is shown to correspond to a single OPE coefficient due to the exchange of the twist-six and spin-two double-stress tensor $: T_\mu{}^\rho T_{\rho\nu} :$.

We will now motivative an ansatz for the subleading contribution to the stress tensor sector at $\mathcal{O}(\mu^2)$. Let us focus first on corrections due to the leading lightcone contribution of twist-four double-stress tensors. These corrections originate from subleading terms in the lightcone expansion of the conformal blocks in (3.59). Note however that they are purely kinematical and do not contain any new data. Explicitly, the subleading corrections to the blocks of twist-four double-stress tensors are given by

$$g^{(0,0)}_{4,s}(1-z, 1-\bar{z}) \underset{\bar{z}\to 1}{\approx} (1-\bar{z})^2 \left(1 + (1-\bar{z}) \left(\frac{3-z}{2(1-z)} \right) + \mathcal{O}\left((1-\bar{z})^2\right) \right) f_{\frac{\beta}{2}}(z) \\ -(1-\bar{z})^{s+3} \left(1 + (1-\bar{z}) \left(\frac{s+2}{2} + \frac{1}{1-z} \right) + \mathcal{O}((1-\bar{z})^2) \right) f_1(z). \tag{6.18}$$

Since we are interested in the subleading contribution, i.e. terms that behave as $(1-\bar{z})^3$ as $\bar{z} \to 1$ in (6.18), only the first line in (6.18) needs to be considered. (Note that $s \geq 4$ for minimal-twist double-stress tensors.)

Next, consider the contribution of twist-six double-stress tensors. Recall that the form of the minimal-twist double-stress tensors' contribution to (6.17) can be motivated by decomposing products of the type $f_a(z)f_b(z)$ in terms of the lightcone conformal blocks. This decomposition is explicitly given by [124]:

$$f_a(z)f_b(z) = \sum_{\ell=0}^{\infty} p(a,b,\ell) f_{a+b+2\ell}(z), \tag{6.19}$$

where

$$\begin{aligned} p(a,b,\ell) = &\frac{\Gamma(a+b+2\ell)}{\Gamma\left(a+b+2\ell-\frac{1}{2}\right)} \times \\ &\frac{2^{-4\ell}\Gamma\left(a+\frac{1}{2}\right)\Gamma\left(b+\frac{1}{2}\right)\Gamma\left(\ell+\frac{1}{2}\right)\Gamma(a+\ell)\Gamma(b+\ell)\Gamma\left(a+b+\ell-\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(a)\Gamma(b)\Gamma(\ell+1)\Gamma\left(a+\ell+\frac{1}{2}\right)\Gamma\left(b+\ell+\frac{1}{2}\right)\Gamma(a+b+\ell)}. \end{aligned} \tag{6.20}$$

Using the leading behavior of the conformal blocks (6.18) in the lightcone limit, it was found that $a+b+2\ell$ should be identified with $\frac{\beta}{2} = \frac{\Delta+s}{2}$. In order to reproduce twist-six double-stress tensors of the form $:T_{\mu\nu}\partial^2\partial_{\alpha_1}\dots\partial_{\alpha_{2\ell}}T_{\rho\sigma}:$ we should therefore consider products f_af_b with $a+b=7$. Likewise, to take into account operators of the form $:T_{\mu\beta}\partial_{\alpha_1}\dots\partial_{\alpha_{2\ell}}T^{\beta}{}_{\nu}:$ we include products f_af_b with $a+b=5$.

From the arguments above, we make the following ansatz for the subleading correction in the lightcone expansion due to double-stress tensors:

$$\begin{aligned} \mathcal{G}^{(2,1)}(z) = &\frac{1}{(1-z)^{\Delta_L}}\Bigg(\left(\frac{3-z}{2(1-z)}\right)(a_{33}f_3(z)^2 + a_{24}f_2(z)f_4(z) \\ &+ a_{15}f_1(z)f_5(z)) + (b_{14}f_1(z)f_4(z) + b_{23}f_2(z)f_3(z) + c_{16}f_1(z)f_6(z) \\ &+ c_{25}f_2(z)f_5(z) + c_{34}f_3(z)f_4(z))\Bigg), \end{aligned} \tag{6.21}$$

where b_{ij}, c_{ij} are coefficients that will be determined using lightcone bootstrap and encode the contribution from twist-six double-stress tensors. Once b_{ij} and c_{ij} are determined, one can use the decomposition in (6.19) to read off the OPE coefficients of twist-six double-stress tensors with any given spin. Moreover, a_{ij} in (6.21) are coefficients that can be read off from the minimal-twist contribution in (6.17) and do therefore not contain any new information.

We proceed with the S-channel calculation to fix the unknown coefficients in (6.21). Let us first mention that the products of $f_a(z)$ functions in the second line of (6.21) are not linearly independent as one can see from (A.1), so we set $b_{23} = 0$. Moreover, the coefficients a_{ij} must be the same as in (6.17). We will momentarily keep them undetermined to have an extra consistency check of our calculation.

In the S-channel we have double-trace operators of the form $:\mathcal{O}_H \partial^{2n} \partial^l \mathcal{O}_L:$ with conformal dimension $\Delta = \Delta_H + \Delta_L + 2n + l + \gamma_{n,l}$. The relevant anomalous dimensions $\gamma_{n,l}$ and OPE coefficients are given in (3.25) and (3.69) ($k = 2$ in this case). In the lightcone limit, the dominant contribution comes from operators with large spin l, $l \gg n$. The zeroth order OPE coefficients are given by (3.70). The conformal blocks of these operators in the limit $1 - \bar{z} \ll z \ll 1$ are

$$g_{n,l}^{(\Delta_{HL}, -\Delta_{HL})}(z, \bar{z}) \approx \frac{(z\bar{z})^{\frac{\Delta_H + \Delta_L + \gamma(n,l)}{2}}}{\bar{z} - z} z^n \bar{z}^{l+n+1}. \tag{6.22}$$

We first need to fix the OPE data at $\mathcal{O}(\mu)$. Coefficients $\gamma_n^{(1,p)}$ and $P_n^{(1,p)}$ can be determined for every p and n by matching the S-channel correlator with the correlator in the T-channel at $\mathcal{O}(\mu)$. This is just the stress tensor block times its OPE coefficient and it is known for arbitrary z and $\bar{z}$. As we saw earlier

$$(\bar{z} - z)\mathcal{G}^{(1)}(z, \bar{z}) = \frac{1}{[(1-z)(1-\bar{z})]^{\Delta_L - 1}} \frac{\Delta_L}{120} (f_3(z) - f_3(\bar{z})). \tag{6.23}$$

Expanding (6.23) near $\bar{z} \to 1$ leads to

$$(\bar{z} - z)\mathcal{G}^{(1)}(z, \bar{z}) = \frac{-1 + \bar{z}}{((1-z)(1-\bar{z}))^{\Delta_L}} \Bigg(\Delta_L \left(\frac{3}{4}(1+z) + \frac{1 + z(z+4)}{4(1-z)} \log(z) \right) \\ \sum_{p=1}^{\infty} \frac{\Delta_L (p-2)(p-1)(1-z)}{4p(p+1)(p+2)} (1-\bar{z})^p \Bigg). \tag{6.24}$$

On the other hand, we expand the integrand of (3.67) up to the $\mathcal{O}(\mu)$, integrate this expansion over l, and then expand in the lightcone limit $\bar{z} \to 1$ to obtain a result of the form

$$(\bar{z} - z)\mathcal{G}^{(1)}(z, \bar{z}) = \frac{1}{(1-\bar{z})^{\Delta_L - 1}} \sum_{p=0}^{\infty} \left(\sum_{n=0}^{\infty} r_{n,p}(z) z^n (1-\bar{z})^p \right). \tag{6.25}$$

The functions $r_{n,p}(z)$ can be explicitly calculated. Here $r_{n,0}(z)$, $r_{n,1}(z)$ and $r_{n,2}(z)$ are given by

$$
\begin{aligned}
r_{n,0}(z) &= \frac{\Gamma(\Delta_L+n-1)}{2\Gamma(\Delta_L)\Gamma(n+1)}\left(2P_n^{(1,0)}+\log(z)\gamma_n^{(1,0)}\right),\\
r_{n,1}(z) &= \frac{\Gamma(\Delta_L+n-1)}{2\Gamma(\Delta_L)\Gamma(n+1)(\Delta_L-2)}\Big(2(P_n^{(1,0)}+P_n^{(1,1)})-(\Delta_L-2)\gamma_n^{(1,0)}\\
&\qquad+\log(z)(\gamma_n^{(1,0)}+\gamma_n^{(1,1)})\Big),\\
r_{n,2}(z) &= \frac{\Gamma(\Delta_L+n-1)}{2(\Delta_L-2)(\Delta_L-3)\Gamma(\Delta_L)\Gamma(n+1)}\Big(2(\Delta_L+n-1)P_n^{(1,0)}\\
&\quad+2(\Delta_L+n)P_n^{(1,1)}+2P_n^{(1,2)}-\frac{1}{2}(\Delta_L-3)(\Delta_L\gamma_n^{(1,0)}+2\gamma_n^{(1,1)})\\
&\quad+\log(z)((\Delta_L+n-1)\gamma_n^{(1,0)}+(\Delta_L+n)\gamma_n^{(1,1)}+\gamma_n^{(1,2)})\Big).
\end{aligned}
\tag{6.26}
$$

Similarly, one can calculate any $r_{n,p}(z)$ for arbitrary p. In each $r_{n,p}(z)$ the z-dependence enters only through a single logarithmic term as in (6.26). In order to extract the OPE data we match (6.24) and (6.25) and obtain the following relations

$$
\begin{aligned}
\sum_{n=0}^{\infty} z^n r_{n,0}(z) &= -\frac{\Delta_L}{(1-z)^{\Delta_L}}\left(\frac{3}{4}(1+z)+\frac{1+z(z+4)}{4(1-z)}\log(z)\right),\\
\sum_{n=0}^{\infty} z^n r_{n,p}(z) &= -\frac{\Delta_L}{(1-z)^{\Delta_L}}\frac{(p-2)(p-1)(1-z)}{4p(p+1)(p+2)},
\end{aligned}
\tag{6.27}
$$

for $p \geq 1$. To solve these equations, we start from the first line, expand the right-hand side in $z \to 0$ limit and match term by term on both sides. From terms with $\log(z)$ we extract the $\gamma_n^{(1,0)}$ and from terms without $\log(z)$, we extract the $P_n^{(1,0)}$. We move on to $p=1$ case, where we again expand the right-hand side of the second line in (6.27) in $z \to 0$ limit. Using $\gamma_n^{(1,0)}$ and $P_n^{(1,0)}$, we extract $\gamma_n^{(1,1)}$ and $P_n^{(1,1)}$. Straightforwardly, one can continue this process and extract OPE data for any value of p.

By proceeding with this calculation to high enough values and p one can notice that there is a simple expression for $\gamma_n^{(1,p)}$ given by

$$
\gamma_n^{(1,p)} = (-1)^{p+1}\left(\frac{1}{2}(\Delta_L-1)\Delta_L+3n^2-3(1-\Delta_L)n\right), \tag{6.28}
$$

for all $p \geq 0$ and $n \geq 0$. Note that for $p = 0$ this expression agrees with the one in [128]. There is no similar expression for $P_n^{(1,p)}$ so we list results for first p-s:

$$
\begin{aligned}
P_n^{(1,0)} =& -\frac{3}{4}(\Delta_L - 1)\Delta_L - \frac{3\Delta_L n}{2}, \\
P_n^{(1,1)} =& 3(n-1)n - \frac{1}{4}\Delta_L\left(\Delta_L\left(\Delta_L + 6n - 6\right) + 6(n-4)n + 5\right), \\
P_n^{(1,2)} =& \frac{1}{8}(\Delta_L(\Delta_L(\Delta_L^2 + 8n\Delta_L + 6n(3n-1) - 13) \\
& + 2(n(3n(2n-5) - 25) + 6)) - 12n(2n^2 + n - 3)), \\
P_n^{(1,3)} =& \frac{1}{120}(180n(n(3 - (n-3)n) + 5) - 234)\Delta_L + 3n(n^3 + n^2 - 2) \\
& + \frac{1}{120}\Delta_L^2(-\Delta_L(\Delta_L\left(11\Delta_L + 90n - 20\right) + 90n(3n-1) + 55) \\
& + 90(3 - 4n)n^2 + 280).
\end{aligned}
\tag{6.29}
$$

After the calculation of the OPE data at $\mathcal{O}(\mu)$, one can fix the coefficients in the ansatz (6.21) by expanding the integrand of (3.67) up to $\mathcal{O}(\mu^2)$ and then integrating the obtained expression over l. The result of the integration is expanded near $\bar{z} \to 1$ and we collect the term that behaves as $(1-\bar{z})^{-\Delta_L+3}$. It depends on z, n and OPE data $P_n^{(k,p)}$ and $\gamma_n^{(k,p)}$ for $k = 1, 2$ and $p = 0, 1$, but we are interested only in the part of this term that contains $\log^2(z)$. This part only depends on OPE data at $\mathcal{O}(\mu)$, so it will be completely determined. We collect terms that behave as $(1-\bar{z})^{-\Delta_L+3}\log^2(z)z^m$. By expanding the ansatz (6.21) near $z \to 0$ we can collect terms that behave as $\log^2(z)z^m$ and by matching these to the ones calculated through S-channel, we obtain a system of linear equations for the coefficients in the ansatz. This system will be over-determined by taking m to be large enough. Solving it for $m \leq 20$, we obtain

$$
\begin{aligned}
a_{33} &= \frac{(\Delta_L-4)(\Delta_L-3)\Delta_L}{28800(\Delta_L-2)}, \\
a_{24} &= \frac{(\Delta_L-8)\Delta_L}{13440(\Delta_L-2)}, \\
a_{15} &= \frac{\Delta_L(\Delta_L+1)}{5040(\Delta_L-2)}, \\
c_{16} &= \frac{25}{396}b_{14} + \frac{\Delta_L\left(\Delta_L\left(\Delta_L\left(83-7\Delta_L\right)+158\right)+108\right)}{3193344\left(\Delta_L-3\right)\left(\Delta_L-2\right)}, \\
c_{25} &= -\frac{1}{12}b_{14} + \frac{\Delta_L\left(\Delta_L\left(\Delta_L\left(\Delta_L+19\right)-146\right)-108\right)}{1451520\left(\Delta_L-3\right)\left(\Delta_L-2\right)}, \\
c_{34} &= \frac{\left(\Delta_L-4\right)\Delta_L\left(11\left(\Delta_L-4\right)\Delta_L-27\right)}{2419200\left(\Delta_L-3\right)\left(\Delta_L-2\right)}.
\end{aligned}
\tag{6.30}
$$

As expected, the coefficients a_{mn} are identical to those in (6.17). We are left with one undetermined coefficient. This is perhaps not surprising since we know from [15] that the OPE coefficients of the subleading twist multi-stress tensor operators are not universal. This non-universality is introduced in our correlator through coefficient b_{14}. One can check that after inserting (6.30) to (6.21) the term that multiplies the unknown coefficient b_{14} corresponds to the lightcone limit of the conformal block of the operator with dimension $\Delta = 8$ and spin $s = 2$. We thus conclude that b_{14} is the OPE coefficient of $:T_{\mu\alpha}T^{\alpha}{}_{\nu}:$,

$$
b_{14} = P^{(2)}_{8,2}. \tag{6.31}
$$

Now, using (6.19) we can write the T-channel OPE coefficients for the remaining double-stress tensor operators with twist $\tau_{2,1} = 6$ and conformal spin $\Delta + s \geq 14$. Explicitly, these are found to be given by

$$
\begin{aligned}
P^{(2)}_{10+2\ell,4+2\ell} = & \frac{\sqrt{\pi}2^{-4\ell-17}\Gamma(2\ell+7)}{(\ell+4)(\ell+5)(\ell+6)(2\ell+1)(2\ell+3)(2\ell+5)\Gamma\left(2\ell+\frac{13}{2}\right)} \\
& \times \frac{\Delta_L}{(\Delta_L-3)(\Delta_L-2)}(a_{1,\ell}\Delta_L^3 + b_{1,\ell}\Delta_L^2 + c_{1,\ell}\Delta_L + d_{1,\ell}),
\end{aligned}
\tag{6.32}
$$

where

$$
\begin{aligned}
a_{1,\ell} &= (\ell+2)(2\ell+9)(\ell(2\ell+13)+9), \\
b_{1,\ell} &= 144 - 2\ell(2\ell+13)(\ell(2\ell+13)+12), \\
c_{1,\ell} &= \ell(2\ell+13)(\ell(2\ell+13)+33)+558, \\
d_{1,\ell} &= 216.
\end{aligned}
\tag{6.33}
$$

Here $\ell \geq 0$ and $P^{(2)}_{\Delta,s}$ is the sum of OPE coefficients of all operators with conformal dimension Δ and spin s. There is no way to distinguish operators with the same quantum numbers Δ and s at this level in the large C_T expansion. This type of degeneracy occurs for each conformal spin greater than 10 for twist $\tau_{2,1} = 6$. Also, perfect agreement between (6.32) and all the OPE coefficients of double-stress tensor operators of twist $\tau_{2,1} = 6$ and spin $s > 2$ calculated in [15] is observed. Note that $P^{(2)}_{8,2}$ can not be found from (6.32) by setting $\ell = -1$, this would not agree with the result in [15]. In Section 6.5 we rederive (6.32) using the Lorentzian inversion formula.

6.2.2. Twist-eight double-stress tensors

We follow the same logic as in the previous section in order to write the subsubleading part of the stress tensor sector of the HHLL correlator in the lightcone limit at $\mathcal{O}(\mu^2)$. This part scales as $(1-\bar{z})^{-\Delta_L+4}$. Here, we include contributions coming from operators with twist $\tau_{2,2} = 8$. These operators can be grouped in three families and they are schematically written as $:T_{\mu\nu}(\partial^2)^2\partial_{\alpha_1}\ldots\partial_{\alpha_{2\ell}}T_{\rho\sigma}:$ with $\Delta = 12 + 2\ell$ and $s = 4 + 2\ell$, $:T_{\mu\beta}\partial^2\partial_{\alpha_1}\ldots\partial_{\alpha_{2\ell}}T^{\beta}{}_{\nu}:$ with $\Delta = 10 + 2\ell$ and $s = 2 + 2\ell$ and finally $:T_{\beta\gamma}\partial_{\alpha_1}\ldots\partial_{\alpha_{2\ell}}T^{\beta\gamma}:$ with $\Delta = 8 + 2\ell$ and $s = 2\ell$. Subtleties with regard to the contributions of the different families are discussed in Section 3.4.3.1.

Once more, we need to include the contributions of lower twist operators, i.e. by expanding their conformal blocks as $\bar{z} \to 1$ up to order $(1-\bar{z})^4$ and collect the additional z dependence. Accordingly, we write the following ansatz

$$\begin{aligned}\mathcal{G}^{(2,2)}(z) = \frac{1}{(1-z)^{\Delta_L}}\Bigg(&\left(\frac{z(2z-7)+11}{6(z-1)^2}\right)(a_{33}f_3^2 + a_{24}f_2f_4 + a_{15}f_1f_5) \\ &+ \left(\frac{2-z}{1-z}\right)(b_{14}f_1f_4 + c_{16}f_1f_6 + c_{25}f_2f_6 + c_{34}f_3f_4) \\ &+ (d_{17}f_1f_7 + d_{26}f_2f_6 + d_{35}f_3f_5 + d_{44}f_4^2 + e_{15}f_1f_5 + e_{24}f_2f_4 \\ &+ e_{33}f_3^2 + g_{13}f_1f_3 + g_{22}f_2^2)\Bigg),\end{aligned} \tag{6.34}$$

where f_a means $f_a(z)$. Coefficients a_{mn} and c_{mn} are already calculated, while b_{14} is undetermined from the bootstrap. The linear dependence between certain products of $f_a(z)$ functions (for more details see Appendix B.1, in particular (A.2)) allows us to set three coefficients to zero, e.g., $g_{22} = 0$, $e_{33} = 0$ and $e_{24} = 0$.

To fix the unknown coefficients in (6.34) we match terms that behave as $(1-\bar{z})^{-\Delta_L+4} z^m \log^2 z$ from the S-channel calculation of the correlator to terms with the same behavior in (6.34) for small z. For the S-channel calculation, we need the OPE data at $\mathcal{O}(\mu)$ up to $p=2$, given by (6.28) and (6.29). We obtain an over-constrained system of linear equations, whose solution is

$$\begin{aligned}
d_{17} &= \frac{9e_{15}}{143} + \frac{5g_{13}}{4004} + \frac{\Delta_L\left(\Delta_L\left(\Delta_L\left(\Delta_L\left(232-17\Delta_L\right)+1009\right)+1908\right)+1008\right)}{115315200\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}, \\
d_{26} &= -\frac{e_{15}}{12} + \frac{5g_{13}}{1386} - \frac{\Delta_L\left(\Delta_L\left(\left(\Delta_L-7\right)\Delta_L\left(11\Delta_L-179\right)+3636\right)+2736\right)}{119750400\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}, \\
d_{35} &= -\frac{g_{13}}{180} + \frac{\Delta_L\left(\Delta_L\left(\left(\Delta_L-7\right)\Delta_L\left(37\Delta_L-13\right)+1332\right)+3312\right)}{108864000\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}, \\
d_{44} &= \frac{\left(\Delta_L-6\right)\Delta_L\left(\Delta_L+2\right)}{9408000\left(\Delta_L-2\right)}.
\end{aligned} \tag{6.35}$$

The undetermined coefficients g_{13} and e_{15} are related to the T-channel OPE coefficients $P^{(2)}_{8,0}$ and $P^{(2)}_{10,2}$ by the following relations

$$\begin{aligned}
g_{13} &= P^{(2)}_{8,0}, \\
e_{15} &= P^{(2)}_{10,2} - \frac{5}{252} P^{(2)}_{8,0}.
\end{aligned} \tag{6.36}$$

Here $P^{(2)}_{8,0}$ is the T-channel OPE coefficient of the operator of the schematic form $:T_{\alpha\beta}T^{\alpha\beta}:$, while $P^{(2)}_{10,2}$ is related to the OPE coefficients of the operators $:T_{\alpha\beta}\partial_{\mu_1}\partial_{\mu_2}T^{\alpha\beta}:$ and $:T_{\mu\alpha}\partial^2 T^{\alpha}{}_{\nu}:$ which have the same quantum numbers Δ and s and are thus indistinguishable at this order in large C_T expansion. After inserting (6.36) and (6.35) into (6.34) one can check that both $P^{(2)}_{8,0}$ and $P^{(2)}_{10,2}$ will be multiplied by the relevant lightcone conformal blocks.

Exactly as in the previous section, we can now extract the OPE coefficients $P^{(2)}_{\Delta,s}$ for operators with twist $\tau_{2,2} = 8$ and $\Delta = 12 + 2\ell$, $s = 4 + 2\ell$, for $\ell \geq 0$[19]

$$\begin{aligned}
P^{(2)}_{12+2\ell,4+2\ell} =& \frac{\sqrt{\pi}\Delta_L 2^{-4\ell-19}\Gamma(2\ell+7)}{3(\Delta_L-4)(\Delta_L-3)(\Delta_L-2)(\ell+4)(\ell+5)} \\
&\times \frac{a_{2,\ell}\Delta_L^4 + b_{2,\ell}\Delta_L^3 + c_{2,\ell}\Delta_L^2 + d_{2,\ell}\Delta_L + e_{2,\ell}}{(\ell+6)(\ell+7)(2\ell+1)(2\ell+3)(2\ell+5)\Gamma\left(2\ell+\frac{15}{2}\right)},
\end{aligned} \tag{6.37}$$

[19] For each $\Delta = 12 + 2\ell$ and $s = 4 + 2\ell$ with $\ell \geq 0$ there is a triple degeneracy, because all three families of operators with twist $\tau_{2,2} = 8$ will be mixed.

where

$$\begin{aligned}
a_{2,\ell} &= \ell(2\ell+15)(\ell(2\ell+15)(\ell(2\ell+15)+59)+1084)+6012, \\
b_{2,\ell} &= 14004 - 2\ell(2\ell+15)(\ell(2\ell+15)(\ell(2\ell+15)+32)-131), \\
c_{2,\ell} &= \ell(2\ell+15)(\ell(2\ell+15)(\ell(2\ell+15)+113)+4594)+60984, \\
d_{2,\ell} &= 216(11\ell(2\ell+15)+302), \\
e_{2,\ell} &= 864(\ell(2\ell+15)+34).
\end{aligned} \tag{6.38}$$

It is quite remarkable that these OPE coefficients are fixed purely by the bootstrap.

6.2.3. Twist-ten double-stress tensors

Now we want to go one step further and analyze the subsubsubleading contribution to the stress tensor sector of the HHLL correlator. This contribution scales as $(1-\bar{z})^{-\Delta_L+5}$ in the lightcone limit. We have to take in to account the double-stress tensor operators of twist $\tau_{2,3} = 10$ in order to calculate this contribution. These operators can again be grouped in three families of the schematic form $:T_{\mu\nu}(\partial^2)^3\partial_{\alpha_1}\dots\partial_{\alpha_{2\ell}}T_{\rho\sigma}:$ with $\Delta = 14+2\ell$ and $s = 4+2\ell$, $:T_{\mu\beta}(\partial^2)^2\partial_{\alpha_1}\dots\partial_{\alpha_{2\ell}}T^{\beta}{}_{\nu}:$ with $\Delta = 12+2\ell$ and $s = 2+2\ell$ and finally $:T_{\beta\gamma}\partial^2\partial_{\alpha_1}\dots\partial_{\alpha_{2\ell}}T^{\beta\gamma}:$ with $\Delta = 10+2\ell$ and $s = 2\ell$.

In order to include contributions from lower twist operators we have to expand their conformal blocks up to $(1-\bar{z})^5$ for $\bar{z}\to 1$. The ansatz takes the following form

$$\begin{aligned}
\mathcal{G}^{(2,3)}(z) = \frac{1}{(1-z)^{\Delta_L}} \Bigg(&\left(\frac{z((13-3z)z-23)+25}{12(1-z)^3}\right)(a_{33}f_3^2 + a_{24}f_2f_4 + a_{15}f_1f_5) \\
&+ \left(\frac{1}{(1-z)^2} + \frac{1}{1-z} + \frac{9}{10}\right)(b_{14}f_1f_4 + c_{16}f_1f_6 + c_{25}f_2f_5 + c_{34}f_3f_4) \\
&+ \left(\frac{1}{1-z} + \frac{3}{2}\right)(d_{17}f_1f_7 + d_{26}f_2f_6 + d_{35}f_3f_5 + d_{44}f_4^2 + e_{15}f_1f_5 \\
&+ g_{13}f_1f_3) - g_{13}f_3 + (h_{18}f_1f_8 + h_{27}f_2f_7 + h_{36}f_3f_6 + h_{45}f_4f_5 \\
&+ j_{16}f_1f_6 + j_{25}f_2f_5 + j_{34}f_3f_4 + i_{14}f_1f_4 + i_{23}f_2f_3)\Bigg),
\end{aligned} \tag{6.39}$$

with h_{mn}, j_{mn} and i_{mn}, coefficients that we need to determine, and with b_{14}, e_{15} and g_{13} undetermined from the bootstrap. The term $g_{13}f_3(z)$ in the next-to-last line of the previous equation has its origin in the correction to the conformal block of operator $:T_{\alpha\beta}T^{\alpha\beta}:$. This operator has $\beta = \tau_{2,2} = 8$ which implies that both lines in the following expansion of the conformal block

$$g^{(0,0)}_{8,0}(1-z,1-\bar{z}) = (1-\bar{z})^4\left(1+(1-\bar{z})\left(\frac{3}{2}+\frac{1}{1-z}\right)+\mathcal{O}\left((1-\bar{z})^2\right)\right)f_4(z) \\ -(1-\bar{z})^5\left(1+(1-\bar{z})\left(2+\frac{1}{1-z}\right)+\mathcal{O}((1-\bar{z})^2)\right)f_3(z) \tag{6.40}$$

contribute. The contribution from the first line of (6.40) is included in the third line of (6.39), while we had to explicitly add the contribution from the second line. Using (A.1) and (A.3) we set $i_{23}=0$, $j_{34}=0$ and $j_{25}=0$.

From the S-channel calculation, we collect the terms in the correlator which behave as $(1-\bar{z})^{-\Delta_L+5}\log^2(z)z^m$ and are fixed in terms of OPE data at $\mathcal{O}(\mu)$ for $p \leq 3$. By expanding (6.39) near $z \to 0$ we obtain terms with the same behavior as linear functions of unknown coefficients and by matching them with the terms from the S-channel, we determine the unknown coefficients. These are

$$\begin{aligned} h_{18} =& -\frac{\Delta_L\left(\Delta_L+1\right)\left(\Delta_L\left(\Delta_L\left(\Delta_L\left(47\Delta_L-721\right)-5182\right)-15204\right)-13680\right)}{4942080000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)} \\ &+\frac{49i_{14}}{38610}+\frac{49j_{16}}{780}, \\ h_{27} =& -\frac{\Delta_L\left(\Delta_L+1\right)\left(\Delta_L\left(\Delta_L\left(\Delta_L\left(8\Delta_L-229\right)+1097\right)+7224\right)+10080\right)}{1383782400\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)} \\ &+\frac{5i_{14}}{1404}-\frac{j_{16}}{12}, \\ h_{36} =& \frac{\Delta_L\left(\Delta_L+1\right)\left(\Delta_L\left(\Delta_L\left(\Delta_L\left(34\Delta_L-137\right)-1829\right)+5712\right)+23040\right)}{2661120000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)} \\ &-\frac{i_{14}}{180}, \\ h_{45} =& \frac{\left(\Delta_L-6\right)\Delta_L\left(\Delta_L+1\right)\left(\Delta_L+2\right)}{62720000\left(\Delta_L-3\right)\left(\Delta_L-2\right)}. \end{aligned} \tag{6.41}$$

Our approach does not allow us to determine the coefficients j_{16} and i_{14}. These are related to the T-channel OPE coefficients of operators with twist $\tau_{2,3} = 10$ and minimal conformal spin by

$$\begin{aligned} i_{14} &= P^{(2)}_{10,0}, \\ j_{16} &= P^{(2)}_{12,2} - \frac{2}{99}P^{(2)}_{10,0}. \end{aligned} \tag{6.42}$$

Notice that, despite the fact that the h_{mn} depend on the undetermined OPE data, we are able to extract all the OPE coefficients of double-stress tensors with twist $\tau_{2,3} = 10$ and conformal spin $\Delta + s \geq 18$. Explicitly, they are given by:

$$
\begin{aligned}
P^{(2)}_{14+2\ell,4+2\ell} = & \frac{\sqrt{\pi} 2^{-4\ell-22}\Gamma(2\ell+9)}{5(2\ell+1)(2\ell+3)(2\ell+5)(2\ell+7)\Gamma\left(2\ell+\frac{17}{2}\right)} \\
\times & \frac{\Delta_L(\Delta_L+1)(a_{3,\ell}\Delta_L^4 + b_{3,\ell}\Delta_L^3 + c_{3,\ell}\Delta_L^2 + d_{3,\ell}\Delta_L + e_{3,\ell})}{(\ell+5)(\ell+6)(\ell+7)(\ell+8)(\Delta_L-5)(\Delta_L-4)(\Delta_L-3)(\Delta_L-2)},
\end{aligned}
\tag{6.43}
$$

where

$$
\begin{aligned}
a_{3,\ell} =& \ell(2\ell+17)(\ell(2\ell+17)(\ell(2\ell+17)+70)+1513)+9756, \\
b_{3,\ell} =& 38232 - 2(\ell-1)\ell(2\ell+17)(2\ell+19)(\ell(2\ell+17)+44), \\
c_{3,\ell} =& 196164 + \ell(17+2\ell(11647+\ell(17+2\ell)(196+\ell(17+2\ell)))), \\
d_{3,\ell} =& 504(647+19\ell(17+2\ell)), \\
e_{3,\ell} =& 4320(53+\ell(17+2\ell)).
\end{aligned}
\tag{6.44}
$$

We expect that a similar picture is true for all subleading twist double-stress tensor operators. At $\mathcal{O}(\mu^2)$, the ansatz for $\mathcal{G}^{(2,m)}(z)$ will naturally include products of the type $f_a(z)f_b(z)$, such that $a+b = 6+m$, together with $f_1(z)f_{3+m}(z)$ and $f_1(z)f_{1+m}(z)$. The coefficients of the latter two will be left undetermined from the lightcone bootstrap at every order in the lightcone expansion. Such coefficients will be related to the non-universal OPE coefficients of double-stress tensors with spin $s = 0, 2$ for a given twist. On the other hand, the coefficients of the products $f_a(z)f_b(z)$, with $a + b = 6 + m$, once determined, will allow us to extract the OPE coefficients of all double-stress tensors with conformal spin $\beta \geq 12 + 2m$. We expect them to be universal, despite the fact that the coefficients of the products $f_a(z)f_b(z)$, with $a + b = 6 + m$, will be plagued by the ambiguities present in the determination of the OPE coefficients of operators spin $s = 0, 2$ – just as herein.

6.3. Triple-stress tensors in four dimensions

In this section, we consider the stress tensor sector of the HHLL correlator at $\mathcal{O}(\mu^3)$ in $d = 4$. The operators which contribute in the T-channel are triple-stress tensors. Since we are interested in the lightcone limit $1 - \bar{z} \ll 1$, we consider contributions of operators with low twist. Triple-stress tensors with minimal twist can be written in the schematic form $: T_{\mu\nu}T_{\rho\sigma}\partial_{\alpha_1}\dots\partial_{\alpha_{2\ell}}T_{\eta\xi} :$. These operators have twist $\tau_{3,\text{min}} = 6$ and their contribution to the HHLL correlator in the lightcone limit was found in [13]:

$$\mathcal{G}^{(3,0)}(z) = \frac{1}{(1-z)^{\Delta_L}}\Big(a_{117}f_1(z)^2f_7(z) + a_{126}f_1(z)f_2(z)f_6(z) \\ + a_{135}f_1(z)f_3(z)f_5(z) + a_{225}f_2(z)^2f_5(z) + a_{234}f_2(z)f_3(z)f_4(z) + a_{333}f_3(z)^3\Big), \tag{6.45}$$

where the coefficients a_{ikl} are

$$\begin{aligned} a_{117} &= \frac{5\Delta_L(\Delta_L+1)(\Delta_L+2)}{768768(\Delta_L-2)(\Delta_L-3)}, \\ a_{126} &= \frac{5\Delta_L(5\Delta_L^2-57\Delta_L-50)}{6386688(\Delta_L-2)(\Delta_L-3)}, \\ a_{135} &= \frac{\Delta_L(2\Delta_L^2-11\Delta_L-9)}{1209600(\Delta_L-3)}, \\ a_{225} &= -\frac{\Delta_L(7\Delta_L^2-51\Delta_L-70)}{2903040(\Delta_L-2)(\Delta_L-3)}, \\ a_{234} &= \frac{\Delta_L(\Delta_L-4)(3\Delta_L^2-17\Delta_L+4)}{4838400(\Delta_L-2)(\Delta_L-3)}, \\ a_{333} &= \frac{\Delta_L(\Delta_L-4)(\Delta_L^3-16\Delta_L^2+51\Delta_L+24)}{10368000(\Delta_L-2)(\Delta_L-3)}. \end{aligned} \tag{6.46}$$

6.3.1. Twist-eight triple-stress tensors

We now consider the subleading contributions at $\mathcal{O}(\mu^3)$ coming from triple-stress tensor operators with twist $\tau_{3,1} = 8$. There are two families of such operators, these can be schematically written as $: T_{\mu\nu}T_{\rho\alpha}\partial_{\alpha_1}\dots\partial_{\alpha_{2\ell}}T^{\alpha}{}_{\xi} :$ with $\Delta = 12+2\ell$ and spin $s = 4+2\ell$ and $: T_{\mu\nu}T_{\rho\sigma}\partial^2\partial_{\alpha_1}\dots\partial_{\alpha_{2\ell}}T_{\eta\xi} :$ with $\Delta = 14+2\ell$ and spin $s = 6+2\ell$. The conformal spins of these families are $\beta = 16+4\ell$ and $\beta = 20+4\ell$, respectively, so we expect products of three $f_a(z)$ functions such that their indices add up to 8 and 10. The contribution to the correlator of these operators scales as $(1-\bar{z})^{-\Delta_L+4}$ for $\bar{z}\to 1$. This implies that one needs to include the contribution from the minimal twist triple-stress tensor operators (due to corrections to their conformal blocks).

Our ansatz takes the form

$$\mathcal{G}^{(3,1)}(z) = \frac{1}{(1-z)^{\Delta_L}} \Bigg(\left(\frac{2-z}{1-z}\right) (a_{117} f_1^2 f_7 + a_{126} f_1 f_2 f_6 + a_{135} f_1 f_3 f_5 \\ + a_{225} f_2^2 f_5 + a_{234} f_2 f_3 f_4 + a_{333} f_3^3) + (b_{116} f_6 f_1^2 + b_{134} f_3 f_4 f_1 \\ + b_{125} f_2 f_5 f_1 + b_{233} f_2 f_3^2 + b_{224} f_2^2 f_4 + c_{118} f_8 f_1^2 + c_{145} f_4 f_5 f_1 \\ + c_{136} f_3 f_6 f_1 + c_{127} f_2 f_7 f_1 + c_{244} f_2 f_4^2 + c_{334} f_3^2 f_4 + c_{235} f_2 f_3 f_5 \\ + c_{226} f_2^2 f_6) \Bigg), \tag{6.47}$$

where a_{jkl} are given in (6.46). The linear dependence between products of three f_a functions, with explicit relations given in Appendix B.1, allows us to set the following coefficients to zero

$$b_{125} = b_{134} = b_{224} = b_{233} = c_{136} = 0. \tag{6.48}$$

To fix the coefficients b_{116} and c_{jkl} we perform an S-channel calculation up to $\mathcal{O}(\mu^3)$. The relevant terms now scale as $(1-\bar{z})^{-\Delta_L+4}\log^3(z)z^m$ and $(1-\bar{z})^{-\Delta_L+4}\log^2(z)z^m$ when $\bar{z}\to 1$ and $z \to 0$.

We fix the S-channel OPE data at $\mathcal{O}(\mu^2)$ using the results of the previous section, specifically eqs. (6.21), (6.34) and (6.39). Since the OPE coefficients of double-stress operators of spin 0 and 2 are left undetermined, the S-channel OPE data is fixed in terms of these. Concretely, $\gamma_n^{(2,0)}$ and $P_n^{(2,0)}$ are completely determined since the leading-twist OPE coefficients are known and universal, while $\gamma_n^{(2,1)}$ and $P_n^{(2,1)}$ depend on b_{14}, $\gamma_n^{(2,2)}$ and $P_n^{(2,2)}$ depend on b_{14}, g_{13} and e_{15} and so on.[20]

We were able to fix all the unknown coefficients in the ansatz (6.47) using bootstrap. Crucially, there are no spin $s = 0, 2$ operators that contribute at this level. Here, we list two of the coefficients while all others can be found in Appendix B.2.

[20] Explicit expressions for the S-channel OPE data are too cumbersome to quote here.

$$
\begin{aligned}
b_{116} =& \frac{-\Delta_L (\Delta_L+3)(\Delta_L(\Delta_L(\Delta_L(1001\Delta_L+387)-4326)+13828)+5040)}{10378368000(\Delta_L-4)(\Delta_L-3)(\Delta_L-2)} \\
&+ \frac{b_{14}(\Delta_L(143\Delta_L+427)+540)}{17160(\Delta_L-4)}, \\
c_{118} =& 7(\Delta_L+3) \times \\
& \frac{604800 b_{14}\left(\Delta_L^2-5\Delta_L+6\right)+\Delta_L\left(-21\Delta_L^3+229\Delta_L^2+414\Delta_L+284\right)}{856627200\left(\Delta_L^3-9\Delta_L^2+26\Delta_L-24\right)}.
\end{aligned}
\tag{6.49}
$$

Notice that they depend on b_{14}. This is because the anomalous dimensions at $\mathcal{O}(\mu^2)$, $\gamma_n^{(2,2)}$ depend on it. Moreover, no OPE coefficient of triple-stress tensors with twist $\tau_{3,1} = 10$ is universal since all of them depend on b_{14}. These OPE coefficients can be written in the form of a finite sum, similarly to what happens for the OPE coefficients of leading twist triple-stress tensor, given in [13]. We define $i_1(r,q)$ and $i_2(r,q)$ as

$$
i_1(r,q) = b_{116}p(1,1,r)p(2r+2,6,q), \tag{6.50}
$$

and

$$
\begin{aligned}
i_2(r,q) =& c_{118}p(1,1,r)p(2r+2,8,q) + c_{127}p(1,2,r)p(2r+3,7,q) \\
&+ c_{145}p(1,4,r)p(2r+5,5,q) + c_{226}p(2,2,r)p(2r+4,6,q) \\
&+ c_{235}p(2,3,r)p(2r+5,5,q) + c_{244}p(2,4,r)p(2r+6,4,q) \\
&+ c_{334}p(3,3,r)p(2r+6,4,q),
\end{aligned}
\tag{6.51}
$$

where $p(a,b,\ell)$ are given by (6.20). The OPE coefficients can be written as

$$
P^{(3)}_{14+2\ell,6+2\ell} = \sum_{r=0}^{\ell+1} i_1(r,\ell+1-r) + \sum_{r=0}^{\ell} i_2(r,\ell-r), \tag{6.52}
$$

for $k \geq 0$, while $P^{(3)}_{12,4} = i_1(0,0) = b_{116}$. We give the explicit expressions for some OPE coefficients in Appendix B.4.

6.3.2. Twist-ten triple-stress tensors

Here, we consider the contribution of triple-stress tensor operators of twist $\tau_{3,2} = 10$. These operators can be divided in three families of the schematic form

$: T_{\mu\nu}T_{\alpha\beta}\partial_{\mu_1}\ldots\partial_{\mu_{2\ell}}(\partial^2)^2 T_{\rho\sigma}:$ with conformal dimension $\Delta = 16+2\ell$ and spin $s = 6+2\ell$, $: T_{\mu\nu}T_{\alpha\beta}\partial_{\mu_1}\ldots\partial_{\mu_{2\ell}}\partial^2 T^{\beta}{}_{\rho}:$ with $\Delta = 14+2\ell$ and $s = 4+2\ell$ and finally $: T_{\mu\alpha}T_{\nu\beta}\partial_{\mu_1}\ldots\partial_{\mu_{2\ell}}T^{\alpha\beta}:$ with $\Delta = 12+2\ell$ and $s = 2+2\ell$. One can see that in the last family an operator of spin $s=2$ is included.

An appropriate ansatz in this case is

$$\mathcal{G}^{(3,2)}(z) = \frac{1}{(1-z)^{\Delta_L}}\Bigg(\left(\frac{144z^2-448z+464}{160(z-1)^2}\right)(a_{117}f_1^2 f_7 + a_{126}f_1f_2f_6 + a_{135}f_1f_3f_5 + a_{225}f_2^2f_5 + a_{234}f_2f_3f_4 + a_{333}f_3^3) + \left(\frac{1}{1-z}+\frac{3}{2}\right)(b_{116}f_6f_1^2 + c_{118}f_8f_1^2 + c_{145}f_4f_5f_1 + c_{127}f_2f_7f_1 + c_{244}f_2f_4^2 + c_{334}f_3^2f_4 + c_{235}f_2f_3f_5 + c_{226}f_2^2f_6) + (d_{117}f_1^2f_7 + e_{115}f_1^2f_5 + g_{119}f_1^2f_9 + g_{128}f_1f_2f_8 + g_{155}f_1f_5^2 + g_{227}f_2^2f_7 + g_{236}f_2f_3f_6 + g_{245}f_2f_4f_5 + g_{335}f_3^2f_5 + g_{344}f_3f_4^2)\Bigg), \tag{6.53}$$

where $f_a = f_a(z)$ and we have included only the linearly independent products of these functions.

The lightcone bootstrap fixes all coefficients except e_{115}. One can check that this is exactly the OPE coefficient $P^{(3)}_{12,2}$ of the spin-2 operator $: T_{\mu\alpha}T_{\nu\beta}T^{\alpha\beta}:$ with $\Delta = 12$ and spin $s = 2$

$$e_{115} = P^{(3)}_{12,2}. \tag{6.54}$$

All other coefficients can be found in Appendix B.2. Notice that all coefficients depend on b_{14}, g_{13} and e_{15} because the S-channel OPE data at $\mathcal{O}(\mu^2)$ depend on them.

Again, we write the OPE coefficients for all triple-stress tensor operators with twist $\tau_{3,2} = 10$ and $\beta \geq 18$ in the form of a finite sum. We define $j_1(r,q)$, $j_2(r,q)$ and $j_3(r,q)$ as

$$j_1(r,q) = e_{115}p(1,1,r)p(2r+2,5,q), \tag{6.55}$$

$$j_2(r,q) = d_{117}p(1,1,r)p(2r+2,7,q) \tag{6.56}$$

and

$$
\begin{aligned}
j_3(r,q) = {} & g_{119}p(1,1,r)p(2r+2,9,q) + g_{128}p(1,2,r)p(2r+3,8,q) \\
& + g_{155}p(1,5,r)p(2r+6,5,q) + g_{227}p(2,2,r)p(2r+4,7,q) \\
& + g_{236}p(2,3,r)p(2r+5,6,q) + g_{245}p(2,4,r)p(2r+6,5,q) \\
& + g_{335}p(3,3,r)p(2r+6,5,q) + g_{344}p(3,4,r)p(2r+7,4,q),
\end{aligned} \tag{6.57}
$$

where $p(a,b,\ell)$ is given by (6.20). The OPE coefficients can now be written as

$$
P^{(3)}_{16+2\ell,6+2\ell} = \sum_{r=0}^{\ell+2} j_1(r,\ell+2-r) + \sum_{r=0}^{\ell+1} j_2(r,\ell+1-r) + \sum_{r=0}^{\ell} j(r,\ell-r), \tag{6.58}
$$

for $\ell \geq 0$, while

$$
P^{(3)}_{14,4} = j_1(0,1) + j_1(1,0) + j_2(0,0). \tag{6.59}
$$

Finally, we conclude that the stress tensor sector of the HHLL correlator to all orders in μ and in the lightcone expansion will take a similar form in terms of products of f_a functions. One should be able to completely fix the coefficients, except for terms that correspond to the OPE coefficients of multi-stress tensor operators with spin $s = 0, 2$, using the lightcone bootstrap.

6.4. Holographic phase shift and multi-stress tensors

In this section, we demonstrate how to calculate the T-channel OPE coefficients of spin-2 operators (up to undetermined spin-0 data) which are left undetermined after the lightcone bootstrap, using a gravitational calculation of the scattering phase shift. We are interested in the scattering phase shift – or eikonal phase – resulting from the eikonal resummation of graviton exchanges when a fast particle is scattered by a black hole[21]. Seeking to explore the universality properties of the undetermined OPE coefficients of the previous section, we perform the calculation in Gauss-Bonnet gravity extending the results of [55] to this case. We argue that the phase shift in the large impact parameter limit is independent of higher-derivative corrections to the dual gravitational lagrangian. This is consistent with the universality of the minimal-twist multi-stress tensor sector in the dual CFT. On the other hand, we observe that the

[21] For CFT approach to the Regge scattering of scalar particles in pure AdS see [58-63,126,134].

subleading OPE data of spin-2 multi-stress tensors depend explicitly on the Gauss-Bonnet coupling $\lambda_{\rm GB}$.

The computation involves performing an inverse Fourier transform of the exponential of the phase shift in the large impact parameter expansion, to obtain the HHLL correlator in position space[22]. This is done following the approach of [135]. Comparison with the expressions for the HHLL correlator in the lightcone limit requires analytically continuing the results of Sections 6.2 and 6.3 and taking the limit $z \to 1$. Identifying terms in the HHLL four-point function with the same large impact parameter and $z \to 1$ behavior allows us to extract the spin-2 OPE coefficients of the double- and triple-stress tensor operators (up to undetermined spin zero data).

6.4.1. Universality of the phase shift in the large impact parameter limit

In this subsection, we consider Gauss-Bonnet gravity in $(d+1)$–dimensions and argue that the phase shift obtained by a highly energetic particle traveling in a spherical AdS-Schwarzschild background is independent of the Gauss-Bonnet coupling $\lambda_{\rm GB}$ in the large impact parameter limit.

The action of Gauss-Bonnet gravity in $(d+1)$-dimensional spacetime is

$$S = \frac{1}{16\pi G}\int d^{d+1}\sqrt{-g}\Big(R + \frac{d(d-1)}{\ell^2} + \frac{\tilde{\lambda}_{\rm GB}}{(d-2)(d-3)}(R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta} - 4R_{\mu\nu}R^{\mu\nu} + R^2)\Big), \tag{6.60}$$

where the coupling parameter $\tilde{\lambda}_{\rm GB}$ is measured in units of the cosmological constant ℓ: $\tilde{\lambda}_{\rm GB} = \lambda_{\rm GB}\ell^2$, with $\lambda_{\rm GB}$ being a dimensionless coefficient. The AdS-Schwarzschild black hole metric which is a solution of the Gauss-Bonnet theory is given by [136,137]:

$$ds^2 = -r_{AdS}^2 f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-1}^2, \tag{6.61}$$

where

$$f(r) = 1 + \frac{r^2}{2\lambda_{\rm GB}}\left(1 - \sqrt{1 - 4\lambda_{\rm GB}(1 - \frac{\tilde{\mu}}{r^d})}\right), \tag{6.62}$$

[22] Recall that the exponential of the phase shift corresponds to the Regge limit of HHLL four-point function in momentum space [55].

with

$$\tilde{\mu} = \frac{16\pi GM}{(d-1)\Omega_{d-1}\ell^{d-2}}, \qquad \mu = \frac{\tilde{\mu}}{r_{AdS}^{d-2}\sqrt{1-4\lambda_{\rm GB}}}, \tag{6.63}$$

and

$$r_{AdS} = \left(\frac{1}{2}(1+\sqrt{1-4\lambda_{\rm GB}})\right)^{1/2} \tag{6.64}$$

where Ω_{d-1} is the surface area of a $(d-1)$-dimensional unit sphere embedded in d-dimensional Euclidean space. The metric is normalized such that the speed of light is equal to 1 at the boundary (i.e. $g_{tt}/g_{\phi\phi} \to 1$ as $r \to \infty$) and all dimensionful parameters are measured in units of ℓ. The product (ℓr_{AdS}) is the radius of the asymptotic Anti-de Sitter space.

The two conserved charges along the geodesics, p^t and p^ϕ, are

$$\begin{aligned} p^t &= r_{AdS}^2 f(r)\frac{dt}{d\lambda}, \\ p^\phi &= r^2\frac{d\phi}{d\lambda}. \end{aligned} \tag{6.65}$$

where λ denotes an affine parameter. Null geodesics are described by the following equation,

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{(p^\phi)^2}{2r^2}f(r) = \frac{1}{2}\frac{(p^t)^2}{r_{AdS}^2}. \tag{6.66}$$

similarly to Einstein gravity.

A light particle, starting from the boundary, traversing the bulk and reemerging on the boundary experiences a time delay and a path deflection given by :

$$\begin{aligned} \Delta t &= 2\int_{r_0}^{\infty} \frac{dr}{r_{AdS} f(r)\sqrt{1-\alpha^2\frac{r_{AdS}^2}{r^2}f(r)}}, \\ \Delta\phi &= 2\alpha\, r_{AdS}\int_{r_0}^{\infty} \frac{dr}{r^2\sqrt{1-\alpha^2\frac{r_{AdS}^2}{r^2}f(r)}}, \end{aligned} \tag{6.67}$$

where $\alpha = p^\phi/p^t$ and r_0 the impact parameter determined by $\frac{dr}{d\lambda}|_{r(\lambda)=r_0} = 0$, *i.e.*,

$$1-\alpha^2\frac{r_{AdS}^2}{r_0^2}f(r_0) = 0. \tag{6.68}$$

Defining the phase shift as $\delta = -p\cdot\Delta x = p^t\Delta t - p^\phi\Delta\phi$, we find that

$$\delta = 2\frac{p^t}{r_{AdS}}\int_{r_0}^{\infty}\frac{dr}{f(r)}\sqrt{1-\alpha^2\frac{r_{AdS}^2}{r^2}f(r)}. \tag{6.69}$$

Just as in [55], we are interested in expanding the phase shift order by order in μ. It is easy to see that in terms of CFT data μ can be expressed as

$$\mu = \frac{4}{(d-1)^2}\frac{\Gamma(d+2)}{\Gamma(d/2)^2}\frac{\Delta_H}{C_T}, \tag{6.70}$$

which is consistent with (3.14). Here C_T is the central charge of the dual conformal theory [138]:

$$C_T = \frac{\pi^{\frac{d}{2}-1}}{2(d-1)}\frac{\Gamma(d+2)}{\Gamma(d/2)^3 G}(r_{AdS}\ell)^{d-1}\sqrt{1-4\lambda_{\text{GB}}}, \tag{6.71}$$

and $\Delta_H = M\ell r_{AdS}$.

In order to calculate the phase shift, we introduce a new variable y, given by $y = \frac{r_0}{r}$. Using this variable (6.69) can be written as:

$$\delta = 2\frac{p^t r_0}{r_{AdS}}\int_0^1 \frac{dy}{y^2 f(\frac{r_0}{y})}\left(1-\alpha^2\frac{r_{AdS}^2 y^2}{{r_0}^2}f(\frac{r_0}{y})\right)^{1/2}. \tag{6.72}$$

Expanding the phase shift

$$\delta = \sum_{k=0}^{\infty}\mu^k\delta^{(k)}, \tag{6.73}$$

and solving (6.68) perturbatively in μ reads

$$r_0 = b - \frac{b^{3-d}}{2r_{AdS}^{2-d}}\mu + \frac{b^{3-2d}}{8r_{AdS}^{4-2d}}\left(b^2(3-2d)+\frac{4\lambda_{\text{GB}}}{\sqrt{1-4\lambda_{\text{GB}}}}\right)\mu^2 + \mathcal{O}(\mu^3). \tag{6.74}$$

Generically, we get an expansion of the form

$$r_0 = b + \sum_{k=1}^{\infty} a_k\mu^k, \tag{6.75}$$

where the a_k, which depend on b, in the large impact parameter limit ($b\to\infty$) behave as

$$a_k \propto b\left(\frac{r_{AdS}}{b}\right)^{k(d-2)}. \tag{6.76}$$

Notice that there is no explicit λ_{GB} dependence in the leading term[23], since the metric (6.61) approaches the one in pure GR.

[23] Except the overall dependence on r_{AdS}.

To study the leading behavior of the phase shift for large impact parameters it is convenient to define a function $g(x)$ as

$$g(x) = r_{AdS}^2 \frac{f(x)}{x^2}, \tag{6.77}$$

with f given by (6.62), and denote the integrand of (6.72) by $h\left(g\left(\frac{r_0}{y}\right)\right)$, with

$$h(x) = \frac{1}{x}\sqrt{1-\alpha^2 x}, \tag{6.78}$$

to express (6.72) as

$$\delta = 2p^t\left(\frac{r_{AdS}}{r_0}\right)\int_0^1 h\left(g\left(\frac{r_0}{y}\right)\right) dy. \tag{6.79}$$

In practice, to calculate the phase shift in the large impact parameter limit, we first expand the integrand of (6.79) in powers of μ, perform the integration with respect to y, and then expand the result in powers of b. The b-dependence of $\delta^{(k)}$ is therefore fixed before the integration and the integral just determines the overall numerical factor (assuming that it is convergent).

We can immediately see that $g\left(\frac{r_0}{y}\right)$ depends on μ explicitly and implicitly through $r_0(\mu)$ in (6.74). In order to make this clear we write $g\left(\frac{r_0}{y},\mu\right)$ instead of just $g\left(\frac{r_0}{y}\right)$. Defining $g^{(n,m)}\left(\frac{b}{y},0\right)$ as

$$g^{(n,m)}\left(\frac{b}{y},0\right) = \frac{\partial^n\partial^m}{\partial r_0^n\partial\mu^m} g\left(\frac{r_0}{y},\mu\right)\Big|_{r_0=b,\mu=0}. \tag{6.80}$$

allows us to write the following expansion for $h\left(g\left(\frac{r_0}{y},\mu\right)\right)$:

$$\begin{aligned} h\left(g\left(r_0/y,\mu\right)\right) =& h(g(b/y,0)) + \mu h'(g(b/y,0))\left(g^{(0,1)}(b/y,0) + a_1 g^{(1,0)}(b/y,0)\right) \\ &+ \frac{\mu^2}{2} h''(g(b/y,0))\left(g^{(0,1)}(b/y,0) + a_1 g^{(1,0)}(b/y,0)\right)^2 \\ &+ \frac{\mu^2}{2} h'(g(b/y,0))\Big(g^{(0,2)}(b/y,0) + 2a_2 g^{(1,0)}(b/y,0) \\ &+ 2a_1 g^{(1,1)}(b/y,0) + a_1^2 g^{(2,0)}(b/y,0)\Big) + \mathcal{O}(\mu^3), \end{aligned} \tag{6.81}$$

where a_k are the coefficients appearing in (6.75). It is clear that at each order in the μ-expansion we will have a sum of products composed from derivatives of $h(x)$ and sums of the form

$$\sum_{\{k_i:\sum_{i=1}^{p} k_i \leqslant n\}} a_{k_1} a_{k_2} \dots a_{k_p} g^{(p,n-\sum_{i=1}^{p} k_i)}(b/y,0)\,. \tag{6.82}$$

Notice first that $g(b/y,0)$, $g^{(m,0)}(b/y,0)$ and $g^{(m,1)}(b/y,0)$ do not depend on $\lambda_{\rm GB}$ as can be seen from (6.77). The same is true for $h^{(n)}(g(b/y,0))$ for any n as follows from (6.78). On the contrary, $g^{(m,n)}(b/y,0)$ with $n \geq 2$ depend explicitly on $\lambda_{\rm GB}$. It is then evident that any dependence on $\lambda_{\rm GB}$ will come from terms like the ones in parenthesis in (6.81) which are of the type (6.82). We will now show that all the terms in such sums which contain $\lambda_{\rm GB}$, are subleading in the large impact parameter limit.

Recall that $a_k \propto b^{1-k(d-2)}$ for $k \geq 1$. Using (6.77) one can check that $g^{(m,n)}(b/y,0) \propto b^{-m-nd}$ for $n > 0$ and $g^{(m,0)}(b/y,0) \propto b^{-m-2}$. We thus need to speparately consider two cases: products of the form $a_{k_1} a_{k_2} \dots a_{k_p} g^{(p,n-q)}(b/y,0)$, with $q = \sum_{i=1}^{p} k_i$ and $q < n$ and products of the form $a_{k_1} a_{k_2} \dots a_{k_p} g^{(p,0)}(b/y,0)$ for which $q = n$.

The former behave as

$$a_{k_1} a_{k_2} \dots a_{k_p} g^{(p,n-q)}(b/y,0) \propto \frac{1}{b^{nd-2q}}. \tag{6.83}$$

Clearly, the leading behavior in the large impact parameter regime corresponds in this case to $q = n-1$, recall, however, that $g^{(p,1)}$ does not depend on $\lambda_{\rm GB}$. The behavior of the latter terms is

$$a_{k_1} a_{k_2} \dots a_{k_p} g^{(p,0)}(b/y,0) \propto \frac{1}{b^{nd-2(n-1)}}. \tag{6.84}$$

which is again independent of $\lambda_{\rm GB}$. The conclusion is that the leading behavior in the large impact parameter regime comes from terms containing $g^{(p,0)}(b/y,0)$ and $g^{(p,1)}(b/y,0)$ that do not contain $\lambda_{\rm GB}$.

One can extend these considerations straightforwardly to any gravitational theory that contains a spherical black hole with a metric given by

$$ds^2 = -(1+r^2\tilde{f}(r))dt^2 + \frac{dr^2}{1+r^2\tilde{h}(r)} + r^2 d\Omega_{d-1}^2 \tag{6.85}$$

where the functions $\tilde{f}(r)$ and $\tilde{h}(r)$ admit an expansion of the following form in the large r limit:

$$\begin{aligned}\tilde{f}(r) &= 1 - \sum_{n=0}^{\infty} \frac{\tilde{f}_{nd}}{r^{nd}} = 1 - \frac{\tilde{f}_0}{r^d} - \frac{\tilde{f}_d}{r^{2d}} - \dots \\ \tilde{h}(r) &= 1 - \sum_{n=0}^{\infty} \frac{\tilde{h}_{nd}}{r^{nd}} = 1 - \frac{\tilde{h}_0}{r^d} - \frac{\tilde{h}_d}{r^{2d}} - \dots,\end{aligned} \tag{6.86}$$

for some constants $\tilde{f}_{nd}$ and $\tilde{h}_{nd}$ (these are the spherical black hole metrics considered in eqs. (5.1) and (5.10) in [15]).

6.4.2. Spin-2 multi-stress tensor OPE data from the gravitational phase shift

The gravitational phase shift in a black hole background is related to the lightcone HHLL four-point function discussed extensively in this article. In the following, we will exploit the precise relationship between the two to extract the OPE data of multi-stress tensor operators of spin-2 in the dual conformal field theory (modulo spin zero data). While the explicit procedure can be worked out for arbitrary multi-stress tensors, we will herein focus on double and triple-stress tensor operators, which control the $\mathcal{O}(\mu^2)$ and $\mathcal{O}(\mu^3)$ lightcone behavior of the HHLL correlation function.

6.4.2.3. The phase shift in Gauss-Bonnet gravity to $\mathcal{O}(\mu^3)$.

In this section, we focus on the gravity side and determine the phase shift order by order in μ up to $\mathcal{O}(\mu^3)$ relevant for this article. Starting from $\mathcal{O}(\mu^0)$ we consider the following expression

$$\delta^{(0)} = 2b\, p^t\, r_{AdS} \sqrt{1-\alpha^2} \int_0^1 \frac{\sqrt{1-y^2}}{b^2 + r_{AdS}^2 y^2} dy. \tag{6.87}$$

Evaluating this integral and using the following notation $p^{\pm} = p^t \pm p^{\phi}$, $-p^2 = p^+ p^-$, leads to

$$\delta^{(0)} = \pi p^-. \tag{6.88}$$

This is of course none other but the “phase shift” in pure AdS space.

At $\mathcal{O}(\mu)$ the result is the same as in [55], where Einstein gravity was considered,

$$\delta^{(1)} = \sqrt{-p^2}\left(\frac{b}{r_{AdS}}\right)^{1-d}\left(\frac{d-1}{2}\right)B\Big[\frac{d-1}{2},\frac{3}{2}\Big]{}_2F_1\Big(1,\frac{d-1}{2},\frac{d}{2}+1,-\frac{r_{AdS}^2}{b^2}\Big). \tag{6.89}$$

At this order, the phase shift depends only on the single graviton exchange, which is unaffected by the higher derivative terms in the gravitational action. According to the holographic dictionary, the exchange of a single graviton is related to the exchange of a single stress tensor in the T-channel. The corresponding OPE coefficient is fixed by the Ward identity, so it does not depend on the details of the theory.

We now consider the phase shift at higher orders in μ. For convenience herein all results are presented in $d=4$. At $\mathcal{O}(\mu^2)$, using the technique presented in the previous subsection, we find that:

$$\begin{aligned}\delta^{(2)} =&\frac{7\pi}{8}\sqrt{-p^2}\Bigg(5\frac{b}{r_{AdS}}(\sqrt{1+\frac{r_{AdS}^2}{b^2}}-1)-\frac{5}{2}\frac{r_{AdS}}{b}+\frac{5}{4}\frac{r_{AdS}^3}{b^3}\\ &+\frac{\lambda_{\rm GB}}{r_{AdS}^2\sqrt{1-4\lambda_{\rm GB}}}\Big(4\frac{b}{r_{AdS}}(\sqrt{1+\frac{r_{AdS}^2}{b^2}}-1)-2\frac{r_{AdS}}{b}+\frac{1}{2}\frac{r_{AdS}^3}{b^3}\\ &-\frac{1}{4}\frac{r_{AdS}^5}{b^5}\Big)\Bigg).\end{aligned} \tag{6.90}$$

In the lightcone limit ($b\to\infty$) this reduces to

$$\delta^{(2)} \underset{b\to\infty}{\approx} \frac{35\pi\sqrt{-p^2}r_{AdS}^5}{128b^5}-\frac{35\pi\sqrt{-p^2}r_{AdS}^7}{1024b^7}\left(5+\frac{4\lambda_{\rm GB}}{r_{AdS}^2\sqrt{1-4\lambda_{\rm GB}}}\right)+\ldots. \tag{6.91}$$

We explicitly see that the leading contribution does not depend on $\lambda_{\rm GB}$, while the subleading does.

Let us denote $\delta^{(2)}_{\rm GR}$ to be equal to (6.90) when $\lambda_{\rm GB}=0$,

$$\delta^{(2)}_{\rm GR} = \frac{35\pi r_{AdS}^5\sqrt{-p^2}}{128b^5}{}_2F_1(1,\frac{5}{2},4,-\frac{r_{AdS}^2}{b^2}), \tag{6.92}$$

which is the pure Einstein gravity result for the phase shift at $\mathcal{O}(\mu^2)$. Then $\delta^{(2)}$ can be written as

$$\delta^{(2)} = \delta^{(2)}_{\rm GR}\left(1+\frac{4\lambda_{\rm GB}}{5r_{AdS}^2\sqrt{1-4\lambda_{\rm GB}}}\right)-\frac{7\pi\sqrt{-p^2}\lambda_{\rm GB}}{32r_{AdS}^2\sqrt{1-4\lambda_{\rm GB}}}\left(\frac{r_{AdS}}{b}\right)^5. \tag{6.93}$$

The phase shift at $\mathcal{O}(\mu^3)$ is given by

$$\begin{aligned}\delta^{(3)} =&\delta^{(3)}_{GR}\left(1+\frac{12\lambda_{\rm GB}}{7r_{AdS}^2\sqrt{1-4\lambda_{\rm GB}}}+\frac{16\lambda_{\rm GB}^2}{21r_{AdS}^4(1-4\lambda_{\rm GB})}\right)\\ &-\sqrt{-p^2}\left(\frac{r_{AdS}}{b}\right)^7\left(\frac{495\pi\lambda_{\rm GB}}{512r_{AdS}^2\sqrt{1-4\lambda_{\rm GB}}}+\frac{55\pi\lambda_{\rm GB}^2}{128r_{AdS}^4(1-4\lambda_{\rm GB})}\right)\\ &+\sqrt{-p^2}\left(\frac{r_{AdS}}{b}\right)^9\frac{77\pi\lambda_{\rm GB}^2}{256r_{AdS}^4(1-4\lambda_{\rm GB})},\end{aligned} \tag{6.94}$$

where

$$\delta^{(3)}_{GR}=\frac{231r_{AdS}^7}{16b^7}\sqrt{-p^2}B\left(\frac{7}{2},\frac{3}{2}\right){}_2F_1(1,\frac{7}{2},5,-\frac{r_{AdS}^2}{b^2}). \tag{6.95}$$

By expanding (6.94) in the large impact parameter limit, one again explicitly sees that the leading term does not depend on $\lambda_{\rm GB}$.

6.4.2.4. Inverse Fourier transform of the phase shift at $\mathcal{O}(\mu^2)$.

To make contact with the position space HHLL correlation function, one needs to perform a Fourier transform of the phase shift. According to [55], the HHLL four-point function in the Regge limit $\sqrt{-p^2}\gg 1$ is given by

$$\tilde{\mathcal{G}}(x)=\int\frac{d^dp}{(2\pi)^d}e^{ipx}\mathcal{B}(p), \tag{6.96}$$

where $\tilde{\mathcal{G}}(x)=\langle\mathcal{O}_H(x_1)\mathcal{O}_L(x_2)\mathcal{O}_L(x_3)\mathcal{O}_H(x_4)\rangle_{\text{Regge limit}}$ and $\mathcal{B}(p)=\mathcal{B}_0(p)e^{i\delta}$. The factor $\mathcal{B}_0(p)$ reproduces the disconnected correlator and it is given by

$$\mathcal{B}_0(p)=C(\Delta_L)\theta(p^0)\theta(-p^2)e^{i\pi\Delta_L}(-p^2)^{\Delta_L-\frac{d}{2}}, \tag{6.97}$$

with normalization

$$C(\Delta_L)=\frac{2^{d+1-2\Delta_L}\pi^{1+\frac{d}{2}}}{\Gamma(\Delta_L)\Gamma(\Delta_L-\frac{d}{2}+1)}. \tag{6.98}$$

We expand the integrand of (6.96) in powers of μ using (6.73), explicitly

$$\begin{aligned}\mathcal{B}(p)=\mathcal{B}_0(p)\Bigg(&1+\mu i\delta^{(1)}+\mu^2\left(i\delta^{(2)}-\frac{1}{2}{\delta^{(1)}}^2\right)\\ &+\mu^3\left(i\delta^{(3)}-\delta^{(1)}\delta^{(2)}-\frac{i}{6}{\delta^{(1)}}^3\right)+\mathcal{O}(\mu^4)\Bigg).\end{aligned} \tag{6.99}$$

This generates an expansion for $\tilde{\mathcal{G}}(x)$ from (6.96) as

$$\tilde{\mathcal{G}}(x) = \sum_{k=0}^{\infty} \mu^k \tilde{\mathcal{G}}^{(k)}(x). \tag{6.100}$$

Let us start by studying the correlator at $\mathcal{O}(\mu^2)$. The imaginary part of the correlator in the Regge limit at this order comes from $i\delta^{(2)}$ in (6.99) while the real part comes from $-\frac{1}{2}{\delta^{(1)}}^2$.

Consider first the imaginary part. To perform the inverse Fourier transform it is convenient to first expand $\delta^{(2)}$ as follows:

$$\begin{aligned}\delta^{(2)} = 7\pi^2\sqrt{-p^2}\Bigg(&\frac{5}{2}\Pi_{5,3}(L) + \left(\frac{15}{4} - \frac{5\lambda_{\rm GB}}{r_{AdS}^2\sqrt{1-4\lambda_{\rm GB}}}\right)\Pi_{7,3}(L) \\ &+ \left(5 - \frac{16\lambda_{\rm GB}}{r_{AdS}^2\sqrt{1-4\lambda_{\rm GB}}}\right)\Pi_{9,3}(L) + \dots\Bigg).\end{aligned} \tag{6.101}$$

In (6.101) $b/r_{AdS} = \sinh(L)$ and

$$\Pi_{\Delta-1;d-1}(x) = \frac{\pi^{1-\frac{d}{2}}\Gamma(\Delta-1)}{2\Gamma(\Delta-\frac{d-2}{2})}\, e^{-(\Delta-1)x}\ {}_2F_1(\frac{d}{2}-1, \Delta-1, \Delta-\frac{d-2}{2}, e^{-2x})\,, \tag{6.102}$$

the three-dimensional hyperbolic space propagator of a massive particle with mass square equal to $(\Delta-1)^2$. The dots in (6.101) stand for terms with hyperbolic space propagators with $\Delta > 10$. We can now perform the inverse Fourier transform of (6.101) with the help of eqs. (3.23) in [55] and (3.4) in [135].

The term which contains $\Pi_{5,3}(L)$ includes (after the inverse Fourier transform) the contribution of double-stress tensors with minimal twist $\tau = 4$. As we have already shown it does not depend on $\lambda_{\rm GB}$, which we can also explicitly see in (6.101). The next term, that contains $\Pi_{7,3}(L)$, includes the contribution from the double-stress tensor operators of twist $\tau_{2,1} = 6$. We can use this term to fix the coefficient b_{14} which was left undetermined in (6.21). Similar reasoning applies to all the higher-order terms in the large impact parameter expansion of (6.101). Namely, the term proportional to $\Pi_{2m+1,3}(L)$ is related to double-stress tensor operators of twist $\tau = 2m$.

Performing the inverse Fourier transform following [135] leads to

$$i\text{Im}\left(\tilde{\mathcal{G}}^{(2)}(\sigma,\rho)\right) = \int \frac{d^4p}{(2\pi)^4} e^{ipx}\mathcal{B}_0(p) i\delta^{(2)} = \frac{2i}{\Gamma(\Delta_L)\Gamma(\Delta_L-1)\sigma^{2\Delta_L+1}}$$
$$\times\Big(a_1\Pi_{5,3}(\rho)\Gamma(\Delta_L-2)\Gamma(\Delta_L+2) + b_1\Pi_{7,3}(\rho)\Gamma(\Delta_L-3)\Gamma(\Delta_L+3)$$
$$+ c_1\Pi_{9,3}(\rho)\Gamma(\Delta_L-4)\Gamma(\Delta_L+4) + \dots\Big) + \dots, \tag{6.103}$$

where $a_1 = \frac{35}{2}\pi^2$, $b_1 = 7\pi^2\left(\frac{15}{4} - \frac{5\lambda_{\text{GB}}}{r^2_{AdS}\sqrt{1-4\lambda_{\text{GB}}}}\right)$ and $c_1 = 7\pi^2\left(5 - \frac{16\lambda_{\text{GB}}}{r^2_{AdS}\sqrt{1-4\lambda_{\text{GB}}}}\right)$. The ellipses outside the parenthesis in (6.103) denote contributions due to double-trace operators in the T-channel that are not important for studying the stress tensor sector. The position space coordinates σ and ρ are defined as

$$z = 1 - \sigma e^{\rho}, \qquad \bar{z} = 1 - \sigma e^{-\rho}. \tag{6.104}$$

after the analytic continuation $z \to ze^{-2i\pi}$. Once more, notice that the dominant contribution in the large impact parameter regime, $\rho \to \infty$, comes from the factor $\Pi_{5,3}(\rho)$ in (6.103) which exactly matches the imaginary part of the correlator (6.17) in [55].

6.4.2.5. Comparison with HHLL correlation function in the lightcone limit at $\mathcal{O}(\mu^2)$.

A few simple steps are required before we can finally relate (6.103) with the results of Section 6.2 and determine the OPE coefficients of the spin-2 double-stress tensor operators. As explained in [55], one has to analytically continue $\mathcal{G}^{(2,1)}$, $\mathcal{G}^{(2,2)}$ and $\mathcal{G}^{(2,3)}$ (defined in Section 6.1) around the origin by taking $z \to ze^{-2i\pi}$ and expand the result in the vicinity of $\sigma \to 0$. The relevant term, which corresponds to the imaginary part of the correlator (6.21) as $\sigma \to 0$, reads:

$$i\text{Im}\left((\sigma e^{-\rho})^{3-\Delta_L}\mathcal{G}^{(2,1)}(1-\sigma e^{\rho})\right) = 7i\pi\frac{e^{-7\rho}}{\sigma^{2\Delta_L+1}}\Big(12600 b_{14}$$
$$+ \frac{\Delta_L\left(\Delta_L\left(\Delta_L\left(123-7\Delta_L\right)+78\right)-12\right)}{16\left(\Delta_L-3\right)\left(\Delta_L-2\right)}\Big). \tag{6.105}$$

Comparing this with the subleading term of (6.103) as $\rho \to \infty$, *i.e.*,

$$i\mathrm{Im}\left(\tilde{\mathcal{G}}^{(2)}(\sigma,\rho)\right)|_{e^{-7\rho}} = \\ -\frac{35i\pi e^{-7\rho}\Delta_L\left(\Delta_L+1\right)\left(8\lambda_{\mathrm{GB}}+\Delta_L\left(4\lambda_{\mathrm{GB}}-5\sqrt{1-4\lambda_{\mathrm{GB}}}r_{AdS}^2\right)\right)}{4\sigma^{2\Delta_L+1}\sqrt{1-4\lambda_{\mathrm{GB}}}r_{AdS}^2\left(\Delta_L^2-5\Delta_L+6\right)}+\dots, \tag{6.106}$$

with the ellipses again denoting double-trace operators, allows one to obtain the following expression for the unknown parameter b_{14}:

$$\begin{aligned} b_{14} = P_{8,2}^{(2)} =& \frac{\Delta_L\left(\Delta_L\left(\Delta_L\left(7\Delta_L-23\right)+22\right)+12\right)}{201600\left(\Delta_L-3\right)\left(\Delta_L-2\right)} \\ &-\frac{\lambda_{\mathrm{GB}}\Delta_L\left(\Delta_L+1\right)\left(\Delta_L+2\right)}{2520\sqrt{1-4\lambda_{\mathrm{GB}}}r_{AdS}^2\left(\Delta_L-3\right)\left(\Delta_L-2\right)}. \end{aligned} \tag{6.107}$$

Note that this precisely matches the OPE coefficient of the double trace operator of conformal dimension $\Delta = 8$ and $s = 2$ calculated in [15] from gravity by other means. As expected, the OPE coefficient in (6.107) explicitly depends on λ_{GB}.

Let us now go one step further and fix $P_{10,2}^{(2)}$ contributing to $\mathcal{G}^{(2,2)}(z)$ through (6.36). Analytically continuing (6.34) and taking the limit $\sigma \to 0$, yields

$$i\mathrm{Im}\left(\left(\sigma e^{-\rho}\right)^{4-\Delta_L}\mathcal{G}^{(2,2)}(1-\sigma e^{\rho})\right) = i\frac{49}{400}\frac{\pi e^{-9\rho}}{\sigma^{2\Delta_L+1}}\left(720000b_{14}+11404800\frac{P_{10,2}^{(2)}}{\mu^2}\right. \\ \left.+\frac{\Delta_L\left(\Delta_L\left(\Delta_L\left(\Delta_L\left(6327-362\Delta_L\right)+749\right)+12888\right)+12288\right)}{7\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}\right). \tag{6.108}$$

For reasons that will be explained later, we only consider here the imaginary part of the subsubleading term in the correlator. To extract the OPE data we need to compare (6.108) with the subsubleading contribution in the large impact parameter limit of (6.103), which is

$$i\mathrm{Im}\left(\tilde{\mathcal{G}}^{(2)}(\sigma,\rho)\right)|_{e^{-9\rho}} = i\frac{7}{4}\frac{\pi e^{-9\rho}}{\sigma^{2\Delta_L+1}}\left(\frac{10\Delta_L\left(\Delta_L+1\right)}{\Delta_L-2}\right. \\ \left.-\frac{7\Delta_L\left(\Delta_L+1\right)\left(\Delta_L+2\right)\left(16\lambda_{\mathrm{GB}}+\Delta_L\left(12\lambda_{\mathrm{GB}}-5\sqrt{1-4\lambda_{\mathrm{GB}}}r_{AdS}^2\right)\right)}{\sqrt{1-4\lambda_{\mathrm{GB}}}r_{AdS}^2\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}\right). \tag{6.109}$$

Substituting (6.107) in (6.108) and matching to (6.109) enables us to determine the OPE coefficient $P^{(2)}_{10,2}$,

$$\begin{aligned}P^{(2)}_{10,2} =&\frac{\Delta_L\left(\Delta_L\left(\Delta_L\left(\Delta_L\left(187\Delta_L-552\right)+901\right)+1012\right)+912\right)}{79833600\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}\\ &-\frac{\lambda_{\rm GB}\Delta_L\left(\Delta_L+1\right)\left(\Delta_L+2\right)\left(\Delta_L+3\right)}{12474\sqrt{1-4\lambda_{\rm GB}}r^2_{AdS}\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}.\end{aligned} \tag{6.110}$$

This precisely matches the one calculated in [15].

Similarly, one can match the CFT expression for $\mathrm{Im}\left((\sigma e^{-\rho})^{5-\Delta_L}\mathcal{G}^{(2,3)}(1-\sigma e^{\rho})\right)$ in (6.39), to its gravitational counterpart $\mathrm{Im}\left(\mathcal{G}^{(2)}(x)\right)|_{e^{-11\rho}}$, by expanding (6.101) and (6.103) up to $\mathcal{O}(e^{-11\rho})$. This allows one to additionally determine $P^{(2)}_{12,2}$ in (6.42)

$$\begin{aligned}P^{(2)}_{12,2} =&-\frac{5\lambda_{\rm GB}\Delta_L\left(\Delta_L+1\right)\left(\Delta_L+2\right)\left(\Delta_L+3\right)\left(\Delta_L+4\right)}{453024\sqrt{1-4\lambda_{\rm GB}}r^2_{AdS}\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}+\\ &\frac{\Delta_L\left(\Delta_L+1\right)\left(\Delta_L\left(\Delta_L\left(\Delta_L\left(6721\Delta_L-15603\right)+46474\right)+100828\right)+143760\right)}{44396352000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}.\end{aligned} \tag{6.111}$$

Notice that we did not use the real part of $\tilde{\mathcal{G}}^{(2)}(\sigma,\rho)$, which comes from the term $-\frac{1}{2}{\delta^{(1)}}^2$ in (6.99) and behaves as $\sigma^{-2\Delta_L-2}$ for $\sigma\to 0$. This term matches the corresponding term with the same σ behavior in the correlator. It does not give us any new information, because it is independent of the OPE coefficients of operators with spin $s=0,2$.

6.4.2.6. Extracting OPE data from the gravitational phase shift at $\mathcal{O}(\mu^3)$.

Let us now consider the $\mathcal{O}(\mu^3)$ terms in the correlator. Focusing on the gravity side, we start by performing an inverse Fourier transform. (6.99) instructs us to consider three terms $i\delta^{(3)}$, $\delta^{(1)}\delta^{(2)}$ and $i(\delta^{(1)})^3$, which give rise to terms that behave as $\sigma^{-2\Delta_L-1}$, $\sigma^{-2\Delta_L-2}$ and $\sigma^{-2\Delta_L-3}$, respectively. Performing the relevant computations, we observe that $\delta^{(1)}\delta^{(2)}$ and $i(\delta^{(1)})^3$ do not provide additional information because the corresponding terms in the correlators are already fixed by bootstrap (these terms simply give us an extra consistency check). Focusing on the inverse Fourier transform of $i\delta^{(3)}$, we expand (6.94) in terms of the hyperbolic space propagators, $\Pi_{m,3}(L)$,

$$\delta^{(3)}=\sqrt{-p^2}\Big(a_2\Pi_{7,3}(L)+b_2\Pi_{9,3}(L)+c_2\Pi_{11,3}(L)+\dots\Big), \tag{6.112}$$

where

$$
\begin{aligned}
a_2 &= \frac{1155}{8}\pi^2, \\
b_2 &= 231\pi^2 \left(-\frac{3\lambda_{\rm GB}}{r_{AdS}^2\sqrt{1-4\lambda_{\rm GB}}} + 2 \right), \\
c_2 &= \frac{231\pi^2}{8} \left(\frac{32\lambda_{\rm GB}^2}{r_{AdS}^4(1-4\lambda_{\rm GB})} - \frac{120\lambda_{\rm GB}}{r_{AdS}^2\sqrt{1-4\lambda_{\rm GB}}} + 35 \right),
\end{aligned}
\tag{6.113}
$$

which leads to

$$
\begin{aligned}
&i\mathrm{Im}\left(\tilde{\mathcal{G}}^{(3)}(\sigma,\rho)\right)\Big|_{\frac{1}{\sigma^{2\Delta_L+1}}} = \int \frac{d^4p}{(2\pi)^4} e^{ipx}\mathcal{B}_0(p) i\delta^{(3)} = \frac{2i}{\Gamma(\Delta_L)\Gamma(\Delta_L-1)\sigma^{2\Delta_L+1}} \\
&\times \Big(a_2\Pi_{7,3}(\rho)\Gamma(\Delta_L-3)\Gamma(\Delta_L+3) + b_2\Pi_{9,3}(\rho)\Gamma(\Delta_L-4)\Gamma(\Delta_L+4) \\
&+ c_2\Pi_{11,3}(\rho)\Gamma(\Delta_L-5)\Gamma(\Delta_L+5) + \dots \Big) + \text{double traces},
\end{aligned}
\tag{6.114}
$$

The leading and subleading contributions in the large impact parameter limit $\rho \to \infty$ come from $\Pi_{7,3}(\rho)$ and $\Pi_{9,3}(\rho)$ and behave as $\frac{i\pi e^{-7\rho}}{\sigma^{2\Delta_L+1}}$ and $\frac{i\pi e^{-9\rho}}{\sigma^{2\Delta_L+1}}$, respectively. They are precisely matched by the relevant terms in (6.45) in the vicinity of $\sigma \to 0$ after analytic continuation [135]. This is another sanity check of the procedure described herein, since these terms do not incorporate contributions from spin-2 operators.

To extract further OPE data, we proceed to match the subsubleading correction of (6.114) in the large impact parameter limit to the term in (6.53) which behaves as $\sim \frac{i\pi e^{-11\rho}}{\sigma^{2\Delta_L+1}}$. This allows us to determine the coefficient $e_{115} = P^{(3)}_{12,2}$ in (6.53) which corresponds to the OPE coefficient of the triple-stress tensors of spin $s=2$ with conformal dimension $\Delta = 12$:

$$
\begin{aligned}
e_{115} =& -\frac{117\Delta_L^6 - 439\Delta_L^5 + 407\Delta_L^4 + 859\Delta_L^3 + 202\Delta_L^2 + 696\Delta_L}{172972800(\Delta_L-2)(\Delta_L-3)(\Delta_L-4)(\Delta_L-5)} \\
&- \frac{\lambda_{\rm GB}(143\Delta_L^6 - 231\Delta_L^5 - 3597\Delta_L^4 - 9489\Delta_L^3 - 11186\Delta_L^2 - 4920\Delta_L)}{43243200 r_{AdS}^2\sqrt{1-4\lambda_{\rm GB}}(\Delta_L-2)(\Delta_L-3)(\Delta_L-4)(\Delta_L-5)} \\
&+ \frac{\lambda_{\rm GB}^2\Delta_L(\Delta_L+1)(\Delta_L+2)(\Delta_L+3)(\Delta_L+4)}{24024 r_{AdS}^4(1-4\lambda_{\rm GB})(\Delta_L-2)(\Delta_L-3)(\Delta_L-4)(\Delta_L-5)} \\
&+ P^{(2)}_{8,0}\frac{76 + \frac{400}{\Delta_L-5} + 11\Delta_L}{1320}.
\end{aligned}
\tag{6.115}
$$

Notice that e_{115} is not completely determined by the above procedure since the spin-0 OPE data, $P^{(2)}_{8,0}$, is not fixed. Summarising, we conclude that we are able to fix all coefficients in the ansatz except those that correspond to the OPE coefficients of operators of spin-0. However, using the expression for $P^{(2)}_{8,0}$ found in [15] one finds

$$
\begin{aligned}
&P^{(3)}_{12,2} = \\
&\quad \frac{1001\Delta_L^7 - 6864\Delta_L^6 + 12615\Delta_L^5 - 3980\Delta_L^4 - 6156\Delta_L^3 - 11736\Delta_L^2 - 1440\Delta_L}{3459456000(\Delta_L-2)(\Delta_L-3)(\Delta_L-4)(\Delta_L-5)} \\
&\quad - \frac{\lambda_{\rm GB}(143\Delta_L^6 - 206\Delta_L^5 - 1631\Delta_L^4 - 3622\Delta_L^3 - 3540\Delta_L^2 - 1200\Delta_L)}{28828800 r_{AdS}^2\sqrt{1-4\lambda_{\rm GB}}(\Delta_L-2)(\Delta_L-3)(\Delta_L-4)(\Delta_L-5)} \\
&\quad + \frac{\lambda_{\rm GB}^2\Delta_L(\Delta_L+1)(\Delta_L+2)(\Delta_L+3)(\Delta_L+4)}{24024 r_{AdS}^4(1-4\lambda_{\rm GB})(\Delta_L-2)(\Delta_L-3)(\Delta_L-4)(\Delta_L-5)}.
\end{aligned}
\tag{6.116}
$$

6.5. Lorentzian inversion formula

It was recently shown in [128] that one can obtain the OPE coefficients of minimal twist double and triple-stress tensors using the Lorentzian inversion formula. Here, we review this method and show how it can be generalized to extract the OPE coefficients of twist-six double-stress tensors. In principle, it can also be generalized to multi-stress tensors of arbitrarily high twist.

6.5.1. Twist-four double-stress tensors

Consider the correlation function

$$
(w\bar{w})^{-\Delta_L}\hat{\mathcal{G}}(w,\bar{w}) = \langle \mathcal{O}_H(\infty)\mathcal{O}_H(1)\mathcal{O}_L(w,\bar{w})\mathcal{O}_L(0)\rangle. \tag{6.117}
$$

The Lorentzian inversion formula is given by [73,78]

$$
\begin{aligned}
c(\tau,\beta) = \frac{1+(-1)^{\frac{\beta-\tau}{2}}}{2}\kappa_\beta \int_0^1 & dw d\bar{w}\mu^{(0,0)}(w,\bar{w}) \\
&\times g^{(0,0)}_{-\tau+2(d-1),\frac{\beta+\tau}{2}-d+1}(w,\bar{w}) \mathrm{dDisc}[\hat{\mathcal{G}}(w,\bar{w})],
\end{aligned}
\tag{6.118}
$$

where

$$
\mu^{(0,0)}(w,\bar{w}) = \frac{|w-\bar{w}|^{d-2}}{(w\bar{w})^d}, \tag{6.119}
$$

$$\kappa_\beta = \frac{\Gamma(\frac{\beta}{2})^4}{2\pi^2\Gamma(\beta)\Gamma(\beta-1)}, \tag{6.120}$$

where $\tau = \Delta - s$ and $\beta = \Delta + s$. Here $g^{(0,0)}_{\tau,s}$ is a conformal block given with $\Delta \to s+d-1$ and $s \to \Delta - d + 1$ and in $d=4$ is given by (3.57). Moreover, dDisc denotes the double-discontinuity of $\hat{\mathcal{G}}(w,\bar{w})$ in (6.117), which is equal to the correlator of a double commutator, and it is given by

$$\text{dDisc}[\hat{\mathcal{G}}(w,\bar{w})] = \hat{\mathcal{G}}(w,\bar{w}) - \frac{1}{2}\hat{\mathcal{G}}^{\circlearrowleft}(w,\bar{w}) - \frac{1}{2}\hat{\mathcal{G}}^{\circlearrowright}(w,\bar{w})\,. \tag{6.121}$$

Here $\hat{\mathcal{G}}^{\circlearrowleft}$ and $\hat{\mathcal{G}}^{\circlearrowright}$ correspond to the same correlator analytically continued in two different ways around $w=1$, namely $(1-w) \to (1-w)e^{\pm 2\pi i}$. The OPE data, $P_{\frac{\tau'+\beta}{2},\frac{\beta-\tau'}{2}}$, can be extracted from $c(\tau,\beta)$ via[24]

$$P_{\frac{\tau'+\beta}{2},\frac{\beta-\tau'}{2}} = -\text{Res}_{\tau=\tau'}c(\tau,\beta), \tag{6.122}$$

where τ' and β denote the twist and conformal spin of operators in the physical spectrum of the theory exchanged in the channel $\mathcal{O}_L \times \mathcal{O}_L \to \mathcal{O}_{\tau',J'} \to \mathcal{O}_H \times \mathcal{O}_H$.

We would like to apply the Lorentzian inversion formula to the HHLL correlator to extract the OPE data of the double-stress tensors. To this end, we will use information of the correlator from the channel where $\mathcal{O}_H\mathcal{O}_L$ merge. The function $\hat{\mathcal{G}}(z,\bar{z})$ can be obtained from $\mathcal{G}(z,\bar{z})$ via

$$\hat{\mathcal{G}}(w,\bar{w}) = (w\bar{w})^{\Delta_L}\mathcal{G}(1-w,1-\bar{w}). \tag{6.123}$$

To apply the Lorentzian inversion formula we first need to calculate $\mathcal{G}(z,\bar{z})$ using the S-channel operator product expansion (3.52). First, let us start with the leading contribution of $\mathcal{G}(z,\bar{z})$ in the lightcone limit $\bar{z} \to 1$ at $\mathcal{O}(\mu^2)$. These give the leading contributions when $\bar{w} \to 0$ in $G(w,\bar{w})$. After the integration with respect to $\bar{w}$ in (6.118), these contributions fix the position of the pole and residue of $c(\tau,\beta)$ that corresponds to lowest-twist double-stress tensors. Subleading contributions in $\bar{z} \to 1$ (or $\bar{w} \to 0$) only create new poles, without changing the residue of existing ones, therefore, they do not affect the OPE

[24] In principle there is an extra term in this relation when $\tau - d = 0, 1, 2, \ldots$ [73], however, it vanishes in the cases considered.

coefficients of lowest-twist operators. The leading contribution in the $(1-\bar{z})$-expansion comes from the leading contribution of the $1/l$-expansion of the S-channel OPE data. Only the term proportional to $\log^2(z)$ contributes to the double-discontinuity and we denote it by $\mathcal{G}^{(2)}(z,\bar{z})\big|_{\log^2(z)}$. The number in the superscript denotes the power of μ in which we are working. Substituting in to (3.67) equations (3.25), (3.70), (5.10) and (3.69), we find that

$$\mathcal{G}^{(2)}(z,\bar{z})\big|_{\log^2(z)} = \log^2(z\bar{z})\int_0^\infty dl \sum_{n=0}^{\infty} \frac{(z\bar{z})^n l^{\Delta_L-3}\left(z^{l+1}-\bar{z}^{l+1}\right)\Gamma\left(n+\Delta_L-1\right)}{8(z-\bar{z})\Gamma(n+1)\Gamma\left(\Delta_L-1\right)\Gamma\left(\Delta_L\right)} \times\left(\left(\gamma_n^{(1,0)}\right)^2+\mathcal{O}\left(\frac{1}{l}\right)\right). \tag{6.124}$$

In the lightcone limit, the dominant contribution to this expression comes from operators with large spin $l \gg 1$, we can, therefore, approximate the sum over l by an integral. Note that only $\mathcal{O}(\mu)$ OPE data, *i.e.*, $\gamma_n^{(1,0)}$, appears in (6.124). Using (6.28) we evaluate (6.124) and collect the leading term as $\bar{z}\to 1$,

$$\begin{aligned}&\mathcal{G}^{(2)}(z,\bar{z})\big|_{\log^2(z)} = \log^2(z)\frac{(1-\bar{z})^{2-\Delta_L}(1-z)^{-\Delta_L-4}}{32\left(\Delta_L-2\right)}\times\\ &\Delta_L\left(\Delta_L\left((z(z+4)+1)^2\Delta_L+z(z(54-(z-28)z)+28)-1\right)+72z^2\right)\\ &+\mathcal{O}\left((1-\bar{z})^{3-\Delta_L}\right).\end{aligned} \tag{6.125}$$

With the help of (6.123) one obtains

$$\begin{aligned}&\hat{\mathcal{G}}^{(2)}(w,\bar{w})\big|_{\log^2(1-w)} = \frac{\Delta_L\bar{w}^2\log^2(1-w)}{32w^4(\Delta_L-2)}\times\\ &\left(\Delta_L\left(((w-6)w+6)^2\Delta_L-w(w(w(w+24)-132)+216)+108\right)+72(w-1)^2\right)\\ &+\mathcal{O}(\bar{w}^3),\end{aligned} \tag{6.126}$$

which agrees with (4.12) in [128]. Now, it is easy to see that

$$\begin{aligned}&\mathrm{dDisc}[\hat{\mathcal{G}}^{(2)}(w,\bar{w})] = \frac{\pi\bar{w}^2\Delta_L}{8w^4(\Delta_L-2)}\times\\ &\left(\Delta_L\left(((w-6)w+6)^2\Delta_L-w(w(w(w+24)-132)+216)+108\right)\right.\\ &\left.+72(w-1)^2\right)+\mathcal{O}(\bar{w}^3).\end{aligned} \tag{6.127}$$

To compute the integral (6.118) we substitute

$$\mu^{(0,0)}(w,\bar{w}) = \frac{1}{w^2\bar{w}^4}+\mathcal{O}\left(\frac{1}{\bar{w}^3}\right), \tag{6.128}$$

$$g^{(0,0)}_{-\tau+2(d-1),\frac{\tau+\beta}{2}-d+1}(w,\bar{w}) = \bar{w}^{3-\frac{\tau}{2}}\left(f_{\frac{\beta}{2}}(1-w)+\mathcal{O}(\bar{w})\right), \tag{6.129}$$

valid in the lightcone limit $\bar{w}\to 0$ (or $\bar{z}\to 1$), and set $(-1)^{\frac{\beta-\tau}{2}}=1$ since only even-spin operators contribute. Combining the above we arrive at the following expression for $c(\tau,\beta)$

$$\begin{aligned} c_0(\tau,\beta) = &-\frac{\sqrt{\pi}2^{-\beta+1}\Delta_L\Gamma\left(\frac{\beta}{2}\right)}{(\tau-4)(\beta-10)(\beta-6)(\beta-2)\beta(\beta+4)}\times \\ &\left(\frac{384\left(\Delta_L-7\right)\Delta_L+4608}{(\beta+8)\left(\Delta_L-2\right)\Gamma\left(\frac{1}{2}(\beta-1)\right)}+\right. \\ &\left.+\frac{(\beta-2)\beta\Delta_L\left((\beta-2)\beta\left(\Delta_L-1\right)-56\Delta_L+200\right)}{(\beta+8)\left(\Delta_L-2\right)\Gamma\left(\frac{1}{2}(\beta-1)\right)}\right), \end{aligned} \tag{6.130}$$

where the subscript denotes that this result is obtained in the leading order of the lightcone expansion. The OPE coefficients of the minimal-twist double-stress tensors are given by

$$P^{(2)}_{\frac{\beta}{2}+2,\frac{\beta}{2}-2} = -\mathrm{Res}_{\tau=4}c_0(\tau,\beta), \tag{6.131}$$

where $\beta=12+4\ell$, $\ell\geq 0$, and are in precise agreement with (1.6) in [124] and (4.15) in [128].

6.5.2. Twist-six double-stress tensors

Here we use the same method to obtain the OPE coefficients of double-stress tensors with twist $\tau_{2,1}=6$. We first need to compute the subleading contribution in the lightcone limit to eqs. (6.127), (6.128) and (6.129). Specifically, the integration measure

$$\mu^{(0,0)}(w,\bar{w}) = \frac{1}{w^2\bar{w}^4}-\frac{2}{w^3\bar{w}^3}+\mathcal{O}\left(\bar{w}^{-2}\right), \tag{6.132}$$

and the conformal block,

$$g^{(0,0)}_{-\tau+2(d-1),\frac{\tau+\beta}{2}-d+1}(w,\bar{w}) = \bar{w}^{3-\frac{\tau}{2}}f_{\frac{\beta}{2}}(1-w)\left(1+\bar{w}\left(1-\frac{\tau}{4}+\frac{1}{w}\right)+\mathcal{O}(\bar{w}^2)\right), \tag{6.133}$$

were obtained from the explicit expressions given in (6.119) and (3.57).

To evaluate the subleading term in $\mathrm{dDisc}[\hat{\mathcal{G}}^{(2)}(w,\bar{w})]$ we reconsider the S-channel computation. Similarly to the case of leading twist, only the part of the correlator with $\log^2(z)$ contributes to the discontinuity. However, we now have to include the subleading corrections in the $1/l$-expansion of the S-channel OPE data. With the help of (3.67), (3.25), (5.10), (3.69) and (3.70) one finds that

$$\begin{aligned}
\mathcal{G}^{(2)}(z,\bar{z})\big|_{\log^2(z)} &= \frac{\log^2(z\bar{z})}{16(z-\bar{z})\Gamma(\Delta_L)\Gamma(\Delta_L-1)}\sum_{n=0}^{\infty}(z\bar{z})^n\frac{\Gamma(\Delta_L-1+n)}{\Gamma(n+1)} \\
&\int_0^{\infty} dl\, l^{\Delta_L-6}\left(z^{l+1}-\bar{z}^{l+1}\right)\left(2(l-2n)+\Delta_L\left(\Delta_L+2n-1\right)\right)\left(l\gamma_n^{(1,0)}+\gamma_n^{(1,1)}\right)^2 \\
&+\mathcal{O}\left(l^{\Delta_L-7}\right).
\end{aligned} \tag{6.134}$$

To proceed, one evaluates (6.134) using (6.28) and collects the leading and subleading contributions as $\bar{z}\to 1$, which behave as $(1-\bar{z})^{2-\Delta_L}$ and $(1-\bar{z})^{3-\Delta_L}$ respectively. Using (6.123) it is then simple to obtain $\hat{\mathcal{G}}^{(2)}(w,\bar{w})\big|_{\log^2(1-w)}$ up to $\mathcal{O}(\bar{w}^4)$ and evaluate its double-discontinuity:

$$\begin{aligned}
\mathrm{dDisc}[\hat{\mathcal{G}}^{(2)}(w,\bar{w})] &= -\frac{\pi^2\bar{w}^2\Delta_L}{8w^5\left(\Delta_L-3\right)\left(\Delta_L-2\right)}\Big(-3w^5\Delta_L-72w^4\Delta_L \\
&+324w^3\Delta_L-504w^2\Delta_L+252w\Delta_L+216w^3-432w^2+216w+4w^5\Delta_L^2 \\
&-12w^4\Delta_L^2+12w^3\Delta_L^2-36w\Delta_L^3-w^5\Delta_L^3+12w^4\Delta_L^3-48w^3\Delta_L^3+72w^2\Delta_L^3 \\
&+\bar{w}(-144\Delta_L+612w\Delta_L+216w^3-432w^2+216w-w^5\Delta_L-52w^4\Delta_L \\
&+324w^3\Delta_L-744w^2\Delta_L+540w\Delta_L^2-216\Delta_L^2-72\Delta_L^3+w^5\Delta_L^2-18w^4\Delta_L^2 \\
&+156w^3\Delta_L^2-456w^2\Delta_L^2+144w\Delta_L^3-2w^4\Delta_L^3+24w^3\Delta_L^3-96w^2\Delta_L^3)\Big) \\
&+\mathcal{O}(\bar{w}^4).
\end{aligned} \tag{6.135}$$

Substituting (6.132), (6.133) and (6.135) in (6.118) and integrating leads to an analytic expression for $c(\tau,\beta)$. The relevant part of this expression – the one with non-zero residue at $\tau = 6$ – turns out to be:

$$c_1(\tau,\beta) = -\frac{2^{4-\beta}\sqrt{\pi}\Gamma\left(\frac{\beta}{2}\right)\Delta_L}{(\beta-12)(\beta-8)(\beta-4)(\tau-10)(\tau-8)(\tau-6)(\tau-4)}$$
$$\times\left(\frac{\beta^4\Delta_L - 4\beta^3\Delta_L - 68\beta^2\Delta_L - 960\beta\Delta_L^2 + 144\beta\Delta_L - 14976\Delta_L^2}{(\beta+2)(\beta+6)(\beta+10)\Gamma\left(\frac{\beta-1}{2}\right)(\Delta_L-3)(\Delta_L-2)}\right.$$
$$+\frac{\beta^4\Delta_L^3 - 2\beta^4\Delta_L^2 - 4\beta^3\Delta_L^3 + 8\beta^3\Delta_L^2 - 116\beta^2\Delta_L^3 + 472\beta^2\Delta_L^2}{(\beta+2)(\beta+6)(\beta+10)\Gamma\left(\frac{\beta-1}{2}\right)(\Delta_L-3)(\Delta_L-2)}$$
$$\left.+\frac{240\beta\Delta_L^3 + 2304\Delta_L^3 + 19584\Delta_L + 13824}{(\beta+2)(\beta+6)(\beta+10)\Gamma\left(\frac{\beta-1}{2}\right)(\Delta_L-3)(\Delta_L-2)}\right) + \ldots, \tag{6.136}$$

where the ellipsis stands for the terms with zero residue at $\tau = 6$ and 1 in the subscript denotes that this expression is obtained in the subleading order of the lightcone expansion.

It is now straightforward to read off the OPE coefficients of double-stress tensors with twist $\tau_{2,1} = 6$ from

$$P^{(2)}_{\frac{\beta}{2}+3,\frac{\beta}{2}-3} = -\mathrm{Res}_{\tau=6}c_1(\tau,\beta). \tag{6.137}$$

For $\beta = 14 + 4\ell$, eq. (6.32) is reproduced. It is already stated in Section 6.2 that this formula does not reproduce the right OPE coefficient $P^{(2)}_{8,2}$ for $\ell = -1$. Thus, we explicitly see that the Lorentzian inversion formula does not allow us to obtain the OPE data of spin-2 double-stress tensors with twist $\tau = 6$.

In general, to determine for which operators at $\mathcal{O}(\mu^k)$ the Lorentzian inversion formula can be applied, one has to consider the behavior of the correlator in the Regge limit. At $\mathcal{O}(\mu^k)$ the correlator in the Regge limit behaves like $1/\sigma^{2\Delta_L+k}$. Therefore, the Lorentzian inversion formula correctly produces the OPE coefficients of multi-stress tensor operators with spin $s > k+1$. Accordingly, already at order $\mathcal{O}(\mu^3)$, fixing the OPE coefficients by combining an ansatz for the correlator with the crossing symmetry (or Lorentzian inversion formula) appears more powerful than the Lorentzian inversion formula alone. Namely, we were able to fix the OPE coefficients of spin-4 operators and the one with twist $\tau = 8$ is given by (D.1), while using the Lorentzian inversion formula one can only fix the OPE coefficients of operators with spin $s > 4$.

In this section, we consider the stress tensor sector of a four-point function of pairwise identical scalars in a class of CFTs with a large central charge. It is completely determined by the OPE coefficients of multi-stress tensor operators, which can be read off the result for a heavy-heavy-light-light correlator. The stress tensor sector of the HHLL correlator is naturally expanded perturbatively in $\mu \sim \frac{\Delta_H}{C_T}$, where Δ_H is the scaling dimension of the heavy operator. The power of μ counts the number of stress tensors within the exchanged multi-stress tensor operators. By further expanding the HHLL stress tensor sector in the lightcone limit, the multi-stress tensor operators can be organized into sectors of different twists. Similarly to the minimal-twist sector, combining an appropriate ansatz with the lightcone bootstrap, we show that the contribution from the non-minimal twist multi-stress tensors is almost completely determined. Unlike the minimal twist case, a few coefficients are not fixed by the bootstrap – these correspond to the OPE coefficients of multi-stress tensors with spin $s = 0, 2$.

An extra check is provided by applying the Lorentzian OPE inversion formula (see [128] for an earlier application of the inversion formula in this context). It gives the same results but has less predictive power than the ansatz.

The OPE coefficients for double-stress tensors are particularly simple and we provide closed-form expressions for those with twist $\tau = 4, 6, 8, 10$ and any spin greater than 2. All of these OPE coefficients are completely fixed by the bootstrap. This is related to their independence of the higher-derivative terms in the dual bulk gravitational Lagrangian. The OPE coefficients for double-stress tensors with spin $s = 0, 2$ are not fixed by the bootstrap and do depend on such higher derivative terms. It is interesting that at the level of double-stress tensors, only the OPE coefficients with spin $s = 0, 2$ are not fixed by the bootstrap (non-universal). On the other hand, all non-minimal twist triple-stress tensor OPE coefficients are non-universal[25] .

Assuming a holographic dual, we show that the OPE coefficients for spin-2 multi-stress tensors can be determined by studying the large impact parameter

[25] Here we use universality and "fixed by the bootstrap" terms interchangeably. However, it remains to be determined what is the universality class and whether it the same as the set of unitary holographic theories.

regime of the Regge limit, following [55,12,135] (modulo the spin zero OPE data). This is done explicitly in Einstein Hilbert+Gauss-Bonnet gravity. Some of these OPE coefficients are known [15] and agree with our results.

It would be interesting if one could compute the spin zero and spin two multi stress tensor OPE coefficients with CFT techniques. Perhaps the conglomeration approach first discussed in [38] or the more recent work [139,140] will be useful in this direction.

The regime of applicability of the ansatz (and the exact meaning of universality) used in this section remains unsettled (the ansatz seems to work in holographic CFTs, but does it also apply for other CFTs with a large central charge?). This question appears already in the leading twist case studied in [13]. To address this issue, it would be interesting to investigate the OPE coefficients of multi-stress tensors in CFTs with a large central charge, but not necessarily holographic. A related question is the existence of an infinite-dimensional algebra responsible for the form of the near-lightcone correlator. In two dimensions the relevant algebra is simply the Virasoro algebra. The Virasoro vacuum block has been computed in several ways [40,105-108,110,141]. Recently an algebraic way of reproducing the near lightcone contribution of the stress tensor was discussed in [142] – it would be interesting to investigate this further.

Returning to holographic theories, one interesting question would be to understand the critical behavior of geodesics in the vicinity of the circular light orbit, recently studied in [143], from the CFT point of view. This corresponds to the situation where the deflection angle is very large. The deflection angle φ in asymptotically flat Schwarzschild geometries is supposed to be related to the eikonal phase δ via

$$2\sin\frac{\varphi}{2} = -\frac{1}{E}\frac{\partial\delta}{\partial b} \tag{6.138}$$

where E is the incoming particle energy and b is the impact parameter (see e.g. [144] for a recent discussion). This agrees with eq. (B.5.1) for small deflection angles, but deviations might occur for large deflection angles. It would be interesting to investigate this further.

7. Thermalization in large-N CFTs

7.1. Introduction and summary

Holography [8-10] provides us with a useful tool to study d-dimensional CFTs at large central charge C_T, especially when combined with modern CFT techniques (see e.g. [24-26] for reviews). One of the basic objects in this setup is a Witten diagram with a single graviton exchange which contributes to four-point functions. It can be decomposed into the conformal blocks of the stress-tensor and of the double-trace operators made out of external fields [43].

When a pair of the external operators denoted by $\mathcal{O}_H$ is taken to be heavy, with the conformal dimension $\Delta_H \sim C_T$, and the other pair denoted by $\mathcal{O}_L$ stays light, the resulting heavy-heavy-light-light (HHLL) correlator describes a light probe interacting with a heavy state. In this case, operators which are comprised out of many stress tensors (multi stress tensor operators) contribute, together with the multi-trace operators involving $\mathcal{O}_L$. As we review below, the OPE coefficients of the scalar operators with a (unit-normalized) multi stress tensor operator $T^k_{\tau,s}$, which contains k stress tensors and has twist τ and spin s, scale like $\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T^k_{\tau,s}} \sim \Delta^k / C_T^{k/2}$ for large Δ.

The contribution of a given multi stress tensor operator to the HHLL four-point function $\langle \mathcal{O}_H \mathcal{O}_L \mathcal{O}_L \mathcal{O}_H \rangle$ can be compared to the contribution of the same operator to the corresponding two-point function at finite temperature[26] β^{-1}, $\langle \mathcal{O}_L \mathcal{O}_L \rangle_\beta$. In this section we argue that they are the same in generic large-C_T CFTs. As we explain later, this means that OPE coefficients of $T^k_{\tau,s}$ with the two heavy operators $\mathcal{O}_H$, $\langle \mathcal{O}_H T^k_{\tau,s} \mathcal{O}_H \rangle$, are equal to their finite temperature expectation values, $\langle T^k_{\tau,s} \rangle_\beta$. The relation between the inverse temperature β and the conformal dimension Δ_H is set by considering the stress tensor ($k = 1, \tau = d-2, s = 2$), but the equality between the thermal expectation values and the OPE coefficients for all other multi stress tensor operators is a nontrivial statement. We call it "the thermalization of the stress tensor sector" [27]. It is

[26] See [29,35,82,145-156] for some previous work on finite temperature conformal field theories in $d > 2$.

[27] We show this explicitly for certain primary heavy operators $\mathcal{O}_H$ in free CFTs. We also observe that other light operators do not satisfy the thermalization property that the stress tensor sector enjoys.

directly related to the Eigenstate Thermalization Hypothesis (ETH) [157-161], as we review below. Hence, we argue that all multi stress tensor operators in the large-C_T CFTs satisfy the ETH. In $d = 2$ the ETH and thermalization have been studied in e.g. [105,108,111-117,162-181].

Here we want to address the $d > 2$ case. In holographic theories CFT and bootstrap techniques provide a lot of data which indicates that the thermalization of the stress tensor sector happens [12-15,54,55,124-126,128,135,182]. Some of the OPE coefficients in holographic CFTs were computed using two-point functions in a black hole background [15] – these are thermal correlators according to the standard holographic dictionary. It is also worth noting that the leading Δ behavior of the OPE coefficients in holographic models does not depend on the coefficients of the higher derivative terms in the bulk lagrangian [14] (this should not be confused with the universality of the OPE coefficients of the minimal-twist multi stress tensors [15]). Such a universality follows from the thermalization of the stress tensor sector as we discuss below.

A natural question is whether the thermalization of the stress tensor sector is just a property of holographic CFTs or if it holds more generally. In this section we argue for the latter scenario. We compute the OPE coefficients (and the thermal expectation values) for a number of multi stress tensor operators in a free CFT and observe thermalization as well as universality of OPE coefficients. We also provide a bootstrap argument for all CFTs with a large central charge.

The rest of the section is organized as follows. In Section 7.2, we begin by considering the thermalization of multi stress tensor operators $T^k_{\tau,s}$. The heavy state we consider is created by a scalar operator $\mathcal{O}_H$ with dimension $\Delta_H \sim C_T$ and by thermalization of a multi stress tensor operator we mean[28]

$$\langle \mathcal{O}_H | T^k_{\tau,s} | \mathcal{O}_H \rangle \Big|_{\frac{\Delta_H^k}{C_T^{k/2}}} = \lambda_{\mathcal{O}_H \mathcal{O}_H T^k_{\tau,s}} \Big|_{\frac{\Delta_H^k}{C_T^{k/2}}} = \langle T^k_{\tau,s} \rangle_\beta, \tag{7.1}$$

where the heavy state $|\mathcal{O}_H\rangle$ on the sphere of unit radius is created by the operator $\mathcal{O}_H$, $\lambda_{\mathcal{O}_H \mathcal{O}_H T^k_{\tau,s}}$ are the OPE coefficients of $T^k_{\tau,s}$ in the $\mathcal{O}_H \times \mathcal{O}_H$ OPE and $|_{\Delta_H^k / C_T^{k/2}}$ means we keep only leading terms that scale like $\Delta_H^k / C_T^{k/2} \sim C_T^{k/2}$.

[28] Here we are suppressing the tensor structure. Note that all terms scale like $C_T^{k/2}$ which is consistent with $T^k_{\tau,s}$ being unit-normalized.

In (7.1) $\langle T^k_{\tau,s}\rangle_\beta$ is the one-point function on the sphere at finite temperature β^{-1}. Note that the OPE coefficients involving the stress tensor are fixed by the Ward identity, and hence eq. (7.1) for the stress tensor establishes a relation between the temperature β^{-1} and Δ_H. By the large-C_T factorization[29], the thermal one-point functions of multi stress tensors can be related to the thermal one-point function of the stress tensor itself. Explicitly,

$$\langle T^k_{\tau,s}\rangle_\beta = c^k_{\tau,s}(\langle T^1_{d-2,2}\rangle_\beta)^k = c^k_{\tau,s}(\lambda_{\mathcal{O}_H\mathcal{O}_H T^1_{d-2,2}})^k, \tag{7.2}$$

where $c^k_{\tau,s}$ are theory-independent coefficients that appear because of the index structure in $\langle T^k_{\tau,s}\rangle_\beta$. In the second equality in (7.2) we used (7.1) for the stress tensor. Note that (7.1) and (7.2) imply that the leading Δ_H behavior of the multi stress tensor OPE coefficients is universal, i.e. it does not depend on the theory[30]. We provide a bootstrap argument for this universality in all large-C_T theories. Also note that (7.2) is written for multi-trace operators $T^k_{\tau,s}$ which do not contain derivatives, but the presence of derivatives does not affect the statement of universality.

In Section 7.3, we check the universality by computing a number of the multi stress tensor OPE coefficients in a free $SU(N)$ adjoint scalar theory in $d = 4$ dimensions. We compare the leading Δ_H behavior in the free theory with results from holography/bootstrap and find perfect agreement in all cases listed below. After fixing the coefficients for the stress tensor case in Section 7.3.1, we look at the first nontrivial case, $T^2_{4,4}$ in Section 7.3.2. Section 7.3.3 is devoted to the double stress tensor with two derivatives, $T^2_{4,6}$. This is an operator whose finite temperature expectation value vanishes in the large volume limit (on the plane), but is finite on the sphere. In Section 7.3.4 we consider minimal twist multi stress tensors of the type $T^k_{2k,2k}$. Section 7.3.5 is devoted to multi stress tensors with non-minimal twist, $T^2_{6,2}$ and $T^2_{8,0}$.

[29] See [183] for a general discussion of large-N factorization and [184,185] and [35] for the discussion in the context of gauge theories and CFTs respectively. The factorization holds in adjoint models in the 't Hooft limit at finite temperature, but there are counterexamples, like e.g. a direct product of low-C_T CFTs. However the factorization of multi stress tensors would still apply in these models.

[30] This amounts to the large-C_T factorization of correlators $\langle \mathcal{O}_H | T_{\mu\nu} \dots T_{\alpha\beta} | \mathcal{O}_H\rangle$ in heavy states.

In Section 7.4, we verify that (7.1) holds in the free adjoint scalar theory for a variety of operators. In this section we again consider $d = 4$, but in addition, take the infinite volume limit. This is for technical reasons – it is easier to compute a finite temperature expectation value on the plane than on the sphere. We spell out the index structure in (7.1) in detail and go over all the examples discussed in the previous section. In addition, we discuss some triple stress tensor operators.

We continue in Section 7.5 by studying thermal two-point functions in the free adjoint scalar model in $d = 4$. By decomposing the correlator into thermal blocks we read off the product of thermal one-point functions and the OPE coefficients for several operators of low dimension and observe agreement with the results of Sections 7.3 and 7.4. Due to the presence of multiple operators with the same dimension and spin, we have to solve a mixing problem to find which operators contribute to the thermal two-point function.

In Section 7.6 we explain the relation between our results and the Eigenstate Thermalization Hypothesis. We observe that unlike multi stress tensors, other light operators explicitly violate the Eigenstate Thermalization Hypothesis and do not thermalize. We end with a discussion in Section 7.7.

Appendices C.1, C.2, and C.3 contain explicit calculations of OPE coefficients while in Appendices C.4 and C.5 thermal one-point functions are calculated. In Appendix C.6 we review the statement that the thermal one-point functions of multi-trace operators with derivatives vanish on $S^1 \times \mathbf{R}^{d-1}$. In Appendix C.7 we study a free scalar in two dimensions and calculate thermal two-point functions of certain quasi-primary operators. In Appendix C.8 we consider a free scalar vector model in four dimensions. Appendix C.9 discusses the factorization of multi-trace operators in the large volume limit.

7.2. Thermalization and universality

In the following we consider large-C_T CFTs on a $(d-1)$-dimensional sphere of radius R, which we set to unity for most of this section. As reviewed in [14], the stress tensor sector of conformal four-point functions consists of the contributions of the stress tensor and all its composites (multi stress tensors). The HHLL correlators we consider involve two heavy operators inserted at $x_E^0 = \pm\infty$ and two light operators inserted on the Euclidean cylinder, with angular separation φ and time separation x_E^0. The correlator in a heavy state (the HHLL correlator on the cylinder) is related to the correlator on the plane by a conformal transformation

$$\langle \mathcal{O}_H|\mathcal{O}(x_E^0,\varphi)\mathcal{O}(0)|\mathcal{O}_H\rangle = \lim_{x_4\to\infty} x_4^{2\Delta_H}(z\bar{z})^{-\Delta/2}\langle \mathcal{O}_H(x_4)\mathcal{O}(1)\mathcal{O}(z,\bar{z})\mathcal{O}_H(0)\rangle, \tag{7.3}$$

where the cross-ratios $(z,\bar{z})$ on the plane are related to the coordinates (x_E^0,φ) via

$$z = e^{-x_E^0 - i\varphi}, \qquad \bar{z} = e^{-x_E^0 + i\varphi}. \tag{7.4}$$

The stress tensor sector of the HHLL correlator is given by

$$\mathcal{G}(z,\bar{z}) = \lim_{x_4\to\infty} x_4^{2\Delta_H}\langle \mathcal{O}_H(x_4)\mathcal{O}(1)\mathcal{O}(z,\bar{z})\mathcal{O}_H(0)\rangle\Big|_{\text{multi stress tensors}} \tag{7.5}$$

and can be expanded in conformal blocks

$$\mathcal{G}(z,\bar{z}) = \frac{1}{[(1-z)(1-\bar{z})]^{\Delta}} \sum_{T^k_{\tau,s}} P^{(HH,LL)}_{T^k_{\tau,s}} g^{(0,0)}_{\tau,s}(1-z,1-\bar{z}), \tag{7.6}$$

where τ, s, k label the twist, spin, and multiplicity of multi stress tensors. We are interested in the double scaling limit where the central charge and the dimension of $\mathcal{O}_H$ are large, $C_T, \Delta_H \to \infty$ with their ratio $\mu \propto \Delta_H/C_T$ fixed. In this limit the products of the OPE coefficients which appear in (7.6) are given by

$$P^{(HH,LL)}_{T^k_{\tau,s}} = \left(-\frac{1}{2}\right)^s \lambda_{\mathcal{O}\mathcal{O}T^k_{\tau,s}} \lambda_{\mathcal{O}_H\mathcal{O}_H T^k_{\tau,s}}\Bigg|_{\left(\frac{\Delta_H}{C_T}\right)^k}, \tag{7.7}$$

where we only keep the leading, $\left(\frac{\Delta_H}{\sqrt{C_T}}\right)^k$ term in the OPE coefficients $\lambda_{\mathcal{O}_H\mathcal{O}_H T^k_{\tau,s}}$, but retain all terms in the OPE coefficients of the light operators $\lambda_{\mathcal{O}\mathcal{O}T^k_{\tau,s}}$. The contribution of the conformal family of a multi stress operator $T^k_{\tau,s}$ to the HHLL correlator is therefore

$$\langle \mathcal{O}_H|\mathcal{O}(x_E^0,\varphi)\mathcal{O}(0)|\mathcal{O}_H\rangle|_{T^k_{\tau,s}} = \frac{P^{(HH,LL)}_{T^k_{\tau,s}} g^{(0,0)}_{\tau,s}(1-z,1-\bar{z})}{[\sqrt{z\bar{z}}(1-z)(1-\bar{z})]^{\Delta}}. \tag{7.8}$$

We now consider these CFTs at finite temperature β^{-1}. To isolate the contribution of the conformal family associated with $T^k_{\tau,s}$, we can write the thermal correlator as

$$\begin{aligned}
\langle \mathcal{O}(x^0_E,\varphi)\mathcal{O}(0)\rangle_\beta =& \frac{1}{Z(\beta)}\sum_i e^{-\beta\Delta_i}\langle \mathcal{O}_i|\mathcal{O}(x^0_E,\varphi)\mathcal{O}(0)|\mathcal{O}_i\rangle \\
=&\frac{1}{[\sqrt{z\bar{z}}(1-z)(1-\bar{z})]^\Delta}\sum_{T^k_{\tau,s}}\left(-\frac{1}{2}\right)^s \lambda_{\mathcal{O}\mathcal{O}T^k_{\tau,s}} g^{(0,0)}_{\tau,s}(1-z,1-\bar{z})\ \langle T^k_{\tau,s}\rangle_\beta \\
&+\ldots,
\end{aligned} \tag{7.9}$$

where

$$\langle T^k_{\tau,s}\rangle_\beta = \frac{1}{Z(\beta)}\sum_i e^{-\beta\Delta_i}\lambda_{\mathcal{O}_i\mathcal{O}_i T^k_{\tau,s}} \tag{7.10}$$

is the finite temperature one-point function on the sphere of the $T^k_{\tau,s}$ operator and the dots denote contributions from other operators. In (7.10) $Z(\beta)$ is the partition function and the sum runs over all operators, including descendants[31]. Note that

$$\langle T^k_{\tau,s}\rangle_\beta = \beta^{-(\tau+s)} f^k_{\tau,s}(\beta). \tag{7.11}$$

Here and below the indices are suppressed (see e.g. [151] for the explicit form) and $f^k_{\tau,s}(\beta) \sim C_T^{k/2}$ is a theory-dependent nontrivial function of β which approaches a constant $f^k_{\tau,s}(0)$ in the large volume ($\beta \to 0$) limit.

Consider the thermalization of the stress tensor sector:

$$\langle \mathcal{O}_H|T^k_{\tau,s}|\mathcal{O}_H\rangle\Big|_{\frac{\Delta^k_H}{C_T^{k/2}}} = \lambda_{\mathcal{O}_H\mathcal{O}_H T^k_{\tau,s}}\Big|_{\frac{\Delta^k_H}{C_T^{k/2}}} = \langle T^k_{\tau,s}\rangle_\beta. \tag{7.12}$$

Note that $T^k_{\tau,s}$ is unit-normalized, so all terms in (7.12) scale like $C_T^{k/2}$. Eq. (7.12) implies the equality between (7.8) and the corresponding term in (7.9). Note that the left-hand side of (7.12) is a function of the energy density while the right-hand side is a function of temperature. The relationship is fixed by considering the stress tensor case: the corresponding function $f^1_{d-2,2}(\beta)$ is determined by the free energy on the sphere (see Section 7.6).

[31] The corresponding conformal blocks can be obtained in the usual way by applying the quadratic conformal Casimir and solving the resulting differential equation [31].

In the following, we will first discuss the case where the multi stress operators $T^k_{\tau,s}$ do not have any derivatives inserted, and then show that the derivatives do not change the conclusions. Assuming large-C_T factorization, the leading C_T behavior of $\langle T^k_{\tau,s}\rangle_\beta$ on the sphere is determined by that of the stress tensor. Schematically,

$$\langle T^k_{\tau,s}\rangle_\beta = c^k_{\tau,s}\ (\langle T^1_{d-2,2}\rangle_\beta)^k + \dots, \tag{7.13}$$

where $c^k_{\tau,s}$ are numerical coefficients, which depend on k, τ, s, but are independent of the details of the theory and the dots stand for terms subleading in C_T^{-1}. By combining (7.13) and (7.12), one can formulate a universality condition

$$\lambda_{\mathcal{O}_H\mathcal{O}_H T^k_{\tau,s}}\Big|_{\frac{\Delta^k_H}{C_T^{k/2}}} = c^k_{\tau,s}\ (\lambda_{\mathcal{O}_H\mathcal{O}_H T^1_{d-2,2}})^k = c^k_{\tau,s}\left(\frac{d}{1-d}\right)^k \frac{\Delta^k_H}{C_T^{\frac{k}{2}}}, \tag{7.14}$$

where the last equality follows from the stress tensor Ward identity for the three-point function which fixes $\lambda_{\mathcal{O}_H\mathcal{O}_H T^1_{d-2,2}}$ ($T^1_{d-2,2}$ here is unit-normalized). In other words, thermalization and large-C_T factorization imply that the leading $\Delta^k/C_T^{k/2}$ behavior of the multi stress tensor OPE coefficients is completely fixed and given by (7.14) in all large-C_T CFTs.

In the paragraph above we considered multi stress tensor operators that did not contain any derivatives in them. However, the story largely remains the same when the derivatives are included, as long as their number does not scale with C_T. Indeed, the three-point function involving the stress-tensor with added derivatives, $\partial_\alpha \dots \partial_\beta T_{\mu\nu}$ still behaves like $\lambda_{\mathcal{O}_H\mathcal{O}_H\partial_\alpha\dots\partial_\beta T_{\mu\nu}} \simeq \Delta_H/\sqrt{C_T}$ up to a theory-independent coefficient. Hence, (7.14) still holds, provided thermalization and large-C_T factorization hold on the sphere.

Note that due to conformal invariance, correlators on the sphere depend on R only through the ratio β/R. Moreover, in the large volume limit, factors of R need to drop out of (7.8) and (7.9) to have a well defined limit. To see this we use that $(1-z) \to 0$ and $(1-\bar{z}) \to 0$ when $R \to \infty$ and the conformal blocks behave as (see e.g. [24])

$$\begin{aligned} g^{(0,0)}_{\tau,s}(1-z, 1-\bar{z}) &\sim \mathcal{N}_{d,s}[(1-z)(1-\bar{z})]^{\frac{\tau+s}{2}} C_s^{(d/2-1)}\Big(\frac{(1-z)+(1-\bar{z})}{2\sqrt{(1-z)(1-\bar{z})}}\Big) \\ &\sim \mathcal{N}_{d,s}\frac{|x|^{\tau+s}}{R^{\tau+s}} C_s^{(d/2-1)}\Big(\frac{x^0_E}{|x|}\Big), \end{aligned} \tag{7.15}$$

where $|x| = \sqrt{(x_E^0)^2 + \mathbf{x}^2}$, $C_s^{(d/2-1)}(\frac{x_E^0}{|x|})$ is a Gegenbauer polynomial and $\mathcal{N}_{d,s} = \frac{s!}{(d/2-1)_s}$. Including the factor $[(1-z)(1-\bar{z})]^{-\Delta}$ from (7.8) in (7.15) this agrees with the thermal block on $S^1 \times \mathbf{R}^{d-1}$ in [82]. Now from the thermalization of the stress tensor we will find in the large volume limit that

$$\frac{\Delta_H}{C_T} \propto \left(\frac{R}{\beta}\right)^d, \tag{7.16}$$

and from (7.14) and (7.15) it follows that

$$g_{\tau,s}^{(0,0)}(1-z, 1-\bar{z})\lambda_{\mathcal{O}\mathcal{O}T_{\tau,s}^k}\lambda_{\mathcal{O}_H\mathcal{O}_H T_{\tau,s}^k}\Big|_{\frac{\Delta_H^k}{C_T^k}} \propto R^{dk-(\tau+s)}\beta^{-dk}. \tag{7.17}$$

The dimension of multi stress tensors $T_{\tau,s}^k$ is given by $\tau + s = dk + n$ where $n = 0, 2, \ldots$. Therefore, the only multi stress tensors that contribute in the large volume limit have dimensions dk. Restoring R in (7.8)-(7.9) and inserting (7.17) one finds that R drops out in the large volume limit. The correct dependence $\beta^{-(\tau+s)}$ from (7.11) in the $R \to \infty$ limit is also recovered in (7.8) using (7.17). The multi stress tensor operators that contribute in the large volume limit are therefore of the schematic form $T_{\mu_1\nu_1}T_{\mu_2\nu_2}\cdots T_{\mu_k\nu_k}$ with arbitrarily many contractions and no derivatives.

In holographic theories thermalization and the Wilson line prescription for the correlator allows one to compute the universal part of the OPE coefficients (see [124,186] for explicit computations in the $d = 4$ case). It is also easy to check explicitly that the universality (7.14) holds for holographic theories with a Gauss-Bonnet gravitational coupling added. While the statement was shown to be true for the leading twist OPE coefficients in [15], it was not immediately obvious for multi stress tensors of non-minimal twist. Some such OPE coefficients were computed in [15,14]. (See e.g. eqs. (5.48), (5.51), (5.52), (5.57) and (D.1)-(D.5) in [14]). Indeed, the leading $\Delta^k/C_T^{k/2}$ behavior of these OPE coefficients is independent of the Gauss-Bonnet coupling.

What about a general large-C_T theory? We first consider the OPE coefficients of double-stress tensors. To this end, consider the four point function[32] $\langle \mathcal{O}T_{\mu\nu}T_{\rho\sigma}\mathcal{O}\rangle$ where $\mathcal{O}$ is a scalar operator with scaling dimension Δ.

[32] This correlator for finite Δ was recently considered in holographic CFTs with $\Delta_{\text{gap}} \gg 1$ and $\Delta \ll \Delta_{\text{gap}}$ in [54].

In the direct channel $\mathcal{O} \times \mathcal{O} \rightarrow \mathcal{O}' \rightarrow T_{\mu\nu} \times T_{\rho\sigma}$ for finite Δ and large C_T, the leading contribution in the large-C_T limit comes from the identity operator $\mathcal{O} \times \mathcal{O} \rightarrow \mathbf{1} \rightarrow T_{\mu\nu} \times T_{\rho\sigma}$. The subleading contributions in the direct-channel are due to single trace operators as well as double trace operators made out of the external operators of the schematic form $T^2_{\tau,s}$ and $[\mathcal{O}\mathcal{O}]_{n,l} =: \mathcal{O}\partial^{2n}\partial_1 \ldots \partial_l \mathcal{O} :$. The exchange of the identity operator is reproduced in the cross-channel $\mathcal{O} \times T_{\mu\nu} \rightarrow [\mathcal{O}T_{\alpha\beta}]_{n,l} \rightarrow \mathcal{O} \times T_{\rho\sigma}$ by mixed double-trace operators $[\mathcal{O}T_{\alpha\beta}]_{n,l}$ with OPE coefficients fixed by the MFT [38,56-57]. The subleading contributions in $1/C_T$ are then due to corrections to the anomalous dimension and OPE coefficients of $[\mathcal{O}T_{\alpha\beta}]_{n,l}$ and single trace operators in the $\mathcal{O} \times T_{\mu\nu}$ OPE. An important example of the latter is the exchange of the single trace operator $\mathcal{O}$, whose contribution is universally fixed by the stress tensor Ward identity to be $(\lambda_{\mathcal{O}T^1_{d-2,2}\mathcal{O}})^2 \propto \Delta^2/C_T$ times the conformal block. This gives a universal contribution to $\lambda_{\mathcal{O}\mathcal{O}T^2_{\tau,s}}$ as was also noted in [54].

We now want to consider the case where $\Delta \sim C_T$ and study the OPE coefficients of the double-stress tensor operators in the $\mathcal{O} \times \mathcal{O}$ OPE. Firstly, note that the contribution from $T^2_{\tau,s}$ to the four-point function expanded in the direct channel is proportional to $\lambda_{\mathcal{O}\mathcal{O}T^2_{\tau,s}}\lambda_{TTT^2_{\tau,s}}$. The OPE coefficients $\lambda_{TTT^2_{\tau,s}}$ are fixed by the MFT and are independent of Δ and therefore the dependence on the scaling dimension comes solely from the OPE coefficients $\lambda_{\mathcal{O}\mathcal{O}T^2_{\tau,s}}$. In the cross-channel, we analyze two kinds of contributions: from the exchanged operator $\mathcal{O}$ and from all other operators $\mathcal{O}' \neq \mathcal{O}$. From the operator $\mathcal{O}$ we get a universal contribution to the OPE coefficients in the direct channel $\lambda_{\mathcal{O}\mathcal{O}T^2_{\tau,s}}$, that we denote by $\lambda^{(1)}_{\mathcal{O}\mathcal{O}T^2_{\tau,s}}$. This contribution is universal since it only depends on $(\lambda_{\mathcal{O}T^1_{d-2,2}\mathcal{O}})^2 \propto \Delta^2/C_T$ in the cross-channel, which is fixed by the Ward identity. The contributions from other operators $\mathcal{O}'$ to the same OPE coefficient will be denoted by $\lambda^{(2)}_{\mathcal{O}\mathcal{O}T^2_{\tau,s}}$, such that $\lambda_{\mathcal{O}\mathcal{O}T^2_{\tau,s}} = \lambda^{(1)}_{\mathcal{O}\mathcal{O}T^2_{\tau,s}} + \lambda^{(2)}_{\mathcal{O}\mathcal{O}T^2_{\tau,s}}$. Note that it also follows from the stress tensor Ward identity that the only scalar primary that appears in the cross-channel is $\mathcal{O}$. The operator $\mathcal{O}'$ therefore necessarily has spin $s \neq 0$.

To prove universality we need to show that $\lambda^{(2)}_{\mathcal{O}\mathcal{O}T^2_{\tau,s}} \ll \Delta^2/C_T$ in limit $1 \ll \Delta \propto C_T$ by studying the Δ dependence of the OPE coefficients $\lambda_{\mathcal{O}T^1_{d-2,2}\mathcal{O}'}$ in the cross-channel. For operators $\mathcal{O}'$, such that $\Delta_{\mathcal{O}'} \ll \Delta$, we expect that these OPE coefficients are heavily suppressed. It would be interesting to understand if one could put a general bound on the contribution of these operators in the cross-channel in any large-C_T theory. On the other hand, assuming thermalization, the OPE coefficients due to operators $\mathcal{O}'$ such that $\Delta_{\mathcal{O}'} \sim \Delta$ have been calculated in [154]. The obtained results are in agreement with our expectation, namely, these OPE coefficients are suppressed in $1 \ll \Delta \propto C_T$ limit. Additionally, in the cross-channel we have double-trace operators $[\mathcal{O}T_{\alpha\beta}]_{n,l}$, whose OPE is fixed by the MFT and it does not get Δ-enhanced.

One can iteratively extend the argument given here to multi stress tensors operators (with $k > 2$) by considering multi stress tensors as external operators. For example, to argue the universality of $\lambda_{\mathcal{O}\mathcal{O}T^3_{\tau,s}}$ one may consider $\langle \mathcal{O}T^1_{d-2,2}T^2_{\tau,s}\mathcal{O}\rangle$. The bootstrap argument above can be applied again by using the fact that OPE coefficients $\lambda_{\mathcal{O}\mathcal{O}T^2_{\tau,s}}$ are universal, and the OPE coefficients $\lambda_{\mathcal{O}T^2_{\tau,s}\mathcal{O}'}$ are again expected to be subleading.

7.3. OPE coefficients in the free adjoint scalar model

In this section we consider a four-dimensional theory of a free scalar in the adjoint representation of $SU(N)$, see [187-192] for related work. The relation between N and the central charge C_T in this theory is [18]

$$C_T = \frac{4}{3}(N^2-1), \tag{7.18}$$

and we consider the large-N (large-C_T) limit. The propagator for the scalar field ${\phi^i}_j$ is given by

$$\langle {\phi^i}_j(x){\phi^k}_l(y)\rangle = \left({\delta^i}_l{\delta^k}_j - \frac{1}{N}{\delta^i}_j{\delta^k}_l\right)\frac{1}{|x-y|^2}. \tag{7.19}$$

A single trace scalar operator with dimension Δ is given by

$$\mathcal{O}_\Delta(x) = \frac{1}{\sqrt{\Delta}N^{\frac{\Delta}{2}}} : Tr(\phi^\Delta) : (x), \tag{7.20}$$

where $: \ldots :$ denotes the oscillator normal ordering and the normalization is fixed by

$$\langle \mathcal{O}_\Delta(x)\mathcal{O}_\Delta(y)\rangle = \frac{1}{|x-y|^{2\Delta}}. \tag{7.21}$$

The CFT data that we compute in this section are the OPE coefficients of multi stress tensors in the $\mathcal{O}_\Delta \times \mathcal{O}_\Delta$ OPE. Assuming we can take $\Delta \to \Delta_H \sim C_T$, the large-$\Delta$ limit of these OPE coefficients is shown to be universal. One may worry that for $\Delta_H \sim C_T$ we can no longer trust the planar expansion, but, as we show in Appendix C.3, the large-Δ limit of the planar result yields the correct expression even for $\Delta_H \sim C_T$.

7.3.1. Stress tensor

The stress tensor operator is given by

$$T_{\mu\nu}(x) = \frac{1}{3\sqrt{C_T}} : Tr\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\phi\partial_\mu\partial_\nu\phi - (\text{trace})\right) : (x), \tag{7.22}$$

with the normalization

$$\langle T^{\mu\nu}(x)T_{\rho\sigma}(0)\rangle = \frac{1}{|x|^8}\left(I^{(\mu}{}_\rho(x)I^{\nu)}{}_\sigma(x) - (\text{traces})\right), \tag{7.23}$$

where $I^\mu{}_\nu(x) := \delta^\mu{}_\nu - \frac{2x^\mu x_\nu}{|x|^2}$. The OPE coefficient is fixed by the stress tensor Ward identity to be

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^1_{2,2}} = -\frac{4\Delta}{3\sqrt{C_T}}. \tag{7.24}$$

It is also useful to find (7.24) using Wick contractions since an analogous calculation will be necessary for multi stress tensors. We do this explicitly in Appendix C.1.

7.3.2. Double-stress tensor with minimal twist

In this section we study the minimal-twist composite operator made out of two stress tensors

$$(T^2)_{\mu\nu\rho\sigma}(x) = \frac{1}{\sqrt{2}} : T_{(\mu\nu}T_{\rho\sigma)} : (x) - (\text{traces}), \tag{7.25}$$

with the normalization

$$\langle (T^2)^{\mu\nu\rho\sigma}(x)(T^2)_{\kappa\lambda\delta\omega}(0)\rangle = \frac{1}{|x|^{16}}\left(I^{(\mu}{}_\kappa I^\nu{}_\lambda I^\rho{}_\delta I^{\sigma)}{}_\omega - (\text{traces})\right). \tag{7.26}$$

Consider the following three-point function

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)(T^2)_{\mu\nu\rho\sigma}(x_3)\rangle = \frac{\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{4,4}}}{|x_{12}|^{2\Delta-4}|x_{13}|^4|x_{23}|^4}\left(Z_\mu Z_\nu Z_\rho Z_\sigma - (\text{traces})\right), \tag{7.27}$$

where $Z^\mu = \frac{x^\mu_{13}}{|x_{13}|^2} - \frac{x^\mu_{12}}{|x_{12}|^2}$. It is shown in Appendix C.1 that the OPE coefficient $\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{4,4}}$ is given at leading order in the large-C_T limit by

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{4,4}} = \frac{8\sqrt{2}\Delta(\Delta-1)}{9C_T}. \tag{7.28}$$

Evaluating $P^{(HH,LL)}_{T^2_{4,4}}$ defined by (7.7) in the large-Δ limit[33], we obtain

$$\begin{aligned} P^{(HH,LL)}_{T^2_{4,4}} &= \left(-\frac{1}{2}\right)^4 \lambda_{\mathcal{O}_H\mathcal{O}_H T^2_{4,4}} \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{4,4}}\Big|_{\left(\frac{\Delta_H}{C_T}\right)^2} \\ &= \frac{8}{81}\frac{\Delta_H^2}{C_T^2}\left(\Delta^2 + \mathcal{O}\left(\Delta\right)\right) = \mu^2\left(\frac{\Delta^2}{28800} + \mathcal{O}(\Delta)\right), \end{aligned} \tag{7.29}$$

where we use the following relation

$$\mu = \frac{160}{3}\frac{\Delta_H}{C_T}. \tag{7.30}$$

The result (7.29) agrees with the leading behavior of the corresponding OPE coefficients computed using holography in [15] and bootstrap in [124,13].

7.3.3. Double-stress tensor with minimal twist and spin $s = 6$

We consider double-stess tensor operator with two (uncontracted) derivatives inserted

$$\begin{aligned} (T^2)_{\mu\nu\rho\sigma\eta\kappa}(x) = \frac{1}{2\sqrt{182}} : \Big(& T_{(\mu\nu}\partial_\rho\partial_\sigma T_{\eta\kappa)}(x) - \frac{7}{6}\left(\partial_{(\rho}T_{\mu\nu}\right)\left(\partial_\sigma T_{\eta\kappa)}\right)(x) \\ & - (\text{traces})\Big) : . \end{aligned} \tag{7.31}$$

Using the conformal algebra eq. (C.2), it is straightforward to check that this operator is primary. It is unit-normalized such that

$$\langle (T^2)^{\mu\nu\rho\sigma\eta\kappa}(x)(T^2)_{\alpha\beta\gamma\delta\xi\epsilon}(0)\rangle = \frac{1}{|x|^{20}}\left(I^{(\mu}{}_\alpha I^\nu{}_\beta I^\rho{}_\gamma I^\sigma{}_\delta I^\eta{}_\xi I^{\kappa)}{}_\epsilon - (\text{traces})\right). \tag{7.32}$$

By a calculation similar to those summarized in Appendix C.1, we observe that the OPE coefficient of $(T^2)_{\mu\nu\rho\sigma\eta\kappa}$ in the $\mathcal{O}_\Delta \times \mathcal{O}_\Delta$ OPE is given at leading order in the large-C_T limit by

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{4,6}} = \frac{8}{3}\sqrt{\frac{2}{91}}\frac{\Delta(\Delta-1)}{C_T}. \tag{7.33}$$

[33] By the large-Δ limit, we strictly speaking mean $1 \ll \Delta \ll C_T$. However in this section we often extrapolate this to the $\Delta \sim C_T$ regime.

Evaluating $P^{(HH,LL)}_{T^2_{4,6}}$, defined by (7.7), in the large-Δ limit, we obtain

$$\begin{aligned} P^{(HH,LL)}_{T^2_{4,6}} &= \left(-\frac{1}{2}\right)^6 \lambda_{\mathcal{O}_H\mathcal{O}_H T^2_{4,6}} \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{4,6}} \Big|_{\left(\frac{\Delta_H}{C_T}\right)^2} \\ &= \frac{2}{819}\frac{\Delta_H^2}{C_T^2}\left(\Delta^2 + \mathcal{O}(\Delta)\right) = \mu^2\left(\frac{\Delta^2}{1164800} + \mathcal{O}(\Delta)\right). \end{aligned} \tag{7.34}$$

The result (7.34) agrees with the leading behavior of the corresponding OPE coefficients computed using holography in [15] and bootstrap in [124,13].

7.3.4. Minimal-twist multi stress tensors

We now consider multi stress tensors $T^k_{2k,2k}$. Just like the double stress tensor ($k = 2$), we show that these have universal OPE coefficients in the large-Δ limit for any k.

Consider the unit-normalized minimal-twist multi stress tensor operator given by

$$(T^k)_{\mu_1\mu_2\ldots\mu_{2k}}(x) = \frac{1}{\sqrt{k!}} :T_{(\mu_1\mu_2}T_{\mu_3\mu_4}\cdots T_{\mu_{2k-1}\mu_{2k})}:(x) - (\text{traces}). \tag{7.35}$$

The OPE coefficient of $(T^k)_{\mu_1\mu_2\ldots\mu_{2k}}$ in the $\mathcal{O}_\Delta \times \mathcal{O}_\Delta$ OPE, in the large-C_T limit is given by[34]

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^k_{2k,2k}} = \left(-\frac{4}{3}\right)^k \frac{1}{\sqrt{k!}C_T^{k/2}}\frac{\Gamma(\Delta+1)}{\Gamma(\Delta-k+1)}. \tag{7.36}$$

First, we write $P^{(HH,LL)}_{T^3_{6,6}}$, defined by (7.7), in the large-Δ limit. We obtain this OPE coefficient from (7.36) for $k = 3$,

$$\begin{aligned} P^{(HH,LL)}_{T^3_{6,6}} &= \left(-\frac{1}{2}\right)^6 \lambda_{\mathcal{O}_H\mathcal{O}_H T^3_{6,6}} \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^3_{6,6}} \Big|_{\left(\frac{\Delta_H}{C_T}\right)^3} \\ &= \frac{32}{2187}\frac{\Delta_H^3}{C_T^3}\left(\Delta^3 + \mathcal{O}(\Delta^2)\right) = \mu^3\left(\frac{\Delta^3}{10368000} + \mathcal{O}(\Delta^2)\right). \end{aligned} \tag{7.37}$$

The result (7.37) agrees with the leading behavior of the corresponding OPE coefficients computed using holography in [15] and bootstrap in [13].

---

[34] See Appendix C.1 for detailed computations of similar OPE coefficients.

Additionally, we consider the OPE coefficient $P^{(HH,LL)}_{T^k_{2k,2k}}$ in the large-Δ limit for general k,

$$
\begin{aligned}
P^{(HH,LL)}_{T^k_{2k,2k}} &= \left(-\frac{1}{2}\right)^{2k} \lambda_{\mathcal{O}_H\mathcal{O}_H T^k_{2k,2k}} \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^k_{2k,2k}}\Big|_{\left(\frac{\Delta_H}{C_T}\right)^k} \\
&= \frac{1}{k!}\left(\frac{2}{3}\right)^{2k} \frac{\Delta_H^k}{C_T^k}\left(\Delta^k + \mathcal{O}\left(\Delta^{k-1}\right)\right) = \mu^k \left(\frac{\Delta^k}{120^k k!} + \mathcal{O}(\Delta^{k-1})\right).
\end{aligned}
\tag{7.38}
$$

If we consider the limit $1-\bar{z} \ll 1-z \ll 1$, such that $\mu(1-\bar{z})(1-z)^3$ is held fixed, only operators $T^k_{2k,2k}$ contribute to the heavy-heavy-light-light four-point function given by eq. (7.5). The conformal blocks of $T^k_{2k,2k}$ in this limit are given by

$$
g^{(0,0)}_{2k,2k}(1-z, 1-\bar{z}) \approx (1-\bar{z})^k (1-z)^{3k}, \tag{7.39}
$$

and we can sum all contributions in eq. (7.6) explicitly to obtain

$$
\mathcal{G}(z,\bar{z}) \approx \frac{1}{((1-z)(1-\bar{z}))^{\Delta}} e^{\frac{\mu\Delta}{120}(1-\bar{z})(1-z)^3}. \tag{7.40}
$$

Notice that the term in the exponential is precisely the stress-tensor conformal block in the limit $1-\bar{z} \ll 1-z \ll 1$ times its OPE coefficient. Therefore, the OPE coefficients (7.36) imply the exponentiation of stress-tensor conformal block. We conclude that these OPE coefficients are the same as the ones computed using holography and bootstrap in the limit of large Δ.

7.3.5. Double-stress tensors with non-minimal twist

So far we have shown that the minimal-twist multi stress tensor OPE coefficients are universal in the limit of large Δ. In this subsection, we extend this to show that the simplest non-minimal twist double-stress tensors also have universal OPE coefficients at large Δ.

The subleading twist double-stress tensor with twist $\tau = 6$ is of the schematic form $:T^{\mu}{}_{\alpha}T^{\alpha\nu}:$ and has dimension $\Delta = 8$ and spin $s = 2$. It is given by

$$
(T^2)_{\mu\nu}(x) = \frac{1}{\sqrt{2}} :T_{\mu\alpha}T^{\alpha}{}_{\nu}:(x) - (\text{trace}). \tag{7.41}
$$

The normalization in (7.41) is again chosen such that $(T^2)_{\mu\nu}$ is unit-normalized, see Appendix C.2 for details.

The OPE coefficient of $(T^2)_{\mu\nu}$ in the $\mathcal{O}_\Delta \times \mathcal{O}_\Delta$ OPE is found from the three-point function in the large-C_T limit, for details see Appendix C.2,

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)(T^2)^{\mu\nu}(x_3)\rangle = \frac{4\sqrt{2}\Delta(\Delta-1)}{9C_T}\frac{Z^\mu Z^\nu - (\text{trace})}{|x_{12}|^{2\Delta-6}|x_{13}|^6|x_{23}|^6}, \quad (7.42)$$

from which we read off the OPE coefficient

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{6,2}} = \frac{4\sqrt{2}\Delta(\Delta-1)}{9C_T}. \quad (7.43)$$

Evaluating $P^{(HH,LL)}_{T^2_{6,2}}$, defined by (7.7), in the large-Δ limit, we obtain

$$\begin{aligned} P^{(HH,LL)}_{T^2_{6,2}} &= \left(-\frac{1}{2}\right)^2 \lambda_{\mathcal{O}_H\mathcal{O}_H T^2_{6,2}}\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{6,2}}\Big|_{\left(\frac{\Delta_H}{C_T}\right)^2} \\ &= \frac{8}{81}\frac{{\Delta_H}^2}{C_T^2}\left(\Delta^2 + \mathcal{O}\left(\Delta\right)\right) = \mu^2\left(\frac{\Delta^2}{28800} + \mathcal{O}(\Delta)\right). \end{aligned} \quad (7.44)$$

The result (7.44) agrees with the leading behavior of the corresponding OPE coefficients computed using holography in [15] and bootstrap in [14].

We further consider the scalar double-stress tensor with $\Delta = 8$ and spin $s = 0$ which is given by

$$(T^2)(x) = \frac{1}{3\sqrt{2}} : T_{\mu\nu}T^{\mu\nu} : (x). \quad (7.45)$$

The three point function $\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)(T^2)(x_3)\rangle$ is found in Appendix C.2 to be

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)(T^2)(x_3)\rangle = \frac{2\sqrt{2}\Delta(\Delta-1)}{9C_T}\frac{1}{|x_{12}|^{2\Delta-8}|x_{13}|^8|x_{23}|^8}, \quad (7.46)$$

from which we read off the OPE coefficient

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{8,0}} = \frac{2\sqrt{2}\Delta(\Delta-1)}{9C_T}. \quad (7.47)$$

We write $P^{(HH,LL)}_{T^2_{8,0}}$ in the large-Δ limit

$$\begin{aligned} P^{(HH,LL)}_{T^2_{8,0}} &= \lambda_{\mathcal{O}_H\mathcal{O}_H T^2_{8,0}}\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{8,0}}\Big|_{\left(\frac{\Delta_H}{C_T}\right)^2} \\ &= \frac{8}{81}\frac{{\Delta_H}^2}{C_T^2}\left(\Delta^2 + \mathcal{O}(\Delta)\right) = \mu^2\left(\frac{\Delta^2}{28800} + \mathcal{O}(\Delta)\right). \end{aligned} \quad (7.48)$$

The result (7.48) agrees with the leading behavior of the corresponding OPE coefficients computed using holography in [15] and bootstrap in [14].

7.4. Thermal one-point functions in the free adjoint scalar model

In this section we explicitly show that multi stress tensor operators thermalize in the free theory by calculating the thermal one-point function of some of these operators on $S^1 \times \mathbf{R}^3$. One-point functions of primary symmetric traceless operators at finite temperature are fixed by symmetry up to a dimensionless coefficient $b_{\mathcal{O}}$ (see e.g. [35,82])

$$\langle \mathcal{O}_{\mu_1 \cdots \mu_{s_{\mathcal{O}}}} \rangle_\beta = \frac{b_{\mathcal{O}}}{\beta^{\Delta_{\mathcal{O}}}} \Big(e_{\mu_1} \cdots e_{\mu_{s_{\mathcal{O}}}} - (\text{traces}) \Big) . \tag{7.49}$$

Here e_μ is a unit vector along the thermal circle.

To compare the thermal one-point functions and OPE coefficients from the previous section, we need to derive a relation between $\frac{\Delta_H}{C_T}$ and the temperature[35] β^{-1}. Here $\Delta_H \sim N^2$ refers to the scaling dimension of a heavy operator $\mathcal{O}_H$ with OPE coefficients given by the large-Δ limit of those obtained in Section 7.3. One can relate the inverse temperature β to the parameter $\mu = \frac{160}{3}\frac{\Delta_H}{C_T}$ using the Stefan-Boltzmann's law $E/\text{vol}(S^3) = N^2\pi^2/30\beta^4$. The energy of the state E is related to its conformal dimension Δ via $E = \Delta/R$. One can then use $\text{vol}(S^3) = 2\pi^2 R^3$ and the relation between N and C_T given by (7.18), to find

$$\mu = \frac{160}{3}\frac{\Delta_H}{C_T} = \frac{160}{3} E \frac{R}{C_T} = \frac{8}{3}\Big(\frac{\pi R}{\beta}\Big)^4 . \tag{7.50}$$

7.4.1. Stress tensor

The thermal one-point function for the stress tensor $T^1_{2,2} = T_{\mu\nu}$ is calculated in Appendix C.4 where we find that $b_{T^1_{2,2}}$ is given by

$$b_{T^1_{2,2}} = -\frac{2\pi^4 N}{15\sqrt{3}} . \tag{7.51}$$

Using (7.50) and (7.51) one arrives at

$$b_{T^1_{2,2}} \beta^{-4} = \lambda_{\mathcal{O}_H \mathcal{O}_H T^1_{2,2}} . \tag{7.52}$$

7.4.2. Double-stress tensor with minimal twist

In this section we calculate the thermal one-point function of the double-stress tensor operator with $\tau = 4$ and spin $s = 4$. The operator is written explicitly in

[35] See also Section 7.6 and Appendix C.4 for alternative derivations.

(7.25). The leading contribution to the thermal one-point function of $(T^2)_{\mu\nu\rho\sigma}$ follows from the large-N factorization and is given by

$$\begin{aligned}\langle (T^2)_{\mu\nu\rho\sigma}\rangle_\beta &= \frac{1}{\sqrt{2}}\langle T_{(\mu\nu}\rangle_\beta \langle T_{\rho\sigma)}\rangle_\beta - (\text{traces}) \\ &= \frac{2\sqrt{2}\pi^8 N^2}{675\beta^8}\left(e_\mu e_\nu e_\rho e_\sigma - (\text{traces})\right).\end{aligned} \tag{7.53}$$

Using the relation (7.50) and the OPE coefficient (7.28), we observe the thermalization of this operator,

$$b_{T^2_{4,4}}\beta^{-8} = \lambda_{\mathcal{O}_H\mathcal{O}_H T^2_{4,4}}\Big|_{\frac{\Delta_H^2}{C_T}}. \tag{7.54}$$

7.4.3. Minimal-twist multi stress tensors

Consider now multi stress tensors $T^k_{2k,2k}$ with twist $\tau = 2k$ and spin $s = 2k$. We show that these operators thermalize for any k by calculating their thermal one-point functions:

$$\langle (T^k)_{\mu_1\mu_2\ldots\mu_{2k}}\rangle_\beta = \frac{b_{T^k_{2k,2k}}}{\beta^{4k}}(e_{\mu_1}e_{\mu_2}\cdots e_{\mu_{2k}} - (\text{traces})), \tag{7.55}$$

where the leading behavior of $b_{T^k_{2k,2k}}$ follows from the large-N factorization:

$$b_{T^k_{2k,2k}} = \frac{1}{\sqrt{k!}}(b_{T^1_{2,2}})^k = \frac{(-\frac{2}{5})^k N^k \pi^{4k}}{3^{\frac{3k}{2}}\sqrt{k!}}. \tag{7.56}$$

Eqs. (7.50) and (7.56) may be combined to yield

$$b_{T^k_{2k,2k}}\beta^{-4k} = \lambda_{\mathcal{O}_H\mathcal{O}_H T^k_{2k,2k}}\Big|_{\frac{\Delta_H^k}{C_T^{k/2}}}. \tag{7.57}$$

7.4.4. Double-stress tensors with non-minimal twist

The subleading twist double-stress tensor is of the schematic form $:T^\mu{}_\alpha T^{\alpha\nu}:$ and has twist $\tau = 6$ and spin $s = 2$. The explicit form can be found in (7.41). The leading term in the thermal one-point function is given by

$$\begin{aligned}\langle (T^2)^{\mu\nu}\rangle_\beta &= \frac{1}{\sqrt{2}}\langle T^{\mu\alpha}\rangle_\beta \langle T^\nu{}_\alpha\rangle_\beta - (\text{trace}) \\ &= \frac{b^2_{T^1_{2,2}}}{2\sqrt{2}\beta^8}(e^\mu e^\nu - \frac{1}{4}\delta^{\mu\nu}) \\ &= \frac{\sqrt{2}N^2\pi^8}{675\beta^8}(e^\mu e^\nu - \frac{1}{4}\delta^{\mu\nu}),\end{aligned} \tag{7.58}$$

therefore,

$$b_{T^2_{6,2}} = \frac{\sqrt{2}N^2\pi^8}{675}. \tag{7.59}$$

Taking the large-Δ limit of the OPE coefficient in (7.43) and substituting (7.50), we observe thermalization,

$$b_{T^2_{6,2}}\beta^{-8} = \lambda_{\mathcal{O}_H\mathcal{O}_H T^2_{6,2}}\Big|_{\frac{\Delta_H^2}{C_T}}. \tag{7.60}$$

We further consider the scalar double-stress tensor with $\tau = 8$ and $s = 0$ which is given by (7.45). The thermal one-point function for this operator is

$$\begin{aligned}\langle (T^2)\rangle_\beta &= \frac{1}{3\sqrt{2}}\langle T_{\mu\nu}\rangle_\beta \langle T^{\mu\nu}\rangle_\beta \\ &= \frac{1}{3\sqrt{2}}\frac{3}{4}b^2_{T^1_{2,2}}\beta^{-8} = \frac{\pi^8 N^2}{675\sqrt{2}\beta^8},\end{aligned} \tag{7.61}$$

where the factor of $\frac{3}{4}$ in the first line comes from the index contractions. Hence,

$$b_{T^2_{8,0}} = \frac{\pi^8 N^2}{675\sqrt{2}}. \tag{7.62}$$

Using (7.62), (7.47) and (7.50), we again observe thermalization,

$$b_{T^2_{8,0}}\beta^{-8} = \lambda_{\mathcal{O}_H\mathcal{O}_H T^2_{8,0}}\Big|_{\frac{\Delta_H^2}{C_T}}. \tag{7.63}$$

7.4.5. Triple-stress tensors with non-minimal twist

We consider the triple stress tensors with $\tau = 8, s = 4$ and $\tau = 10, s = 2$. The unit-normalized triple stress tensor with $\tau = 8$ can be written as

$$(T^3)_{\mu\nu\rho\sigma}(x) = \frac{1}{\sqrt{3}}\left(: T_{(\mu\nu}T_{\rho|\alpha|}T^{\alpha}{}_{\sigma)} : (x) - (\text{traces})\right), \tag{7.64}$$

where $|\alpha|$ denotes that index α is excluded from the symmetrization. The thermal one-point function follows from large-N factorization

$$\begin{aligned}\langle (T^3)_{\mu\nu\rho\sigma}\rangle_\beta &= \frac{1}{\sqrt{3}}\left(\langle T_{(\mu\nu}\rangle_\beta \langle T_{\rho|\alpha|}\rangle_\beta \langle T^{\alpha}{}_{\sigma)}\rangle_\beta - (\text{traces})\right) \\ &= \frac{1}{2\sqrt{3}}\frac{b^3_{T^1_{2,2}}}{\beta^{12}}\left(e_\mu e_\nu e_\rho e_\sigma - (\text{traces})\right) \\ &= -\frac{4\pi^{12}N^3}{30375\beta^{12}}\left(e_\mu e_\nu e_\rho e_\sigma - (\text{traces})\right),\end{aligned} \tag{7.65}$$

therefore,

$$b_{T^3_{8,4}} = -\frac{4\pi^{12}N^3}{30375}. \tag{7.66}$$

The OPE coefficient of the operator with same quantum numbers ($\Delta = 12$, $s = 4$) is calculated holographically and is given by (D.1) in [14]. In the large-Δ limit it can be written as

$$P^{(HH,LL)}_{T^3_{8,4}} = \left(-\frac{1}{2}\right)^4 \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^3_{8,4}}\lambda_{\mathcal{O}_H\mathcal{O}_H T^3_{8,4}}\Big|_{\left(\frac{\Delta_H}{C_T}\right)^3} = \frac{64}{2187}\frac{\Delta_H^3\Delta^3}{C_T^3} + \mathcal{O}(\Delta^2). \tag{7.67}$$

Now, one can easily read-off $\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^3_{8,4}}$ in the large-Δ limit

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^3_{8,4}} = -\frac{32\Delta^3}{27\sqrt{3}C_T{}^{3/2}} + \mathcal{O}(\Delta^2) = -\frac{4\Delta^3}{9N^3} + \mathcal{O}(\Delta^2), \tag{7.68}$$

where we use the relation between central charge C_T and N given by (7.18). Using (7.50) one can obtain

$$b_{T^3_{8,4}}\beta^{-12} = \lambda_{\mathcal{O}_H\mathcal{O}_H T^3_{8,4}}\Big|_{\frac{\Delta_H^3}{C_T^{3/2}}}. \tag{7.69}$$

We also consider the triple stress tensors with quantum numbers $\Delta = 12$ and $s = 2$. There are two linearly independent such operators that schematically can be written as $:T_{\alpha\beta}T^{\alpha\beta}T_{\mu\nu}:$ and $:T_{\mu\alpha}T^{\alpha\beta}T_{\beta\nu}:$. We write the following linear combinations of these operators

$$(T^3)_{\mu\nu}(x) = \frac{1}{10\sqrt{2}}\left(:T_{\alpha\beta}T^{\alpha\beta}T_{\mu\nu}:(x) + 4:T_{\mu\alpha}T^{\alpha\beta}T_{\beta\nu}:(x) - (\text{trace})\right), \tag{7.70}$$

$$(\tilde{T}^3)_{\mu\nu}(x) = \frac{7}{20}\left(:T_{\alpha\beta}T^{\alpha\beta}T_{\mu\nu}:(x) - \frac{12}{7}:T_{\mu\alpha}T^{\alpha\beta}T_{\beta\nu}:(x) - (\text{trace})\right). \tag{7.71}$$

Both $(T^3)_{\mu\nu}$ and $(\tilde{T}^3)_{\mu\nu}$ are unit-normalized and their overlap vanishes in the large-N limit

$$\langle(T^3)_{\mu\nu}(x)(\tilde{T}^3)^{\rho\sigma}(y)\rangle = \mathcal{O}(1/N^2). \tag{7.72}$$

The thermal one-point functions of these operators, obtained by large-N factorization, in the large-N limit are given by

$$\begin{aligned}\langle(T^3)_{\mu\nu}\rangle_\beta &= -\sqrt{\frac{2}{3}}\frac{N^3\pi^{12}}{10125\beta^{12}}\left(e_\mu e_\nu - (\text{trace})\right),\\ \langle(\tilde{T}^3)_{\mu\nu}\rangle_\beta &= \mathcal{O}(N),\end{aligned} \tag{7.73}$$

therefore,

$$\begin{aligned} b_{T^3_{10,2}} &= -\sqrt{\frac{2}{3}}\frac{N^3\pi^{12}}{10125}, \\ b_{\tilde{T}^3_{10,2}} &= 0. \end{aligned} \tag{7.74}$$

The holographic OPE coefficient of the operator with the same quantum numbers ($\Delta = 12$, $s = 2$), with external scalar operators is given by (5.57) in [14]. In the large-Δ limit it can be written as

$$P^{(HH,LL)}_{T^3_{10,2}} = \left(-\frac{1}{2}\right)^2 \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^3_{10,2}}\lambda_{\mathcal{O}_H\mathcal{O}_H T^3_{10,2}}\Big|_{\left(\frac{\Delta_H}{C_T}\right)^3} = \frac{32}{729}\frac{\Delta_H^3\Delta^3}{C_T^3} + \mathcal{O}(\Delta^2). \tag{7.75}$$

We can read-off $\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^3_{10,2}}$:

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^3_{10,2}} = -\frac{8\sqrt{2}}{27}\frac{\Delta^3}{C_T^{3/2}} + \mathcal{O}(\Delta^2) = -\frac{\sqrt{2}}{3\sqrt{3}}\frac{\Delta^3}{N^3} + \mathcal{O}(\Delta^2). \tag{7.76}$$

Again, using (7.50), one can confirm that this operator thermalizes

$$b_{T^3_{10,2}}\beta^{-12} = \lambda_{\mathcal{O}_H\mathcal{O}_H T^3_{10,2}}\Big|_{\frac{\Delta_H^3}{C_T^{3/2}}}. \tag{7.77}$$

7.5. Thermal two-point function and block decomposition

In this section we study the thermal two-point function $\langle\mathcal{O}_\Delta\mathcal{O}_\Delta\rangle_\beta$ and decompose it in thermal blocks. We determine the contributions of a few low-lying operators, including the stress tensor $T^1_{2,2}$ and the double stress tensor $T^2_{4,4}$. They exactly match the corresponding OPE coefficients and thermal expectation values computed in previous sections. Due to the presence of multiple operators with equal scaling dimension and spin, there is a mixing problem which we solve explicitly in a few cases. Related appendices include Appendix C.6, where we review the statement that the thermal one-point functions of multi-trace operators with derivatives vanish on $S^1 \times \mathbf{R}^{d-1}$ and Appendix C.7, where we consider two-dimensional thermal two-point functions. In Appendix C.8 we do a similar analysis for the vector model in four dimensions.

7.5.1. Thermal two-point function of a single trace scalar operator

The correlator at finite temperature β^{-1} in the free theory can be calculated by Wick contractions using the propagators on $S^1 \times \mathbf{R}^3$. Explicitly, the two-point function at finite temperature is given by[36]

$$\langle \mathcal{O}_\Delta(x)\mathcal{O}_\Delta(0)\rangle_\beta = \tilde{g}(x_E^0, |\mathbf{x}|)^\Delta + \frac{\pi^4\Delta(\Delta-2)}{9\beta^4}\tilde{g}(x_E^0, |\mathbf{x}|)^{\Delta-2} + \ldots, \tag{7.78}$$

where

$$\begin{aligned}\tilde{g}(x_E^0, |\mathbf{x}|) &= \sum_{m=-\infty}^{\infty} \frac{1}{(x_E^0 + m\beta)^2 + \mathbf{x}^2} \\ &= \frac{\pi}{2\beta|\mathbf{x}|}\Big[\mathrm{Coth}\Big(\frac{\pi}{\beta}(|\mathbf{x}| - ix_E^0)\Big) + \mathrm{Coth}\Big(\frac{\pi}{\beta}(|\mathbf{x}| + ix_E^0)\Big)\Big].\end{aligned} \tag{7.79}$$

The dots in (7.78) contain contributions due to further self-contractions which will not be important below[37]. Taking the $\beta \to \infty$ limit of (7.78) we can read off the decomposition of the two-point function in terms of thermal conformal blocks on $S^1 \times \mathbf{R}^3$ with coordinates $x = (x_E^0, \mathbf{x})$.

Following [82], if $|x| = \sqrt{(x_E^0)^2 + \mathbf{x}^2} \leq \beta$ the two-point function can be evaluated using the OPE:

$$\langle \mathcal{O}_\Delta(x)\mathcal{O}_\Delta(0)\rangle_\beta = \sum_{\mathcal{O}} \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}} |x|^{\tau-2\Delta} x_{\mu_1}\cdots x_{\mu_{s_\mathcal{O}}} \langle \mathcal{O}^{\mu_1\cdots\mu_{s_\mathcal{O}}}\rangle_\beta, \tag{7.80}$$

where $\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}}$ is the OPE coefficient, τ and $s_\mathcal{O}$ is the twist and spin of $\mathcal{O}$, respectively. Using (7.49) together with (7.80), the two-point function on $S^1 \times \mathbf{R}^3$ can be organized in the following way [82]:

$$\langle \mathcal{O}_\Delta(x)\mathcal{O}_\Delta(0)\rangle_\beta = \sum_{\mathcal{O}_{\tau,s}\in\mathcal{O}_\Delta\times\mathcal{O}_\Delta} \frac{a_{\mathcal{O}_{\tau,s}}}{\beta^{\Delta_\mathcal{O}}} \frac{1}{|x|^{2\Delta-\tau+s}} C_s^{(1)}\left(\frac{x_E^0}{|x|}\right), \tag{7.81}$$

[36] Here and below we assume that $\Delta > 4$. We further drop the disconnected term $\langle \mathcal{O}_\Delta\rangle_\beta^2 \sim N^2$.

[37] These terms will be proportional to $\beta^{-2a}\tilde{g}(x_E^0, |\mathbf{x}|)^{\Delta-a}$, with $a \geq 4$. When decomposed into thermal blocks, these will not affect the operators with dimension $\Delta < 8$ or $\Delta = 8$ with non-zero spin $s \neq 0$.

where we sum over primary operators $\mathcal{O}_{\tau,s}$, with twist τ and spin s, appearing in the OPE $\mathcal{O}_\Delta \times \mathcal{O}_\Delta \sim \mathcal{O}_{\tau,s} + \ldots$. In (7.81) $C_s^{(1)}(x_E^0/|x|)$ is a Gegenbauer polynomial which, together with a factor of $|x|^{-2\Delta+\tau-s}$, forms a thermal conformal block in $d=4$ dimensions and the coefficients $a_{\mathcal{O}_{\tau,s}}$ are given by

$$a_{\mathcal{O}_{\tau,s}} = \left(\frac{1}{2}\right)^s \lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}_{\tau,s}} b_{\mathcal{O}_{\tau,s}}. \tag{7.82}$$

Expanding (7.78) for $\beta \to \infty$ one finds:

$$\begin{aligned}\langle \mathcal{O}_\Delta(x)\mathcal{O}_\Delta(0)\rangle_\beta =& \frac{1}{|x|^{2\Delta}}\Big[1 + \frac{\pi^2\Delta}{3\beta^2}|x|^2 \\ &+ \frac{\pi^4\Delta}{90\beta^4}|x|^2(3\mathbf{x}^2(5\Delta-9) + (15\Delta-19)(x_E^0)^2) + \mathcal{O}(\beta^{-6})\Big].\end{aligned} \tag{7.83}$$

From the expansion (7.83), we can read off the coefficients $a_{\tau',s'} := \sum_{\mathcal{O}_{\tau',s'}} a_{\mathcal{O}_{\tau',s'}}$ where we sum over all operators with twist τ' and spin s':

$$\begin{aligned} a_{2,0} &= \frac{\pi^2\Delta}{3}, \\ a_{4,0} &= \frac{\pi^4\Delta(3\Delta-5)}{18}, \\ a_{2,2} &= \frac{\pi^4\Delta}{45}. \end{aligned} \tag{7.84}$$

For future reference, expanding (7.78) to $\mathcal{O}(\frac{1}{\beta^8})$ one finds

$$\begin{aligned} a_{2,4} &= \frac{2\pi^6\Delta}{945}, \\ a_{4,4} &= \frac{\pi^8\Delta(\Delta-1)}{1050}. \end{aligned} \tag{7.85}$$

Note that due to the mixing of operators with the same twist and spin, $a_{\tau,s}$ generically contains the contribution from multiple operators. In the following section we calculate the OPE coefficients and thermal one-point functions of operators which are not multi stress tensors but contribute to (7.84) and (7.85).

7.5.2. CFT data of scalar operators with dimensions two and four

We explicitly calculate the thermal one-point functions $\langle\mathcal{O}\rangle_\beta = b_{\mathcal{O}}\beta^{-\Delta_{\mathcal{O}}}$ and OPE coefficients $\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}}$ for scalar operators $\mathcal{O}$ with twist $\tau'=2$ and $\tau'=4$

using Wick contractions. This is done to find which operators contribute to the thermal two-point function and to resolve a mixing problem.

For $\tau' = 2$ there is only one such operator, the single trace operator $\mathcal{O}_2(x) = \frac{1}{\sqrt{2}N} : Tr(\phi^2) : (x)$ given in (7.20). The OPE coefficient is found by considering the three-point correlator

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\mathcal{O}_2(x_3)\rangle = \frac{\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}_2}}{|x_{12}|^{2\Delta-2}|x_{13}|^2|x_{23}|^2}. \tag{7.86}$$

The three-point function is calculated in Appendix C.1, in the large-N limit, and it is given by

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\mathcal{O}_2(x_3)\rangle = \frac{\sqrt{2}\Delta}{N}\frac{1}{|x_{12}|^{2\Delta-2}|x_{13}|^2|x_{23}|^2}, \tag{7.87}$$

and therefore $\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}_2} = \frac{\sqrt{2}\Delta}{N}$ to leading order in $1/N$. To calculate the thermal one-point function $\propto \langle Tr(\phi^2)\rangle_\beta$, we include self-contractions, i.e. contractions of fundamental fields within the same composite operator separated by a distance $m\beta$ along the thermal circle for $m \neq 0$ and integer. Explicitly, the one-point function of $\mathcal{O}_2$ is given by

$$\langle \mathcal{O}_2(x)\rangle_\beta = \frac{1}{\sqrt{2}N}\sum_{m\neq 0}\frac{N^2}{(m\beta)^2} = \frac{\pi^2 N}{3\sqrt{2}\beta^2}, \tag{7.88}$$

therefore,

$$b_{\mathcal{O}_2} = \frac{\pi^2 N}{3\sqrt{2}}. \tag{7.89}$$

The contribution to the thermal two-point function $a_{\mathcal{O}_2}$ is found using (7.87) and (7.89)

$$a_{2,0} = b_{\mathcal{O}_2}\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}_2} = \frac{\pi^2\Delta}{3}. \tag{7.90}$$

This agrees with (7.84) which was obtained from the thermal two-point function.

We now continue with scalar operators of twist four. There are two such linearly independent operators appearing in the $\mathcal{O}_\Delta \times \mathcal{O}_\Delta$ OPE. In order to construct an orthonormal basis, consider the following single and double trace operators:

$$\begin{aligned}\mathcal{O}_4(x) &= \frac{1}{2N^2} : Tr(\phi^4) : (x),\\ \mathcal{O}_{4,\mathrm{DT}}(x) &= \frac{1}{2\sqrt{2}N^2} : Tr(\phi^2)Tr(\phi^2) : (x).\end{aligned} \tag{7.91}$$

We further construct the operator $\tilde{\mathcal{O}}_4$ that has vanishing overlap with $\mathcal{O}_{4,\mathrm{DT}}(x)$ as follows:

$$\tilde{\mathcal{O}}_4 = \mathcal{N}\Big[\mathcal{O}_4 - c_{\mathcal{O}_4\mathcal{O}_{4,\mathrm{DT}}}\mathcal{O}_{4,\mathrm{DT}}\Big], \tag{7.92}$$

with $\mathcal{N}$ a normalization constant and $c_{\mathcal{O}_4\mathcal{O}_{4,\mathrm{DT}}}$ is the overlap defined by

$$\langle \mathcal{O}_4(x)\mathcal{O}_{4,\mathrm{DT}}(y)\rangle = \frac{c_{\mathcal{O}_4\mathcal{O}_{4,\mathrm{DT}}}}{|x-y|^8}. \tag{7.93}$$

Explicit calculation gives $c_{\mathcal{O}_4\mathcal{O}_{4,\mathrm{DT}}} = \frac{2\sqrt{2}}{N}$ and $\mathcal{N} = \frac{1}{\sqrt{2}}$ in the large-N limit, and the scalar dimension four operator orthogonal to the double trace operator $\mathcal{O}_{4,\mathrm{DT}}$ is therefore

$$\tilde{\mathcal{O}}_4 = \frac{1}{\sqrt{2}}\Big[\mathcal{O}_4 - \frac{2\sqrt{2}}{N}\mathcal{O}_{4,\mathrm{DT}}\Big]. \tag{7.94}$$

Note that even though the second term in (7.92) is suppressed by $1/N$, it can still contribute to the thermal two-point function due to the scaling of OPE coefficients and one-point function of a k-trace operator $\mathcal{O}^{(k)}$:

$$\begin{aligned} b_{\mathcal{O}^{(k)}} &\sim N^k, \\ \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}^{(k)}} &\sim \frac{1}{N^k}, \end{aligned} \tag{7.95}$$

in the limit $N\to\infty$.

The one-point function and the OPE coefficient for $\mathcal{O}_4$ is found analogously to that of $\mathcal{O}_2$ in the large-N limit

$$\begin{aligned} b_{\mathcal{O}_4} &= \frac{\pi^4 N}{9}, \\ \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}_4} &= \frac{2\Delta}{N}. \end{aligned} \tag{7.96}$$

Consider now the double trace operator given in (7.91). The one-point function factorizes in the large-N limit:

$$\begin{aligned} \langle \mathcal{O}_{4,\mathrm{DT}}(x)\rangle_\beta &= \frac{1}{\sqrt{2}}(\langle \mathcal{O}_2(x)\rangle_\beta)^2 \\ &= \frac{\pi^4 N^2}{18\sqrt{2}\beta^4}. \end{aligned} \tag{7.97}$$

Likewise, the OPE coefficient can be computed in the large-N limit (see Appendix C.1)

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}_{4,\mathrm{DT}}} = \frac{\sqrt{2}\Delta(3\Delta-5)}{N^2}. \tag{7.98}$$

Consider now the thermal one-point function of $\tilde{\mathcal{O}}_4$ in (7.94)

$$\begin{aligned}\langle\tilde{\mathcal{O}}_4\rangle_\beta =& \frac{1}{\sqrt{2}\beta^4}\left[b_{\mathcal{O}_4} - \frac{2\sqrt{2}}{N} b_{\mathcal{O}_{4,\mathrm{DT}}}\right] \\ &= \mathcal{O}(N^{-1}),\end{aligned} \tag{7.99}$$

where we have used (7.96) and (7.97). Since the corresponding OPE coefficient is suppressed by N^{-1}, it follows that the only scalar operator with dimension four contributing to the thermal two-point function is the double trace operator $\mathcal{O}_{4,\mathrm{DT}}$. From the OPE coefficient and thermal one-point function of this double trace operator, using (7.97) and (7.98), we find the following contribution to the thermal two-point function

$$a_{4,0} = \frac{\pi^4\Delta(3\Delta-5)}{18}, \tag{7.100}$$

which agrees with (7.84).

7.5.3. CFT data of single-trace operator with twist two and spin four

The primary single trace operator $\Xi = \mathcal{O}_{2,4}$ with twist $\tau = 2$ and spin $s = 4$ is given by

$$\begin{aligned}\Xi_{\mu\nu\rho\sigma}(x) = \frac{1}{96\sqrt{35}N} :Tr\big(&\phi(\partial_\mu\partial_\nu\partial_\rho\partial_\sigma\phi) - 16(\partial_{(\mu}\phi)(\partial_\nu\partial_\rho\partial_{\sigma)}\phi) \\ &+ 18(\partial_{(\mu}\partial_\nu\phi)(\partial_\rho\partial_{\sigma)}\phi) - (\text{traces})\big) : (x).\end{aligned} \tag{7.101}$$

The relative coefficients follow from requiring that the operator is a primary, see Appendix C.5 for details.

The thermal one-point function of this operator is found from Wick contractions in the large-N limit to be

$$\langle\Xi_{\mu\nu\rho\sigma}\rangle_\beta = \frac{8\pi^6 N}{27\sqrt{35}\beta^6}\left(e_\mu e_\nu e_\rho e_\sigma - (\text{traces})\right). \tag{7.102}$$

Moreover, the OPE coefficient in the $\mathcal{O}_\Delta \times \mathcal{O}_\Delta$ OPE can again be calculated using Wick contractions similarly to how it was done for $T^2_{4,4}$ in Appendix C.1. By explicit calculation one finds

$$\langle\mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\Xi_{\mu\nu\rho\sigma}(x_3)\rangle = \frac{4\Delta}{\sqrt{35}N}\frac{Z_\mu Z_\nu Z_\rho Z_\sigma - (\text{traces})}{|x_{12}|^{2\Delta-2}|x_{13}|^2|x_{23}|^2}, \tag{7.103}$$

and therefore the OPE coefficient $\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}_{2,4}}$ is given by

$$\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}_{2,4}} = \frac{4\Delta}{\sqrt{35}N}. \tag{7.104}$$

Now, it is easy to check that

$$\frac{1}{2^4}\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}_{2,4}} b_{\mathcal{O}_{2,4}} = \frac{2\pi^6\Delta}{945}, \tag{7.105}$$

which agrees with $a_{2,4}$ in (7.85).

7.5.4. CFT data of double-trace operators with twist and spin equal to four

To find the full contribution to the thermal two-point function from the operators with $\tau = 4$ and $s = 4$ we need to take into account the contribution of all operators with these quantum numbers and solve a mixing problem. In addition to the double-stress tensor operator with these quantum numbers, the other double trace primary operator which contributes is given by

$$\begin{aligned}\mathcal{O}^{\mathrm{DT}}_{\mu\nu\rho\sigma}(x) = \frac{1}{96\sqrt{70}N^2} &: Tr(\phi^2)\Big(Tr(\phi\partial_\mu\partial_\nu\partial_\rho\partial_\sigma\phi) - 16Tr(\partial_{(\mu}\phi\partial_\nu\partial_\rho\partial_{\sigma)}\phi) \\ &+18Tr(\partial_{(\mu}\partial_\nu\phi\partial_\rho\partial_{\sigma)}\phi)(x) - (\text{traces})\Big) : (x),\end{aligned} \tag{7.106}$$

where the operator is unit-normalized. Notice that this is the double trace operator obtained by taking the normal ordered product of two single trace operators, the scalar operator with dimension 2 and the single trace spin-4 operator with dimension 6. There are more double trace operators with these quantum numbers which are, however, not simply products of single trace operators. These do not contribute to the thermal two-point function to leading order in $\frac{1}{N^2}$ (see Appendix C.6).

Note that it follows from large-N factorization that the overlap of this operator with $(T^2)_{\mu\nu\rho\sigma}$ is suppressed by powers of $\frac{1}{N}$; since both of these are double trace operators and obey the scaling (7.95), to study the thermal two-point function to leading order in N^2, one can therefore neglect this overlap.

The thermal one-point function of $\mathcal{O}^{\mathrm{DT}}_{\mu\nu\rho\sigma}$ follows from the large-N factorization and we find that

$$b_{\mathcal{O}^{\mathrm{DT}}_{4,4}} = \sqrt{\frac{2}{35}}\frac{4\pi^8 N^2}{81}, \tag{7.107}$$

where we used the thermal one-point functions for each single trace operator given by (7.88) and (7.102). The OPE coefficient is calculated in Appendix C.1,

$$\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}^{\mathrm{DT}}_{4,4}} = \sqrt{\frac{2}{35}}\frac{4\Delta(\Delta-1)}{N^2}. \tag{7.108}$$

Using the thermal one point function and the OPE coefficient in (7.107) and (7.108) respectively, it is found that it the operator $\mathcal{O}^{\mathrm{DT}}_{\mu\nu\rho\sigma}$ gives the following contribution to the thermal two point function:

$$a_{\mathcal{O}^{\mathrm{DT}}_{4,4}} = \left(\frac{1}{2}\right)^4 b_{\mathcal{O}^{\mathrm{DT}}_{4,4}} \lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}^{\mathrm{DT}}_{4,4}} = \frac{2\pi^8\Delta(\Delta-1)}{2835}. \tag{7.109}$$

The total contribution from $T^2_{4,4}$ together with that of $\mathcal{O}^{\mathrm{DT}}_{4,4}$, using (7.28), (7.53) and (7.109), is

$$a_{4,4} = (a_{T^2_{4,4}} + a_{\mathcal{O}^{\mathrm{DT}}_{4,4}}) = \frac{\pi^8\Delta(\Delta-1)}{1050}. \tag{7.110}$$

This agrees with $a_{4,4}$ in (7.85).

7.6. Comparison with the eigenstate thermalization hypothesis

In this section we discuss the relation of our results to the eigenstate thermalization hypothesis (ETH). We argue that the stress tensor sector of the free $SU(N)$ adjoint scalar theory in $d = 4$ satisfies the ETH to leading order in $C_T \sim N^2 \gg 1$. We explain the equivalence of the micro-canonical and canonical ensemble when $\Delta_H \sim C_T$ in large-C_T theories. In this regime, the diagonal part of the ETH is (up to exponentially suppressed terms which we do not consider), equivalent to thermalization. Note that in two dimensions the Virasoro descendants of the identity satisfy the ETH (see e.g. [165] for a recent discussion).

We begin by showing the equivalence between the micro-canonical and the canonical ensemble on $S^1_\beta \times S^{d-1}$ when $\Delta_H \sim C_T \gg 1$. See [29,148] for a similar discussion at infinite volume as well as [116] in the two-dimensional case. The expectation value in the micro-canonical ensemble of an operator $\mathcal{O}$, which we take to be a scalar for simplicity, at energy $E = \Delta_H/R$ is given by

$$\langle \mathcal{O} \rangle^{(\mathrm{micro})}_E = \frac{1}{N(E)} \sum_{\tilde{\mathcal{O}}} \langle \tilde{\mathcal{O}} | \mathcal{O} | \tilde{\mathcal{O}} \rangle, \tag{7.111}$$

where we sum over states $|\tilde{\mathcal{O}}\rangle$ with energy $(E, E+\delta E)$ and $N(E)$ is the number of states in this interval. On the other hand, consider the partition function at inverse temperature β given by

$$Z(\beta) = \sum_{\tilde{\mathcal{O}}} e^{-\frac{\beta\tilde{\Delta}}{R}} = \int d\tilde{\Delta}\rho(\tilde{\Delta})e^{-\frac{\beta\tilde{\Delta}}{R}}, \tag{7.112}$$

where we sum over all states in the theory. In the second line in (7.112) we have approximated the sum of delta-functions by a continuous function $\rho(\tilde{\Delta})$. Expectation values in the canonical ensemble is then computed by[38]

$$\langle\mathcal{O}\rangle_\beta = Z(\beta)^{-1}\int d\tilde{\Delta}\rho(\tilde{\Delta})\langle\mathcal{O}\rangle_E^{(\text{micro})}e^{-\frac{\beta\tilde{\Delta}}{R}}. \tag{7.113}$$

Consider the partition function in (7.112) with a free energy $F = -\beta^{-1}\log Z(\beta)$. By an inverse Laplace transform of (7.112) we find the density of states

$$\rho(\Delta_H) = \frac{1}{2\pi i R}\int d\beta' e^{\beta'(\frac{\Delta_H}{R}-F(\beta'))}. \tag{7.114}$$

For $\Delta_H \sim C_T$ and a large free energy[39] $F \sim C_T$, we can evaluate (7.114) using a saddlepoint approximation with the saddle at β given by

$$\frac{\Delta_H}{R} = \partial_{\beta'}(\beta' F)|_\beta. \tag{7.115}$$

Consider now the thermal expectation value in (7.113), multiplying both sides by $Z(\beta)$ and doing an inverse Laplace transform evaluated at $\Delta_H \sim C_T$ we find

$$\rho(\Delta_H)\langle\mathcal{O}\rangle_{\Delta_H/R}^{(\text{micro})} = \frac{1}{2\pi i R}\int d\beta'\langle\mathcal{O}\rangle_{\beta'}e^{\beta'(\frac{\Delta_H}{R}-F(\beta'))}. \tag{7.116}$$

For $F \sim C_T \gg 1$ we again use a saddlepoint approximation to evaluate (7.116) with the saddle at β determined by (7.115), assuming $\langle\mathcal{O}\rangle_{\beta'}$ does not grow exponentially with C_T. The RHS of (7.116) is therefore the thermal expectation value $\langle\mathcal{O}\rangle_\beta$ multiplied by the saddlepoint approximation of the density of states in (7.114). It then follows that

$$\langle\mathcal{O}\rangle_{\Delta_H/R}^{(\text{micro})} \approx \langle\mathcal{O}\rangle_\beta, \tag{7.117}$$

[38] It was argued in [148] that the existence of the thermodynamic limit implies that we only need to sum over operators with low spin.

[39] We consider a CFT in a high temperature phase.

with β determined by (7.115). In particular, in the infinite volume limit $R \to \infty$, the free energy is given by[40]

$$F = \frac{b_{T^{(\mathrm{can})}_{\mu\nu}} S_d R^{d-1}}{d\beta^d}, \tag{7.118}$$

where $S_d = Vol(S^{d-1}) = 2\pi^{\frac{d}{2}}/\Gamma(\frac{d}{2})$. Inserting (7.118) in (7.115) we find [29]

$$\frac{\beta}{R} = \left(\frac{-(d-1) b_{T^{(\mathrm{can})}_{\mu\nu}} S_d}{d\Delta_H}\right)^{\frac{1}{d}}. \tag{7.119}$$

We can use (7.117) to see the thermalization of the stress tensor. The free energy is related to the expectation value of the stress tensor $T^{(\mathrm{can})}_{\mu\nu}$ [25]

$$\langle T^{(\mathrm{can})}_{00}\rangle_\beta = \frac{1}{S_d R^{d-1}} \partial_\beta(-\beta F(\beta)). \tag{7.120}$$

On the other hand, the expectation value of $T^{(\mathrm{can})}_{00}$ in a heavy state $|\mathcal{O}_H\rangle$ is fixed by the Ward identity to be

$$\langle \mathcal{O}_H | T^{(\mathrm{can})}_{00} | \mathcal{O}_H\rangle = -\frac{\Delta_H}{S_d R^d}. \tag{7.121}$$

Multiplying (7.115) with $(S_d R^{d-1})^{-1}$ and comparing with (7.120)-(7.121) we find that

$$\langle \mathcal{O}_H | T^{(\mathrm{can})}_{00} | \mathcal{O}_H\rangle = \langle T^{(\mathrm{can})}_{00}\rangle_\beta. \tag{7.122}$$

This shows the thermalization of the stress tensor in heavy states where $F \sim \Delta_H \sim C_T$ in large-C_T theories. Note that this follows from (7.117) since we can replace the micro-canonical expectation value at $E = \Delta_H/R$, on the LHS, with the expectation value in any single heavy state with dimension Δ_H due to the Ward identity, independent of the heavy state. Put differently, the stress tensor satisfies the ETH as we will review below.

We now consider the eigenstate thermalization hypothesis for CFTs at finite temperature on the sphere S^{d-1} of radius R. The diagonal part of the ETH is given by

$$\langle \mathcal{O}_H | \mathcal{O}_{\tau,s} | \mathcal{O}_H\rangle = \langle \mathcal{O}_{\tau,s}\rangle^{(\mathrm{micro})}_E + \mathcal{O}\left(e^{-S(E)}\right), \tag{7.123}$$

[40] Here we denote the canonically normalized stress tensor by $T^{(\mathrm{can})}_{\mu\nu}$, whose two-point function is given by $\langle T^{(\mathrm{can})\mu\nu}(x) T^{(\mathrm{can})}_{\rho\sigma}(y)\rangle = \frac{C_T}{S_d^2}(I^\mu{}_{(\rho} I^\nu{}_{\sigma)} - (\mathrm{trace}))$.

where $\mathcal{O}_H$ and $\mathcal{O}_{\tau,s}$ are local primary operators and $\langle \mathcal{O}_{\tau,s}\rangle_E^{(\text{micro})}$ is the expectation value of $\mathcal{O}_{\tau,s}$ in the micro-canonical ensemble at energy $E = \frac{\Delta_H}{R}$. Here we assume that the operator $\mathcal{O}_H$ is a heavy scalar operator with large conformal dimension $\Delta_H \propto C_T \gg 1$. The operator $\mathcal{O}_{\tau,s}$ on the other hand can have non-zero spin.[41]. In (7.123), $e^{S(E)}$ is the density of states at energy $E = \Delta_H/R$. As shown in (7.117), in the limit $\Delta_H \sim C_T \gg 1$, the micro-canonical ensemble is equivalent to the canonical ensemble at inverse temperature β determined by (7.115). It then follows from (7.123) that the diagonal part of the ETH can written in terms of OPE coefficients and thermal one-point functions:

$$\frac{\lambda_{\mathcal{O}_H\mathcal{O}_H\mathcal{O}_{\tau,s}}}{R^{\tau+s}} = \frac{b_{\mathcal{O}_{\tau,s}} f_{\mathcal{O}_{\tau,s}}\left(\beta/R\right)}{\beta^{\tau+s}} + \mathcal{O}\left(e^{-S(E)}\right), \tag{7.124}$$

where $f_{\mathcal{O}_{\tau,s}}$ also appears in (7.11). This is equivalent to the statement of thermalization discussed in the rest of the section.

In this section we observed that the multi stress tensor operators satisfy (7.124). One can also ask if (7.124) holds for any operator in the specific heavy state we considered. By comparing eqs. (7.87) and (7.88) using (7.50), one can check that operator $\mathcal{O}_2 = \frac{1}{\sqrt{2}N} : Tr(\phi^2) :$ does not satisfy (7.124). Since this is a free theory, it is not a surprise that the ETH is not satisfied by all operators in the spectrum which is seen explicitly in this case.

7.7. Discussion

In this section we argued that multi stress tensor operators $T^k_{\tau,s}$ in CFTs with a large central charge C_T thermalize: their expectation values in heavy states are the same as their thermal expectation values. This is equivalent to the statement that multi stress tensor operators in higher-dimensional CFTs satisfy the diagonal part of the ETH in the thermodynamic limit. The analogous statement in the $d = 2$ case is that the Virasoro descendants of the identity satisfy the ETH condition in the large-C_T limit.

We observed that the operator $\mathcal{O}_2 = \frac{1}{\sqrt{2}N} : Tr(\phi^2) :$ does not satisfy the ETH. This is seen by comparing eqs. (7.87) and (7.88) using (7.50). While this operator does not thermalize in the heavy states we considered, the OPE coefficient averaged over all operators with $\Delta_H \sim C_T$ is expected to be proportional

[41] The tensor structure in (7.123) is suppressed.

to the thermal one-point function. The averaged OPE coefficients should therefore scale like $\sim \sqrt{\Delta_H}$ compared to $\lambda_{\mathcal{O}_H\mathcal{O}_H\mathcal{O}_2} \sim \Delta_H/\sqrt{C_T}$ for the heavy states we considered. It would be interesting to exhibit heavy operators that produce the former scaling.

We provided a bootstrap argument in favor of the thermalization of multi stress tensor operators. One should be able to refine it to give an explicit form for leading behavior of the multi stress tensor OPE coefficients – we leave it for future work. The holographic/bootstrap OPE coefficients for the leading twist double stress tensor operators can be found in e.g. [124] – they are nontrivial functions of the spin. As explained in [124,13], the leading Δ behavior of the minimal-twist double- and triple-stress tensor OPE coefficients is consistent with the exponentiation of the near lightcone stress tensor conformal block. One can go beyond the leading twist multi stress tensors. In holographic HHLL correlators each term of the type $(\Delta\mu)^k \sim (\Delta\Delta_H/C_T)^k$ comes from the exponentiation of the stress-tensor block – this follows from the Wilson line calculation of the correlator in the AdS-Schwarzschild background [193,124,186].

In this section we argue that this behavior is universal, and is not just confined to holographic theories. Hence, one can formulate another statement equivalent to the thermalization of multi stress tensor operators. Namely, scalar correlators of pairwise identical operators of dimensions $\Delta_{1,2}$ in large-C_T theories in the limit $\Delta_{1,2} \gg 1$, $\Delta_1\Delta_2/C_T$ fixed are given by the exponentials of the stress tensor conformal block[42]. This is similar to what happens in two-dimensional CFTs.

Note that the universality of the OPE coefficients is naively in tension with the results of [54], where finite gap (Δ_{gap}) corrections to the multi stress tensor OPE coefficients were considered. In particular, for double stress tensors, such corrections behave like $\Delta^3/\Delta_{\text{gap}}$ which is clearly at odds with the universality statement. Of course, the results of [54] are obtained in the limit $\Delta \ll \Delta_{\text{gap}}$, while in this section we consider the opposite regime $\Delta \gg \Delta_{\text{gap}}$.

One may also wonder what happens with the universality of the OPE coefficients beyond leading order in Δ. In particular, in [186], it was shown that

[42] See [105] for previous work on the eikonalization of the multi stress tensor OPE coefficients at large spin.

the bootstrap result for the HHLL correlator exactly matches the holographic Wilson line calculation (in the double scaling limit where only the minimal twist multi stress tensor operators contribute). This corresponds to including terms beyond the exponential of the stress tensor block – one needs to compute the HHLL correlator, take a logarithm of the result, divide by Δ, and then take the large-Δ limit. The result is sensitive to terms subleading in the large-Δ limit of the multi stress tensor OPE coefficients. In four spacetime dimensions the result in [186] is given by an elliptic integral – is it applicable beyond holography?

In [124] terms subleading in Δ were shown to be important for the computation of the phase shift. The simplest nontrivial case in two spacetime dimensions is the operator Λ_4 which is a level four Virasoro descendant of the identity (see e.g.[194]). One could also get it by using the CFT normal ordering and imposing the quasi-primary condition [195]. Consider now the case of minimal twist (twist four) operators in four dimensions. How do we determine the analog of Λ_4? There is no Virasoro algebra now.

Presumably, one can reconstruct the analog of Λ_4 in four spacetime dimensions by considering a CFT normal ordered product of stress tensors, and adding a single trace term to ensure that the resulting operator is a primary and is orthogonal to the stress tensor itself. Note that the CFT normal ordering differs from the oscillator normal ordering in a free theory by the addition of a single trace operator, as reviewed in Appendix C.7. This procedure can then be generalized to other multi-trace operators. We leave it for future work.

It is also helpful to imagine what happens in a theory like $\mathcal{N} = 4$ Super Yang-Mills, where there is a marginal line connecting the weak and the strong coupling (the latter admits a holographic description). Presumably, as the coupling is turned on, only one operator remains light (with dimension eight and spin four), while others get anomalous dimensions. It would be interesting to see this explicitly even to the leading nontrivial order in the 't Hooft coupling. It would also be interesting to see how the corresponding OPE coefficient interpolates between its free and strong coupling values.

Using crossing symmetry, we argued that the universality of multi stress tensor OPE coefficients is related to the OPE coefficients $\lambda_{\mathcal{O}_H T_{\mu\nu} \mathcal{O}'}$, with $\mathcal{O}' \neq \mathcal{O}_H$ being either heavy or light, present in the cross-channel expansion. Such OPE coefficients with at least one operator being heavy were recently studied in

[154,196]. It would be interesting to further study the connection of our results to this work.

Another interesting question concerns the fate of the double trace operators of the schematic form $[\mathcal{O}_\Delta \mathcal{O}_\Delta]_{n,l}$. Consider the $d=4$ case in the large volume limit and $n,l=0$, for simplicity. We expect that the corresponding OPE coefficients in the free theory behave like $\lambda_{\mathcal{O}_H\mathcal{O}_H[\mathcal{O}_\Delta\mathcal{O}_\Delta]_{0,0}} \propto \Delta_H^2/C_T \propto C_T\mu^2$,[43] while their thermal one-point functions behave like $\langle[\mathcal{O}_\Delta\mathcal{O}_\Delta]_{0,0}\rangle_\beta \propto C_T\beta^{-2\Delta}$. Comparing the two results with the help of (7.50) one observes that such operators do not thermalize in the free theory for generic Δ. The situation is more nontrivial in holography where we do not know the large μ behavior of the OPE coefficients[44]. As pointed out in [15], the contribution of double-trace operators to thermal two-point functions is different from that of multi stress tensors. The latter is only sensitive to the behavior of the metric near the boundary, but the former knows about the full black hole metric. This seems to indicate that the thermalization of the double trace operators in holographic theories is also unlikely[45].

It is a natural question how generic are the heavy states for which the stress tensor sector thermalizes. The our results seem to suggest that such thermalization is more generic than the thermalization of other light operators[46]. Other interesting questions include generalizations to the case of finite but large central charge and to non-conformal quantum field theories.

[43] This scaling is obtained by computing the OPE coefficient $\lambda_{\mathcal{O}_H\mathcal{O}_H[\mathcal{O}_\Delta\mathcal{O}_\Delta]_{0,0}}$ for $1 \ll \Delta_H \ll C_T$ and extrapolating it to the $\Delta_H \sim C_T$ regime.

[44] Note that the large-N scaling in holography is different. Both the OPE coefficients and the thermal expectation values behave like C_T^0 as opposed to $C_T \sim N^2$.

[45] A simple way to decouple such operators is to take the large-Δ limit.

[46] A closely related question of finding "typical" states where the stress tensor sector thermalizes in the large volume limit in $d=2$ was recently discussed in [179]. There it was observed that such states are Virasoro descendants when the central charge is finite.

8. Discussion and conclusions

We started this research to learn more about the whole class of holographic conformal field theories and their implications for the dual gravity. The conformal bootstrap in the Regge and lightcone limit allowed us to gain a better insight into the four-point correlation functions at higher orders of the inverse central charge. We specifically focus on the stress tensor sector of the conformal field theory that consists of the stress tensor itself and the multi-trace primary operators made from the stress tensor. The importance of this sector lies in the fact that it is necessarily present in all holographic CFTs that have a dual theory with dynamical gravity. The gravitational analog of multi stress tensor operators that contribute to the correlators in CFTs are the graviton loops in the corresponding Witten diagrams. In the four-point correlation functions where two operators $\mathcal{O}_H$ have the large conformal dimension Δ_H, the contributions of the stress tensor sector get enhanced by factors of Δ_H compared to the contributions of other, generic, multi-trace operators, and therefore decouple from them. That is the reason why the heavy-heavy-light-light correlation function is the perfect setup for studying the stress tensor sector contributions.

We confirm that the OPE coefficients of the multi stress tensor operators with a minimal twist at each order in μ are universal, i.e. they are the same in all holographic CFTs, as claimed in [15]. Introducing the finite gap in the theory breaks this universality, as shown in [54]. One can still ask is it possible to add up all contributions of the minimal twist subset of the stress tensor sector and obtain the correlator exact in μ. As the contributions of these operators dominate in the lightcone limit, one could hope to obtain the exact non-perturbative correlator in this limit by adding up all of them. This result would be the higher-dimensional analog of the Virasoro vacuum block in two-dimensional CFT. So far, it has not been established how to write this non-perturbative correlator and we leave this for future work. The general idea is that there exists some Virasoro-like symmetry algebra generated by the components of the stress tensor that emerges near the lightcone limit and protects the OPE coefficients of the minimal twist multi stress tensors in the holographic CFTs [127,142,197-199]. If we had the closed algebra of this type, computing the contributions of its irreducible representations to the correlator would greatly simplify the problem.

The important property of the OPE coefficients of multi-trace operators in the stress tensor sector with external scalars is that they have poles at the integer values of Δ_L. These poles are the consequence of the mixing of multi stress tensors with double-trace operators $[\mathcal{O}_L\mathcal{O}_L]_{n,l}$ when Δ_L are integers. It would be interesting to find a way to solve this mixing problem and obtain finite contributions for integer Δ_L. According to the statement in [15], to calculate the OPE coefficients of these double trace operators using the dual gravity approach, one would have to analytically solve the equation of motion of the light probe in the black hole background all the way to the horizon, therefore, the perturbative $1/r$ calculation that works for the stress tensor sector is not applicable here. One comes to a similar conclusion when trying to compute the OPE coefficients of $[\mathcal{O}_L\mathcal{O}_L]_{n,l}$ operators using the conformal bootstrap technique. Namely, in this case one would have to solve the bootstrap equation non-perturbatively in large spin, for which the non-perturbative results for the anomalous dimensions of the operators in the S-channel are needed. Therefore, we leave this task for future work as well.

The multi stress tensor operators whose OPE coefficients are not fixed by the conformal bootstrap are those with spin $s = 0, 2$. As they are not fixed by the consistency conditions of the underlying CFT these OPE coefficients can be viewed as the parameters in the class of holographic CFTs. They depend on the other parameters of the holographic CFT, besides Δ_L and μ, which is the reason we say they are not universal. They introduce these parameters in the correlation function and then the bootstrap forces the other OPE coefficients of multi stress tensor operators with the non-minimal twist and higher spin to depend on them. The universality of the OPE coefficients of the minimal twist multi stress tensors at all orders in μ can be explained by the fact that there are no multi stress tensor operators with a minimal twist and spin $s = 0, 2$ that would introduce additional parameters in the OPE coefficients of the minimal twist subset of the stress tensor sector.

One could ask if there is some other consistency condition in the CFT that could fix or bound the coefficients of multi stress tensors with spin $s = 0, 2$, for example, obtained from the bootstrap of the higher-point correlation functions or correlation functions of spinning operators. We leave these questions for future research. We showed how one can fix the OPE coefficients of spin-2 multi stress tensor operators using the wave function phase shift calculation in the gravity dual. In our case, this is done for Gauss-Bonnet gravity dual and the results obtained match with those from [15] whenever available in the latter. The only way, known so far, to fix the OPE coefficients of the spin zero multi stress tensors through the gravity dual is the one developed in [15].

One obvious question to ask is which part of the formalism developed for the stress tensor sector would apply to the case when we have additional conserved currents in the holographic CFT, for example, $U(1)$ current J_μ. In this case, one would have to consider the contributions of the stress tensor sector together with the contributions of the conserved current sector and multi-trace operators created from both single-traces $T_{\mu\nu}$ and J_μ. We plan to tackle this problem in the near future.

Finally, by studying the thermalization properties of the stress tensor sector, we demonstrated that it satisfies the diagonal part of the eigenstate thermalization hypothesis in the thermodynamic limit of large-N CFTs. This justifies using the pure heavy state $\mathcal{O}_H$ as the CFT analog of the black hole when we are interested in the graviton contributions to the correlation functions in holographic CFTs. We also showed that the other operators generically present in the large-N CFTs do not thermalize in this sense. Here, one could again ask what happens in the large-N theory with additional conserved charges, whether the conserved current sector thermalizes the same way the stress tensor sector does. Additionally, it would be interesting to check what happens with the off-diagonal elements of the eigenstate thermalization hypothesis for both the stress tensor sector and the conserved current sector.

Appendix A.1. Details on the conformal bootstrap

Below we review some of the details of the confomal bootstrap calculations. Explicitly, we will show that exchanges of heavy-light double-trace operators in the S-channel reproduce the disconnected correlator at $\mathcal{O}(\mu^0)$ and the stress tensor exchange at $\mathcal{O}(\mu)$.

A.1.1. Solving the crossing equation to $\mathcal{O}(\mu)$ in $d = 4$

We start with the leading $\mathcal{O}(\mu^0)$ term in the S-channel that should reproduce the disconnected propagator in the T-channel. This is given in $d = 4$ by

$$G(z,\bar{z})|_{\mu^0} = \frac{C_{\Delta_L}}{z-\bar{z}} \int_0^\infty dh \int_0^h d\bar{h} (h\bar{h})^{\Delta_L - 2}(h-\bar{h})(z^{h+1}\bar{z}^{\bar{h}} - z^{\bar{h}}\bar{z}^{h+1}). \quad \text{(A.1.1)}$$

Let us look at the following piece of (A.1.1):

$$\begin{aligned} &-\int_0^\infty dh \int_0^h d\bar{h} (h\bar{h})^{\Delta_L-2}(h-\bar{h}) z^{\bar{h}} \bar{z}^{h+1} = \\ &-\int_0^\infty d\bar{h} \int_{\bar{h}}^\infty dh (h\bar{h})^{\Delta_L-2}(h-\bar{h}) z^{\bar{h}} \bar{z}^{h+1} = \\ &\frac{\bar{z}}{z}\int_0^\infty dh \int_h^\infty d\bar{h} (h\bar{h})^{\Delta_L-2}(h-\bar{h}) z^{h+1} \bar{z}^{\bar{h}} \,. \end{aligned} \quad \text{(A.1.2)}$$

Setting $\bar{z}/z = 1$ to leading order in the Regge limit, we find that the S-channel expression reproduces the disconnected correlator:

$$\begin{aligned} G(z,\bar{z})|_{\mu^0} &= \frac{zC_{\Delta_L}}{z-\bar{z}} \int_0^\infty dh \int_0^\infty d\bar{h} (h\bar{h})^{\Delta_L-2}(h-\bar{h}) z^h \bar{z}^{\bar{h}} \\ &= \frac{zC_{\Delta_L}}{z-\bar{z}} \frac{(\log\bar{z} - \log z)}{(\log z \log \bar{z})^{\Delta_L}} \Gamma(\Delta_L)\Gamma(\Delta_L - 1) \simeq \frac{1}{(1-z)^{\Delta_L}(1-\bar{z})^{\Delta_L}} \,. \end{aligned} \quad \text{(A.1.3)}$$

Notice that to arrive in the last equality we expanded $(z, \bar{z})$ around unity and substituted $C_{\Delta_L} = (\Gamma(\Delta_L)\Gamma(\Delta_L - 1))^{-1}$.

Consider now the imaginary part at $\mathcal{O}(\mu)$ in the S-channel. For convenience we define

$$I^{(d=4)} \equiv \text{Im}(G(z,\bar{z}))|_\mu \,, \quad \text{(A.1.4)}$$

which is then equal to:

$$\begin{aligned} I^{(d=4)} = &\frac{-i\pi C_{\Delta_L}}{\sigma(e^{-\rho} - e^{\rho})} \int_0^\infty dh \int_0^h d\bar{h} (h\bar{h})^{\Delta_L-2}(h-\bar{h}) \gamma^{(1)}_{\bar{h},h-\bar{h}} \\ &\times \Big((1-\sigma e^\rho)^{h+1}(1-\sigma e^{-\rho})^{\bar{h}} - (h \leftrightarrow \bar{h}) \Big) \,. \end{aligned} \quad \text{(A.1.5)}$$

Notice that we used the variables (σ, ρ) defined as $z = 1-\sigma e^{\rho}$ and $\bar{z} = 1-\sigma e^{-\rho}$.

Consider the following ansatz for $\gamma^{(1)}_{\bar{h},h-\bar{h}} = \frac{c h^a \bar{h}^b}{h-\bar{h}}$, where (a, b, c) are numbers to be determined by the crossing equation. Substituting into (A.1.5) and collecting the leading singularity σ^{-k} as $\sigma \to 0$ with $k = 2\Delta_L + a + b - 1$ leads to

$$\begin{aligned}
I^{(d=4)}|_{\sigma^{-k}} =& \frac{-ic\pi C_{\Delta_L}}{(e^{-\rho}-e^{\rho})}\Big(\Gamma(\Delta_L+a-1)\Gamma(\Delta_L+b-1)(e^{(b-a)\rho}-e^{(a-b)\rho})+ \\
&+\frac{\Gamma(2\Delta_L+a+b-2)}{(\Delta_L+a-1)e^{(2\Delta_L+a+b-2)\rho}}\times \\
&{}_2F_1(\Delta_L+a-1, 2\Delta_L+a+b-2, \Delta_L+a, -e^{-2\rho}) \\
&-\frac{\Gamma(2\Delta_L+a+b-2)}{(\Delta_L+a-1)e^{-(2\Delta_L+a+b-2)\rho}}\times \\
&{}_2F_1(\Delta_L+a-1, 2\Delta_L+a+b-2, \Delta_L+a, -e^{2\rho})\Big).
\end{aligned} \tag{A.1.6}$$

Note that in order to do these integrals we need $\Delta_L + a > 1$ and $\Delta_L + b > 1$. Using the following identity of the hypergeometric function

$$\begin{aligned}
{}_2F_1(a,b,c,x) =& \frac{\Gamma(b-a)\Gamma(c)}{\Gamma(b)\Gamma(c-a)}(-x)^{-a}{}_2F_1(a, a-c+1, a-b+1, \frac{1}{x}) \\
&+\frac{\Gamma(a-b)\Gamma(c)}{\Gamma(a)\Gamma(c-b)}(-x)^{-b}{}_2F_1(b, b-c+1, -a+b+1, \frac{1}{x}),
\end{aligned} \tag{A.1.7}$$

the third line in (A.1.6) can be simplified and we are left with

$$\begin{aligned}
I^{(d=4)}|_{\sigma^{-k}} =& \frac{ic\pi C_{\Delta_L}}{(e^{2\rho}-1)}\Big(-\Gamma(\Delta_L+a-1)\Gamma(\Delta_L+b-1)e^{(a-b+1)\rho} \\
&+\frac{\Gamma(2\Delta_L+a+b-2)}{\Delta_L+a-1}e^{-(2\Delta_L+a+b-3)\rho}\times \\
&{}_2F_1(\Delta_L+a-1, 2\Delta_L+a+b-2, \Delta_L+a, -e^{-2\rho}) \\
&+\frac{\Gamma(2\Delta_L+a+b-2)}{\Delta_L+b-1}e^{-(2\Delta_L+a+b-3)\rho}\times \\
&{}_2F_1(\Delta_L+b-1, 2\Delta_L+a+b-2, \Delta_L+b, -e^{-2\rho})\Big).
\end{aligned} \tag{A.1.8}$$

On the other hand, the Regge limit in the T-channel is dominated by operators of maximal spin. In a holographic CFT, we have $J = 2$. If we further take the lightcone limit, $\rho \gg 1$, the dominant contribution is due to the stress tensor exchange and behaves as $\sigma^{-1}e^{-(d-1)\rho}$. To reproduce this behavior from the S-channel, we must set $a = 0$ and $b = 2$ and make an appropriate choice for the overall constant c. Substituting the designated values of (a, b, c) revals that the first term in (A.1.8) precisely matches the T-channel stress tensor contribution, which in the Regge limit (after analytic continuation) behaves like:

$$g_{\Delta,J} \propto \frac{1}{\sigma^{J-1}} \frac{e^{-(\Delta-3)\rho}}{(e^{2\rho}-1)} + \dots, \tag{A.1.9}$$

with $\Delta = d$ and $J = 2$. Furthermore, the remaining terms correspond to the exchange of operators with spin 2 and dimension $2\Delta_L + 2 + 2n$; these are the double-trace operators $[\mathcal{O}_L \mathcal{O}_L]_{n,l=2}$.

A.1.2. Integrating the S-channel result at $\mathcal{O}(\mu^2)$ in $d = 4$

Below we describe how to use the results for the anomalous dimensions at $\mathcal{O}(\mu^2)$ in order to recover the imaginary part of the correlator to the same order. Using the obtained expressions for the anomalous dimensions (4.8) and (4.24), we note that the integrand in (4.11) can be written as

$$\begin{aligned} &P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}} \left(\gamma^{(2)}_{\bar{h},h-\bar{h}} - \frac{\gamma^{(1)}_{\bar{h},h-\bar{h}}}{2} (\partial_h + \partial_{\bar{h}}) \gamma^{(1)}_{\bar{h},h-\bar{h}} \right) \\ &= -\frac{35\bar{h}^3(2h-\bar{h})}{4(h-\bar{h})^3} P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}} = -\frac{35 h^{\Delta_L-3} \bar{h}^{\Delta_L+1}}{2\Gamma(\Delta_L-1)\Gamma(\Delta_L)} \sum_{n=0}^{\infty} \left(\frac{\bar{h}}{h}\right)^n (1+\frac{n}{2}). \end{aligned} \tag{A.1.10}$$

Therefore we see that (4.11) can be written as an infinite sum of integrals of the same form that appeared at $\mathcal{O}(\mu)$ in (A.1.5). It then follows that the full S-channel result can be integrated in order to obtain the correlator in position space. Especially, the lightcone result is obtained by setting $k = 0$ in (A.1.10) and taking $\rho \to \infty$ which gives

$$\mathrm{Im}(G(z,\bar{z}))|_{\mu^2} = \frac{i35\pi\Delta_L(\Delta_L+1)}{2(\Delta_L-2)} \frac{e^{-3\rho}}{\sigma^{2\Delta_L+1}(e^{2\rho}-1)} + \dots, \tag{A.1.11}$$

with ... denoting terms that are subleading in the lightcone limit. The result (A.1.11) has a form consistent with the contribution of an operator with spin-2 and $\Delta = 6$. The full result (beyond the lightcone limit) further contains an infinite number of operators with spin-2 of dimension $\Delta = 6 + 2n$ and $\Delta = 2\Delta_L + 2n + 2$.

A.1.3. Solving the crossing equation to $\mathcal{O}(\mu)$ in $d = 2$

Here we review the calculations needed for the $d = 2$ case explained in Appendix D. To $\mathcal{O}(\mu^0)$ the S-channel (3.37) is given by

$$G(z,\bar{z})|_{\mu^0} = \frac{1}{\Gamma(\Delta_L)^2} \int_0^\infty \int_0^h d\bar{h}(h\bar{h})^{\Delta_L - 1}(z^h \bar{z}^{\bar{h}} + (z \leftrightarrow \bar{z})). \tag{A.1.12}$$

The integrand in (A.1.12) is symmetric w.r.t. $h \leftrightarrow \bar{h}$ and can thus be rewritten as

$$G(z,\bar{z})|_{\mu^0} = \frac{1}{\Gamma(\Delta_L)^2} \int_0^\infty \int_0^\infty d\bar{h}(h\bar{h})^{\Delta_L - 1} z^h \bar{z}^{\bar{h}}, \tag{A.1.13}$$

which can easily be seen to reproduce the disconnected correlator $[(1-z)(1-\bar{z})]^{-\Delta_L}$ in the Regge limit.

As in the previous subsection we proceed to consider the imaginary part of the correlator in the S-channel expansion to $\mathcal{O}(\mu)$. Using a similar notation,

$$I^{(d=2)} \equiv \mathrm{Im}(G(z,\bar{z}))|_\mu \,, \tag{A.1.14}$$

combined with the ansatz $\gamma^{(1)}_{\bar{h},h-\bar{h}} = c\, h^a \bar{h}^b$, allows us to write:

$$I^{(d=2)} = -\frac{ic\,\pi}{\Gamma(\Delta_L)^2} \int_0^\infty \int_0^h d\bar{h}(h\bar{h})^{\Delta_L - 1} h^a \bar{h}^b (z^h \bar{z}^{\bar{h}} + (z \leftrightarrow \bar{z})). \tag{A.1.15}$$

The integrals in (A.1.15) can be easily performed given that $a + \Delta_L > 0$ and $b + \Delta_L > 0$. Changing variables to $z = 1 - \sigma e^\rho$, $\bar{z} = 1 - \sigma e^{-\rho}$ and collecting the most singular term σ^{-k}, with $k = 2\Delta_L + a + b$, leads to

$$\begin{aligned} I^{(d=2)}|_{\sigma^{-k}} = &\frac{ic\pi}{\Gamma(\Delta_L)^2}\Big(\Gamma(a+\Delta_L)\Gamma(b+\Delta_L)(-e^{\rho(b-a)} - e^{\rho(a-b)}) \\ &+ \frac{\Gamma(a+b+2\Delta_L)e^{-\rho(a+b+2\Delta_L)}}{a+\Delta_L}{}_2F_1(a+\Delta_L, a+b+2\Delta_L, 1+a+\Delta_L, -e^{-2\rho}) \\ &+ \frac{\Gamma(a+b+2\Delta_L)e^{\rho(a+b+2\Delta_L)}}{a+\Delta_L}{}_2F_1(a+\Delta_L, a+b+2\Delta_L, 1+a+\Delta_L, -e^{2\rho})\Big). \end{aligned} \tag{A.1.16}$$

Using again (A.1.7) we express (A.1.16) as follows

$$\begin{aligned} I^{(d=2)}|_{\sigma^{-k}} = &\frac{ic\pi}{\Gamma(\Delta_L)^2}\Big(- \Gamma(a+\Delta_L)\Gamma(b+\Delta_L)e^{\rho(a-b)} \\ &+ \frac{\Gamma(a+b+2\Delta_L)e^{-\rho(a+b+2\Delta_L)}}{a+\Delta_L}{}_2F_1(a+\Delta_L, a+b+2\Delta_L, 1+a+\Delta_L, -e^{-2\rho}) \\ &- \frac{\Gamma(a+b+2\Delta_L)e^{-(a+b+2\Delta_L)\rho}}{b+\Delta_L}{}_2F_1(b+\Delta_L, a+b+2\Delta_L, 1+b+\Delta_L, -e^{-2\rho})\Big). \end{aligned} \tag{A.1.17}$$

In matching (A.1.17) with the T-channel expansion, following the same logic as in the previous subsection we deduce that $a = 0$ and $b = 1$ and fix c. The first line in (A.1.17) then reproduces the exchange of the stress tensor in the T-channel. The other two lines match the contribution of double-trace operators $[\mathcal{O}_L\mathcal{O}_L]_{n,l=2}$ with dimension $\Delta = 2\Delta_L + 2n + 2$ and spin 2 in the T-channel expansion.

Appendix A.2. Details on the impact parameter representation in $d = 4$

Here we will see how the impact parameter representation in four dimensions leads to the expression for the disconnected correlator in the Regge limit, in terms of the integral over $h, \bar{h}$.

The objective of this section is to explicitly see that the disconnected contribution of the correlator in the Regge limit

$$\frac{1}{((1-z)(1-\bar{z}))^{\Delta}} = \frac{1}{\Gamma(\Delta)\Gamma(\Delta-1)} \int_0^\infty dh \int_0^h d\bar{h} (h\bar{h})^{\Delta-2} \frac{h-\bar{h}}{z-\bar{z}} \left(z^{h+1}\bar{z}^{\bar{h}} - z^{\bar{h}}\bar{z}^{h+1}\right), \tag{A.2.1}$$

can be equivalently written as

$$\int_0^\infty dh \int_0^h d\bar{h}\, \mathcal{I}_{h,\bar{h}}\,, \tag{A.2.2}$$

with

$$\mathcal{I}_{h,\bar{h}} \equiv C(\Delta) \int_{M^+} \frac{d^4p}{(2\pi)^4} (-p^2)^{\Delta-2} e^{-ipx} (h-\bar{h}) \delta(p\cdot\bar{e} + h + \bar{h})\, \delta\left(\frac{p^2}{4} + h\bar{h}\right). \tag{A.2.3}$$

where M^+ is the upper Milne wedge with $\{p^2 \leq 0,\ p^0 \geq 0\}$ and

$$C(\Delta) \equiv \frac{2^{d+1-2\Delta}\pi^{1+\frac{d}{2}}}{\Gamma(\Delta)\Gamma(\Delta - \frac{d}{2} + 1)}\,, \tag{A.2.4}$$

with d the dimensionality of the spacetime, here $d = 4$.

In practice, we need to perform the integral over p in (A.2.3). To do so, we will use spherical polar coordinates and write:

$$\mathcal{I}_{h,\bar{h}} = \frac{C(\Delta)}{(2\pi)^3} \int_{-\infty}^{\infty} dp^0 \int_0^{\infty} dp^r\,(p^r)^2 \int_{-1}^{1} d(\cos\theta)\;(-p^2)^{\Delta-2}\,\theta(p^0)\theta(-p^2)\times$$
$$e^{ip^0x^0}\,e^{-irp^r\cos\theta}\left(\delta\left(\frac{p^0+p^r}{2}-h\right)\delta\left(\frac{p^0-p^r}{2}-\bar{h}\right)+h\leftrightarrow\bar{h}\right). \tag{A.2.5}$$

The overall factor of (2π) is simply the result of the integration with respect to the angular variable ϕ. Next we perform the integral over $\cos\theta$:

$$\mathcal{I}_{h,\bar{h}} = \frac{C(\Delta)}{(2\pi)^3} \int_{-\infty}^{\infty} dp^0 \int_0^{\infty} dp^r\,(p^r)^2\,(-p^2)^{\Delta-2}\,e^{ip^0x^0}\left(\frac{e^{-irp^r}-e^{irp^r}}{-irp^r}\right)\times$$
$$\theta(p^0)\theta(-p^2)\,(\delta\,\delta), \tag{A.2.6}$$

where we set

$$(\delta\,\delta)\equiv\delta\left(\frac{p^0+p^r}{2}-h\right)\delta\left(\frac{p^0-p^r}{2}-\bar{h}\right)+(h\leftrightarrow\bar{h}). \tag{A.2.7}$$

Notice that

$$\int_0^{\infty} dp^r\,\frac{p^r}{ir}\,(-p^2)^{\Delta-2}\,e^{irp^r}(\delta\,\delta)-\int_0^{\infty} dp^r\,\frac{p^r}{ir}\,(-p^2)^{\Delta-2}\,e^{-irp^r}(\delta\,\delta)=$$
$$=\int_{-\infty}^{\infty} dp^r\,\frac{p^r}{ir}\,(-p^2)^{\Delta-2}\,e^{irp^r}(\delta\,\delta). \tag{A.2.8}$$

Hence we can write (A.2.6) as follows

$$\mathcal{I}_{h,\bar{h}} = \frac{C(\Delta)}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{dp^+\,dp^-}{2}\,\frac{p^+-p^-}{i(x^+-x^-)}\,(-p^2)^{\Delta-2}\,e^{\frac{i}{2}(p^+x^-+p^-x^+)}\times$$
$$\theta(p^+)\theta(p^-)\,(\delta\,\delta). \tag{A.2.9}$$

Performing the last two integrals is trivial due to the delta-functions. The result is

$$\mathcal{I}_{h,\bar{h}} = \frac{1}{\Gamma(\Delta)\Gamma(\Delta-1)}\,\frac{h-\bar{h}}{i(x^+-x^-)}\,(h\bar{h})^{\Delta-2}\,(e^{ihx^+}e^{i\bar{h}x^-}-e^{i\bar{h}x^+}e^{ihx^-}), \tag{A.2.10}$$

which allows us to write (A.2.2) as follows:

$$\int_0^{\infty} dh\int_0^{h} d\bar{h}\,\mathcal{I}_{h,\bar{h}} = \frac{1}{\Gamma(\Delta)\Gamma(\Delta-1)}\int_0^{\infty} dh\int_0^{h} d\bar{h}\frac{h-\bar{h}}{i(x^+-x^-)}\,(h\bar{h})^{\Delta-2}\times$$
$$(z^h\bar{z}^{\bar{h}}-z^{\bar{h}}\bar{z}^h). \tag{A.2.11}$$

Here we also used the identification $(z = e^{ix^+}, \bar{z} = e^{ix^-})$.

Observe that (A.2.11) is equal to (A.2.1) in the Regge limit, where

$$\frac{z}{z-\bar{z}} \simeq \frac{1}{i(x^+ - x^-)}, \quad \frac{\bar{z}}{z-\bar{z}} \simeq \frac{1}{i(x^+ - x^-)}. \tag{A.2.12}$$

However, when considering next order corrections in (x^+, x^-) the impact parameter represention may require corrections. Below we show that these are irrelevant for the questions we are interested in.

A.2.1. Exact Fourier transform

Here we will compute the Fourier transform for the S-channel expression with the identification $(z = e^{ix^+}, \bar{z} = e^{ix^-})$ and show that the leading order results in the Regge limit given in the previous section do not miss any important contributions.

The generic term in the S-channel which we would like to Fourier transform looks like:

$$\int dh\, d\bar{h}\, g(x^+, x^-) \tilde{f}(h, \bar{h}), \tag{A.2.13}$$

where

$$g(x^+, x^-) = \frac{e^{i(1+h)x^+} e^{i\bar{h}x^-} - e^{i\bar{h}x^+} e^{i(h+1)x^-}}{(e^{ix^+} - e^{ix^-})}, \tag{A.2.14}$$

and

$$\tilde{f}(h, \bar{h}) = i\pi (h\bar{h})^{\Delta-2}(h - \bar{h}) f(h, \bar{h}), \tag{A.2.15}$$

where $f(h, \bar{h})$ stands for all the contributions in the S-channel to a given order.

The Fourier transform is:

$$\int d^4x\, e^{ipx} \int dh\, d\bar{h}\, g(x^+, x^-) \tilde{f}(h, \bar{h}) = \int dh\, d\bar{h} \tilde{f}(h, \bar{h}) \int d^4x\, e^{ipx} g(x^+, x^-), \tag{A.2.16}$$

where we simply reversed the order of integration. Our focus in what follows will be the integral:

$$I \equiv \int d^4x\, e^{ipx} g(x^+, x^-). \tag{A.2.17}$$

Since $x^+ = t + r$ and $x^- = t - r$, it is convenient to use spherical polar coordinates to perform the integration. The angular integration over ϕ gives us an overall factor of (2π) as the integrand is independent of ϕ. Next we perform the integration over the other angular variable. Similar to what was discussed in the previous section,

$$\int_{-1}^{1} d(\cos\theta)\, e^{ip^r r\cos\theta} = \frac{e^{irp^r} - e^{-irp^r}}{irp^r}\,. \tag{A.2.18}$$

Combining the above we can write:

$$I = 2\pi \int_{-\infty}^{\infty} dte^{-itp^t} \int_{0}^{\infty} drr\, \frac{e^{irp^r} - e^{-irp^r}}{ip^r}\, g(t,r)\,. \tag{A.2.19}$$

It is easy to see that $g(t,r) = g(t,-r)$ and as a result:

$$\int_{0}^{\infty} dr\, re^{-irp^r}\, g(t,r) = -\int_{-\infty}^{0} dr\, re^{irp^r}\, g(t,r)\,, \tag{A.2.20}$$

which allows us to write the integral as:

$$I = 2\pi \int_{-\infty}^{\infty} \frac{dx^+dx^-}{2} e^{ip\cdot x} \frac{x^+ - x^-}{i(p^+ - p^-)} g(x^+,x^-)\,. \tag{A.2.21}$$

Here $e^{ip\cdot x} = e^{-\frac{i}{2}(p^+x^- + p^-x^+)}$ and the above integral can be thought of as a two-dimensional Fourier transform.

To proceed we need the explicit form of $g(x^+,x^-)$ which we write as

$$g(x^+,x^-) = \frac{e^{ihx^+}e^{i\bar{h}x^-}}{1 - e^{-i(x^+ - x^-)}} + (x^+ \leftrightarrow x^-) \tag{A.2.22}$$

and then expand the denominator in the Regge limit

$$\frac{1}{1 - e^{-i(x^+ - x^-)}} = \frac{1}{i(x^+ - x^-)} \left(1 - \frac{i}{2}(x^+ - x^-) + \cdots\right)\,. \tag{A.2.23}$$

Substituting into (A.2.21) leads to:

$$I = 2\pi \frac{1}{(-p^+ + p^-)} \int \frac{dx^+dx^-}{2} e^{ip\cdot x} \left(e^{ihx^+}e^{i\bar{h}x^-} \left(1 - \frac{i}{2}(x^+ - x^-) + \cdots\right) + (x^+ \leftrightarrow x^-) \right)\,. \tag{A.2.24}$$

Let us compute the integral term by term. The leading term in the Regge limit yields the standard delta functions:

$$
\begin{aligned}
I_0 &= 2^2\pi^3 \, \frac{1}{p^- - p^+}\delta(\frac{p^+}{2} - \bar{h})\delta(\frac{p^-}{2} - h) + (p^+ \leftrightarrow p^-) = \\
&= 2\pi^3 \, \frac{1}{h - \bar{h}} \left\{ \delta(\frac{p^+}{2} - \bar{h})\delta(\frac{p^-}{2} - h) + (p^+ \leftrightarrow p^-) \right\} = \\
&= 2\pi^3 \, \frac{1}{h - \bar{h}}\delta(p \cdot \bar{e} + h + \bar{h})\delta(\frac{p^2}{4} + h\bar{h}) \, .
\end{aligned}
\tag{A.2.25}
$$

The subleading terms on the other hand produce the same result except that the delta functions are replaced with derivatives of themselves with respect to $p^r = \frac{p^+ - p^-}{2}$.

Let us now consider the full result which up to an overall numerical coefficient can be written as:

$$
\int dh \, d\bar{h} \, \tilde{f}(h, \bar{h}) \left(1 - \frac{\partial}{\partial p^r} + \cdots \right) \delta(p \cdot \bar{e} + h + \bar{h})\delta(\frac{p^2}{4} + h\bar{h}) \, . \tag{A.2.26}
$$

To evaluate the terms with derivatives of the delta function we need to integrate by parts. Now recall that we are interested in the imaginary piece of the S-channel whose leading behaviour is $\sim \sqrt{-p^2}$ (this dependence is hidden in what we called $\tilde{f}$). It is obvious that the derivatives will produce subleading terms which we are not interested in.

What about the other pieces in the S-channel which are not imaginary? To $\mathcal{O}(\mu^2)$ in this case, we know that the leading behaviour grows like $\sim (\sqrt{-p^2})^2$, so by differentiation, a term of the order $\sqrt{-p^2}$ may be produced. However, it is clear that this term will never contribute to the *imaginary* term of the S-channel (note that the coefficient in the first term in the parenthesis in (A.2.26) is real). We thus deduce that the subleading terms in (A.2.24) are irrelevant for our study.

Appendix A.3. Impact parameter representation in general spacetime dimension d

Here we want to prove the following equation for general spacetime dimension d:

$$\mathcal{I}_{h,\bar{h}} = (z\bar{z})^{-\frac{(\Delta_H+\Delta_L)}{2}} P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}} g^{\Delta_{HL},-\Delta_{HL}}_{h,\bar{h}}(z,\bar{z}), \tag{A.3.1}$$

using the form of conformal blocks given in (4.31). We start with the definition of $\mathcal{I}_{h,\bar{h}}$ that is given as:

$$\mathcal{I}_{h,\bar{h}} = C(\Delta_L) \int_{M^+} \frac{d^d p}{(2\pi)^d} (-p^2)^{\Delta_L - \frac{d}{2}} e^{-ipx} (h-\bar{h}) \delta(p\cdot\bar{e} + h + \bar{h}) \delta(\frac{p^2}{4} + h\bar{h}), \tag{A.3.2}$$

where:

$$C(\Delta_L) \equiv \frac{2^{d+1-2\Delta_L} \pi^{1+\frac{d}{2}}}{\Gamma(\Delta_L)\Gamma(\Delta_L - \frac{d}{2} + 1)}\,. \tag{A.3.3}$$

Using spherical coordinates we write (A.3.2) as:

$$\begin{aligned}\mathcal{I}_{h,\bar{h}} &= C(\Delta_L) \int_{-\infty}^{\infty} dp^t e^{ip^t t} \int_0^\infty dp^r (p^r)^{d-2} \int_{S_{d-2}} \sin^{d-3}\phi_1 d\phi_1\, d\Omega_{d-3} \\ &\times e^{-ip^r r\cos\phi_1} (-p^2)^{\Delta_L - \frac{d}{2}} \theta(-p^2)\theta(p^t) \Big\{ \delta\left(\frac{p^t + p^r}{2} - h\right) \delta\left(\frac{p^t - p^r}{2} - \bar{h}\right) \\ &+ (h \leftrightarrow \bar{h}) \Big\}\,,\end{aligned} \tag{A.3.4}$$

where $\Omega_{d-3} = \frac{2\pi^{\frac{d-2}{2}}}{\Gamma(\frac{d-2}{2})}$ denotes the area of the unit $(d-3)$-dimensional hypersphere.

Notice now that

$$\int_0^\pi \sin^{d-3}\phi_1 e^{-ip^r r\cos\phi_1} d\phi_1 = \sqrt{\pi}\Gamma(\frac{d}{2}-1) {}_0F_1(\frac{d-1}{2}; -\frac{1}{4}(p^r)^2 r^2)\,. \tag{A.3.5}$$

Substituting (A.3.5) back in to (A.3.4), one is left with integrals with respect to p^t and p^r only. These integrals are trivial due to the presence of delta functions.[47] When these integrals are calculated, the expression for $\mathcal{I}_{h,\bar{h}}$ is given as:

$$\begin{aligned}\mathcal{I}_{h,\bar{h}} = \frac{2^{3-d}\sqrt{\pi}}{\Gamma(\Delta_L)\Gamma(\Delta_L - \frac{d}{2}+1)} e^{it(h+\bar{h})} (h-\bar{h})^{d-2} (h\bar{h})^{\Delta_L - \frac{d}{2}} \times \\ {}_0F_{1R}(\frac{d-1}{2}; -\frac{1}{4}(h-\bar{h})^2 r^2),\end{aligned} \tag{A.3.6}$$

[47] One only needs to remember that $h \geq \bar{h} \geq 0$.

where ${}_0F_{1R}(a,x) = \Gamma(a)^{-1}{}_0F_1(a,x)$. Relations between coordinates t and r with x^+ and x^- are given as: $x^+ = t + r$ and $x^- = t - r$.

On the other hand, using the explicit form for conformal blocks (4.31) and OPE coefficients in the Regge limit (3.40) one finds that:

$$\begin{aligned}&(z\bar{z})^{-\frac{(\Delta_H+\Delta_L)}{2}} P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}} g^{\Delta_{HL},-\Delta_{HL}}_{h,\bar{h}}(z,\bar{z}) = \\ &= \frac{\Gamma(\frac{d}{2}-1)}{\Gamma(\Delta_L)\Gamma(\Delta_L-\frac{d}{2}+1)}(h\bar{h})^{\Delta_L+\frac{d}{2}}(h-\bar{h})(z\bar{z})^{\frac{h+\bar{h}}{2}} C^{(\frac{d}{2}-1)}_{h-\bar{h}}\Big(\frac{z+\bar{z}}{2\sqrt{z\bar{z}}}\Big).\end{aligned} \tag{A.3.7}$$

Using the relations between coordinates r, t and $z, \bar{z}$ it is easy to see that $(z\bar{z})^{\frac{h+\bar{h}}{2}} = e^{it(h+\bar{h})}$. Next, one can use the relation between Gegenbauer polynomials and hypergeometric functions:

$$C^{(\alpha)}_n(z) = \frac{(2\alpha)_n}{n!}{}_2F_1(-n, 2\alpha+n, \alpha+\frac{1}{2}; \frac{1-z}{2}), \tag{A.3.8}$$

which for $h - \bar{h} = l \gg 1$ gives:

$$C^{(\frac{d}{2}-1)}_l\Big(\frac{z+\bar{z}}{2\sqrt{z\bar{z}}}\Big) = \frac{l^{d-3}}{\Gamma(d-2)}{}_2F_1(-l, l+d-2, \frac{d-1}{2}; \frac{1}{2} - \frac{1}{2}(\frac{z+\bar{z}}{2\sqrt{z\bar{z}}})). \tag{A.3.9}$$

With the help of the following properties of hypergeometric functions:

$$\begin{aligned}{}_2F_1(a,b,c;z) &= (1-z)^{-b}{}_2F_1(c-a,b,c;\frac{z}{z-1}), \\ \lim_{m,n\to\infty}{}_2F_1(m,n,b;\frac{z}{mn}) &= {}_0F_1(b;z).\end{aligned} \tag{A.3.10}$$

Using these, together with the assumption that in the Regge limit the values of x^+l and x^-l are fixed constants: $x^+l = a_1$ and $x^-l = a_2$ while $l \to \infty$, one can easily see[48] that (A.3.6) reproduces (A.3.1). This confirms the validity of the impact parameter representation.

---

[48] By noting that:

$$\Gamma(x-\frac{1}{2}) = 2^{2-2x}\sqrt{\pi}\frac{\Gamma(2x-1)}{\Gamma(x)}. \tag{A.3.11}$$

Appendix A.4. Anomalous dimensions of heavy-light double-trace operators in $d = 2$

The OPE data of the heavy-light double trace operators in $d = 2$ dimensions can be directly obtained from the heavy-light Virasoro vacuum block [40,105]. For completeness, in this appendix we investigate the anomalous dimension of $[\mathcal{O}_H\mathcal{O}_L]_{\bar{h},h-\bar{h}}$ in $d = 2$ following the discussion in Section 4.2. As in $d = 4$, we introduce an impact parameter representation following [55]. We calculate the anomalous dimensions to $\mathcal{O}(\mu)$ by solving the crossing equation and then use the impact parameter representation to relate them to the bulk phase shift. We find a precise agreement between the two. Using the bulk phase shift we furthermore determine the anomalous dimension to second order in μ. Much of the discussion follows closely the four-dimensional case and will be briefer.

A.4.1. Anomalous dimensions in the Regge limit using bootstrap

The conformal blocks in two dimension are given by [30,24]

$$g_{\Delta,J}^{\Delta_{12},\Delta_{34}}(z,\bar{z}) = k_{\Delta+J}(z)k_{\Delta-J}(\bar{z}) + (z \leftrightarrow \bar{z})\,, \tag{A.4.1}$$

where $k_\beta(z)$ was defined in (4.3). Similar to the four dimensional case, the blocks for heavy-light double-trace operators simplify in the heavy limit ($\Delta_H \sim C_T$)

$$g_{[\mathcal{O}_H\mathcal{O}_L]_{h,\bar{h}}}^{\Delta_{HL},-\Delta_{HL}}(z,\bar{z}) = (z\bar{z})^{\frac{1}{2}(\Delta_H+\Delta_L)}(z^h\bar{z}^{\bar{h}} + (z \leftrightarrow \bar{z}))\,. \tag{A.4.2}$$

Inserting this form of the conformal blocks in (3.37) together with the OPE coefficients in the Regge limit (3.40) and approximating the sums with integrals, one can due to symmetry extend the region of integration and it is easily found that the disconnected correlator in the T-channel is reproduced.

Similar to the four-dimensional case the stress tensor dominates at order μ in the T-channel. The block of the stress tensor after analytic continuation in the Regge limit is given by

$$g_{T_{\mu\nu}} = \frac{24i\pi e^{-\rho}}{\sigma} + \dots\,, \tag{A.4.3}$$

where $\dots$ denote non-singular terms. As in the four-dimensional case, this has to be reproduced in the S-channel by the term in (3.37) proportional to $-i\pi\gamma^{(1)}$.

With the conformal blocks (A.4.2), the imaginary part in the S-channel to $\mathcal{O}(\mu)$ is given by

$$\mathrm{Im}(G(z,\bar{z}))|_{\mu} = -i\pi C_{\Delta_L} \int_0^\infty dh \int_0^h d\bar{h} (h\bar{h})^{\Delta_L - 1} \gamma^{(1)}_{\bar{h},h-\bar{h}} \left(z^h \bar{z}^{\bar{h}} + z^{\bar{h}} \bar{z}^h \right). \tag{A.4.4}$$

Using the ansatz $\gamma^{(1)}_{\bar{h},h-\bar{h}} = c_1 h^a \bar{h}^b$ we find that the T-channel contribution is reproduced for $a = 0$ and $b = 1$ (see Appendix A.2 for details). We thus find using (3.41)

$$\gamma^{(1)}_{\bar{h},h-\bar{h}} = -\frac{6\lambda_{\mathcal{O}_H\mathcal{O}_H T_{\mu\nu}} \lambda_{\mathcal{O}_L\mathcal{O}_L T_{\mu\nu}}}{\mu \Delta_L} \bar{h} = -\bar{h}. \tag{A.4.5}$$

To $\mathcal{O}(\mu^2)$ we can use (4.11) to find the following contribution to the purely imaginary terms in the S-channel

$$\begin{aligned} \mathrm{Im}(G(z,\bar{z}))|_{\mu^2} = -i\pi C_{\Delta_L} \int_0^\infty dh \int_0^h d\bar{h} (h\bar{h})^{\Delta_L - 1} \left(\gamma^{(2)}_{\bar{h},h-\bar{h}} - \frac{c_1^2 \bar{h}}{2} \right) \times \\ (z^h \bar{z}^{\bar{h}} + z^{\bar{h}} \bar{z}^h). \end{aligned} \tag{A.4.6}$$

A.4.2. 2d impact parameter representation and relation to bulk phase shift

Similar to the four-dimensional case we introduce an impact parameter representation in order to relate the anomalous dimension with the bulk phase shift. The impact parameter representation in $d = 2$ is given by

$$\mathcal{I}_{h,\bar{h}} \equiv C(\Delta_L) \int_{M^+} d^2p (-p^2)^{\Delta - 1} e^{-ipx} (h - \bar{h}) \delta(p \cdot \bar{e} + h + \bar{h})\, \delta\left(\frac{p^2}{4} + h\bar{h} \right), \tag{A.4.7}$$

with straightforward generalization of the $d = 4$ case explained above. This is again chosen such that when the impact parameter represetation is integrated over $h, \bar{h}$ the disconnected correlator is reproduced:

$$\int_0^\infty dh \int_0^h \mathcal{I}_{h,\bar{h}} = \frac{1}{[(1-z)(1-\bar{z})]^{\Delta_L}}. \tag{A.4.8}$$

The discussion of the phase shift is completely analogous to the four-dimensional case, as in (4.21) we find the following relation between the bulk phase shift and the anomalous dimension to second order in μ

$$\begin{aligned} \gamma^{(1)}_{\bar{h},h-\bar{h}} &= -\frac{\delta^{(1)}}{\pi} \\ \tilde{\gamma}^{(2)}_{\bar{h},h-\bar{h}} - \frac{c_1^2 p^-}{4} &= -\frac{\delta^{(2)}}{\pi}. \end{aligned} \tag{A.4.9}$$

In [55] the phase shift in $d = 2$ was found to be

$$\delta^{(1)} = \frac{\pi}{2}\sqrt{-p^2}e^{-L}$$
$$\delta^{(2)} = \frac{3\pi}{8}\sqrt{-p^2}e^{-L}. \tag{A.4.10}$$

Using the identification $p^+ = 2h$ and $p^- = 2\bar{h}$ together with (4.23) we find for the anomalous dimension in the Regge limit

$$\gamma^{(1)}_{\bar{h},h-\bar{h}} = -\bar{h}$$
$$\gamma^{(2)}_{\bar{h},h-\bar{h}} = -\frac{1}{4}\bar{h}. \tag{A.4.11}$$

We thus see that the first order result agrees with that obtained from bootstrap (A.4.5). Furthermore, the second order correction agrees also in $d = 2$ with the result (6.40) in [55].

Appendix A.5. Discussion of the boundary term integrals

There are a few integrals containing total derivative terms that we have ignored throughout this section and we analyze more carefully here. Let us start with a total derivative term which shows up in the real part of the correlator at $\mathcal{O}(\mu)$. It is given by[49]:

$$I_1 = \frac{1}{2}(z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)} \int_0^{+\infty} dl \Big[P^{(HL,HL);\mathrm{MFT}}_{n,l} \gamma^{(1)}_{n,l} g^{\Delta_{HL},-\Delta_{HL}}_{n+l,n}(z,\bar{z})\Big]^{n\to\infty}_{n=0}. \tag{A.5.1}$$

Let us focus on the integrand: $\Big[P^{(HL,HL);\mathrm{MFT}}_{n,l} \gamma^{(1)}_{n,l} g^{\Delta_{HL},-\Delta_{HL}}_{n+l,n}(z,\bar{z})\Big]^{n\to\infty}_{n=0}$. When $n = 0$, the expression within the brackets trivially vanishes. On the other hand, when $n \to \infty$, it takes the form $n^{2\Delta_L-2}(z\bar{z})^n \times f(l)$, where f is some function of l only. We are instructed here to take the limit $n \to \infty$ independently of all other limits (recall that the Regge limit is taken after the integration). For generic values $0 < (z, \bar{z}) < 1$ it is clear that $\lim_{n\to\infty} \Big[P^{(HL,HL);\mathrm{MFT}}_{n,l} \gamma^{(1)}_{n,l} g^{\Delta_{HL},-\Delta_{HL}}_{n+l,n}(z,\bar{z})\Big] = \lim_{n\to\infty} n^{2\Delta_L-2}(z\bar{z})^n \times f(l) \to$

[49] We are again using variables n and l, one can notice that $n = \bar{h}$ and $l = h - \bar{h}$. It is trivial to prove that $\partial_n = \partial_h + \partial_{\bar{h}}$.

0. In other words, the expression $\left[P_{n,l}^{(HL,HL);\mathrm{MFT}}\gamma_{n,l}^{(1)}g_{n+l,n}^{\Delta_{HL},-\Delta_{HL}}(z,\bar{z})\right]_{n=0}^{n\to\infty}\to$ 0, and we conclude that the integral (A.5.1) does not contribute to the S-channel expansion of the correlator.

There are a few more integrals of similar kind that appear at $\mathcal{O}(\mu^2)$. We will analyse one of them here:

$$I_2=\frac{-i\pi}{2}(z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)}\int_0^{+\infty}dl\left[P_{n,l}^{(HL,HL);\mathrm{MFT}}(\gamma_{n,l}^{(1)})^2 g_{n+l,n}^{\Delta_{HL},-\Delta_{HL}}(z,\bar{z})\right]_{n=0}^{n\to\infty}. \tag{A.5.2}$$

The same logic can be applied here. Again, the value of the expression in brackets at $n=0$ is trivially zero, while for large n it behaves like: $n^{2\Delta_L+d-4}(z\bar{z})^n\tilde{f}(l)$. As long as $(z,\bar{z})<1$, this vanishes exponentially in the limit $n\to\infty$. One concludes therefore that the integral (A.5.2) vanishes. The same logic is valid for all other integrals of similar total derivative terms that appear at $\mathcal{O}(\mu^2)$.

Appendix A.6. An identity for the bulk phase shift.

The aim is to elaborate on the results of [55] for the bulk phase shift in a black hole background as computed in gravity. Firstly, let us note the following identity involving hypergeometric functions:

$$\sum_{n=0}^{\infty}a(n)x^n\,{}_2F_1[\tau_0+2n+1,\frac{d}{2}-1,\tau_0+2n-\frac{d}{2}+3,x]={}_2F_1[\tau_0+1,\frac{\tau_0}{2},\frac{\tau_0}{2}+2,x]$$
$$a(n)=\frac{2^{2n}}{n!}\frac{\tau_0+2}{\tau_0+2+2n}\frac{(\frac{\tau_0}{2}+1-\frac{d}{2})_n\left(\frac{\tau_0+1}{2}\right)_n}{(\tau_0+n+2-\frac{d}{2})_n},\quad \tau_0\neq 0\,. \tag{A.6.1}$$

Given that both sides of the equality can be expressed as an infinite series expansion around $x=0$, one simply needs to show that the expansion coefficients match to all orders in x. This is proven in Appendix G.

Consider now the case $\tau_0=k(d-2)$ where $k\in N^\star$. Setting $x\equiv e^{-2L}$ and multiplying both sides with $e^{-[k(d-2)+1]L}$ yields:

$$\Pi_{k(d-2)+1,k(d-2)+1}(L)=\sum_{n=0}^{\infty}\beta_n\Pi_{k(d-2)+2n+1,d-1}(L)$$
$$\beta(n)\equiv\pi^{\frac{(1-k)(d-2)}{2}}\frac{a(n)}{(k(d-2)+1)_n}\frac{\Gamma\left[k(d-2)-\frac{d}{2}+2n+3\right]}{\Gamma\left[\frac{k(d-2)}{2}+2\right]}\,. \tag{A.6.2}$$

The left hand side represents the hyperbolic space propagator for a scalar field of squared mass equal to $k(d-2)+1$ in a hyperbolic space of dimensionality $k(d-2)+1$ and is proportional to the k-th order expression for the bulk phase shift computed in gravity in [55], where

$$\delta^{(k)}(S,L) = \frac{1}{k!}\frac{2\Gamma\left(\frac{dk+1}{2}\right)}{\Gamma\left(\frac{k(d-2)+1}{2}\right)}\frac{\pi^{1+\frac{k(d-2)}{2}}}{\Gamma\left(\frac{k(d-2)}{2}+1\right)} S\,\Pi_{k(d-2)+1,k(d-2)+1}(L)\,. \quad \text{(A.6.3)}$$

On the other hand, the right-hand side of (A.6.2) expresses the k-th order term of the bulk phase shift as an infinite sum of $(d-1)$-dimensional hyperbolic space propagators for fields with mass-squared equal to $m^2 = k(d-2)+1+2n$.

It can be shown [132,74] that the analytically continued T-channel scalar conformal block in the Regge limit behaves like:

$$g_{\Delta,J}(\sigma,\rho) = i\,c_{\Delta,J}\frac{\Pi_{\Delta-1,d-1}(\rho)}{\sigma^{J-1}}\,, \quad \text{(A.6.4)}$$

where

$$c_{\Delta,J} = \frac{4^{\Delta+J-1}\Gamma\left(\frac{\Delta+J-1}{2}\right)\Gamma\left(\frac{\Delta+J+1}{2}\right)}{\Gamma(\frac{\Delta+J}{2})^2}\frac{2\Gamma\left(\Delta-\frac{d}{2}+1\right)}{\pi^{1-\frac{d}{2}}\Gamma\left(\Delta-1\right)}\,. \quad \text{(A.6.5)}$$

Here $\Pi_{\Delta-1,d-1}$ denotes as usual the $(d-1)$-dimensional hyperbolic space propagator for a massive scalar of mass-squared $m^2 = (\Delta-1)$.

It follows that the k-th order term in the μ-expansion of the bulk phase shift in a black hole background can be expressed as an infinite sum of conformal blocks corresponding to operators of twist $\tau = \tau_0(k)+2n = k(d-2)+2n$ and spin $J=2$ in the Regge limit. In other words, we can write:

$$\begin{aligned} i\,\delta^{(k)}(S,L) =&\ f(k)\sum_{n=0}^{\infty}\lambda_k(n)\ g^R_{\tau_0(k)+2n+2,2}(S,L) \\ \lambda_k(n) =&\ a(n)\,\frac{2^{-4n}\left[\left(\frac{\tau_0(k)+4}{2}\right)_n\right]^2}{\left(\frac{\tau_0(k)+3}{2}\right)_n\left(\frac{\tau_0(k)+5}{2}\right)_n}, \quad \tau_0(k) = k(d-2) \end{aligned} \quad \text{(A.6.6)}$$

where

$$f(k) \equiv \frac{\sqrt{\pi}}{64}\ \frac{1}{2^{k(d-2)}\ k!}\ \frac{\Gamma\left(\frac{kd+1}{2}\right)\Gamma\left(\frac{k(d-2)+4}{2}\right)}{\Gamma\left(\frac{k(d-2)+5}{2}\right)\Gamma\left(\frac{k(d-2)+3}{2}\right)}\,, \quad \text{(A.6.7)}$$

and

$$g^R_{\Delta,J}(S,L) = ic_{\Delta,J}\,S^{J-1}\,\Pi_{\Delta-1,d-1}(L)\,. \quad \text{(A.6.8)}$$

Appendix A.7. An identity for hypergeometric functions.

Here we will show that for $q \neq 0$,

$$\sum_{n=0}^{\infty} a(n)x^n \, {}_2F_1[q+2n+1, \frac{d}{2}-1, q+2n-\frac{d}{2}+3, x] = {}_2F_1[q+1, \frac{q}{2}, \frac{q}{2}+2, x]$$
$$a(n) = \frac{2^{2n}}{n!} \frac{q+2}{q+2+2n} \frac{(\frac{q}{2}+1-\frac{d}{2})_n \left(\frac{q+1}{2}\right)_n}{(q+n+2-\frac{d}{2})_n}, \quad q \neq 0 \,. \tag{A.7.1}$$

Given that both sides of the equality can be expressed as an infinite series expansion around $x = 0$, one simply needs to show that the expansion coefficients match to all orders in x. Let us first set:

$$\begin{aligned} b(n,m) &\equiv \frac{1}{m!} \frac{(q+1+2n)_m \left(\frac{d}{2}-1\right)_m}{\left(q-\frac{d}{2}+2n+3\right)_m} \\ c(\ell) &\equiv \frac{1}{\ell!} \frac{(q+1)_\ell \left(\frac{q}{2}\right)_\ell}{\left(\frac{q}{2}+2\right)_\ell} = \frac{(q+1)_\ell}{\ell!} \frac{q(q+2)}{(q+2\ell)(q+2\ell+2)} \,, \end{aligned} \tag{A.7.2}$$

such that:

$$\begin{aligned} {}_2F_1[q+2n+1, \frac{d}{2}-1, q+2n-\frac{d}{2}+3, x] &= \sum_{m=0}^{\infty} b(n,m)x^m, \\ {}_2F_1[q+1, \frac{q}{2}, \frac{q}{2}+2, x] &= \sum_{\ell=0}^{\infty} c(\ell)x^\ell. \end{aligned} \tag{A.7.3}$$

It is easy to check that the coefficients of the first few powers of x precisely match. Indeed, e.g.,

$$\begin{aligned} a(0)b(0,0) - c(0) &= 0 \\ a(1)b(1,0) + a(0)b(0,1) - c(1) &= 0 \\ a(2)b(2,0) + a(1)b(1,1) + a(0)b(0,2) - c(2) &= 0. \end{aligned} \tag{A.7.4}$$

To show that the above identity is true for all powers of x we must show that:

$$\sum_{k=0}^{\ell} a(k)b(k,\ell-k) = c(\ell) \,, \tag{A.7.5}$$

for all $\ell \in N$. The left-hand side of (A.7.5) can be easily summed to yield:

$$\sum_{k=0}^{\ell} a(k)b(k,\ell-k) = \frac{1}{\ell!} \frac{\Gamma[q+1+\ell]}{\Gamma[q]} \frac{(q+2)}{(q+2\ell)(2+2\ell+q)} \,, \tag{A.7.6}$$

which can be trivially shown to be equal to $c(\ell)$.

Appendix B.1. Linear relations between products of $f_a(z)$ functions

Here we list some linear relations between products of the $f_a(z)$ functions used in the main text.

$$f_1(z)f_4(z) + \frac{1}{15}f_3(z)f_4(z) - \frac{4}{63}f_2(z)f_5(z) - f_2(z)f_3(z) = 0, \tag{B.1.1}$$

$$\begin{aligned}
&\frac{308}{25}f_2^2(z) - \frac{308}{25}f_1(z)f_3(z) + \frac{5929}{375}f_3^2(z) - \frac{2673}{2500}f_4^2(z) - \frac{396}{25}f_1(z)f_5(z)\\
&+ f_2(z)f_6(z) = 0,\\
&245f_2^2(z) - 245f_1(z)f_3(z) - \frac{7}{12}f_3^2(z) - \frac{81}{80}f_4^2(z) + f_3(z)f_5(z) = 0,\\
&\frac{140}{9}f_2^2(z) - \frac{140}{9}f_1(z)f_3(z) - \frac{28}{27}f_3^2(z) + f_2(z)f_4(z) = 0,
\end{aligned} \tag{B.1.2}$$

$$\begin{aligned}
&\frac{3991680}{16000}f_2(z)f_3(z) - \frac{99}{125}f_4(z)f_3(z) + f_6(z)f_3(z) - \frac{6237}{25}f_1(z)f_4(z)\\
&- \frac{891}{875}f_4(z)f_5(z) = 0,
\end{aligned} \tag{B.1.3}$$

$$\begin{aligned}
&f_2(z)f_7(z) + \frac{7007}{500}f_2(z)f_3(z) + \frac{39611}{2500}f_4(z)f_3(z) - \frac{7007}{500}f_1(z)f_4(z)\\
&- \frac{4719}{4375}f_4(z)f_5(z) - \frac{143}{9}f_1(z)f_6(z) = 0,
\end{aligned}$$

$$\begin{aligned}
&-\frac{1}{15}f_6(z)f_2(z)^2 + \frac{297}{4375}f_4(z)^2f_2(z) + f_1(z)f_5(z)f_2(z) + \frac{44}{625}f_3(z)f_5(z)f_2(z)\\
&+ \frac{9}{143}f_1(z)f_7(z)f_2(z) - \frac{44}{625}f_3(z)^2f_4(z) - \frac{297}{4375}f_1(z)f_4(z)f_5(z)\\
&- f_1(z)f_1(z)f_6(z) = 0,
\end{aligned} \tag{B.1.4}$$

$$\begin{aligned}
&- f_6(z)f_1(z)^2 + f_3(z)f_4(z)f_1(z) - \frac{297}{4375}f_4(z)f_5(z)f_1(z) + \frac{9}{143}f_2(z)f_7(z)f_1(z)\\
&+ \frac{9}{2500}f_2(z)f_4(z)^2 - \frac{7}{1875}f_3(z)^2f_4(z) + \frac{7}{1875}f_2(z)f_3(z)f_5(z)\\
&- \frac{7}{1980}f_2(z)^2f_6(z) = 0,
\end{aligned} \tag{B.1.5}$$

$$\begin{aligned}
&- f_6(z)f_1(z)^2 + \frac{9}{143}f_2(z)f_7(z)f_1(z) - \frac{297}{4375}f_4(z)f_5(z)f_1(z) + \frac{297}{4375}f_2(z)f_4(z)^2\\
&+ f_2(z)^2f_4(z) - \frac{44}{625}f_3(z)^2f_4(z) + \frac{7}{1875}f_2(z)f_3(z)f_5(z) - \frac{7}{1980}f_2(z)^2f_6(z)\\
&+ \frac{297}{4375}f_2(z)f_4(z)^2 - \frac{7}{1980}f_2(z)^2f_6(z) = 0,
\end{aligned} \tag{B.1.6}$$

$$
\begin{aligned}
&- f_6(z)f_1(z)^2 + \frac{9}{143}f_2(z)f_7(z)f_1(z) - \frac{297}{4375}f_4(z)f_5(z)f_1(z) + f_2(z)f_3(z)^2 \\
&+ \frac{9}{2500}f_2(z)f_4(z)^2 - \frac{44}{625}f_3(z)^2f_4(z) + \frac{2647}{39375}f_2(z)f_3(z)f_5(z) \\
&- \frac{7}{1980}f_2(z)^2f_6(z) = 0,
\end{aligned}
\tag{B.1.7}
$$

$$
\begin{aligned}
-f_6(z)f_2(z)^2 + \frac{891}{875}f_4(z)^2f_2(z) + \frac{132}{125}f_3(z)f_5(z)f_2(z) - \frac{132}{125}f_3(z)^2f_4(z) \\
- \frac{891}{875}f_1(z)f_4(z)f_5(z) + f_1(z)f_3(z)f_6(z) = 0,
\end{aligned}
\tag{B.1.8}
$$

Appendix B.2. Coefficients in $\mathcal{G}^{(3,1)}(z)$

Here we list the coefficients in $\mathcal{G}^{(3,1)}(z)$:

$$
\begin{aligned}
b_{116} &= -\frac{\Delta_L(\Delta_L+3)(\Delta_L(\Delta_L(\Delta_L(1001\Delta_L+387)-4326)+13828)+5040)}{10378368000(\Delta_L-4)(\Delta_L-3)(\Delta_L-2)} \\
&+\frac{b_{14}(\Delta_L(143\Delta_L+427)+540)}{17160(\Delta_L-4)}, \\
c_{118} &= 7(\Delta_L+3)\times \\
&\frac{\left(604800b_{14}\left(\Delta_L^2-5\Delta_L+6\right)+\Delta_L\left(-21\Delta_L^3+229\Delta_L^2+414\Delta_L+284\right)\right)}{856627200(\Delta_L^3-9\Delta_L^2+26\Delta_L-24)}, \\
c_{127} &= \frac{\Delta_L(\Delta_L(\Delta_L(\Delta_L(\Delta_L(14\Delta_L-15)+6040)-36125)-75814)-49620)}{2306304000(\Delta_L-4)(\Delta_L-3)(\Delta_L-2)} \\
&-\frac{3b_{14}(\Delta_L(2\Delta_L+3)+135)}{11440(\Delta_L-4)}, \\
c_{145} &= \frac{3b_{14}(\Delta_L(257\Delta_L-2227)+510)}{700000(\Delta_L-4)}+\Delta_L\times \\
&\frac{(\Delta_L(\Delta_L(\Delta_L((32680-1183\Delta_L)\Delta_L-183605)+34900)+570808)+436440)}{47040000000(\Delta_L-4)(\Delta_L-3)(\Delta_L-2)}, \\
c_{226} &= \frac{b_{14}(\Delta_L(22\Delta_L-267)+960)}{39600(\Delta_L-4)}+\Delta_L\times \\
&\frac{(\Delta_L(\Delta_L(\Delta_L((40020-1337\Delta_L)\Delta_L-274845)+96350)+2323212)+1910160)}{71850240000(\Delta_L-4)(\Delta_L-3)(\Delta_L-2)}, \\
c_{235} &= \frac{b_{14}((10283-1153\Delta_L)\Delta_L-5790)}{900000(\Delta_L-4)} \\
&+\frac{\Delta_L\left(51463\Delta_L^5-846480\Delta_L^4+1320405\Delta_L^3\right)}{1632960000000(\Delta_L^3-9\Delta_L^2+26\Delta_L-24)} \\
&+\frac{\Delta_L\left(22381100\Delta_L^2-46886088\Delta_L-46446840\right)}{1632960000000(\Delta_L^3-9\Delta_L^2+26\Delta_L-24)}, \\
c_{244} &= \frac{9b_{14}(\Delta_L(71-11\Delta_L)+270)}{175000(\Delta_L-4)}+\Delta_L\times \\
&\frac{(\Delta_L(\Delta_L(\Delta_L(\Delta_L(1337\Delta_L-32145)+160095)+19525)-266712)-182160)}{70560000000(\Delta_L-4)(\Delta_L-3)(\Delta_L-2)}, \\
c_{334} &= \frac{b_{14}(\Delta_L(11\Delta_L-71)-270)}{18750(\Delta_L-4)}+\Delta_L\times \\
&\frac{(\Delta_L(\Delta_L(\Delta_L(\Delta_L(509\Delta_L-1515)+83415)-808325)+823116)+902880)}{90720000000(\Delta_L-4)(\Delta_L-3)(\Delta_L-2)}.
\end{aligned}
\tag{B.2.1}
$$

Appendix B.3. Coefficients in $\mathcal{G}^{(3,2)}(z)$

Here we list the coefficients in $\mathcal{G}^{(3,2)}(z)$:

$$
\begin{aligned}
g_{119} &= \frac{g_{13}\left(7\Delta_L\left(128-77\Delta_L\right)+6720\right)}{16409250\left(\Delta_L-5\right)} \\
&+ \frac{49b_{14}\left(\Delta_L\left(\Delta_L\left(170-11\Delta_L\right)+981\right)+1620\right)}{16409250\left(\Delta_L-5\right)\left(\Delta_L-4\right)} \\
&+ \frac{196e_{115}}{49725} + \frac{539\Delta_L^7-15386\Delta_L^6+54215\Delta_L^5+951510\Delta_L^4+2911426\Delta_L^3}{472586400000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)} \\
&+ \frac{98e_{15}\left(\Delta_L+4\right)}{16575\left(\Delta_L-5\right)} + \frac{3737076\Delta_L^2+1779120\Delta_L}{472586400000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}, \\
g_{128} &= -\frac{7g_{13}\left(\Delta_L\left(4\Delta_L-469\right)+930\right)}{12355200\left(\Delta_L-5\right)} \\
&- \frac{7b_{14}\left(\Delta_L\left(22\Delta_L^2-64\Delta_L+4197\right)+11745\right)}{6177600\left(\Delta_L-5\right)\left(\Delta_L-4\right)} + \\
&\frac{462\Delta_L^7-24203\Delta_L^6+1044630\Delta_L^5-3466005\Delta_L^4-24181012\Delta_L^3-39855972\Delta_L^2}{1779148800000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)} \\
&- \frac{49e_{15}\left(\Delta_L\left(\Delta_L+2\right)+102\right)}{93600\left(\Delta_L-5\right)} \\
&- \frac{61201\Delta_L}{4942080000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}, \\
g_{155} &= \frac{11e_{15}\left(\Delta_L\left(278\Delta_L-2789\right)+126\right)}{2756250\left(\Delta_L-5\right)} \\
&+ \frac{11g_{13}\left(\Delta_L\left(2279\Delta_L-7400\right)-8370\right)}{231525000\left(\Delta_L-5\right)} \\
&- \frac{3146e_{115}}{275625} + \frac{b_{14}\left(12063\Delta_L^3-88048\Delta_L^2-131165\Delta_L+196110\right)}{77175000\left(\Delta_L-5\right)\left(\Delta_L-4\right)} \\
&+ \frac{-244401285\Delta_L^4+853023786\Delta_L^3+2178372216\Delta_L^2+1399907880\Delta_L}{233377200000000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)} \\
&+ \frac{-1406986\Delta_L^7+28367309\Delta_L^6-123035140\Delta_L^5}{233377200000000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}, \\
g_{227} &= \frac{e_{15}\left(\Delta_L\left(52\Delta_L-751\right)+3234\right)}{93600\left(\Delta_L-5\right)} \\
&- \frac{e_{115}}{240} + \frac{g_{13}\left(\Delta_L\left(1051\Delta_L-12370\right)-52530\right)}{86486400\left(\Delta_L-5\right)} \\
&+ \frac{b_{14}\left(\Delta_L\left(\Delta_L\left(3131\Delta_L-33896\right)-62985\right)+1236870\right)}{86486400\left(\Delta_L-5\right)\left(\Delta_L-4\right)} \\
&+ \frac{-213549\Delta_L^7+6031106\Delta_L^6-23990385\Delta_L^5-205647690\Delta_L^4}{87178291200000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)} \\
&+ \frac{853227874\Delta_L^3+2135805744\Delta_L^2+1445776920\Delta_L}{87178291200000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)},
\end{aligned}
\tag{B.3.1}
$$

$$
\begin{aligned}
g_{245} =& -\frac{99e_{15}\left(\Delta_L\left(83\Delta_L-754\right)-1064\right)}{4900000\left(\Delta_L-5\right)} \\
&+\frac{g_{13}\left(73\Delta_L\left(275-274\Delta_L\right)+170060\right)}{137200000\left(\Delta_L-5\right)} \\
&+\frac{5577e_{115}}{245000}+\frac{b_{14}\left(\Delta_L\left(\Delta_L\left(79801-14981\Delta_L\right)+410980\right)-55320\right)}{68600000\left(\Delta_L-5\right)\left(\Delta_L-4\right)} \\
&+\frac{1300313\Delta_L^7-22489422\Delta_L^6+63989995\Delta_L^5+399569530\Delta_L^4}{138297600000000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)} \\
&+\frac{-690996588\Delta_L^3-2276065528\Delta_L^2-1491467040\Delta_L}{138297600000000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}, \\
g_{335} =& \frac{1144e_{115}}{5315625}+\frac{g_{13}\left(\Delta_L\left(6426275-894839\Delta_L\right)+685170\right)}{17860500000\left(\Delta_L-5\right)} \\
&-\frac{11e_{15}\left(\Delta_L\left(11143\Delta_L-143659\right)+451206\right)}{212625000\left(\Delta_L-5\right)} \\
&-\frac{b_{14}\left(\Delta_L\left(\Delta_L\left(446853\Delta_L-4788638\right)+4992635\right)+44234910\right)}{5953500000\left(\Delta_L-5\right)\left(\Delta_L-4\right)} \\
&+\frac{43544683\Delta_L^7-877022702\Delta_L^6+4877336920\Delta_L^5-1356232020\Delta_L^4}{9001692000000000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)} \\
&+\frac{-28767381333\Delta_L^3-34411007748\Delta_L^2-12217009140\Delta_L}{9001692000000000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}, \\
g_{344} =& \frac{11e_{15}\left(\Delta_L\left(278\Delta_L-2789\right)+126\right)}{2625000\left(\Delta_L-5\right)} \\
&+\frac{g_{13}\left(\Delta_L\left(17194\Delta_L-10525\right)-249570\right)}{220500000\left(\Delta_L-5\right)} \\
&-\frac{1573e_{115}}{131250}+\frac{b_{14}\left(\Delta_L\left(\Delta_L\left(9438\Delta_L-48673\right)-325415\right)+511110\right)}{73500000\left(\Delta_L-5\right)\left(\Delta_L-4\right)} \\
&+\frac{-1593347\Delta_L^7+27045868\Delta_L^6-6670280\Delta_L^5-1193221320\Delta_L^4}{444528000000000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)} \\
&+\frac{1878076947\Delta_L^3+5698801932\Delta_L^2+3877115760\Delta_L}{444528000000000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}.
\end{aligned}
\tag{B.3.2}
$$

$$
\begin{aligned}
d_{117} = &-\frac{9}{220}e_{115} + \frac{84+\Delta_L(53+13\Delta_L)}{1560(\Delta_L-5)}e_{15} \\
&+\frac{13\Delta_L\left(209\Delta_L+409\right)+8340}{7207200\left(\Delta_L-5\right)}g_{13} \\
&-\frac{4641\Delta_L^7+22727\Delta_L^6+44901\Delta_L^5+67569\Delta_L^4+519742\Delta_L^3}{290594304000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)} \\
&-\frac{828876\Delta_L^2+333648\Delta_L}{290594304000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)} \\
&+\frac{\Delta_L\left(\Delta_L\left(5317\Delta_L+18140\right)+68763\right)+69660}{7207200\left(\Delta_L-5\right)\left(\Delta_L-4\right)}b_{14}.
\end{aligned}
\tag{B.3.3}
$$

$$
\begin{aligned}
g_{236} = &\frac{e_{15}\left(\left(15074-1223\Delta_L\right)\Delta_L-39816\right)}{6804000\left(\Delta_L-5\right)} \\
&+\frac{g_{13}\left(\Delta_L\left(186926\Delta_L-1951295\right)+5891220\right)}{6286896000\left(\Delta_L-5\right)} \\
&+\frac{143e_{115}}{340200}+\frac{b_{14}\left(\Delta_L\left(\Delta_L\left(23001\Delta_L-469741\right)+3383740\right)-7782480\right)}{1047816000\left(\Delta_L-5\right)\left(\Delta_L-4\right)} \\
&-\frac{9324749\Delta_L^7-433851406\Delta_L^6+5233472135\Delta_L^5-21967190310\Delta_L^4}{6337191168000000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)} \\
&-\frac{10644674676\Delta_L^3+72859312056\Delta_L^2+65903302080\Delta_L}{6337191168000000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)},
\end{aligned}
\tag{B.3.4}
$$

Appendix B.4. OPE coefficients of twist-eight triple-stress tensors

Here we list a few OPE coefficients of twist-eight triple-stress tensors which are found using (6.52):

$$
\begin{aligned}
P^{(3)}_{12,4} = &\frac{P^{(2)}_{8,2}\left(\Delta_L\left(143\Delta_L+427\right)+540\right)}{17160\left(\Delta_L-4\right)} \\
&-\frac{1001\Delta_L^6+3390\Delta_L^5-3165\Delta_L^4+850\Delta_L^3+46524\Delta_L^2+15120\Delta_L}{10378368000\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)},
\end{aligned}
\tag{B.4.1}
$$

$$
\begin{aligned}
P^{(3)}_{14,6} = &\frac{9P^{(2)}_{8,2}\left(\Delta_L\left(13\Delta_L+11\right)+12\right)}{544544\left(\Delta_L-4\right)} \\
&+\frac{7917\Delta_L^6+38174\Delta_L^5+140795\Delta_L^4+266390\Delta_L^3+253908\Delta_L^2+97776\Delta_L}{548900352000\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)},
\end{aligned}
\tag{B.4.2}
$$

$$P^{(3)}_{16,8} = \frac{5P^{(2)}_{8,2}\left(\Delta_L\left(17\Delta_L+2\right)+6\right)}{9876048\left(\Delta_L-4\right)} + \frac{362593\Delta_L^6+881129\Delta_L^5+2782307\Delta_L^4}{438022480896000\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)} \frac{4155839\Delta_L^3+3518084\Delta_L^2+1198176\Delta_L}{438022480896000\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}, \tag{B.4.3}$$

$$P^{(3)}_{18,10} = \frac{P^{(2)}_{8,2}\left(\Delta_L\left(323\Delta_L-77\right)+54\right)}{823727520\left(\Delta_L-4\right)} + \frac{17413253\Delta_L^6+23717684\Delta_L^5+79039447\Delta_L^4}{377794389772800000\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)} + \frac{92754344\Delta_L^3+73231064\Delta_L^2+22535496\Delta_L}{377794389772800000\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}. \tag{B.4.4}$$

Assuming Einstein-Hilbert + Gauss-Bonnet gravity in the bulk, the OPE coefficient $P^{(2)}_{8,2}$ was derived in (6.107) and can be inserted in (B.4.1)-(B.4.4).

Appendix B.5. Derivation of the deflection angle from the phase shift.

Here we simply show that the bulk phase shift, defined as $\delta = p^t(\Delta t) - p^\phi(\Delta\phi)$ in [55] is consistent with the standard equation relating the eikonal phase and the scattering angle

$$\frac{\partial \delta}{\partial b} = -p^t\,\Delta\phi \tag{B.5.1}$$

obtained with the use of the stationary phase approximation for small scattering angles. Our discussion is focused on asymptotically flat space. In this case, the formulas in classical gravity which provide the deflection angle and the time delay are:

$$\begin{aligned} \Delta t &= 2\int_{r_0}^{\infty} \frac{dr}{f\sqrt{1-\frac{b^2 f}{r^2}}} \\ \Delta\phi &= 2b\int_{r_0}^{\infty} \frac{dr}{r^2\sqrt{1-\frac{b^2 f}{r^2}}}\,. \end{aligned} \tag{B.5.2}$$

They can be obtained from eq. (2.9) in [55] with the substitution $\frac{p^\phi}{p^t} = b$ (and the appropriate definition of the blackening factor $f(r)$). Note that the equation for the turning point of the geodesic, r_0, reduces in Schwarzschild geometry to:

$$1 - \frac{b^2}{r^2 f(r_0)} = 0 \tag{B.5.3}$$

Defining the bulk phase shift via $\delta = p^t(\Delta t) - p^\phi(\Delta\phi)$, leads to

$$\delta = p^t(\Delta t) - p^\phi(\Delta\phi) = p^t\left(\Delta t - b\Delta\phi\right) = 2p^t \int_{r_0}^{\infty} \frac{dr}{f}\sqrt{1-\frac{b^2 f}{r^2}} \tag{B.5.4}$$

Differentiating the bulk phase shift with respect to the impact parameter yields:

$$\frac{\partial\delta}{\partial b} = -2p^t\, b \int_{r_0}^{\infty} \frac{dr}{r^2\sqrt{1-\frac{b^2 f}{r^2}}} - 2p^t \frac{1}{f(r_0)}\sqrt{1-\frac{b^2 f(r_0)}{r_0^2}} = -p^t(\Delta\phi)\,, \tag{B.5.5}$$

where to arrive at the last equality we used the equation satisfied by the turning point r_0. Hence,

$$\Delta\phi = -\frac{1}{p^t}\frac{\partial\delta}{\partial b}\,. \tag{B.5.6}$$

Finally note that assuming the classical relation $J \equiv p_\phi = b\,p^t$, the deflection angle can also be computed through

$$\Delta\phi = -\frac{\partial\delta}{\partial J}\,. \tag{B.5.7}$$

Appendix B.6. Anomalous dimensions and phase shift at $\mathcal{O}(\mu^2)$

We give explicit expressions for $\gamma_n^{(2,0)}$, $\gamma_n^{(2,1)}$ and $\gamma_n^{(2,2)}$ from (3.69)

$$\begin{aligned}\gamma_n^{(2,0)} =& -\frac{1}{8}\left(\Delta_L-1\right)\Delta_L\left(4\Delta_L+1\right) - \frac{51}{4}n^2\left(\Delta_L-1\right)\\ &+\frac{1}{4}n\left(3\left(11-7\Delta_L\right)\Delta_L-17\right) - \frac{17}{2}n^3,\end{aligned} \tag{B.6.1}$$

$$\begin{aligned}\gamma_n^{(2,1)} =& \frac{1}{8\sqrt{1-4\lambda_{\rm GB}}r_{AdS}^2}\Big(\lambda_{\rm GB}(4\Delta_L^4+8\Delta_L^3-4\Delta_L^2-8\Delta_L+560n^3\Delta_L\\ &+360n^2\Delta_L^2-600n^2\Delta_L+80n\Delta_L^3-120n\Delta_L^2+200n\Delta_L+280n^4\\ &-560n^3+440n^2-160n)+r_{AdS}^2\sqrt{1-4\lambda_{\rm GB}}(-\Delta_L^4+6\Delta_L^3-5\Delta_L^2\\ &-140n^3\Delta_L-90n^2\Delta_L^2+354n^2\Delta_L-20n\Delta_L^3+114n\Delta_L^2\\ &-182n\Delta_L-70n^4+276n^3-314n^2+108n)\Big),\end{aligned} \tag{B.6.2}$$

$$
\begin{aligned}
\gamma_n^{(2,2)} = &\frac{1}{8\sqrt{1-4\lambda_{\rm GB}}r_{AdS}^2}\Big(\lambda_{\rm GB}(16\Delta_L^3 - 16\Delta_L + 840n^4\Delta_L + 720n^3\Delta_L^2 \\
&- 2880n^3\Delta_L + 240n^2\Delta_L^3 - 1440n^2\Delta_L^2 + 3720n^2\Delta_L + 24n\Delta_L^4 - 192n\Delta_L^3 \\
&+ 888n\Delta_L^2 - 1536n\Delta_L + 336n^5 - 1680n^4 + 3440n^3 - 3120n^2 + 1024n) \\
&+ r_{AdS}^2\sqrt{1-4\lambda_{\rm GB}}(3\Delta_L^4 - 10\Delta_L^3 + 6\Delta_L^2 + \Delta_L + 420n^3\Delta_L + 270n^2\Delta_L^2 \\
&- 876n^2\Delta_L + 60n\Delta_L^3 - 264n\Delta_L^2 + 420n\Delta_L + 210n^4 - 704n^3 + 756n^2 \\
&- 262n)\Big),
\end{aligned}
\tag{B.6.3}
$$

where we use the expression for $P^{(2)}_{8,0}$, found in [15], to fix $\gamma_n^{(2,2)}$. If one considers limit $1 \ll l, n \ll \Delta_H$ one gets

$$
\gamma^{(2)}_{n,l} \underset{l,n\to\infty}{\approx} -\frac{17n^3}{2l^2} - \frac{35n^4}{4l^3}\left(1 - \frac{4\lambda_{\rm GB}}{r_{AdS}^2\sqrt{1-4\lambda_{\rm GB}}}\right) + \frac{42\lambda_{\rm GB}n^5}{\sqrt{1-4\lambda_{\rm GB}}l^4 r_{AdS}^2} + \ldots, \tag{B.6.4}
$$

where ... denote terms that come from $\gamma_n^{(2,m)}$ for $m > 2$ and they have higher powers of l (and n) as well as terms that are subleading in the given limit and behave as $\mathcal{O}(1)$.

By using the following relations from [55,12]

$$
\sinh(L) = \frac{b}{r_{AdS}}, \quad \cosh(L) = \frac{p^+ + p^-}{2\sqrt{-p^2}}, \tag{B.6.5}
$$

with

$$
-p^2 = p^+p^-, \quad p^+ = 2h, \quad p^- = 2\bar{h}, \tag{B.6.6}
$$

where

$$
h = n + l, \quad \bar{h} = n, \tag{B.6.7}
$$

one obtains $\delta^{(2)}$ from (6.93) in terms of the S-channel variables n and l

$$
\delta^{(2)} = \frac{7\pi n^3}{4l^5}\left(10l^3 + 5l^2 n - \frac{4n(5l^2 + 6ln + 2n^2)\lambda_{\rm GB}}{r_{AdS}^2\sqrt{1-4\lambda_{\rm GB}}}\right). \tag{B.6.8}
$$

From (6.28) and (3.69) one concludes that the leading behavior in the large-l and large-n limit ($1 \ll n, l \ll \Delta_H$) of $\gamma^{(1)}_{n,l}$ is

$$
\gamma^{(1)}_{n,l} \underset{l,n\to\infty}{\approx} -\frac{3n^2}{l} + \mathcal{O}(1). \tag{B.6.9}
$$

Now, one can evaluate (1.5) from [12] using (B.6.8) and (B.6.9)

$$\begin{aligned}\gamma^{(2)}_{n,l} \underset{l,n\to\infty}{\approx} & -\frac{\delta^{(2)}}{\pi}+\frac{1}{2}\gamma^{(1)}_{n,l}\partial_n\gamma^{(1)}_{n,l}\\ \underset{l,n\to\infty}{\approx} & -\frac{17n^3}{2l^2}-\frac{35n^4}{4l^3}\left(1-\frac{4\lambda_{\rm GB}}{r^2_{AdS}\sqrt{1-4\lambda_{\rm GB}}}\right)+\frac{42\lambda_{\rm GB}n^5}{\sqrt{1-4\lambda_{\rm GB}}l^4r^2_{AdS}}\\ & +\frac{14\lambda_{\rm GB}n^6}{\sqrt{1-4\lambda_{\rm GB}}l^5r^2_{AdS}}+\mathcal{O}(1).\end{aligned} \tag{B.6.10}$$

We see that first three terms in (B.6.10) precisely matches with terms in (B.6.4), which explicitly confirms the validity of relation (1.5) in [12]. One would expect that term $\frac{14\lambda_{\rm GB}n^6}{\sqrt{1-4\lambda_{\rm GB}}l^5r^2_{AdS}}$ is due to $\frac{\gamma_n^{(2,3)}}{l^5}$ in (3.69), while all other $\frac{\gamma_n^{(2,k)}}{l^{2+k}}$, for $k>3$, should behave as $\mathcal{O}(1)$ in $1\ll n,l\ll\Delta_H$ limit for (1.5) from [12] to be true.

Appendix C.1. OPE coefficients from Wick contractions

In this appendix we go through the calculations needed for finding the OPE coefficients of various operators using Wick contractions. This mainly amounts to counting the number of contractions leading to a planar diagram. For simplicity, the figures are shown for external operators with $\Delta = 4$ while we write down the result for general Δ as this is needed for the main body of the section.

To begin with, since we consider a large-N matrix theory, it is convenient to use the double-line notation for fundamental field propagators. In Fig. 1 the two-point function $\langle : Tr(\phi^4) :: Tr(\phi^4) : \rangle$ is visualised.

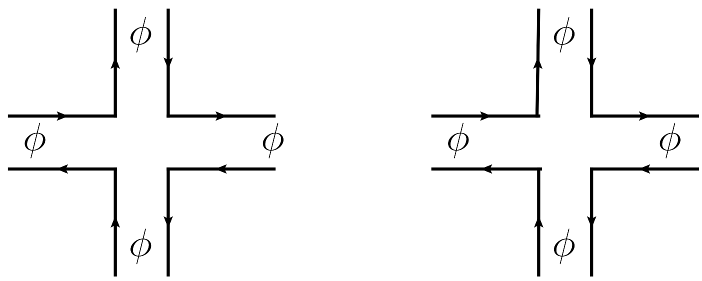

Fig. 1: The two-point function $\langle : Tr(\phi^4) :: Tr(\phi^4) : \rangle$ before any contractions.

In Fig. 2, the planar diagram is shown for $\Delta = 4$ and there are Δ number of such contractions giving a planar diagram

$$P_{\langle :Tr(\phi^{\Delta})::Tr(\phi^{\Delta}):\rangle} = \Delta, \tag{C.1.1}$$

where the $P_{\langle ... \rangle}$ denotes the number of planar diagrams for $\langle ... \rangle$.

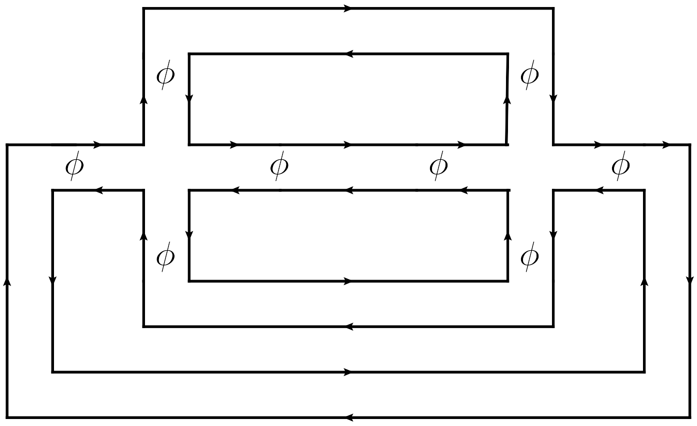

Fig. 2: The two-point function $\langle : Tr(\phi^4) :: Tr(\phi^4) : \rangle$ completely contracted.

We further need the OPE coefficient $\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}_2}$. This is shown in Fig. 3 for $\Delta = 4$ and there are 2Δ possibilities for step (1), Δ number of possibilites for step (2) after which everything is fixed assuming that the diagram is planar. This gives

$$P_{\langle :Tr(\phi^{\Delta})::Tr(\phi^{\Delta})::Tr(\phi^2):\rangle} = 2\Delta^2. \tag{C.1.2}$$

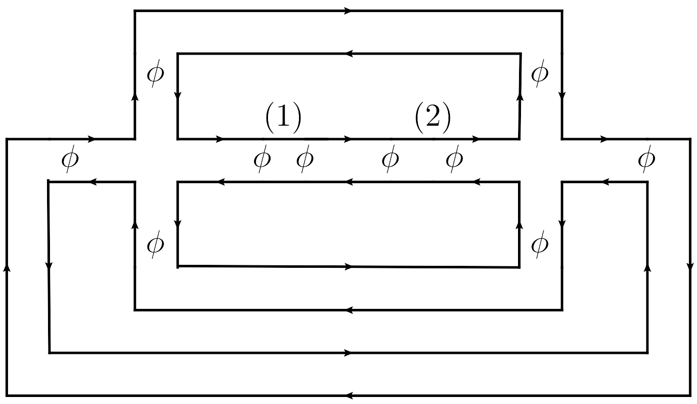

Fig. 3: The three-point function $\langle : Tr(\phi^4) :: Tr(\phi^4) :: Tr(\phi^2) : \rangle$ completely contracted.

In Fig. 4 the three-point function $\langle : Tr(\phi^{\Delta}) :: Tr(\phi^{\Delta}) :: Tr(\phi^4) : \rangle$ for $\Delta = 4$ is shown. For the first contraction (1) there are 2Δ possibilites, for the second contraction there are Δ and for step (3) there are two possibilites. This gives overall

$$P_{\langle :Tr(\phi^{\Delta})::Tr(\phi^{\Delta})::Tr(\phi^4): \rangle} = 4\Delta^2. \quad \text{(C.1.3)}$$

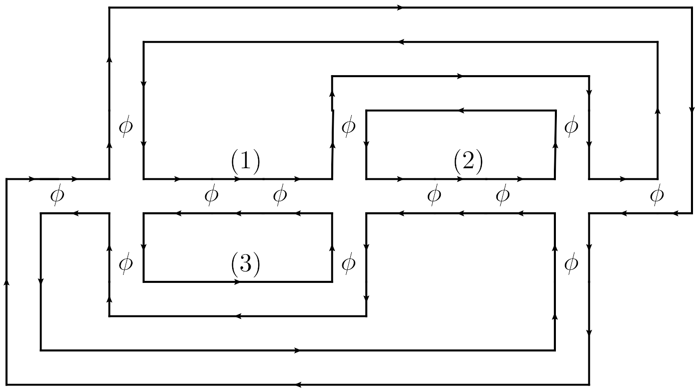

Fig. 4: The three-point function $\langle : Tr(\phi^4) :: Tr(\phi^4) :: Tr(\phi^4) : \rangle$ completely contracted.

In Fig. 5 and Fig. 6, the three-point function $\langle : Tr(\phi^4) :: Tr(\phi^4) : Tr(\phi^2)Tr(\phi^2) : \rangle$ is shown. The reason for there being two different types of diagrams is because each trace term in the double trace operator $: Tr(\phi^2)Tr(\phi^2) :$ can either be contracted with the same $: Tr(\phi^4) :$ (Fig. 5, type B), or to both (Fig. 6, type A).

Consider first the type of diagrams in Fig. 5. For the first contraction there are 2Δ such terms and the second contraction gives another factor of 2. Contraction (3) and (4) contributes factors of Δ and 2 respectively. What remains is equivalent to the two-point function $\langle : Tr(\phi^{\Delta-2}) :: Tr(\phi^{\Delta-2}) : \rangle$ which further give a factor of $(\Delta - 2)$ and therefore there are $8\Delta^2(\Delta - 2)$ contractions of type B in Fig. 5.

Continuing with Fig. 6, the first contraction gives a factor of 2Δ, the second contraction Δ and the third one a factor of $2(\Delta - 1)$. What remains is then fixed by imposing that the diagram is planar. The type A diagrams in Fig. 6 therefore further contributes $4\Delta^2(\Delta - 1)$ planar diagrams to $\langle : Tr(\phi^{\Delta}) :: Tr(\phi^{\Delta}) : Tr(\phi^2)Tr(\phi^2) : \rangle$. It is therefore found that

$$P_{\langle :Tr(\phi^{\Delta})::Tr(\phi^{\Delta}):Tr(\phi^2)Tr(\phi^2): \rangle} = 4\Delta^2(3\Delta - 5). \quad \text{(C.1.4)}$$

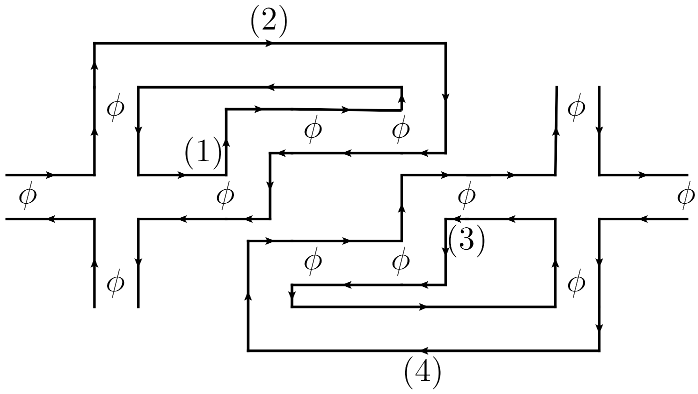

Fig. 5: The three-point function $\langle: Tr(\phi^4) :: Tr(\phi^4) :: Tr(\phi^2)Tr(\phi^2) : \rangle$. There are two such types of contractions that give planar diagrams, here it shown when each $: Tr(\phi^2) :$ connect to a separate $: Tr(\phi^4) :$.

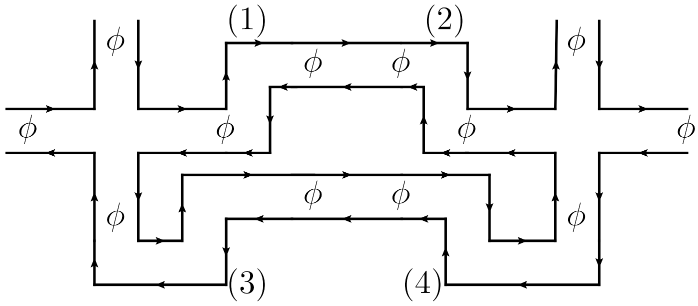

Fig. 6: The three-point function $\langle: Tr(\phi^4) :: Tr(\phi^4) :: Tr(\phi^2)Tr(\phi^2) : \rangle$. There are two such types of contractions that give planar diagrams, here it shown when each $: Tr(\phi^2) :$ connect to both $: Tr(\phi^4) :$ operators.

Consider now the stress tensor OPE coffiecient $\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T_{\mu\nu}}$ where

$$T_{\mu\nu}(x) = \frac{1}{2\sqrt{3}N} : Tr\left(\partial_\mu \phi \partial_\nu \phi - \frac{1}{2}\phi \partial_\mu \partial_\nu \phi - (\text{trace})\right) : (x) \tag{C.1.5}$$

and the three-point function $\langle \mathcal{O}_\Delta \mathcal{O}_\Delta T_{\mu\nu} \rangle$:

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)T_{\mu\nu}(x_3)\rangle = \lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T_{\mu\nu}} \frac{Z_\mu Z_\nu - \text{traces}}{|x_{12}|^{2\Delta-2}|x_{23}|^2|x_{13}|^2}, \tag{C.1.6}$$

where $Z_\mu = \frac{x_{13\mu}}{|x_{13}|^2} - \frac{x_{12\mu}}{|x_{12}|^2}$. From the definition of $T_{\mu\nu}$ in (C.1.5) it is clear that the only term that contributes to term $x_{13\mu}x_{13\nu}$ comes from the second term in (C.1.5) that is of the form $\propto Tr(\phi\partial_\mu\partial_\nu\phi)$. Up to the derivatives, the diagram will look like those visualised in Fig. 3. The number of diagrams is half of that given in (C.1.2) since we restrict to terms proportional to $x_{13\mu}x_{13\nu}$:

$$P_{\langle \mathcal{O}_\Delta \mathcal{O}_\Delta T_{\mu\nu}\rangle|_{x_{13\mu}x_{13\nu}}} = \Delta^2, \tag{C.1.7}$$

from which we reproduce (7.24).

Now we want to find the OPE coefficient $\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{4,4}}$ for the double-stress tensor $T^2_{4,4}$. This is done similarly to the way the stress tensor OPE coefficient was found. First, the operator $(T^2)_{\mu\nu\rho\sigma}$ was given in (7.25) to be

$$(T^2)_{\mu\nu\rho\sigma}(x) = \frac{1}{\sqrt{2}} : T_{(\mu\nu}T_{\rho\sigma)} : (x) - (\text{traces}) \tag{C.1.8}$$

and the three-point function $\langle \mathcal{O}_\Delta \mathcal{O}_\Delta (T^2)_{\mu\nu\rho\sigma}\rangle$ is fixed by conformal symmetry to be

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)(T^2)_{\mu\nu\rho\sigma}(x_3)\rangle = \frac{\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{4,4}}}{|x_{12}|^{2\Delta-4}|x_{13}|^4|x_{23}|^4}\left(Z_\mu Z_\nu Z_\rho Z_\sigma - (\text{traces})\right). \tag{C.1.9}$$

Consider the term in (C.1.9) proportional to $x_{13\mu}x_{13\nu}x_{13\rho}x_{13\sigma}$. This will be due to the term in $(T^2)_{\mu\nu\rho\sigma}$ of the form $Tr(\phi\partial_{(\mu}\partial_\nu\phi)Tr(\phi\partial_\rho\partial_{\sigma)}\phi)$. Using this we find that

$$\begin{aligned}\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)(T^2)_{\mu\nu\rho\sigma}(x_3)\rangle|_{x_{13\mu}x_{13\nu}x_{13\rho}x_{13\sigma}} &= \frac{1}{\Delta N^\Delta}\frac{1}{\sqrt{2}}\left(\frac{-1}{4\sqrt{3}N}\right)^2 8^2 N^\Delta \\ &\times \frac{P_{\langle \mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{4,4}\rangle|_{x_{13\mu}x_{13\nu}x_{13\rho}x_{13\sigma}}}}{|x_{12}|^{2(\Delta-2)}|x_{23}|^4|x_{13}|^{12}}.\end{aligned} \tag{C.1.10}$$

The number of contractions giving a planar diagram, $P_{\langle \mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{4,4}\rangle|_{x_{13\mu}x_{13\nu}x_{13\rho}x_{13\sigma}}}$, come from diagrams of the form given in Fig. 6. Since we are considering the term proportional $x_{13\mu}x_{13\nu}x_{13\rho}x_{13\sigma}$, the number of such diagrams are reduced compared to scalar double trace operator. Instead the first contraction, (1) in Fig. 6, give a factor of Δ, the second contraction, (2), a factor of $(\Delta-1)$, the third contraction (3) gives a further factor Δ after which everything is fixed by imposing that the diagram is planar. We therefore find that

$$P_{\langle \mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{4,4}\rangle|_{x_{13\mu}x_{13\nu}x_{13\rho}x_{13\sigma}}} = \Delta^2(\Delta-1), \tag{C.1.11}$$

and inserting this in (C.1.10) gives

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{4,4}} = \frac{2\sqrt{2}\Delta(\Delta-1)}{3N^2}, \tag{C.1.12}$$

and therefore reproduces (7.28).

Similar to the double-stress tensor, consider the dimension-eight spin-four double trace operator

$$\mathcal{O}^{\rm DT}_{\mu\nu\rho\sigma}(x) = \frac{1}{96\sqrt{70}N^2} : Tr(\phi^2)\Big(Tr(\phi\partial_\mu\partial_\nu\partial_\rho\partial_\sigma\phi) - 16Tr(\partial_{(\mu}\phi\partial_\nu\partial_\rho\partial_{\sigma)}\phi) + 18Tr(\partial_{(\mu}\partial_\nu\phi\partial_\rho\partial_{\sigma)}\phi)(x) - (\text{traces})\Big) : (x). \tag{C.1.13}$$

The three-point function $\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\mathcal{O}^{\rm DT}_{\mu\nu\rho\sigma}(x_3)\rangle$ is given by

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\mathcal{O}^{\rm DT}_{\mu\nu\rho\sigma}(x_3)\rangle = \frac{\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}^{\rm DT}_{\mu\nu\rho\sigma}}}{|x_{12}|^{2\Delta-4}|x_{13}|^4|x_{23}|^4}\left(Z_\mu Z_\nu Z_\rho Z_\sigma - (\text{traces})\right). \tag{C.1.14}$$

By again considering terms in (C.1.14) proportional to $x_{13\mu}x_{13\nu}x_{13\rho}x_{13\sigma}$ we find that each term in (C.1.13) will contribute planar diagram of the type in Fig. 5, while only the term $\sim Tr(\phi\partial^4\phi)$ also give a contribution of the type in Fig. 6. Considering first the terms coming from the diagram in Fig. 5, one finds that this contribution vanishes. The remaining contribution to the term (C.1.14) proportional to $x_{13\mu}x_{13\nu}x_{13\rho}x_{13\sigma}$ comes from the first term in (C.1.13) and the planar diagram pictured in Fig. 6; there are $2\Delta^2(\Delta-1)$ contractions giving such a planar diagram leading to

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\mathcal{O}^{\rm DT}_{\mu\nu\rho\sigma}(x_3)\rangle|_{x_{13\mu}x_{13\nu}x_{13\rho}x_{13\sigma}} = \frac{1}{\Delta N^\Delta}\frac{384}{96\sqrt{70}N^2}N^\Delta \times \frac{2\Delta^2(\Delta-1)}{|x_{12}|^{2(\Delta-2)}|x_{23}|^4|x_{13}|^{12}}, \tag{C.1.15}$$

where the 384 in the numerator come from the derivatives. This gives the OPE coefficient:

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}^{\rm DT}_{\mu\nu\rho\sigma}} = \sqrt{\frac{2}{35}}\frac{4\Delta(\Delta-1)}{N^2} + \mathcal{O}(N^{-4}). \tag{C.1.16}$$

Appendix C.2. Subleading twist double-stress tensors

In this Appendix we study the subleading twist double-stress tensors, both with dimension 8 and spin $s=0,2$ denoted (T^2) and $(T^2)^{\mu\nu}$ respectively. The calculations needed to find the OPE coefficient in the $\mathcal{O}_\Delta\times\mathcal{O}_\Delta$ OPE are reviewed as well as the normalization of $(T^2)^{\mu\nu}$.

The $(T^2)^{\mu\nu}$ was defined in (7.41) which we repeat here:

$$(T^2)^{\mu\nu}(x) = \frac{1}{\sqrt{2}} : T^{\mu}{}_{\alpha} T^{\alpha\nu} : (x) - \frac{\delta^{\mu\nu}}{4\sqrt{2}} : T^{\beta}{}_{\alpha} T^{\alpha}{}_{\beta} : (x). \tag{C.2.1}$$

The operator $(T^2)^{\mu\nu}$ can be seen to be unit-normalized to leading order in N:

$$\begin{aligned} \langle (T^2)^{\mu\nu}(x_1)(T^2)_{\rho\sigma}(x_2)\rangle =& \frac{1}{\sqrt{2}}\langle T^{\mu\alpha}(x_1)T_{\rho\beta}(x_2)\rangle\langle T^{\nu}{}_{\alpha}(x_1)T^{\beta}{}_{\sigma}\rangle \\ &+ (\rho \longleftrightarrow \sigma) - (\text{traces}) + \mathcal{O}(N^{-2}). \end{aligned} \tag{C.2.2}$$

Using the two-point function of the stress tensor in (7.23) and $I^{\mu}{}_{\alpha} I^{\alpha}{}_{\rho} = \delta^{\mu}{}_{\rho}$ one finds

$$\langle (T^2)^{\mu\nu}(x_1)(T^2)_{\rho\sigma}(x_2)\rangle = \frac{1}{|x|^{16}} \left(I^{(\mu}{}_{\rho} I^{\nu)}{}_{\sigma} - (\text{traces}) \right), \tag{C.2.3}$$

from which it is seen that $(T^2)^{\mu\nu}$ is unit-normalised.

We now want to find the OPE coefficient of $(T^2)^{\mu\nu}$ in the $\mathcal{O}_\Delta \times \mathcal{O}_\Delta$ OPE. It can be found from the basic objects $I^{(1)}_{\mu\nu\rho\sigma}$, $I^{(2)}_{\mu\nu\rho\sigma}$ and $I^{(3)}_{\mu\nu\rho\sigma}$ which we calculate below.

We first consider a similar quantity $J^{(1)^{\mu\nu\rho\sigma}}$:

$$\begin{aligned} J^{(1)^{\mu\nu\rho\sigma}} &= \langle : Tr(\phi^\Delta) : (x_1) : Tr(\phi^\Delta) : (x_2) :: Tr(\partial_\mu\phi\partial_\nu\phi)Tr(\partial_\rho\phi\partial_\sigma\phi) : (x_3)\rangle \\ = & \frac{2^4 N^\Delta}{|x_{13}|^8|x_{23}|^8|x_{12}|^{2\Delta-4}} \times \Big[(2\Delta)^2(\Delta-2)(x^\mu_{13}x^\nu_{13}x^\rho_{23}x^\sigma_{23} + x^\mu_{23}x^\nu_{23}x^\rho_{13}x^\sigma_{13}) + \\ & \Delta^2(\Delta-1)(x^\mu_{13}x^\nu_{23}(x^\rho_{13}x^\sigma_{23} + x^\rho_{23}x^\sigma_{13}) + x^\mu_{23}x^\nu_{13}(x^\rho_{13}x^\sigma_{23} + x^\rho_{23}x^\sigma_{13}))\Big]. \end{aligned} \tag{C.2.4}$$

Definining $X^{\mu\nu}_{13} = \frac{1}{|x_{13}|^4}(-\delta^{\mu\nu} + 4\frac{x^\mu_{13}x^\nu_{13}}{|x_{13}|^2})$ we then study $J^{(2)^{\mu\nu\rho\sigma}}$:

$$\begin{aligned} J^{(2)^{\mu\nu\rho\sigma}} &= \langle : Tr(\phi^\Delta) : (x_1) : Tr(\phi^\Delta) : (x_2) :: Tr(\phi\partial_\mu\partial_\nu\phi)Tr(\phi\partial_\rho\partial_\sigma\phi) : (x_3)\rangle \\ =& \frac{N^\Delta}{|x_{12}|^{2\Delta-4}}\Big[\Delta^2(\Delta-1)2^2\Big(X^{\mu\nu}_{13}\frac{1}{|x_{23}|^2}X^{\rho\sigma}_{13}\frac{1}{|x_{23}|^2} + X^{\mu\nu}_{13}\frac{1}{|x_{23}|^2}X^{\rho\sigma}_{23}\frac{1}{|x_{13}|^2}\Big) \\ &+((2\Delta)^2(\Delta-2))2^2X^{\mu\nu}_{13}\frac{1}{|x_{13}|^2}X^{\rho\sigma}_{23}\frac{1}{|x_{23}|^2} \\ &+(13) \longleftrightarrow (23)\Big]. \end{aligned} \tag{C.2.5}$$

And lastly $J^{(3)\mu\nu\rho\sigma}$:

$$\begin{aligned} J^{(3)\mu\nu\rho\sigma} =& \langle : Tr(\phi^\Delta) : (x_1) : Tr(\phi^\Delta) : (x_2) :: Tr(\phi\partial_\mu\partial_\nu\phi)Tr(\partial_\rho\phi\partial_\sigma\phi) : (x_3)\rangle \\ =& \frac{N^\Delta}{|x_{12}|^{2\Delta-4}}\Big[((2\Delta)^2(\Delta-2))2^3 X_{13}^{\mu\nu}\frac{1}{|x_{13}|^2}\frac{x_{23}^\rho x_{23}^\sigma}{|x_{23}|^8} + \\ &+ \Delta^2(\Delta-1)2^3 X_{13}^{\mu\nu}\frac{1}{|x_{23}|^2}\frac{x_{13}^\rho x_{23}^\sigma + x_{23}^\rho x_{13}^\sigma}{|x_{13}|^4|x_{23}|^4} \\ &+ (13) \longleftrightarrow (23)\Big]. \end{aligned} \tag{C.2.6}$$

We further need to make (C.2.4)-(C.2.6) traceless in the pairs (μ,ν) and (ρ,σ) and therefore define $I^{(i)\mu\nu\rho\sigma}$ as

$$I^{(i)\mu\nu\rho\sigma} = J^{(i)\mu\nu\rho\sigma} - \frac{\delta^{\mu\nu}}{4}J^{(i)\alpha}{}_{\alpha}{}^{\rho\sigma} - \frac{\delta^{\rho\sigma}}{4}J^{(i)\mu\nu\alpha}{}_{\alpha} + \frac{\delta^{\mu\nu}\delta^{\rho\sigma}}{16}J^{(i)\alpha}{}_{\alpha}{}^{\gamma}{}_{\gamma}. \tag{C.2.7}$$

From (C.2.4)-(C.2.6), the three-point function $\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)(T^2)^{\mu\nu}(x_3)\rangle$ is given by

$$\begin{aligned} \langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)(T^2)^{\mu\nu}\rangle =& \frac{1}{12\sqrt{2}\Delta N^{\Delta+2}}(I^{(1)(\mu|\alpha}{}_{\alpha}{}^{|\nu)} - I^{(3)(\mu|\alpha}{}_{\alpha}{}^{|\nu)} \\ &+ \frac{1}{4}I^{(2)(\mu|\alpha}{}_{\alpha}{}^{|\nu)} - (\text{trace})). \end{aligned} \tag{C.2.8}$$

Explicitly we find that

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)(T^2)^{\mu\nu}(x_3)\rangle = \frac{\sqrt{2}\Delta(\Delta-1)}{3N^2}\frac{Z^\mu Z^\nu - (\text{trace})}{|x_{12}|^{2\Delta-6}|x_{13}|^6|x_{23}|^6} + \mathcal{O}(N^{-4}). \tag{C.2.9}$$

Consider now the scalar operator (T^2) defined by

$$(T^2)(x) = \frac{1}{36\sqrt{2}N^2} : T_{\mu\nu}T^{\mu\nu} : (x). \tag{C.2.10}$$

The three-point function $\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)(T^2)(x_3)\rangle$ can be found using $I^{(i)}$ defined in (C.2.7) as follows

$$\begin{aligned} \langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)(T^2)(x_3)\rangle =& \frac{1}{36\sqrt{2}\Delta N^{2+\Delta}}(I^{(1)\mu\nu}{}_{\mu\nu} - I^{(3)\mu\nu}{}_{\mu\nu} + \frac{1}{4}I^{(2)\mu\nu}{}_{\mu\nu}) \\ + \mathcal{O}(N^{-4}) =& \frac{\Delta(\Delta-1)}{3\sqrt{2}N^2}\frac{1}{|x_{12}|^{2\Delta-8}|x_{13}|^8|x_{23}|^8} + \mathcal{O}(N^{-4}). \end{aligned} \tag{C.2.11}$$

Appendix C.3. Single trace operator with dimension $\Delta \sim C_T$

In this appendix we study the single trace scalar operator $\mathcal{O}_{\Delta_H}$ given by

$$\mathcal{O}_H(x) = \frac{1}{\sqrt{\mathcal{N}_{\Delta_H}}} :Tr(\phi^{\Delta_H}):(x), \tag{C.3.1}$$

with $\Delta_H \sim C_T$ and $\mathcal{N}_{\Delta_H}$ a normalization constant[50]. When calculating the normalization constant $\mathcal{N}_{\Delta_H}$ as well as the three-point functions $\langle \mathcal{O}_H(x_1)\mathcal{O}_H(x_2)\mathcal{O}(x_3)\rangle$, non-planar diagrams generically gets enhanced by powers of Δ_H and therefore invalidates the naive planar expansion. The goal of this appendix is to show that

$$\langle \mathcal{O}_H(x_1)\mathcal{O}_H(x_2)\hat{\mathcal{O}}(x_3)\rangle = \langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\hat{\mathcal{O}}(x_3)\rangle|_{\Delta=\Delta_H}, \tag{C.3.2}$$

where $\hat{\mathcal{O}}$ is either $:Tr(\phi^2):$ or, more importantly, minimal-twist multi stress tensors with any spin. Moreover, note that the LHS in (C.3.2) is in principle exact in $C_T \sim N^2$ while the RHS is obtained by keeping only planar diagrams with $\Delta \ll C_T$ and then setting $\Delta = \Delta_H$ in the end.

The propagator for the field ϕ was given in (7.19) by

$$\langle \phi^i{}_j(x)\phi^k{}_l(y)\rangle = \left(\delta^i{}_l\delta^k{}_j - \frac{1}{N}\delta^i{}_j\delta^k{}_l\right)\frac{1}{|x-y|^2}. \tag{C.3.3}$$

Consider now the three-point function $\langle :Tr(\phi^{\Delta_H}):(x_1) :Tr(\phi^{\Delta_H}):(x_2) :Tr(\phi^2):(x_3)\rangle$. Due to the normal ordering, one ϕ field in $:Tr(\phi^2):(x_3)$ need to be contracted with $:Tr(\phi^{\Delta_H}):(x_1):$ and the other one with $:Tr(\phi^{\Delta_H}):(x_2):$. Note that for this contraction the second term in (C.3.3) give a contribution proportional to $Tr(\phi(x_3)) = 0$. It is therefore seen that

$$\langle :Tr(\phi_1^{\Delta_H})::Tr(\phi_2^{\Delta_H})::Tr(\phi_3^2):\rangle = 2\Delta_H \langle :Tr(\phi_3\phi_1^{\Delta_H-1})::Tr(\phi_2^{\Delta_H}):\rangle, \tag{C.3.4}$$

where we introduced the notation $\phi_i = \phi(x_i)$ and dropped the $|x_{ij}|^{-2}$ coming from (C.3.3). The position dependence is easily restored in the end. Now it is seen that the RHS of (C.3.4) is proportional to the two-point function[51] of $\mathcal{O}_H$ and we therefore find that

$$\langle :Tr(\phi_1^{\Delta_H})::Tr(\phi_2^{\Delta_H})::Tr(\phi_3^2):\rangle = 2\Delta_H \mathcal{N}_{\Delta_H}, \tag{C.3.5}$$

[50] Mixing with other operators with $\Delta \sim C_T$ is not important for this discussion.

[51] Up to the position dependence.

which is exact to all orders in C_T. Including the normalization factor of $\mathcal{O}_H$ in (C.3.1) and $\mathcal{O}_2$ from (7.20) we find that

$$\langle \mathcal{O}_H(x_1)\mathcal{O}_H(x_2)\mathcal{O}_2(x_3)\rangle = \frac{\sqrt{2}\Delta_H}{N}\frac{1}{|x_{12}|^{2\Delta_H-2}|x_{13}|^2|x_{23}|^2} + \mathcal{O}(N^{-3}). \quad \text{(C.3.6)}$$

By comparing (C.3.6) with (7.87) we find that

$$\lambda_{\mathcal{O}_H\mathcal{O}_H\mathcal{O}_2} = \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}_2}|_{\Delta=\Delta_H}. \quad \text{(C.3.7)}$$

Note that in (C.3.6) the normalization of $\mathcal{O}_H$ cancels the contribution from non-planar diagrams in limit $\Delta_H \sim C_T$. For $\Delta = 2$ in (7.20), it is trivial to compute the normalization exact in N to get the correction to $\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}_2}$ in (C.3.6).

Consider now the stress tensor operator defined in (7.22) and the three-point function $\langle \mathcal{O}_H(x_1)\mathcal{O}_H(x_2)T_{\mu\nu}(x_3)\rangle$. This is fixed by the Ward identity but is an instructive example before considering more general multi stress tensors. In the same way as the OPE coefficient was found in the $\mathcal{O}_\Delta \times \mathcal{O}_\Delta$ OPE, due to the tensor structure being fixed by conformal symmetry, we consider the term proportional to $x_{13}^\mu x_{13}^\nu$ in the three-point function. This comes from the $-\frac{1}{6\sqrt{C_T}}Tr(\phi\partial_\mu\partial_\nu\phi)$ term in the stress tensor when $\partial_\mu\partial_\nu\phi$ is contracted with one of the Δ_H number of $\phi(x_1)$ fields. Doing this contraction we therefore see that

$$\begin{aligned}&\langle : Tr(\phi_1^{\Delta_H}) :: Tr(\phi_2^{\Delta_H}) :: Tr(\phi_3\partial_\mu\partial_\nu\phi_3) :\rangle|_{x_{13}^\mu x_{13}^\nu} = \\ &8\Delta_H\langle : Tr(\phi_3\phi_1^{\Delta_H-1}) :: Tr(\phi_2^{\Delta_H}) :\rangle,\end{aligned} \quad \text{(C.3.8)}$$

where the factor 8 comes from the derivatives and we again suppress the space-time dependence. The RHS of (C.3.8) is also proportional to the normalization constant of $\mathcal{O}_H$. Including the normalization factor of the stress tensor in (7.22) and that of $\mathcal{O}_H$ in (C.3.1), the three-point function $\langle \mathcal{O}_H\mathcal{O}_H T_{\mu\nu}\rangle$ can be obtained from (C.3.8) from which we read off the OPE coefficient

$$\lambda_{\mathcal{O}_H\mathcal{O}_H T_{\mu\nu}} = -\frac{4\Delta_H}{3\sqrt{C_T}}. \quad \text{(C.3.9)}$$

This agrees with (7.24).

We now want to show that is true for minimal-twist multi stress tensors with any spin. For simplicity, consider the double-stress tensor with spin 4 defined in (7.25)

$$(T^2)_{\mu\nu\rho\sigma}(x) = \frac{1}{\sqrt{2}} : T_{(\mu\nu}T_{\rho\sigma)} : (x) - (\text{traces}). \quad \text{(C.3.10)}$$

Similarly to the calculation of the three-point function with the stress tensor, we can obtain the three-point function $\langle \mathcal{O}_H(x_1)\mathcal{O}_H(x_2)(T^2)_{\mu\nu\rho\sigma}(x_3)\rangle$ by considering the term proportional to $x_{13}^\mu x_{13}^\nu x_{13}^\rho x_{13}^\sigma$. This will be due to the term $\frac{1}{\sqrt{2}6^2 C_T} Tr(\phi\partial_\mu\partial_\nu\phi)Tr(\phi\partial_\rho\partial_\sigma\phi)$ when contracting $\partial_\mu\partial_\nu\phi$ with some $\phi(x_1)$ and likewise contracting $\partial_\rho\partial_\sigma\phi$ with some other $\phi(x_1)$. The number of such contractions is given by $\Delta_H(\Delta_H - 1)$ and we find that

$$\begin{aligned}\langle : Tr(\phi_1^{\Delta_H}) :: Tr(\phi_2^{\Delta_H}) : : Tr(\phi_3\partial_\mu\partial_\nu\phi_3)Tr(\phi_3\partial_\rho\partial_\sigma\phi_3) :\rangle|_{x_{13}^\mu x_{13}^\nu x_{13}^\rho x_{13}^\sigma} \\ = 8^2\Delta_H(\Delta_H - 1)\langle : Tr(\phi_3^2\phi_1^{\Delta_H - 2}) :: Tr(\phi_2^{\Delta_H}) :\rangle,\end{aligned} \tag{C.3.11}$$

where the factor of 8^2 again is due to acting with the derivatives and note that the position of the ϕ_3 fields in in the last line is not important. It is again seen that the RHS of (C.3.11) is proportional to the normalization constant of $\mathcal{O}_H$. Including the normalization in (7.25) and (C.3.1) we find the three-point function $\langle \mathcal{O}_H\mathcal{O}_H(T^2)_{\mu\nu\rho\sigma}\rangle$ and read off the OPE coefficient:

$$\lambda_{\mathcal{O}_H\mathcal{O}_H T^2_{4,4}} = \frac{8\sqrt{2}\Delta_H(\Delta_H - 1)}{9C_T} + \mathcal{O}(C_T^{-3/2}). \tag{C.3.12}$$

which is seen to agree with (7.28) when setting $\Delta_H = \Delta$. Note that the corrections in (C.3.12) are solely due to corrections in the normalization of $T^2_{4,4}$ and therefore $\lambda_{\mathcal{O}_H\mathcal{O}_H T^2_{4,4}} = \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{4,4}}$ to all orders in C_T. These arguments generalize straightforwardly to minimal-twist multi stress tensor with any spin such that the results are the same as those obtained in the planar limit for $\Delta \ll C_T^2$ in Section 7.3 by setting $\Delta_H = \Delta$. The only correction in C_T is then due to the normalization of the multi stress tensor.

The same argument applies to any scalar primary multi-trace operator $\mathcal{O}_\Delta$, without any derivatives, with OPE coefficients given by (C.3.6), (C.3.9) and (C.3.12).

Appendix C.4. Stress tensor thermal one-point function

In order to calculate thermal one-point functions in the free adjoint scalar model we use the fact that the thermal correlation function is related to the zero-temperature case by summing over images. Consider now the thermal one-point function of the stress tensor. Generally, the one-point function of a spin-s symmetric traceless operator with dimension $\Delta_{\mathcal{O}}$ on $S^1 \times \mathbf{R}^{d-1}$ is given by [82]

$$\langle \mathcal{O}^{\mu_1 \dots \mu_s}(x) \rangle_\beta = \frac{b_{\mathcal{O}}}{\beta^{\Delta_{\mathcal{O}}}} (e^{\mu_1} \dots e^{\mu_s} - (\text{traces})), \tag{C.4.1}$$

where e^{μ_1} is a unit-vector along the thermal circle. Consider first the canonically normalized stress tensor given by $T^{(\text{can})}_{\mu\nu} = \frac{1}{3S_d}(Tr(\partial_\mu \phi \partial_\mu \phi) - \frac{1}{2} Tr(\phi \partial_\mu \partial_\nu \phi) - (\text{traces}))$. In order to find the one-point function, use the following:

$$\langle Tr(\partial^{(x)}_\mu \phi(x) \partial^{(y)}_\nu \phi(y)) \rangle = \frac{2(N^2-1)}{|x-y|^4} (\delta_{\mu\nu} - 4(y-x)_\mu (y-x)_\nu \frac{1}{|x-y|^2}) \tag{C.4.2}$$

and

$$\langle Tr(\partial^{(x)}_\mu \partial^{(x)}_\nu \phi(x) \phi(y)) \rangle = \frac{2(N^2-1)}{|x-y|^4} (-\delta_{\mu\nu} + 4(y-x)_\mu (y-x)_\nu \frac{1}{|x-y|^2}). \tag{C.4.3}$$

To get the thermal correlator, we use (C.4.2) and (C.4.3) with x, y along the thermal circle separated by a distance $m\beta$, with m integer, and sum over $m \neq 0$. The reason for summing over $m \neq 0$ is the normal ordering of the operators. Namely, at $T = 0$, the normal ordering means that the correlation functions are computed without the self contractions, which would give the divergent contributions to the correlator. Removing the self-contractions in the correlation functions at $T = 0$ is the same as removing the divergent term with $m = 0$ when computing the thermal expectation value. The relevant terms for calculating the one-point functions in terms of fundamental fields are therefore

$$\begin{aligned} \langle Tr(\partial_\mu \phi \partial_\nu \phi) \rangle_{\beta,m} &= -\frac{8(N^2-1)}{(m\beta)^4} e^\mu e^\nu + \frac{2(N^2-1)}{(m\beta)^4} \delta_{\mu\nu}, \\ \langle Tr(\partial_\mu \partial_\nu \phi \phi) \rangle_{\beta,m} &= \frac{8(N^2-1)}{(m\beta)^4} e^\mu e^\nu - \frac{2(N^2-1)}{(m\beta)^4} \delta_{\mu\nu}, \end{aligned} \tag{C.4.4}$$

where we note that only the first term in each equation in (C.4.4) contribute to the non-trace term in (C.4.1).

We therefore find for the stress tensor one-point function:

$$\begin{aligned} \langle T^{(\text{can})}_{\mu\nu} \rangle_\beta &= \frac{1}{3S_d} (\langle Tr(\partial_\mu \phi \partial_\nu \phi) \rangle_\beta - \frac{1}{2} \langle Tr(\partial_\mu \partial_\nu \phi \phi) \rangle_\beta - \text{trace}) \\ &= \frac{-12(N^2-1)}{3S_d} \frac{2\zeta(4)}{\beta^4} (e_\mu e_\nu - (\text{trace})), \end{aligned} \tag{C.4.5}$$

where the $2\zeta(4)$ comes from summing over images and we therefore have

$$b_{T^{(\text{can})}_{\mu\nu}} = -\frac{4(N^2-1)}{S_d} 2\zeta(4) = -\frac{4\pi^4}{45 S_d}(N^2-1). \tag{C.4.6}$$

This agrees with $f = \frac{b_{T^{(\text{can})}_{\mu\nu}}}{d}$ in eq. (2.17) in [82] for (N^2-1) free scalar fields. This also agrees with $a_{2,2} = \frac{\pi^4 \Delta}{45}$ found from the two-point thermal correlator using:

$$a_{2,2} = \frac{\pi^4 \Delta}{45} = \left(\frac{1}{2}\right)^2 \frac{\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T^{(\text{can})}} b_{T^{(\text{can})}_{\mu\nu}}}{\frac{C_T}{S_d^2}}, \tag{C.4.7}$$

using $\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T^{(\text{can})}} = -\frac{4\Delta}{3S_d}$ in this normalization and $C_T = \frac{4}{3}(N^2-1)$. This is simply related to the one-point function for the unit-normalized stress tensor by (to leading order in N)

$$\begin{aligned} b_{T_{\mu\nu}} &= \frac{b_{T^{(\text{can})}_{\mu\nu}}}{\frac{\sqrt{C_T}}{S_d}} \\ &\approx -\frac{2\pi^4 N}{15\sqrt{3}}. \end{aligned} \tag{C.4.8}$$

Let us now consider the thermalization of the stress tensor, keeping all the index structures. To compare the thermal two-point function with the heavy-heavy-light-light correlator, we want to relate the dimension of the heavy operator, Δ_H, to the inverse temperature β. Consider the expectation value of the stress tensor in a heavy state created by $\mathcal{O}_H$ on the cylinder $\mathbf{R} \times S^3$

$$\langle \mathcal{O}_H | T^{\mu\nu}(x^0_{E,2}, \hat{n}) | \mathcal{O}_H \rangle_{\text{cyl}} = \lim_{x_3 \to \infty} |x_3|^{2\Delta_H} |x_2|^4 \lambda_{\mathcal{O}_H \mathcal{O}_H T_{\mu\nu}} \frac{Z^\mu Z^\nu - \frac{1}{4}\delta^{\mu\nu} Z^\rho Z_\rho}{|x_{13}|^{2\Delta_H - 2} |x_{23}|^2 |x_{12}|^2}, \tag{C.4.9}$$

where the RHS is found by a conformal transformation to the plane with $Z^\mu = \left(\frac{x^\mu_{12}}{|x_{12}|^2} + \frac{x^\mu_{23}}{|x_{23}|^2}\right)$. When $x_1 = 0$ and $x_3 \to \infty$, it is seen that $Z^\mu = -\frac{x^\mu_2}{|x_2|^2}$ and (C.4.9) only depends on $\hat{x}^\mu = \frac{x^\mu_{21}}{|x_{21}|} = \hat{r}$, where $\hat{r}$ is a radial unit vector. In radial quantization it follows that

$$\langle \mathcal{O}_H | T^{\mu\nu}(x^0_{E,2}, \hat{n}) | \mathcal{O}_H \rangle_{\text{cyl}} = \frac{\lambda_{\mathcal{O}_H \mathcal{O}_H T_{\mu\nu}}}{R^4} (\hat{e}_\mu \hat{e}_\nu - \frac{1}{4}\delta_{\mu\nu}) \tag{C.4.10}$$

where we reintroduced the radius of the sphere R, $\lambda_{\mathcal{O}_H \mathcal{O}_H T_{\mu\nu}}$ is the OPE coefficient of $T_{\mu\nu}$ in the $\mathcal{O}_H \times \mathcal{O}_H$ OPE and $\hat{e}_\mu = (1, 0, 0, 0)$.

The thermal one-point function of an operator $\mathcal{O}_{\tau,s}$, with twist τ and spin s, on $S^1 \times S^3$ is fixed by conformal symmetry [82]

$$\langle \mathcal{O}_{\tau,s}(x) \rangle_\beta = \frac{b_{\mathcal{O}_{\tau,s}} f_{\mathcal{O}_{\tau,s}}(\frac{\beta}{R})}{\beta^{\tau+s}} (e^{\mu_1} \cdots e^{\mu_s} - (\text{traces})), \tag{C.4.11}$$

where $f_{\mathcal{O}_{\tau,s}}(0) = 1$ and $e^\mu = (1, 0, 0, 0)$.

We assume thermalization of the stress tensor in the heavy state:

$$\langle \mathcal{O}_H | T_{\mu\nu}(x) | \mathcal{O}_H \rangle = \langle T_{\mu\nu}(x) \rangle_\beta \tag{C.4.12}$$

where $\langle T^{\mu\nu}(x) \rangle_\beta$ is the thermal one-point function at inverse temperature β evaluated on $S^1 \times S^3$, with R being the radius of S^3. Using (C.4.10)-(C.4.12) we find

$$\frac{\lambda_{\mathcal{O}_H \mathcal{O}_H T_{\mu\nu}}}{R^4} = \frac{b_{T_{\mu\nu}} f_{T_{\mu\nu}}(\frac{\beta}{R})}{\beta^4}. \tag{C.4.13}$$

Using (C.4.13) for $R \to \infty$ in the free adjoint scalar theory, together with the one-point function $b_{T_{\mu\nu}} = -\frac{2\pi^4 N}{15\sqrt{3}}$ and the OPE coefficient $\lambda_{\mathcal{O}_H \mathcal{O}_H T_{\mu\nu}} = -\frac{4\Delta_H}{3\sqrt{C_T}}$, one finds the following relation between $\mu = \frac{160\Delta_H}{3C_T}$ and the inverse temperature β:

$$\mu = \frac{8}{3} \left(\frac{\pi R}{\beta} \right)^4. \tag{C.4.14}$$

This agrees with (7.50).

Appendix C.5. Dimension-six spin-four single trace operator

We want to calculate the contribution of the single trace operator with $\tau = 2$ and $s = 4$. The unit-normalised $\mathcal{O}_{2,4}$ operator is given by[52]

$$\begin{aligned} \Xi_{\mu\nu\rho\sigma}(x) = \frac{1}{96\sqrt{35}N} : Tr\big(&\phi(\partial_\mu \partial_\nu \partial_\rho \partial_\sigma \phi) - 16(\partial_{(\mu}\phi)(\partial_\nu \partial_\rho \partial_{\sigma)}\phi) \\ &+ 18(\partial_{(\mu}\partial_\nu \phi)(\partial_\rho \partial_{\sigma)}\phi) - (\text{traces})\big) : (x). \end{aligned} \tag{C.5.1}$$

The relative coefficients are fixed by demanding that it is a primary operator $[K_\alpha, \Xi_{\mu\nu\rho\sigma}] = 0$. Explictily, this is done using the conformal algebra

$$\begin{aligned} [K_\mu, P_\nu] &= 2i(\eta_{\mu\nu} D - M_{\mu\nu}), \\ [M_{\mu\nu}, P_\rho] &= -i(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu), \end{aligned} \tag{C.5.2}$$

[52] We denote this operator either as $\mathcal{O}_{2,4}$ or $\Xi_{\mu\nu\rho\sigma}$ depending whether we want to explicitly list the indices or not.

and the action on the fundamental field ϕ

$$\begin{aligned} P_\mu \phi(0) &= -i\partial_\mu \phi(0), \\ D\phi(0) &= i\phi(0). \end{aligned} \tag{C.5.3}$$

The relevant commutators in order to fix $\Xi_{\mu\nu\rho\sigma}$ are

$$\begin{aligned} [K_\alpha, P_\mu \phi] =& -2\eta_{\alpha\mu}\phi, \\ [K_\alpha, P_\mu P_\nu \phi] =& -4\eta_{\alpha\mu}P_\nu\phi - 4\eta_{\alpha\nu}P_\mu\phi + 2\eta_{\mu\nu}P_\alpha\phi, \\ [K_\alpha, P_\mu P_\nu P_\rho \phi] =& -6\eta_{\alpha\mu}P_\nu P_\rho\phi - 6\eta_{\alpha\nu}P_\mu P_\rho\phi - 6\eta_{\alpha\rho}P_\nu P_\mu\phi \\ &+ 2\eta_{\mu\nu}P_\rho P_\alpha\phi + 2\eta_{\rho\nu}P_\mu P_\alpha\phi + 2\eta_{\mu\rho}P_\nu P_\alpha\phi, \\ [K_\alpha, P_\mu P_\nu P_\rho P_\sigma \phi] =& -8\eta_{\alpha\mu}P_\nu P_\rho P_\sigma\phi - 8\eta_{\alpha\nu}P_\mu P_\rho P_\sigma\phi - 8\eta_{\alpha\rho}P_\nu P_\mu P_\sigma\phi \\ &+ 2\eta_{\mu\nu}P_\rho P_\sigma P_\alpha\phi + 2\eta_{\mu\rho}P_\nu P_\sigma P_\alpha\phi + 2\eta_{\mu\sigma}P_\rho P_\nu P_\alpha\phi \\ &+ 2\eta_{\nu\rho}P_\mu P_\sigma P_\alpha\phi + 2\eta_{\nu\sigma}P_\mu P_\rho P_\alpha\phi + 2\eta_{\rho\sigma}P_\mu P_\nu P_\alpha\phi \\ &- 8\eta_{\alpha\sigma}P_\nu P_\rho P_\mu\phi, \end{aligned} \tag{C.5.4}$$

which can also be found in e.g. Appendix F in [34].

The thermal one-point function of this operator is found from Wick contractions to be

$$\langle \Xi_{\mu\nu\rho\sigma} \rangle_\beta = \frac{8(\pi T)^6 N}{27\sqrt{35}} \left(e_\mu e_\nu e_\rho e_\sigma - (\text{traces}) \right). \tag{C.5.5}$$

Moreover, the three-point function with operators $\mathcal{O}_\Delta(x) = \frac{1}{\sqrt{\Delta N^\Delta}} : Tr\left(\phi^\Delta\right) : (x)$ can again be calculated using Wick contractions similarly to how it was done for $T^2_{\mu\nu\rho\sigma}$ in Appendix A. By explicit calculation one finds

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\Xi_{\mu\nu\rho\sigma}(x_3) \rangle = \frac{4\Delta}{\sqrt{35}N} \frac{Z_\mu Z_\nu Z_\rho Z_\sigma - (\text{traces})}{|x_{12}|^{2\Delta-2}|x_{13}|^2|x_{23}|^2}, \tag{C.5.6}$$

and therefore the OPE coefficient $\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}_{2,4}}$ is given by

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}_{2,4}} = \frac{4\Delta}{\sqrt{35}N}. \tag{C.5.7}$$

Now, it is easy to check that

$$\frac{1}{2^4}\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}_{2,4}} b_{\mathcal{O}_{2,4}} = \frac{2\pi^6\Delta}{945}, \tag{C.5.8}$$

which agrees with $a_{2,4}$ in (7.85).

Appendix C.6. Thermal one-point functions of multi-trace operators in the large-N limit

In (7.110), it was shown that $a_{4,4}$ was due to double trace operators which were normal ordered products of single trace operators without any derivatives. There are, however, other double trace operators that have the same quantum numbers and are schematically represented as $[\mathcal{O}_a\mathcal{O}_b]_{n,l}$. Concretely, the double trace operators with twist and spin four besides $(T^2)_{\mu\nu\rho\sigma}$ and $(\mathcal{O}^{\mathrm{DT}})_{\mu\nu\rho\sigma}$ are $[\mathcal{O}_2\mathcal{O}_2]_{0,4}$ and $[\mathcal{O}_2 T_{\mu\nu}]_{0,2}$. We argue that the thermal one-point functions of these operators are subleading in the large-N limit when evaluated on the plane.

Consider the thermal one-point function of a double trace operator $[\mathcal{O}_a\mathcal{O}_b]_{n,l} = \mathcal{O}_a\partial^{2n}\partial^l\mathcal{O}_b + \ldots$, where $\mathcal{O}_a$ and $\mathcal{O}_b$ are single trace primary operators and dots represent terms where derivatives acts on $\mathcal{O}_a$ as well, in order to make $[\mathcal{O}_a\mathcal{O}_b]_{n,l}$ a primary operator. The term in the thermal one-point function that behaves as N^k (N^2 for double trace operators) comes from contracting the fundamental field within each trace separately. Therefore we have

$$\langle \mathcal{O}_a\partial^{2n}\partial^l\mathcal{O}_b\rangle_\beta \approx \langle \mathcal{O}_a\rangle_\beta \langle \partial^{2n}\partial^l\mathcal{O}_b\rangle_\beta + \mathcal{O}(1), \tag{C.6.1}$$

which is simply due to large-N factorization. As $\partial^{2n}\partial^l\mathcal{O}_b$ is a descendant of $\mathcal{O}_b$, it is easy to explicitly show that $\langle \partial^{2n}\partial^l\mathcal{O}_b\rangle_\beta = 0$ for $n \neq 0$ or $l \neq 0$, from which it follows that

$$\langle \mathcal{O}_a\partial^{2n}\partial^l\mathcal{O}_b\rangle_\beta = \mathcal{O}(1). \tag{C.6.2}$$

Similar reasoning holds for all terms in $[\mathcal{O}_a\mathcal{O}_b]_{n,l}$, so we conclude for $n \neq 0$ or $l \neq 0$ that

$$\langle [\mathcal{O}_a\mathcal{O}_b]_{n,l}\rangle_\beta = \mathcal{O}(1). \tag{C.6.3}$$

It is easy to generalise (n and/or l non-zero)

$$\langle [\mathcal{O}_{a_1}\ldots\mathcal{O}_{a_k}]_{n,l}\rangle_\beta = \mathcal{O}(N^{k-2}). \tag{C.6.4}$$

Using the canonical scaling for the OPE coefficients (7.95) it is found that these multi-trace operators give a suppressed contribution to the thermal two point function in the large-N limit:

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta[\mathcal{O}_{a_1}\ldots\mathcal{O}_{a_k}]_{n,l}} \langle [\mathcal{O}_{a_1}\ldots\mathcal{O}_{a_k}]_{n,l}\rangle_\beta = \mathcal{O}\Big(\frac{1}{N^2}\Big). \tag{C.6.5}$$

The conclusion is that these operators with $n \neq 0$ or $l \neq 0$ do not contribute to the thermal two-point functions to leading order in N. Note that for $n = l = 0$, the operator is just $:\mathcal{O}_{a_1}\mathcal{O}_{a_2}\dots\mathcal{O}_{a_k}:$ and it does contribute to the thermal 2pt function since

$$\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta [\mathcal{O}_{a_1}\dots\mathcal{O}_{a_k}]_{n=0,l=0}} \langle [\mathcal{O}_{a_1}\dots\mathcal{O}_{a_k}]_{n=0,l=0}\rangle_\beta = \mathcal{O}(1). \tag{C.6.6}$$

From (C.6.5) it is seen that multi stress tensor operators of the schematic form $[T^k]_{n,l}$ with either n or l, or both, being non-zero will not contribute to the thermal correlator to leading order in N on the plane.

Appendix C.7. Free boson in two dimensions

In this appendix we discuss free scalars in two dimensions. We first consider a single scalar and then the case of the $SU(N)$ adjoint scalar. We compute two-point functions of a particular class of quasi-primary operators at finite temperature $1/\beta$. These two-point functions are not determined by the conformal symmetry, because the quasi-primary operators do not transform covariantly from the plane to the cylinder. They transform covariantly only with respect to the global conformal transformations. The only operators that have the non-zero thermal one-point functions are the Virasoro descendants of the vacuum and therefore, only these operators contribute to the thermal two-point function of the quasi-primary operators[53]. Virasoro descendants of the vacuum have different OPE coefficients with external quasi-primary operators compared with the case when primary external operators are considered.[54]

C.7.1. Review free boson in two dimensions

We consider single free boson $\phi(z)$ in two dimensions. The stress tensor can be written in terms of Virasoro modes as

$$T(z) = \sqrt{2}\sum_n z^{-n-2} L_n. \tag{C.7.1}$$

[53] We check this explicitly up to the $\mathcal{O}(1/\beta^4)$.

[54] Deviation from the Virasoro vacuum block in the Regge limit of four-point HHLL correlator is observed in [200] as well.

This stress tensor is unit-normalized

$$\langle T(z)T(w)\rangle = \frac{1}{(z-w)^4}. \tag{C.7.2}$$

The fundamental field can be expressed as Laurent series

$$\partial\phi(z) = \sum_{n=-\infty}^{+\infty} z^{-n-1}\alpha_n, \tag{C.7.3}$$

where oscillators α_n obey the following algebra

$$[\alpha_n, \alpha_m] = n\delta_{n+m,0}. \tag{C.7.4}$$

They act on the vacuum as

$$\alpha_n|0\rangle = 0, \qquad n \geq 0. \tag{C.7.5}$$

The two-point function of the fundamental fields is given by

$$\langle \partial\phi(z)\partial\phi(w)\rangle = \frac{1}{(z-w)^2}. \tag{C.7.6}$$

The unit-normalized stress tensor can be expressed in terms of the fundamental field as

$$T(z) = \frac{1}{\sqrt{2}} : \partial\phi\partial\phi : (z) = \frac{1}{\sqrt{2}}\sum_{m,n} z^{-m-n-2} : \alpha_m\alpha_n :, \tag{C.7.7}$$

where $: ab :$ denotes product of operators a and b with the corresponding free theory oscillators being normally ordered such that the operators annihilating the vacuum are put at the rightmost position. Then, it follows

$$L_n = \frac{1}{2}\sum_m : \alpha_{n-m}\alpha_m := \frac{1}{2}\left(\sum_{m\geq 0}\alpha_{n-m}\alpha_m + \sum_{m<0}\alpha_m\alpha_{n-m}\right). \tag{C.7.8}$$

C.7.2. Thermal two-point function of quasi-primary operator

We are interested in computing the thermal two-point function of quasi-primary operators at temperature $1/\beta$. Quasi-primary operators $\mathcal{O}(z)$ are defined as $[L_1, \mathcal{O}(z)] = 0$, or equivalently, in therms of their asymptotic in-states $\mathcal{O}(0)|0\rangle = |\mathcal{O}\rangle$, as $L_1|\mathcal{O}\rangle = 0$. We denote the quantum numbers of quasi-primary operators that correspond to eigenvalues of L_0 and $\bar{L}_0$ by $(h, \bar{h})$. We consider the following unit-normalized quasi-primary operator with quantum numbers $(h, 0)$

$$\mathcal{O}_h(z) = \frac{1}{\sqrt{h!}} : (\partial\phi)^h : (z) = \frac{1}{\sqrt{h!}} \sum_{m_1, m_2, \ldots, m_h} z^{-\sum_{i=1}^h m_i - h} : \alpha_{m_1} \ldots \alpha_{m_h} :, \tag{C.7.9}$$

which is properly defined when h is a positive integer. Its asymptotic in-state is given by

$$|\mathcal{O}_h\rangle = \mathcal{O}_h(0)|0\rangle = \frac{1}{\sqrt{h!}}(\alpha_{-1})^h|0\rangle. \tag{C.7.10}$$

One can check that this operator is a quasi-primary but not a Virasoro primary.

The thermal two-point function of this operator for even h is given by

$$\begin{aligned}\langle \mathcal{O}_h(z)\mathcal{O}_h(0)\rangle_\beta = &\sum_{n=0}^{\frac{1}{2}(h-2)} \frac{h!}{4^n(h-2n)!}\left(\frac{2\zeta(2)}{\beta^2}\right)^{2n}\left(\sum_{m=-\infty}^{\infty}\frac{1}{(z+m\beta)^2}\right)^{h-2n} \\ &+ \frac{2^h\pi}{\Gamma\left(\frac{1}{2}-\frac{h}{2}\right)^2\Gamma(h+1)}\left(\frac{2\zeta(2)}{\beta^2}\right)^h.\end{aligned} \tag{C.7.11}$$

This expression is obtained by writing all possible Wick contractions between fundamental fields $\partial\phi$, including those that belong to same operator $\mathcal{O}_h$, that we call self-contractions. Fundamental fields are separated along the thermal circle in all Wick contractions. Factors $\left(\frac{2\zeta(2)}{\beta^2}\right)$ are due to the self-contractions,

$$\sum_{m=-\infty, m\neq 0}^{\infty} \frac{1}{\beta^2 m^2} = \left(\frac{2\zeta(2)}{\beta^2}\right). \tag{C.7.12}$$

The sum over n comes from doing n self-contractions within each of the external operators. Term $\frac{h!}{4^n(h-2n)!}$ counts the number of Wick contractions with n self-contractions for each external operator, including $1/\sqrt{h!}$ normalization factors. The term in the second line of (C.7.11) is due to the case when we take $n = h/2$ self-contractions in both external operators, i.e. it represents the disconnected contribution.

Since the state $\mathcal{O}_h$ is quasi-primary, it transforms properly only with respect to the global conformal transformation. These are just the Möbius transformations in two-dimensional spacetime $z \to \frac{az+b}{cz+d}$, with $ad - bc = 1$. On the other hand, the usual way to calculate the thermal two-point function of primary operators in two dimensions is to do a conformal transformation from the plane to the cylinder with radius β, $z \to \frac{\beta}{2\pi}\log(z)$. This transformation is clearly not one of the Möbius transformations and that is why we can not use this method to compute the thermal two-point functions of quasi-primary operators.

Expanding (C.7.11) for $T = \frac{1}{\beta} \to 0$ one finds

$$z^{2h}\langle \mathcal{O}_h(z)\mathcal{O}_h(0)\rangle_\beta = 1 + \frac{h}{3}\frac{(\pi z)^2}{\beta^2} + \frac{h(h-\frac{1}{5})}{12}\frac{(\pi z)^4}{\beta^4} + \mathcal{O}\left(\frac{1}{\beta^6}\right). \quad \text{(C.7.13)}$$

C.7.3. Quasi-primaries, OPE coefficients, and thermal one-point functions

In expansion (C.7.13), terms $\mathcal{O}(z^{h_1})$ are due to the quasi-primary operator with quantum numbers $(h_1, 0)$ in the operator product expansion $\mathcal{O}_h \times \mathcal{O}_h$. Identity in the expansion is due to the identity operator. We show that the second term on the RHS is due to the stress tensor. The quantum numbers of stress tensor $T(z)$ are $(2, 0)$. First, we evaluate the thermal one-point function of the stress tensor

$$\langle T\rangle_\beta = \frac{1}{\sqrt{2}}\sum_{m=-\infty, m\neq 0}^{\infty}\frac{1}{\beta^2 m^2} = \frac{\pi^2}{3\sqrt{2}\beta^2}. \quad \text{(C.7.14)}$$

This is obtained by the Wick contractions of fundamental fields in the stress tensor, that are separated along the thermal circle. The same result can be obtained by the transform of the stress tensor from the plane to the cylinder using the Schwarzian derivative.

We define the OPE coefficient of unit-normalized operator $\mathcal{O}$, with quantum numbers $(h_\mathcal{O}, 0)$, with two $\mathcal{O}_h$ operators as

$$\langle \mathcal{O}_h(z_1)\mathcal{O}_h(z_2)\mathcal{O}(z_3)\rangle = \frac{\lambda_{\mathcal{O}_h\mathcal{O}_h\mathcal{O}}}{(z_1-z_3)^{h_\mathcal{O}}(z_2-z_3)^{h_\mathcal{O}}(z_1-z_2)^{2h-h_\mathcal{O}}}. \quad \text{(C.7.15)}$$

Next, we evaluate its OPE coefficient of the stress tensor with $\mathcal{O}_h$ by doing the Wick contractions between fundamental fields

$$\langle \mathcal{O}_h(z_1)\mathcal{O}_h(z_2)T(z_3)\rangle = \sqrt{2}h\frac{1}{(z_1-z_3)^2(z_2-z_3)^2(z_1-z_2)^{2(h-1)}}, \quad \text{(C.7.16)}$$

therefore $\lambda_{\mathcal{O}_h\mathcal{O}_h T} = \sqrt{2}h$. This OPE coefficient is fixed by the Ward identity. Now, it follows

$$z^2\lambda_{\mathcal{O}_h\mathcal{O}_h T}\langle T\rangle_\beta = \frac{h}{3}\frac{(\pi z)^2}{\beta^2}, \quad \text{(C.7.17)}$$

which reproduces the second term on the RHS of (C.7.13).

We are now interested in the contributions of quasi-primary operators with quantum numbers $(4,0)$. There are only two linearly independent operators with these quantum numbers given by[55]

$$:TT:(z) = \frac{1}{\sqrt{24}} : (\partial\phi)^4 : (z) = \frac{1}{\sqrt{24}} \sum_{a,b,c,d} z^{-a-b-c-d-4} : \alpha_a \alpha_b \alpha_c \alpha_d :, \tag{C.7.18}$$

$$\begin{aligned} \Lambda_4(z) = \sqrt{\frac{10}{27}} \Big(& \sum_{m,n=-\infty}^{\infty} z^{-m-n-4} * L_m L_n * \\ & - \frac{3}{10} \sum_{m=-\infty}^{\infty} z^{-m-4} (m+2)(m+3) L_m \Big), \end{aligned} \tag{C.7.19}$$

where $*ab*$ denotes the product where the relevant Virasoro generators are normally ordered. It should be noted that the operator $\Lambda_4(z)$ is Virasoro descendant of unity, while $:TT:(z)$ is not. The relevant asymptotic in-states are given by

$$\begin{aligned} |:TT:\rangle =: TT:(0)|0\rangle &= \frac{1}{\sqrt{24}} (\alpha_{-1})^4 |0\rangle, \\ |\Lambda_4\rangle = \Lambda_4(0)|0\rangle &= \sqrt{\frac{10}{27}} \left(L_{-2}^2 - \frac{3}{5} L_{-4} \right) |0\rangle. \end{aligned} \tag{C.7.20}$$

In terms of oscillators, $|\Lambda_4\rangle$ state can be represented as

$$|\Lambda_4\rangle = \sqrt{\frac{10}{27}} \left(\frac{1}{4} (\alpha_{-1})^4 + \frac{2}{5} \alpha_{-1} \alpha_{-3} - \frac{3}{10} (\alpha_{-2})^2 \right) |0\rangle. \tag{C.7.21}$$

From eqs. (C.7.20) and (C.7.21) one can see that $|:TT:\rangle$ and $|\Lambda_4\rangle$ are the only quasi-primary states with quantum numbers $(4,0)$. Namely, all such states have to be linear combinations of the following states

$$\alpha_{-4}|0\rangle, \qquad \alpha_{-3}\alpha_{-1}|0\rangle, \qquad \alpha_{-2}^2|0\rangle, \qquad \alpha_{-2}\alpha_{-1}^2|0\rangle, \qquad \alpha_{-1}^4|0\rangle, \tag{C.7.22}$$

because

$$L_0 \left(\prod_{i=1}^{N} \alpha_{-k_i} \right) |0\rangle = \left(\sum_{i=1}^{N} k_i \right) \left(\prod_{i=1}^{N} \alpha_{-k_i} \right) |0\rangle, \tag{C.7.23}$$

[55] Both of them are unit-normalized.

where $k_i > 0$. It is straightforward to check

$$\begin{aligned} L_1\alpha_{-4}|0\rangle &= 4\alpha_{-3}|0\rangle, \\ L_1\alpha_{-3}\alpha_{-1}|0\rangle &= 3\alpha_{-2}\alpha_{-1}|0\rangle, \\ L_1\alpha_{-2}^2|0\rangle &= 4\alpha_{-2}\alpha_{-1}|0\rangle, \\ L_1\alpha_{-2}\alpha_{-1}^2|0\rangle &= 2\alpha_{-1}^3|0\rangle, \\ L_1\alpha_{-1}^4|0\rangle &= 0. \end{aligned} \tag{C.7.24}$$

It follows that $\alpha_{-1}^4|0\rangle$ is already quasi-primary and one can make only one more as $\alpha_{-3}\alpha_{-1}|0\rangle - \frac{3}{4}\alpha_{-2}\alpha_{-2}|0\rangle$.[56] $|:TT:\rangle$ and $|\Lambda_4\rangle$ are just the linear combination of these two states with overall normalization.

Now, one can calculate the overlap of $|:TT:\rangle$ and $|\Lambda_4\rangle$ states as

$$\langle 0|\Lambda_4(0):TT:(0)|0\rangle = \frac{\sqrt{5}}{3}. \tag{C.7.25}$$

The state orthogonal to $|\Lambda_4\rangle$ can be written as

$$|\tilde{\Lambda}_4\rangle = \frac{3}{2}\left(:TT:(0) - \frac{\sqrt{5}}{3}\Lambda_4(0)\right)|0\rangle. \tag{C.7.26}$$

Using (C.7.20) and (C.7.21), it can be written in terms of free theory oscillators.

We compute the OPE coefficients of $:TT:$ and Λ_4 with two $\mathcal{O}_h$ operators. We express all states in terms of free theory oscillators and use algebra (C.7.4) to find

$$\lambda_{\mathcal{O}_h\mathcal{O}_h:TT:} = \langle\mathcal{O}_h|\mathcal{O}_h(1)|:TT:\rangle = \frac{\sqrt{6}}{2}h(h-1), \tag{C.7.27}$$

$$\lambda_{\mathcal{O}_h\mathcal{O}_h\Lambda_4} = \langle\mathcal{O}_h|\mathcal{O}_h(1)|\Lambda_4\rangle = \sqrt{\frac{5}{6}}h\left(h-\frac{1}{5}\right), \tag{C.7.28}$$

$$\lambda_{\mathcal{O}_h\mathcal{O}_h\tilde{\Lambda}_4} = \langle\mathcal{O}_h|\mathcal{O}_h(1)|\tilde{\Lambda}_4\rangle = \frac{2}{\sqrt{6}}h\,(h-2)\,. \tag{C.7.29}$$

Now, we evaluate the thermal one-point functions of Λ_4 and $\tilde{\Lambda}_4$. From (3.4) in [179] we have

$$\langle *T^2*\rangle_\beta = \frac{3\pi^4}{20\beta^4}, \tag{C.7.30}$$

[56] These states are not unit-normalized.

which is the thermal one-point function of the first term on the RHS of (C.7.19). The second term can be written as $-\frac{3}{10}\sum_{m=-\infty}^{\infty} z^{-m-4}(m+2)(m+3)L_m = -\frac{3}{10\sqrt{2}}\partial^2 T(z)$. It is clear that it will not affect the thermal one-point function of $\Lambda_4(z)$, as $\langle \partial^2 T\rangle_\beta = 0$.

Therefore, from (C.7.19), we have

$$\langle \Lambda_4\rangle_\beta = \sqrt{\frac{10}{27}}\langle *T^2*\rangle_\beta = \frac{\pi^4}{2\sqrt{30}\beta^4}. \tag{C.7.31}$$

Now, it follows

$$z^4\langle \Lambda_4\rangle_\beta \lambda_{\mathcal{O}_h\mathcal{O}_h\Lambda_4} = \frac{\pi^4 z^4}{12\beta^4}h\left(h-\frac{1}{5}\right), \tag{C.7.32}$$

which is the third therm at the RHS of (C.7.13). On the other hand, we can evaluate the thermal one-point function of $:TT:(z)$ operator by Wick contractions of fundamental fields separated along the thermal circle

$$\langle :TT:\rangle_\beta = \frac{\pi^4}{6\sqrt{6}\beta^4}. \tag{C.7.33}$$

Using (C.7.26), it is straightforward to confirm that $\langle \tilde{\Lambda}_4\rangle_\beta = 0$. Therefore, as we expected, operator $\tilde{\Lambda}_4$ does not contribute to the thermal two-point function of $\mathcal{O}_h$ operators, even thought it is present in the operator product expansion $\mathcal{O}_h \times \mathcal{O}_h$.

This is a general property of two-dimensional CFTs, that only the operators in the Virasoro vacuum module have non-zero expectation value on the cylinder.

C.7.4. Free adjoint scalar model in two dimensions

In this subsection we study a large-c theory. Consider the free adjoint $SU(N)$ scalar in 2d with

$$\partial\phi(z)^a{}_b = \sum_m z^{-m-1}(\alpha_m)^a{}_b \tag{C.7.34}$$

with

$$[(\alpha_m)^a{}_b, (\alpha_n)^c{}_d] = m\delta_{m+n}\Big(\delta^a{}_d\delta^c{}_b - \frac{1}{N}\delta^a{}_b\delta^c{}_d\Big). \tag{C.7.35}$$

The thermal two point of the quasi-primary operator $\mathcal{O}_h = \frac{1}{\sqrt{hN^h}} : Tr((\partial\phi)^h):$ follows immediately from the result in four dimensions upon replacing the propagator of fundamental fields. We find that

$$\langle \mathcal{O}_h(z)\mathcal{O}_h(0)\rangle_\beta = g_{2d}(z)^h + \frac{\pi^4 h(h-2)}{9\beta^4}g_{2d}(z)^{h-2} + \ldots, \tag{C.7.36}$$

where

$$g_{2d}(z) = \sum_{m=-\infty}^{\infty} \frac{1}{(z+m\beta)^2} \\ = \Big(\frac{\pi}{\beta\sin(\pi z/\beta)}\Big)^2. \tag{C.7.37}$$

Expanding (C.7.36) for $\beta \to \infty$ we find

$$\langle \mathcal{O}_h(z)\mathcal{O}_h(0)\rangle_\beta = z^{-2h}\Big[1 + \frac{\pi^2 h}{3\beta^2}z^2 + \frac{\pi^4 h(15h-19)}{90\beta^4}z^4 + \mathcal{O}(\beta^{-6})\Big]. \tag{C.7.38}$$

Consider first the normalized stress tensor which is given by

$$T = \frac{1}{\sqrt{2}N} : Tr(\partial\phi\partial\phi) :, \tag{C.7.39}$$

with $c = N^2$ so that $\langle T(z)T(0)\rangle = \frac{1}{z^4}$. By calculating the OPE coefficient with $\mathcal{O}_h$ and the thermal one-point function of T, one finds that these are the same as those for the scalar $Tr(\phi^2)$ operator in four dimensions so that $\langle T\rangle_\beta = \frac{\pi^2 N}{3\sqrt{2}\beta^2}$ and $\lambda_{\mathcal{O}_h\mathcal{O}_h T} = \frac{\sqrt{2}h}{N}$, and the product reproduces the weight two term in (C.7.38):

$$\langle T\rangle_\beta \lambda_{\mathcal{O}_h\mathcal{O}_h T} = \frac{\pi^2 h}{3\beta^2}. \tag{C.7.40}$$

Consider now $*TT*$ defined by

$$*TT*(0) = \lim_{z\to 0} T(z)T(0) - (\text{sing. terms}). \tag{C.7.41}$$

The OPE of the stress tensor in (C.7.39) can be found in the free theory by first performing Wick contractions

$$T(z)T(0) = \frac{1}{2N^2} : Tr(\partial\phi(z)\partial\phi(z)) :: Tr(\partial\phi(0)\partial\phi(0)) : \\ =: TT : (0) + \ldots + \frac{2}{N^2 z^2} : Tr(\partial\phi(z)\partial\phi(0)) : + \frac{1}{z^4}, \tag{C.7.42}$$

and expanding the second term in (C.7.42) for $z \to 0$ we find

$$T(z)T(0) =: TT : (0) + \ldots + \frac{2}{N^2 z^2} : Tr(\partial\phi(0)\partial\phi(0)) : \\ + \frac{2}{N^2 z} : Tr(\partial^2\phi(0)\partial\phi(0)) : + \frac{1}{N^2} : Tr(\partial^3\phi(0)\partial\phi(0)) : + \ldots \\ + \frac{1}{z^4}, \tag{C.7.43}$$

where the dots refer to higher order terms in z. Inserting the OPE (C.7.43) in (C.7.41) we find that

$$*TT*(0) =: TT:(0) + \frac{1}{N^2} : Tr(\partial^3\phi(0)\partial\phi(0)) : . \tag{C.7.44}$$

Consider the state $*TT*(0)|0\rangle$, which is given in terms of oscillator modes by

$$*TT*(0)|0\rangle = \frac{1}{2N^2} Tr(\alpha_{-1}^2) Tr(\alpha_{-1}^2)|0\rangle + 2\frac{1}{N^2} Tr(\alpha_{-3}\alpha_{-1})|0\rangle. \tag{C.7.45}$$

Now $Tr(\alpha_{-1}^m)|0\rangle$ is a quasi-primary while $Tr(\alpha_{-3}\alpha_{-1})|0\rangle$ is not. One way to make it a quasi-primary is to simply remove the second term in (C.7.45) and then we get a quasi-primary state which is just $:TT:|0\rangle$. Another option is to remove a descendant of the stress tensor to construct $|\Lambda_4\rangle$. To do the latter we need to remove the descendant of the stress tensor with weight 4 given by $\partial^2 T$

$$\partial^2 T = \frac{\sqrt{2}}{N} : Tr(\partial^3\phi\partial\phi) : + \frac{\sqrt{2}}{N} : Tr(\partial^2\phi\partial^2\phi) : . \tag{C.7.46}$$

Acting on the vacuum we find

$$\partial^2 T(0)|0\rangle = \frac{2\sqrt{2}}{N} Tr(\alpha_{-3}\alpha_{-1})|0\rangle + \frac{\sqrt{2}}{N} Tr(\alpha_{-2}^2)|0\rangle. \tag{C.7.47}$$

Consider now $L_1 = \frac{\sqrt{2}}{N}(Tr(\alpha_{-1}\alpha_2) + Tr(\alpha_{-2}\alpha_3 + \ldots))$ which acts as $L_1 Tr(\alpha_{-2}^2)|0\rangle = \frac{4\sqrt{2}}{N} Tr(\alpha_{-1}\alpha_{-2})|0\rangle$ and as $L_1 Tr(\alpha_{-3}\alpha_{-1})|0\rangle = \frac{3\sqrt{2}}{N} Tr(\alpha_{-1}\alpha_{-2})|0\rangle$. We can therefore construct a quasi-primary state annihilated by L_1: $Tr(\alpha_{-3}\alpha_{-1})|0\rangle - \frac{3}{4} Tr(\alpha_{-2}^2)|0\rangle$. The quasi-primary $|\Lambda_4\rangle$ is then given by:

$$\begin{aligned} |\Lambda_4\rangle &= \frac{1}{\sqrt{2}}\Big(*TT*(0)|0\rangle - \frac{3}{5\sqrt{2}N}\partial^2 T(0)|0\rangle\Big) \\ &= \frac{1}{2\sqrt{2}N^2}\Big(Tr(\alpha_{-1}^2)Tr(\alpha_{-1}^2)|0\rangle - \frac{6}{5}Tr(\alpha_{-2}^2)|0\rangle + \frac{8}{5}Tr(\alpha_{-1}\alpha_{-3})|0\rangle\Big) \end{aligned} \tag{C.7.48}$$

There are two more weight 4 single trace quasi-primary operators given by

$$\begin{aligned} \mathcal{O}^{(1)} &= \frac{1}{2N^2} Tr((\partial\phi)^4) \\ \mathcal{O}^{(2)} &= \frac{n_{\mathcal{O}^{(2)}}}{N}(Tr(\partial^3\phi\partial\phi) - \frac{3}{2}Tr(\partial^2\phi\partial^2\phi)), \\ &= \frac{n_{\mathcal{O}^{(2)}}}{N}(\frac{1}{2}\partial^2 Tr(\partial\phi\partial\phi) - \frac{5}{2}Tr(\partial^2\phi\partial^2\phi)), \end{aligned} \tag{C.7.49}$$

where $n_{\mathcal{O}^{(2)}}$ is some N-independent normalization constant. The state $|\Lambda_4\rangle$ can be written in terms of $:TT:(0)|0\rangle + a\mathcal{O}_2(0)|0\rangle$ in the following way

$$|\Lambda_4\rangle = \frac{1}{\sqrt{2}}\Big(:TT:(0)|0\rangle + \frac{2}{5Nn_{\mathcal{O}^{(2)}}}\mathcal{O}^{(2)}|0\rangle\Big). \tag{C.7.50}$$

The OPE coefficient for $:TT:$ is up to a normalization the same as the scalar dimension 4 double trace operator in 4d and is given by

$$\begin{aligned}\langle \mathcal{O}_h\mathcal{O}_h :TT:\rangle &= \frac{1}{hN^h}\frac{1}{2N^2}4h^2(3h-5)N^h\frac{1}{z_{13}^4 z_{23}^4 z_{12}^{2h-4}}\\ &= \frac{1}{N^2}2h(3h-5)\frac{1}{z_{13}^4 z_{23}^4 z_{12}^{2h-4}},\end{aligned} \tag{C.7.51}$$

where $4h^2(3h-5)$ come from the number of contractions giving planar diagrams. Consider now the OPE coefficient for $\mathcal{O}^{(2)}$. One finds

$$\begin{aligned}\langle \mathcal{O}_h\mathcal{O}_h\mathcal{O}^{(2)}\rangle = &\frac{n_{\mathcal{O}^{(2)}}N^h}{hN^{h+1}z_{13}^4 z_{23}^4 z_{12}^{2h-2}}\Big((-2)(-3)h^2(z_{13}^2+z_{23}^2)\\ &-\frac{3}{2}2h^2(-2)^2)z_{13}z_{23}\Big) = \frac{6hn_{\mathcal{O}^{(2)}}}{Nz_{13}^4 z_{23}^4 z_{12}^{2h-4}}.\end{aligned} \tag{C.7.52}$$

Using (C.7.51), (C.7.52) and (C.7.50) we find the OPE coefficient for $|\Lambda_4\rangle$

$$\langle \mathcal{O}_h\mathcal{O}_h\Lambda_4\rangle = \frac{\sqrt{2}h(15h-19)}{5N^2}. \tag{C.7.53}$$

Note that the h dependence matches that of the weight 4 term in the two-point function (C.7.38). Additionally, the OPE coefficient given by (C.7.53) can not be extrapolated to the limit when $h \sim C_T$, as in this limit the planar expansion used for calculating (C.7.53) breaks down. For this reason, we can not test the thermalization of Λ_4 in heavy state $\mathcal{O}_{h_H}$. Let us consider the thermal one-point function which is given by

$$\langle \Lambda_4\rangle_\beta = \Big[\frac{1}{\sqrt{2}}b_T^2 + \mathcal{O}(1)\Big] = \frac{\pi^4 N^2}{18\sqrt{2}\beta^4}, \tag{C.7.54}$$

where the term $\propto \frac{1}{N}\langle\mathcal{O}^{(2)}\rangle_\beta$ is subleading since it is single trace. We find that

$$\langle \Lambda_4\rangle_\beta \lambda_{\mathcal{O}_h\mathcal{O}_h\Lambda_4} = \frac{\pi^4 h(15h-19)}{90\beta^4}, \tag{C.7.55}$$

which agrees with the weight 4 term in (C.7.38).

Note that it is explicitly seen that one can write Λ_4 either as $*TT*$ +(desc. of T) or as $:TT: + \frac{1}{N}\mathcal{O}_{ST}$ with $\mathcal{O}_{ST}$ a quasi-primary single trace operator. In this case the single trace operator which one needs to add to $:TT:$ to get Λ_4 can be written as a sum of descendants $\mathcal{O}^{(2)} \propto \partial^2 T - \frac{5}{\sqrt{2}} Tr(\partial^2\phi\partial^2\phi)$. Explicitly, we have

$$\begin{aligned}|\Lambda_4\rangle &= \frac{1}{\sqrt{2}}\Big[*TT*(0) - \frac{3}{5\sqrt{2}N}\partial^2 T(0)\Big]|0\rangle \\ &= \frac{1}{\sqrt{2}}\Big[:TT:(0) + \frac{2}{5Nn_{\mathcal{O}^{(2)}}}\mathcal{O}^{(2)}\Big]|0\rangle.\end{aligned} \tag{C.7.56}$$

As we saw above, using the second line in (C.7.56) it is straightforward to calculate correlation functions using Wick contractions to see that Λ_4 gives the full weight four contributions to the thermal two-point function for large-N theories.

Now, we consider the following quasi-primary operator

$$\mathcal{O}_\Delta(z,\bar{z}) = \frac{\sqrt{2}}{\sqrt{\Delta}N^{\Delta/2}} : Tr\left((\partial\phi\bar{\partial}\bar{\phi})^{\frac{\Delta}{2}}\right) : (z,\bar{z}), \tag{C.7.57}$$

where we denote the anti-holomorphic part of the free field by $\bar{\phi} = \bar{\phi}(\bar{z})$. The thermal two-point function of this operator, up to the terms subleading in large-N expansion, is given by

$$\begin{aligned}\langle\mathcal{O}_\Delta(z,\bar{z})\mathcal{O}_\Delta(0,0)\rangle_\beta &= \frac{\pi^{2\Delta}}{\beta^{2\Delta}\sin^\Delta\left(\frac{\pi z}{\beta}\right)\sin^\Delta\left(\frac{\pi\bar{z}}{\beta}\right)} \\ &= \frac{1}{(z\bar{z})^\Delta}\Big(1 + \frac{\pi^2\Delta(z^2+\bar{z}^2)}{6\beta^2} + \frac{\pi^4\Delta(5\Delta+2)}{360\beta^4}(z^4+\bar{z}^4) \\ &+ \frac{\pi^4\Delta^2}{36\beta^4}z^2\bar{z}^2 + \mathcal{O}\left(\frac{1}{\beta^6}\right)\Big).\end{aligned} \tag{C.7.58}$$

One can easily check that the OPE coefficients of stress tensor T and its anti-holomorphic partner $\bar{T}$ with $\mathcal{O}_\Delta$ are given by

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T} = \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\bar{T}} = \frac{\Delta}{\sqrt{2}N}, \tag{C.7.59}$$

while their thermal one-point function are given by

$$\langle T\rangle_\beta = \langle\bar{T}\rangle_\beta = \frac{\pi^2 N}{3\sqrt{2}\beta^2}. \tag{C.7.60}$$

It is easy to check that terms proportional to β^{-2} in (C.7.58) are contributions of T and $\bar{T}$ operators

$$\langle T\rangle_\beta \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T} z^2 + \langle \bar{T}\rangle_\beta \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta \bar{T}} \bar{z}^2 = \frac{\pi^2\Delta(z^2+\bar{z}^2)}{6\beta^2}. \tag{C.7.61}$$

We compute the OPE coefficient of operators Λ_4, defined by (C.7.48), and its anti-holomorphic partner $\bar{\Lambda}_4$ with $\mathcal{O}_\Delta$ and obtain

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\Lambda_4} = \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\bar{\Lambda}_4} = \frac{\Delta(5\Delta+2)}{10\sqrt{2}N^2}, \tag{C.7.62}$$

which agrees with (C.26) in [124]. Its thermal one-point function (which is the same as $\langle\bar{\Lambda}_4\rangle_\beta$) is given by (C.7.54). Another operator that contributes to thermal two-point function (C.7.58) is $:T\bar{T}:$. Its OPE coefficient with $\mathcal{O}_\Delta$ and thermal one-point function are given by

$$\begin{aligned} \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta :T\bar{T}:} &= \frac{\Delta^2}{2N^2} \\ \langle :T\bar{T}:\rangle_\beta &= \frac{\pi^4 N^2}{18\beta^4}. \end{aligned} \tag{C.7.63}$$

Again, it is easy to check

$$\begin{aligned} &\langle\Lambda_4\rangle_\beta \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\Lambda_4} z^4 + \langle\bar{\Lambda}_4\rangle_\beta \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\bar{\Lambda}_4} \bar{z}^4 + \langle :T\bar{T}:\rangle_\beta \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta :T\bar{T}:} z^2\bar{z}^2 = \\ &= \frac{\pi^4\Delta(5\Delta+2)}{360\beta^4}(z^4+\bar{z}^4) + \frac{\pi^4\Delta^2}{36\beta^4} z^2\bar{z}^2, \end{aligned} \tag{C.7.64}$$

which matches with the corresponding terms in (C.7.58).

The OPE coefficients $\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\Lambda_4}$, $\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\bar{\Lambda}_4}$, and $\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta :T\bar{T}:}$ can be extrapolated to the limit $\Delta \sim N^2$, by the same logic as in Appendix C. Then, we can explicitly check the thermalization property of Λ_4, $\bar{\Lambda}_4$, and $:T\bar{T}:$. To establish a relation between the inverse temperature β and the conformal dimension Δ_H of heavy state $\mathcal{O}_H = \mathcal{O}_{\Delta\sim N^2}$, we assume the thermalization of stress tensor

$$\langle T\rangle_\beta = \lambda_{\mathcal{O}_H\mathcal{O}_H T}, \tag{C.7.65}$$

which implies

$$\frac{\Delta_H}{N^2} = \frac{\pi^2}{3\beta^2}. \tag{C.7.66}$$

Using this relation, it is easy to show

$$\begin{aligned}\langle\Lambda_4\rangle_\beta &= \lambda_{\mathcal{O}_H\mathcal{O}_H\Lambda_4}\Big|_{\frac{\Delta_H^2}{N^2}},\\ \langle\bar{\Lambda}_4\rangle_\beta &= \lambda_{\mathcal{O}_H\mathcal{O}_H\bar{\Lambda}_4}\Big|_{\frac{\Delta_H^2}{N^2}},\\ \langle :T\bar{T}:\rangle_\beta &= \lambda_{\mathcal{O}_H\mathcal{O}_H:T\bar{T}:}\Big|_{\frac{\Delta_H^2}{N^2}}.\end{aligned} \tag{C.7.67}$$

This means that operators Λ_4, $\bar{\Lambda}_4$, and $:T\bar{T}:$ thermalize in the quasi-primary state $\mathcal{O}_H$ similarly to the thermalization in a Virasoro primary states in large-c theory, that was analyzed in [165].

Appendix C.8. Vector model

In this section we study the free scalar vector model at large-N. Consider the scalar operator

$$\mathcal{O}_\Delta = \frac{1}{\sqrt{\mathcal{N}(\Delta)}} : (\varphi^i\varphi^i)^{\frac{\Delta}{2}} : (x), \tag{C.8.1}$$

where $\mathcal{N}(\Delta)$ is a normalization constant which to leading order in N is given by

$$\mathcal{N}(\Delta) \approx (\Delta)!!N^{\frac{\Delta}{2}}. \tag{C.8.2}$$

The thermal two-point function is given by

$$\langle\mathcal{O}_\Delta(x)\mathcal{O}_\Delta(0)\rangle_\beta = \tilde{g}(x_E^0,|\mathbf{x}|)^\Delta + \left(\frac{\Delta}{2}\right)^2\frac{1}{\Delta}\tilde{g}(x_E^0,|\mathbf{x}|)^{\Delta-2} + \dots, \tag{C.8.3}$$

where

$$\begin{aligned}\tilde{g}(x_E^0,|\mathbf{x}|) &= \sum_{m=-\infty}^{\infty}\frac{1}{(x_E^0+m\beta)^2+\mathbf{x}^2}\\ &= \frac{\pi}{2\beta|\mathbf{x}|}\Big[\mathrm{Coth}\Big(\frac{\pi}{\beta}(|\mathbf{x}|-ix_E^0)\Big)+\mathrm{Coth}\Big(\frac{\pi}{\beta}(|\mathbf{x}|+ix_E^0)\Big)\Big].\end{aligned} \tag{C.8.4}$$

The thermal $a_{\tau,J}$ coefficients $a_{2,2}$ and $a_{4,4}$ are the same as in the adjoint model (this is so since the second term in (C.8.3) does not affect these):

$$\begin{aligned}a_{2,2} &= \frac{\pi^4\Delta}{45},\\ a_{4,4} &= \frac{\pi^8\Delta(\Delta-1)}{1050}.\end{aligned} \tag{C.8.5}$$

The unit-normalized stress tensor is given by

$$T_{\mu\nu}(x) = \frac{1}{3\sqrt{C_T}} : \left(\partial_\mu\varphi^i\partial_\nu\varphi^i - \frac{1}{2}\varphi^i\partial_\mu\partial_\nu\varphi^i - (\text{trace})\right) : (x), \tag{C.8.6}$$

where $C_T = \frac{4}{3}N$. The OPE coefficient of the stress tensor is again found by Wick contractions to be

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T_{\mu\nu}} = -\frac{4\Delta}{3\sqrt{C_T}}, \tag{C.8.7}$$

in agreement with the stress tensor Ward identity. The double-stress tensor is given by

$$T^2_{\mu\nu\rho\sigma} = \frac{1}{\sqrt{2}} : T_{(\mu\nu}T_{\rho\sigma)} : -(\text{traces}), \tag{C.8.8}$$

and the OPE coefficient is calculated precisely as for the adjoint model and we find

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{4,4}} = \frac{8\sqrt{2}}{9C_T}\Delta(\Delta-1). \tag{C.8.9}$$

There is another double-trace operator with twist 4 and spin 4 and takes the same form $:\mathcal{O}_2\mathcal{O}_{2,4}:$ as for the adjoint model

$$\begin{aligned}\mathcal{O}^{\text{DT}}_{\mu\nu\rho\sigma}(x) = \frac{1}{96\sqrt{70N}} : \varphi^i\varphi^i\Big(&\varphi^j\partial_\mu\partial_\nu\partial_\rho\partial_\sigma\varphi^j - 16\partial_{(\mu}\varphi^j\partial_\nu\partial_\rho\partial_{\sigma)}\varphi^j \\ &+18\partial_{(\mu}\partial_\nu\varphi^j\partial_\rho\partial_{\sigma)}\varphi^j - (\text{traces})\Big) : (x).\end{aligned} \tag{C.8.10}$$

The OPE coefficient and the thermal one-point function yields the same result as for the corresponding operator in the adjoint model[57]. It then follows that the $a_{4,4}$ extracted from (C.8.3) is reproduced by the sum of the double stress tensor and (C.8.10).

Appendix C.9. Factorization of thermal correlators

In this appendix we argue for the factorization of thermal expectation values of multi-trace operators in large-C_T theories on $S^1\times\mathbf{R}^{d-1}$. Consider the thermal two-point function of a scalar operator $\mathcal{O}$ with dimension Δ:

$$\langle\mathcal{O}(x)\mathcal{O}(0)\rangle_\beta = \langle\mathcal{O}\rangle_\beta\langle\mathcal{O}\rangle_\beta + \langle\mathcal{O}(x)\mathcal{O}(0)\rangle_{\beta,c}, \tag{C.9.1}$$

[57] Note that this is not true for all operators but is in line with the fact that $a_{4,4}$ is unaffected by the second term in (C.8.3).

where the second term consist of the connected part of the correlator. Note that the disconnected term in (C.9.1) is independent of the position x. On the other hand we can evaluate (C.9.1) using the OPE on the plane which takes the form

$$\mathcal{O}(x)\mathcal{O}(0) = \frac{1}{|x|^{2\Delta}} + \sum_{n,l} \lambda_{\mathcal{O}\mathcal{O}[\mathcal{O}\mathcal{O}]_{n,l}} x^{2n+l} [\mathcal{O}\mathcal{O}]_{n,l} + \dots, \tag{C.9.2}$$

when written in terms of primaries and the dots refer to terms surpressed in the large-C_T limit. Note that $\lambda_{\mathcal{O}\mathcal{O}[\mathcal{O}\mathcal{O}]_{n,l}}$ are the MFT OPE coefficient which are of order 1. The term in (C.9.2) that is independent of x is due to the $n = l = 0$ term in (C.9.2) and inserting the OPE on the LHS of (C.9.2), we find that

$$\lambda_{\mathcal{O}\mathcal{O}[\mathcal{O}\mathcal{O}]_{0,0}} \langle [\mathcal{O}\mathcal{O}]_{0,0} \rangle_\beta = \langle \mathcal{O} \rangle_\beta^2. \tag{C.9.3}$$

When $[\mathcal{O}\mathcal{O}]_{0,0}$ is unit-normalized the OPE coefficient is given by $\lambda_{\mathcal{O}\mathcal{O}[\mathcal{O}\mathcal{O}]_{0,0}} = \sqrt{2}$ and it follows that

$$\langle [\mathcal{O}\mathcal{O}]_{0,0} \rangle_\beta = \frac{1}{\sqrt{2}} \langle \mathcal{O} \rangle_\beta^2. \tag{C.9.4}$$

We therefore see the that the thermal one-point function of the double-trace operator factorizes on the plane. We expect a similar argument to hold for multi stress tensors.

References

[1] J. D. Bekenstein, "Black Holes and Entropy," Phys. Rev. D **7**, 2333 (1973).

[2] S. W. Hawking, "Black Holes and Thermodynamics," Phys. Rev. D **13**, 191 (1976).

[3] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975), Erratum: [Commun. Math. Phys. **46**, 206 (1976)].

[4] G. 't Hooft, "Dimensional reduction in quantum gravity," Conf. Proc. C **930308**, 284 (1993). [gr-qc/9310026].

[5] L. Susskind, "The World as a hologram," J. Math. Phys. **36**, 6377 (1995). [hep-th/9409089].

[6] G. 't Hooft, "A Planar Diagram Theory for Strong Interactions," Nucl. Phys. B **72**, 461 (1974)..

[7] J. D. Brown and M. Henneaux, "Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity," Commun. Math. Phys. **104**, 207 (1986).

[8] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," Int. J. Theor. Phys. **38**, 1113 (1999), [Adv. Theor. Math. Phys. **2**, 231 (1998)]. [hep-th/9711200].

[9] E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2**, 253 (1998). [hep-th/9802150].

[10] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "Gauge theory correlators from noncritical string theory," Phys. Lett. B **428**, 105 (1998). [hep-th/9802109].

[11] I. Heemskerk, J. Penedones, J. Polchinski and J. Sully, "Holography from Conformal Field Theory," JHEP **0910**, 079 (2009). [arXiv:0907.0151 [hep-th]].

[12] R. Karlsson, M. Kulaxizi, A. Parnachev and P. Tadić, "Black Holes and Conformal Regge Bootstrap," JHEP **1910**, 046 (2019). [arXiv:1904.00060 [hep-th]].

[13] R. Karlsson, M. Kulaxizi, A. Parnachev and P. Tadić, "Leading Multi-Stress Tensors and Conformal Bootstrap," JHEP **2001**, 076 (2020). [arXiv:1909.05775 [hep-th]].

[14] R. Karlsson, M. Kulaxizi, A. Parnachev and P. Tadić, "Stress tensor sector of conformal correlators operators in the Regge limit," JHEP **2007**, 019 (2020). [arXiv:2002.12254 [hep-th]].

[15] A. L. Fitzpatrick and K. W. Huang, "Universal Lowest-Twist in CFTs from Holography," [arXiv:1903.05306 [hep-th]].

[16] R. Karlsson, A. Parnachev and P. Tadić, "Thermalization in Large-N CFTs," [arXiv:2102.04953 [hep-th]].

[17] A. M. Polyakov, "Conformal symmetry of critical fluctuation," JETP Lett. **12**, 381-383 (1970).

[18] H. Osborn and A. C. Petkou, "Implications of conformal invariance in field theories for general dimensions," Annals Phys. **231**, 311 (1994). [hep-th/9307010].

[19] A. B. Zamolodchikov, "Irreversibility of the Flux of the Renormalization Group in a 2D Field Theory," JETP Lett. **43**, 730 (1986), [Pisma Zh. Eksp. Teor. Fiz. **43**, 565 (1986)]..

[20] S. Ferrara, A. F. Grillo and R. Gatto, "Tensor representations of conformal algebra and conformally covariant operator product expansion," Annals Phys. **76**, 161 (1973)..

[21] A. M. Polyakov, "Nonhamiltonian approach to conformal quantum field theory," Zh. Eksp. Teor. Fiz. **66**, 23 (1974), [Sov. Phys. JETP **39**, 9 (1974)]..

[22] R. Rattazzi, V. S. Rychkov, E. Tonni and A. Vichi, "Bounding scalar operator dimensions in 4D CFT," JHEP **0812**, 031 (2008). [arXiv:0807.0004 [hep-th]].

[23] S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin and A. Vichi, "Solving the 3D Ising Model with the Conformal Bootstrap," Phys. Rev. D **86**, 025022 (2012). [arXiv:1203.6064 [hep-th]].

[24] D. Poland, S. Rychkov and A. Vichi, "The Conformal Bootstrap: Theory, Numerical Techniques, and Applications," [arXiv:1805.04405 [hep-th]].

[25] D. Simmons-Duffin, "The Conformal Bootstrap," [arXiv:1602.07982 [hep-th]].

[26] S. Rychkov, "EPFL Lectures on Conformal Field Theory in D¿= 3 Dimensions," [arXiv:1601.05000 [hep-th]].

[27] J. Polchinski, "String theory. Vol. 1: An introduction to the bosonic string,"

[28] J. D. Qualls, "Lectures on Conformal Field Theory," [arXiv:1511.04074 [hep-th]].

[29] D. Pappadopulo, S. Rychkov, J. Espin and R. Rattazzi, "OPE Convergence in Conformal Field Theory," Phys. Rev. D **86**, 105043 (2012). [arXiv:1208.6449 [hep-th]].

[30] F. A. Dolan and H. Osborn, "Conformal four point functions and the operator product expansion," Nucl. Phys. B **599**, 459 (2001). [hep-th/0011040].

[31] F. A. Dolan and H. Osborn, "Conformal partial waves and the operator product expansion," Nucl. Phys. B **678**, 491 (2004). [hep-th/0309180].

[32] I. Heemskerk and J. Sully, "More Holography from Conformal Field Theory," JHEP **1009**, 099 (2010). [arXiv:1006.0976 [hep-th]].

[33] A. L. Fitzpatrick, E. Katz, D. Poland and D. Simmons-Duffin, "Effective Conformal Theory and the Flat-Space Limit of AdS," JHEP **1107**, 023 (2011). [arXiv:1007.2412 [hep-th]].

[34] J. Penedones, "Writing CFT correlation functions as AdS scattering amplitudes," JHEP **1103**, 025 (2011). [arXiv:1011.1485 [hep-th]].

[35] S. El-Showk and K. Papadodimas, "Emergent Spacetime and Holographic CFTs," JHEP **1210**, 106 (2012). [arXiv:1101.4163 [hep-th]].

[36] A. L. Fitzpatrick, J. Kaplan, J. Penedones, S. Raju and B. C. van Rees, "A Natural Language for AdS/CFT Correlators," JHEP **1111**, 095 (2011). [arXiv:1107.1499 [hep-th]].

[37] A. L. Fitzpatrick and J. Kaplan, "Analyticity and the Holographic S-Matrix," JHEP **1210**, 127 (2012). [arXiv:1111.6972 [hep-th]].

[38] A. L. Fitzpatrick and J. Kaplan, "Unitarity and the Holographic S-Matrix," JHEP **1210**, 032 (2012). [arXiv:1112.4845 [hep-th]].

[39] A. L. Fitzpatrick and J. Kaplan, "AdS Field Theory from Conformal Field Theory," JHEP **1302**, 054 (2013). [arXiv:1208.0337 [hep-th]].

[40] A. L. Fitzpatrick, J. Kaplan and M. T. Walters, "Universality of Long-Distance AdS Physics from the CFT Bootstrap," JHEP **1408**, 145 (2014). [arXiv:1403.6829 [hep-th]].

[41] V. Gonalves, J. Penedones and E. Trevisani, "Factorization of Mellin amplitudes," JHEP **1510**, 040 (2015). [arXiv:1410.4185 [hep-th]].

[42] L. F. Alday, A. Bissi and T. Lukowski, "Lessons from crossing symmetry at large N," JHEP **1506**, 074 (2015). [arXiv:1410.4717 [hep-th]].

[43] E. Hijano, P. Kraus, E. Perlmutter and R. Snively, "Witten Diagrams Revisited: The AdS Geometry of Conformal Blocks," JHEP **1601**, 146 (2016). [arXiv:1508.00501 [hep-th]].

[44] J. Maldacena, D. Simmons-Duffin and A. Zhiboedov, "Looking for a bulk point," JHEP **1701**, 013 (2017). [arXiv:1509.03612 [hep-th]].

[45] L. F. Alday and A. Bissi, "Unitarity and positivity constraints for CFT at large central charge," JHEP **1707**, 044 (2017). [arXiv:1606.09593 [hep-th]].

[46] L. Rastelli and X. Zhou, "Mellin amplitudes for $AdS_5 \times S^5$," Phys. Rev. Lett. **118**, no. 9, 091602 (2017). [arXiv:1608.06624 [hep-th]].

[47] O. Aharony, L. F. Alday, A. Bissi and E. Perlmutter, "Loops in AdS from Conformal Field Theory," JHEP **1707**, 036 (2017). [arXiv:1612.03891 [hep-th]].

[48] L. F. Alday, "Large Spin Perturbation Theory for Conformal Field Theories," Phys. Rev. Lett. **119**, no. 11, 111601 (2017). [arXiv:1611.01500 [hep-th]].

[49] L. F. Alday, A. Bissi and E. Perlmutter, "Holographic Reconstruction of AdS Exchanges from Crossing Symmetry," JHEP **1708**, 147 (2017). [arXiv:1705.02318 [hep-th]].

[50] L. F. Alday and A. Bissi, "Loop Corrections to Supergravity on $AdS_5 \times S^5$," Phys. Rev. Lett. **119**, no. 17, 171601 (2017). [arXiv:1706.02388 [hep-th]].

[51] F. Aprile, J. M. Drummond, P. Heslop and H. Paul, "Quantum Gravity from Conformal Field Theory," JHEP **1801**, 035 (2018). [arXiv:1706.02822 [hep-th]].

[52] L. Rastelli and X. Zhou, "How to Succeed at Holographic Correlators Without Really Trying," JHEP **1804**, 014 (2018). [arXiv:1710.05923 [hep-th]].

[53] L. F. Alday and S. Caron-Huot, "Gravitational S-matrix from CFT dispersion relations," JHEP **1812**, 017 (2018). [arXiv:1711.02031 [hep-th]].

[54] A. L. Fitzpatrick, K. W. Huang, D. Meltzer, E. Perlmutter and D. Simmons-Duffin, "Model-Dependence of Minimal-Twist OPEs in $d > 2$ Holographic CFTs," [arXiv:2007.07382 [hep-th]].

[55] M. Kulaxizi, G. S. Ng and A. Parnachev, "Black Holes, Heavy States, Phase Shift and Anomalous Dimensions," [arXiv:1812.03120 [hep-th]].

[56] A. L. Fitzpatrick, J. Kaplan, D. Poland and D. Simmons-Duffin, "The Analytic Bootstrap and AdS Superhorizon Locality," JHEP **1312**, 004 (2013). [arXiv:1212.3616 [hep-th]].

[57] Z. Komargodski and A. Zhiboedov, "Convexity and Liberation at Large Spin," JHEP **1311**, 140 (2013). [arXiv:1212.4103 [hep-th]].

[58] L. Cornalba, M. S. Costa, J. Penedones and R. Schiappa, "Eikonal Approximation in AdS/CFT: From Shock Waves to Four-Point Functions," JHEP **0708**, 019 (2007). [hep-th/0611122].

[59] L. Cornalba, M. S. Costa, J. Penedones and R. Schiappa, "Eikonal Approximation in AdS/CFT: Conformal Partial Waves and Finite N Four-Point Functions," Nucl. Phys. B **767**, 327 (2007). [hep-th/0611123].

[60] L. Cornalba, M. S. Costa and J. Penedones, "Eikonal approximation in AdS/CFT: Resumming the gravitational loop expansion," JHEP **0709**, 037 (2007). [arXiv:0707.0120 [hep-th]].

[61] L. Cornalba, "Eikonal methods in AdS/CFT: Regge theory and multi-reggeon exchange," [arXiv:0710.5480 [hep-th]].

[62] L. Cornalba, M. S. Costa and J. Penedones, "Deep Inelastic Scattering in Conformal QCD," JHEP **1003**, 133 (2010). [arXiv:0911.0043 [hep-th]].

[63] M. S. Costa, V. Goncalves and J. Penedones, "Conformal Regge theory," JHEP **1212**, 091 (2012). [arXiv:1209.4355 [hep-th]].

[64] R. C. Brower, J. Polchinski, M. J. Strassler and C. I. Tan, "The Pomeron and gauge/string duality," JHEP **0712**, 005 (2007). [hep-th/0603115].

[65] J. Bartels, A.-M. Mischler and M. Salvadore, “Four point function of R-currents in N=4 SYM in the Regge limit at weak coupling,” Phys. Rev. D **78**, 016004 (2008). [arXiv:0803.1423 [hep-ph]].

[66] J. Bartels, J. Kotanski, A.-M. Mischler and V. Schomerus, “Regge limit of R-current correlators in AdS Supergravity,” Nucl. Phys. B **830**, 153 (2010). [arXiv:0908.2301 [hep-th]].

[67] T. Banks and G. Festuccia, “The Regge Limit for Green Functions in Conformal Field Theory,” JHEP **1006**, 105 (2010). [arXiv:0910.2746 [hep-th]].

[68] M. Giordano, R. Peschanski and S. Seki, “Eikonal Approach to N=4 SYM Regge Amplitudes in the AdS/CFT Correspondence,” Acta Phys. Polon. B **43**, 1289 (2012). [arXiv:1110.3680 [hep-th]].

[69] M. S. Costa, J. Drummond, V. Goncalves and J. Penedones, “The role of leading twist operators in the Regge and Lorentzian OPE limits,” JHEP **1404**, 094 (2014). [arXiv:1311.4886 [hep-th]].

[70] J. Maldacena, S. H. Shenker and D. Stanford, “A bound on chaos,” JHEP **1608**, 106 (2016). [arXiv:1503.01409 [hep-th]].

[71] N. Afkhami-Jeddi, T. Hartman, S. Kundu and A. Tajdini, “Einstein gravity 3-point functions from conformal field theory,” JHEP **1712**, 049 (2017). [arXiv:1610.09378 [hep-th]].

[72] C. Cardona, Y. t. Huang and T. H. Tsai, “On the linearity of Regge trajectory at large transfer energy,” [arXiv:1611.05797 [hep-th]].

[73] S. Caron-Huot, “Analyticity in Spin in Conformal Theories,” JHEP **1709**, 078 (2017). [arXiv:1703.00278 [hep-th]].

[74] M. Kulaxizi, A. Parnachev and A. Zhiboedov, “Bulk Phase Shift, CFT Regge Limit and Einstein Gravity,” JHEP **1806**, 121 (2018). [arXiv:1705.02934 [hep-th]].

[75] D. Li, D. Meltzer and D. Poland, “Conformal Bootstrap in the Regge Limit,” JHEP **1712**, 013 (2017). [arXiv:1705.03453 [hep-th]].

[76] M. S. Costa, T. Hansen and J. Penedones, “Bounds for OPE coefficients on the Regge trajectory,” JHEP **1710**, 197 (2017). [arXiv:1707.07689 [hep-th]].

[77] N. Afkhami-Jeddi, T. Hartman, S. Kundu and A. Tajdini, “Shockwaves from the Operator Product Expansion,” [arXiv:1709.03597 [hep-th]].

[78] D. Simmons-Duffin, D. Stanford and E. Witten, “A spacetime derivation of the Lorentzian OPE inversion formula,” JHEP **1807**, 085 (2018). [arXiv:1711.03816 [hep-th]].

[79] D. Jafferis, B. Mukhametzhanov and A. Zhiboedov, “Conformal Bootstrap At Large Charge,” JHEP **1805**, 043 (2018). [arXiv:1710.11161 [hep-th]].

[80] D. Meltzer and E. Perlmutter, "Beyond $a = c$: gravitational couplings to matter and the stress tensor OPE," JHEP **1807**, 157 (2018). [arXiv:1712.04861 [hep-th]].
[81] M. Lemos, P. Liendo, M. Meineri and S. Sarkar, "Universality at large transverse spin in defect CFT," JHEP **1809**, 091 (2018). [arXiv:1712.08185 [hep-th]].
[82] L. Iliesiu, M. Kologlu, R. Mahajan, E. Perlmutter and D. Simmons-Duffin, "The Conformal Bootstrap at Finite Temperature," JHEP **1810**, 070 (2018). [arXiv:1802.10266 [hep-th]].
[83] P. Kravchuk and D. Simmons-Duffin, "Light-ray operators in conformal field theory," JHEP **1811**, 102 (2018). [arXiv:1805.00098 [hep-th]].
[84] N. Afkhami-Jeddi, S. Kundu and A. Tajdini, "A Conformal Collider for Holographic CFTs," JHEP **1810**, 156 (2018). [arXiv:1805.07393 [hep-th]].
[85] D. Meltzer, "Higher Spin ANEC and the Space of CFTs," [arXiv:1811.01913 [hep-th]].
[86] N. Afkhami-Jeddi, S. Kundu and A. Tajdini, "A Bound on Massive Higher Spin Particles," [arXiv:1811.01952 [hep-th]].
[87] A. Kaviraj, K. Sen and A. Sinha, "Analytic bootstrap at large spin," JHEP **1511**, 083 (2015). [arXiv:1502.01437 [hep-th]].
[88] G. Vos, "Generalized Additivity in Unitary Conformal Field Theories," Nucl. Phys. B **899**, 91 (2015). [arXiv:1411.7941 [hep-th]].
[89] L. F. Alday, A. Bissi and T. Lukowski, "Large spin systematics in CFT," JHEP **1511**, 101 (2015). [arXiv:1502.07707 [hep-th]].
[90] P. Dey, A. Kaviraj and K. Sen, "More on analytic bootstrap for O(N) models," JHEP **1606**, 136 (2016). [arXiv:1602.04928 [hep-th]].
[91] C. Sleight and M. Taronna, "Spinning Mellin Bootstrap: Conformal Partial Waves, Crossing Kernels and Applications," Fortsch. Phys. **66**, no. 8-9, 1800038 (2018). [arXiv:1804.09334 [hep-th]].
[92] O. Aharony, L. F. Alday, A. Bissi and R. Yacoby, "The Analytic Bootstrap for Large N Chern-Simons Vector Models," JHEP **1808**, 166 (2018). [arXiv:1805.04377 [hep-th]].
[93] C. Cardona and K. Sen, "Anomalous dimensions at finite conformal spin from OPE inversion," JHEP **1811**, 052 (2018). [arXiv:1806.10919 [hep-th]].
[94] C. Sleight and M. Taronna, "Anomalous Dimensions from Crossing Kernels," JHEP **1811**, 089 (2018). [arXiv:1807.05941 [hep-th]].
[95] C. Cardona, S. Guha, S. K. Kanumilli and K. Sen, "Resummation at finite conformal spin," JHEP **1901**, 077 (2019). [arXiv:1811.00213 [hep-th]].
[96] Z. Komargodski and A. Schwimmer, "On Renormalization Group Flows in Four Dimensions," JHEP **1112**, 099 (2011). [arXiv:1107.3987 [hep-th]].

[97] D. M. Hofman and J. Maldacena, "Conformal collider physics: Energy and charge correlations," JHEP **0805**, 012 (2008). [arXiv:0803.1467 [hep-th]].

[98] T. Hartman, S. Jain and S. Kundu, "Causality Constraints in Conformal Field Theory," JHEP **1605**, 099 (2016). [arXiv:1509.00014 [hep-th]].

[99] D. Li, D. Meltzer and D. Poland, "Conformal Collider Physics from the Lightcone Bootstrap," JHEP **1602**, 143 (2016). [arXiv:1511.08025 [hep-th]].

[100] Z. Komargodski, M. Kulaxizi, A. Parnachev and A. Zhiboedov, "Conformal Field Theories and Deep Inelastic Scattering," Phys. Rev. D **95**, no. 6, 065011 (2017). [arXiv:1601.05453 [hep-th]].

[101] T. Hartman, S. Jain and S. Kundu, "A New Spin on Causality Constraints," JHEP **1610**, 141 (2016). [arXiv:1601.07904 [hep-th]].

[102] D. M. Hofman, D. Li, D. Meltzer, D. Poland and F. Rejon-Barrera, "A Proof of the Conformal Collider Bounds," JHEP **1606**, 111 (2016). [arXiv:1603.03771 [hep-th]].

[103] T. Faulkner, R. G. Leigh, O. Parrikar and H. Wang, "Modular Hamiltonians for Deformed Half-Spaces and the Averaged Null Energy Condition," JHEP **1609**, 038 (2016). [arXiv:1605.08072 [hep-th]].

[104] M. Kologlu, P. Kravchuk, D. Simmons-Duffin and A. Zhiboedov, "The light-ray OPE and conformal colliders," [arXiv:1905.01311 [hep-th]].

[105] A. L. Fitzpatrick, J. Kaplan and M. T. Walters, "Virasoro Conformal Blocks and Thermality from Classical Background Fields," JHEP **1511**, 200 (2015). [arXiv:1501.05315 [hep-th]].

[106] E. Hijano, P. Kraus and R. Snively, "Worldline approach to semi-classical conformal blocks," JHEP **1507**, 131 (2015). [arXiv:1501.02260 [hep-th]].

[107] E. Hijano, P. Kraus, E. Perlmutter and R. Snively, "Semiclassical Virasoro blocks from AdS_3 gravity," JHEP **1512**, 077 (2015). [arXiv:1508.04987 [hep-th]].

[108] A. L. Fitzpatrick, J. Kaplan, M. T. Walters and J. Wang, "Hawking from Catalan," JHEP **1605**, 069 (2016). [arXiv:1510.00014 [hep-th]].

[109] J. Cotler and K. Jensen, "A theory of reparameterizations for AdS_3 gravity," JHEP **1902**, 079 (2019). [arXiv:1808.03263 [hep-th]].

[110] S. Collier, Y. Gobeil, H. Maxfield and E. Perlmutter, "Quantum Regge Trajectories and the Virasoro Analytic Bootstrap," [arXiv:1811.05710 [hep-th]].

[111] A. L. Fitzpatrick and J. Kaplan, "Conformal Blocks Beyond the Semi-Classical Limit," JHEP **1605**, 075 (2016). [arXiv:1512.03052 [hep-th]].

[112] A. L. Fitzpatrick, J. Kaplan, D. Li and J. Wang, "On information loss in AdS_3/CFT_2," JHEP **1605**, 109 (2016). [arXiv:1603.08925 [hep-th]].

[113] T. Anous, T. Hartman, A. Rovai and J. Sonner, "Black Hole Collapse in the 1/c Expansion," JHEP **1607**, 123 (2016). [arXiv:1603.04856 [hep-th]].
[114] A. L. Fitzpatrick and J. Kaplan, "On the Late-Time Behavior of Virasoro Blocks and a Classification of Semiclassical Saddles," JHEP **1704**, 072 (2017). [arXiv:1609.07153 [hep-th]].
[115] H. Chen, C. Hussong, J. Kaplan and D. Li, "A Numerical Approach to Virasoro Blocks and the Information Paradox," JHEP **1709**, 102 (2017). [arXiv:1703.09727 [hep-th]].
[116] T. Faulkner and H. Wang, "Probing beyond ETH at large c," JHEP **1806**, 123 (2018). [arXiv:1712.03464 [hep-th]].
[117] C. T. Asplund, A. Bernamonti, F. Galli and T. Hartman, "Holographic Entanglement Entropy from 2d CFT: Heavy States and Local Quenches," JHEP **1502**, 171 (2015). [arXiv:1410.1392 [hep-th]].
[118] P. Caputa, J. Simón, A. Štikonas and T. Takayanagi, "Quantum Entanglement of Localized Excited States at Finite Temperature," JHEP **1501**, 102 (2015). [arXiv:1410.2287 [hep-th]].
[119] B. Chen and J. q. Wu, "Holographic Entanglement Entropy For a Large Class of States in 2D CFT," JHEP **1609**, 015 (2016). [arXiv:1605.06753 [hep-th]].
[120] B. Chen, J. Wu and J. Zhang, "Holographic Description of 2D Conformal Block in Semi-classical Limit," [arXiv:1609.00801 [hep-th]].
[121] T. Hartman, "Entanglement Entropy at Large Central Charge," [arXiv:1303.6955 [hep-th]].
[122] T. Faulkner, "The Entanglement Renyi Entropies of Disjoint Intervals in AdS/CFT," [arXiv:1303.7221 [hep-th]].
[123] M. Beccaria, A. Fachechi and G. Macorini, "Virasoro vacuum block at next-to-leading order in the heavy-light limit," JHEP **1602**, 072 (2016). [arXiv:1511.05452 [hep-th]].
[124] M. Kulaxizi, G. S. Ng and A. Parnachev, "Subleading Eikonal, AdS/CFT and Double Stress Tensors," [arXiv:1907.00867 [hep-th]].
[125] Y. Z. Li, Z. F. Mai and H. L, "Holographic OPE Coefficients from AdS Black Holes with Matters," JHEP **1909**, 001 (2019). [arXiv:1905.09302 [hep-th]].
[126] A. L. Fitzpatrick, K. W. Huang and D. Li, "Probing Universalities in $d > 2$ CFTs: from Black Holes to Shockwaves," [arXiv:1907.10810 [hep-th]].
[127] K. W. Huang, "Stress-tensor commutators in conformal field theories near the lightcone," Phys. Rev. D **100**, no. 6, 061701 (2019). [arXiv:1907.00599 [hep-th]].

[128] Y. Z. Li, "Heavy-light Bootstrap from Lorentzian Inversion Formula," [arXiv:1910.06357 [hep-th]].

[129] L. F. Alday and E. Perlmutter, "Growing Extra Dimensions in AdS/CFT," JHEP **1908**, 084 (2019). [arXiv:1906.01477 [hep-th]].

[130] M. S. Costa, J. Penedones, D. Poland and S. Rychkov "Spinning Conformal Blocks," JHEP **1111**, 154 (2015). [arXiv:1109.6321 [hep-th]].

[131] M. F. Paulos, J. Penedones, J. Toledo, B. C. van Rees and P. Vieira, "The S-matrix Bootstrap I: QFT in AdS," [arXiv:1607.06109 [hep-th]].

[132] D. Simmons-Duffin, "Projectors, Shadows, and Conformal Blocks," JHEP **1404**, 146 (2014). [arXiv:1204.3894 [hep-th]].

[133] L. Cornalba, M. S. Costa and J. Penedones, "Eikonal Methods in AdS/CFT: BFKL Pomeron at Weak Coupling," JHEP **0806**, 048 (2008). [arXiv:0801.3002 [hep-th]].

[134] D. Meltzer, "AdS/CFT Unitarity at Higher Loops: High-Energy String Scattering," [arXiv:1912.05580 [hep-th]].

[135] R. Karlsson, "Multi-stress tensors and next-to-leading singularities in the Regge limit," [arXiv:1912.01577 [hep-th]].

[136] D. G. Boulware and S. Deser, "String Generated Gravity Models," Phys. Rev. Lett. **55**, 2656 (1985).

[137] R. G. Cai, "Gauss-Bonnet black holes in AdS spaces," Phys. Rev. D **65**, 084014 (2002). [hep-th/0109133].

[138] A. Buchel, J. Escobedo, R. C. Myers, M. F. Paulos, A. Sinha and M. Smolkin, "Holographic GB gravity in arbitrary dimensions," JHEP **1003**, 111 (2010). [arXiv:0911.4257 [hep-th]].

[139] D. Carmi and S. Caron-Huot, "A Conformal Dispersion Relation: Correlations from Absorption," [arXiv:1910.12123 [hep-th]].

[140] A. Bissi, P. Dey and T. Hansen, "Dispersion Relation for CFT Four-Point Functions," [arXiv:1910.04661 [hep-th]].

[141] M. Besken, S. Datta and P. Kraus, "Semi-classical Virasoro blocks: proof of exponentiation," JHEP **2001**, 109 (2020). [arXiv:1910.04169 [hep-th]].

[142] K. W. Huang, "A Lightcone Commutator and Stress-Tensor Exchange in d=4 CFTs," [arXiv:2002.00110 [hep-th]].

[143] M. Bianchi, A. Grillo and F. Morales, "Chaos at the rim of black hole and fuzzball shadows," [arXiv:2002.05574 [hep-th]].

[144] Z. Bern, H. Ita, J. Parra-Martinez and M. S. Ruf, "Universality in the classical limit of massless gravitational scattering," [arXiv:2002.02459 [hep-th]].

[145] N. Lashkari, A. Dymarsky and H. Liu, "Eigenstate Thermalization Hypothesis in Conformal Field Theory," J. Stat. Mech. **1803**, no. 3, 033101 (2018). [arXiv:1610.00302 [hep-th]].

[146] N. Lashkari, A. Dymarsky and H. Liu, "Universality of Quantum Information in Chaotic CFTs," JHEP **1803**, 070 (2018). [arXiv:1710.10458 [hep-th]].

[147] E. Katz, S. Sachdev, E. S. Srensen and W. Witczak-Krempa, "Conformal field theories at nonzero temperature: Operator product expansions, Monte Carlo, and holography," Phys. Rev. B **90**, no. 24, 245109 (2014). [arXiv:1409.3841 [cond-mat.str-el]].

[148] Y. Gobeil, A. Maloney, G. S. Ng and J. q. Wu, "Thermal Conformal Blocks," SciPost Phys. **7**, no. 2, 015 (2019). [arXiv:1802.10537 [hep-th]].

[149] L. V. Delacrtaz, T. Hartman, S. A. Hartnoll and A. Lewkowycz, "Thermalization, Viscosity and the Averaged Null Energy Condition," JHEP **1810**, 028 (2018). [arXiv:1805.04194 [hep-th]].

[150] A. C. Petkou and A. Stergiou, "Dynamics of Finite-Temperature Conformal Field Theories from Operator Product Expansion Inversion Formulas," Phys. Rev. Lett. **121**, no. 7, 071602 (2018). [arXiv:1806.02340 [hep-th]].

[151] L. Iliesiu, M. Kologlu and D. Simmons-Duffin, "Bootstrapping the 3d Ising model at finite temperature," JHEP **1912**, 072 (2019). [arXiv:1811.05451 [hep-th]].

[152] S. Banerjee, K. Papadodimas, S. Raju, P. Samantray and P. Shrivastava, "A Bound on Thermal Relativistic Correlators at Large Spacelike Momenta," SciPost Phys. **8**, no. 4, 064 (2020). [arXiv:1902.07203 [hep-th]].

[153] A. Manenti, "Thermal CFTs in momentum space," JHEP **2001**, 009 (2020). [arXiv:1905.01355 [hep-th]].

[154] L. V. Delacretaz, "Heavy Operators and Hydrodynamic Tails," SciPost Phys. **9**, no. 3, 034 (2020). [arXiv:2006.01139 [hep-th]].

[155] N. Chai, S. Chaudhuri, C. Choi, Z. Komargodski, E. Rabinovici and M. Smolkin, "Thermal Order in Conformal Theories," Phys. Rev. D **102**, no. 6, 065014 (2020). [arXiv:2005.03676 [hep-th]].

[156] L. F. Alday, M. Kologlu and A. Zhiboedov, "Holographic Correlators at Finite Temperature," [arXiv:2009.10062 [hep-th]].

[157] J. M. Deutsch, "Quantum statistical mechanics in a closed system," Phys. Rev. A, **43** (1991), 2046.

[158] M. Srednicki, "Chaos and quantum thermalization," Phys. Rev. E, **50** (1994), 888.

[159] M. Rigol, V. Dunjko, M. Olshanii, "Thermalization and its mechanism for generic isolated quantum systems," Nature, **452** (2008), 854.

[160] L. D'Alessio, Y. Kafri, A. Polkovnikov and M. Rigol, "From quantum chaos and eigenstate thermalization to statistical mechanics and thermodynamics," Adv. Phys. **65**, no. 3, 239 (2016). [arXiv:1509.06411 [cond-mat.stat-mech]].

[161] J. M. Deutsch, "Eigenstate Thermalization Hypothesis," Rept.Prog.Phys., **81** (2018), 082001.

[162] J. de Boer and D. Engelhardt, "Remarks on thermalization in 2D CFT," Phys. Rev. D **94**, no. 12, 126019 (2016). [arXiv:1604.05327 [hep-th]].

[163] P. Kraus and A. Maloney, "A cardy formula for three-point coefficients or how the black hole got its spots," JHEP **1705**, 160 (2017). [arXiv:1608.03284 [hep-th]].

[164] H. H. Lai and K. Yang, "Entanglement entropy scaling laws and eigenstate typicality in free fermion systems," Phys. Rev. B **91**, no. 8, 081110 (2015). [arXiv:1409.1224 [cond-mat]].

[165] P. Basu, D. Das, S. Datta and S. Pal, "Thermality of eigenstates in conformal field theories," Phys. Rev. E **96**, no. 2, 022149 (2017). [arXiv:1705.03001 [hep-th]].

[166] S. He, F. L. Lin and J. j. Zhang, "Subsystem eigenstate thermalization hypothesis for entanglement entropy in CFT," JHEP **1708**, 126 (2017). [arXiv:1703.08724 [hep-th]].

[167] S. He, F. L. Lin and J. j. Zhang, "Dissimilarities of reduced density matrices and eigenstate thermalization hypothesis," JHEP **1712**, 073 (2017). [arXiv:1708.05090 [hep-th]].

[168] E. M. Brehm, D. Das and S. Datta, "Probing thermality beyond the diagonal," Phys. Rev. D **98**, no. 12, 126015 (2018). [arXiv:1804.07924 [hep-th]].

[169] A. Romero-Bermdez, P. Sabella-Garnier and K. Schalm, "A Cardy formula for off-diagonal three-point coefficients; or, how the geometry behind the horizon gets disentangled," JHEP **1809**, 005 (2018). [arXiv:1804.08899 [hep-th]].

[170] Y. Hikida, Y. Kusuki and T. Takayanagi, "Eigenstate thermalization hypothesis and modular invariance of two-dimensional conformal field theories," Phys. Rev. D **98**, no. 2, 026003 (2018). [arXiv:1804.09658 [hep-th]].

[171] W. Z. Guo, F. L. Lin and J. Zhang, "Note on ETH of descendant states in 2D CFT," JHEP **1901**, 152 (2019). [arXiv:1810.01258 [hep-th]].

[172] A. Maloney, G. S. Ng, S. F. Ross and I. Tsiares, "Thermal Correlation Functions of KdV Charges in 2D CFT," JHEP **1902**, 044 (2019). [arXiv:1810.11053 [hep-th]].

[173] A. Maloney, G. S. Ng, S. F. Ross and I. Tsiares, "Generalized Gibbs Ensemble and the Statistics of KdV Charges in 2D CFT," JHEP **1903**, 075 (2019). [arXiv:1810.11054 [hep-th]].

[174] A. Dymarsky and K. Pavlenko, "Generalized Gibbs Ensemble of 2d CFTs at large central charge in the thermodynamic limit," JHEP **1901**, 098 (2019). [arXiv:1810.11025 [hep-th]].

[175] A. Dymarsky and K. Pavlenko, "Exact generalized partition function of 2D CFTs at large central charge," JHEP **1905**, 077 (2019), [JHEP **2019**, 077 (2020)]. [arXiv:1812.05108 [hep-th]].

[176] E. M. Brehm and D. Das, "Kortewegde Vries characters in large central charge CFTs," Phys. Rev. D **101**, no. 8, 086025 (2020). [arXiv:1901.10354 [hep-th]].

[177] A. Dymarsky and K. Pavlenko, "Generalized Eigenstate Thermalization Hypothesis in 2D Conformal Field Theories," Phys. Rev. Lett. **123**, no. 11, 111602 (2019). [arXiv:1903.03559 [hep-th]].

[178] T. Anous and J. Sonner, "Phases of scrambling in eigenstates," SciPost Phys. **7**, 003 (2019). [arXiv:1903.03143 [hep-th]].

[179] S. Datta, P. Kraus and B. Michel, "Typicality and thermality in 2d CFT," JHEP **1907**, 143 (2019). [arXiv:1904.00668 [hep-th]].

[180] M. Besken, S. Datta and P. Kraus, "Quantum thermalization and Virasoro symmetry," J. Stat. Mech. **2006**, 063104 (2020). [arXiv:1907.06661 [hep-th]].

[181] S. Collier, A. Maloney, H. Maxfield and I. Tsiares, "Universal dynamics of heavy operators in CFT_2," JHEP **2007**, 074 (2020). [arXiv:1912.00222 [hep-th]].

[182] Y. Z. Li and H. Y. Zhang, "More on Heavy-Light Bootstrap up to Double-Stress-Tensor," [arXiv:2004.04758 [hep-th]].

[183] L. G. Yaffe, "Large n Limits as Classical Mechanics," Rev. Mod. Phys. **54**, 407 (1982)..

[184] F. R. Brown and L. G. Yaffe, "The Coherent State Variational Algorithm: A Numerical Method for Solving Large N Gauge Theories," Nucl. Phys. B **271**, 267 (1986)..

[185] T. A. Dickens, U. J. Lindqwister, W. R. Somsky and L. G. Yaffe, "The Coherent State Variational Algorithm. 2. Implementation and Testing," Nucl. Phys. B **309**, 1 (1988)..

[186] A. Parnachev, "Near Lightcone Thermal Conformal Correlators and Holography," [arXiv:2005.06877 [hep-th]].

[187] B. Sundborg, "The Hagedorn transition, deconfinement and N=4 SYM theory," Nucl. Phys. B **573**, 349 (2000). [hep-th/9908001].

[188] P. Haggi-Mani and B. Sundborg, "Free large N supersymmetric Yang-Mills theory as a string theory," JHEP **0004**, 031 (2000). [hep-th/0002189].

[189] O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas and M. Van Raamsdonk, "The Hagedorn - deconfinement phase transition in weakly coupled large N gauge theories," Adv. Theor. Math. Phys. **8**, 603 (2004). [hep-th/0310285].

[190] M. Brigante, G. Festuccia and H. Liu, "Inheritance principle and non-renormalization theorems at finite temperature," Phys. Lett. B **638**, 538 (2006). [hep-th/0509117].

[191] I. Amado, B. Sundborg, L. Thorlacius and N. Wintergerst, "Probing emergent geometry through phase transitions in free vector and matrix models," JHEP **1702**, 005 (2017). [arXiv:1612.03009 [hep-th]].

[192] I. Amado, B. Sundborg, L. Thorlacius and N. Wintergerst, "Black holes from large N singlet models," JHEP **1803**, 075 (2018). [arXiv:1712.06963 [hep-th]].

[193] H. Maxfield, "A view of the bulk from the worldline," [arXiv:1712.00885 [hep-th]].

[194] E. Perlmutter, "Virasoro conformal blocks in closed form," JHEP **1508**, 088 (2015). [arXiv:1502.07742 [hep-th]].

[195] P. Di Francesco, P. Mathieu and D. Senechal, "Conformal Field Theory,"

[196] A. Belin and J. de Boer, "Random Statistics of OPE Coefficients and Euclidean Wormholes," [arXiv:2006.05499 [hep-th]].

[197] K. W. Huang, "$d > 2$ stress-tensor operator product expansion near a line," Phys. Rev. D **103**, no. 12, 121702 (2021), [Phys. Rev. D **103**, L121702 (2021)]. [arXiv:2103.09930 [hep-th]].

[198] A. Belin, D. M. Hofman, G. Mathys and M. T. Walters, "On the stress tensor light-ray operator algebra," JHEP **2105**, 033 (2021). [arXiv:2011.13862 [hep-th]].

[199] M. Besken, J. De Boer and G. Mathys, "On Local and Integrated Stress-Tensor Commutators," [arXiv:2012.15724 [hep-th]].

[200] S. Giusto, M. R. R. Hughes and R. Russo, "The Regge limit of AdS_3 holographic correlators," [arXiv:2007.12118 [hep-th]].