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On the correspondence between gravity fields and CFT operators

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Abstract

It is shown that a nonlinear derivative-dependent transformation of gravity fields changes correlation functions in a boundary CFT, and, therefore, corresponds to a change of a basis of operators in the CFT. It is argued that only non-renormalized structures in correlation functions can be changed by such a field transformation, and that the study of the response of correlation functions to gravity field transformations allows one to find them. In the case of 4-point functions of CPOs in SYM_4 several non-renormalized structures are found, including the extremal and subextremal ones. It is also checked that quartic couplings of scalar fields s^I that are dual to extended chiral primary operators vanish in the subextremal case, as dictated by the non-renormalization theorem for the subextremal 4-point functions and the AdS/CFT correspondence.

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1 Introduction

According to the AdS/CFT correspondence [1, 2, 3] fields of type IIB supergravity on the $AdS_5 \times S^5$ background are dual to gauge invariant operators in $D = 4$, $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM$_4$) which belong to short representations of the conformal superalgebra $SU(2,2|4)$ and have protected scale dimensions. The short representations are generated by chiral primary operators (CPOs) transforming in the k-traceless symmetric representations of $SO(6)$. It is well-known that single-trace operators $O_k^I = \operatorname{tr}(\phi^{(i_1} \cdots \phi^{i_k)})$ are chiral, and it was shown in [4] that multi-trace operators of the form $O^I_{k_1 \cdots k_n} = \operatorname{tr}(\phi^{(i_1} \cdots \phi^{i_{k_1}}) \cdots \operatorname{tr}(\phi^{j_1} \cdots \phi^{j_{k_n}}))$ are chiral too. There are also CPOs which are normal-ordered products of single- and multi-trace CPOs and their descendents. Thus, in general, CPOs are admixtures of single- and multi-trace operators with the same (protected) conformal dimension.

On the other hand the particle spectrum of type IIB supergravity on $AdS_5 \times S^5$ [5, 6] contains only one set of fields which can couple to CPOs. These fields s^I are mixtures of the five form field strength and the trace of the graviton on the sphere. Thus, one should understand which linear combinations of CPOs are dual to the gravity fields s^I. Although, it is customarily believed that they are dual to single-trace operators O_k^I, no complete reliable proof of this fact is known.

A way to solve the problem is to compute correlation functions in free field theory and in the supergravity approximation, and to compare them. Of course, one can compare only correlation functions subject to non-renormalization theorems. According to [2, 3], to compute n-point functions in SYM$_4$ one has to know the type IIB supergravity action on $AdS_5 \times S^5$ up to the n-th order. The quadratic action for physical fields was found in [7] by using the "covariant" action of [8, 9]. The first step in finding interaction vertices was made in [10] where quadratic and cubic actions for the scalars s^I were found by expanding the covariant equations of motion [11, 12, 13] for type IIB supergravity up to the second order. By using the actions, all 3-point functions of normalized CPOs dual to s^I were computed, and, for generic values of conformal dimensions of the CPOs, appeared to coincide with 3-point functions of the single-trace CPOs O_k^I calculated in the free field theory. It was conjectured in [10] that the 3-point functions are not renormalized, and this was later proven in [14]. One might conclude on the basis of this coincidence that the fields s^I are dual to the single-trace CPOs. However, as was noted in [15], a 3-point function of CPOs computed in the supergravity approximation vanishes in the extremal case, for which the sum of conformal dimensions of two operators equals the conformal dimension of the third operator, e.g. $k_1 = k_2 + k_3$, because of the vanishing of the cubic couplings of the dual scalar fields.

There were proposed three different ways to resolve the puzzle. According to [16], to compute extremal 3-point functions, one should first analytically continue in the conformal dimensions k_1, k_2, k_3. Then, since the gravity coupling is proportional to $k_2+k_3-k_1$, and the AdS integral [17] behaves itself as $1/(k_2+k_3-k_1)$, one obtains a finite extremal 3-point function. However, from the computational point of view the procedure of analytical continuation looks superfluous, because no actual singularity is involved. An extremal 3-point function vanishes due to the absence of the corresponding cubic coupling, and one does not have to evaluate any AdS integral.

In [18] we explained the vanishing of the extremal cubic couplings by noting that the scalars s^I, and, in general, supergravity fields, may be dual to extended CPOs which are admixtures of single- and multi-trace CPOs. Nevertheless, the fact that the analytical continuation procedure seems to work in all known examples,[1] allows one to assume that, in the large N limit and for generic values of conformal dimensions, correlation functions of extended CPOs coincide with the ones of the single-trace CPOs. However, it is also clear that the analytical continuation procedure may work only in the large N limit, because for finite N only the single-trace CPOs O^I_k with $2 \leq k \leq N$ are independent, and a single-trace CPO with $k > N$ is equal to a linear combination of multi-trace CPOs. This also shows that the appearence of multi-trace CPOs is unavoidable for finite N.

Other arguments in favour of the proposal come from the study of quartic couplings of the scalars s^I performed in [20]. It is shown there that the quartic couplings vanish in the extremal case for which $k_1 = k_2 + k_3 + k_4$. As was pointed out in [18] the vanishing of extremal couplings is dictated by the AdS/CFT correspondence because in this case contact Feynman diagrams are ill-defined, and therefore, non-vanishing extremal quartic couplings would contradict to the AdS/CFT correspondence. By the same reason 2- and 4-derivative quartic couplings have to vanish in the subextremal case for which $k_1 = k_2+k_3+k_4-2$, and 4-derivative quartic couplings should vanish in the sub-subextremal case when $k_1 = k_2+k_3+k_4-4$. The vanishing of extremal couplings means that 4-point extremal correlators of CPOs dual to the scalars s^I vanish, and, therefore, the scalars correspond not to single-trace CPOs but to extended CPOs.

Then, it is shown in [20] that the quartic action is consistent with the Kaluza-Klein (KK) reduction down to five dimensions, and admits a truncation to the massless multiplet, which can be identified with the field content of the gauged $\mathcal{N} = 8$, $d = 5$ supergravity [21, 22]. Consistency means that there is no term linear in massive KK modes in the untruncated supergravity action, so that all massive KK fields can be put to zero without any contradiction with equations of motion. From the AdS/CFT correspondence point of view the consistent truncation implies that *any* n-point correlation function of $n-1$ operators dual to the fields from the massless

[1] In particular, this procedure works in the case of 3-point functions of operators dual to two scalars s^I and a supergravity field, computed in [18], where the cubic couplings [18, 19] also vanish in extremal cases.

multiplet and one operator dual to a massive KK field vanishes because, as one can easily see there is no exchange Feynman diagram in this case. This in particular implies that the scalars s^I are dual to extended CPOs. Indeed, if we assume that the scalars s^I correspond to the single-trace CPOs O^I_k, we derive from the consistency of the KK reduction that correlators of the form $\langle O_2^{I_1} O_2^{I_2} \cdots O_2^{I_{n-1}} O_k^{I_n} \rangle$ vanish for $k \geq 4$, that is not the case for such correlators of single-trace CPOs.

Finally, the third way of solving the puzzle was proposed in [23], where it was noted that since the scalars s^I used in [10] differ from the original scalars appearing in the covariant equations of motion for type IIB supergravity, one could obtain a nonvanishing extremal 3-point function by using an action for the original scalars. This was demonstrated for fields from the descendent sector where the relevant part of the type IIB supergravity action is known. The action used in [23] contains higher-derivative terms, and although the bulk extremal couplings vanish on shell, there appear boundary terms which provide nonvanishing contribution to extremal 3-point functions. However, as was also noted in [23], one can make a nonlinear off-shell transformation of the gravity fields and remove all higher-derivative terms and all nonvanishing (off-shell) extremal couplings. No boundary terms appear as a result of the field transformation, and the transformed action leads to vanishing extremal correlators.

Thus, these results seem to indicate that although the original gravity fields may be dual to single-trace operators, the transformed fields are already dual to mixtures of single- and multi-trace operators. From this point of view a redefinition of the gravity fields corresponds to a change of an operator basis in CFT.

To justify this point of view we study how 4-point correlation functions are changed under derivative-dependent gravity fields redefinitions of the form used in [20] to reduce the non-Lagrangian equations of motion to a Lagrangian form. The field transformations discussed in [23] are their particular case. We begin with the quartic action for scalars s^I found in [20] and show that these transformations indeed can change some correlators, in particular, the extremal 3- and 4-point functions and the subextremal 4-point functions for which $k_1 = k_2 + k_3 + k_4 - 2$.

The subextremal correlators are of special interest because, as has been shown in [24], and checked in [25] to first order in perturbation theory, they are not renormalized. Thanks to the non-renormalization theorem one can also employ subextremal 4-point correlators to test the AdS/CFT correspondence. In particular the non-renormalization implies that the subextremal quartic couplings vanish, and we show that this is indeed the case. This fact together with the absence of the exchange Feynman diagrams leads to the vanishing of the subextremal 4-point functions of extended CPOs dual to the scalars s^I.

We show that any field redefinition induces a change of correlation functions which is always given by a product of 2- and 3-point functions. By this reason, and due to the non-renormalization theorems for extremal and subextremal correlators, it seems possible to find such a field transformation that the extremal and subextremal 3- and 4-point functions coincide with the ones of single-trace CPOs.

As a by-product of our study, we also find that if conformal dimensions of at least two CPOs do not coincide then some structures in a 4-point function of these CPOs can be also changed by a field redefinition. Thus, although such a 4-point function in general is not protected by a non-renormalization theorem, this seems to be an indication that the coefficients of the changing structures of the 4-point function are not renormalized.

The plan of the paper is as follows. In section 2 we recall the definition of normalized single-trace CPOs and extended CPOs, and discuss the general properties of the supergravity action used to compute 3- and 4-point functions of the extended CPOs. In section 3 we study how the 3- and 4-point correlation functions change under a derivative-dependent field redefinition. In appendix we show that the quartic couplings of [20] vanish in the subextremal case, and that 4-derivative couplings vanish in the sub-subextremal case.

2 Extended CPOs and quartic supergravity action

We follow [10] defining the normalized single-trace CPOs as

$$O^I(\vec{x}) = \frac{(2\pi)^k}{\sqrt{k\lambda^k}} C^I_{i_1\cdots i_k}\,\mathrm{tr}\,(\,\phi^{i_1}(\vec{x})\cdots\phi^{i_k}(\vec{x})), \tag{2.1}$$

where $C^I_{i_1\cdots i_k}$ are totally symmetric traceless rank k orthonormal tensors of $SO(6)$: $\langle C^I C^J\rangle = C^I_{i_1\cdots i_k}C^J_{i_1\cdots i_k} = \delta^{IJ}$, and ϕ^i are scalars of SYM$_4$.

The two- and three-point functions of CPOs can be easily computed in free field theory and in the large N limit, and are given by [10]

$$\langle O^I(\vec{x})O^J(\vec{y})\rangle = \frac{\delta^{IJ}}{|\vec{x}-\vec{y}|^{2k}}, \tag{2.2}$$

$$\langle O^{I_1}(\vec{x})O^{I_2}(\vec{y})O^{I_3}(\vec{z})\rangle = \frac{1}{N}\frac{C^{I_1I_2I_3}}{|\vec{x}-\vec{y}|^{2\alpha_3}|\vec{y}-\vec{z}|^{2\alpha_1}|\vec{z}-\vec{x}|^{2\alpha_2}}, \tag{2.3}$$

where $\alpha_i = \frac{1}{2}(k_j + k_l - k_i)$, $j \neq l \neq i$, $C^{I_1I_2I_3} = \sqrt{k_1k_2k_3}\langle C^{I_1}C^{I_2}C^{I_3}\rangle$, and $\langle C^{I_1}C^{I_2}C^{I_3}\rangle$ is the unique $SO(6)$ invariant obtained by contracting α_1 indices between C^{I_2} and C^{I_3}, α_2 indices between C^{I_3} and C^{I_1}, and α_3 indices between C^{I_2} and C^{I_1}. As was discussed in the Introduction, single-trace CPO cannot be dual to the scalar fields s^I used in [10] to compute their 3-point functions. However, as was shown in [18], one can define an extended CPO which corresponds to a scalar s^I by adding to a single-trace CPO a proper combination of multi-trace CPOs:

$$\tilde{O}^{I_1} = O^{I_1} - \frac{1}{2N} \sum_{I_2+I_3=I_1} C^{I_1 I_2 I_3} O^{I_2} O^{I_3}. \tag{2.4}$$

One can easily check that in the large N limit these operators have the normalized two-point functions (2.2), the three-point functions (2.3) in the non-extremal case, and vanishing three-point functions in the extremal case. Note that these operators require further modification to be consistent with all n-point functions computed in the framework of the AdS/CFT correspondence. In general, an extended CPO is a linear combination of a CPO and chiral primary composite operators which are normal-ordered products of CPOs and their descendants.

The quartic action for the scalars s^I dual to the extended CPOs was found in [20], and the part of the action depending only on the scalars can be written in the form

$$\begin{aligned} S &= \int_{AdS_5} \Big(-\frac{1}{2}(\nabla_a s_I \nabla^a s_I + m_I^2 s_I^2) + \lambda_{IJK} s_I s_J s_K + \lambda^{(0)}_{IJKL} s_I s_J s_K s_L \\ &+ \lambda^{(2)}_{IJKL} \nabla_a s_I \nabla^a s_J s_K s_L + \lambda^{(4)}_{IJKL} \nabla_a s_I \nabla^a s_J \nabla_b s_K \nabla^b s_L \Big) \\ &= \int_{AdS_5} \mathcal{L}(s^I). \end{aligned} \tag{2.5}$$

Since the action does not contain higher-derivative terms, the Hamiltonian reformulation of the quartic action is straightforward, and, therefore, as was shown in [26], there is no need to add boundary terms.

There are also cubic terms describing the interaction of the scalars s^I with other scalars, with vector fields, and with massive symmetric tensor fields of the second rank, but we omit them for the sake of simplicity.

Considering the contribution of contact Feynman diagrams to 3- and 4-point functions, one can easily observe that the integrals over the AdS_5 space diverge in several cases: (i) if cubic couplings do not vanish in the extremal case for which, e.g. $k_1 = k_2 + k_3$, (ii) if quartic couplings do not vanish in the extremal case when $k_1 = k_2 + k_3 + k_4$, (iii) if 4-derivative and 2-derivative quartic couplings do not vanish in the subextremal case when $k_1 = k_2 + k_3 + k_4 - 2$, and (iv) if 4-derivative quartic couplings do not vanish in the sub-subextremal case for which $k_1 = k_2 + k_3 + k_4 - 4$. Thus the AdS/CFT correspondence requires vanishing all these couplings. Moreover, although the AdS integral involved in the subextremal non-derivative quartic graph does not diverge, the non-derivative quartic couplings also have to vanish in the subextremal case, because as was proven in [24] the subextremal 4-point functions are non-renormalized, and, therefore, have a free field form (a product of 2- and 3-point functions of a free CFT). On the other hand a nonvanishing quartic subextremal coupling would lead to a 4-point function which does not have a free field form, and, this would contradict to the AdS/CFT correspondence.

Since one can easily show that all exchange Feynman diagrams vanish in the extremal and subextremal cases, the vanishing of the quartic couplings means that extremal and subextremal 4-point functions of operators dual to the scalars s^I in (2.5) also vanish. This is certanly not the case for the correlators of the single-trace CPOs O_k^I, and, therefore, we interpret the scalars s^I as to be dual to the extended CPOs of the form (2.4). However, to obtain action (2.5) a number of nonlinear derivative-dependent field redefinitions was performed. Thus, a natural question arises whether it is possible to make such a field redefinition of the scalars s^I that the redefined scalars $\mathbf{s}^I$ would correspond to the single-trace CPOs. In the next section we study the response of 3- and 4-point correlation functions to such changes and show that the desirable field redefinitions may exist.

3 Field redefinitions and 4-point functions

According to the proposal by [2, 3], the generating functional of connected Green functions in SYM_4 at large N and at strong 't Hooft coupling coincides with the on-shell value of the type IIB supergravity action on $AdS_5 \times S^5$ subject to the Dirichlet boundary conditions imposed on supergravity fields at the boundary of $AdS_5 \times S^5$. To have a well-defined functional of the boundary fields we cut the AdS space[2] off at $z = \varepsilon$ and consider the part of AdS with $z \geq \varepsilon$. We impose the Dirichlet boundary conditions on the scalars s^I: $s^I(\varepsilon, \vec{x}) \equiv s^I(\vec{x})$, and denote the on-shell value of the action (2.5) as $S(s)$. To compute 3- and 4-point functions we only need equations of motion for the scalars s_I decomposed up to the second order in fields:

$$(\nabla_a^2 - m_I^2)s_I + 3\lambda_{IJK}s_J s_K = 0. \tag{3.1}$$

The solution of the equation that satisfies the Dirichlet boundary conditions can be written in the form

$$s_I = s_I^{(0)} + s_I^{(1)}. \tag{3.2}$$

Here $s_I^{(0)}$ solves the linear part of (3.1) with the Dirichlet boundary conditions at $z = \varepsilon$, and $s_I^{(1)}$ has the vanishing boundary conditions at $z = \varepsilon$, and solves the equation

$$(\nabla_a^2 - m_I^2)s_I^{(1)} + 3\lambda_{IJK}s_J^{(0)} s_K^{(0)} = 0. \tag{3.3}$$

[2] We use the AdS metric of the form: $ds^2 = \frac{1}{z^2}(dz^2 + dx_i^2)$.

The on-shell value of action (2.5) is obtained by substituting (3.2) into it:

$$S(s) = \int_\varepsilon^\infty dz \int d\vec{x}\, z^{-5} \mathcal{L}(s_I^{(0)} + s_I^{(1)}). \tag{3.4}$$

Let us now consider the following off-shell transformation of the fields s_I

$$\begin{aligned} s_I &= \mathbf{s}_I + C^{(0)}_{IJK}\mathbf{s}_J\mathbf{s}_K + C^{(2)}_{IJK}\nabla_a\mathbf{s}_J\nabla^a\mathbf{s}_K \\ &+ C^{(0)}_{IJKL}\mathbf{s}_J\mathbf{s}_K\mathbf{s}_L + C^{(2)}_{IJKL}\nabla_a\mathbf{s}_J\nabla^a\mathbf{s}_K\mathbf{s}_L + C^{(4)}_{IJKL}\nabla_a\mathbf{s}_J\nabla_b\mathbf{s}_K\nabla^a\nabla^b\mathbf{s}_L \\ &= \mathbf{s}_I + (\delta s)_I. \end{aligned} \tag{3.5}$$

This transformation is of the same form as the most general s-dependent field redefinition that was used in [20] to reduce the original non-Lagrangian equations of motion to a Lagrangian form. We also assume that the constants $C^{(0)}_{IJK}$ and $C^{(2)}_{IJK}$ do not vanish only if any of the conformal dimensions does not exceed its extremal value: $\Delta_I \le \Delta_J + \Delta_K$.

The equations of motion for the redefined fields look as follows

$$(\nabla_a^2 - m_I^2)\mathbf{s}_I + 3\lambda_{IJK}\mathbf{s}_J\mathbf{s}_K + (\nabla_a^2 - m_I^2)(\delta_2 s)_I = 0, \tag{3.6}$$

where

$$(\delta_2 s)_I = C^{(0)}_{IJK}\mathbf{s}_J\mathbf{s}_K + C^{(2)}_{IJK}\nabla_a\mathbf{s}_J\nabla^a\mathbf{s}_K. \tag{3.7}$$

The simplest way to study the influence of the field redefinition (3.5) on the 3- and 4-point functions is to impose on the redefined fields $\mathbf{s}_I$ the same boundary conditions as on s_I, and to compare the on-shell values of the original and transformed actions. Thus we write the solution to (3.6) in the form

$$\mathbf{s}_I = s_I^{(0)} + \mathbf{s}_I^{(1)} = s_I^{(0)} + s_I^{(1)} - (\delta_2 s)_I^{(0)} + \sigma_I^{(0)}. \tag{3.8}$$

Here

$$(\delta_2 s)_I^{(0)} = C^{(0)}_{IJK}s_J^{(0)}s_K^{(0)} + C^{(2)}_{IJK}\nabla_a s_J^{(0)}\nabla^a s_K^{(0)}, \tag{3.9}$$

and $\sigma_I^{(0)}$ solves the linear part of the equation (3.1), and satisfies the following boundary condition

$$\sigma_I^{(0)}|_{z=\varepsilon} = (\delta_2 s)_I^{(0)}|_{z=\varepsilon}. \tag{3.10}$$

Substituting (3.8) into (3.5), we find

$$\mathbf{s}_I + (\delta s)_I = s_I^{(0)} + s_I^{(1)} + \sigma_I^{(0)} + (\delta_3 s)_I^{(0)} + 2C^{(0)}_{IJK}s_J^{(0)}\mathbf{s}_K^{(1)} + 2C^{(2)}_{IJK}\nabla_a s_J^{(0)}\nabla^a\mathbf{s}_K^{(1)},$$

where

$$(\delta_3 s)_I^{(0)} = C^{(0)}_{IJKL}s_J^{(0)}s_K^{(0)}s_L^{(0)} + C^{(2)}_{IJKL}\nabla_a s_J^{(0)}\nabla^a s_K^{(0)}s_L^{(0)} + C^{(4)}_{IJKL}\nabla_a s_J^{(0)}\nabla_b s_K^{(0)}\nabla^a\nabla^b s_L^{(0)}.$$

Thus the on-shell value of the transformed Lagrangian is given by

$$
\begin{aligned}
\tilde{\mathcal{L}}(\mathbf{s}_I) &= \mathcal{L}(\mathbf{s}_I + (\delta s)_I) = \mathcal{L}(s_I^{(0)} + s_I^{(1)}) \\
&- \nabla_a \left(s_I^{(0)} + s_I^{(1)}\right) \nabla^a \left(\sigma_I^{(0)} + (\delta_3 s)_I^{(0)} + 2C_{IJK}^{(0)} s_J^{(0)} \mathbf{s}_K^{(1)} + 2C_{IJK}^{(2)} \nabla_b s_J^{(0)} \nabla^b \mathbf{s}_K^{(1)}\right) \\
&- m_I^2 \left(s_I^{(0)} + s_I^{(1)}\right) \left(\sigma_I^{(0)} + (\delta_3 s)_I^{(0)} + 2C_{IJK}^{(0)} s_J^{(0)} \mathbf{s}_K^{(1)} + 2C_{IJK}^{(2)} \nabla_a s_J^{(0)} \nabla^a \mathbf{s}_K^{(1)}\right) \\
&- \frac{1}{2} \nabla_a \sigma_I^{(0)} \nabla^a \sigma_I^{(0)} - \frac{1}{2} m_I^2 \sigma_I^{(0)} \sigma_I^{(0)} + 3\lambda_{IJK} s_I^{(0)} s_J^{(0)} \sigma_K^{(0)}.
\end{aligned}
\tag{3.11}
$$

The first term on the r.h.s. of this equation is equal to the on-shell value of the original Lagrangian (2.5), and other terms represent relevant corrections to it. By using equations of motion we can rewrite (3.11) as follows

$$
\begin{aligned}
\tilde{\mathcal{L}}(\mathbf{s}_I) &= \mathcal{L}(s_I^{(0)} + s_I^{(1)}) \\
&- \nabla_a \Big(\nabla^a \left(s_I^{(0)} + s_I^{(1)}\right) \left(\sigma_I^{(0)} + (\delta_3 s)_I^{(0)} + 2C_{IJK}^{(0)} s_J^{(0)} \mathbf{s}_K^{(1)} + 2C_{IJK}^{(2)} \nabla_b s_J^{(0)} \nabla^b \mathbf{s}_K^{(1)}\right)\Big) \\
&- \frac{1}{2} \nabla_a \Big(\nabla^a \sigma_I^{(0)} \sigma_I^{(0)}\Big).
\end{aligned}
\tag{3.12}
$$

Thus the on-shell values of the original action and the redefined one only differ by a boundary term. Omitting nonessential terms, which cannot change the 3- and 4-point functions, this boundary term can be written in the form

$$
\begin{aligned}
I = \int d\vec{x}\, \varepsilon^{-d+1} \Big(& \varepsilon^2 C_{IJK}^{(2)} \partial_0 s_I^{(0)} \partial_0 s_J^{(0)} \partial_0 s_K^{(0)} + \varepsilon^2 C_{IJK}^{(2)} \partial_0 s_I^{(1)} \partial_0 s_J^{(0)} \partial_0 s_K^{(0)} \\
&+ \partial_0 s_I^{(0)} \left(C_{IJKL}^{(2)} \nabla_a s_J^{(0)} \nabla^a s_K^{(0)} s_L^{(0)} + C_{IJKL}^{(4)} \nabla_a s_J^{(0)} \nabla^b s_K^{(0)} \nabla^a \nabla_b s_L^{(0)}\right) \\
&+ 2\varepsilon^2 C_{IJK}^{(2)} \partial_0 s_I^{(0)} \partial_0 s_J^{(0)} \partial_0 \left(s_K^{(1)} - (\delta_2 s)_K^{(0)} + \sigma_K^{(0)}\right) + \frac{1}{2} \partial_0 \sigma_I^{(0)} \sigma_I^{(0)} \Big).
\end{aligned}
\tag{3.13}
$$

The first term on the r.h.s. of (3.13) represents the change in a 3-point function induced by the field redefinition (3.5). It was shown in [23] that this term gives a nonvanishing contribution only to the extremal 3-point functions, and always leads to a contribution of the free-field form. In particular, one can choose a field redefinition of such a form that all 3-point functions will coincide with the 3-point functions of the single-trace CPOs.

We are going to study the influence of the field redefinition on the 4-point functions and begin with considering the simplest term[3]

$$
I_1 = \int d\vec{x}\, \varepsilon^{-d+1} C_{IJKL}^{(2)} \partial_0 s_I^{(0)} \nabla_a s_J^{(0)} \nabla^a s_K^{(0)} s_L^{(0)}. \tag{3.14}
$$

It is obvious that only derivatives in the radial direction can give a nonvanishing contribution to a 4-point function, thus we can replace I_1 by

$$
I_1 = \int d\vec{x}\, \varepsilon^{-d+3} C_{IJKL}^{(2)} \partial_0 s_I^{(0)} \partial_0 s_J^{(0)} \partial_0 s_K^{(0)} s_L^{(0)}. \tag{3.15}
$$

[3]For the sake of generality we consider the boundary term (3.13) in d dimensions, and for arbitrary scalars s_I dual to operators of conformal dimensions Δ_I with the only restriction $\Delta_I \geq d/2$.

It is well-known that the Fourier transform of the solution of the Dirichlet problem is given by (see, e.g. [23])

$$s_I^{(0)}(z,\vec{k}) = K_I(z,\vec{k})s_I(\vec{k}),$$

where

$$K_I(z,\vec{k}) = \left(\frac{z}{\varepsilon}\right)^{d/2}\frac{\mathcal{K}_\nu(kz)}{\mathcal{K}_\nu(k\varepsilon)}, \quad \nu = \Delta_I - \frac{d}{2} \tag{3.16}$$

and $\mathcal{K}_\nu(kz)$ is a Macdonald function. By using this formula, we find

$$z\partial_0 K_I(z,\vec{k})|_{z=\varepsilon} = d - \Delta_I + a_I(k\varepsilon)^{2(\Delta_I-\frac{d}{2})}\log(k\varepsilon) + \cdots, \tag{3.17}$$

where $\cdots$ refer to terms which do not contribute to 4-point functions in the limit $\varepsilon \to 0$. Thus a relevant contribution of I_1 to the Fourier transform of a 4-point function is proportional to

$$\delta(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)\varepsilon^{2\Delta_I+2\Delta_J+2\Delta_K-4d}k_1^{\Delta_I-\frac{d}{2}}k_2^{\Delta_J-\frac{d}{2}}k_3^{\Delta_K-\frac{d}{2}}\log(k_1\varepsilon)\log(k_2\varepsilon)\log(k_3\varepsilon).$$

This expression is always the Fourier transform of a product of three 2-point functions. It gives a contribution to a 4-point function only if [4]

$$2\Delta_I + 2\Delta_J + 2\Delta_K - 4d = \Delta_I + \Delta_J + \Delta_K + \Delta_L - 4d,$$

i.e., in the extremal case $\Delta_I + \Delta_J + \Delta_K = \Delta_L$. Note that in the non-extremal cases $\Delta_L < \Delta_I + \Delta_J + \Delta_K$, in particular in the subextremal one, the boundary term scales too fast to give a contribution.

The second integral to be considered is

$$I_2 = \int d\vec{x}\,\varepsilon^{-d+3}C^{(2)}_{IJL}\partial_0 s_I^{(0)}\partial_0 s_J^{(0)}\partial_0\sigma_L^{(0)}. \tag{3.18}$$

To analyse the integral we use

$$\begin{aligned}\sigma_L^{(0)}(z,\vec{k}) = K_L(z,\vec{k})\int d\vec{k}_3 d\vec{k}_4\,\delta(\vec{k}_3+\vec{k}_4-\vec{k})\Big(&C^{(2)}_{LMN}\varepsilon^2\partial_0 s_M^{(0)}(\varepsilon,\vec{k}_3)\partial_0 s_N^{(0)}(\varepsilon,\vec{k}_4)\\ &+ C^{(2)}_{LMN}\varepsilon^2\partial_i s_M(\vec{k}_3)\partial_i s_N(\vec{k}_4) + C^{(0)}_{LMN}s_M(\vec{k}_3)s_N(\vec{k}_4)\Big).\end{aligned} \tag{3.19}$$

One can easily see that only the first term can give a nonvanishing contribution to a 4-point function. Combining (3.18) with (3.19) we get that this contribution to a 4-point function is proportional to

$$C^{(2)}_{IJL}C^{(2)}_{LMN}\delta(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)\varepsilon^{-d+5}\partial_0 K_I(\varepsilon,\vec{k}_1)\partial_0 K_J(\varepsilon,\vec{k}_2)\partial_0 K_L(\varepsilon,\vec{k}_3+\vec{k}_4)$$

[4] Note that the correct scaling behaviour of a 4-point function is $\mathcal{O}\left(\varepsilon^{\Delta_I+\Delta_J+\Delta_K+\Delta_L-4d}\right)$.

$$\times\partial_0 K_M(\varepsilon,\vec{k}_3)\partial_0 K_N(\varepsilon,\vec{k}_4). \tag{3.20}$$

By using (3.17), we see that there are several cases when we can get a nonlocal contribution: (i) the five-logs case, (ii) the four-logs case, and (iii) the three-logs case. It is not difficult to show that there is no contribution in the five- and four-logs cases. Three-logs case has three subcases. The first one is

$$\log(k_3+k_4)\log k_3\log k_4\delta(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)$$

In this case we get $\delta(x_2-x_1)$ after integrating over momenta.

The second case is

$$\log k_2\log k_3\log k_4\delta(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)$$

It is obvious that in this case we get a product of three 2-point functions, and a nonvanishing contribution will be only in the extremal case $\Delta_J+\Delta_M+\Delta_N=\Delta_I$.

The third case is

$$\log(k_3+k_4)\log k_2\log k_4\delta(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)$$

One can easily see that in this case we also obtain a product of three 2-point functions, and a nonvanishing contribution will be only if

$$-\Delta_I+\Delta_J+2\Delta_L-\Delta_M+\Delta_N=0.$$

Taking into account that

$$-\Delta_I+\Delta_J+\Delta_L\geq 0,\qquad -\Delta_M+\Delta_N+\Delta_L\geq 0$$

we get that there is a solution to this equation if

$$\Delta_L=\Delta_I-\Delta_J=\Delta_M-\Delta_N.$$

This is a new case, and it is tempting to assume that the coefficient of the corresponding structure in the 4-point function is not renormalized. To understand why this may be so it is instructive to write down the changing term in the 4-point function:

$$\langle O^I(\vec{x}_1)O^J(\vec{x}_2)O^M(\vec{x}_3)O^N(\vec{x}_4)\rangle\sim\frac{1}{x_{12}^{2\Delta_J}x_{13}^{2\Delta_L}x_{34}^{2\Delta_N}}+\cdots.$$

We see that this term is obtained by plunging the operators O^J and O^N into the operators O^I and O^M respectively. The perturbative non-renormalization of the Feynman diagrams of such a type was checked in [25] to first order in perturbation theory where it was noted that this effectively is equivalent to the proof of the non-renormalization of 2-point functions of CPOs given in [27].

The next integral to be considered is

$$I_3 = \int d\vec{x}\, \varepsilon^{-d+3} C^{(2)}_{IJL} \partial_0 s_I^{(0)} \partial_0 s_J^{(0)} \partial_0 s_L^{(1)}. \tag{3.21}$$

Taking into account that $s_L^{(1)}$ solves the equation (3.3), we obtain the formula

$$s_L^{(1)}(x_0, \vec{x}) = 3\lambda_{LMN} \int d^{d+1}y\, \sqrt{g} G_L^\varepsilon(x,y) s_M^{(0)}(y) s_N^{(0)}(y),$$

where the Green function can be found in [28], and satisfies

$$\frac{\partial}{\partial x_0} G_L^\varepsilon(x,y)|_{x0=\varepsilon} = -\varepsilon^{d-1} \int \frac{d\vec{k}}{(2\pi)^d} e^{-i\vec{k}(\vec{x}-\vec{y})} K_L(y_0, \vec{k}) = -\varepsilon^{\Delta_L - 1} K_{\Delta_L}(y_0, \vec{x}-\vec{y}),$$

where $K_{\Delta_L}(x_0, \vec{x}-\vec{y})$ is the bulk-to-boundary propagator defined in [17].

It is convenient to analyse (3.21) in the x-space, where the solution $s_I^{(0)}$ can be written as

$$s_I^{(0)}(x_0, \vec{x}) = \frac{1}{\varepsilon^{d-\Delta_I}} \left(\int d\vec{y} K_{\Delta_I}(x_0, \vec{x}-\vec{y}) s(\vec{y}) + o(\varepsilon) \right). \tag{3.22}$$

Thus, for $s_L^{(1)}$ we have

$$\partial_0 s_L^{(1)}(\varepsilon, \vec{x}) = -3\lambda_{LMN} \varepsilon^{\Delta_L + \Delta_M + \Delta_N - 2d - 1} \times \tag{3.23}$$
$$\int d\vec{y}_3 d\vec{y}_4 s_M(\vec{y}_3) s_N(\vec{y}_4) \left(\int d^{d+1}y \sqrt{g} K_{\Delta_L}(y_0, \vec{x}-\vec{y}) K_{\Delta_M}(y_0, \vec{y}-\vec{y}_3) K_{\Delta_N}(y_0, \vec{y}-\vec{y}_4) + o(\varepsilon) \right).$$

Since the cubic couplings λ_{LMN} vanish in the extremal case, the integral

$$\int_\varepsilon^\infty dy_0 \int d\vec{y} \sqrt{g} K_{\Delta_L}(y_0, \vec{x}-\vec{y}) K_{\Delta_M}(y_0, \vec{y}-\vec{y}_3) K_{\Delta_N}(y_0, \vec{y}-\vec{y}_4),$$

which appears in evaluation of a 3-point function, is finite in the limit $\varepsilon \to 0$ (it diverges only in the extremal case) and, therefore, can be approximated as

$$\Lambda_{LMN}(\vec{x}; \vec{y}_3, \vec{y}_4) + o(\varepsilon),$$

where $\Lambda_{LMN}(\vec{x}, \vec{y}_3, \vec{y}_4)$ is defined as

$$\Lambda_{LMN}(\vec{x}; \vec{y}_3, \vec{y}_4) = \int_0^\infty dy_0 \int d\vec{y} \sqrt{g} K_{\Delta_L}(y_0, \vec{x}-\vec{y}) K_{\Delta_M}(y_0, \vec{y}-\vec{y}_3) K_{\Delta_N}(y_0, \vec{y}-\vec{y}_4).$$

Thus, $\partial_0 s_L^{(1)}$ behaves itself as

$$\partial_0 s_L^{(1)}(\varepsilon, \vec{x}) = -3\lambda_{LMN} \varepsilon^{\Delta_L + \Delta_M + \Delta_N - 2d - 1} \left(A_{LMN}(\vec{x}) + o(\varepsilon) \right) \tag{3.24}$$

with $A_{LMN}(\vec{x}) = \int d\vec{y}_3 d\vec{y}_4 s(\vec{y}_3) s(\vec{y}_4) \Lambda_{LMN}(\vec{x}; \vec{y}_3, \vec{y}_4)$ and we, therefore, find

$$I_3 = -3C^{(2)}_{IJL} \lambda_{LMN} \varepsilon^{\Delta_L + \Delta_M + \Delta_N - 3d} \int d\vec{x}\ \ \varepsilon \partial_0 s_I^{(0)} \varepsilon \partial_0 s_J^{(0)} (A_{LMN}(\vec{x}) + o(\varepsilon)\ ).$$

The last formula allows one to determine the behavior of the corresponding correlation function in the momentum space. Namely, the leading in ε contribution to the 4-point correlation function is proportional to

$$C^{(2)}_{IJL}\lambda_{LMN}\varepsilon^{\Delta_L+\Delta_M+\Delta_N-3d} \quad \varepsilon\partial_0 K_I(\varepsilon,\vec{k}_1)\varepsilon\partial_0 K_J(\varepsilon,\vec{k}_2)\Lambda_{LMN}(\vec{k}_1+\vec{k}_2;\vec{k}_3,\vec{k}_4), \tag{3.25}$$

where function $\Lambda_{LMN}(\vec{k};\vec{k}_3,\vec{k}_4)$ stands now for the Fourier transform of $\Lambda_{LMN}(\vec{x};\vec{y}_3,\vec{y}_4)$ in all its arguments.

To find the relevant contribution to the 4-point function we use (3.17) so that the leading contribution (3.25) is given by the sum of two different terms: the first one contains only one log, while the second one contains the product of two logs. We first consider the one-log case and show that it provides in particular a contribution to a subextremal 4-point correlation function. For definiteness we pick up here the following term

$$C^{(2)}_{IJL}\lambda_{LMN}\varepsilon^{\Delta_L+\Delta_M+\Delta_N+2\Delta_J-4d}k_2^{2\Delta_J-d}\log k_2\Lambda_{LMN}(\vec{k}_1+\vec{k}_2;\vec{k}_3,\vec{k}_4), \tag{3.26}$$

which gives a nonvanishing contribution if

$$2\Delta_J+\Delta_L+\Delta_M+\Delta_N-4d=\Delta_I+\Delta_J+\Delta_M+\Delta_N-4d$$

i.e. $\Delta_I=\Delta_J+\Delta_L$. Clearly, this equality is not too restrictive and it allows in particular the solution for the subextremal case[5], i.e., when $\Delta_I=\Delta_J+\Delta_M+\Delta_N-2$ and $\Delta_L=\Delta_M+\Delta_N-2$. Let us now show that for these values of conformal dimensions (3.26) indeed represents the relevant momentum space structure of a subextremal 4-point correlation function.

Due to the non-renormalization theorem a subextremal 4-point correlation function of single-trace CPOs is given by the sum of products of two-point functions that is further restricted by the conformal invariance to the form

$$\langle O^I(\vec{x}_1)O^J(\vec{x}_2)O^M(\vec{x}_3)O^N(\vec{x}_4)\rangle=\frac{A_{IJMN}}{x_{12}^{2(\Delta_J+\alpha-1)}x_{13}^{2(\Delta_M+\beta-1)}x_{14}^{2(\Delta_N+\gamma-1)}x_{23}^{2\gamma}x_{24}^{2\beta}x_{34}^{2\alpha}}, \tag{3.27}$$

where $\Delta_I=\Delta_J+\Delta_M+\Delta_N-2$ and α,β,γ are integers obeying the condition $\alpha+\beta+\gamma=1$, so that only one of them is non-zero and equals to 1. Thus we have three different subextremal structures which have in general three different coefficients A.

Consider the structure with $\beta=\gamma=0$ and perform the Fourier transform. The corresponding structure in the momentum space looks as

$$\langle O^I(\vec{k}_1)O^J(\vec{k}_2)O^M(\vec{k}_3)O^N(\vec{k}_4)\rangle\sim\int d\vec{x}_1d\vec{x}_2d\vec{x}_3d\vec{x}_4\frac{e^{i\vec{k}_1\vec{x}_1+i\vec{k}_2\vec{x}_2+i\vec{k}_3\vec{x}_3+i\vec{k}_4\vec{x}_4}}{x_{12}^{2\Delta_J}x_{13}^{2(\Delta_M-1)}x_{14}^{2(\Delta_N-1)}x_{34}^2}$$
$$\sim\delta(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)k_2^{2\Delta_J-d}\log k_2\int dvdw\frac{e^{-i\vec{k}_3v-i\vec{k}_4w}}{v^{2(\Delta_M-1)}w^{2(\Delta_N-1)}(v-w)^2},$$

[5]The extremal case is of no interest here since the coupling λ_{LMN} vanishes.

where new integration variables $v = x_{13}$ and $w = x_{14}$ were introduced. Coming back to (3.26) it remains to note that in the x-space $\Lambda_{LMN}(\vec{x};\vec{y}_3,\vec{y}_4)$ is fixed by conformal invariance to be

$$\Lambda_{LMN}(\vec{x};\vec{y}_3,\vec{y}_4) = \frac{C}{(\vec{x}-\vec{y}_3)^{2(\Delta_M-1)}(\vec{x}-\vec{y}_4)^{2(\Delta_N-1)}(\vec{y}_3-\vec{y}_4)^2},$$

where we have used subextremality condition $\Delta_L = \Delta_M + \Delta_N - 2$ and C is the numerical (non-zero) constant. Transforming this expression to the momentum space we therefore find

$$\Lambda_{LMN}(\vec{k}_1+\vec{k}_2;\vec{k}_3,\vec{k}_4) = \delta(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)\int dvdw\frac{e^{-i\vec{k}_3 v - i\vec{k}_4 w}}{v^{2(\Delta_M-1)}w^{2(\Delta_N-1)}(v-w)^2}.$$

Thus, we have shown that the field redefinition induces a non-trivial contribution to the subextremal 4-point functions.

The general case $\Delta_I = \Delta_J + \Delta_L$ is considered in the same way, and we get that the changing term in a 4-point function has the form

$$\langle O^I(\vec{x}_1)O^J(\vec{x}_2)O^M(\vec{x}_3)O^N(\vec{x}_4)\rangle \sim \frac{1}{x_{12}^{2\Delta_J}x_{13}^{\Delta_L+\Delta_M-\Delta_N}x_{14}^{\Delta_L+\Delta_N-\Delta_M}x_{34}^{\Delta_M+\Delta_N-\Delta_L}} + \cdots.$$

We see that this term is obtained by plunging the operator O^J into the operator O^I. The perturbative non-renormalization of the Feynman diagrams of such a type seems to be equivalent to the proof of the non-renormalization of 2- and 3-point functions of CPOs given in [27].

Consideration of the term involving two logs shows that it scales too fast and by this reason does not lead to any contribution to 4-point functions.

The last integral to be considered is

$$I_4 = \int d\vec{x}\,\varepsilon^{-d+1}C^{(4)}_{IJKL}\partial_0 s_I^{(0)}\nabla_a s_J^{(0)}\nabla^b s_K^{(0)}\nabla^a\nabla_b s_L^{(0)} \tag{3.28}$$

The only case when the integral gives a contribution to a 4-point function is $a = b = 0$. However, in this case we can use the equations of motion for scalars s^I to express $\nabla^0\nabla_0 s_L^{(0)}$ as $(m_L^2 - \nabla^i\nabla_i)s_L^{(0)}$. Thus, this integral is equivalent to the integral I_1 (3.14), and can give a nonvanishing contribution to an extremal 4-point function.

This completes our consideration of the boundary terms (3.13).

4 Conclusion

In this paper we studied nonlinear derivative-dependent transformations of gravity fields, and showed that they change 3- and 4-point functions in a boundary CFT. We interpreted such a change of correlation functions as a manifestation of an operator basis transformation in CFT. Thus, a derivative-dependent field redefinition invokes a transformation of operators in CFT, and, as the consequence, a transformation of correlation functions. However, this transformation has a very restrictive form as by a derivative-dependent field redefinition it is possible to change only the coefficients of non-renormalized structures of correlation functions. In particular, one probably can find such a field redefinition of the gravity fields that the redefined scalars s^I would be dual to the single-trace CPOs. Still the analysis performed in the paper does not allow one to conclude this definitely. The point is that we do not have enough parameters in the field transformations because we only considered field redefinitions of the scalars s^I. In general, one should take into account the scalar dependent redefinitions of vector and tensor fields as well. This would give us enough parameters to transform the correlation functions of the extended CPOs into the ones of the single-trace CPOs.

5 Appendix

In [20] the quartic action for scalars s^I was found in the form

$$S(s) = \int_{AdS_5} \left(\mathcal{L}_4^{(4)} + \mathcal{L}_4^{(2)} + \mathcal{L}_4^{(0)} \right),$$

where the quartic terms contain the 4-derivative couplings

$$\mathcal{L}_4^{(4)} = \left(S^{(4)}_{I_1I_2I_3I_4} + A^{(4)}_{I_1I_2I_3I_4} \right) s^{I_1} \nabla_a s^{I_2} \nabla_b^2 (s^{I_3} \nabla^a s^{I_4}),$$

the 2-derivative couplings

$$\mathcal{L}_4^{(2)} = \left(S^{(2)}_{I_1I_2I_3I_4} + A^{(2)}_{I_1I_2I_3I_4} \right) s^{I_1} \nabla_a s^{I_2} s^{I_3} \nabla^a s^{I_4}$$

and the couplings without derivatives

$$\mathcal{L}_4^{(0)} = S^{(0)}_{I_1I_2I_3I_4} \; s^{I_1} s^{I_2} s^{I_3} s^{I_4}.$$

The corresponding vertices have the following symmetry properties

$$\begin{aligned} S^{(4)}_{I_1I_2I_3I_4} &= S^{(4)}_{I_2I_1I_3I_4} = S^{(4)}_{I_3I_4I_1I_2}, \quad A^{(4)}_{I_1I_2I_3I_4} = -A^{(4)}_{I_2I_1I_3I_4} = A^{(4)}_{I_3I_4I_1I_2}, \\ S^{(2)}_{I_1I_2I_3I_4} &= S^{(2)}_{I_2I_1I_3I_4} = S^{(2)}_{I_3I_4I_1I_2}, \quad A^{(2)}_{I_1I_2I_3I_4} = -A^{(2)}_{I_2I_1I_3I_4} = A^{(2)}_{I_3I_4I_1I_2} \end{aligned}$$

and their explicit values are given in [20]. What is important for our discussion here is that all the couplings are represented as sums of the $SO(6)$ tensors of three different types:

$$F(I_5) a_{I_1I_2I_5} a_{I_3I_4I_5}, \quad F(I_5) t_{I_1I_2I_5} t_{I_3I_4I_5}, \quad F(I_5) p_{I_1I_2I_5} p_{I_3I_4I_5},$$

where $F(I_5)$ is a function of I_5 and the sum over I_5 is assumed. There also appear tensors obtained from these ones by different permutation of indices. Recall that $a_{I_1I_2I_3}$, $t_{I_1I_2I_3}$ and $p_{I_1I_2I_3}$ represent the following integrals involving the scalar Y^I, the vector Y^I_α and the tensor $Y^I_{(\alpha\beta)}$ spherical harmonics respectively

$$a_{I_1I_2I_3} = \int_{S^5} Y^{I_1}Y^{I_2}Y^{I_3}, \quad t_{I_1I_2I_3} = \int_{S^5} \nabla^\alpha Y^{I_1}Y^{I_2}Y^{I_3}_\alpha, \quad p_{I_1I_2I_3} = \int_{S^5} \nabla^\alpha Y^{I_1}\nabla^\beta Y^{I_2}Y^{I_3}_{(\alpha\beta)}.$$

To prove the vanishing of the couplings in the subextremal case as well as the vanishing of the 4-derivative couplings in the sub-subextremal case we find convenient to pass to the 4-derivative vertices of the form (2.5). This is achieved by using the following relations valid on-shell:

$$\begin{aligned}
A^{(4)}_{1234}\int s_1\nabla_a s_2\nabla^2_b(s_3\nabla^a s_4) &= -2A^{(4)}_{1234}\int \nabla_a s_1\nabla_b s_2\nabla^a s_3\nabla^b s_4 \\
&- 4A^{(4)}_{1234}\int s_1\nabla_a s_2 s_3\nabla^a s_4 \\
&- \frac{1}{4}A^{(4)}_{1234}(m_1^2-m_2^2)(m_3^2-m_4^2)\int s_1s_2s_3s_4. \\
S^{(4)}_{1234}\int s_1\nabla_a s_2\nabla^2_b(s_3\nabla^a s_4) &= -S^{(4)}_{1234}\int \nabla_a s_1\nabla^a s_2\nabla_b s_3\nabla^b s_4 \\
&+ S^{(4)}_{1234}\left(m_1^2+m_2^2+m_3^2+m_4^2-4\right)\int s_1\nabla_a s_2 s_3\nabla^a s_4 \\
&+ \frac{1}{4}S^{(4)}_{1234}(m_1^2+m_2^2)(m_3^2+m_4^2)\int s_1s_2s_3s_4,
\end{aligned} \tag{5.1}$$

where here and below we write concisely the summation index I_1 simply as 1 and similar for the others, m denotes the AdS mass of a scalar field.

First we consider the subextremal case and assume for definiteness that $k_1 = k_2+k_3+k_4-2$. It is easy to show, by using the description of spherical harmonics as restrictions of functions, vectors and tensors on the $\mathbf{R}^6$ in which the sphere S^5 is embedded [10, 18], that the tensor[6] $t_{125}t_{345}$ does not vanish in the subextremal case only for $k_5 = k_3+k_4-1$ while for $p_{125}p_{345}$ it is the case only if $k_5 = k_3+k_4-2$. As for the tensor $a_{125}a_{345}$, it differs from zero in two cases: when $k_5 = k_3+k_4$ or when $k_5 = k_3+k_4-2$. Analogously, the only non-trivial values of k_5 for $a_{135}a_{245}$ are $k_5 = k_2+k_4$ and $k_5 = k_2+k_4-2$, and for $a_{145}a_{235}$ they are $k_5 = k_2+k_3$ and $k_5 = k_2+k_3-2$. Thus in all vertices we can replace k_5 by a corresponding function of k_2, k_3, k_4, and, then the only dependence on k_5 is in tensors $t_{125}t_{345}$, $p_{125}p_{345}$, $a_{125}a_{345}$, $a_{135}a_{245}$ and $a_{145}a_{235}$. However, not all of these tensors are independent. Indeed, $a_{125}a_{345}$, $a_{135}a_{245}$ and $a_{145}a_{235}$ are subjected to the following three identities [20]:

$$\begin{aligned}
&a_{125}a_{345} = a_{135}a_{245} = a_{145}a_{235}, \\
&f_5(a_{125}a_{345}+a_{135}a_{245}+a_{235}a_{145}) = (f_1+f_2+f_3+f_4)a_{125}a_{345},
\end{aligned} \tag{5.2}$$

[6]We do not assume here summation over I_5.

where $f_i = f(k_i) = k_i(k_i+4)$. For the sake of simplicity it is useful to introduce the notation

$$\begin{aligned} l_1 &= a_{125}a_{345}|_{k_5=k_3+k_4}, & l_2 &= a_{125}a_{345}|_{k_5=k_3+k_4-2}, \\ m_1 &= a_{145}a_{235}|_{k_5=k_2+k_3}, & m_2 &= a_{145}a_{235}|_{k_5=k_2+k_3-2}, \\ n_1 &= a_{135}a_{245}|_{k_5=k_2+k_4}, & n_2 &= a_{135}a_{245}|_{k_5=k_2+k_4-2}, \end{aligned}$$

where, e.g., l_1 denotes tensor $a_{125}a_{345}$ for the value of k_5 equal to k_3+k_4. Hence we have six tensors corresponding to different values of k_5 and to different order of indices, which are confined by three relations (5.2). Therefore, restricting eqs.(5.2) to the subextremal case, i.e., putting $k_1 = k_2+k_3+k_4-2$ one can solve them for any three tensors. If we choose here l_1, m_1 and n_1 as independent variables, then l_2, m_2 and n_2 are expressed as

$$\begin{aligned} l_2 &= \frac{(m_1+n_1-l_1)(k_2+1)+m_1k_3+n_1k_4}{k_1+k_2+k_3+2}, \\ m_2 &= \frac{(n_1+l_1-m_1)(k_4+1)+l_1k_3+n_1k_2}{k_1+k_2+k_3+2}, \\ n_2 &= \frac{(m_1-n_1+l_1)(k_3+1)+l_1k_4+m_1k_2}{k_1+k_2+k_3+2}. \end{aligned} \tag{5.3}$$

For $t_{125}t_{345}$ and $p_{125}p_{345}$ we will need the following three identities found in [20]:

$$t_{125}t_{345} = -\frac{(f_1-f_2)(f_3-f_4)}{4f_5}a_{125}a_{345} + \frac{1}{4}f_5(a_{145}a_{235} - a_{245}a_{135}), \tag{5.4}$$

$$\begin{aligned} (1-f_5)t_{125}t_{345} &= \frac{1}{4}(f_5^2 - f_5(f_1+f_2+f_3+f_4-4))(a_{145}a_{235} - a_{135}a_{245}) \\ &\quad -\frac{4-f_5}{4f_5}(f_1-f_2)(f_3-f_4)a_{125}a_{345}, \end{aligned} \tag{5.5}$$

$$\begin{aligned} p_{125}p_{345} &= -\frac{(f_1-f_2)(f_3-f_4)}{2(f_5-5)}t_{125}t_{345} - \frac{5}{4f_5(f_5-5)}d_{125}d_{345} \\ &\quad -\frac{1}{20}(f_1+f_2-f_5)(f_3+f_4-f_5)a_{125}a_{345} + \frac{1}{8}(f_1+f_3-f_5)(f_2+f_4-f_5)a_{135}a_{245} \\ &\quad +\frac{1}{8}(f_1+f_4-f_5)(f_2+f_3-f_5)a_{145}a_{235}, \end{aligned} \tag{5.6}$$

where

$$d_{123} = \int_{S^5} \nabla^{(\alpha}\nabla^{\beta)}Y^{I_3}\nabla_\alpha Y^{I_1}\nabla_\beta Y^{I_2} = \left(\frac{1}{10}f_2f_3 + \frac{1}{10}f_1f_3 + \frac{1}{2}f_1f_2 - \frac{1}{4}f_1^2 - \frac{1}{4}f_2^2 + \frac{3}{20}f_3^2\right)a_{125}.$$

Since in the subextremal case $t_{125}t_{345}$ is non-zero only for one value of k_5 we may use formula (5.4) and eqs.(5.3) to express $t_{125}t_{345}$ in terms of l_1, m_1 and n_1. Similarly, combining eq.(5.6) with (5.4) and with eqs.(5.3) one obtains an analogous representation for $p_{125}p_{345}$. In this way we have expressed all the quartic vertices via independent tensors l_1, m_1 and n_1.

Now we single out the field s^{I_1} and write the relevant part of the quartic 4-derivative vertices as functions of l_1, m_1 and n_1 in the form

$$Ł^{(4)} = 4 \sum_{I_2,I_3,I_4} \left(-S^{(4)}_{I_1I_2I_3I_4} - A^{(4)}_{I_1I_3I_2I_4} + A^{(4)}_{I_1I_4I_3I_2} \right) \nabla_a s^{I_1} \nabla^a s^{I_2} \nabla_b s^{I_3} \nabla^b s^{I_4}, \tag{5.7}$$

where we sum over the representations satisfying the subextremality condition. Now, we substitute the values of k_5 discussed above, and $k_1 = k_2 + k_3 + k_4 - 2$ in the quartic couplings, and obtain zero.

To analyse 2-derivative terms we represent the 2-derivative Lagrangian as follows

$$\begin{aligned} Ł^{(2)} = 4 \sum_{I_2,I_3,I_4} \Bigg(& \left(-\frac{1}{2} \tilde{S}^{(2)}_{I_1I_3I_2I_4} + \tilde{A}^{(2)}_{I_1I_2I_3I_4} \right) s^{I_1} \nabla^a s^{I_2} s^{I_3} \nabla_a s^{I_4} \\ & + \frac{1}{4} \left(\tilde{A}^{(2)}_{I_1I_2I_3I_4} (m_4^2 - m_3^2) - \tilde{S}^{(2)}_{I_1I_2I_3I_4} (m_4^2 + m_3^2) \right) s^{I_1} s^{I_2} s^{I_3} s^{I_4} \Bigg), \end{aligned}$$

where using (5.1) we define

$$\begin{aligned} \tilde{S}^{(2)}_{I_1I_2I_3I_4} &= S^{(2)}_{I_1I_2I_3I_4} + S^{(4)}_{I_1I_2I_3I_4} \left(m_1^2 + m_2^2 + m_3^2 + m_4^2 - 4 \right), \\ \tilde{A}^{(2)}_{I_1I_2I_3I_4} &= A^{(2)}_{I_1I_2I_3I_4} - 4 A^{(4)}_{I_1I_2I_3I_4}. \end{aligned}$$

This time substituting k_5 and k_1 and symmetrizing the expression obtained in I_2 and I_4, we get a non-zero function which is, however, completely symmetric in I_2, I_3 and I_4. Thus we can remove the 2-derivative term by using the shift

$$s^{I_1} \to s^{I_1} - \frac{2}{3\kappa_1} \left(-\frac{1}{2} \tilde{S}^{(2)}_{I_1I_3I_2I_4} + \tilde{A}^{(2)}_{I_1I_2I_3I_4} \right) s^{I_2} s^{I_3} s^{I_4},$$

where $\kappa_1 = \frac{32k_1(k_1-1)(k_1+2)}{k_1+1}$. This shift also produces an additional contribution to the non-derivative terms which is equal to

$$-\frac{2}{3} \left(-\frac{1}{2} \tilde{S}^{(2)}_{I_1I_3I_2I_4} + \tilde{A}^{(2)}_{I_1I_2I_3I_4} \right) (m_2^2 + m_3^2 + m_4^2 - m_1^2) s^{I_1} s^{I_2} s^{I_3} s^{I_4}.$$

After accounting this contribution the non-derivative terms acquire the form

$$\begin{aligned} Ł^{(0)} = 4 \sum_{I_2,I_3,I_4} \Bigg(& S^{(0)}_{I_1I_2I_3I_4} - \frac{1}{6} \left(-\frac{1}{2} \tilde{S}^{(2)}_{I_1I_3I_2I_4} + \tilde{A}^{(2)}_{I_1I_2I_3I_4} \right) (m_2^2 + m_3^2 + m_4^2 - m_1^2) \\ & + \frac{1}{4} \left(\tilde{A}^{(2)}_{I_1I_2I_3I_4} (m_4^2 - m_3^2) - \tilde{S}^{(2)}_{I_1I_2I_3I_4} (m_4^2 + m_3^2) \right) \\ & + \frac{1}{4} S^{(4)}_{1234} (m_1^2 + m_2^2)(m_3^2 + m_4^2) - \frac{1}{4} A^{(4)}_{1234} (m_1^2 - m_2^2)(m_3^2 - m_4^2) \Bigg) s^{I_1} s^{I_2} s^{I_3} s^{I_4}. \end{aligned}$$

Substituting k_5 and k_1 and symmetrizing the coefficient obtained in I_2, I_3 and I_4 we end up with zero. Thus, we have shown that after the additional field redefinition all subextremal quartic couplings vanish.

The treatment of the 4-derivative quartic couplings in the sub-subextremal case is quite analogous to the previous one. For definiteness we assume that $k_1 = k_2 + k_3 + k_4 - 4$. Then $a_{125}a_{345}$ is non-zero in three cases: $k_5 = k_3 + k_4$, $k_5 = k_3 + k_4 - 2$ and $k_5 = k_3 + k_4 - 4$. Similarly $a_{135}a_{245}$ is non-zero only for k_5 equal to $k_2 + k_4$, $k_2 + k_4 - 2$ or $k_2 + k_4 - 4$, while $a_{145}a_{235}$ admits for k_5 one of the following values $k_2 + k_3$, $k_2 + k_3 - 2$ or $k_2 + k_3 - 4$. Denote

$$\begin{aligned}
l_1 &= a_{125}a_{345}|_{k_5=k_3+k_4}, & l_2 &= a_{125}a_{345}|_{k_5=k_3+k_4-2}, & l_3 &= a_{125}a_{345}|_{k_5=k_3+k_4-4},\\
m_1 &= a_{145}a_{235}|_{k_5=k_2+k_3}, & m_2 &= a_{145}a_{235}|_{k_5=k_2+k_3-2}, & m_3 &= a_{145}a_{235}|_{k_5=k_2+k_3-4},\\
n_1 &= a_{135}a_{245}|_{k_5=k_2+k_4}, & n_2 &= a_{135}a_{245}|_{k_5=k_2+k_4-2}, & n_3 &= a_{135}a_{245}|_{k_5=k_2+k_4-4}.
\end{aligned}$$

Then identities (5.2) allow one to express l_3, m_3, n_3 via six independent tensors l_1, m_1, n_1 and l_2, m_2, n_2, e.g.,

$$\begin{aligned}
l_3 = &-\frac{1}{k_2+k_3+k_4+2}\Big(m_2 + n_2 + l_2 + (-2m_1 - m_2 + l_2)k_3 \\
&+ (-2m_1 - m_2 - 2n_1 - n_2 + 2l_1 + 2l_2)k_2 + (-2n_1 - n_2 + l_2)k_4\Big).
\end{aligned} \tag{5.8}$$

The formulas for m_3 and n_3 are obtained from this one by permutations of indices.

Except the tensor structures we have just considered the quartic couplings of 4-derivative vertices contain a tensor $(f_5 - 1)^2 t_{125}t_{345}$ (see Appendix A of [20]). In the sub-subextremal case $t_{125}t_{345}$ differs from zero only for $k_5 = k_3 + k_4 - 1$ or $k_5 = k_3 + k_4 - 3$. It is then suitable to represent

$$(f_5 - 1)^2 t_{125}t_{345} = ((f_5 - \alpha)(f_5 - \beta) + a(f_5 - 1) + b)\, t_{125}t_{345}, \tag{5.9}$$

where

$$\begin{aligned}
a &= \alpha + \beta - 2,\\
b &= -(\alpha - 1)(\beta - 1)
\end{aligned}$$

and pick up for α and β the following values $\alpha = f(k_3+k_4-1)$ and $\beta = f(k_3+k_4-3)$. Clearly, in the sub-subextremal case the first term in the r.h.s. of (5.9) is absent and we may use identities (5.4) and (5.5) to rewrite $(f_5 - 1)^2 t_{125}t_{345}$ via $l_1, \ldots, n_2$. Hence, as in the subextremal case, we reduced all quartic couplings of 4-derivative vertices to the independent tensor structures.

Now upon substituting in eq.(5.7) the 4-derivative quartic couplings evaluated for the proper values of k_5 and putting $k_1 = k_2 + k_3 + k_4 - 4$ we obtain zero.

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