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# Ergodicity Breaking in Quantum Discrete Time Models

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A thesis submitted in partial fulfilment of the requirements towards the degree of  
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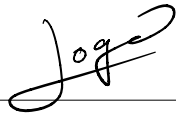
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## Abstract

In recent years, we have witnessed astonishing technological progress in the capabilities of digital quantum simulators, which hold the potential to solve certain problems that are classically intractable. A particularly promising application is the simulation of the dynamics of quantum many-body systems. Although these devices inherently realise discrete-time models, they can also be programmed to approximate continuous-time evolution. Moreover, in certain circuit models, such as dual-unitary or random unitary circuits, one can analytically compute various quantities of interest. Therefore, there are two aspects to studying discrete-time models. On the one hand, they are inherently realised in digital quantum processors, and on the other hand, they can enable exact calculations.

In this thesis, we focus on studying quantum discrete-time models which display ergodicity breaking mechanisms. The first major result is the demonstration of ergodicity breaking in dual-unitary circuits (DUCs) via quantum many-body scars (QMBS). We present the main properties of DUCs and key results from the literature, followed by a systematic approach to obtain DUCs that host QMBS states, and we highlight the consequences using numerical evidence.

We next consider a different toy model that hosts so-called asymptotic quantum many-body scars (AQMBS), which are exact eigenstates only in the thermodynamic limit. However, their effects can also be observed for finite system sizes, in which they lead to anomalous thermalisation. The considered system hosts both exact QMBS and AQMBS states. The latter are shown to arise from the remnants of gapless excitations that participate in a ground-state localisation transition. The repercussions of these states are demonstrated using classical tensor network simulations and experimental results from a trapped-ion quantum computer, which constitute the first experimental detection of AQMBS.

In the final results chapter we present Complete Hilbert Space Ergodicity (CHSE), a notion for ergodicity in a unitarily evolving quantum system, closely aligned with the classical concept. CHSE can be achieved by breaking time translational invariance in discrete-time models, moving beyond the Floquet paradigm by constructing aperiodic drives. We investigate how conserved quantities, both local and non-local, affect CHSE. We find that in the presence of conserved quantities, whether from QMBS, Hilbert space fragmentation, or symmetries, a system can still exhibit CHSE within dynamically decoupled subspaces.

In conclusion, we have studied several discrete-time models exhibiting ergodicity breaking. Exploiting known properties of DU circuits, we demonstrated the presence of QMBS in provably chaotic and thermalising systems. We also presented the first experimental demonstration of AQMBS states, on quantum computers, underscoring their relevance for investigating new physical phenomena.



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# Glossary and Notation

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## Acronyms

- **ETH**: Eigenstate Thermalisation Hypothesis
- **HSF**: Hilbert Space Fragmentation
- **MBL**: Many-Body Localisation
- **QMBS**: Quantum Many-Body Scars
- **AQMBS**: Asymptotic Quantum Many-Body Scars
- **DU**: Dual-unitary

## Systems and states

- $N$ : System size (e.g. number of lattice sites or spins)
- $\mathcal{H}$ : Hilbert space of the full system;  $\dim(\mathcal{H}) = d^N$  for local dimension  $d$
- $\hat{H}$ : Hamiltonian (time-independent unless stated otherwise)
- $|\psi\rangle$ : Pure quantum state;  $|\psi(t)\rangle$  denotes the time-evolved state
- $\hat{U}(t)$ : Time-evolution operator,  $\hat{U}(t) = e^{-i\hat{H}t}$  (setting  $\hbar = 1$ ).
- $|E_\alpha\rangle$ : Energy eigenstate with eigenvalue  $E_\alpha$ , i.e.  $\hat{H}|E_\alpha\rangle = E_\alpha|E_\alpha\rangle$

## Subsystems and density matrices

- $A$ : Subsystem (spatial region);  $\bar{A}$  is its complement
- $\hat{\rho}$ : Density matrix of the full system;  $\hat{\rho} = |\psi\rangle\langle\psi|$  for a pure state
- $\hat{\rho}_A$ : Reduced density matrix on  $A$ ,  $\hat{\rho}_A = \text{Tr}_{\bar{A}}(\hat{\rho})$
- $\langle\hat{O}\rangle$ : Expectation value of observable  $\hat{O}$ :  $\langle\hat{O}\rangle = \text{Tr}(\hat{\rho}\hat{O})$
- $S_A$ : Bipartite entanglement (von Neumann) entropy of subsystem  $A$ :

$$S_A = -\text{Tr}(\hat{\rho}_A \log \hat{\rho}_A)$$

## Pauli matrices

The Pauli matrices are:

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \hat{\mathbb{I}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We also use the equivalent notation:

$$\hat{X} \equiv \hat{\sigma}_x, \quad \hat{Y} \equiv \hat{\sigma}_y, \quad \hat{Z} \equiv \hat{\sigma}_z.$$

Operators acting on site  $n$  are written as  $\hat{\sigma}_\mu^n$  or  $\hat{O}_n$ .



## List of publications

The results exposed in this thesis are based, primarily, on the following publications:

- L. Logarić, S. Dooley, S. Pappalardi, and J. Goold, *Quantum many-body scars in dual-unitary circuits*, [Phys. Rev. Lett. \*\*132\*\*, 010401 \(2024\)](#).
- L. Logarić, J. Goold, and S. Dooley, *Hilbert subspace ergodicity*, [Phys. Rev. B \*\*111\*\*, 144310 \(2025\)](#)

And the following preprint:

- L. Logarić, J. Goold, and S. Dooley, *Dynamical signatures of conventional and asymptotic quantum many-body scars on a trapped ion simulator*, [arXiv:2604.12296 \(2026\)](#)



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## Introduction

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In our everyday lives, the notion of temperature and associated concepts from thermodynamics play a key role in navigating the world around us. The laws of thermodynamics have, evidently, stood the test of time and are therefore held as foundational for our understanding of nature.

However, there is an inherent tension between the unitary evolution of quantum-mechanical systems and the intuitive concept of temperature from our daily experience. How can unitary, time-reversible evolution lead to thermal states? This apparent conflict was first addressed by the Eigenstate Thermalisation Hypothesis (ETH) [1, 2]. Informally, it states that for chaotic quantum systems starting in an arbitrary pure state, local observables appear to be thermal in the long-time limit, with the temperature determined by the energy contained in the initial state. This can be related to the so-called “thermal” structure of eigenstates. The ETH thereby provides a microscopic mechanism by which thermodynamics emerges from the structure of eigenstates of closed quantum systems, rather than requiring an external environment.

Since its inception, several experimental and numerical tests have corroborated its validity, with analytical proofs in special cases [2]. After its proposal, discoveries of mechanisms for its violation also emerged. These include Many-body localisation

(MBL) [3–5], in which the presence of a strong disorder leads to the absence of thermalisation and Hilbert space fragmentation (HSF) [6, 7], where kinetic constraints prevent thermalisation. ETH may also be violated by so-called quantum many-body scars (QMBS) [8–10], in which isolated non-thermal eigenstates are embedded in an otherwise thermalising spectrum, leading to the absence of thermalisation for particular initial conditions.

In parallel to these developments, we have witnessed the steady improvement of analogue and digital quantum simulators. The former have been crucial for the discovery of QMBS [8] and HSF [11], while the latter offer a versatile platform for probing physical phenomena and are anticipated to potentially solve certain classically intractable problems [12, 13].

Although digital quantum simulators are frequently used to investigate continuous-time dynamics, they inherently implement discrete-time dynamics, which correspond to certain quantum circuits [13]. However, discrete-time dynamics can give rise to phenomena with no direct continuous-time analogue, such as Floquet phases of matter [14–16]. In some circuit models, such as dual-unitary or random unitary circuits, one can analytically compute various quantities of interest [17–19]. Therefore, there are two complementary aspects to studying discrete-time models. On the one hand, they are inherently implemented in digital quantum processors, making them directly relevant for simulation on near-term quantum hardware. And on the other hand, they may enable exact calculations, providing benchmarks for experimental results as well as insights into phenomena such as thermalisation and information scrambling [19].

In this thesis, the focus is on discrete-time models which display ergodicity breaking mechanisms. We provide a detailed introduction to motivate the research undertaken in Chapter 2, where we formally introduce the Eigenstate thermalisation hypothesis, discuss the mechanisms for its violation, and present some basic notation and properties of digital quantum simulators.

In Chapter 3, we introduce dual-unitary circuits and some of their properties [18]. We provide a systematic approach to find instances of these models which host QMBS states. By exploiting the fact that these models are provably ergodic, we find that

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the embedded QMBS states lead to a weak violation of the ETH.

This is followed by discussion on a related phenomenon, the so-called asymptotic quantum many-body scars (AQMBS) [20] in Chapter 4. These share several properties with QMBS but become exact QMBS only in the thermodynamic limit. We present a direct connection between AQMBS and gapless excitations in a specific class of models. We then consider in depth a particular model that hosts both regular QMBS and AQMBS states. Using classical numerical techniques, we highlight how AQMBS lead to anomalous thermalisation in finite systems, in both continuous- and discrete-time models. For the latter, we perform an experiment on current digital quantum simulators and observe slower thermalisation as the system size is increased. We contrast this with the behaviour of a non-scar state, which thermalises rapidly at any system size, and a regular scar, which evades thermalisation.

Finally, in Chapter 5, we go beyond the Floquet picture. In models with aperiodic driving, it has been shown that one can recover a notion of ergodicity reminiscent of the classical one. Namely, the system may explore all of its Hilbert space, a phenomenon termed Complete Hilbert space ergodicity (CHSE) [21]. We explore how ETH-violating mechanisms affect these models, finding that CHSE may still be observed within the decoupled subspaces. We conclude the thesis with final remarks in Chapter 6.



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### Background

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#### 2.1 Thermalisation in Quantum Systems: Introducing the Eigenstate Thermalisation Hypothesis

The concept of ergodicity has been one of the underpinning foundations for thermalisation in classical systems, ever since first introduced in the late 19th century [22]. A classical system is considered ergodic if, over sufficiently long timescales, it explores all accessible points in phase space with equal probability. Extending the same notion to closed quantum systems, undergoing unitary dynamics governed by the Schrödinger equation however [23], appears to be in strong tension with the existence of stationary solutions, the eigenstates of the Hamiltonian.

The general, time-dependent Schrödinger equation can be written as [23, 24]:

$$i\hbar \frac{d}{dt} |\Psi(t)\rangle = \hat{H}(t) |\Psi(t)\rangle, \quad (2.1)$$

where  $|\Psi(t)\rangle$  is the state vector describing the state of the system at time  $t$ , and  $\hat{H}(t)$  is the Hamiltonian governing the time evolution. In the case of time-independent Hamiltonians, it reduces to  $i\hbar \frac{d}{dt} |\Psi(t)\rangle = \hat{H} |\Psi(t)\rangle$ . This allows for a closed form

expression for the time-evolution propagator [24]:

$$\hat{U}(t) = e^{-it\hat{H}/\hbar}. \quad (2.2)$$

From here on, we set  $\hbar = 1$ . The time-independent case also leads to well-defined energy eigenstates, which satisfy [24]:

$$\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle, \quad (2.3)$$

where  $E_n$  is the eigenenergy corresponding to the state  $|\psi_n\rangle$ . The eigenstates acquire only a global phase while undergoing time evolution. If we start in some state  $|\psi_n\rangle$  at time  $t = 0$ , then at any later time  $t > 0$ , the state of the system will be  $e^{-itE_n}|\psi_n\rangle$ . Similarly, if we start in some arbitrary state, we may expand it in the energy eigenbasis as  $|\Psi(t=0)\rangle = \sum_{n=1}^{\dim(\mathcal{H})} c_n|E_n\rangle$ , and the state at any time  $t$  will still have a simple form:  $|\Psi(t)\rangle = \sum_{n=1}^{\dim(\mathcal{H})} e^{-itE_n} c_n|E_n\rangle$ . We see that for an arbitrary initial state, the system will have fixed, time-independent amplitudes  $c_n$ , only acquiring relative phases during the time evolution. Therefore the time-evolved state will be constrained to a tiny corner of the full Hilbert space  $\mathcal{H}$ , as any state with different amplitudes will be inaccessible. Viewing the Hilbert space of quantum states that can be occupied by the system in analogy to classical phase space, it becomes evident that our understanding of ergodicity for classical systems cannot be directly translated to the quantum case.

Therefore, we need to approach defining ergodicity for quantum systems from a different angle. One potential choice would be to focus on *local* observables, with support on a small, arbitrary subregion, denoted by  $A$ , of the full system. This is motivated both from an experimental stand-point, as *local* observables are accessible in contrast to global ones [2], as well as from the theoretical view, which will be elaborated upon throughout this section. For concreteness, we consider a discrete lattice with  $N$  sites,  $N_A = |A| \ll N$ , and denote the rest of the sites with  $B$ . For the purposes of the following discussion, we will also limit ourselves to translationally invariant systems. As the full system evolves unitarily, the local density matrix over  $A$  will evolve to:  $\hat{\rho}_A(t) = \text{Tr}_B[e^{-i\hat{H}t} |\psi(0)\rangle \langle\psi(0)| e^{+i\hat{H}t}]$ . The system is deemed ergodic if, in the long-

time limit,  $\hat{\rho}_A$  evolves to the Gibbs density matrix dictated by the energy of the initial state [1, 2, 25]:

$$\lim_{t \rightarrow \infty} \hat{\rho}_A(t) = \text{Tr}_B[\hat{\rho}_{\text{Th}}(\beta_0)], \quad \hat{\rho}_{\text{Th}}(\beta_0) \equiv \frac{1}{Z} e^{-\beta_0 \hat{H}}, \quad (2.4)$$

where  $Z = \text{Tr}[e^{-\beta_0 \hat{H}}]$  is the partition function,  $\hat{\rho}_{\text{Th}}(\beta_0)$  denotes the equilibrium, thermal state of the full system at inverse temperature  $\beta_0$ , which is uniquely determined by the energy of the initial state via:  $\langle \psi(0) | \hat{H} | \psi(0) \rangle = E_0 = \text{Tr}(\hat{H} e^{-\beta_0 \hat{H}}) / Z(\beta_0)$ . Note that there is also a small caveat: in the presence of symmetries, Eq. (2.4) needs to be adjusted to take those into account [26]. Eq. (2.4) is equivalent to the statement that, at the level of local observables, the pure time evolved state  $\hat{\rho}(t) = e^{-i\hat{H}t} |\psi(0)\rangle \langle \psi(0)| e^{i\hat{H}t}$  is indistinguishable from a thermal state at inverse temperature  $\beta_0$ . Loosely speaking, we can also interpret this as  $B$  acting as a bath on  $A$ .

While the energy of the total system is conserved, there are no constraints on the energy contained within the different subsystems. Therefore, even though the dynamics of the full system is unitary and it stays in a pure state throughout the time evolution, the states describing the subsystems generally become mixed [1, 27]. Provided that  $B$  is sufficiently larger than  $A$ ,  $N_B \gg N_A$ , it can be seen as thermal reservoir for  $A$ , as any transfer of energy from  $B$  to  $A$  will not have a significant impact on the total energy contained within  $B$  [28]. But, even though  $A$  will generically become a mixed state, there is no guarantee that it will approach the thermal state as dictated by Eq. (2.4) in the long time limit. However, we can try to infer what properties need to be satisfied by the eigenstates of the full Hamiltonian in order to obey Eq. (2.4) [1].

Firstly we notice that, if any initial state  $|\psi(0)\rangle$  at a given energy (density) reproduces the thermal expectation values for local observables after sufficiently long time evolution, this directly implies that this is also true for any eigenstate at the same energy (density) [29]. However, as eigenstates are stationary under Hamiltonian dynamics by definition, they must already be “thermal”, in the sense that they are indistinguishable from thermal states with the right temperature through the lens of local

observables [1, 27]. From this point on, whenever we refer to pure states as thermal, we mean this specific indistinguishability. The thermal nature of Hamiltonian eigenstates is the first, albeit a bit loose, statement of the ETH, which we will discuss in more detail below.

When probing thermalisation experimentally, or numerically, one would typically start in a product state,  $|\psi(0)\rangle = \bigotimes_{j=0}^{N-1} |\phi\rangle_j$ , as these are easy to prepare [26]. Considering local Hamiltonians of the form  $\hat{H} = \sum_j^N \hat{h}_j$ , where  $\hat{h}_j$  are bounded local operators centered at  $j$ , the total energy bandwidth  $W$  (range of eigenenergies) is extensive:  $W \sim N$ , assuming that all of the individual local terms have similar norms [2]. However, the energy standard deviations of initial product states scale as  $\sim \sqrt{N}$ . In order to see this, we can write out the energy variance as:  $(\Delta E)^2 = \langle \hat{H}^2 \rangle - \langle \hat{H} \rangle^2 = \sum_{i,j}^N \langle \hat{h}_i \hat{h}_j \rangle - \langle \hat{h}_i \rangle \langle \hat{h}_j \rangle$ . Since the individual  $\hat{h}_j$  terms are local, they can act on sites up to a set distance  $d$  away from site  $j$ . Noting that, for product states we have that  $\langle \hat{h}_i \hat{h}_j \rangle = \langle \hat{h}_i \rangle \langle \hat{h}_j \rangle$  for  $|i - j| > d$ , the corresponding contributions to the energy variance will vanish. And since the local operators are bounded as well, the energy variance will scale as  $(\Delta E)^2 \sim Nd$ . Usually we have  $d = \mathcal{O}(1)$ , so we can take  $(\Delta E)^2 \sim N \rightarrow \Delta E \sim \sqrt{N}$  [2].

Contrasting the scaling of  $\Delta E$  for product states with the total energy bandwidth, we see that  $\frac{\Delta E}{W} \sim N^{-1/2}$ . We can also expand the initial product state in the energy eigenbasis of  $\hat{H}$ :  $|\psi(0)\rangle = \sum_{\alpha} c_{\alpha} |E_{\alpha}\rangle$ . Due to the expression for the variance  $\Delta E$ , we see that the magnitudes of the coefficients  $\{|c_{\alpha}|\}$  will only be significant in the energy range  $E_{\alpha} \in [\bar{E} - \Delta E, \bar{E} + \Delta E]$ , with  $\bar{E} = \langle \psi(0) | \hat{H} | \psi(0) \rangle$  [2, 26]. Using this expansion in the energy eigenbasis, we also have:  $\langle \hat{O}(t) \rangle = \langle \psi(t) | \hat{O} | \psi(t) \rangle = \sum_{\alpha} |c_{\alpha}|^2 O_{\alpha\alpha} + \sum_{\alpha \neq \beta} c_{\alpha}^* c_{\beta} O_{\alpha\beta} \exp[i(E_{\alpha} - E_{\beta})t]$ , with  $O_{\alpha\beta} \equiv \langle E_{\alpha} | \hat{O} | E_{\beta} \rangle$  [25]. Now, if we take the average over a long time window, the off-diagonal terms will cancel out, assuming there are no degeneracies [26]. Hence, the long-time average will be [2]:

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt \langle \hat{O}(t) \rangle = \sum_{\alpha} |c_{\alpha}|^2 O_{\alpha\alpha}. \quad (2.5)$$

As stated before, the coefficients  $c_{\alpha}$  will only be significant for the energy eigenstates within a small energy range dictated by  $\bar{E}$  and  $\Delta E$ . We can therefore ne-

glect the contributions outside this energy window, such that [2, 25]:  $\sum_{\alpha} |c_{\alpha}|^2 O_{\alpha\alpha} = \sum_{E_{\alpha} \in [\bar{E}-\Delta E, \bar{E}+\Delta E]} |c_{\alpha}|^2 O_{\alpha\alpha}$ .

In order to obtain the diagonal part of the ETH ansatz, we can now look at what properties need to be satisfied by  $O_{\alpha\alpha}$ . It has been experimentally observed that local observables  $\hat{O}$  tend to an average dictated solely by the macroscopic mean energy  $\bar{E}$  [28, 30]. However, in Eq. (2.5), we see an explicit dependence on the specific  $c_{\alpha}$  determined by the initial state. Preparing two initial states with the same macroscopic energy  $\bar{E}$ , but significantly different  $c_{\alpha}$  should still yield the same value in the long time limit for Eq. (2.5) to be satisfied.

In order to guarantee this, the sum  $\sum_{\alpha} |c_{\alpha}|^2 O_{\alpha\alpha}$  needs to be independent of the specific distribution of  $c_{\alpha}$ , which holds only if the diagonal elements do not vary significantly between the eigenstates  $|E_{\alpha}\rangle$ . Therefore they must be a smooth function of the energy, similar for all eigenstates within the energy window:  $O_{\alpha\alpha} = O(\bar{E})$ , which is the diagonal part of the ETH [1, 27]. We can see that if this is satisfied, the long time average for any initial state will be:

$$\sum_{E_{\alpha} \in [\bar{E}-\Delta E, \bar{E}+\Delta E]} |c_{\alpha}|^2 O_{\alpha\alpha} = \sum_{E_{\alpha} \in [\bar{E}-\Delta E, \bar{E}+\Delta E]} |c_{\alpha}|^2 O(\bar{E}) = O(\bar{E}), \quad (2.6)$$

as desired for the thermal averages of local observables in the long time limit. The diagonal part of the ETH thus guarantees the thermalisation of local observables as stated in Eq.(2.5).

We will now motivate the off-diagonal part of the ETH. We can think about the expansion of the energy eigenstates in a product basis, which we denote as  $\{|p\rangle\}$ . As discussed above, product states have significant overlap with only those energy eigenstates whose eigenvalues are close to the expectation value  $\langle p | \hat{H} | p \rangle$ :

$$|p\rangle = \sum_{\alpha, \text{ with } E_{\alpha} \approx \langle p | \hat{H} | p \rangle} c_{\alpha} |E_{\alpha}\rangle. \quad (2.7)$$

Now, conversely, we can see the energy eigenstates  $|E_{\alpha}\rangle$  as composed of those  $|p\rangle$ , for which the energy expectation values are compatible,  $\langle p | \hat{H} | p \rangle \approx E_{\alpha}$ , such that:

$|E_\alpha\rangle = \sum_{p, \text{ with } E_\alpha \approx \langle p|\hat{H}|p\rangle} c_p |p\rangle$  [1, 27]. The number of such product states at energy  $E$  is roughly  $\Omega(E)$ , which represents the density of states. Under the additional assumption that all contributing basis states have equal overlaps with the energy eigenstates, we expect the complex amplitudes to scale as  $|\langle p|E_\alpha\rangle| \sim 1/\sqrt{\Omega(E_\alpha)}$  for the relevant product states [31].

We now use this scaling to obtain an ansatz for  $O_{\alpha\beta}$ . Inserting the identities,  $\hat{\mathbb{I}} = \sum_p |p\rangle\langle p|$ , we obtain:  $O_{\alpha\beta} = \sum_{p,q} \langle E_\alpha|p\rangle \langle p|\hat{O}|q\rangle \langle q|E_\beta\rangle$  [1, 2]. Due to locality, only a constant number of  $\langle p|\hat{O}|q\rangle$  terms will be non-zero. Additionally, for these non-vanishing contributions, we also expect  $E_\alpha$  and  $E_\beta$  to be close (i.e., within the same energy window), as  $\hat{O}$  cannot significantly change the energy density of a given  $|p\rangle$  [2]. We now invoke  $|\langle p|E_\alpha\rangle| \sim 1/\sqrt{\Omega(E_\alpha)}$  to obtain:  $|\langle p|E_\alpha\rangle||\langle q|E_\beta\rangle| \sim 1/\sqrt{\Omega(E_\alpha)\Omega(E_\beta)} \approx 1/\Omega(\bar{E})$ , where we have used  $\bar{E} = (E_\alpha + E_\beta)/2$  and that  $E_\alpha \approx E_\beta$  in the last approximation. In the above discussion we have omitted the phases,  $\phi_{p,\alpha}$  in the overlaps:  $\langle p|E_\alpha\rangle = e^{i\phi_{p,\alpha}}|\langle p|E_\alpha\rangle|$ . Including these, the expression for the off-diagonal elements becomes:  $O_{\alpha\beta} = \sum_{p,q} e^{i\phi_{p,\alpha}} e^{-i\phi_{q,\beta}} \frac{1}{\Omega(\bar{E})} \langle p|\hat{O}|q\rangle$ , where we have inserted  $|\langle p|E_\alpha\rangle| = |\langle q|E_\beta\rangle| = 1/\sqrt{\Omega(\bar{E})}$ .

Assuming that the phases  $\phi_{\alpha,p}, \phi_{\beta,q}$  are random and independent [1], we can think of the sum as a random walk in the complex plane, with each step of the walk having length  $1/\Omega(\bar{E})$  and a random phase  $\phi_{\alpha,p} - \phi_{\beta,q}$ . Due to the presence of  $\langle p|\hat{O}|q\rangle$ , the sums over  $p$  and  $q$  are not independent. Since the operator  $\hat{O}$  is local, there will be a constant number of states  $|p\rangle$  giving non-vanishing contributions  $\langle p|\hat{O}|q\rangle$  for each  $q$ . Since there are approximately  $\Omega(\bar{E})$  states (corresponding to  $q$ ) contributing to this sum, there will be  $\Omega(\bar{E})$  steps in total. Putting everything together,  $O_{\alpha,\beta}$  is a random walk of  $\Omega(\bar{E})$  steps, with length  $\frac{1}{\Omega(\bar{E})}$  and a random phase  $\phi_{p,\alpha} - \phi_{q,\beta} \equiv \phi_n$ , such that:  $O_{\alpha\beta} \sim \frac{1}{\Omega(\bar{E})} \sum_n^{\Omega(\bar{E})} e^{i\phi_n}$ . Now, to get the expected displacement after  $\Omega(\bar{E})$  steps we

use:

$$\begin{aligned}
 |O_{\alpha,\beta}|^2 &\sim \frac{1}{\Omega(\bar{E})^2} \sum_n^{\Omega(\bar{E})} \sum_{n'}^{\Omega(\bar{E})} e^{i\phi_n} e^{-i\phi_{n'}} = \frac{1}{\Omega(\bar{E})^2} \sum_{n,n'=1}^{\Omega(\bar{E})} e^{i(\phi_n - \phi_{n'})} \\
 &= \frac{1}{\Omega(\bar{E})^2} \left[ \Omega(\bar{E}) + \sum_{n,n'=1, n \neq n'}^{\Omega(\bar{E})} e^{i(\phi_n - \phi_{n'})} \right] \\
 \implies \langle |O_{\alpha,\beta}|^2 \rangle &= \frac{1}{\Omega(\bar{E})} + \frac{1}{\Omega(\bar{E})^2} \left\langle \sum_{n,n'=1, n \neq n'}^{\Omega(\bar{E})} e^{i(\phi_n - \phi_{n'})} \right\rangle. \quad (2.8)
 \end{aligned}$$

Now, since  $\phi_{n/n'}$  are equally likely to point in any direction in the complex plane, we have  $\langle \sum_{n,n'=1, n \neq n'}^{\Omega(\bar{E})} e^{i(\phi_n - \phi_{n'})} \rangle = 0$ , giving  $\langle |O_{\alpha,\beta}|^2 \rangle = \frac{1}{\Omega(\bar{E})} \implies \langle |O_{\alpha,\beta}| \rangle = \frac{1}{\sqrt{\Omega(\bar{E})}}$ . Hence, we finally arrive at the approximation  $O_{\alpha\beta} \sim 1/\sqrt{\Omega(\bar{E})}$  [1, 2]. Note that this dependence can also be written in terms of the thermodynamic entropy [32]:  $S(E) = \ln[\Omega(E)] \rightarrow O_{\alpha\beta} \sim e^{-S(\bar{E})/2}$ .

Apart from the dependence on the density of states, we also expect  $O_{\alpha\beta}$  to depend on the energy difference  $\omega \equiv E_\alpha - E_\beta$ . For large  $\omega$ , the off-diagonal elements  $O_{\alpha\beta}$  should be suppressed, since local operators cannot introduce significant energy changes [2]. We can encapsulate this dependence in terms of a smooth operator dependent function  $f_O(\bar{E}, \omega)$  [1]. Finally, we note that there is some inherent randomness in the structure of energy eigenstates  $|E_\alpha\rangle$ , which we also need to take into account [33]. This can be done by using a random variable  $R_{\alpha\beta}$ , which has zero mean  $\langle R_{\alpha\beta} \rangle = 0$  and unit variance  $\langle |R_{\alpha\beta}|^2 \rangle = 1$ . Therefore, for  $\alpha \neq \beta$ , we arrive at the conjecture  $O_{\alpha\beta} = e^{-S(\bar{E})/2} f_O(\bar{E}, \omega) R_{\alpha\beta}$  [1].

Combining the above with the diagonal case,  $O_{\alpha\alpha}$ , we obtain the full ETH ansatz for local observables in the energy eigenbasis [1, 2]:

$$O_{\alpha\beta} = \langle E_\alpha | \hat{O} | E_\beta \rangle = O(\bar{E}) \delta_{\alpha\beta} + e^{-S(\bar{E})/2} f_O(\bar{E}, \omega) R_{\alpha\beta}, \quad (2.9)$$

where  $\hat{O}$  is an arbitrary local operator,  $\bar{E} = \frac{E_m + E_n}{2}$ ,  $\omega = E_m - E_n$ ,  $f_O(\bar{E}, \omega)$  is a smooth function of  $\bar{E}$  and  $\omega$ ,  $S = \ln \Omega(E)$  is the thermodynamic microcanonical entropy with  $\Omega(E)$  representing the density of states, and  $R_{mn}$  is a random variable accounting for the chaotic structure of the off-diagonal matrix elements. The first part

of the ansatz,  $\langle E_\alpha | \hat{O} | E_\beta \rangle = O(\bar{E}) \delta_{\alpha\beta} + \dots$  is referred to as the diagonal ETH, with the second term representing the so-called off-diagonal ETH. With respect to the diagonal ETH, one can further make a distinction between its *strong* version and *weak* version. The former requires that  $\langle E_n | \hat{O} | E_n \rangle = O(\bar{E})$  is satisfied for all eigenstates, whereas the latter permits a small fraction of “non-thermal” states, which is vanishing in the thermodynamic limit [34].

The “thermal” structure of eigenstates also has a direct impact on other quantities, such as the entanglement entropy. The von Neumann entropy for the reduced density matrix describing the state of subsystem  $A$  is defined as [13]:

$$S_A^{vN}(|\psi_\alpha\rangle) = -\text{Tr}[\hat{\rho}_A \ln(\hat{\rho}_A)], \quad \hat{\rho}_A \equiv \text{Tr}_B[|\psi_\alpha\rangle \langle \psi_\alpha|]. \quad (2.10)$$

Now, due to the thermal structure we expect  $\hat{\rho}_A$  to be approximately equal to a thermal density matrix [2]. This also implies that the von Neumann entropy of subsystem  $A$  will tend to the thermodynamic entropy at inverse temperature  $\beta_\alpha$ , as dictated by the eigenstate energy  $E_\alpha$ . Hence, for a subsystem of size  $|A|$ , the ETH predicts volume law scaling for its von Neumann entropy [1]:

$$S_A^{vN}(|\psi_\alpha\rangle) \approx s(E_\alpha)|A|, \quad s(E_\alpha) \equiv \ln[\Omega(E_\alpha)]/N, \quad (2.11)$$

where  $s(E_\alpha)$  is the microcanonical thermodynamic entropy density, for states in the middle of the energy spectrum (up to corrections which vanish in the thermodynamic limit).

Chronologically in parallel to these developments, Page [35] has computed the expected bipartite entanglement entropy for a random pure state, uniformly sampled from the full Hilbert space  $\mathcal{H}$ , also called a Haar random state. The system is considered along a bipartition into  $A$  and  $B$ , with corresponding Hilbert space dimensions  $\dim(\mathcal{H}_A) = D_A$  and  $\dim(\mathcal{H}_B) = D_B$ , setting  $D_A \leq D_B$  without loss of generality. The main result of Ref. [35] in this setup is:

$$S_A^{\text{Page}} = \ln[D_A] - \frac{D_A}{2D_B} + \mathcal{O}(D_B^{-2}). \quad (2.12)$$

Therefore, in the case  $D_B \gg D_A$  we get a further simplification:  $S_A^{\text{Page}} \approx \ln[D_A]$ . In the common setting where each site has a local Hilbert space with dimension  $d$  this reduces to  $S_A^{\text{Page}} \approx |A| \ln(d)$ , also displaying a volume-law scaling, which is also characteristic for the “thermal” states according to the ETH.

However, the eigenstates of a Hamiltonian, assumed to obey the ETH, are also constrained by energy conservation. Therefore they cannot be random states with respect to the full Hilbert space, but are rather random states restricted to the allowed energy shell, with corresponding dimension  $\Omega(E) = e^{S(E)}$  [2]. Comparing the von Neumann entropy of a mid-spectrum thermal state,  $S_A^{vN}(|\psi_\alpha\rangle) \approx s(E_\alpha)|A|$ , to the approximate Page entropy,  $S_A^{\text{Page}} \approx |A| \ln(d)$ , we see that the Page entropy can also be interpreted as the entropy of an infinite temperature state, with the maximal thermodynamic entropy density  $s_{\text{max}} = \ln(d^N)/N = \ln(d)$ . Vice versa, the von Neumann entropies of thermal states have a “Page” value, rescaled to the appropriate microcanonical entropy density:  $S_A^{vN}(|\psi_\alpha\rangle) \approx \frac{s(E_\alpha)}{\ln(d)} S_A^{\text{Page}}$  [2, 35].

This connection gives us two further diagnostic tools for testing whether a given system obeys the ETH. Firstly, looking at the bipartite entanglement entropy of the eigenstates, we should see them saturating along the curve given by the maximum allowed by the microcanonical entropy density at that energy:  $S_A^{vN}(|\psi_\alpha\rangle) \approx \frac{s(E_\alpha)}{\ln(d)} S_A^{\text{Page}}$  [2]. This is routinely performed in numerical tests. However, in experimental settings there are no protocols corresponding to exact diagonalisation. An alternative probe is the growth of the entanglement entropy, starting from different product states [26]. The entanglement entropy is expected to grow until it saturates at some maximum value, dictated by the energy density of the initial state. While measuring this quantity is not straightforward, it can be done in both synthetic matter such as ultracold atoms [8, 36, 28], as well as in digital quantum simulators [37, 38].

The ergodic properties of quantum many-body systems can be probed via connected dynamical two-point correlation functions [39],

$$C_{jk}^{mm}(t) = \langle \hat{o}_j^n(t) \hat{o}_k^m(0) \rangle - \langle \hat{o}_j^n \rangle \langle \hat{o}_k^m \rangle, \quad (2.13)$$

where  $\langle \cdot \rangle = \text{Tr}(\hat{\rho}, \cdot)$  denotes expectation with respect to a reference state  $\hat{\rho}$ ,  $n, m$  label lattice sites, and  $\hat{o}_j$  is an orthonormal operator basis on a single site with  $\hat{o}_0 = \mathbb{I}$  and  $\text{Tr}(\hat{o}_j) = 0$  for  $j > 0$ . In many cases it is convenient to choose  $\hat{\rho}$  to be the infinite-temperature state, as the disconnected (second) term vanishes for traceless operators. For the purposes of discussion, we will restrict ourselves to the infinite temperature case.

For ergodic dynamics these correlations are expected to decay to zero at long times, reflecting loss of memory of local initial conditions [2, 26]. In the presence of conserved quantities, correlations decay according to a power-law [40, 41]. Correlations that decay to constant non-zero values, by contrast, indicate non-ergodic behaviour, preserving some memory of the initial state [6, 42].

For a more quantitative analysis, this can be directly related to the off-diagonal ETH [2]. In order to see the connection, we express Eq. (2.13) for the infinite temperature state in the energy eigenbasis as:  $C_{jk}^{nm}(t) = \frac{1}{D} \sum_{\alpha} \langle E_{\alpha} | \hat{o}_j^n(t) \hat{o}_k^m(0) | E_{\alpha} \rangle$ . Now, inserting the time evolution operator and a resolution of the identity:

$$\begin{aligned} C_{jk}^{nm}(t) &= \frac{1}{D} \sum_{\alpha, \beta} \langle E_{\alpha} | e^{i\hat{H}t} \hat{o}_j^n | E_{\beta} \rangle \langle E_{\beta} | e^{-i\hat{H}t} \hat{o}_k^m | E_{\alpha} \rangle \\ &= \frac{1}{D} \sum_{\alpha, \beta} e^{i(E_{\alpha} - E_{\beta})t} \langle E_{\alpha} | \hat{o}_j^n | E_{\beta} \rangle \langle E_{\beta} | \hat{o}_k^m | E_{\alpha} \rangle. \end{aligned} \quad (2.14)$$

We now see that the two-point correlation functions can be directly related to the ETH ansatz, written in Eq. (2.9). For clarity, we split the sum into the two cases  $\alpha = \beta$ , which is related to the diagonal ETH, and  $\alpha \neq \beta$ , which is dictated by the off-diagonal one. Using  $\omega = E_{\beta} - E_{\alpha}$ , we obtain:

$$\begin{aligned} C_{jk}^{nm}(t) &= \frac{1}{D} \sum_{\alpha} \langle E_{\alpha} | \hat{o}_j^n | E_{\alpha} \rangle \langle E_{\alpha} | \hat{o}_k^m | E_{\alpha} \rangle \\ &\quad + \frac{1}{D} \sum_{\alpha \neq \beta} e^{-i\omega t} \langle E_{\alpha} | \hat{o}_j^n | E_{\beta} \rangle \langle E_{\beta} | \hat{o}_k^m | E_{\alpha} \rangle. \end{aligned} \quad (2.15)$$

Now, for the diagonal part, we note that  $\langle E_{\alpha} | \hat{o}_j^n | E_{\alpha} \rangle = o_j(E_{\alpha}) + e^{-S(E_{\alpha})/2} R_{j,\alpha}$ , with  $R_{j,\alpha}$  as a random variable. The thermal expectation value for  $o_j(E_{\alpha})$  at infinite

temperature is 0 for a traceless operator. Since the same also holds for  $\hat{o}_k^m$ , the diagonal part reduces to:

$$\begin{aligned} \frac{1}{D} \sum_{\alpha} \langle E_{\alpha} | \hat{o}_j^n | E_{\alpha} \rangle \langle E_{\alpha} | \hat{o}_k^m | E_{\alpha} \rangle &= \frac{1}{D} \sum_{\alpha} e^{-S(E_{\alpha})} R_{j,\alpha} R_{k,\alpha} \\ &\sim \frac{1}{D} \sum_{\alpha} e^{-S(E_{\alpha})} \sim \frac{1}{D}. \end{aligned} \quad (2.16)$$

Hence, the contribution from the diagonal parts is exponentially suppressed as the system size is increased. We recall from Eq. (2.9) that for the off-diagonal elements we have  $\langle E_{\alpha} | \hat{o}_j^n | E_{\beta} \rangle = e^{-S(\bar{E})/2} f_{o_j}(\bar{E}, \omega) R_{j,\alpha\beta}$ . Using this expression for the off-diagonal contributions in Eq. (2.15):

$$\begin{aligned} &\frac{1}{D} \sum_{\alpha \neq \beta} e^{-i\omega t} \langle E_{\alpha} | \hat{o}_j^n | E_{\beta} \rangle \langle E_{\beta} | \hat{o}_k^m | E_{\alpha} \rangle \\ &= \frac{1}{D} \sum_{\alpha \neq \beta} e^{-i\omega t} [e^{-S(\bar{E})/2} f_{o_j}(\bar{E}, \omega) R_{j,\alpha\beta}] [e^{-S(\bar{E})/2} f_{o_k}(\bar{E}, -\omega) R_{k,\beta\alpha}] \\ &= \frac{1}{D} \sum_{\alpha \neq \beta} e^{-i\omega t} e^{-S(\bar{E})} f_{o_j}(\bar{E}, \omega) f_{o_k}(\bar{E}, -\omega) R_{j,\alpha\beta} R_{k,\beta\alpha}. \end{aligned} \quad (2.17)$$

For a large system size, where the spectrum becomes sufficiently dense (i.e. in the thermodynamic limit), we can approximate the sum over energy levels by an integral over the density of states:  $\frac{1}{D} \sum_{\alpha} \rightarrow \int_{E_{\alpha}} d\rho(E_{\alpha})$ . Since the summand in Eq. (2.17) depends explicitly on  $\bar{E}$  and  $\omega$ , we also perform the change of variables:  $\int_{E_{\alpha}} d\rho(E_{\alpha}) \int_{E_{\beta}} d\rho(E_{\beta}) \rightarrow \int_{\bar{E}} d\bar{E} \int_{\omega} d\omega \rho(\bar{E} + \omega/2) \rho(\bar{E} - \omega/2)$ . For a dense spectrum, we also expect the random contributions from  $R_{j,\alpha\beta}$  and  $R_{j,\alpha\beta}$  to average out to  $\sim 1$ . Using these assumptions, we obtain:

$$\begin{aligned} &\frac{1}{D} \sum_{\alpha \neq \beta} e^{-i\omega t} e^{-S(\bar{E})} f_{o_j}(\bar{E}, \omega) f_{o_k}(\bar{E}, -\omega) R_{j,\alpha\beta} R_{k,\beta\alpha} \\ &= \int_{\bar{E}, \omega} d\bar{E} d\omega e^{-i\omega t} \rho(\bar{E} + \omega/2) \rho(\bar{E} - \omega/2) e^{-S(\bar{E})} f_{o_j}(\bar{E}, \omega) f_{o_k}(\bar{E}, -\omega). \end{aligned} \quad (2.18)$$

We now simplify  $\rho(\bar{E} + \omega/2) \rho(\bar{E} - \omega/2) e^{-S(\bar{E})}$ . We first use  $\rho(\bar{E} \pm \omega/2) = e^{S(\bar{E} \pm \omega/2)}$  and then use the Taylor expansion:  $S(\bar{E} + \omega/2) + S(\bar{E} - \omega/2) = 2S(\bar{E}) + S''(\bar{E}) \frac{\omega^2}{4} + \mathcal{O}(\omega^4)$ . Under the mild assumption that  $S(\bar{E})$  is not varying rapidly,  $S''(\bar{E}) \omega^2/4$  will

significantly differ from 1 only at large frequencies. However, at large frequencies, the spectral functions  $f_o$  decay exponentially to 0. Therefore, we neglect  $S''(\bar{E})\omega^2/4$  and higher order terms. Now, Eq. (2.18) reduces to:

$$\begin{aligned} \int_{\bar{E}, \omega} d\bar{E} d\omega e^{-i\omega t} e^{S(\bar{E})} f_{o_j}(\bar{E}, \omega) f_{o_k}(\bar{E}, -\omega) \\ = \int_{\omega} d\omega e^{-i\omega t} F_{jk}(\omega), \end{aligned} \quad (2.19)$$

where we have introduced  $F_{jk}(\omega) = \int_{\bar{E}} d\bar{E} e^{S(\bar{E})} f_{o_j}(\bar{E}, \omega) f_{o_k}(\bar{E}, -\omega)$ . In other words, the two point correlation function can be thought of as the Fourier transform, up to the relevant time  $t$ , of  $F_{jk}(\omega)$ , which is determined by the spectral functions  $f_o$ . Now, by the ETH smoothness assumption for  $f_o$ , their exponential decay, and smoothness of  $e^{S(\bar{E})} = \rho(\bar{E})$  for a dense spectrum, the function  $F_{jk}(\omega)$  should cover a finite region over its domain. In more mathematical terms, we are assuming that  $F_{jk}(\omega)$  is absolutely integrable, such that  $\int d\omega |F_{jk}(\omega)|$  is finite. By the Riemann-Lebesgue lemma [43], its Fourier transform tends to 0:

$$\lim_{t \rightarrow \infty} \left[ \int_{\omega} d\omega e^{-i\omega t} F_{jk}(\omega) \right] = 0, \quad (2.20)$$

which leads to the decay of two-point correlations functions. From a physical perspective, the density of the spectrum for a large system size, smoothness and decay of  $f_o(\bar{E}, \omega)$ , and the dephasing between different contributions guarantee that an ergodic system will lose its memory of the initial configuration in the long-time limit.

In the special case of  $j = k$ ,  $n = m$  we obtain the autocorrelation function, which is frequently chosen to probe the ergodic nature of a given system. From an experimental point of view, these are the easiest two-point correlations to probe. We also note that, while we have focused on the infinite temperature correlations, similar conclusions can be obtained for other relevant cases, such as the microcanonical ensemble.

From the experimental standpoint, these two-point correlation functions are indispensable and directly accessible. In cold atomic systems, they can be measured via time-of-flight imaging and noise correlations [44], or more recently through quantum

gas microscopes which allow spatially resolved measurements of density-density and spin-spin correlators [45]. In spin chain systems, dynamical structure factors, which are the Fourier transform of two-point correlators, are directly probed by inelastic neutron scattering experiments [46]. Finally, in digital quantum simulators, two-point correlation functions can be extracted via projective measurements [47]. All of the mentioned protocols require only forward time evolution dictated by the system Hamiltonian, as opposed to some more sophisticated schemes required for other signatures of chaos and ergodicity [48]. In stark contrast, however, these are usually analytically intractable for interacting, including integrable, many-body systems [49, 50].

Finally, the last numerical signature of ETH that we discuss is the level spacing statistics of the energy eigenvalues  $\{E_\alpha\}$ , whose origin lies in the initial investigations of quantum chaos. Similar to thermalisation and ergodicity, transferring the classical notion of chaos to the quantum domain is not straightforward. Loosely speaking, the former is defined by an exponential growth of separation in phase space, even for very small deviations in the initial conditions [51]. Attempting to find analogous behaviour for quantum models then appears to be futile, due to unitary time evolution.

This requires us to define chaos in quantum systems differently. Similarly to ergodicity, we can instead base it on the properties of the eigenstates and associated energy eigenvalues. While not directly working on quantum chaos, Wigner was numerically studying the spectra of complex atomic nuclei [52]. Interestingly, it was observed that those spectra can be very well modelled by random matrices, constrained only by the symmetries of the system. Ordering the obtained eigenvalues in increasing order  $E_{\alpha+1} \geq E_\alpha$ , the level spacing is the difference between consecutive eigenvalues:  $s_\alpha = E_{\alpha+1} - E_\alpha$ . The probability distribution of these level spacings appeared to be the same for eigenvalues of atomic nuclei and for random matrices.

While this was only an interesting empirical finding at first, it later obtained firmer foundations via calculations on quantum systems with a classical limit. The first key result was derived by Bohigas, Giannoni and Schmit [53]. Computing the eigenspectra of quantum billiards whose classical counterparts are chaotic, they found that

level spacing statistics follow the Wigner-Dyson distribution, which led to the BGS (Bohigas-Giannoni-Schmit) conjecture. Informally, it states that for quantum models whose classical limit exhibits chaotic dynamics, the energy levels will match those of random matrices. More specifically, they will match the predictions from either the Gaussian Orthogonal (GOE), Gaussian unitary (GUE), or Gaussian symplectic ensembles (GSE), as consistent with the systems symmetries. Note that the latter is not common, therefore we omit it for the rest of the discussion. The probability distribution of the level spacings for the GOE and GUE [2]:

$$P_{\text{WD}}(s) = A_{\beta} s^{\beta} \exp[-B_{\beta} s^2], \quad (2.21)$$

where  $A_{\beta}$  and  $B_{\beta}$  are determined by fixing the mean level spacing and normalisation. For GOE, we have  $\{\beta = 1, A_{\beta} = \pi/2, B_{\beta} = \pi/4\}$  and for GUE we have  $\{\beta = 2, A_{\beta} = 32/\pi^2, B_{\beta} = 4/\pi\}$ . In contrast to chaotic systems, the level spacing distributions of integrable systems are expected to follow the Poisson distribution,  $P_{\text{Poi}}(s) = e^{-s}$ , as stated by the Berry-Tabor (BT) conjecture [54]. It is also worth noting is an alternative measure to distinguish chaos and integrability, the mean spacing ratio  $\langle r \rangle$ , defined as the average of  $r_n \equiv \min(s_n, s_{n+1})/\max(s_n, s_{n+1})$  [4]. What is neat about this quantity is that it is a scalar, and even more importantly, does not require us to take into account the density of states. When performing level spacing statistics, we need to numerically take into account the fact that the density of states varies across the spectrum (also called “unfolding” [2]), but this is not needed when using  $\langle r \rangle$ . For integrable models we expect  $\langle r \rangle_{\text{Poi}} \approx 0.386$ , and for chaotic ones either  $\langle r \rangle_{\text{GOE}} \approx 0.531$  or  $\langle r \rangle_{\text{GUE}} \approx 0.603$  [55].

These exact results have been derived for a handful of cases analytically, but we can also motivate these distributions heuristically for generic systems. For integrable Hamiltonians, the existence of an extensive number of conserved quantities severely restricts the structure of eigenstates and the eigenenergies are generically expected to be uncorrelated [54]. The level spacings are therefore well approximated by the Poisson distribution. Hence, level spacing statistics are also routinely used to check

for (non)-integrability in the many-body setting [2]

The spectral properties of a system can also be quantified via the spectral form factor (SFF), which can be compactly written as [17, 56, 57]:

$$K(t) = \langle |\text{Tr}[\hat{U}(t)]|^2 \rangle \quad \text{or} \quad K(t) = \langle |\text{Tr}[\hat{U}^t]|^2 \rangle, \quad (2.22)$$

where the former expression is used for continuous-time models, with  $\hat{U}(t) = e^{-i\hat{H}t}$ , and the latter for discrete-time ones. The average is taken with respect to disorder realisations over an appropriate ensemble, since single instances can exhibit significant fluctuations [58, 59]. Similarly to the level spacing statistics, random matrix theory makes precise statements for the behaviour of  $K(t)$  for the different ensembles, with all of them displaying a linear ramp (with the GOE having logarithmic corrections in time, which may dominate at early times) [33, 57]. While this quantity can be computed exactly for random matrix ensembles, it is generically intractable for many-body systems. As it is a dynamical quantity in contrast to the level spacing statistics, it can fortunately be probed experimentally [60].

The random matrix structure of chaotic models does not have an impact only on the level spacing statistics of eigenvalues, but also on the structure of eigenstates [2]. Using a fixed basis,  $\{|i\rangle\}$ , one can express the eigenstates of the system as  $|E_\alpha\rangle = \sum_i^N c_{\alpha,i} |i\rangle$ . In the limit  $N \rightarrow \infty$ , the real valued components of  $c_{\alpha,i}$  are well approximated by (independent and identically distributed) Gaussian random variables with zero mean [61, 62]. The variance of the real-valued components depends on the exact ensemble of the random matrices:

$$\sigma_\beta^2 = 1/(N\beta), \quad (2.23)$$

with  $\beta = 1, 2$  for the GOE and GUE, respectively [61, 62]. Loosely speaking, eigenstates of random matrices appear as random states in the Hilbert space, constrained by the symmetries of the system, which is also a requirement for the ETH ansatz.

It is also important to point out that, for the “chaotic” quantum models, one can sometimes also find eigenstates that do not have this random structure [63]. More precisely, these states have large amplitudes along unstable classical periodic orbits,

hence do not have the random structure as governed by RMT, and were dubbed *quantum scars*. Since they represent a small fraction of the full eigenspectrum, they do not have a significant impact on the level spacing distributions and other quantitative signatures of chaos.

While the development of quantum chaos and random matrix theory predates the ETH, ergodicity and chaos have now effectively become equivalent in the quantum domain [34]. This is due to the fact that, chaos is a key ingredient to produce ETH in generic interacting systems, as it gives the “randomness” of eigenstates, required for their thermal appearance in terms of local observables. Therefore, level spacing statistics is a standard diagnostic tool for ETH as well. Note that, in the presence of conventional symmetries (such as  $\mathbb{Z}_2$ ,  $U(1)$ , etc.), these need to be resolved first, and the level spacing statistics are subsequently performed for the eigenstates within different symmetry sectors [2].

All of the discussion so far has been focused on models with static Hamiltonians  $\hat{H}$ , but the predictions due to ETH and quantum chaos considerations can also be extended to periodically driven Floquet models. For the latter, we can consider a time dependent Hamiltonian  $\hat{H}(t)$ , driven with some period  $T$  such that  $\hat{H}(t+T) = \hat{H}(t)$  and the Floquet operator is then  $\hat{U}_F(T) = \mathcal{T}[\exp(-i \int_0^T \hat{H}(t) dt)]$ , where  $\mathcal{T}$  is the time ordering operator. In this setting, the energy eigenstates are replaced by Floquet eigenstates,  $|E_\alpha\rangle \rightarrow |\varphi_\alpha\rangle$  with associated eigenphases  $\varphi_\alpha$ :

$$\hat{U}(T) |\varphi_\alpha\rangle = e^{-i\varphi_\alpha} |\varphi_\alpha\rangle. \quad (2.24)$$

Instead of eigenphases one can also use “quasi-energies”  $\epsilon_\alpha$ , with the two quantities related via  $\varphi_\alpha \equiv \epsilon_\alpha T$ , although sometimes these terms are used interchangeably.

Due to the periodic driving, energy is not conserved in Floquet models, directly impacting all of the signatures of the ETH. Firstly, since energy is not conserved, any initial state will heat up to infinite temperature, from the perspective of local observables. For chaotic Floquet systems we therefore expect the eigenstates to be “thermal”, but with an inverse temperature of  $\beta = 0$ . Secondly, this also implies

that all of the eigenstates will have an entanglement entropy approaching the Page value. Lastly, the eigenphases are also expected to obey Wigner-Dyson statistics for chaotic, and Poisson statistics for integrable cases. Strictly speaking, there are subtle differences between random matrix theory for Hermitian and unitary operators, with the former described by Gaussian ensembles (GUE/GOE/GSE), and the latter by circular ensembles (CUE/COE/CSE) [64]. Nonetheless, these two sets of ensembles give rise to indistinguishable probability distributions for the level spacing statistics (where we use the eigenphase  $\varphi_\alpha$  for Floquet models), provided they obey the same symmetries [64].

It is important to point out that, while the diagonal ETH becomes trivial for Floquet models, due to lack of energy conservation, the off-diagonal ETH does not. In general, the off-diagonal elements in Eq. (2.9) cannot be evaluated exactly for arbitrary Floquet models. Recalling the connection between ETH and two-point correlation functions from Eq. (2.14), this also implies that two-point correlation functions are non-trivial, in contrast to traceless local observables, for which the expectation values decay to 0.

While the eigenstate thermalisation hypothesis makes several quantitative predictions, it is important to keep in mind that it is, as included in the name, still a hypothesis. While it has been extensively studied numerically and there have been experimental demonstrations as well, there are few models for which thermalisation is rigorously proven [2]. This leaves open the possibility of ergodicity-breaking, whereby some systems may violate the ETH. Furthermore, based on the *strong* and *weak* versions of the diagonal ETH, we can also distinguish between *weak* and *strong* ergodicity breaking mechanisms. We provide an extensive overview of those in the next section.

## 2.2 Overview of Ergodicity-Breaking Mechanisms

We now provide an extensive overview of ergodicity breaking mechanisms, their signatures, underpinning theory, and experimental/numerical demonstrations. Most of the thesis is concerned with quantum many-body scars, while also covering asymptotic

scars and fragmentation. Therefore, the subsection on integrability and many-body localisation is not required for understanding, but provided in order to complete the picture on ergodicity breaking mechanisms.

### 2.2.1 Integrability and Many-Body Localisation

As discussed in the previous section, the ETH requires that the eigenstates of a Hamiltonian behave like random states within an energy shell, a feature characteristic of quantum-chaotic systems with Wigner-Dyson level statistics. In contrast, integrable systems possess an extensive set of conserved quantities and therefore cannot satisfy the ETH. Integrability thus provides the clearest example of ergodicity breaking in quantum systems. Because all eigenstates fail to obey the ETH structure, it constitutes a form of *strong* ergodicity breaking.

In the case of non-interacting systems, such as free fermions, integrability is straightforward to see, as their time evolution is completely determined by the single particle solutions. This is in contrast to generic interacting systems, which are widely believed to be chaotic. Therefore, integrability in quantum interacting systems is associated with substantial fine-tuning and is known to be unstable under generic perturbations [65, 66]. Many interacting integrable models are solvable by means of the Bethe ansatz, with the one-dimensional XXZ model and the one-dimensional Hubbard model as paradigmatic examples.

While integrability requires a high degree of fine-tuning, Many-Body Localisation (MBL) is believed to emerge generically in systems with strong disorder [67]. MBL constitutes another form of *strong* ergodicity breaking and can be seen as the many-body generalisation of Anderson localisation. The latter describes the absence of diffusion for a single particle on a lattice. Considering a particle starting on an arbitrary site in the system, it can be shown that it will remain localised in the presence of sufficiently strong disorder, which depends on the lattice dimensions [68]. This phenomenon underpins insulating phases in certain weakly interacting electronic systems and has been observed in several experimental settings, such as doped semiconductors and amorphous metal-non-metal composites among others [69].

This naturally raises the question whether such localisation behaviour can persist in the presence of interactions, which was already hypothesised in the original work in Ref. [68]. Seminal works published roughly five decades later [70, 3, 4] have addressed this issue in more detail and proposed the existence of the MBL phase, in which strong disorder suppresses transport and thereby prevents thermalisation, even for interacting systems. The key mechanism behind this behaviour are the so-called Local Integrals of Motion (LIOMs) [71], which cause the system to retain memory of its initial conditions for an infinite amount of time. As a result, the ETH is violated for all eigenstates, displaying *strong* ergodicity breaking. Therefore the MBL phase should be observable at any temperature [5]. The strength of the disorder can then be seen as driving the transition between the ETH-obeying ergodic phase and an MBL phase.

This has spurred a flurry of experimental and, even more prominently, numerical investigations. Examples of the former include tests in a (quasi)-random one-dimensional optical lattice of fermions [72], a two-dimensional optical lattice of interacting bosons [73] and a disordered Bose-Hubbard chain [74]. The canonical model for numerical studies of MBL is the disordered Heisenberg model:

$$H = J \sum_{i=1}^{N-1} (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + S_i^z S_{i+1}^z) + \sum_{i=1}^N h_i S_i^z, \quad (2.25)$$

where  $S_i^\alpha$  ( $\alpha = x, y, z$ ) are spin- $\frac{1}{2}$  operators on site  $i$ ,  $J$  is the exchange coupling, and  $h_i \in [-h, h]$  are random on-site magnetic fields drawn from a uniform distribution. It is hypothesised that for  $h \gg J$  this model exhibits the MBL phase, while for  $h \ll J$  it enters the ergodic regime.

The existence of the MBL phase also makes testable predictions about the average gap ratio, wave function properties, entanglement growth, transport behaviour as a function of disorder strength, etc. All of these have been extensively tested for finite sizes and finite times for various systems, such as the one presented in Eq. (2.25), the disordered transverse-field Ising model, disordered Bose-Hubbard and Fermi-Hubbard models, and disordered constrained models [42].

Despite the several proposed systems hosting an MBL phase and the plethora of investigations in finite-size simulations, its existence in the thermodynamic limit remains elusive. It has been argued that the MBL phase does not exist in the thermodynamic limit due to the onset of quantum chaos [75]. To date, there is no affirmative analytic proof, or other conclusive evidence, that confirms or rules out the MBL phase.

Finally, it is important to point out that digital quantum simulators, based on trapped-ion [76] as well as superconducting qubits [77] have already been used to probe the phenomenon of MBL. While such use cases will most likely stay in the shadow of analogue simulators, which allow for significantly larger system sizes and longer coherence times, their application in this domain naturally leads into questions about the phenomenon in Floquet models. Studying MBL in these was first proposed in Ref. [78], showing that sufficiently weak driving does not destroy the key dynamical signature of logarithmic entanglement growth. This initial proposal was then extended in Ref. [79], establishing that the stability of MBL in periodically driven systems depends on the drive frequency. The ergodic to MBL phase transition in Floquet systems was studied in [80, 81]. Similarly to the case of Hamiltonian models, the existence of the MBL phase in driven systems remains an active area of research, without affirmative answers in the thermodynamic limit.

### 2.2.2 Quantum Many-Body Scars

In the case of integrability and MBL, the ETH is violated by all eigenstates in the spectrum, displaying a form of *strong* ergodicity breaking. This naturally raises the question of whether it is also possible to observe *weak* ergodicity breaking in quantum systems, in which only a small, vanishing fraction, of eigenstates violates the ETH. While these should not have a significant impact on the spectral signatures of chaos and ergodicity, such as the spectral form factor, level-spacing statistics, etc., their presence could be observed in dynamical signatures, such as the evolution of fidelity, entanglement growth and thermalisation of local observables provided that the initial state has a sufficiently large overlap with the non-thermal eigenstates.

This was first observed in a seminal experiment in an analog Rydberg simulator

[8]. The Rydberg atoms were arranged in a one-dimensional chain. Although there are multiple energy levels that the Rydberg atoms can occupy, the ones relevant for the dynamics are the ground and the Rydberg (excited) states, denoted by  $|g\rangle$  and  $|r\rangle$ , respectively. The states at different sites interact via a repulsive van der-Waals interaction, while also acted upon by laser pulses which lead to a global detuning and Rabi oscillation terms. The full Hamiltonian dictating the time evolution of the system is:

$$\hat{H} = \sum_i \left[ -\Delta \hat{n}_i + \frac{\Omega}{2} (\hat{\sigma}_i^+ + \hat{\sigma}_i^-) \right] + \sum_{i < j} V_{ij} \hat{n}_i \hat{n}_j, \quad (2.26)$$

where  $\hat{n}_i = |r\rangle \langle r|_i$  is the occupation operator at site  $i$ ,  $\hat{\sigma}_i^+ = |r\rangle \langle g|$ ,  $\hat{\sigma}_i^- = [\hat{\sigma}_i^+]^\dagger$  are the local raising and lowering operators, respectively,  $\Delta$  is the global detuning strength, and  $\Omega$  is the effective Rabi frequency. The inter-atomic interaction strength decays rapidly with distance,  $V_{ij} = C_6/r_{ij}^6$ , where  $C_6 = 5,420,503 \mu\text{m}^6\text{rad} / \mu\text{s}$  is a constant and the interatomic distance is  $r_{ij} = a|i - j|$  in terms of the lattice spacing  $a$  for a one-dimensional chain.

For the experiment performed in Ref. [8], the system was initialised in various product states and then the Hamiltonian was quenched to specific values of  $\{\Delta, \Omega\}$ . The expectation was that any of the probed initial states would give rapidly decaying fidelities, and consequently quickly decaying observables. In striking opposition to the ETH predictions the Neel state  $|\mathbb{Z}_2\rangle = |grgrgr\dots\rangle$  was observed to give oscillations in the number of domain walls, as depicted in Fig. 2.1, reproduced from Ref. [8], with further numerical evidence showing slow entanglement growth. This prompted a surge of theoretical work aimed at uncovering the underlying mechanism for these observations.

The resolution to this open question was presented in [9]. The anomalous behaviour of the fidelity and entanglement growth was found to be a consequence of the initial state having significant overlaps with special eigenstates of the spectrum, possessing low bipartite entanglement entropy, and defy the diagonal part of the ETH ansatz. These can be seen as forming a small decoupled subspace from the rest of the “thermal” states and therefore any coherent superposition of the former can give rise to stable

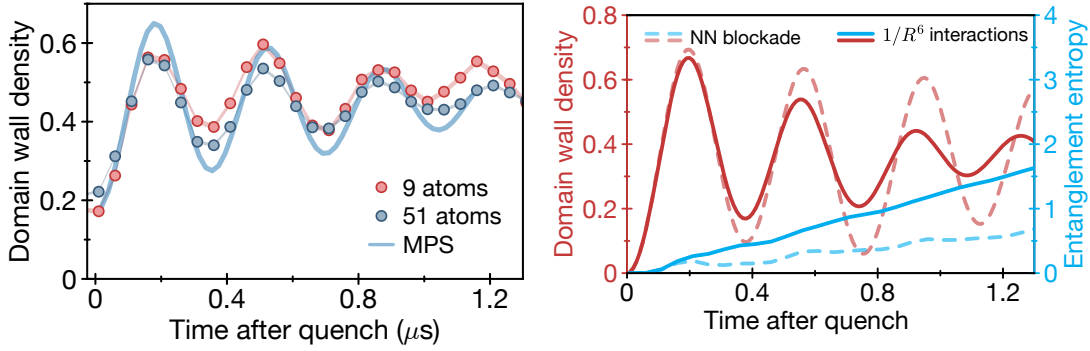


Figure 2.1: Dynamics starting from the  $|\mathbb{Z}_2\rangle$  state in a linear chain of Rydberg atoms. LEFT: Density of domain walls (number of kinks where the states at nearest neighbouring sites are different:  $|\dots gr\dots\rangle$ ), with the experimental data for  $N = 9, 51$  and numerical results for  $N = 51$ . RIGHT: Domain wall density and entanglement growth from  $|\mathbb{Z}_2\rangle$ , computed numerically for  $N = 25$ . The full lines represent dynamics under the exact Hamiltonian, while the dashed ones result from considering only nearest neighbour interactions. Reproduced from Ref. [8], with permission from Springer Nature © (2017).

periodic oscillations. In analogy to the stable periodic orbits of single particle, chaotic Hamiltonians - *quantum scars*, these states were dubbed *quantum many-body scars* (QMBS). In the particular case of the experiment in Ref. [8], the dynamics can be effectively described by the  $PXP$  Hamiltonian in the used parameter regime [9]:

$$\hat{H} = \sum_i \hat{P}_{i-1} \hat{X}_i \hat{P}_{i+1}, \quad (2.27)$$

where  $\hat{P}_i = |g\rangle\langle g|$  are projectors onto the ground state and  $\hat{X}_i = |r\rangle\langle g| + |g\rangle\langle r|$  flips the “qubit” at  $i$  between the ground and rydberg states. This effective Hamiltonian arises due to the fact that, for the given parameter set  $\{a, \Delta, \Omega\}$ , the nearest neighbour repulsion between rydberg states is orders of magnitude stronger than the other terms in the Hamiltonian,  $V_{i,i+1} \gg \Omega, \Delta$ , and therefore exciting the atom at any site is energetically prohibitive if any of the neighbouring ones are in the  $|r\rangle$  state. An exact derivation of the  $PXP$  model from the Rydberg Hamiltonian in this regime can be found in [82].

Numerically computing the eigenstates in the subspace containing the  $|\mathbb{Z}_2\rangle$  (the other subspaces are energetically separated by multiples of the interaction strength  $V_{i,i+1}$ ), and their respective bipartite entanglement entropies reveals the presence of QMBS in the spectrum [9, 83]. The latter lie distinctly apart from the remainder of the spectrum

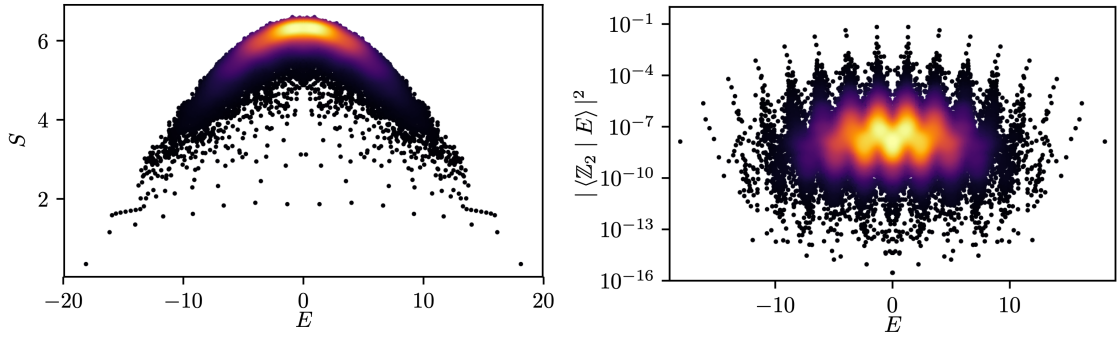


Figure 2.2: LEFT: Bipartite entanglement entropy  $S$  versus the energy  $E$  for the  $|E\rangle$  eigenstates of the  $PXP$  Hamiltonian with  $N = 30$  sites, within the sector containing  $|\mathbb{Z}_2\rangle$ . While most of the eigenstates in the middle of the spectrum are obeying the volume law, we can clearly see the low entropy QMBS states well separated from them. RIGHT: Overlaps of  $|\mathbb{Z}_2\rangle$  with the  $|E\rangle$  states, showing greater overlaps with the QMBS states. Reproduced from Ref. [83], © (2019) American Physical Society.

because their entropies follow an area law, rather than the volume law characteristic of thermal states. Furthermore, computing the overlaps of  $|\mathbb{Z}_2\rangle$  with the QMBS states shows that it is mostly a coherent superposition of those, leading to the observed fidelity oscillations. In Fig. 2.2 we display the reproduced results from [83], showing the bipartite entanglement entropy of the  $PXP$  model energy eigenstates, as well as their overlaps with  $|\mathbb{Z}_2\rangle$ .

As the number of QMBS states represents a small fraction of the full Hilbert space, the presence of QMBS states in a spectrum leads to *weak* ergodicity breaking, since the vast majority of initial states will still lead to thermalisation of local observables. This revelation naturally raised the question if there are any other models that host QMBS states.

The first one of these is the celebrated AKLT (Affleck-Kennedy-Lieb-Tasaki) model [84], which was also the first one for which the ground state was explicitly shown to have a matrix product state (MPS) form. QMBS states were also revealed in a Fermi-Hubbard model with added hopping-density interactions in the form of  $\eta$ -pairing states [85], as well as in the spin-1  $XY$  model in arbitrary dimensions [86]. While the QMBS states in these models are *exact*, admitting closed form algebraic expressions for any system size  $N$ , the scars in the  $PXP$  model are *approximate* and appear to be unstable with increasing system size [83].

The presence of scars in various different toy models prompts the question of whether

these can be understood within a unified framework. Rather surprisingly, the first systematic approach to find models with QMBS was already described in 2017 by Shiraishi and Mori [87].

The method is quite intuitive and widely applicable. For concreteness, consider an arbitrary quantum system on a lattice of  $N$  sites labelled with  $j$ . Now, if the system were evolving under  $\hat{H}' = \sum_j \hat{h}_j$ , where  $\hat{h}_j$  are arbitrary, bounded, multi-site operators in the neighbourhood of site  $j$ , the dynamics will be “chaotic”, as indicated by the level spacing statistics, thermalisation of local observables, etc. However, we want to “embed” a small fraction of states which will violate the ETH. For this purpose, one can define a set of local projectors,  $\hat{P}_j$ , which are not necessarily mutually commuting. These projectors also uniquely define the target space  $\mathcal{T}$ :

$$\mathcal{T} = \{|\psi\rangle : \hat{P}_j |\psi\rangle = 0 \quad \forall j\}, \quad (2.28)$$

which is the space spanned by the states which are simultaneously annihilated by all of the projectors. Using these ingredients, we now construct a “scarred” Hamiltonian:

$$\hat{H}_{\text{QMBS}} = \sum_j \hat{P}_j \hat{h}_j \hat{P}_j + \hat{H}_0, \quad [\hat{H}_0, \hat{P}_j] = 0. \quad (2.29)$$

It is important to highlight three main properties of  $\hat{H}_{\text{QMBS}}$ . For generic choices of  $\hat{h}_j$ , the level-spacing statistics will indicate non-integrability and most of the eigenstates in the spectrum will be thermal, similar to  $\hat{H}'$ . However, since the states in the target space are annihilated by the projectors, they will be invariant under the action of the  $\hat{P}_j \hat{h}_j \hat{P}_j$  terms. And since  $[\hat{H}_0, \hat{P}_j] = 0$ ,  $\hat{P}_j$  and  $\hat{H}_0$  will share an eigenbasis, and  $\hat{H}_0$  can be diagonalised within  $\mathcal{T}$ , with the eigenstates in it found in the middle of the energy spectrum. While it is not guaranteed, we can always choose  $\mathcal{T}$  and  $\hat{H}_0$  in such a way that the eigenstates  $|E_{\text{QMBS}, \alpha}\rangle \in \mathcal{T}$  will have a sub-volume-law scaling of the bipartite entanglement entropy. Therefore they will be so-called quantum many-body scars - eigenstates in the middle of the spectrum with low entanglement entropy in comparison to the bulk of the thermal states.

While the QMBS states in the spin-1  $XY$  model and the  $\eta$ -pairing QMBS in the Fermi-Hubbard model have been discovered independently of this projector-embedding framework, they can be recast in the form of Eq. (2.29) [86, 85]. Thus the “projector-embedding” formalism presents the first framework encompassing multiple models hosting QMBS. Several other paradigms aimed at describing the formation of these states have been presented: the spectrum generating algebra (SGA) [88], restricted spectrum generating algebra [89], as well as the group invariant formalism [90]. Since we primarily use the SM projector-embedding formalism, we refer the interested reader to the listed references for more details.

It is also important to note that, while the general projector embedding method appears to also include the  $PXP$  Hamiltonian, there is a major difference. While the states in the target subspace  $\mathcal{T}$ , which contains the QMBS states, are annihilated by the individual local projectors  $\hat{P}_j$  by construction, this does not hold for the QMBS subspace in the  $PXP$  model. Directly related to this, the embedded QMBS states of Eq. (2.29) are *exact* scars by construction, while the experimentally observed QMBS in the  $PXP$  model form *approximate* scars, which do not correspond to a target space  $\mathcal{T}$  associated with the projectors.

Secondly, the embedded QMBS states of Eq. (2.29) are *exact* scars, while the experimentally observed QMBS in the  $PXP$  model form *approximate* scars, which do not correspond to a target space  $\mathcal{T}$  associated with the projectors. Due to these deviations we usually do not consider  $PXP$  to be an instance of  $\hat{H}_{\text{QMBS}}$  from Eq. (2.29).

Similarly to the phenomenology of MBL, QMBS states have also been explored in Floquet systems. Within this context, it has been found that the Floquet version of the  $PXP$  Hamiltonian can host additional QMBS states supported at specific drive frequencies [91], while not all scar states in the Hamiltonian model survive in the driven model [92]. The Floquet  $PXP$  has also been implemented on current digital quantum simulators, with the signatures of QMBS clearly visible on IBM’s superconducting devices [93]. Moving beyond solely the  $PXP$  model, it is important to point out that the “projector-embedding” formalism can be directly applied to generic circuit models, since local unitaries of the form  $\exp[i\hat{P}_j\hat{h}_j\hat{P}_j]$  will also annihilate the target states in

$\mathcal{T}$ , a property which will be exploited on multiple occasions throughout this thesis.

### 2.2.3 Hilbert Space Fragmentation

The last ergodicity breaking mechanism that completes this overview is Hilbert space fragmentation (HSF), which was also the last one to be observed experimentally [94]. Usually fragmentation arises as a consequence of severe constraints in the dynamics, which prevent many transitions between basis states, thus leading to many sub-blocks within the Hamiltonian, which are the so-called “fragments”. These dynamical constraints play a similar role in fragmentation as disorder for the MBL phenomenon.

In order to define HSF more formally, consider some system with Hilbert space  $\mathcal{H}$ , whose dynamics is dictated by the Hamiltonian  $\hat{H}$ . We can always decompose the Hilbert space into a set of dynamically decoupled subspaces as [34]:

$$\mathcal{H} = \bigoplus_j^{N_{\mathcal{K}}} \mathcal{K}_j, \quad \mathcal{K}_j = \text{span}_t \{e^{-i\hat{H}t} |\psi_j\rangle\}, \quad (2.30)$$

where  $\text{span}_t \{e^{-i\hat{H}t} |\psi_j\rangle\} \equiv \text{span}\{|\psi_j\rangle, \hat{H}|\psi_j\rangle, \hat{H}^2|\psi_j\rangle, \dots\}$  is the subspace spanned through the time evolution of  $|\psi_j\rangle$  and is also referred to as a Krylov subspace, and  $N_{\mathcal{K}}$  is the number of these subspaces. We stress that this term is used differently in numerical applications, where the subspace refers to the space spanned by applying the Hamiltonian a finite number of times. Note that, somewhat tautologically, the states  $|\psi_j\rangle$  are chosen such that the  $\mathcal{K}_j$  are distinct, and are usually allowed to be only computational basis states, since they are experimentally easy to prepare. Otherwise, one could always set the  $|\psi_j\rangle$  as the energy eigenstates and the Hilbert space would be trivially decoupled. Hilbert space fragmentation refers to the phenomenon where the number of decoupled subspaces scales exponentially with system size  $N_{\mathcal{K}} \sim \exp(N)$ , where  $N$  is the number of sites in the system. This is to be contrasted with chaotic systems, in which  $N_{\mathcal{K}} = 1$  and systems which possess conventional symmetries, for which the number of subspaces either grows polynomially with system size or stays constant. A schematic depiction of these different structures is displayed in Fig. 2.3. The dimensions of these subspaces can also vary significantly, ranging from simple

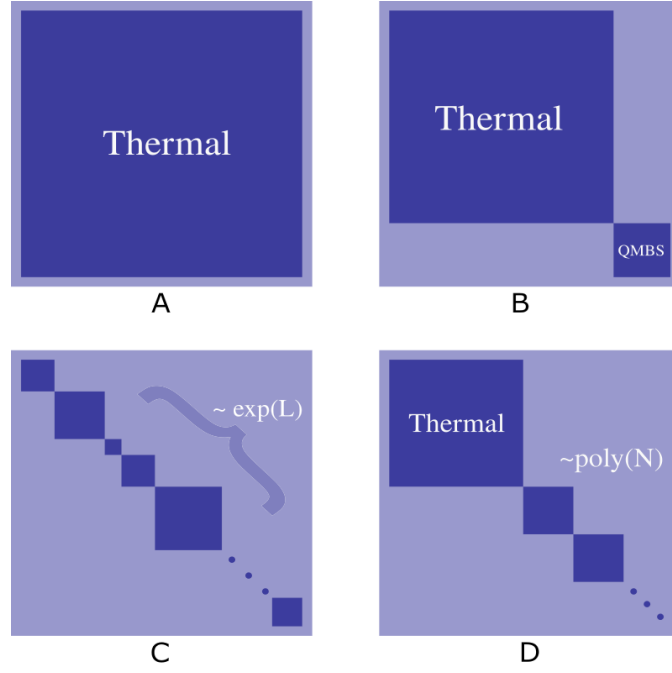


Figure 2.3: Schematic representation of dynamically decoupled Krylov subspaces for chaotic systems (A), systems with QMBS (B), Hilbert-space fragmentation (C), and conventional symmetries (D). (A) In chaotic systems, all states are thermal. (B) For systems hosting QMBS, most states are thermal apart from a vanishingly small non-thermal fraction. For HSF induced decoupling (C), the number of subspaces grows exponentially with system size, while conventional symmetries (D) yield only a constant or polynomial number of subspaces.

one-dimensional spaces containing solely one “frozen” state, up to exponentially larger ones. Within the non-trivial subspaces, models can also display level spacing statistics corresponding to either chaotic or integrable dynamics [34].

Based on the dimensionality of the largest subspace, we may further subdivide fragmentation into *strong* and *weak* fragmentation. Denoting the dimension of the largest subspace as  $D_{\max} = \max_j \{\dim(\mathcal{K}_j)\}$  and the full Hilbert space dimension as  $D = \dim(\mathcal{H})$ , a model exhibits *strong* fragmentation if  $\lim_{N \rightarrow \infty} (D_{\max}/D) = 0$  and *weak* fragmentation if  $\lim_{N \rightarrow \infty} (D_{\max}/D) = 1$  [6]. These are referred to as *strong* and *weak* fragmentation as they violate the weak and the strong variation of the ETH, respectively. In the latter case, one can still observe thermalisation for many initial states, as the size of the largest subspace allows for it, analogously to models hosting QMBS states.

Note that one may also subdivide HSF along different lines, contrasting *classical* and *quantum* fragmentation. Most models present examples of the former, where the “fragmented” structure is apparent already in the computational basis, and can be

designed using classical cellular automata rules. The latter on the other hand refers to the case where fragmentation occurs in an entangled basis, and has been shown to occur in one-dimensional models based on the Temperley-Lieb algebra [7].

In order to make it more concrete, we will here described a spin-chain model exhibiting strong HSF in the computational basis in more detail, since we also use it in one of the upcoming sections, and we will provide an overview of models and some experimental realisations afterwards.

The model we consider is a length  $N$  spin chain, defined by the Hamiltonian [95]:

$$\hat{H}_{PF} = \sum_{j=0}^{N-2} \sum_{a,b=0}^{d-1} h_{ab}^j |aa\rangle \langle bb|_{j,j+1} + \sum_{j=0}^{N-1} \sum_{a \neq b}^{d-1} \phi_{ab}^j |ab\rangle \langle ab|, \quad (2.31)$$

where  $a, b$  label the local basis states,  $h_{ab}^j$  are the elements of the Hamiltonian acting on nearest neighbouring sites  $j$  and  $j+1$ , and the  $\phi_{ab}^j$  are additional terms that are added in order to break any accidental, undesired symmetries. The terms in any  $h^j$  operator clearly allow two nearest neighbours to change their states if they are in the same local basis state, hence the name pair-flip model. It is straightforward to formulate this as a Floquet model, which will share the same Hilbert space structure. First we note that for  $d=2$ , this model is not fragmented, but has a conventional  $U(1)$  symmetry. Since only the transitions  $|\dots 0_j 0_{j+1} \dots\rangle \leftrightarrow |\dots 1_j 1_{j+1} \dots\rangle$  are allowed for any two neighbouring sites, the staggered magnetisation  $\hat{M}_S = \sum_{j=0}^{N-1} (-1)^j \hat{\sigma}_z^j$  will be conserved. However, for  $d > 2$ , this model will exhibit *strong* Hilbert space fragmentation.

We can verify this by counting the number of frozen states, showing that they scale exponentially with system size, and provide a lower bound for the number of subspaces, and comparing the size of the largest subspace to the full Hilbert space dimension. To find the number of frozen states, we notice that any global computational basis state where all nearest neighbours are in different local states cannot be changed by the action of the Hamiltonian  $\hat{H}_{PF}$  (up to a global phase). We fix the state on site  $j=0$  to any of the possible  $d$  states, and notice that each subsequent state at  $j+1$  can be in any of the  $d-1$  states different from the one at  $j$ . The number of frozen states is therefore

simply  $d(d-1)^{N-1}$ , scaling exponentially with system size. In prior work, it has been shown that the total number of subspaces is  $N_{\mathcal{K}}(N, d) = [(d-1)^{N+1} - 1]/(d-2)$ , and that the size of the largest subspace for an even number of sites is [96]:

$$|\mathcal{K}_0(N, d)| = d^N \left( 1 + \frac{1}{2} \sum_{j=1}^{N/2} d^{-2j} \binom{1/2}{j} (-1)^j (2\sqrt{d-1})^{2j} \right), \quad (2.32)$$

which is an exponentially vanishing fraction of the total Hilbert space. Therefore, the dynamics under  $\hat{H}_{PF}$  will lead to strong HSF.

Several other spin-based models have been shown to exhibit Hilbert space fragmentation, such as the Motzkin [97], Fredkin [98] and Temperley-Lieb spin-chains [7], as well as the quantum East model [99]. HSF can arise from domain wall confinement in spin- $\frac{1}{2}$  models [100], as well as due to frustration in quasi-1D models [101]. In the specific case of a Fredkin spin-chain it has also been shown to be stable against the presence of certain long-range interactions [102].

The first model in which HSF was shown to lead to ergodicity-breaking is a 1-dimensional chain with three- (and four-) site interactions [6]. More concretely, the authors of Ref. [6] considered Hamiltonians of the form:

$$\hat{H}_3 = \sum_n \left( S_n^+ (S_{n+1}^-)^2 S_{n+2}^+ + \text{H.c.} \right), \quad \hat{H}_4 = \sum_n \left( S_n^+ S_{n+1}^- S_{n+2}^- S_{n+3}^+ + \text{H.c.} \right), \quad (2.33)$$

where  $\hat{S}_n^{+(-)}$  are the spin-1 raising (lowering) operators at site  $n$ . Both  $\hat{H}_3$  and  $\hat{H}_4$  can be shown to conserve the total charge ( $\hat{Q} \equiv \sum_n \hat{S}_n^z$ ) and the dipole moment  $\hat{P}_{n_0} \equiv \sum_n (n - n_0) \hat{S}_n^z$ , where  $n_0$  is an arbitrary, but fixed reference point. While the mere presence of two conserved quantities is not sufficient to induce HSF, their interplay in this system leads to heavily constrained dynamics, which in turn give rise to exponentially many Krylov subspaces. While the bare Hamiltonian  $\hat{H}_3$  was found to lead to *strong* fragmentation, the addition of four-site interaction led to *weak* fragmentation in  $\hat{H}_3 + \hat{H}_4$ . These two can be distinguished by the infinite temperature autocorrelation function for local observables. ETH dictates that it should decay to  $\sim 1/D$  for thermalising systems. The dynamics governed solely by  $\hat{H}_3$  led to the

normalised autocorrelator  $\langle \hat{S}_0^z(t) \hat{S}_0^z(0) \rangle - \frac{2}{3N}$  saturating at a finite value of  $\sim 0.2$  in the large time limit, while it decayed to effectively 0 in the additional presence of  $\hat{H}_4$ . The reproduced results from Ref. [6] are in Fig. 2.4, presenting the distinguishing features of *strong* and *weak* fragmentation. These also include the bipartite entanglement entropy of eigenstates. While the eigenstates of  $\hat{H}_3$  are similar to strongly disordered MBL models, the spectrum arising from  $\hat{H}_3 + \hat{H}_4$  is strongly reminiscent of models with scars. We mention in passing that for HSF models, one can construct *statistically* localised integrals of motion, also called SLIOMs [103], which label the decoupled subspaces and play the analogous role as LIOMs for MBL phenomena. In contrast to the latter however, they are not perfectly localised but can have extensive support and they are not necessarily commuting as the conserved quantities in integrable models.

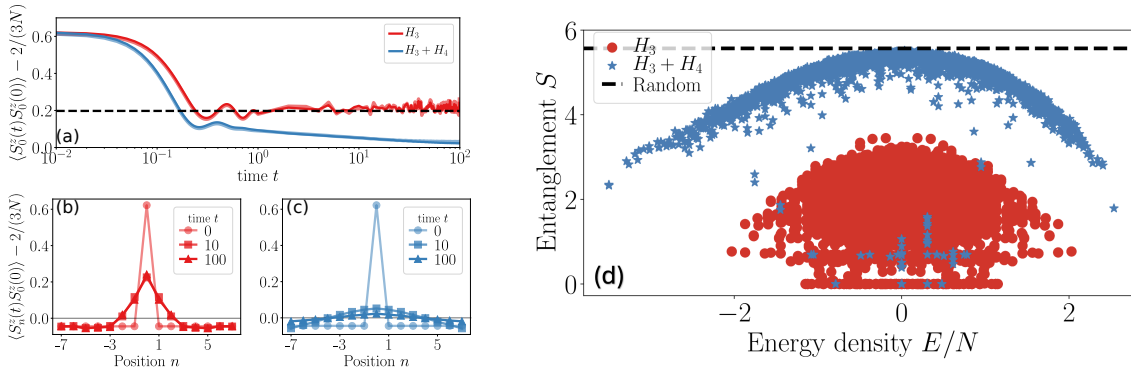


Figure 2.4: Differences between models with *strong* versus *weak* fragmentation. Decay of the autocorrelator on the middle site of the chain (a), and the spatially resolved correlators for time evolution under  $\hat{H}_3$  (b) and  $\hat{H}_3 + \hat{H}_4$  (c). While the autocorrelator decays to  $\sim 0$  for  $\hat{H}_3 + \hat{H}_4$ , it saturates to a finite value for  $\hat{H}_3$ . This is corroborated by the spatially-resolved correlators in (b, c), showing that the system retains memory of the initial state for  $\hat{H}_3$ , establishing the autocorrelator as an accessible probe for *strong* breaking of ergodicity. In (d) we see the entanglement  $S$  plotted against the energy density  $E/N$  for the eigenstates of  $\hat{H}_3$  (red) and of  $\hat{H}_3 + \hat{H}_4$  (blue). The low  $S$  eigenstates of  $\hat{H}_3 + \hat{H}_4$  are strongly reminiscent of the QMBS states, which are displayed in Fig. 2.2, while the spectrum of  $\hat{H}_3$  resembles MBL. Reproduced from Ref. [6], © (2020) American Physical Society.

The same mechanism can also be observed for Bose-Hubbard models. In Ref. [11], the consequences of HSF were shown experimentally in the relaxation of the imbalance following a quench to a tilted Bose-Hubbard model. Different initial states led to the imbalance saturating at distinct finite values, which are dictated by the Krylov subspaces. This can be seen as an immediate consequence of Krylov-restricted thermalisation, by which the thermalisation can occur within non-integrable decoupled

subspaces. This was directly observed for the first time in Ref. [104], in a one-dimensional chain of Rydberg atoms. The authors engineered the Rabi frequency and the global detuning in such a way to obtain the so-called PQXQP models, in which any site is only allowed to flip if nearest neighbours are in the *rydberg* state  $|r\rangle$  and next nearest neighbours are in the *ground* state  $|g\rangle$ . Using different initial product configurations, thermalisation could be observed in real-space sublattice dynamics. While they have used Floquet frequency modulation in order to construct additional terms in the effective Hamiltonian, the dynamics were still dominated by static interaction terms. Although there have been no experimental realisations of fragmentation in periodically driven continuous-time or discrete-time models up to date, it has been extensively studied numerically and analytically in both the former [105–108] and the latter [109–111].

Ultimately, both the QMBS and the HSF phenomena came to be understood from the commutant algebra framework [7, 112]. In a nutshell, the algebra of operators that commute with the individual local terms  $\hat{h}_i$  of any Hamiltonian  $\hat{H} = \sum_i \hat{h}_i$ , dictates the structure of the Hilbert space, as depicted in Fig. 2.3. As such, it provides the most complete paradigm up to date for distinguishing fragmentation from conventional symmetries and QMBS. Since the full discussion on commutant algebras would be very technical and is not essential for the remainder of the thesis, we refer the interested reader to Ref. [7] for an in-depth presentation, and the less enthusiastic reader to Fig. 2.3.

## 2.3 Digital Quantum Simulators

The exploration of the ergodic properties of quantum systems is a very material and time consuming effort. One of the potential solutions to these hurdles comes in the form of quantum simulators, both digital and analogue. Due to their versatility, they could be used to replace a great amount of experimental overhead. However, these devices are still relatively small and noisy. But the progress in the last decade has been, in the authors humble opinion, quite remarkable. Since most of the models discussed

in this thesis are circuit models, we focus on these instead of analogue quantum computers. We first present the basic notation and some of the most important results, which established the universality of these devices. This is followed by a discussion on how these simulators can be used to study the properties of many-body systems, including of ETH.

When introducing quantum circuits as models for digital simulation, we usually use a simplified diagrammatic representation of the operations they are performing. The Hilbert spaces at individual sites are represented by wires and almost always these local spaces are two-dimensional ( $d = 2$ ) and are referred to as qubits. The set of  $N$  individual sites constituting the device is called the quantum register. In order to perform computation on these devices, one has to perform a sequence of unitary operations, also called gates in the quantum computation context, and measurements. We mention in passing that this representation does not encapsulate all of the noisy processes going on in the background of currently available devices, but they serve as a good representation of the idealised operations we want to perform, and hopefully will be able to one day.

For any digital quantum device, we usually have access to only a finite set of gates. For example, one such gate-set could be  $\{R_x(\theta_x), R_z(\theta_z), \text{CNOT}\}$ , with  $\theta_{x,z} \in [0, 2\pi)$ . Here the  $R_{x(z)}(\theta_{x(z)}) \equiv \exp[-i\theta_{x(z)}\hat{\sigma}_{x(z)}]$  gates represent single qubit rotations about the  $x(z)$  axis and the controlled not operation is:  $\text{CNOT} \equiv |0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\sigma}_x$ . While other choices are possible, we focus on this instance for discussion purposes.

Given this limited gate set, the first important issue to address is how to construct an arbitrary single qubit gate. This can be done by a sequence of 3 single qubit operations,  $R_z(\alpha)R_x(\beta)R_z(\gamma)$ , where  $\{\alpha, \beta, \gamma\}$  are known as the Euler angles [113]. Secondly, given arbitrary single qubit gates and the CNOT entangling gates, we want to be able to construct an arbitrary 2-qubit gate, which belongs to the unitary group  $U(4)$ . An arbitrary two-qubit gate can be constructed using at most 3 CNOT operations and a limited number of single qubit operations [114]. As an aside, we also

note that an arbitrary 2-local unitary operator can be written as:

$$\hat{U} = (\hat{u}_1 \otimes \hat{u}_2) \exp[\alpha \hat{\sigma}_x \otimes \hat{\sigma}_x + \beta \hat{\sigma}_y \otimes \hat{\sigma}_y + \gamma \hat{\sigma}_z \otimes \hat{\sigma}_z] (\hat{u}_3 \otimes \hat{u}_4), \quad \forall \hat{U} \in \text{U}(4), \quad (2.34)$$

where  $\hat{u}_k$  are arbitrary single qubit unitaries and  $\alpha, \beta, \gamma \in \mathbb{R}$ . In a nutshell, this decomposition is possible because the different Hermitian generators for two qubit operations, which have the form  $\hat{h} = \hat{\sigma}_j \otimes \hat{\sigma}_k$ ,  $\hat{\sigma}_{j,k} \in \{\hat{\mathbb{I}}, \hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z\}$ , can be shown to have a convenient algebraic structure [115]. Unfortunately, such a decomposition is not possible for arbitrary local Hilbert space dimension  $d$  [116].

Note that for practical purposes usually the number of single qubit gates is not as important, as their fidelities are a lot better than the fidelities of 2-qubit ones. Finally, given that we can perform any arbitrary 2-qubit operation, we can also construct any unitary over the full Hilbert space [117]. The information from the register is then retrieved via local projective measurements, usually assumed to be in the  $Z$  basis. A circuit diagram representing these basic components is in Fig. 2.5.

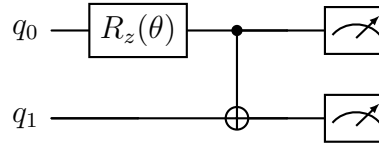


Figure 2.5: Quantum circuit representing some basic operations: the  $R_z(\theta)$  single qubit rotation, CNOT entangling gate  $\text{CNOT} \equiv |0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes \hat{\sigma}_x$ , and local projective measurements.

Now, given a fault-tolerant digital quantum simulator, one can perform certain tasks which are intractable using classical devices. The most celebrated example is Shor's algorithm [12], using which one can decipher currently used cryptographic protocols. Other cases include Grover's algorithm, used to speed up unsorted search [118] and quantum phase estimation, with which one can efficiently compute eigenstate energies [119]. As we are still far away from the required number of error corrected qubits to execute these, there appear to be other applications which are more amenable to the current devices and the near future. A very obvious target would be the direct simulation of quantum many-body systems [120].

The general protocol for any many-body or condensed matter simulation can be boiled down to three steps: preparing a state of interest, performing the time evolution, and finally extracting the information of interest from the device. We cover these three aspects in the next three paragraphs. We assume that the Hamiltonian  $\hat{H}$  describes a system of spin- $\frac{1}{2}$  particles (qubits), which is natively implemented on the devices, but this is not a major limitation as we can map fermionic Hamiltonians onto qubits via the Jordan-Wigner transformation [121], as well as bosons by truncating the otherwise infinite-dimensional Hilbert space [122].

The register of qubits is usually initialised in the  $|0\rangle^{\otimes N}$  state by default. In order to transform this state into some desired initial state,  $|\psi(0)\rangle$ , we need to perform a sequence of unitaries. For some specific states of interest from the quantum information perspective, such as the  $W$  or GHZ states, there are well known, optimised preparation protocols in literature [123]. Similarly, preparing states whose entanglement entropy obeys the area law can also be done efficiently, in logarithmic circuit depth [124]. There are several other protocols to prepare states of interest, such as adiabatic state preparation [125], quantum imaginary time evolution [126], dissipation based protocols [104], etc. These cover a wide range of potential states of interest, both pure and mixed. Unfortunately, one of the most important tasks, finding the ground state of a  $k$ -local Hamiltonian cannot be solved efficiently in general [127]. But we are still hopeful to find good algorithms for physical systems of interest, since these are not arbitrary Hermitian operators, but have specific structures which can be exploited [128].

Once we have prepared  $|\psi(0)\rangle$ , we now want to evolve it according to the appropriate system Hamiltonian  $\hat{H} = \sum_j \hat{h}_j$ , which gives the propagator  $\hat{U}(t) = \exp[-i\hat{H}t]$  when evolving for a given time  $t$ . Implementing this operator faithfully using the limited gate set on a digital simulator is not trivial and is still an ongoing research endeavour. The most widely known, and currently used, approach is the Suzuki-Trotter decomposition [129]. The main idea is to group the local terms in the Hamiltonian into mutually commuting sets, and then propagate in time by alternatively propagating according to each one of them for a small time step  $\delta t$ . Since these sets do not com-

mute, this method will incur an error, which can be controlled by suitably adjusting  $\delta t$ . To provide a clear, illustrative and common example, we consider a linear spin chain with 2-local interactions,  $\hat{H} = \sum_j \hat{h}_{j,j+1}$ , with each term acting strictly on sites  $j$  and  $j+1$  only. The local interactions can then be split into odd and even bond terms that are mutually commuting:  $\hat{H} = \hat{A} + \hat{B}$ ,  $\hat{A} = \sum_{\text{even } j} \hat{h}_{j,j+1}$ ,  $\hat{B} = \sum_{\text{odd } j} \hat{h}_{j,j+1}$ . Then we have  $\hat{U}(t) = \exp[-it(\hat{A} + \hat{B})]$ , with its first order approximation given by [130]:

$$\exp[-it(\hat{A} + \hat{B})] \approx (\exp[-i\delta t \hat{A}] \exp[-i\delta t \hat{B}])^n + \mathcal{O}\left(\frac{t^2}{n}\right), \quad \delta t \equiv \frac{t}{n}, \quad (2.35)$$

where the error scaling can be deduced from the Baker-Campbell-Hausdorff (BCH) formula [131]. This can be extended to higher order product formulae, with the second order approximation given by [129]:

$$\exp[-it(\hat{A} + \hat{B})] \approx (\exp[-i\frac{\delta t}{2} \hat{A}] \exp[-i\delta t \hat{B}] \exp[-i\frac{\delta t}{2} \hat{A}])^n + \mathcal{O}\left(\frac{t^3}{n^2}\right), \quad \delta t \equiv \frac{t}{n}. \quad (2.36)$$

Using either of the decompositions in Eq. (2.35) or Eq. (2.36) we can then perform the time evolution on a circuit by applying local gates corresponding to  $e^{-i(\delta t)\hat{h}_{j,j+1}}$  or  $e^{-i(\delta t/2)\hat{h}_{j,j+1}}$ , depicted in Fig. 2.6. Note that the second order approach is most commonly used in current devices, due to the fact that one can combine the  $\exp[-i\frac{\delta t}{2}\hat{A}]$  terms from two different time-steps, resulting in negligible depth increase in comparison to the first order approximation. An interesting feature is that, in practice, the Suzuki-Trotter approximation gives lower errors than expected based on the general error scaling [132–134]. There are several other algorithms for implementing time evolution, such as qubitization [135], linear combination of unitaries (LCU) [136], quantum-walk based methods [137], etc., but these are only advantageous in the fault-tolerant regime.

Now that we have covered how to simulate the dynamics, we are faced with measuring some desired properties of the state prepared on the device, which we denote using the corresponding density matrix as  $\hat{\rho}_{\text{out}} = |\psi_{\text{out}}\rangle \langle \psi_{\text{out}}|$ . Digital simulators allow us to perform local projective measurements, assumed to be in the  $Z$  basis, therefore we can easily extract the expectation values of the associated observables  $\langle \hat{Z}_j \rangle$ . Secondly,

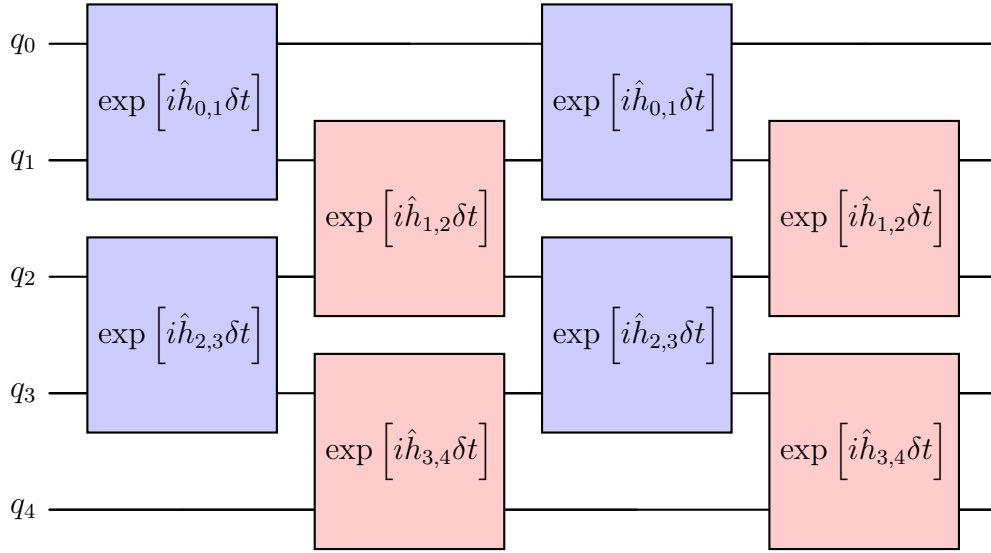


Figure 2.6: Quantum circuit representing the first two time-steps of the Trotterized evolution for  $\hat{H} = \sum_{j=0} \hat{h}_{j,j+1}$ , with  $N = 5$ .

by using single qubit rotations, we can also estimate  $\langle \hat{X}_j \rangle$  or  $\langle \hat{Y}_j \rangle$ . If we wanted to extract arbitrary properties of  $\hat{\rho}_{\text{out}}$ , we would have to perform full state tomography, which refers to the reconstruction of  $\hat{\rho}_{\text{out}}$ . We can write out the output state as  $\hat{\rho}_{\text{out}} = \sum_i \alpha_i \mathcal{P}_i$  in terms of length  $N$  Pauli strings  $\mathcal{P}_i$  and the prefactors  $\alpha_i$  which define the state. Full state tomography refers to the computation of  $\alpha_i$  up to some desired precision. For  $N$  qubits, the number of different Pauli bases we would have to measure is  $\mathcal{O}(4^N)$ , corresponding to the different combinations of  $\{\hat{\mathbb{I}}, \hat{X}, \hat{Y}, \hat{Z}\}$  at the local sites [13]. Clearly, such scaling is not favourable. Fortunately full state tomography is usually not necessary, since we are interested only in a few properties of  $\hat{\rho}$ , for example its fidelity with respect to the initial state, energy expectation values, Rényi entropies, etc.

One very promising approach that emerged recently is the estimation of these quantities via randomised measurements [138]. These can be leveraged to efficiently estimate the fidelity of the output state  $\hat{\rho}_{\text{out}}$  with respect to some reference state  $\hat{\sigma}$  ( $F = \text{Tr}[\hat{\rho}_{\text{out}}\hat{\sigma}]$ ) [139], the Rényi 2-entropy of a bipartition [37], von Neumann entanglement entropy [38], etc. For a comprehensive overview we refer the interested reader to Refs. [38, 138].

Given that we can efficiently prepare various states, perform the time evolution, as well

as estimate a variety of quantities, digital quantum simulators provide a great platform for probing phenomena displayed by many-body systems, including the validity of ETH. However, we can also see the quantum circuits themselves as models of interest. For example, taking  $\delta t \sim 1$  in the circuits displayed in Fig. 2.6, the resulting dynamics will deviate from the Hamiltonian evolution. We can also directly define the model by choosing some specific local gates  $\hat{U}_{j,j+1}$  on their own merit, without reference to any specific Hamiltonian. A single layer implementing a time-step in the circuit thus defines a Floquet operator  $\hat{U}_F$ , and we can study its properties from the lens of ETH, including its violation, which is the focus of the next chapter.



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# QMBS in Dual-unitary Circuits

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## 3.1 Introduction

*This chapter is based on Ref. [140] and adapted for the purposes of the thesis.*

One of the paradigmatic quantum circuit models that emerged in recent years are so-called *dual-unitary* (DU) circuits. These are a class of quantum circuits constructed from two-qudit gates, which are unitary both in space and time. This peculiar property enables the exact calculation of some properties that would ordinarily be prohibitively hard to compute [17, 18, 141–156].

The exact calculation of the spectral form factor has shown that certain DU circuits are “maximally chaotic,” as their spectral form factor agrees with the predictions of random matrix theory at all timescales [17, 147]. Exact results also indicate that DU circuits are fast scramblers of quantum information, since two-time correlation functions and out-of-time correlators spread at their maximal possible velocities [18, 142]. In a similar spirit, from a certain class of solvable initial states it has been shown that entanglement growth occurs at the maximal rate [144, 146, 153] and that any finite subsystem thermalises to its maximally mixed (i.e., infinite temperature) reduced density matrix in a short finite time [146]. Moreover, the exact results on two-time correlation functions allow a rigorous classification of DU circuits in terms

of their ergodic and mixing properties [18, 150].

The above exact properties seem to indicate that generic DU circuits are rapidly thermalizing systems [157]. This is further supported by the observation that it is impossible to induce MBL through disorder in DU circuits [17, 147], leaving integrability as the only known mechanism of strong ergodicity breaking in DU circuits. However, there are no arguments for why weak ergodicity breaking should not be present in some of these models.

In this chapter, we demonstrate the possibility of weak ergodicity breaking in DU circuits. Adopting the projector-embedding approach [87], presented in Section 2.2, we provide an explicit construction to insert QMBS in DU circuits of arbitrary size  $N$  and arbitrary local Hilbert space dimension  $d$ . We demonstrate our construction with examples that embed a single, two, or exponentially many QMBS in this class of circuits. Our construction shows that provably chaotic, ergodic, and mixing systems can support QMBS. Despite the rapid scrambling properties of such systems, if the system is initialised in the QMBS subspace then all quantum information remains localised in the subspace. Because of the native structure of current digital quantum simulators, some of the proposed models can be adapted directly to current devices.

## 3.2 Definition and Parameterisation

The dual of a unitary is defined via a permutation of its input and output qudits. Writing out a 2-qudit unitary operator  $\hat{U}$  in terms of 4 tensor indices, its *dual*  $\tilde{U}$  is [17]:

$$\langle j | \langle l | \hat{U} | i \rangle | k \rangle = \langle k | \langle l | \tilde{U} | i \rangle | j \rangle. \quad (3.1)$$

If we imagine  $\hat{U}$  as a propagator in  $1 + 1$  dimension, taking the dual corresponds to swapping the spatial and temporal dimension. A gate (or operator) is dual-unitary (DU) if both itself and its dual are unitary:

$$\hat{U}\hat{U}^\dagger = \hat{U}^\dagger\hat{U} = \tilde{U}\tilde{U}^\dagger = \tilde{U}^\dagger\tilde{U} = \hat{\mathbb{I}}_{d^2}. \quad (3.2)$$

Coming back to the picture of reverting the spatial and temporal dimensions, a DU gate is unitary in both the spatial and temporal dimensions. A DU circuit is any quantum circuit composed solely of DU gates, although it does not necessarily exhibit the same space-time duality. We can also represent operators and their complex conjugates pictorially, as shown in Fig. 3.1, where we explicitly label the incoming and outgoing tensor indices aligned with Eq. (3.1) and use the convention where time is evolving from bottom to top, and the sites are labelled in increasing order left to right. We omit the tensor indices in other diagrammatic representations in this section. Using this formalism, the “standard” and *dual* unitarity of operators can be depicted as displayed in Fig. 3.2.

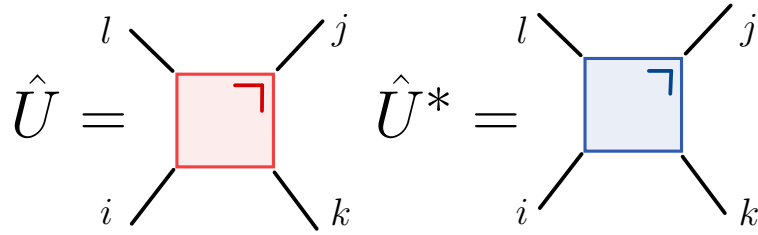


Figure 3.1: Graphic representation of a 2-local unitary  $\hat{U}$  and its complex conjugate  $\hat{U}^*$ . Adapted from Ref. [18], © (2019) American Physical Society.

For certain calculations, such as the one of the two-point correlation functions, it is also useful to introduce the “folded” picture, in which we represent the tensor product  $\hat{U} \otimes \hat{U}^*$ , corresponding to overlapping the two operator representations of top of each other. This is shown in Fig. 3.3, in which we also introduce contraction with the identity, as well as depict unitarity in the temporal and spatial directions. This pictorial calculus is used later on in Section 3.3.

At first sight, the DU condition appears to be quite a strong restriction, and it is reasonable to wonder how “large” the space of DU operators is in contrast to the full manifold formed by unitary operators. In the case of qubits,  $d = 2$ , it was found that the following parameterisation is exhaustive, encompassing all DU gates [18]:

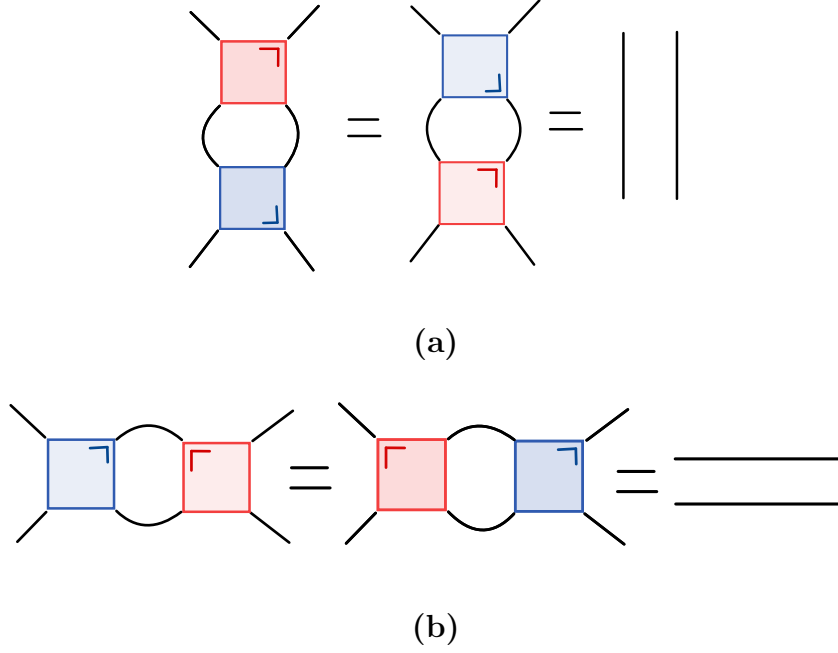


Figure 3.2: Schematic representation of “standard” unitarity **(a)** and *dual* unitarity **(b)**. Note that we have omitted the tensor indices. Adapted from Ref. [18], © (2019) American Physical Society.

$$\hat{U}_{d=2}^{\text{DU}} = e^{i\phi} (\hat{u}^+ \otimes \hat{u}^-) \hat{V}[J] (\hat{v}^- \otimes \hat{v}^+), \quad (3.3)$$

$$\hat{V}[J] = \exp \left[ -i \left( \frac{\pi}{4} \hat{\sigma}_x \otimes \hat{\sigma}_x + \frac{\pi}{4} \hat{\sigma}_y \otimes \hat{\sigma}_y + J \hat{\sigma}_z \otimes \hat{\sigma}_z \right) \right], \quad (3.4)$$

where  $J \in \mathbb{R}$  and  $\hat{u}^\pm, \hat{v}^\pm$  are arbitrary single qubit rotations. Alternatively, the entangling gate  $\hat{V}$  can also be written as:

$$\hat{V}[J] = \hat{S} \exp[iJ \hat{\sigma}_z \otimes \hat{\sigma}_z]. \quad (3.5)$$

Naively brute force counting the free parameters, there appear to be 14 ( $\phi$ ,  $J$  and three due to each single qubit rotation). However, this is an overestimation [50]: Since the local rotations around the  $Z$  axis commute with the interaction term  $\hat{V}$ , two parameters are redundant, and the degrees of freedom can be reduced further down to 12. Therefore, DU operators form a 12-dimensional subspace of the 16-dimensional space of all two-qubit unitaries. Unfortunately, such an exhaustive parameterisation has not been identified for greater local dimensions,  $d > 2$ . One major limitation

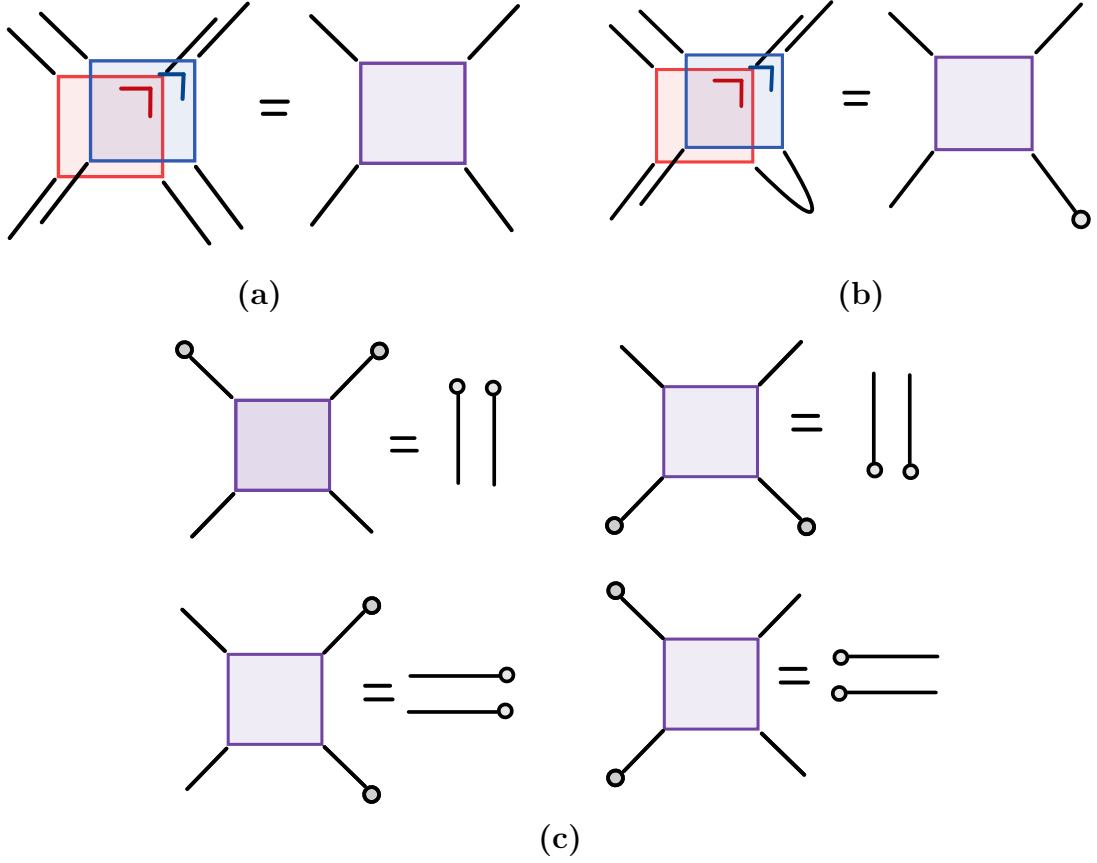


Figure 3.3: **(a)** Representation of  $\hat{U} \otimes \hat{U}^*$  in the folded picture. **(b)** Contraction with the identity, which is relevant for “infinite temperature” calculations. **(c)** Representation of unitarity in the temporal (top) and spatial (bottom) directions. Adapted from Ref. [158], © (2021) American Physical Society.

is also that there is no analogue to the generic 2-qubit gate written using the Ising interaction, presented in Eq. (2.34), for  $d > 2$ . Noting that the swap operator is DU for any  $d$ , one possible option are dressed swap gates [150]:

$$\hat{U} = (\hat{u}^+ \otimes \hat{u}^-) \hat{S} \hat{D} (\hat{v}^- \otimes \hat{v}^+), \quad (3.6)$$

with  $\hat{u}^\pm, \hat{v}^\pm$  now representing arbitrary single qudit operations and  $\hat{D}$  is a diagonal unitary operator. A more exhaustive parameterisation was presented in Ref. [159]:

$$\hat{U}^{\text{DU},1} = (\hat{u}^+ \otimes \hat{u}^-) \hat{S} \left[ \sum_{j=0}^{d-1} |j\rangle \langle j| \otimes \exp[i\hat{h}^{(j)}] \right] (\hat{v}^- \otimes \hat{v}^+), \quad (3.7)$$

with  $\exp[i\hat{h}^{(j)}] \in U(d)$ . We can thus interpret the additional term,  $[\sum_{j=0}^{d-1} |j\rangle \langle j| \otimes \exp(i\hat{h}^{(j)})]$ , as a sequence of controlled unitaries. Another parameterisation can be

obtained by exchanging the roles of control and target qudits:

$$\hat{U}^{\text{DU},2} = \hat{S}\hat{U}^{\text{DU},1}\hat{S} = (\hat{u}^+ \otimes \hat{u}^-) \hat{S} \left[ \sum_{j=0}^{d-1} \exp[i\hat{h}^{(j)}] \otimes |j\rangle \langle j| \right] (\hat{v}^- \otimes \hat{v}^+), \quad (3.8)$$

which is overlapping with, but not identical to the  $\hat{U}^{\text{DU},1}$  set. These two constructions have a number of free parameters that scales as  $\sim d^3$ , which agrees with the numerical scaling obtained in Ref. [50] for arbitrary  $d$ . Nonetheless, it still fails to encapsulate all DU gates with  $d > 2$ , and several other families have been proposed [159–164].

### 3.3 Some Exact Results

In this section we go into more detail with respect to some specific properties of DU circuits. For a comprehensive review, we refer the interested reader to Ref. [19].

A significant property of DU circuits is that they are the first class of quantum many-body models for which the spectral form factor, as defined in Eq. (2.22), has been computed exactly [17, 147]. The first model for which it was explicitly calculated, the self-dual kicked Ising model [17], was also the first instance in which the DU property was noticed as crucial for the analytical tractability. Averaging over the longitudinal “kick” fields, the SFF can be directly related to the partition function of a two-dimensional classical Ising model on a rectangular lattice. Exploiting this property at the self-dual points of the model, where the spatial transfer matrix becomes unitary, the SFF can be computed exactly at odd integer times  $t$ . At these points, the SFF reproduces the random matrix theory prediction for the COE. This not only proves the chaotic nature of the model, but also demonstrates the inability to exhibit MBL. As pointed out and discussed in Section 2.1, the values of two point correlation functions are intimately linked to the ergodic properties of a model. Therefore it is important to highlight that these can be computed exactly in the case of DU circuits in a brickwork layout [18]. For the demonstrations in this chapter, we label all sites with integers  $n$  and construct the circuit in such a manner that the first layers of unitaries acts on pairs of even and odd-labelled sites, in that order. For a more detailed

description of the cases, and aesthetically sophisticated diagrams, we refer the reader to Ref. [19].

First we recall the definition of two-point correlation functions as written in Eq. (2.13), and writing it specifically for traceless operators with respect to the infinite temperature state  $\hat{\rho}_{T=\infty} = \frac{1}{\dim(\mathcal{H})} \hat{\mathbb{I}}$ :

$$C_{jk}^{nm}(t) = \langle \hat{o}_j^n(t) \hat{o}_k^m(0) \rangle = \langle \hat{\mathbb{U}}^t \hat{o}_j^n \hat{\mathbb{U}}^{-t} \hat{o}_k^m \rangle, \quad (3.9)$$

where  $\langle \cdot \rangle = \frac{1}{\dim(\mathcal{H})} \text{Tr}[\cdot \hat{\mathbb{I}}]$ . Using the pictorial representation introduced in 3.2 and the folded representation in 3.3, we can represent the time-evolved operator  $\hat{\mathbb{U}}^t \hat{o}_j^n \hat{\mathbb{U}}^{-t}$  as shown in Fig. 3.4.

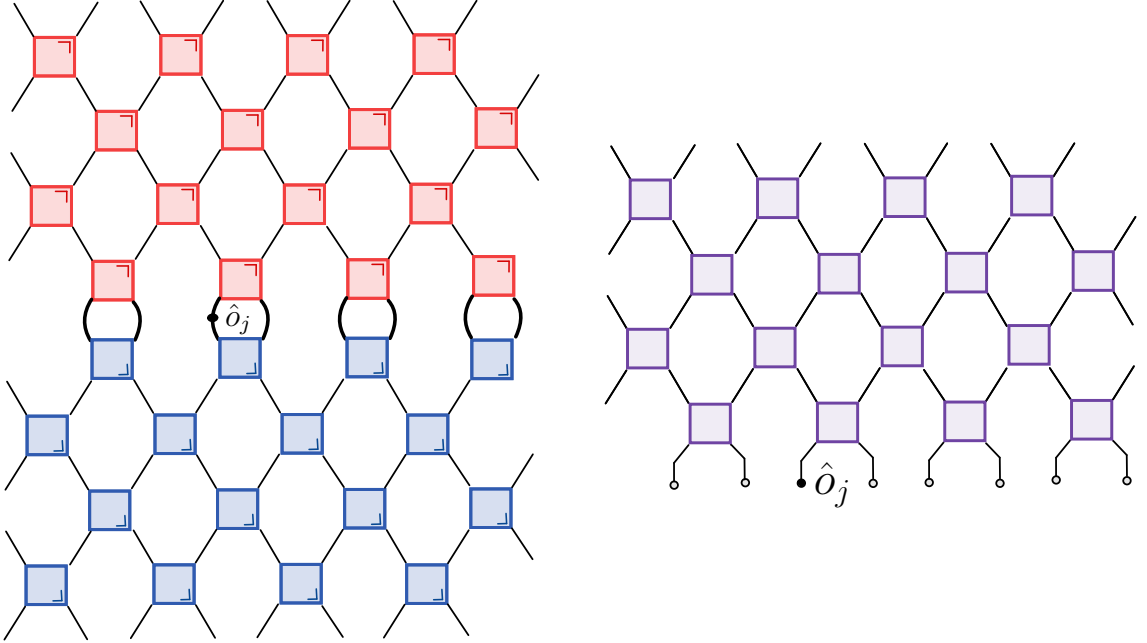


Figure 3.4: Representation of the time evolved operator  $\hat{\mathbb{U}}^t \hat{o}_j^n \hat{\mathbb{U}}^{-t}$  using the individual local gates (LEFT), as well as the folded representation (RIGHT). Adapted from Ref. [158], © (2021) American Physical Society.

Note that the infinite temperature state is the physically relevant one, as there is no energy conservation in these models. Now, we consider three separate cases for the values of these two-point correlation functions, and we ignore the trivial operators where  $j(k) = 0$ , such that  $\hat{o}_{j(k)} = \hat{\mathbb{I}}$ . The first case is when  $|n - m| > t$ , which is also referred to as the “outside the light-cone” scenario. In this case, we can use the unitarity of the operators, shown in Fig. 3.3, in the folded circuit shown in Fig.

3.4. Leveraging unitarity we then iteratively simplify the diagrammatic expression, as displayed in Fig. 3.5.

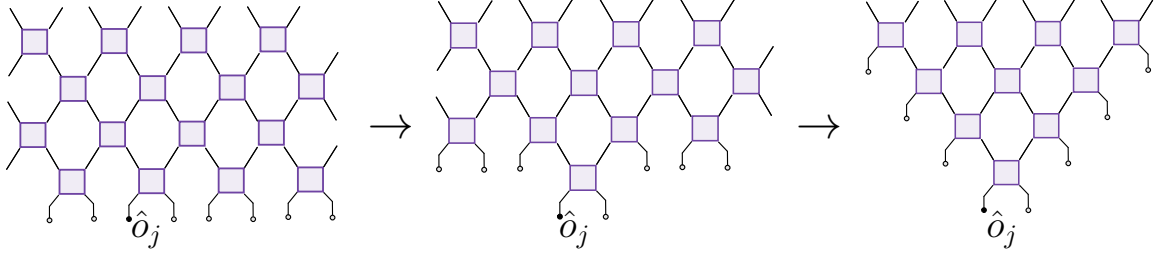


Figure 3.5: Iterative simplification of the time evolved operator (left to right) by using unitarity. We can infer that this operator has support growing only linearly with  $t$ , and is therefore not causally connected to any operator on sites  $k$  which satisfy  $|n - m| > t$ . Adapted from Ref. [19].

This process leads us to the conclusion that this expression has to vanish in the case  $|n - m| > t$ . As we have only invoked the usual unitarity of the operators, this finding is not valid only for DU circuits, but for any brickwork structured circuit. This is also physically as expected, as there is no way for behaviour at site  $n$  to causally influence  $m$  if  $t < |n - m|$ .

We now turn to the case where  $|n - m| < t$ , i.e., the “inside of light-cone” scenario. Similarly to the first case, we can iteratively simplify the diagrammatic representation for this case, as shown in Fig. 3.5. We now use the unitarity of the *dual*. Similarly to the “outside of light-cone” scenario, the two-point correlation functions will vanish in this case. However, in contrast to the former, this is a special feature of solely DU circuits and not of generic brickwork ones. Another way to view this fact is that, analogously to the unitarity in time leading to vanishing correlations outside the light-cone, unitarity in space will lead to vanishing correlations inside of it.

This brings us to the final, “along the light-cone” case, which occurs for  $|n - m| = t$ . As for the other two, one can leverage the properties of dual-unitary operators to simplify this expression, also displayed in Fig. 3.7. This is the only scenario where the two-point correlations are non-trivial, but are dictated by a CPTP map  $\mathcal{M}_+$ , whose graphic representation is in Fig. 3.8.

Using the same procedure, it can be deduced that along the  $m = n - t$  light-cone, the correlations will be dictated by  $\mathcal{M}_-$ . The  $\pm$  subscripts correspond to the two cases

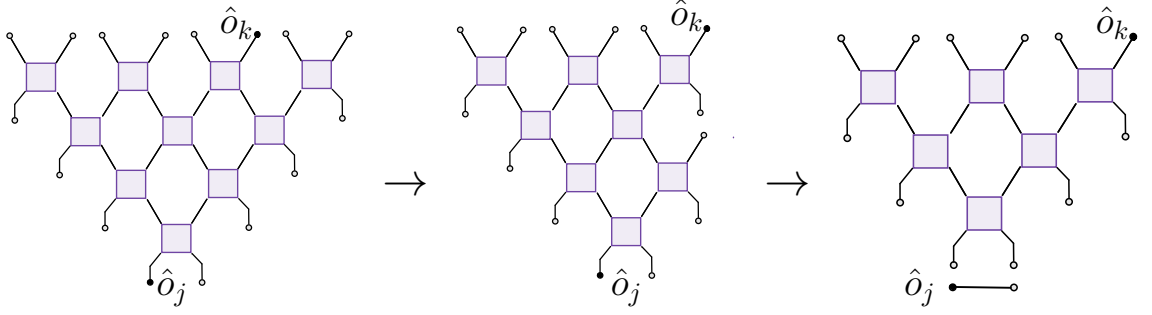


Figure 3.6: Iterative simplification of a two-point correlation function representation (up to a normalisation factor) in the “inside of light-cone” scenario, by using dual-unitarity (left to right). Note that we have set the sites  $n = 0, m = t - 2$ , but similar calculations can be done for other choices. Adapted from Ref. [19].

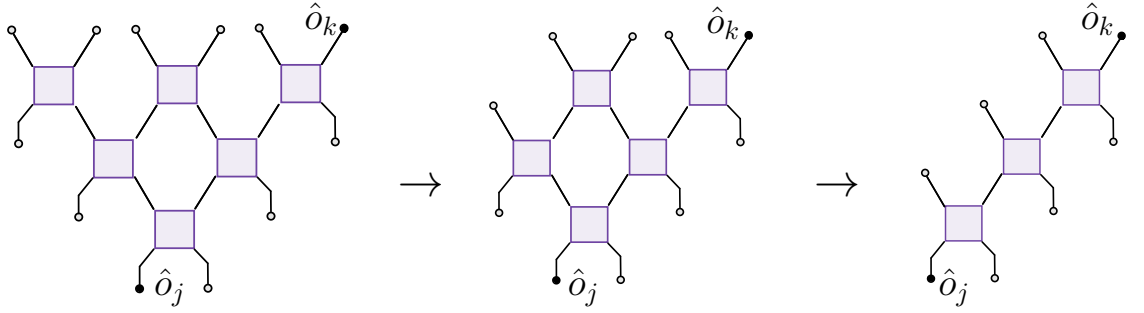


Figure 3.7: Diagrammatic representation of correlations spreading along the light-cone. Note that an analogous representation can be obtained for  $m = n - t$  at time  $t$ . Adapted from Ref. [19].

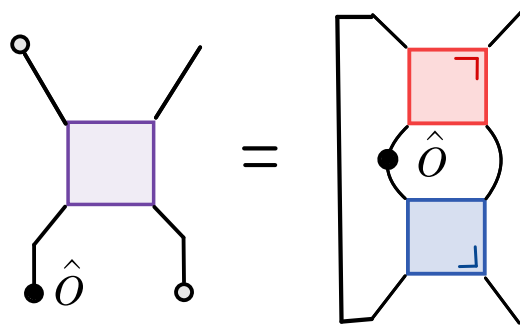


Figure 3.8: Unfolding of the channel dictating the evolution of  $\hat{O}$ , giving the form of  $\mathcal{M}_+$ , written in Eq. (3.10). Adapted from Ref. [18], © (2019) American Physical Society.

$m - n = \pm t$ . Using the calculus of these diagrammatic representations, the  $\mathcal{M}_\pm$  maps can be written as:

$$\mathcal{M}_+(\hat{o}) = \frac{1}{d} \text{Tr}_1[(\hat{U}^{\text{DU}})^\dagger(\hat{o} \otimes \hat{\mathbb{I}})\hat{U}^{\text{DU}}], \quad \mathcal{M}_-(\hat{o}) = \frac{1}{d} \text{Tr}_2[(\hat{U}^{\text{DU}})^\dagger(\hat{\mathbb{I}} \otimes \hat{o})\hat{U}^{\text{DU}}], \quad (3.10)$$

where  $\hat{U}^{\text{DU}}$  represents the local DU operator. And the value of the two-point correlation functions is then [19]:

$$C_{j,k}^{n,n\pm t}(t) = \frac{1}{d} \text{Tr}[\mathcal{M}_\pm^t(\hat{o}_j^n)\hat{o}_k^{n\pm t}], \quad (3.11)$$

for the non-vanishing correlations where  $m = n \pm t$  and  $j, k \neq 0$ . We can also think of these CPTP maps in the standard “vectorisation” picture, mapping the operators  $\hat{o}$  to states and the  $\mathcal{M}_\pm$  channels to operators [13]. We notice that due to unitarity of  $\hat{U}^{\text{DU}}$  in Eq. (3.10), the identity  $\hat{\mathbb{I}}$  will always be an eigenoperator of  $\mathcal{M}_\pm$  with eigenvalue 1. The other eigenvalues will become important for determining the ergodicity of specific models, a point that we return to shortly. Additional mathematical properties of  $\mathcal{M}_\pm$  guarantee that all of its eigenvalues lie on or within a unit circle, centered at the origin of the complex plane.

The analytic tractability of Eq. (3.9) cannot be overstated. By exploiting a relatively “simple” property of the local operators, it is possible to compute the infinite temperature, two-point correlations exactly, for a model that is generically non-integrable [19]. And, as pointed out in Section 2.1, their properties directly determine the ergodic nature of many-body systems.

Therefore, the properties of the  $\mathcal{M}_\pm$  maps can be directly linked to the ergodic behaviour of DU circuits, with different classes found for any local Hilbert space dimension  $d$  [150]. We will use this classification in order to verify that the models hosting QMBS are ergodic. We note that the ergodic properties indicated by  $\mathcal{M}_+$  may differ by the ones revealed through  $\mathcal{M}_-$ , therefore it is important to evaluate both. Based on the eigenvalues of  $\mathcal{M}_\pm$ , the DU circuits can be divided into 5 ergodicity classes:

1. **Non-interacting:** All of the  $d^2$  eigenvalues are equal to 1, implying that the

dynamical correlations must remain constant.

2. **Interacting, non-ergodic:** There is more than 1, but fewer than  $d^2$  eigenvalues equal to 1. Therefore some, but not all of the correlations, remain constant.
3. **Ergodic and non-mixing:** All  $d^2 - 1$  non-trivial eigenvalues are different from 1, but at least one of them has unit magnitude. The correlations oscillate around a time-averaged value corresponding to the ergodic one, but do not decay.
4. **Ergodic and mixing:** The  $d^2 - 1$  nontrivial eigenvalues satisfy  $|\lambda| < 1$ , implying that all dynamical correlations decay to the ergodic value.
5. **Maximally ergodic:** All non-trivial eigenvalues are zero, and all of the dynamical correlations become 0 in a finite number of time steps.

In a nutshell, by finding the eigenvalues of  $\mathcal{M}_\pm$ , we will know precisely how the correlations between local observables will behave.

These  $\mathcal{M}_\pm$  maps were also used to verify the validity of the off-diagonal part of the ETH ansatz [149], as presented in Eq. 2.9. As discussed in 2.1, the off-diagonal part of the ETH is directly related to the two-point correlation functions. Since these are dictated by  $\mathcal{M}_\pm$ , the smooth function  $f(\omega)$  can be computed from them. Note that there is no  $\bar{E}$  dependence in  $f$  as there is no energy conservation in these models.

Similarly to the  $\mathcal{M}_\pm$  maps, one can construct channels which dictate the time evolution of operators which are not strictly 1-local [19]. While verifying ergodicity at the level of  $k$ -local observables becomes computationally prohibitively expensive, it has been demonstrated that generic DU circuits for qubits ( $d = 2$ ) are ergodic at the level of these observables as well [165].

Based on the presented results, we might naturally wonder if it is possible to have DU circuits which exhibit weak ergodicity breaking via QMBS. We might still be able to embed them in DU circuits which exhibit ergodicity, as the infinite temperature correlations will not be affected by an exponentially vanishing fraction subspace formed by QMBS. We will focus on this question in the next section.

### 3.4 Circuit Setup

In order to find DU circuits with QMBS states, we will first present the general setup. The 2-local building blocks of the circuits constructed will be either of the form  $\hat{U}^{\text{DU},1} = (\hat{u}^+ \otimes \hat{u}^-) \hat{S} [\sum_{j=0}^{d-1} |j\rangle \langle j| \otimes \exp[i\hat{h}^{(j)}]] (\hat{v}^- \otimes \hat{v}^+)$  or  $\hat{U}^{\text{DU},2} = \hat{S} \hat{U}^{\text{DU},1} \hat{S}$ , as presented in Eqs. (3.7), (3.8). We rewrite the non-entangling unitaries as:

$$\hat{u}^+ \otimes \hat{u}^- = \exp\{i(\hat{f}^+ \otimes \hat{\mathbb{I}} + \hat{\mathbb{I}} \otimes \hat{f}^-)\}, \quad (3.12)$$

$$\hat{v}^- \otimes \hat{v}^+ = \exp\{i(\hat{g}^- \otimes \hat{\mathbb{I}} + \hat{\mathbb{I}} \otimes \hat{g}^+)\}, \quad (3.13)$$

in terms of the single-qudit Hermitian operators  $\hat{f}^\pm$  and  $\hat{g}^\pm$ . With this parameterisation, a dual-unitary gate is specified by  $\{\hat{f}^\pm, \hat{g}^\pm, \hat{h}^{(j)}\}$ .

For the circuit construction, we consider an even number  $N$  of qudits, where each qudit has a local Hilbert space  $\mathbb{C}^d$ . The qudit sites are labelled by  $n = 0, 1, 2, \dots, N-1$ , and we impose periodic boundary conditions  $n \equiv n + N$ . The basic building blocks of the dynamics are local DU gates  $\hat{U}_{n,n+1}$ . A single timestep is implemented by a Floquet unitary operator  $\hat{\mathbb{U}}_F = \hat{\mathbb{U}}_o \hat{\mathbb{U}}_e$  which is a layer of DU gates across even-odd bonds  $\hat{\mathbb{U}}_e = \bigotimes_{j=0}^{N/2-1} \hat{U}_{2j,2j+1}$  and a layer of DU gates across odd-even bonds  $\hat{\mathbb{U}}_o = \bigotimes_{j=1}^{N/2} \hat{U}_{2j-1,2j}$ . We set all gates in the even layer to be identical to each other and in the form of Eq. (3.7),  $\hat{U}_{0,1} = \hat{U}_{2,3} = \dots = \hat{U}_{N-2,N-1} = \hat{U}_e \in \hat{U}^{\text{DU},1}$ , and all gates in the odd layer to be identical to each other and in the form of Eq. (3.8), i.e.,  $\hat{U}_{1,2} = \hat{U}_{3,4} = \dots = \hat{U}_{N-1,0} = \hat{U}_o \in \hat{U}^{\text{DU},2}$ . However, this two-site translation invariance is not essential for the presented results. Evolution for  $t \in \mathbb{Z}$  timesteps is generated by powers of the Floquet operator,  $\hat{\mathbb{U}}^t$ .

### 3.5 Embedding Procedure

In order to construct DU circuits with QMBS, we employ a projector embedding method, similar to the construction initially proposed by Shiraishi and Mori for continuous-time dynamics in Ref. [87]. The essential idea is to use projectors in

the generators of the unitary gates in Eqs. (3.7), (3.8), (3.12), (3.13), so that a chosen set of target states evolves by an elementary brickwork circuit of two-qudit swap gates, while states outside the target set evolve by more complicated dynamics.

Let  $\hat{P}_{n,n+1}$  denote two-qudit projectors acting on neighbouring sites  $n$  and  $n+1$ , and define the extended projectors acting on the total system of  $N$  qudits as  $\hat{\mathbb{P}}_{n,n+1} \equiv \hat{\mathbb{I}}_{0,n-1} \otimes \hat{P}_{n,n+1} \otimes \hat{\mathbb{I}}_{n+2,N-1}$ , where  $\hat{\mathbb{I}}_{i,j}$  is the identity acting on all qudits in the range  $i, i+1, \dots, j-1, j$ . The common kernel of all projectors is the set of states that are simultaneously annihilated by all projectors:

$$\mathcal{K} = \{|\psi\rangle : \hat{\mathbb{P}}_{n,n+1} |\psi\rangle = 0, \forall n\}. \quad (3.14)$$

Our target set of states  $\mathcal{T}$ , which we wish to embed in our circuit as QMBS, is the subset of  $\mathcal{K}$  that is invariant under the action of even and odd layers of swap gates:

$$\mathcal{T} = \{|\psi\rangle : |\psi\rangle \in \mathcal{K}, \hat{\mathbb{S}}_e |\psi\rangle \in \mathcal{K}, \hat{\mathbb{S}}_o |\psi\rangle \in \mathcal{K}\}, \quad (3.15)$$

where  $\hat{\mathbb{S}}_e = \bigotimes_{j=0}^{N/2-1} \hat{S}_{2j,2j+1}$  is the even layer of swap gates and  $\hat{\mathbb{S}}_o = \bigotimes_{j=1}^{N/2} \hat{S}_{2j-1,2j}$  is the odd layer.

We now outline a construction to embed  $\mathcal{T}$  as a set of scars in our DU circuit. To do this we impose three conditions on the generators  $\{\hat{f}^\pm, \hat{g}^\pm, \hat{h}^{(j)}\}$  that define the DU gates in Eqs. (3.7)–(3.8) and on the projectors  $\{\hat{P}_{n,n+1}\}$  that define the target subspace:

$$\hat{P}_{n,n+1}(\hat{f}_n^+ \otimes \hat{\mathbb{I}} + \hat{\mathbb{I}} \otimes \hat{f}_{n+1}^-) \hat{P}_{n,n+1} = \hat{f}_n^+ \otimes \hat{\mathbb{I}} + \hat{\mathbb{I}} \otimes \hat{f}_{n+1}^-, \quad (\text{C1})$$

$$\hat{P}_{n,n+1}(\hat{g}_n^- \otimes \hat{\mathbb{I}} + \hat{\mathbb{I}} \otimes \hat{g}_{n+1}^+) \hat{P}_{n,n+1} = \hat{g}_n^- \otimes \hat{\mathbb{I}} + \hat{\mathbb{I}} \otimes \hat{g}_{n+1}^+, \quad (\text{C2})$$

$$\hat{P}_{n,n+1} \left( \sum_{j=0}^{d-1} \hat{h}_n^{(j)} \otimes |j\rangle \langle j|_{n+1} \right) \hat{P}_{n,n+1} = \sum_{j=0}^{d-1} \hat{h}_n^{(j)} \otimes |j\rangle \langle j|_{n+1}. \quad (\text{C3})$$

Clearly, two-qudit gates satisfying conditions (C1)–(C3) are still of the DU form in Eq. (3.7). Conditions (C1), (C2), and (C3) ensure that the unitaries  $\hat{u}_n^+ \otimes \hat{u}_{n+1}^-$ ,  $\hat{v}_n^+ \otimes \hat{v}_{n+1}^-$ , and  $\hat{V}_{n,n+1}$  act trivially on all states in the kernel  $\mathcal{K}$  of the projectors  $\hat{P}_{n,n+1}$ . However, the dynamics are not necessarily closed in the subspace  $\mathcal{K}$ . This is

because the parameterisation of Eq. (3.7) includes the swap operators  $\hat{S}_{n,n+1}$  which can potentially take states out of  $\mathcal{K}$ . So, the QMBS subspace is the subspace  $\mathcal{T} \subset \mathcal{K}$  that is invariant under even  $\hat{S}_e$  and odd  $\hat{S}_o$  layers of swap operators. For any initial state  $|\psi\rangle \in \mathcal{T}$  the circuit will then act as  $\hat{U}|\psi\rangle = \hat{S}|\psi\rangle$ , where  $\hat{S} = \hat{S}_o\hat{S}_e$ , corresponding to integrable dynamics in the target subspace  $\mathcal{T}$ , even if the overall circuit  $\hat{U}$  is non-integrable and results in complicated dynamics for initial states that are not in  $\mathcal{T}$ . The action of  $\hat{U}$  in  $\mathcal{T}$  can thus be seen as a straightforward permutation of the qudits.

This prescription embeds QMBS in a DU circuit, which allows us to use all known exact results for these models. One of the key results for DU circuits is the rigorous classification in terms of their ergodic and mixing properties, based on the spectrum of  $\mathcal{M}_\pm$ , which was presented in Section 3.3. We can show that our construction leads to QMBS in provably ergodic and mixing many-body systems, since all dynamical correlation functions decay to their infinite temperature values at long times.

Before moving on to specific examples demonstrating the existence of QMBS states in DU circuits, it is important to point out two major limitations in this procedure. Firstly, for arbitrary local Hilbert space dimension, an exhaustive parameterisation of all DU gates is not known [19], therefore there could be other instances of DU circuits hosting QMBS states, which are not encompassed by this method. One of the obstacles in obtaining an all encompassing parameterisation is the fact that the equivalent of Eq. (2.34) is not known for  $d > 2$ .

Secondly, Eqs. (C1)–(C3) are to be interpreted as restrictions on both the generators of the gates  $\{\hat{f}^\pm, \hat{g}^\pm, \hat{h}_n^{(j)}\}$ , as well as the used projectors  $\hat{P}_{n,n+1}$ . Some particular forms of projectors are incompatible with the three conditions, as they break the dual-unitarity of the constructed circuits. One such concrete example is:

$$\begin{aligned}\hat{P}_{\text{lin}} &= \hat{\mathbb{I}} - |00\rangle\langle 00| - |11\rangle\langle 11| - |\Phi_-\rangle\langle \Phi_-| \\ |\Phi_-\rangle &\equiv |+-\rangle - |-+\rangle, |\pm\rangle = |0\rangle \pm |1\rangle,\end{aligned}\tag{3.16}$$

which would give a linear number of scar states, but is unfortunately incompatible with the given DU parameterisation.

### 3.6 Examples

In order to illustrate the presented construction, we start with the simplest possible example, for which the set of projectors is:

$$\hat{P}_{n,n+1} = \hat{\mathbb{I}}_{n,n+1} - |0\rangle\langle 0|_n \otimes |0\rangle\langle 0|_{n+1}, \quad (3.17)$$

for all  $n = 0, 1, \dots, N-1$ . The common kernel  $\mathcal{K}$  of all projectors consists solely of the product state  $|0\rangle^{\otimes N}$ . Since the latter is clearly invariant under the action of any swap gates, the target space is the single state  $\mathcal{T} = \{|0\rangle^{\otimes N}\}$ .

We choose  $\{\hat{f}^\pm, \hat{g}^\pm, \hat{h}^{(j)}\}$  to be random Hermitian matrices, apart from some rows and columns  $\langle i|\hat{f}^\pm|0\rangle = \langle i|\hat{g}^\pm|0\rangle = \langle i|\hat{h}^{(0)}|0\rangle = 0$ ,  $i \in \{0, 1, \dots, d-1\}$ , which have to be set to zero to ensure that conditions (C1)–(C3) are satisfied. This choice gives a generic DU gate which we expect to give a rapidly thermalising circuit. By using the  $\mathcal{M}_\pm$  map, it can be shown that this model is ergodic and mixing, which is demonstrated for all the examples at the end of this chapter. However, by construction, the target state  $|0\rangle^{\otimes N}$  will be a non-thermal eigenstate of the circuit:  $\hat{\mathbb{U}}_F |0\rangle^{\otimes N} = |0\rangle^{\otimes N}$ .

Although this construction embeds a QMBS for any system size  $N$ , we demonstrate our results with numerical simulations on finite-size systems. The signatures of the QMBS states can be seen both in the properties of the eigenspectrum, as well as in dynamical signatures.

We first focus on the former. In Fig. 3.9(left) we plot the half-system bipartite entanglement entropy  $S(|\varphi_\alpha\rangle) = -\text{Tr}[\hat{\rho}_\alpha \log(\hat{\rho}_\alpha)]$  of the eigenstates  $|\varphi_\alpha\rangle$  of the Floquet unitary  $\hat{\mathbb{U}}$ . Here  $\hat{\rho}_\alpha = \text{Tr}_{0, \frac{N}{2}-1} |\varphi_\alpha\rangle\langle \varphi_\alpha|$  is the reduced density matrix of the eigenstate obtained by tracing out the first  $N/2$  qudits of the system. We see the single QMBS  $|0\rangle^{\otimes N}$  with zero entanglement, separated from the rest of the spectrum. All the other eigenstates are concentrated around the Page entropy,  $S_{\text{Page}}$ , which is the expected value of entanglement entropy for a random pure state [35].

Similarly, we can compute the expectation values of local observables with respect to the eigenstates. By the ETH we expect thermalising systems to have eigenstate

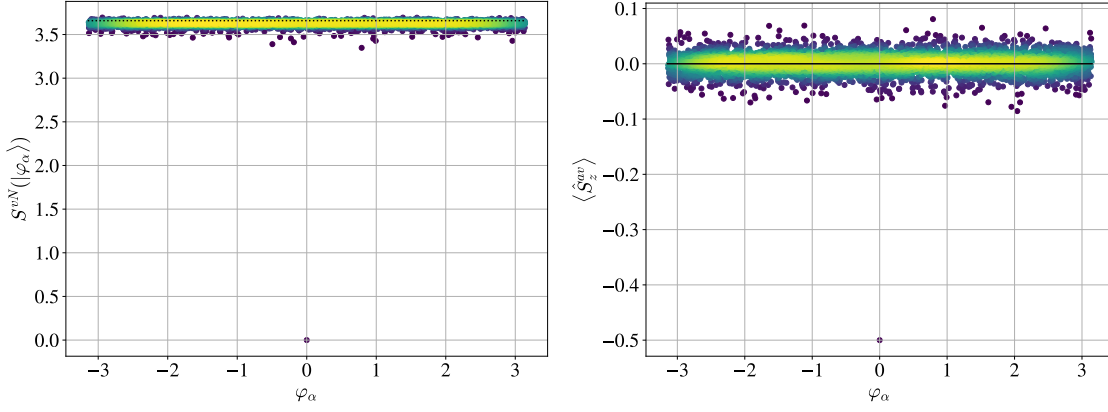


Figure 3.9: Presence of single QMBS in the eigenspectrum of the constructed Floquet operator  $\hat{U}_F$ , with system size  $N = 12$ , and local dimension  $d = 2$ . LEFT: Bipartite entanglement entropy  $S^{vN}(|\varphi_\alpha\rangle)$  vs eigenphase  $\varphi_\alpha$  of the eigenstates. The dotted line represents the Page value,  $S_{\text{Page}}$ . All eigenstates apart from the QMBS state  $|0\rangle^{\otimes N}$  are saturated around  $S_{\text{Page}}$ . The embedded scar can be observed at zero entanglement entropy, well separated from the bulk of the spectrum. RIGHT: Expectation value  $\langle \hat{S}_z^{av} \rangle$ , as defined in Eq. (3.18), with respect to the eigenstates. We see that apart from the QMBS, all eigenstates are in agreement with the “infinite temperature” prediction, dictated by Floquet ETH.

expectation values that are close to the thermal expectation values (for local observables). If all eigenstates are thermal then the corresponding observables will thermalise, starting from a non-equilibrium state. However, non-thermal eigenstates, like QMBS can prevent thermalisation. In the presented model, since energy is not conserved, the thermal expectation values are with respect to the infinite temperature state  $\hat{\rho}_{T=\infty} = \hat{\mathbb{I}}/d^N$ .

In order to probe these statements numerically, we introduce the sum of local  $\hat{S}_n^z$  operators:

$$\hat{S}_z^{av} = \frac{1}{N} \sum_{n=0}^{N-1} \hat{S}_n^z, \quad \hat{S}_n^z = \sum_{i=0}^{d-1} \frac{2i - (d-1)}{d-1} |i\rangle \langle i|_n. \quad (3.18)$$

We then compute the expectation values with respect to the Floquet eigenstates,  $|\varphi_\alpha\rangle$ , which are presented in Fig. 3.9(right) for the single scar DU circuit model. Corroborating the expected results from Floquet ETH, and in alignment with the bipartite entanglement entropy results, the bulk of the eigenstates are concentrated around the thermal value of  $\text{Tr}[\hat{\rho}_T \hat{S}_z^{av}] = \text{Tr}[\frac{1}{d^N} \hat{\mathbb{I}} \hat{S}_z^{av}] \approx 0$ , with the scar found at  $\langle \hat{S}_z^{av} \rangle = -0.5$ .

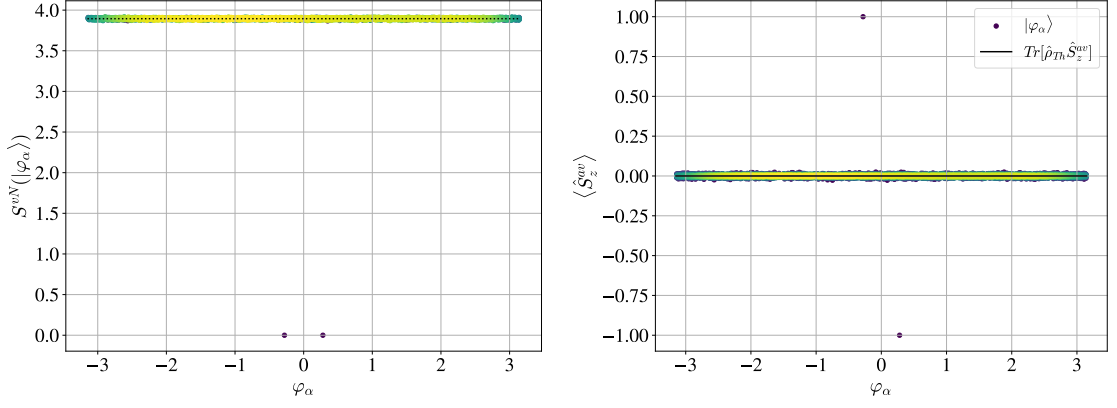


Figure 3.10: Presence of two QMBS in the eigenspectrum of a DU circuit, with  $N = 12$ , and  $d = 3$ . LEFT: Bipartite entanglement entropy  $S^{vN}(|\varphi_\alpha\rangle)$  vs eigenphase  $\varphi_\alpha$  of the eigenstates. The dotted line represents the Page value,  $S_{\text{Page}}$ . All eigenstates apart from the QMBS states  $\{|0\rangle^{\otimes N}, |d-1\rangle^{\otimes N}\}$  are saturated around  $S_{\text{Page}}$ . The embedded scars can be observed at zero entanglement entropy, well separated from the bulk of the spectrum. Note that we have used single qutrit unitaries in order to break the phase degeneracy between QMBS states, as described in the main text. RIGHT: Expectation value  $\langle \hat{S}_z^{av} \rangle$ , as defined in Eq. (3.18), with respect to the eigenstates, corroborating the entanglement entropy results.

In order to embed two QMBS into a DU circuit, we can use:

$$\hat{P}_{n,n+1} = \hat{\mathbb{1}}_{n,n+1} - |0\rangle\langle 0|_n \otimes |0\rangle\langle 0|_{n+1} - |d-1\rangle\langle d-1|_n \otimes |d-1\rangle\langle d-1|_{n+1}, \quad (3.19)$$

which will lead to the common kernel  $\mathcal{K} = \{|0\rangle^{\otimes N}, |d-1\rangle^{\otimes N}\}$ . Note that this will work only for  $d > 2$ , because in the case of qubits, this would lead to the trivial circuit constructed solely in terms of swap gates. Since any state of the form  $|i\rangle^{\otimes N}$  is invariant under the application of a layer of swap gates  $\hat{\mathbb{S}}_{e/o}$ , the target space will be the same as the kernel,  $\mathcal{T} = \mathcal{K}$ .

Note that, using the current construction the two QMBS states will have degenerate eigenphases (both at  $\varphi_\alpha = 0$ ). However, some degeneracies in the QMBS in DU circuits can be broken by adding an additional unitary layer  $\hat{\mathbb{U}}'$  at each timestep of the circuit with the properties  $[\hat{\mathbb{S}}, \hat{\mathbb{U}}'] = 0$  and  $[\hat{P}_{n,n+1}, \hat{\mathbb{U}}'] = 0$  for all  $n$ . These properties ensure that the projectors and the swap layers share a set of simultaneous eigenstates with  $\hat{\mathbb{U}}'$ , including the QMBS in  $\mathcal{K}$ . If  $\hat{\mathbb{U}}'$  can be constructed as a product of one-qudit gates then these gates can be absorbed into the even or odd layer unitaries  $\hat{\mathbb{U}}_{e/o}$  without impacting the brickwork structure or dual unitarity.

In order to break the QMBS eigenphase degeneracy in this case we introduce an

additional layer of single qudit unitaries  $\hat{\mathbb{U}} \rightarrow \hat{\mathbb{U}}\hat{\mathbb{U}}'$ , where  $\hat{\mathbb{U}}' = \hat{u}_\phi^{\otimes N}$  and  $\hat{u}_\phi = \sum_{j=0}^{d-1} \exp\{-i\phi \frac{2j-(d-1)}{d-1}\} |j\rangle\langle j|$  are generalised Pauli  $Z$ -rotations. This additional layer of single-qudit unitaries will add a phase of  $e^{iN\phi}$  to the  $|0\rangle^N$  state, whereas the other QMBS state,  $|d-1\rangle^N$ , will acquire an eigenphase of  $e^{-iN\phi}$ . In our simulations we set  $\phi = 0.01$ .

Finally, before moving on to the last case, we point out that this construction can be extended to embed  $k < d$  QMBS of the form  $\{|i\rangle\}_{i=0}^{k-1}$  into a DU circuit with local Hilbert space dimension  $d$ . The main adjustment is, instead of Eq. (3.17), to use the set of projectors  $\hat{P}_{n,n+1} = \hat{\mathbb{I}}_{n,n+1} - \sum_{i=0}^{k-1} |i, i\rangle\langle i, i|_{n,n+1}$ . Hence, we can embed any constant number of scars for DU circuits, limited solely by the local Hilbert space dimension.

A more complex example is obtained by using the projectors:

$$\begin{aligned} \hat{P}_{n,n+1} = \hat{\mathbb{I}}_{n,n+1} & - |0\rangle\langle 0|_n \otimes |0\rangle\langle 0|_{n+1} \\ & - |0\rangle\langle d-1|_n \otimes |d-1\rangle\langle 0|_{n+1} \\ & - |d-1\rangle\langle 0|_n \otimes |0\rangle\langle d-1|_{n+1} \\ & - |d-1\rangle\langle d-1|_n \otimes |d-1\rangle\langle d-1|_{n+1}, \end{aligned} \quad (3.20)$$

for all  $n$ . We choose the DU gate generators  $\{\hat{f}^\pm, \hat{g}^\pm, \hat{h}^{(j)}\}$  to be random, apart from a few rows and columns that are set to zero to ensure that conditions (C1)–(C3) are satisfied  $[\langle i| \hat{f}^\pm |j\rangle = \langle i| \hat{g}^\pm |j\rangle = \langle i| \hat{h}^{(j)} |j\rangle = 0, j \in \{0, d-1\}, i \in \{0, 1, \dots, d-1\}]$ .

The common kernel  $\mathcal{K}$  of the projectors in Eq. (3.20) is the  $2^N$ -dimensional space spanned by the set of all  $N$ -qudit product states of the form  $|\vec{i}\rangle = |i_0, i_1, \dots, i_{N-1}\rangle$  with  $i_n \in \{0, d-1\}$ . We note that this space, though exponentially large in the system size  $N$ , is still an exponentially small fraction  $(2/d)^N$  of the full Hilbert space when  $d > 2$ . Therefore, the fraction of non-thermal eigenstates constitutes a measure-zero set in the thermodynamic limit and thus leads to weak ergodicity breaking [10, 7]. It is also invariant under the action of  $\hat{\mathbb{S}}_{e/o}$ , so that our target space is  $\mathcal{T} = \mathcal{K}$ . However, this does not mean that the product states  $|\vec{i}\rangle, i_n \in \{0, d-1\}$ , are QMBS since, in general,  $\hat{\mathbb{U}}|\vec{i}\rangle = \hat{\mathbb{S}}|\vec{i}\rangle \neq |\vec{i}\rangle$ . Instead, the QMBS are found by diagonalising the swap

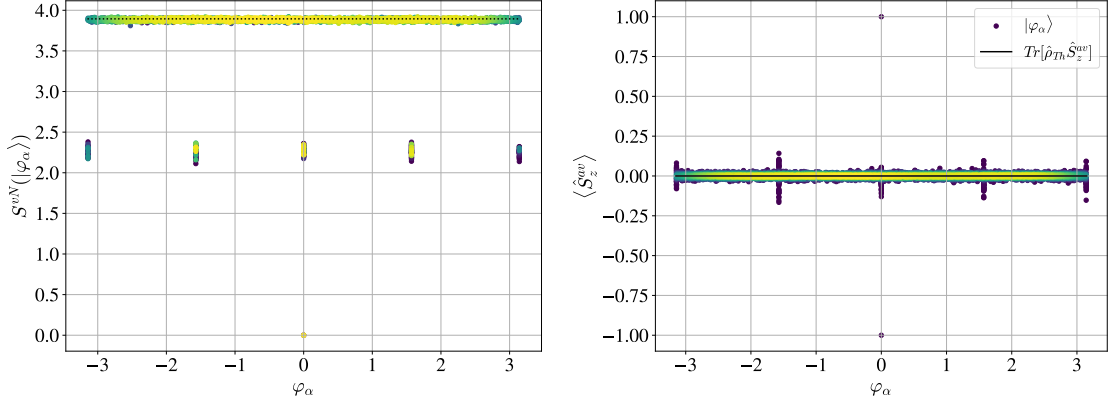


Figure 3.11: Presence of  $2^N$  QMBS in the eigenspectrum of a DU circuit, with  $N = 12$ , and  $d = 3$ . LEFT: Bipartite entanglement entropy  $S^{vN}(|\varphi_\alpha\rangle)$  vs eigenphase  $\varphi_\alpha$  of the eigenstates, where we can notice three trends. Most of the eigenstates are saturated close to the Page value, in accordance with the ETH predictions. Similarly to the cases with a constant number of QMBS, we observe two QMBS at  $(\varphi_\alpha = 0, S^{vN} = 0)$ . In addition, we also find towers of QMBS states at distinct eigenphases, which are dictated by Eq. (3.21) in the main text. RIGHT: Expectation value  $\langle \hat{S}_z^{av} \rangle$ , as defined in Eq. (3.18), with respect to the eigenstates, showcasing the two zero entropy QMBS at  $\langle \hat{S}_z^{av} \rangle = \pm 1$ .

circuit  $\hat{S}$  in the subspace spanned by  $\{|\vec{i}\rangle\}_{i_n \in \{0, d-1\}}$ .

In general, for examples with multiple QMBS they can have a degeneracy in the eigenphase  $\varphi_\alpha$  (where  $\hat{U}|\varphi_\alpha\rangle = e^{i\varphi_\alpha}|\varphi_\alpha\rangle$ ). To determine the different possible values of the eigenphase for QMBS we observe that for QMBS we have  $\hat{U}|\varphi_\alpha\rangle = \hat{S}|\varphi_\alpha\rangle = e^{i\varphi_\alpha}|\varphi_\alpha\rangle$  and that  $\hat{S}^{N/2} = \hat{\mathbb{I}}$ . Hence:

$$\begin{aligned} \hat{S}^{N/2}|\varphi_\alpha\rangle &= \exp\{iN\varphi_\alpha/2\}|\varphi_\alpha\rangle = |\varphi_\alpha\rangle \\ \implies e^{iN\varphi_\alpha/2} &= 1 \\ \implies \varphi_\alpha &= \frac{4\pi k}{N}, \end{aligned} \tag{3.21}$$

where  $k \in \{-\lfloor \frac{N}{4} \rfloor, -\lfloor \frac{N}{4} \rfloor + 1, \dots, \lfloor \frac{N}{4} \rfloor - 1\}$ . Since there are only  $N/2$  distinct values of the eigenphase there must be degeneracies when there are more than  $N/2$  QMBS, for instance. This is precisely the case in this example with  $2^N$  scars.

In Fig. 3.11 (left) we can see the consequences of this degeneracy in the towers of scar states formed. The bipartite entanglement entropy reveals the two product QMBS,  $\{|0\rangle^{\otimes N}, |d-1\rangle^{\otimes N}\}$  at zero entanglement entropy. The rest of the  $2^N - 2$  non-thermal states cluster at  $\varphi_\alpha = \frac{2\pi k}{N}$ ,  $0 \leq k < N$ , as predicted in the general case.

While these towers of states do not have exactly zero bipartite entanglement entropy

they still obey a sub-volume scaling which is characteristic of QMBS states [7, 10], in contrast to thermal eigenstates, which have a volume-law scaling.

Here we show that the entanglement entropy is upper bounded by the logarithm of the volume,  $S \leq \log(N/2)$  [166]. For any state  $|\psi\rangle \in \mathcal{T}$  in the QMBS subspace it follows, by construction, that  $\hat{\mathbb{U}}|\psi\rangle = \hat{\mathbb{S}}|\psi\rangle$ . The QMBS are therefore obtained by diagonalising  $\hat{\mathbb{S}}$  in the subspace  $\mathcal{T}$ . Any of the QMBS can be expressed as a superposition of product states. However, since  $\hat{\mathbb{S}}^{N/2} = \hat{\mathbb{I}}$ , they are superpositions of at most  $N/2$  orthonormal computational basis states. Therefore, the bipartite entanglement is bounded by  $S(|\psi\rangle) \leq \log(N/2)$ .

Note that this bound can be violated in the numerical results in Fig. 3.11 due to an eigenphase degeneracy in the QMBS. The QMBS are formed as random superpositions of degenerate eigenstates states that do not obey the bound. However, it is always possible to choose them so that they do obey  $S(|\psi\rangle) \leq \log(N/2)$ . Alternatively, we could break this degeneracy by using the same trick as invoked in the two QMBS case, by introducing additional layers of single qudit gates.

Similarly to the case with a constant number of QMBS, we can generalise the embedding process presented here to find DU circuits with  $q^N$  scar states, constrained only by the local Hilbert space dimension  $q < d$ . This may be achieved by using  $\hat{P}_{n,n+1} = \hat{\mathbb{I}} - \sum_{j=0}^{q-1} [|j\rangle \langle j|_n \otimes |j\rangle \langle j|_{n+1}] - \sum_{j \neq k}^{q-1} [|j\rangle \langle k|_n \otimes |k\rangle \langle j|_{n+1}]$ .

For all of the considered models, we can demonstrate that they are ergodic via the classification presented in [150]. As we have seen that for DU circuits, the dynamical correlation functions can be calculated exactly [18], and are directly related to the spectral properties the CPTP map  $\mathcal{M}_{\pm}$ . For the used circuit setup, with DU gates  $\hat{U}_e^{\text{DU}}$  making up the even layer and DU gates  $\hat{U}_o^{\text{DU}}$  making up the odd layer, the map can be written as  $\mathcal{M}_{\pm} = \mathcal{M}_{\pm,o} \mathcal{M}_{\pm,e}$ , with:

$$\mathcal{M}_{+,e/o}(\hat{o}) \equiv \frac{1}{d} \text{Tr}_1[(\hat{U}_{e/o}^{\text{DU}})^{\dagger}(\hat{o} \otimes \hat{\mathbb{I}})\hat{U}_{e/o}^{\text{DU}}], \quad (3.22)$$

$$\mathcal{M}_{-,e/o}(\hat{o}) \equiv \frac{1}{d} \text{Tr}_2[(\hat{U}_{e/o}^{\text{DU}})^{\dagger}(\hat{\mathbb{I}} \otimes \hat{o})\hat{U}_{e/o}^{\text{DU}}], \quad (3.23)$$

where  $\hat{o}$  is an arbitrary local observable and  $\text{Tr}_{1/2}$  is a partial trace over the left/right

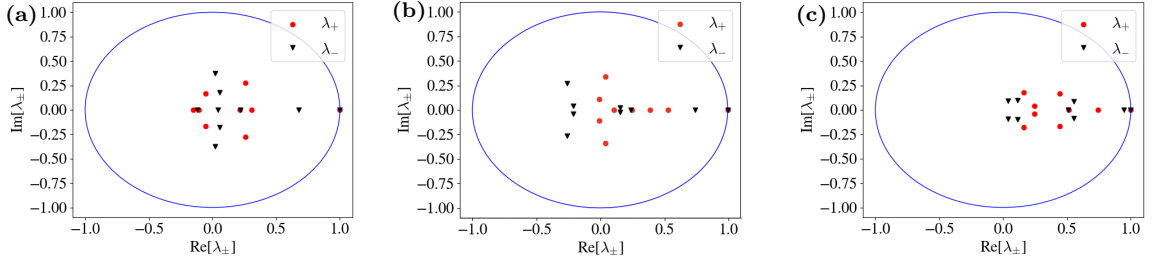


Figure 3.12: Eigenvalues of the  $\mathcal{M}_\pm$  map for the three models presented in this work. The cases corresponding to the single, two, and exponentially many QMBS cases are shown in panels, (a), (b), and (c), respectively. For all of the models we can observe exactly one eigenvalue of  $\lambda_\pm = 1.0$ , corresponding to the identity operator  $\hat{o} = \hat{\mathbb{I}}$ . All of the other eigenvalues of  $\mathcal{M}_\pm$  satisfy  $|\lambda_\pm| < 1$ , thus demonstrating that the presented models are ergodic and mixing [18].

qudit [149].

For the DU circuit setup in this work we have chosen the even layer DU gates to be of the form  $\hat{U}_e^{\text{DU}} \in \hat{U}^{\text{DU},1}$  (Eq. (3.7)) and the odd layer DU gates to be of the form  $\hat{U}_o^{\text{DU}} \in \hat{U}^{\text{DU},2}$  (Eq. (3.8)). With this choice we obtain a map  $\mathcal{M}_\pm$  that has a single eigenvalue with unit modulus, corresponding to  $\hat{\mathbb{I}}$ , and therefore the DU circuit will display ergodic and mixing behaviour according to the classification described above. This is shown numerically in Fig. 3.12 where we plot the eigenvalues  $\lambda_\pm$  of  $\mathcal{M}_\pm$  for each of the three examples. We mention in passing that we have observed that the circuit is also ergodic and mixing if we choose  $\hat{U}_e^{\text{DU}} \in \hat{U}^{\text{DU},2}$  and  $\hat{U}_o^{\text{DU}} \in \hat{U}^{\text{DU},1}$ .

However, it is also possible to embed QMBS in DU circuits that are *non-ergodic*. This can be achieved by choosing even and odd layer gates to be the same [i.e.,  $\hat{U}_e^{\text{DU}} = \hat{U}_o^{\text{DU}} \in \hat{U}^{\text{DU},1}$  or  $\hat{U}_e^{\text{DU}} = \hat{U}_o^{\text{DU}} \in \hat{U}^{\text{DU},2}$ ]. In this case one of the  $\mathcal{M}_\pm$  maps will have 2 additional unit eigenvalues, and thus the DU circuit will be non-ergodic. This is the main reason we have used  $\hat{U}_{e/o}^{\text{DU}} \in \hat{U}^{\text{DU},1/2}$ . This is explained in more detail in Appendix B.

### 3.7 Concluding remarks

Dual-unitary circuits are a paradigmatic model for investigations of many-body phenomena due to the abundance of available exact results. In this chapter, we have shown that these models can host QMBS, which leads to weak ergodicity breaking in

a provably ergodic and chaotic system.

The presented results raise several fundamental questions regarding QMBS, DU circuits, and chaotic quantum many-body systems in general. Since the used parameterisation is not complete for DU gates with  $d > 2$ , we expect that there are more DU circuit instances that can host QMBS. However, it is not certain whether the proposed embedding approach will work for DU circuits constructed with gates that do not fall within the forms used in Eqs. (3.7), (3.8) for  $d > 2$ . Furthermore, even with this parameterisation, the presented embedding approach is most likely not exhaustive. While we have shown that the models are ergodic at the level of one-local observables via the  $\mathcal{M}_{\pm}$  maps, verifying ergodicity of DU circuits for  $k$ -local observables for  $d > 2$  remains an open and challenging problem [19].

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# AQMBS Arising from Remnants of Gapless Excitations

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*This chapter is based on Ref. [167] and adapted for the purposes of the thesis.*

While QMBS states were a significantly counter-intuitive experimental discovery, they are relatively straightforward to understand from the perspective of Hilbert space structure. Although they are usually non-trivial to unveil in any given system, once they are found, they can be understood as a small, decoupled, subspace that is surrounded by a plethora of ETH-obeying eigenstates.

A more recently discovered related phenomenon, the so-called asymptotic quantum many-body scars (AQMBS) [20], is more perplexing from the theoretical perspective. These share some properties of regular QMBS, such as low entanglement entropy and non-thermal structure, but they are not eigenstates of the system for any finite size  $N$ : they are superpositions of an extensive number of thermal eigenstates. However, their variance with respect to the Hamiltonian is vanishing as the system size is increased and thus they become eigenstates in the thermodynamic limit.

In contrast to regular QMBS, which were initially observed in the seminal Rydberg experiment [8], asymptotic scars were first found analytically in the spin-1  $XY$  model [20]. Although these states appear to be quite pathological, follow-up work has revealed they are more ubiquitous than expected [168, 169]. Despite this, they are yet

to be observed in an experimental setting.

In this chapter, we first present a direct connection between *gapless* excitations and *asymptotic* scars. Then we will focus on a concrete model hosting both regular and asymptotic QMBS states. One of the regular QMBS is an edge localised zero mode, and displays a localisation transition at the critical value of the defining parameter, at which its localisation length diverges and it becomes the  $W$  state. At this point, we also observe the presence of AQMBS states, due to the *gapless* excitations in the model in which the edge localised mode is embedded as a ground state.

In order to investigate the thermalisation behaviour, one may simulate the Hamiltonian model, or the stroboscopic, Floquet version. We will numerically probe both, but for experimental aspects, we will focus on the latter. Due to the 2-local nature, it is easily implementable on current digital quantum simulators. For this purpose we use the Quantinuum H1-1 device, based on trapped ions. We probe the dynamics of total magnetisation from various initial conditions: a regular QMBS state, a non-scar state, an AQMBS state, and an edge excitation. In line with expectations, we observe a rapid decay of the initial magnetisation for the non-scar, and a very slow decay from the QMBS state, independently of system size. In stark contrast, when starting in the AQMBS state, the thermalisation becomes slower as the system size is increased, showing the first experimental signatures of AQMBS states.

## 4.1 Properties of Asymptotic Scars

Similarly to regular QMBS, asymptotic scars possess sub-volume law entanglement entropy. As such, they also represent non-thermal states in the sense of ETH. However, in contrast to regular scars, they are not eigenstates of the considered system at any finite size. For concreteness, let us consider some chaotic Hamiltonian  $\hat{H}$ , describing a system of  $N$  sites. We assume  $\hat{H}$  hosts regular QMBS states, which are denoted by  $|\mathcal{S}_j\rangle$ , as well as asymptotic QMBS, which are denoted by  $|\mathcal{A}_k\rangle$ . Both of these classes

of states have sub-volume law scaling of their entanglement entropy:

$$S^{vN}(|\mathcal{S}_j\rangle) \leq \log(N), \quad S^{vN}(|\mathcal{A}_k\rangle) \leq \log(N).$$

These two classes of states are also mutually orthogonal:  $\langle \mathcal{A}_k | \mathcal{S}_j \rangle = 0$ . We now consider the energy variance, which we recall is defined via:

$$\Delta E(|\psi\rangle) \equiv \sqrt{\langle \psi | \hat{H}^2 | \psi \rangle - \langle \psi | \hat{H} | \psi \rangle^2}.$$

As  $|\mathcal{S}_j\rangle$  are eigenstates of  $\hat{H}$ , their variance will vanish for any  $N$ :  $\Delta E(|\mathcal{S}_j\rangle) = 0$ . In contrast,  $|\mathcal{A}_k\rangle$  are not eigenstates of  $\hat{H}$ , but their energy variance is *decreasing* with system size, vanishing in the thermodynamic limit, where these states emerge as scars. This last property is the key distinguishing feature of AQMBS states, which has drastic implications. Firstly, at any finite  $N$ ,  $|\mathcal{A}_k\rangle$  will be superpositions of thermal eigenstates. Secondly, the Mandelstam-Tamm bound [170]:

$$T_{\text{rel}} = \frac{\pi \hbar}{2\Delta E}, \tag{4.1}$$

dictates the fidelity relaxation time  $T_{\text{rel}}$ . This directly implies that when starting in  $|\mathcal{A}_k\rangle$ , the fidelity decay will become slower as  $N$  is increased, eventually diverging as  $N \rightarrow \infty$ . Similarly, the thermalisation of local observables and entanglement growth will also become slower with increasing  $N$ .

It is important to highlight that this scaling of the energy variance, and its consequences, with respect to the system size are highly unusual. We may contrast it with product states, whose energy variance is expected to scale as  $\Delta E \sim \sqrt{N}$ , as argued in Section 2.1.

This stark contrast makes the AQMBS states appear highly pathological. Nonetheless, they have been found in several models. Their presence was first pointed out in the spin-1  $XY$  model [20]. They were also found to be present in spin models with the Dzyaloshinskii-Moriya interaction, which can be simulated using specific setups of Rydberg atoms [171]. A peculiar connection has been pointed out in [168], in which

it was shown that AQMBS can be seen as quasi-Nambu-Goldstone modes, which are present in the spin-1  $XY$ , AKLT, as well as a deformed  $PXP$  model, demonstrating their ubiquity. This point was further underscored in [169], proposing a systematic construction for AQMBS states.

## 4.2 General Framework

A widely celebrated result in the quantum many-body literature is the demonstration that certain one-dimensional Hamiltonians are guaranteed to have ground states with low entanglement entropy and short range correlations [172]. In technical terms, they have the form of matrix product states (MPS). Conversely, for any MPS state, one can construct a Hamiltonian, which will have the given MPS as a ground state. Moreover, using a similar framework, one can find a Hamiltonian where the MPS will be a QMBS [87].

Apart from the ground states, one is also commonly interested in the first excited states in a model, and the energy gap between these two. If this gap remains finite even in the thermodynamic limit, the system is *gapped*, but if it is vanishing, we deem it *gapless*. Since the *gapless* excitations have an energy approaching the ground state one, we can view them as “asymptotic ground states”. In Ref. [20], the authors have introduced asymptotic quantum many-body scars (AQMBS), which emerge as QMBS only in the thermodynamic limit. Now, given that we can embed certain states either as ground states or QMBS into different Hamiltonians, this hints towards a possible connection between the *gapless* excitations (“asymptotic ground states”) and asymptotic QMBS.

Building upon the prior work in [168, 169], we will demonstrate the presence of asymptotic scars in certain models, which arise due to a connection to *gapless* excitations in *frustration free* Hamiltonians. For the argument, we consider Hamiltonians of the form:  $\hat{\mathbb{H}}_+ = \sum_n \hat{P}_{X_n}$  that are sums of local projectors  $\hat{P}_{X_n}$  that act non-trivially only on the neighbouring qubits in the region  $X_n$ . Since each projector in the sum is a non-negative operator,  $\hat{P}_{X_n} \geq 0$  (with at least one eigenvalue of 0) the Hamiltonian

eigenvalues must all be non-negative.

The *frustration-free* property implies that any ground state of the Hamiltonian  $\hat{\mathbb{H}}_+$  lies in the ground-state manifold of each local projector  $\hat{P}_{X_n}$  [173]. Since every projector in the sum is a non-negative operator with minimum eigenvalue zero, any ground state  $|\mathcal{G}\rangle$  of  $\hat{\mathbb{H}}_+$  has zero energy, i.e.,  $\hat{\mathbb{H}}_+ |\mathcal{G}\rangle = 0$ . The Hamiltonian  $\hat{\mathbb{H}}_+$  is said to be *gapped* if a non-zero energy gap separates the ground-state manifold from the first excited states even in the thermodynamic limit  $N \rightarrow \infty$ , and *gapless* if this gap closes in that limit [174]. In the gapless case, there exists a family of low-lying excited states  $|\mathcal{A}\rangle$  whose energies vanish as  $N \rightarrow \infty$ , so that  $\hat{\mathbb{H}}_+ |\mathcal{A}\rangle \rightarrow 0$ . Because each term in  $\hat{\mathbb{H}}_+$  is non-negative, it follows that  $\hat{P}_{X_n} |\mathcal{A}\rangle \rightarrow 0$  for all  $n$  in the thermodynamic limit.

Given any *frustration-free* Hamiltonian  $\hat{\mathbb{H}}_+ = \sum_n \hat{P}_{X_n}$ , we can construct a Hamiltonian  $\hat{\mathbb{H}} = \sum_n \hat{P}_{X_n} \hat{H}_{X_n} \hat{P}_{X_n}$ , where  $\hat{H}_{X_n}$  is an arbitrary Hermitian operator with support on the set  $X_n$ . Since  $\hat{P}_{X_n} |\mathcal{G}\rangle = 0$  for all  $n$ , the ground states of  $\hat{\mathbb{H}}_+$  are also zero-energy eigenstates of  $\hat{\mathbb{H}}$ , though not necessarily the ground states, since the interaction terms  $\hat{P}_{X_n} \hat{H}_{X_n} \hat{P}_{X_n}$  are no longer non-negative in general [87]. Then  $|\mathcal{G}\rangle$  can become regular QMBS of  $\hat{\mathbb{H}}$  in the middle of the spectrum.

Moreover, if the Hamiltonian  $\hat{\mathbb{H}}_+$  is also *gapless*, then  $\hat{\mathbb{H}}_+ |\mathcal{A}\rangle \xrightarrow{N \rightarrow \infty} 0$  implies that  $\hat{\mathbb{H}} |\mathcal{A}\rangle \xrightarrow{N \rightarrow \infty} 0$ , which is shown in more detail in Appendix E. In other words, the asymptotic ground states of  $\hat{\mathbb{H}}_+$  are asymptotic QMBS of  $\hat{\mathbb{H}}$ , establishing a direct link between *gapless* excitations and *asymptotic* scars in a relatively wide range of models, not constrained by dimensionality of the lattice, nor the locality of operators. Interestingly, the states  $|\mathcal{A}\rangle$  are also asymptotic scars for the Floquet circuit models which arise from the Trotterisation of a Hamiltonian  $\hat{\mathbb{H}}$  hosting the same AQMBS, even for a large time-step, which is also shown in Appendix E. Hence, we have some freedom, allowing us to find a model amenable for digital simulation.

### 4.3 Concrete Model

In order to demonstrate a concrete model, both numerically and experimentally, we will focus on a 2-local model. We will use projectors of the form:

$$\hat{P}^{(g)} = \hat{\mathbb{I}} \otimes \hat{\mathbb{I}} - |00\rangle\langle 00| - |\psi_{\perp}^{(g)}\rangle\langle\psi_{\perp}^{(g)}|, \quad |\psi_{\perp}^{(g)}\rangle = \frac{1}{\sqrt{1+|g|^2}}(|01\rangle + g|10\rangle). \quad (4.2)$$

Note that  $|\psi_{\perp}^{(g)}\rangle$  is a parameterised two-qubit state and in general  $g \in \mathbb{C}$ , but we will for most demonstrations restrict ourselves to the case  $g \in \mathbb{R}$ . Using this projector, we can now specify a local interaction term, which satisfies  $\hat{P}^{(g)} \hat{H}^{(g)}(\hat{h}) \hat{P}^{(g)} = \hat{H}^{(g)}(\hat{h})$ . The action of the projectors constraints this term to:

$$\hat{H}^{(g)}(\hat{h}) = h_{00} |\psi^{(g)}\rangle\langle\psi^{(g)}| + h_{11} |11\rangle\langle 11| + h_{01} |\psi^{(g)}\rangle\langle 11| + h_{01}^* |11\rangle\langle\psi^{(g)}|, \quad (4.3)$$

where  $|\psi^{(g)}\rangle = \frac{1}{\sqrt{1+|g|^2}}(g^*|01\rangle - |10\rangle)$  is orthogonal to  $|\psi_{\perp}^{(g)}\rangle$  and  $\hat{h}$  is a 2 by 2 Hermitian operator. Using these interaction terms, we can then build a Hamiltonian model:

$$\hat{\mathbb{H}}^{(g)} = \sum_{n=0}^{N-2} \hat{H}_{n,n+1}^{(g)}(\hat{h}_n), \quad (4.4)$$

where the lower indices at  $\hat{H}$  specify the sites it is acting on, and the index at  $\hat{h}$  is used to express the fact that interactions may differ site to site. We will restrict ourselves to 2-site translationally invariant qubit models, determined by the even and odd generators  $\{\hat{h}_e, \hat{h}_o\}$ . Alternatively, we can also build a Floquet model with these interactions, where the local unitaries have the form:

$$\hat{U}^{(g)}(\hat{h}) = \exp[i\hat{H}^{(g)}(\hat{h})]. \quad (4.5)$$

This unitary can also be rewritten as a controlled operation in a rotated basis:

$$\hat{U}^{(g)}(\hat{h}) = \hat{V}^{(g)\dagger}(|0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes e^{i\hat{h}}) \hat{V}^{(g)}. \quad (4.6)$$

where  $\hat{V}^{(g)}$  encodes a basis transformation:

$$\hat{V}^{(g)} |00\rangle = |00\rangle, \quad \hat{V}^{(g)} |\psi_{\perp}^{(g)}\rangle = |01\rangle, \quad \hat{V}^{(g)} |\psi^{(g)}\rangle = |10\rangle, \quad \hat{V}^{(g)} |11\rangle = |11\rangle. \quad (4.7)$$

We demonstrate that Eq. 4.6 holds in Appendix C.

In order to construct the global unitary, we use two different local operators  $\{\hat{h}_e, \hat{h}_o\}$ , as also done for the Hamiltonian model. These will be used to construct the even and odd layers, respectively, according to:

$$\hat{\mathbb{U}}_e^{(g)} = [\hat{U}^{(g)}(\hat{h}_e)]^{\otimes N/2}, \quad \hat{\mathbb{U}}_o^{(g)} = \hat{\mathbb{I}} \otimes [\hat{U}^{(g)}(\hat{h}_o)]^{\otimes (N/2-1)} \otimes \hat{\mathbb{I}}, \quad (4.8)$$

where  $\hat{\mathbb{I}}$  is the single-qubit identity operator. The time evolution is then dictated by the Floquet operator  $\hat{\mathbb{U}}_F^{(g)} = \hat{\mathbb{U}}_o^{(g)} \hat{\mathbb{U}}_e^{(g)}$ . In both the Floquet and Hamiltonian models we impose open boundary conditions, which are necessary for the existence of a boundary localised regular scar, as well as for observing asymptotic scar states.

## 4.4 Regular QMBS states

By construction of the Hermitian operators in Eq. (4.3), it is evident that any operator of that form will act trivially on  $|00\rangle$  and therefore the state:

$$|\mathcal{V}\rangle = |0\rangle^{\otimes N} \quad (4.9)$$

will be a zero energy eigenstate of  $\hat{\mathbb{H}}$ . By immediate extension, the state  $|\mathcal{V}\rangle$  will also be a zero eigenphase eigenstate of  $\hat{\mathbb{U}}_F^{(g)}$ . It is also possible to show that the state:

$$|\mathcal{B}^{(g)}\rangle = \frac{1}{\mathcal{N}} \sum_{n=0}^{N-1} g^{-n} |0\dots 01_n 0\dots 0\rangle = \frac{1}{\mathcal{N}} \sum_{n=0}^{N-1} g^{-n} \hat{X}_n |0\rangle^{\otimes N}, \quad (4.10)$$

is a zero eigenenergy (eigenphase) eigenstate of  $\hat{\mathbb{H}}^{(g)}$  ( $\hat{\mathbb{U}}_F^{(g)}$ ), with  $\mathcal{N} = |g| \sqrt{\frac{1-|g^{-1}|^{2N}}{1-|g^{-1}|^2}}$  as the normalisation factor. We demonstrate this in Appendix D

Note that  $|\mathcal{B}^{(g)}\rangle$  is a boundary localised state, whose localisation properties depend on  $g$ . Interestingly, at  $g = 1$  it becomes the  $N$  qubit  $W$  state, delocalised over the whole system. We will address some of its properties in more detail later on in this section.

The two scar states can be seen in the eigenspectrum of both the Hamiltonian and the

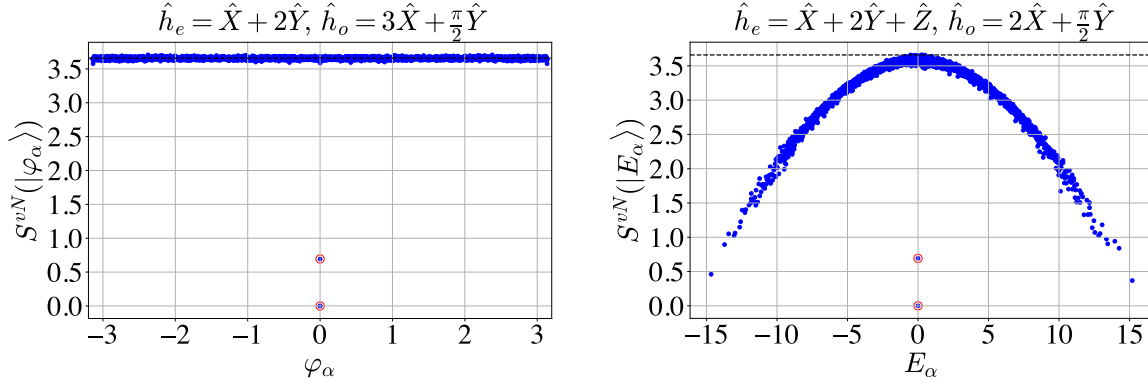


Figure 4.1: Bipartite entanglement entropy of eigenstates for the Floquet (LEFT) and Hamiltonian models (RIGHT), for  $N = 12$ ,  $g = 1.0$ . The dashed line represents the Page entropy,  $S_{\text{Page}} = \frac{1}{2}(N \ln 2 - 1)$  and the QMBS states at  $\varphi_\alpha(E_\alpha) = 0$  are marked in red, with  $S^{vN}(|\mathcal{V}\rangle) = 0$ ,  $S^{vN}(|\mathcal{B}^{(g=1)}\rangle) = \ln(2)$ .

Floquet models. For  $N = 12$  we can find all of the eigenstates and their eigen-energies (-phases) exactly. We can also compute the bipartite entanglement entropy, with the results displayed Fig. 4.1 for  $g = 1$ , where the two exact QMBS states can be seen well below the bulk of the spectrum, at significantly lower bipartite entanglement entropies. It is easy to see that  $|\mathcal{V}\rangle$  will have a bipartite entanglement entropy of 0, whereas  $|\mathcal{B}^{(g)}\rangle$  will in general have a non-zero entropy. At  $g = 1$ , is is  $S^{vN}(|\mathcal{B}^{(g)}\rangle) = \ln(2)$ , as demonstrated later in this section. While the rest of the eigenstates of  $\hat{\mathcal{U}}_F$  is saturated around the Page value, the eigenstates of the Hamiltonian model are saturating at an arch, with the maximum achieved in the middle of the spectrum. This is consistent with the ETH predictions, and aligned with other scarred models, such as the  $PXP$  spectrum, displayed in Fig. 2.2.

As already mentioned, the model hosts two exact QMBS states:  $|\mathcal{V}\rangle$  and  $|\mathcal{B}^{(g)}\rangle$ . The former is both trivial to notice, and trivial from the perspective of its properties. The latter however, is not immediately apparent from the construction of the model, and also displays interesting localisation behaviour.

The state  $|\mathcal{B}^{(g)}\rangle = \frac{1}{\mathcal{N}} \sum_{n=0}^{N-1} g^{-n} \hat{X}_n |0\rangle^{\otimes N}$  can be seen as an “edge localised” QMBS, with the  $|1\rangle$  state representing a local excitation. For  $|g| < 1$ , the magnitude of complex amplitudes is decreasing right to left, and vice versa for  $|g| > 1$ . Therefore, for  $g \neq 1$ , the excitation will be exponentially localised at either the left or right boundary of the chain. To quantify the degree of the localisation, we introduce an

“imbalance” operator, defined as:

$$\hat{\mathcal{I}} = \sum_{i=0}^{N/2-1} \hat{n}_i - \sum_{i=N/2}^{N-1} \hat{n}_i, \quad \hat{n}_i = |1\rangle \langle 1|_i, \quad (4.11)$$

where  $\hat{n}_i$  is the occupation operator at site  $i$ .

The expectation value of the imbalance with respect to the state  $|\mathcal{B}^{(g)}\rangle$  can be computed exactly in terms of  $g$  and  $N$ . For brevity we use  $q = g^{-1}$ :

$$\begin{aligned} \langle \mathcal{B}^{(g)} | \hat{\mathcal{I}} | \mathcal{B}^{(g)} \rangle &= \frac{1}{\mathcal{N}^2} \left( \sum_{n=0}^{N/2-1} |g^{-1}|^{2n} - \sum_{n=N/2}^{N-1} |g^{-1}|^{2n} \right) \\ &= \frac{1}{\mathcal{N}^2} \left( |q|^2 \frac{1 - |q|^N}{1 - |q|^2} - |q|^{N+2} \frac{1 - |q|^N}{1 - |q|^2} \right) \\ &= \frac{1}{|q|^2} \frac{1 - |q|^2}{1 - |q|^{2N}} \frac{1 - |q|^N}{1 - |q|^2} (|q|^2 - |q|^{N+2}) \\ &= \frac{(1 - |q|^N)^2}{1 - |q|^{2N}} = \frac{(1 - |q|^N)^2}{(1 - |q|^N)(1 + |q|^N)} = \frac{1 - |g^{-1}|^N}{1 + |g^{-1}|^N} \end{aligned} \quad (4.12)$$

The plot of this expectation value is depicted in Fig. 4.2, showing a sharper and sharper transition at  $g = 1$ , as the system size  $N$  is increased. In the limit  $N \rightarrow \infty$ , this becomes a step function  $\lim_{N \rightarrow \infty} \langle \mathcal{B}^{(g)} | \hat{\mathcal{I}} | \mathcal{B}^{(g)} \rangle(N) = 2 \Theta(g) - 1$ , where  $\Theta(g)$  is the Heaviside step function. This showcases critical behaviour in the properties of the embedded QMBS state. For very large values of  $N$ ,  $|\mathcal{B}^{(g)}\rangle$  will be exponentially localised on either side of the chain, except at  $g = 1$ , where it becomes an equal superposition of all single excitation states, i.e. the  $N$ -qubit  $W$  state, delocalised over the whole system. Thus, we can identify  $\hat{\mathcal{I}}$  as an order parameter for the localisation transition. The most important aspect of this transition, is whether or not the  $\hat{\mathbb{H}}_+$  model is *gapless* at the critical value of the order parameter, as the *gapless* excitations in  $\hat{\mathbb{H}}_+$  would lead to AQMBS in  $\hat{\mathbb{H}}$ , which will be addressed in the next section.

We can also characterize this transition via the localisation length  $\xi_{\text{loc}}(g)$ , and its divergence around the critical point. First, we note that due to the particularly simple structure of  $|\mathcal{B}^{(g)}\rangle$ , constrained in the single excitation subspace, the localisation length

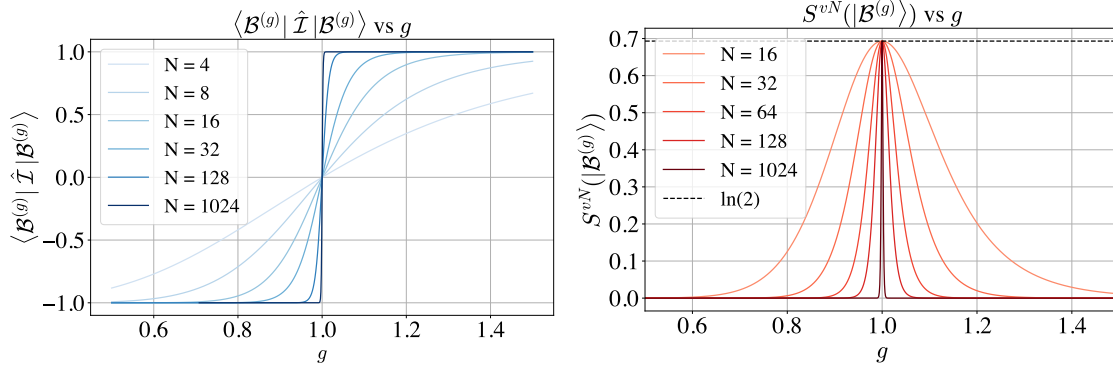


Figure 4.2: LEFT: Expectation value of left-right imbalance for the boundary QMBS state. As the system size is increased, we clearly notice discontinuous behaviour at  $g = 1$ . RIGHT: The bipartite entanglement entropy for the boundary QMBS. Independently of system size, the maximum is always achieved for  $g = 1$ , at which the boundary QMBS becomes the  $N$ -qubit  $W$  state. For any  $g \neq 1$ , it approaches 0 as  $N$  is increased.

scale is the same as the correlation length, for certain two point correlations, which are explicitly calculated in Appendix F. Using the exact form of the state:  $|\mathcal{B}^{(g)}\rangle = \frac{1}{\mathcal{N}} \sum_{n=0}^{N-1} g^{-n} |0\dots 01_n 0\dots 0\rangle$ , and rewriting  $g^{-n} = e^{-\ln(g)n}$ , we can immediately identify the localisation length as  $e^{-n/\xi_{\text{loc}}} = e^{-\ln(g)n} \rightarrow \xi_{\text{loc}} = \frac{1}{|\ln(g)|}$ . Evidently the localisation length diverges at  $g_c = 1$ . By expanding around the latter,  $\ln(g) \approx g - 1 = g - g_c \implies \xi_{\text{loc}} = \frac{1}{|g - g_c|}$ , we see that the critical exponent associated with the localisation (and consequently correlation) length is  $\nu = 1$ .

Furthermore, similarly drastic behavior can be observed in the half chain entanglement entropy as a function of  $g$ . First we recall it is defined as:  $S^{vN}(|\psi\rangle) = -\text{Tr}(\hat{\rho}_A \log \hat{\rho}_A)$ ,  $\hat{\rho}_A = \text{Tr}_{N/2, N-1}[|\psi\rangle \langle \psi|]$ , where  $\text{Tr}_{N/2, N-1}[\cdot]$  denotes the partial trace over all qubits on the right hand side of the chain. This can also be computed exactly arbitrary  $N$  and  $g$  values. We will again use  $q = g^{-1}$  for brevity. The  $|\mathcal{B}^{(g)}\rangle$  state can be split across the bipartition as:

$$\begin{aligned}
 |\mathcal{B}^{(g)}\rangle &= \frac{1}{\mathcal{N}} \sum_{n=0}^{N-1} q^n \hat{X}_n |0\rangle^{\otimes N} \\
 &= \frac{1}{\mathcal{N}} \left[ \left( \sum_{n=0}^{N-1} q^n \hat{X}_n |0\rangle^{\otimes N/2} \right) \otimes |0\rangle^{\otimes N/2} + |0\rangle^{\otimes N/2} \otimes \left( \sum_{n=0}^{N-1} q^{N/2+n} \hat{X}_n |0\rangle^{\otimes N/2} \right) \right]. \quad (4.13)
 \end{aligned}$$

In order to simplify the calculations, we now introduce the two states:

$$|\chi^{(q)}\rangle = \sum_{n=0}^{N-1} q^n \hat{X}_n |0\rangle^{\otimes N/2}, \quad |\emptyset\rangle = |0\rangle^{\otimes [N/2]}. \quad (4.14)$$

Note that  $|\chi^{(q)}\rangle$  is not normalised,  $\langle\chi^{(q)}|\chi^{(q)}\rangle = |q|^{2\frac{1-|q|^N}{1-|q|^2}}$ . Let  $|\tilde{\chi}^{(q)}\rangle = \frac{1}{|q|} \sqrt{\frac{1-|q|^2}{1-|q|^N}} |\chi^{(q)}\rangle$  be the corresponding normalised state. We refer to the first  $N/2$  sites as subsystem  $A$ , and the rest of the chain as subsystem  $B$ . Now:

$$\begin{aligned} |\mathcal{B}^{(g)}\rangle &= \frac{1}{\mathcal{N}} [|\chi^{(q)}\rangle_A \otimes |\emptyset\rangle_B + q^{N/2} |\emptyset\rangle_A \otimes |\chi^{(q)}\rangle_B] \implies \\ \hat{\rho}^{(g)} &= |\mathcal{B}^{(g)}\rangle \langle\mathcal{B}^{(g)}| = \frac{1}{\mathcal{N}^2} [|\chi^{(q)}\rangle \langle\chi^{(q)}|_A \otimes |\emptyset\rangle \langle\emptyset|_B + \\ &\quad |q|^N |\emptyset\rangle \langle\emptyset|_A \otimes |\chi^{(q)}\rangle \langle\chi^{(q)}|_B + \\ &\quad q^{N/2} |\chi^{(q)}\rangle \langle\emptyset|_A \otimes |\emptyset\rangle \langle\chi^{(q)}|_B + \\ &\quad q^{N/2} |\emptyset\rangle \langle\chi^{(q)}|_A \otimes |\chi^{(q)}\rangle \langle\emptyset|_B]. \end{aligned} \quad (4.15)$$

Since  $\langle\chi^{(q)}|\emptyset\rangle = 0$ , it is evident that, when tracing over  $B$ , the last two terms in Eq. (4.15) will vanish. Inserting the normalised state  $|\tilde{\chi}^{(q)}\rangle$  into the obtained expression, we have:

$$\begin{aligned} \hat{\rho}^{(g)} &= \frac{1}{\mathcal{N}^2} [|q|^2 \frac{1-|q|^N}{1-|q|^2} |\tilde{\chi}^{(q)}\rangle \langle\tilde{\chi}^{(q)}|_A \otimes |\emptyset\rangle \langle\emptyset|_B + \\ &\quad |q|^{N+2} \frac{1-|q|^N}{1-|q|^2} |\emptyset\rangle \langle\emptyset|_A \otimes |\tilde{\chi}^{(q)}\rangle \langle\tilde{\chi}^{(q)}|_B + \dots] \implies \\ \hat{\rho}_A^{(g)} &= \text{Tr}_B[\hat{\rho}^{(g)}] = \frac{1}{\mathcal{N}^2} [|q|^2 \frac{1-|q|^N}{1-|q|^2} |\tilde{\chi}^{(q)}\rangle \langle\tilde{\chi}^{(q)}|_A + \\ &\quad |q|^{N+2} \frac{1-|q|^N}{1-|q|^2} |\emptyset\rangle \langle\emptyset|_A] = \\ &\quad p_1 |\tilde{\chi}^{(q)}\rangle \langle\tilde{\chi}^{(q)}|_A + p_2 |\emptyset\rangle \langle\emptyset|_A, \\ \implies S^{vN}(|\mathcal{B}^{(g)}\rangle) &= -p_1 \ln p_1 - p_2 \ln p_2, \end{aligned} \quad (4.16)$$

where :  $p_1 = \frac{1}{1+|g^{-1}|^N}, \quad p_2 = \frac{|g^{-1}|^N}{1+|g^{-1}|^N}.$

From this general expression we can also read off  $S^{vN}(|W\rangle) = S^{vN}(|\mathcal{B}^{(g=1)}\rangle) = \ln 2$ . It is also evident that in the limit  $N \rightarrow \infty$ , we have either  $p_1 \rightarrow 1, p_2 \rightarrow 0$  (in the case

$g < 1$ ) or  $p_1 \rightarrow 0$ ,  $p_2 \rightarrow 1$  (for  $g > 1$ ). In both cases we have  $S^{vN}(|\mathcal{B}^{(g \neq 1)}\rangle) \rightarrow 0$ , which can also be seen in Fig. 4.2. We can again notice sharper and sharper peaks at  $g = 1$  as  $N$  is increased. For very large system sizes, there is no bipartite entanglement entropy for  $g \neq 1$ , and we always have a maximum of  $\ln 2$  at  $g = 1$ .

As a final note, this behaviour in the localisation transition is independent of whether the boundary state is a ground state or an excited state in the bulk of the spectrum. Both of these options are possible, based on whether or not the local Hermitian operators used to construct the model,  $\{\hat{h}_n\}$ , are positive semidefinite or not. In the next section, we will show that in the particular instance ( $\hat{h}_n = \hat{\mathbb{I}}$ ), we have gapless excitations at  $g = 1$ , which lead to asymptotic scars in related models.

## 4.5 Asymptotic QMBS Arising from Gapless Excitations

In order to elucidate how asymptotic scars arise, let us consider a case where the local generators are positive semidefinite, such that the two “projector-embedded” states,  $\{|\mathcal{V}\rangle, |\mathcal{B}^{(g)}\rangle\}$ , are ground states. The Hamiltonian is [175]:

$$\hat{\mathbb{H}}_+^{(g)} = \sum_{n=0}^{N-2} \hat{P}_{n,n+1}^{(g)}, \quad (4.17)$$

where  $\hat{P}_{n,n+1}^{(g)}$  is the projector as defined in Eq. (4.2). Note that this is a special case of the gapless models described in detail in Ref. [176]. This corresponds to the case where we are setting the  $\hat{h}_n = \hat{\mathbb{I}}$ . It is possible to show that the Hamiltonian commutes with the total magnetisation:  $[\hat{\mathbb{H}}_+^{(g)}, \sum_{n=0}^{N-1} \hat{Z}_n] = 0$ , for any value of  $g$ . This is obvious from the structure of individual  $\hat{P}_{n,n+1}^{(g)}$ , which cannot change the total number of excitations in the system.

Therefore the Hamiltonian will be block diagonal in the computational basis. In the single excitation subspace it will, conveniently, form a tight-binding model with energy off-sets at the boundary sites. Note that we will write this sub-block in a different basis, where  $|n\rangle$  corresponds to the state  $\hat{X}_n |0\rangle^{\otimes N}$  of the original computational basis,

which was used for the full Hamiltonian. In the single excitation subspace then:

$$\begin{aligned} \hat{\mathbb{H}}_{+,SE}^{(g)} = \epsilon_0 \sum_{n=0}^{N-1} |n\rangle \langle n| + t \sum_{n=0}^{N-2} (|n\rangle \langle n+1| + |n+1\rangle \langle n|) \\ + \Delta_0 |0\rangle \langle 0| + \Delta_{N-1} |N-1\rangle \langle N-1|, \end{aligned} \quad (4.18)$$

where  $\epsilon_0 = 1$  is the on-site energy,  $t = -\frac{g}{2}$  is the hopping term, and  $\Delta_0 = \frac{g}{2}, \Delta_{N-1} = 1 - \frac{g}{2}$  are the energy off-sets at the left/right hand-side boundaries. At  $g = 1$ , the non-zero eigenvalues and corresponding eigenstates of this Hamiltonian can be found in a closed-form expression [177]:

$$[\mathcal{A}_k]_j = \cos[(N - j - \frac{1}{2})\frac{k\pi}{N}] \quad (4.19)$$

$$E_k = 1 - \cos(\frac{k\pi}{N}), \quad k = 1, 2, \dots, N-1. \quad (4.20)$$

where  $[\mathcal{A}_k]_j$  are the complex amplitudes of  $\hat{X}_j |0\rangle^{\otimes N}$  in the eigenstate  $|\mathcal{A}_k\rangle$  (up to a normalisation constant). We have omitted the lowest energy eigenstate in this subspace, which corresponds to the state  $|\mathcal{B}^{(g=1)}\rangle$ , and has an eigenvalue of  $E_0 = 0$ . Some of these states can be thought of as counter-propagating momentum states, as shown in Appendix G.

Away from  $g = 1$ , we can diagonalise the Hamiltonian within the different subspaces numerically in order to obtain the energy gap between the ground state and the first excited state, shown in Fig. 4.3, where we can see a closing of the gap at  $g = 1$ .

From Eq. (4.20), we can clearly see that for a fixed  $k$ , the gap  $E_k - E_0$  will be closing as  $N \rightarrow \infty$ . Hence,  $|\mathcal{A}_k\rangle$  form *gapless* excitations of  $\hat{\mathbb{H}}_+$ , satisfying  $\hat{\mathbb{H}}_+ |\mathcal{A}_k\rangle \xrightarrow{N \rightarrow \infty} 0$ . This Hamiltonian is also frustration-free since the product state  $|\mathcal{V}\rangle$  and the boundary state  $|\mathcal{B}^{(g)}\rangle$  are zero-energy eigenstates. Therefore, as shown in Appendix E, these states will form asymptotic QMBS for generic  $\hat{\mathbb{H}}$ , with non-positive semi-definite  $\hat{h}_n$ .

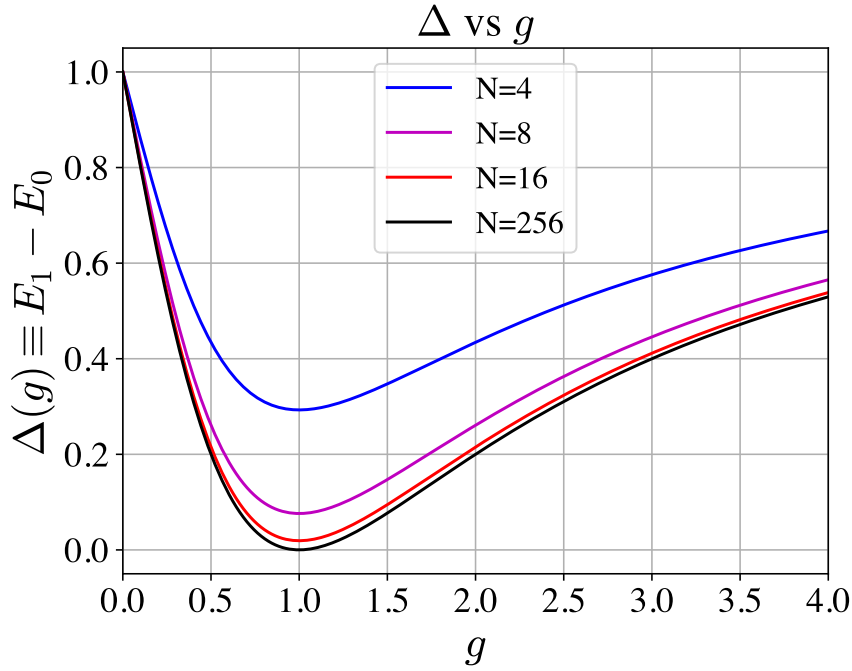


Figure 4.3: Dependence of the energy gap on  $g$ , where we can see a clear closing of the gap at  $g = 1$ , at which the  $|\mathcal{B}^{(g)}\rangle$  state undergoes the localisation transition.

## 4.6 Hamiltonian Model

In order to demonstrate the presence of asymptotic scars, we will first consider two instances of the Hamiltonian model, with different local interaction terms. As stated in the previous sections we need to ensure that  $\{\hat{h}_n\}$  are not positive semi-definite. We also need to ensure the model is not integrable. While there are some models which appear to be integrable (for example setting  $\hat{h}_n = \hat{Z} \forall n$ ), we note that this is generically not the case.

There are two peculiarities important to mention. First, for any 1- or 2-site translationally invariant model, we will have a  $\mathbb{Z}_2$  symmetry at  $g = 1$ . This symmetry is described by  $\hat{\Pi}\hat{R}$ , where  $\hat{R}$  is a reflection about the middle of the chain and  $\hat{\Pi} = e^{i\pi\sum_n|1\rangle\langle 1|_n}$  is the parity operator. The presence of this symmetry is shown formally in Appendix H. It can also be seen from the structure of operators in Eq. (4.3). At  $g = 1$ , we have  $|\psi^{(g=1)}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ . Hence, when reflecting about the middle of the chain, this state acquires a  $-1$  flip. When considering all terms in the Hamiltonian, in the computational basis, it is then easy to see that terms with an odd number of excitations acquire a  $-1$  phase change, while terms with an even number

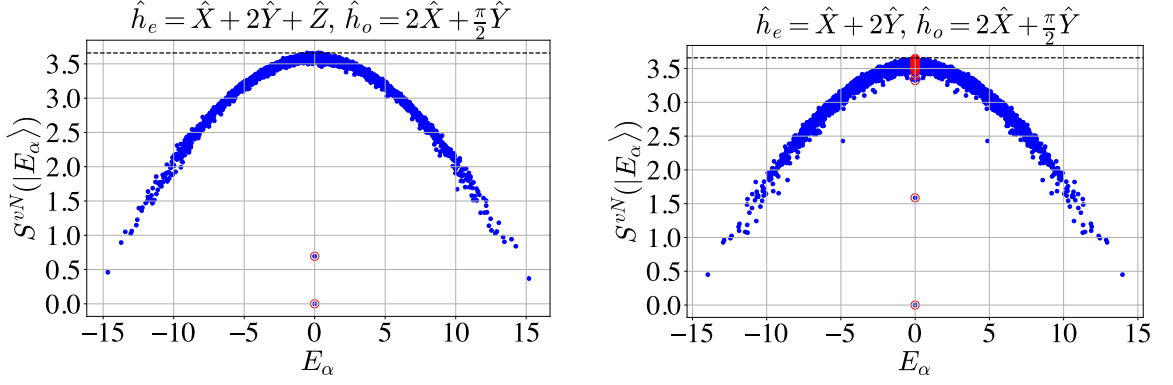


Figure 4.4: Comparison of the eigenspectra for Hamiltonians with (LEFT) and without (RIGHT) the mid-spectrum degeneracy, with the states at  $E_\alpha = 0$  marked in red. We note that, due to the degeneracy, we do not observe  $|\mathcal{B}^{(g=1)}\rangle$  in the subplot on the left, since we numerically obtain arbitrary superpositions of zero energy eigenstates.

of excitations do not. This is taken into account by using  $\hat{\Pi}$ .

Secondly, for particular choices of local generators in the two site translationally invariant case, where  $\{\hat{h}_e = \alpha_x^e \hat{X} + \alpha_y^e \hat{Y}, \hat{h}_o = \alpha_x^o \hat{X} + \alpha_y^o \hat{Y}\}$ , we observe an exponential degeneracy at  $E_\alpha(\varphi_\alpha) = 0$ . We provide further numerical evidence in Appendix I. While this degeneracy could have some other ramifications, from the perspective of our investigations, we just want to ensure it is not impacting the presence of AQMBS states. We therefore consider two instances of the Hamiltonian model, with and without this exponential mid-spectrum degeneracy. Concretely, we will set either  $\{\hat{h}_e = \hat{X} + 2\hat{Y} + \hat{Z}, \hat{h}_o = 2\hat{X} + \frac{\pi}{2}\hat{Y}\}$  or  $\{\hat{h}_e = \hat{X} + 2\hat{Y}, \hat{h}_o = 2\hat{X} + \frac{\pi}{2}\hat{Y}\}$ , with only the latter exhibiting the exponential degeneracy. We note that the sets of eigenstates of these models are almost indistinguishable from the perspective of bipartite entanglement entropy, shown in Fig. 4.4

We now want to investigate the dynamics when starting from AQMBS states,  $|\mathcal{A}_k\rangle$ , as defined in Eq. (4.19). For this purpose, we will use the time evolving block decimation algorithm (TEBD) implemented using the Julia ITensor library [178]. We perform time evolution starting from the states  $|\mathcal{A}_1\rangle$  and  $|\mathcal{A}_2\rangle$ , computing both the fidelity as well as the total magnetisation. The results for the models with and without the degeneracy are displayed in Figs. 4.5 and 4.6, respectively.

In the panels (a) and (b) of these figures we display the overlaps of  $|\mathcal{A}_1\rangle$  and  $|\mathcal{A}_2\rangle$  with the eigenstates  $|E_\alpha\rangle$  of the Hamiltonian for  $N = 12$ . These reveal  $|\mathcal{A}_1\rangle$  and

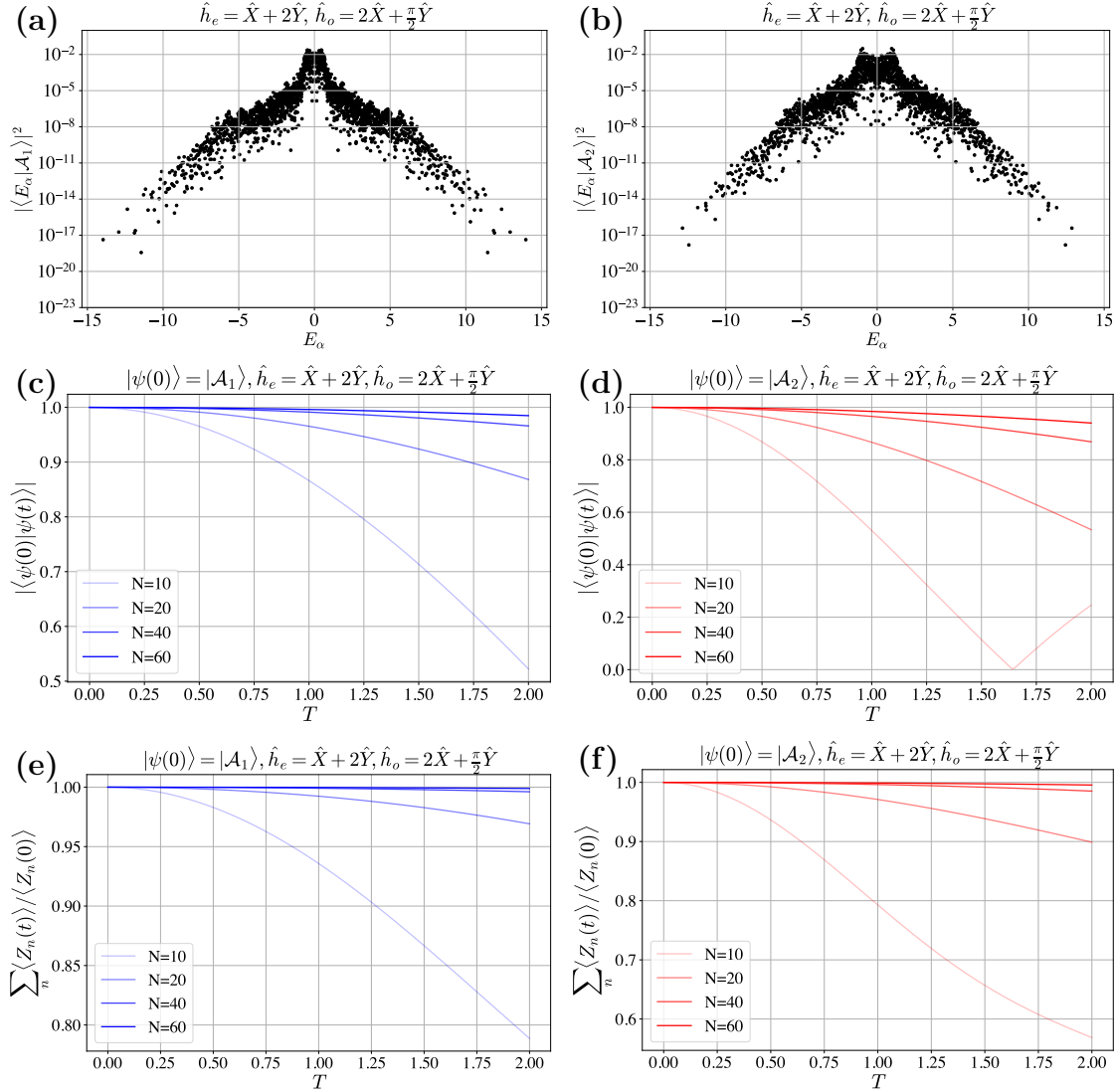


Figure 4.5: Results for the Hamiltonian model with the local generators set to  $\hat{h}_e = \hat{X} + 2\hat{Y}$ ,  $\hat{h}_o = 2\hat{X} + \frac{\pi}{2}\hat{Y}$ . Note that this instance possesses an exponential degeneracy in the middle of the spectrum, as presented in Appendix I. The overlaps of the probed AQMBS with the energy eigenstates, for  $N = 12$ , and  $g = 1$ , are presented in panels (a), (b). We perform TEBD simulations for the two AQMBS states, with the bond dimension set to  $\chi = 200$  and a time-step of  $\delta t = 0.01$ . The results for the fidelity are in panels (c), (d), and the dynamics of the total magnetisation is in (e),(f), and we probe system sizes  $N \in \{10, 20, 40, 60\}$ . The results are in line with the dynamics for the instance in Fig. 4.6, showing slower decays of the fidelity and magnetisation as  $N$  is increased, demonstrating that the mid-spectrum degeneracy is not impacting the asymptotic scarring phenomenon.

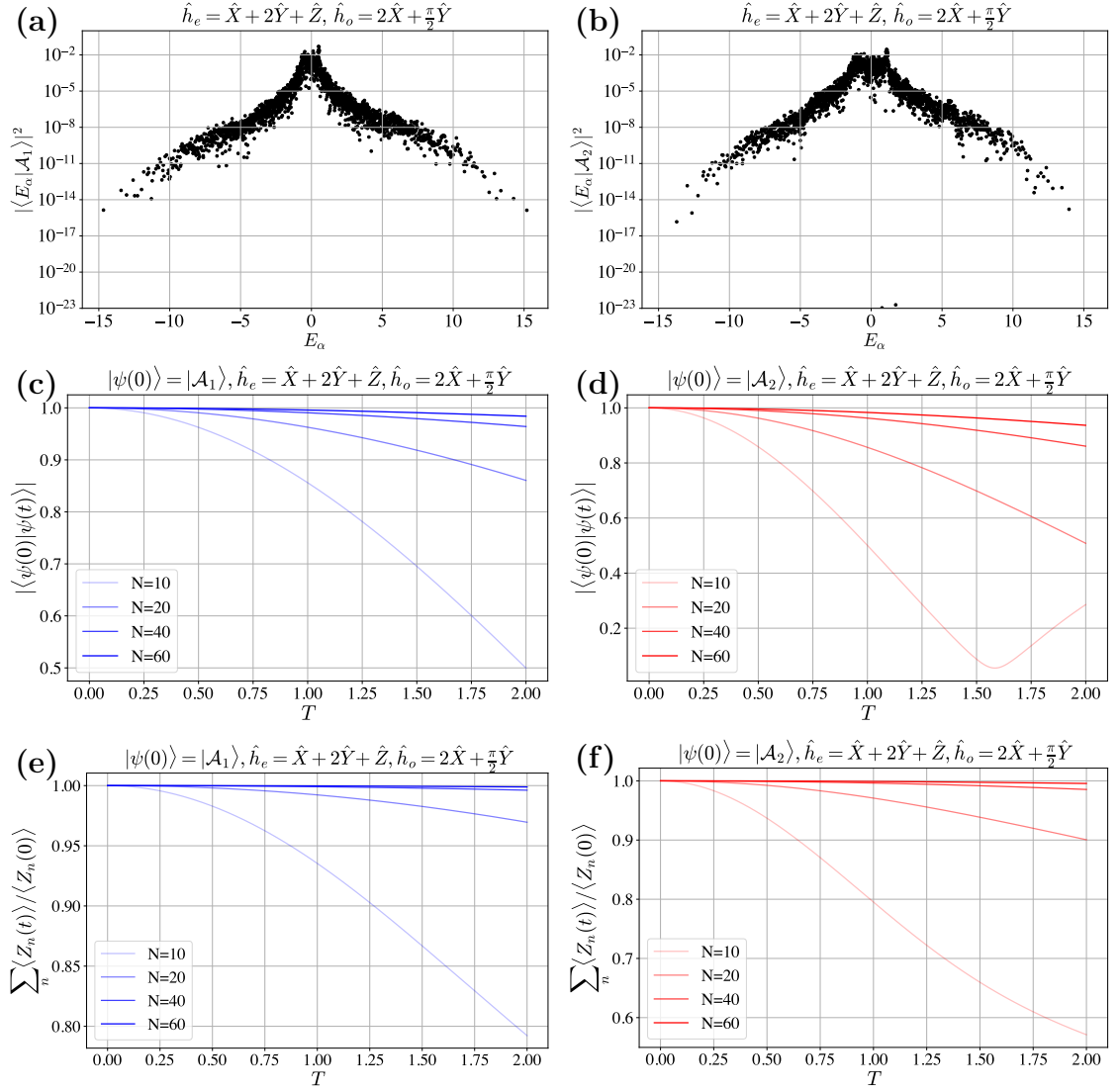


Figure 4.6: Results for the Hamiltonian model with the local generators set to  $\hat{h}_e = \hat{X} + 2\hat{Y} + \hat{Z}$ ,  $\hat{h}_o = 2\hat{X} + \frac{\pi}{2}\hat{Y}$ . The overlaps of the probed AQMBS states with the energy eigenstates, for  $N = 12$ , and  $g = 1$ , are presented in panels (a), (b). We perform TEBD simulations for the two AQMBS states, with the bond dimension set to  $\chi = 200$  and a time-step of  $\delta t = 0.01$ . The results for the fidelity are in panels (c), (f), and the dynamics of the total magnetisation is in (e), (f). Using system sizes  $N \in \{10, 20, 40, 60\}$ , we see slower decays of the fidelity and magnetisation as  $N$  is increased for all states, showing a clear indicator of asymptotic scarring.

$|\mathcal{A}_2\rangle$  to be superpositions of extensively many “thermal” eigenstates. Secondly, we also notice that  $|\langle E_\alpha | \mathcal{A}_1 \rangle|^2$  are saturated closer to  $E_\alpha = 0$ . Heuristically, this is as expected. Since the gap  $E_1 - E_0$  is closing faster than  $E_2 - E_0$  for the  $\hat{\mathbb{H}}_+$  Hamiltonian in 4.17, as described by the energy eigenvalues in Eq. (4.20). Therefore,  $|\mathcal{A}_1\rangle$  should be converging faster to a zero energy eigenstates than  $|\mathcal{A}_2\rangle$ .

In panels (c) and (d) of Figs. 4.5, 4.6, we show the fidelity decay from  $|\mathcal{A}_1\rangle$  and  $|\mathcal{A}_2\rangle$ . We observe that for both models and for both AQMBS states, the fidelity

decay becomes slower as  $N$  is gradually increased through the range  $\{10, 20, 40, 60\}$ . Secondly, for both models and at every individual system size, the fidelity when starting from  $|\mathcal{A}_1\rangle$  decays slower than when the initial state is set to  $|\mathcal{A}_2\rangle$ . This can again be traced back to the fact that, in the  $\hat{\mathbb{H}}_+$  model, the gap  $E_1 - E_0$  is closing faster than  $E_2 - E_0$ . This is further corroborated by the time evolution of the total magnetisation displayed in panes (e) and (f) of Figs. 4.5, 4.6. Increasing the system size  $N$  evidently leads to slower thermalisation of the total magnetisation, for both AQMBS states in both models.

It is worth pointing out that, although AQMBS states emerge as scars only in the thermodynamic limit, their impact in the dynamics can already be observed for (relatively) small system sizes, in the form of anomalous thermalisation and increased stability in terms of the fidelity. The results presented in Figs. 4.5 and 4.6 are supporting numerical evidence for the proof in Appendix E.

## 4.7 Floquet Model

Having presented evidence for AQMBS states in the Hamiltonian model, we now turn our attention to its Floquet version, as defined through Eqs. (4.5), (4.5). This can be seen as a Trotterisation of the Hamiltonian model, taking a large timestep of  $\delta t = 1$ . (Note that, strictly speaking, for a Trotterisation we should have included a  $-$  sign in the exponent of Eq. (4.5). However, we can instead just view it as a Trotterisation of the Hamiltonian with local generators set to  $-\hat{h}_{e/o}$ ).

As in the Hamiltonian case, we will consider instances with and without the exponential degeneracy at  $\varphi_\alpha = 0$ , setting the local generators to either  $\{\hat{h}_e = \hat{X} + 2\hat{Y}, \hat{h}_o = 3\hat{X} + \frac{\pi}{2}\hat{Y}\}$  or  $\{\hat{h}_e = \hat{X} + 2\hat{Y} + \hat{Z}, \hat{h}_o = 3\hat{X} + \frac{\pi}{2}\hat{Y}\}$ . Again we observe that these two models, from the perspective of eigenspectra, are indistinguishable, which is demonstrated in Fig. 4.7.

Secondly, we can also characterise these models via level spacing statistics, in order to verify their non-integrability, as mentioned in Section 2.1. Recalling that we have a  $\mathbb{Z}_2$  symmetry in these models at  $g = 1$  as demonstrated in Appendix H, we perform

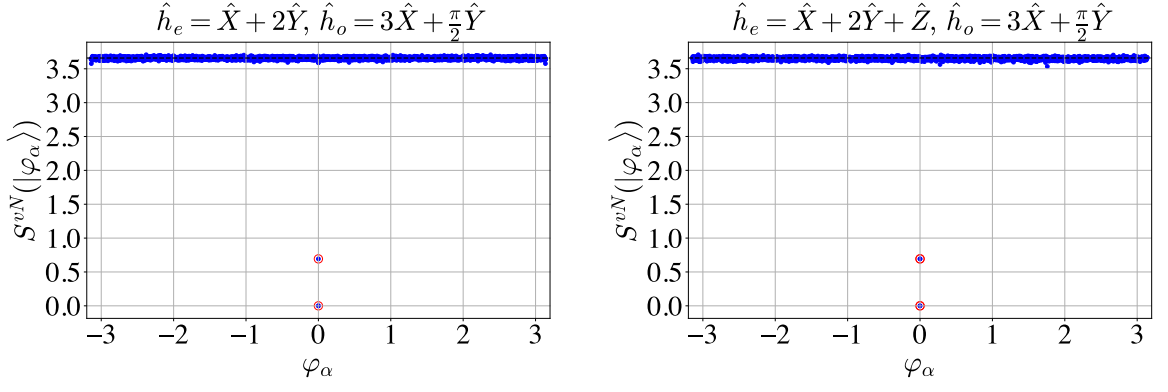


Figure 4.7: Comparison of the eigenspectra for the Floquet models with (LEFT) and without (RIGHT) an exponential degeneracy at  $\varphi_\alpha = 0$ . We note that in both cases, all states apart from  $\{|\mathcal{V}\rangle, |\mathcal{B}^{(g)}\rangle\}$  are saturating at the Page entropy,  $S_{\text{Page}}$ , marked by the dashed black line. We have used  $N = 12$  and  $g = 1.0$  in both instances.

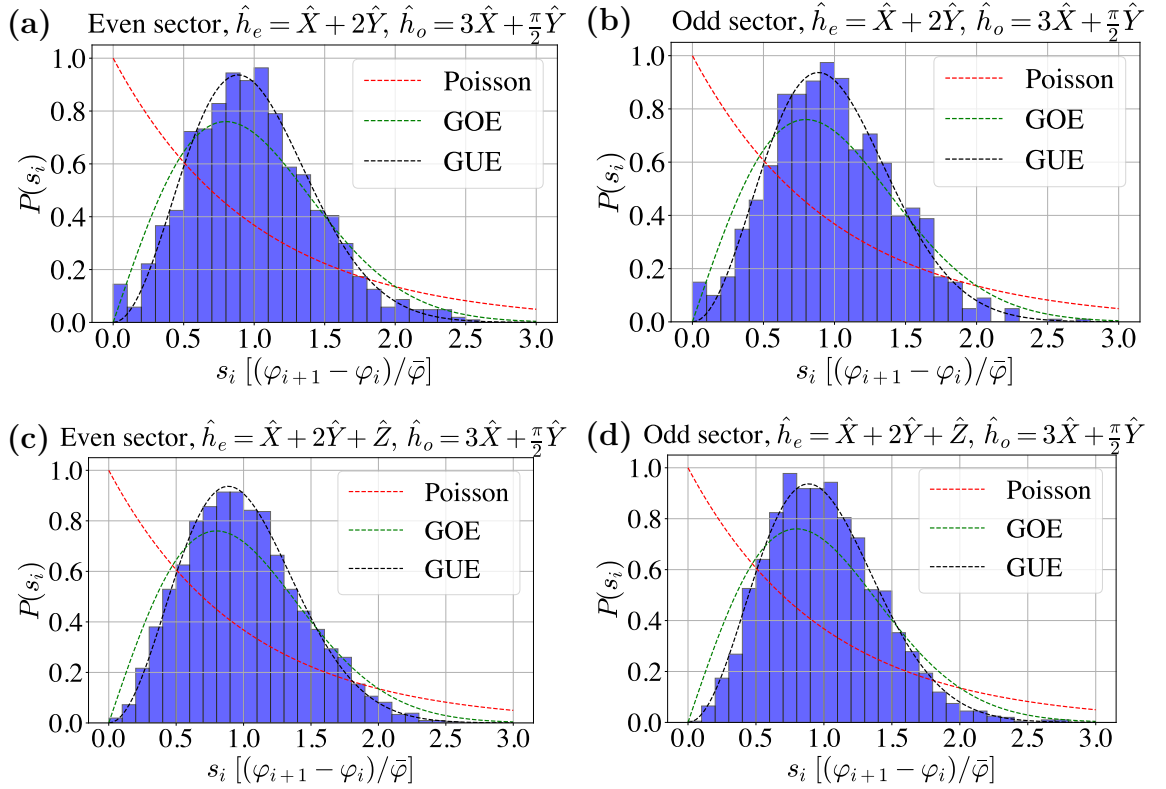


Figure 4.8: Level spacing statistics for the Floquet model within the  $\mathbb{Z}_2$  symmetry sectors. Panels (a,b) correspond to the case with exponential degeneracy at  $\varphi_\alpha = 0$  for even and odd parity sectors, respectively, with  $\bar{r} = 0.597170$  (even) and  $0.603895$  (odd). Panels (c,d) show the case without exponential degeneracy, with  $\bar{r} = 0.590351$  (even) and  $0.612889$  (odd).

the level spacing analysis within the two symmetry sectors. These are presented in Fig. 4.8, for the two cases. Both of these show agreement with the Gaussian unitary ensemble, indicating non-integrability at  $g = 1$ . Finally, it is worth noting that in for  $g = 0$  and  $g \rightarrow \infty$ , the Hamiltonian and circuit models become the kinetically constrained quantum East/West models which is shown in Appendix J.

We now turn our attention to the dynamics when starting in AQMBS states for the Floquet model. We again probe  $|\mathcal{A}_{1,2}\rangle$  for the two models with and without mid-spectrum degeneracy. The results are presented in Figs. 4.9 and 4.10.

Almost in perfect agreement with the Hamiltonian models, we notice that the overlaps  $|\langle\varphi_\alpha|\mathcal{A}_1\rangle|$  are saturated closer to  $\varphi_\alpha = 0$  than  $|\langle\varphi_\alpha|\mathcal{A}_2\rangle|$ , shown in **(a)**, **(b)** panels. In order to compute the dynamics, we again use the Julia ITensor library [178]. The evolution of the fidelity is presented in panels **(c)**, **(d)**, and the evolution of the total magnetisation in panels **(e)**, **(f)** within Figs. 4.9 and 4.10. Building on top of the Hamiltonian results, we observe that the fidelity decay becomes slower as  $N$  is increased, for both AQMBS states and both instances of Floquet models. We also observe slower thermalisation of the total magnetisation, which can be directly linked to the slower fidelity decay.

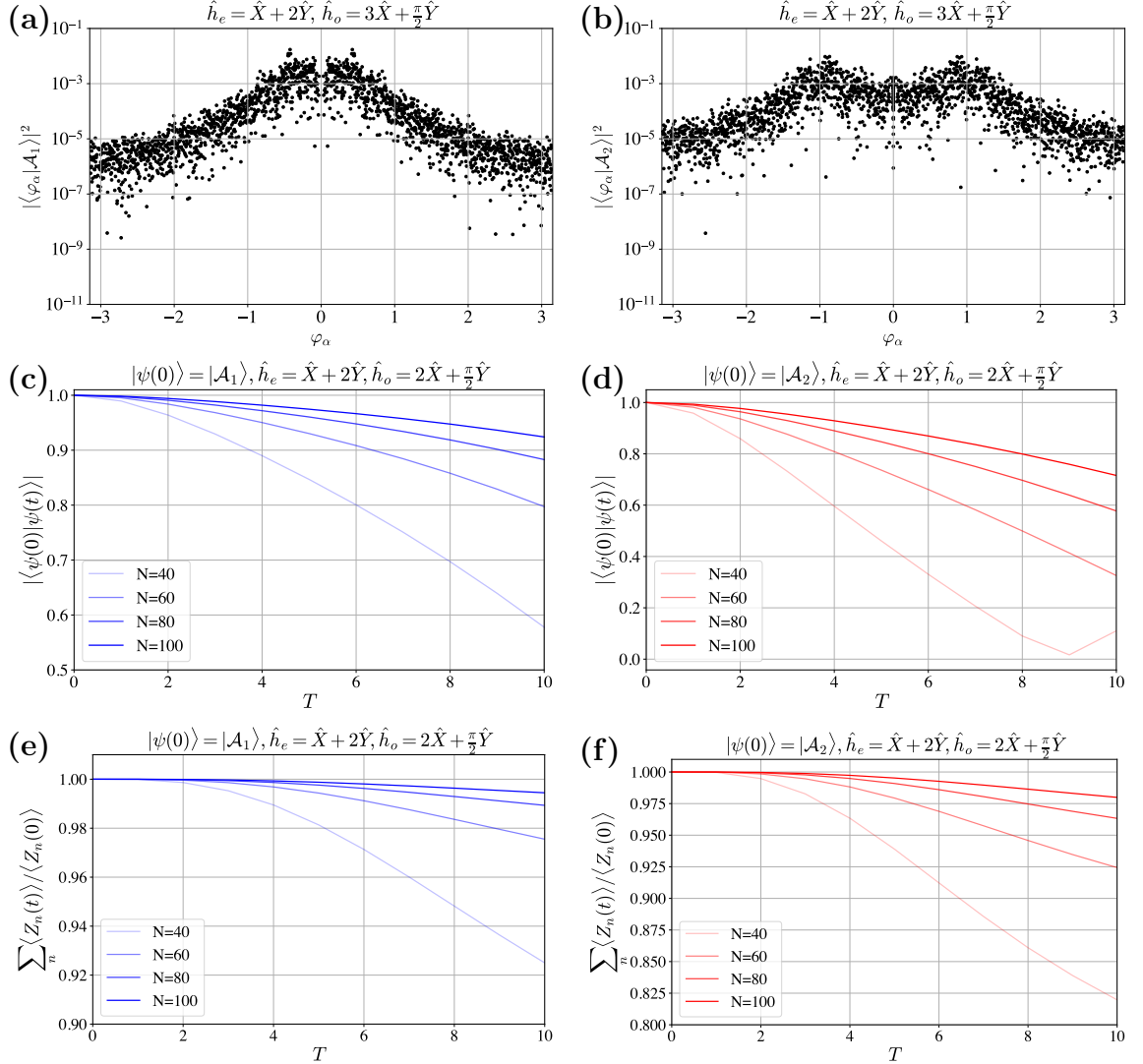


Figure 4.9: Results for the Floquet model with the local generators set to  $\hat{h}_e = \hat{X} + 2\hat{Y}$ ,  $\hat{h}_o = 3\hat{X} + \frac{\pi}{2}\hat{Y}$ . This particular instance has an exponential degeneracy at  $\varphi_\alpha = 0$ , as discussed in Appendix I. The overlaps of the probed AQMBS states with the eigenstates, for  $N = 12$ , and  $g = 1$ , are in (a), (b). In order to investigate the dynamics, we perform TEBD simulations for the two AQMBS states, with the bond dimension set to  $\chi = 200$ , for system sizes  $N \in \{40, 60, 80, 100\}$ . The results for the fidelity are in (c), (d). Similarly to the case without the mid-spectrum degeneracy which is presented in Fig. 4.10, we observe significantly slower decay for  $|\mathcal{A}_1\rangle$ , with the results for  $|\mathcal{A}_2\rangle$  showing a faster decay, aligned with the expectations for increasing  $k$ . This is further corroborated by the magnetisation dynamics, in (e), (f), showing slower thermalisation as the system size is increased.

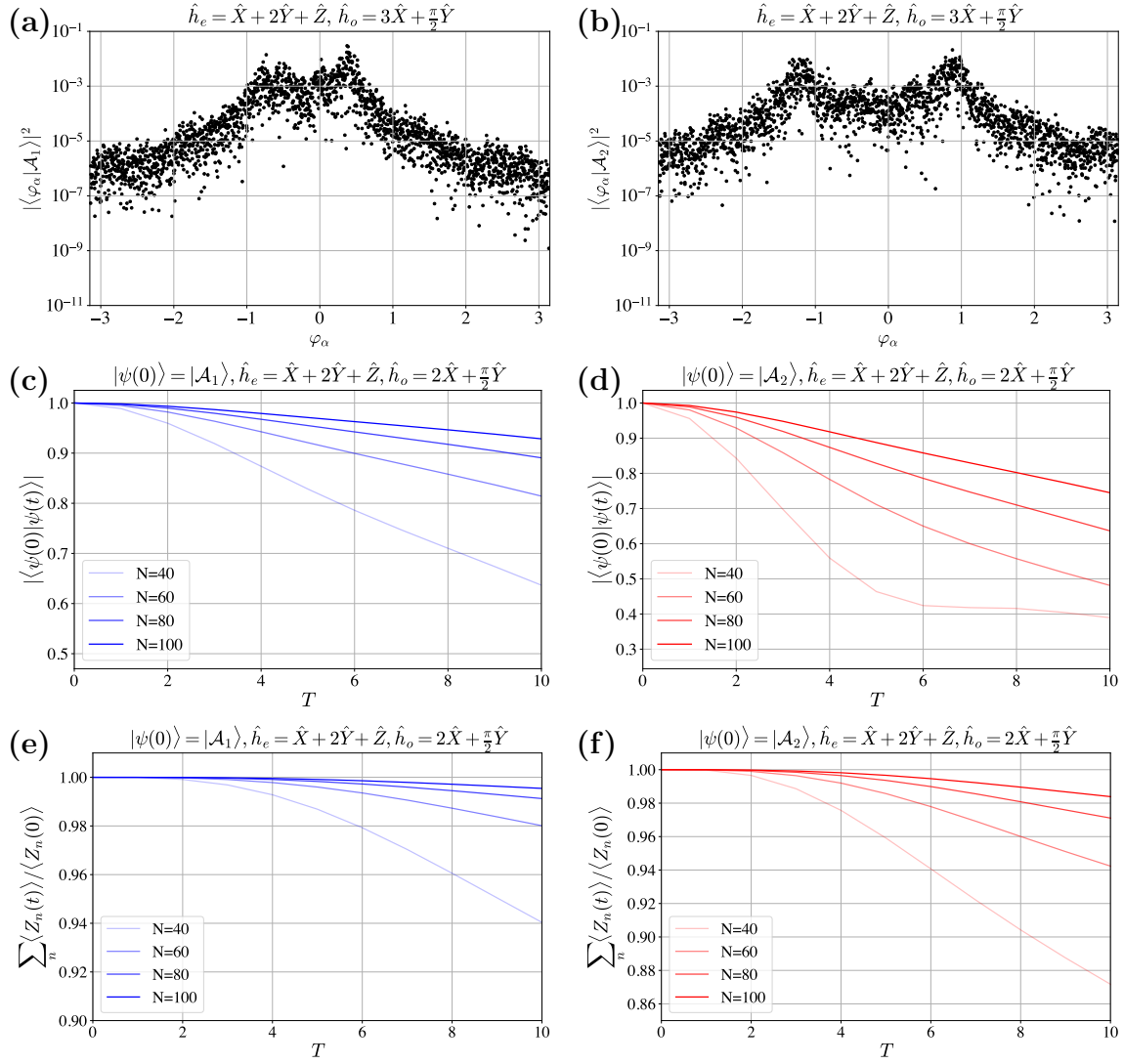


Figure 4.10: Results for the Floquet model with the local generators set to  $\hat{h}_e = \hat{X} + 2\hat{Y} + \hat{Z}$ ,  $\hat{h}_o = 3\hat{X} + \frac{\pi}{2}\hat{Y}$ . The overlaps of the probed AQMBS states with the eigenstates, for  $N = 12$ , and  $g = 1$ , are in (a), (b). In order to investigate the dynamics, we perform TEBD simulations for the two AQMBS states, with the bond dimension set to  $\chi = 200$ , for system sizes  $N \in \{40, 60, 80, 100\}$ . The results for the fidelity are in (c), (d). We observe significantly slower decay for  $|\mathcal{A}_1\rangle$ , with the results for  $|\mathcal{A}_2\rangle$  showing a faster decay, aligned with the expectations for increasing  $k$ . This is further corroborated by the magnetisation dynamics, in (e), (f), showing slower thermalisation as the system size is increased.

## 4.8 Experiment on Trapped Ion Quantum Simulator

Although QMBS states do not have a significant impact on certain spectral properties such as the density of states, spectral form factor, etc., they can have meaningful consequences for the dynamical observations for specific initial states. Starting in a QMBS state, or a state with significant overlap with a QMBS can lead to drastically slowed entanglement growth, fidelity decay [8] and, most importantly, thermalisation timescales. Similarly, for AQMBS states, the fidelity decays slower and slower as the system size is increased. This directly leads to a slower thermalisation of local observables and slower entanglement growth.

In order to demonstrate these effects on the level of local observables, we consider the Floquet model, with the even and odd generators set to  $\hat{h}_e = \hat{X} + 2\hat{Y}$ ,  $\hat{h}_o = 3\hat{X} + \frac{\pi}{2}\hat{Y}$ . Note that we picked these generators since any one with the form  $\hat{h} = \alpha_x \hat{X} + \alpha_y \hat{Y}$ ,  $\alpha_{x/y} \in \mathbb{R}$ , can be decomposed with 2 entangling gates, for any value of  $g$ . This is in contrast to the general case, which requires 3, and thus leads to greater error accumulation. While such a choice of generators gives an exponential degeneracy in the middle of the spectrum, this has no significant impact on the AQMBS states.

The time evolution is then dictated by 4.8, with one application of  $\hat{\mathcal{U}}_F$  corresponding to one time-step. An example showing the time evolution circuit for  $N = 8$ ,  $T = 2$  is displayed in Fig. 4.11 (a), with the decompositions of  $\hat{U}_e \equiv \hat{U}^{(g)}(\hat{h}_e)$  and  $\hat{U}_o \equiv \hat{U}^{(g)}(\hat{h}_o)$  displayed in (b) and (c), respectively. We perform the simulations on the Quantinuum H1-1 trapped ion quantum simulator [179, 180], with some further device specifications and experimental details presented in appendix K.

In order to demonstrate the effects of starting in an AQMBS state, we perform experiments for various system sizes,  $N \in \{8, 12, 16, 20\}$ , up to  $T = 10$  time-steps. We perform these for three different initial states: an exact, regular QMBS ( $|\mathcal{V}\rangle = |0\rangle^{\otimes N}$ ), a non-scar state ( $|1\rangle^{\otimes N}$ ), and an AQMBS state ( $|\mathcal{A}_1\rangle$ ). Note that, while the former two can be prepared trivially on any digital simulator, the latter requires a longer

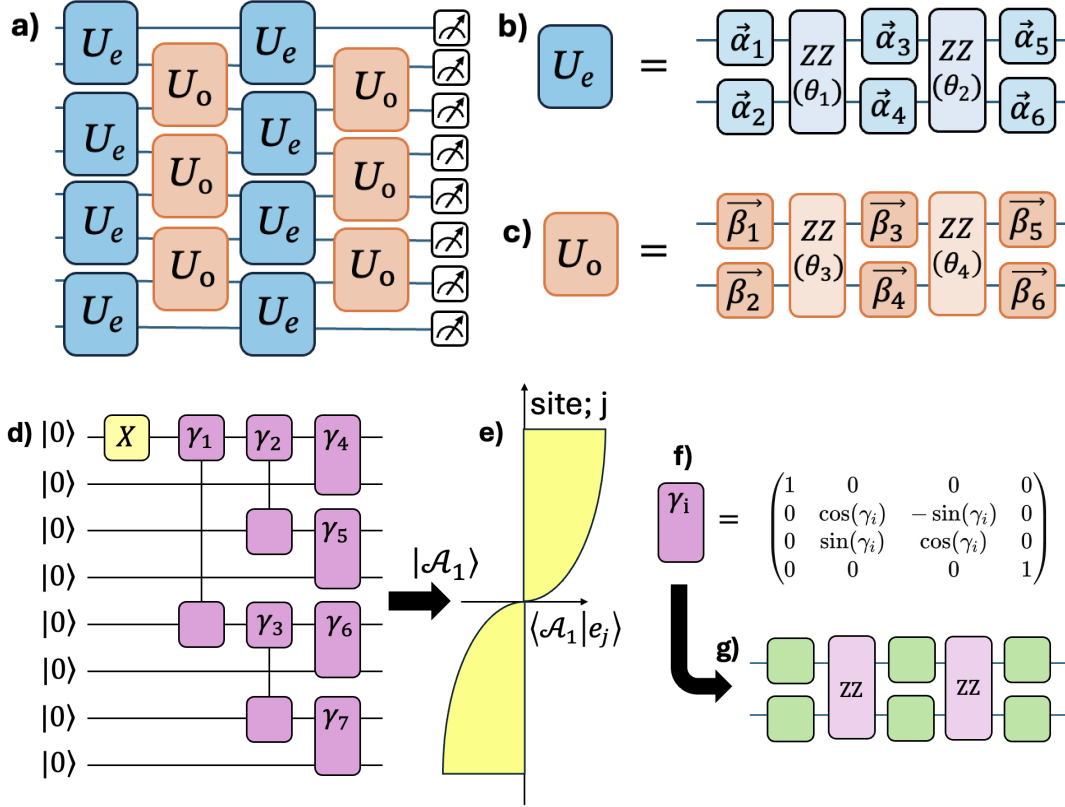


Figure 4.11: Circuit constructions used for performing the simulation on the Quantinuum H1-1 processor. **a)** Depiction of the implemented time evolution circuit, for  $N = 8$  qubits and  $T = 2$  Floquet layers. **b), c)** The presented choice of generators,  $\{\hat{h}_e = \hat{X} + 2\hat{Y}, \quad \hat{h}_o = 3\hat{X} + \frac{\pi}{2}\hat{Y}\}$ , allows for an efficient decomposition for  $U^{(g)}(\hat{h}_{e/o})$ , in terms of  $ZZ(\theta)$  rotation gates, which are the native entangling gates of the device. Note that this holds for any value of  $g$ . The  $\vec{\alpha}_i$  labelled gates represent arbitrary single qubit operations in terms of the Euler angles in a Z-X-Z rotation sequence:  $R_z(\alpha_i^1)R_x(\alpha_i^2)R_z(\alpha_i^3)$ , and similarly for the  $\vec{\beta}_i$  labelled gates. **d)** Preparation of the  $|\mathcal{A}_1\rangle$  AQMBS state for  $N = 8$ . Following a preparation of a single excitation on the first qubit in the chain, we can generate an arbitrary state in the single excitation subspace by suitably choosing the rotation parameters  $\gamma_i$ . This construction can be easily generalised to more qubits, and requires a circuit depth of  $\mathcal{O}(\log_2(N))$ , with  $N - 1$  general  $U(1)$  symmetric gates. **e)** Schematic representation of the prepared  $|\mathcal{A}_1\rangle$  AQMBS state, with the  $y$ -axis representing the qubit positions (directly corresponding to circuit in **d)** and the  $x$  axis to the amplitudes  $\langle \mathcal{A}_1 | \hat{X}_j | 0 \rangle^{\otimes N}$ , where  $j$  is the site index of the qubits. The matrix decomposition of the  $\gamma_i$  labelled gates is in **f)**, and its decomposition in terms of  $ZZ$  rotation gates and arbitrary single qubit gates is in **g)**. Note that all of the parameters used for time evolution  $\{\alpha_i, \beta_i\}$  and the state preparation  $\{\gamma_i\}$  can be found in the Appendices [L](#) and [M](#), respectively.

state preparation protocol. Fortunately, due to being constrained to the single excitation subspace,  $|\mathcal{A}_1\rangle$  can be prepared in logarithmic circuit depth on devices with all-to-all connectivity. The state preparation protocol, for  $N = 8$ , is displayed in Fig. 4.11 (d), with the matrix representation of the deployed gates shown in (f) and their gate decomposition in (g). This construction can be easily generalised to any  $N = 2^k$ , requiring  $\sim N - 1$  entangling gates and a depth of  $\sim \log_2 N = k$ . For  $N \neq 2^k$ , the state preparation circuit depth will be  $\sim \lceil \log_2 N \rceil$ . These protocols are described in more detail in appendix M.

The dynamics of the total magnetisation, when starting in  $|0\rangle^{\otimes N}$  or  $|1\rangle^{\otimes N}$ , is displayed in Fig. 4.12 (left). We observe that independently of system size, starting in  $|1\rangle^{\otimes N}$ , leads to a rapid decay, reaching  $\approx 0$  after 3 discrete time steps, which is in accordance with the Floquet generalisation of the eigenstate thermalisation hypothesis, consistent with the “infinite” temperature state. In contrast, when the qubit register is initialised in the QMBS state  $|\mathcal{V}\rangle$ , the magnetisation decays very slowly. Note that, it should stay constant by construction, but the accumulation of gate infidelities and other errors gives rise to a deviation from the predictions. It is evident that the error is also increasing as we increase the system size through the range  $N = 8, 12, 16, 20$ . This is expected since the dominant sources of errors are the 2-qubit gate infidelities. This is a clear indication of lack of thermalisation for the QMBS state, which can be observed despite of the device imperfections.

The corresponding results for the AQMBS state are displayed on the right-hand side of Fig. 4.12. For  $N = 8, 12$  we see strong agreement between the classical simulation and experimental results. For greater system sizes,  $N = 16, 20$ , we see a more noticeable deviation due to greater accumulation of errors, as already observed for the exact QMBS state. Despite of greater errors, one can still observe slower thermalisation as system size is increased, a distinct feature of AQMBS states. This is to be contrasted with the lack of thermalisation for  $|\mathcal{V}\rangle$  and the rapid decay in the case of  $|1\rangle^{\otimes N}$ , both of which are independent of system size. As the  $N$  is increased, the state  $|\mathcal{A}_1\rangle$  is behaving progressively closer to the QMBS state. This is, to the best of our knowledge, the first experimental demonstration of asymptotic scars.

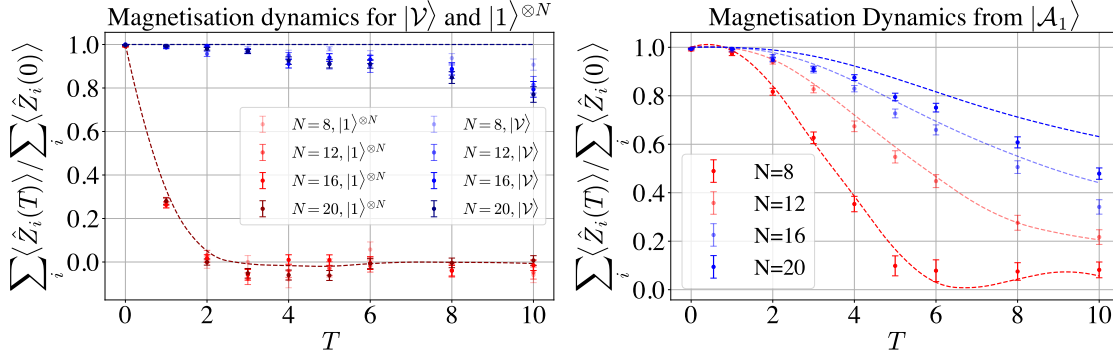


Figure 4.12: LEFT: Expectation value of the total magnetisation  $\sum_i^N \hat{Z}_i$ , normalised by the initial magnetisation, as a function of Floquet time steps for the exact QMBS state,  $|\mathcal{V}\rangle = |0\rangle^{\otimes N}$  and a non-scar state  $|1\rangle^{\otimes N}$ . Note that all experiments are performed with  $C = 100$  shots. The scatter plots correspond to experimental data, with the spreads displaying the standard error of the mean. The blue dashed line represents the exact results for  $|0\rangle^{\otimes N}$ , and the red dashed line the tensor network simulation for  $|1\rangle^{\otimes N}$ , with  $N = 20$ . RIGHT: Magnetisation decay for the AQMBS state  $|\mathcal{A}_1\rangle$  for system sizes  $N = 8, 12, 16, 20$ . The dashed lines represent classical simulation results. Notably, as the system size is increased, the AQMBS is thermalising slower, emerging as a QMBS.

We now turn to investigate the impact of the edge localised state on the thermalisation dynamics. In order to demonstrate the impact of  $|\mathcal{B}^{(g)}\rangle$  experimentally, we first highlight that the initial state  $|L\rangle = |1\rangle \otimes |0\rangle^{\otimes(N-1)}$  has a large overlap with the boundary QMBS when  $|g| \gg 1$ , and a small overlap when  $|g| \ll 1$ . Time-evolution starting from this initial state should therefore have a large component that is approximately stationary when  $|g| \gg 1$ , with the approximation improving as  $|g|$  increases. Therefore we investigate how the site-resolved and total magnetisation evolve for different  $g$  values when starting from  $|L\rangle$ , with the results displayed in Fig. 4.13. Note that we use the same local even and odd generators as used in the previous set of experiments,  $\hat{h}_e = \hat{X} + 2\hat{Y}$   $\hat{h}_o = 3\hat{X} + \frac{\pi}{2}\hat{Y}$ , and we use 5 different values of  $g$ :  $\{0.5, \frac{2}{3}, 1.0, 1.5, 2.0\}$ . At  $g = 2.0$ , we see that the excitation remains strongly localised at the boundary, which can be directly seen in the spin profile as a function of time, in panel (a), as well as the slow decay of the total magnetisation, shown in (f).

As we decrease  $g$  to 1.5, we see that the excitation is not as strongly localised, in line with the expectations based on the properties of  $|\mathcal{B}^{(g)}\rangle$ . As  $g$  is decreased, the overlap between  $|\mathcal{B}^{(g)}\rangle$  and  $|L\rangle$  will decrease as well, leading to weaker localisation, as displayed in Fig. 4.13 (b). This is further corroborated by the time evolution of the total magnetisation, displayed in Fig. 4.13 (f), showing a slightly faster decay in

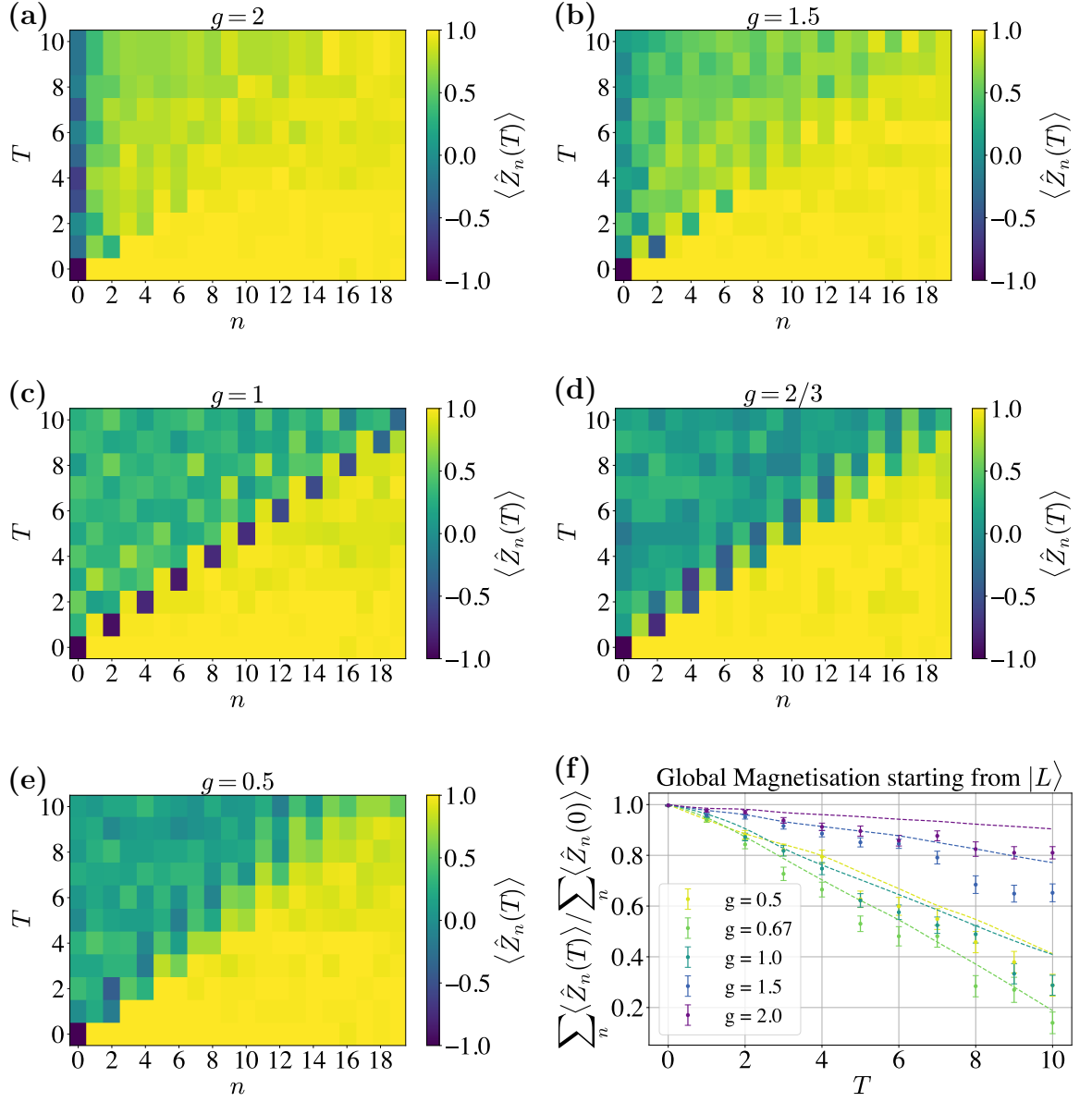


Figure 4.13: Magnetisation profiles when starting from an edge excitation  $|L\rangle = |1\rangle \otimes |0\rangle^{\otimes N-1}$  at system size  $N = 20$  probed for  $g \in \{0.5, \frac{2}{3}, 1.0, 1.5, 2.0\}$ , presented in (a)–(e), respectively. As per analytical predictions, the degree of localisation becomes stronger as  $g$  is increased. Interestingly, at  $g = 1$ , we notice almost perfect spin transport along a “light-cone”. The comparison of the total magnetisation evolution for different values of  $g$  is shown in (f). The dashed lines represent results obtained via classical methods, while the scatter plots show the mean of the experimental data with the associated standard error of the mean. In line with expectations, we observe slowest thermalisation for the  $g = 1.5, 2.0$  cases.

comparison to  $g = 2.0$ .

At  $g = 1.0$ ,  $|\mathcal{B}^{(g)}\rangle$  becomes the  $W$  state, delocalised over the whole chain of qubits, and thus has an extensively small overlap with the initial state we probed. This leads to a faster thermalisation in contrast to the two cases with  $g > 1$ , seen again both in the spin profile as a function of time and the decay of the total magnetisation, displayed in Fig. 4.13, (c) and (f). Most interestingly, we observe a strong signal along the “light-cone”, showing an almost perfect transport of the spin excitation.

Lastly, for the two values which are below 1,  $g = 0.5$  and  $g = 2/3$ , the edge scar is expected to be localised on the other side of the chain in relation to our prepared excitation. Therefore we expect relatively fast thermalisation, relative to  $g > 1$ , which is confirmed by the experimental data shown in (d), (e), and (f).

## 4.9 Concluding remarks

To summarise, we first showed a direct relationship between low-lying energy states in gapless Hamiltonians and the existence of AQMBS states. We then presented a 2-local model hosting both regular and asymptotic scars. One of the QMBS states displays edge localisation and becomes the  $N$ -qubit  $W$  state at  $g_c = 1$ . At this critical value, we also observe AQMBS states, corresponding to gapless excitations in a model where the exact QMBS states are embedded as ground states.

The model’s specific features made a digital quantum simulation feasible. We exploited the freedom in the 2-local unitaries to find efficient gate decompositions, and the structure of the probed AQMBS state  $|\mathcal{A}_1\rangle$  to prepare it in logarithmic depth. Performing simulations starting from  $|\mathcal{A}_1\rangle$  at various system sizes on the Quantinuum H1-1 trapped ion simulator, we observe longer thermalisation timescales as  $N$  is increased. This is the first direct experimental detection of anomalous thermalisation due to AQMBS states and one of the rare occasions where a new phenomenon was observed for the first time in digital simulators. While the signatures of asymptotic scars can also be seen, even more cleanly, in the classical simulation, this demonstrates the value of digital simulators as versatile experimental platforms. We can also expect

that with more resources, and more involved error mitigation methods [181], it would be possible to achieve greater accuracy in the quantum simulation. Furthermore, we also observed direct evidence of the product scar  $|\mathcal{V}\rangle$  and the edge localised scar state  $|\mathcal{B}^{(g)}\rangle$ .

This work opens up several research pathways. We have considered a relatively simple model with solely nearest-neighbour interactions in one dimension to facilitate digital quantum simulation. However, we expect that the same arguments could be generalised to higher-order interactions of similar form, as well as to two or three dimensions [182], giving rise to a wide family of models that could potentially host AQMBS states. Furthermore, for some sets of local generators, we can observe an exponential degeneracy in the middle of the spectrum, which might include additional QMBS states, as found in the spin-1  $XY$  model [183]. On the experimental side, with more resources, it would also be possible and of great interest to investigate the fidelity decay or growth of Rényi entropies from AQMBS via randomised measurement techniques [37, 139]. We have also established a direct link between AQMBS states and gapless excitations in a relatively simple ground localisation transition. This suggests that there are also AQMBS arising from other transitions, for example, at the critical point of the Rokhar-Kivelson model [184], at which the Hamiltonian is both *frustration-free* and *gapless*.



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# Hilbert Subspace Ergodicity

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## 5.1 Introduction

*This chapter is based on Ref. [185] and adapted for the purposes of the thesis.*

In the previous chapters, the focus has been on the ETH definition of ergodicity in closed quantum systems, and its violation via exact QMBS and asymptotic QMBS states. A contrasting conceptual framework has been put forward recently, which corresponds almost directly to our classical framework of ergodicity, and has been termed complete Hilbert space ergodicity (CHSE) [21]. We begin by providing a brief overview of CHSE, its main claims and how it can be achieved via aperiodic driving. The original example of a single qubit is provided as reference point, and then generalised to a generic circuit model. By imposing different constraints on the local unitaries, we numerically probe the effects of local and non-local conserved quantities on systems which otherwise exhibit CHSE, finding that they may still exhibit CHSE in the dynamically decoupled subspaces.

To formally introduce CHSE, consider an arbitrary initial state  $|\psi(0)\rangle$  in a  $D$ -dimensional Hilbert space undergoing unitary dynamics,  $|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle$ . For concreteness, we assume discrete-time dynamics where  $|\psi(t)\rangle = \hat{U}_t|\psi(t-1)\rangle$  propagates the system forward one time-step from  $t-1$  to  $t$  and  $\hat{U}(t) = \hat{U}_t\hat{U}_{t-1}\dots\hat{U}_1$ . Equivalently, we

may also have continuous-time dynamics, using unitaries which arise from specifically chosen Hamiltonians,  $\hat{\mathbb{U}}_t = \exp[-i\alpha\hat{H}_t]$ , where  $t$  labels the different operators chosen for different times, and the duration of the time evolution is some fixed value  $\alpha$ .

In close analogy with the classical notion of ergodicity, the concept of CHSE describes the case in which a quantum system, over sufficiently long timescales, explores all of its Hilbert space with equal likelihood. To make this idea of “equal likelihood” more precise, we need to introduce measure on the Hilbert space that allows us to formally define a uniform distribution. The natural choice is the Haar measure, which we will denote by  $d\phi$ , which is the unique measure invariant under all unitary transformations. This invariance under arbitrary unitaries ensures the equal distribution over the Hilbert space. Formally, the CHSE property can be written as [21]:

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt f(|\psi(t)\rangle \langle \psi(t)|) = \int d\phi f(|\phi\rangle \langle \phi|), \quad (5.1)$$

for any initial state  $|\psi(0)\rangle$  and any integrable function  $f$  on the space of pure states. In other words, the system’s long-term behaviour cannot be distinguished from that of a random pure state drawn uniformly from the Hilbert space.

In practice one cannot test Eq. (5.1) for all integrable functions  $f$  and it is more convenient to restrict to polynomial functions of degree  $k$ , which can always be computed using a limited set of moments. The  $k$ ’th moment of the ensemble of states in the timeseries  $\{|\psi(t)\rangle\}_{t=0}^{T-1}$  is:

$$\hat{\rho}_T^{(k)} = \frac{1}{T} \sum_{t=0}^{T-1} (|\psi(t)\rangle \langle \psi(t)|)^{\otimes k}, \quad (5.2)$$

and the  $k$ ’th moment of the Haar ensemble of states is:

$$\hat{\rho}_{\text{Haar}}^{(k)} = \int d\phi (|\phi\rangle \langle \phi|)^{\otimes k}. \quad (5.3)$$

The latter can be computed exactly and has a closed form expression [186]:

$$\hat{\rho}_{\text{Haar}}^{(k)} = \frac{\sum_{\pi \in \mathcal{S}_k} \sum_{\alpha_1, \dots, \alpha_k} |\alpha_1, \alpha_2, \dots, \alpha_k\rangle \langle \alpha_{\pi(1)}, \dots, \alpha_{\pi(k)}|}{D(D+1)\dots(D+k-1)}, \quad (5.4)$$

where  $\mathcal{S}_k$  is the permutation group of order  $k$ ,  $\pi$  are the permutation group elements and  $\{|\alpha_i\rangle\}_{i=1}^D$  is an orthonormal basis for the Hilbert space of dimension  $D$ .

We denote the Hilbert-Schmidt distance between  $\hat{\rho}_T^{(k)}$  and  $\hat{\rho}_{\text{Haar}}^{(k)}$  as [13]:

$$D_{\text{HS}}(\hat{\rho}_{\text{Haar}}^{(k)}, \hat{\rho}_T^{(k)}) = \|\hat{\rho}_{\text{Haar}}^{(k)} - \hat{\rho}_T^{(k)}\|_2^2, \quad (5.5)$$

where  $\|\hat{X}\|_2 = \sqrt{\text{Tr}(\hat{X}^\dagger \hat{X})}$  is the Hilbert-Schmidt matrix norm. Similarly, the trace distance between these two states is given by [13]:

$$\Delta_T^{(k)} = D_{\text{Tr}}(\hat{\rho}_{\text{Haar}}^{(k)}, \hat{\rho}_T^{(k)}) = \frac{1}{2} \sum_i^{d^N} |\lambda_i|, \quad (5.6)$$

where  $\lambda_i$  are the eigenvalues of  $\hat{\rho}_T^{(k)} - \hat{\rho}_{\text{Haar}}^{(k)}$ .

Note that when originally introduced in Ref. [21], the CHSE was probed using the trace distance as a metric in Eq. (5.6), which has the advantage of a clear operational meaning in quantum state discrimination in contrast to the Hilbert-Schmidt norm. However, the latter is significantly computationally cheaper to compute, as it does not require diagonalisation of the difference matrix. It can also be used to upper bound the trace distance [187] as:

$$D_{\text{Tr}}(\rho, \sigma)^2 \leq \frac{D}{4} D_{\text{HS}}(\rho, \sigma). \quad (5.7)$$

Therefore we will use the Hilbert-Schmidt norm for the calculations presented in this chapter, apart from the single qubit example. Now, the system is CHSE if  $\Delta_\infty^{(k)} = 0$  or  $D_{\text{HS}}(\hat{\rho}_{\text{Haar}}^{(k)}, \hat{\rho}_T^{(k)}) = 0$  for all  $k$  and for all initial states  $|\psi(0)\rangle$  [21].

Furthermore, in the case of a time-independent Hamiltonian for  $k \geq 2$ , the trace distance to the Haar moments cannot be lower than [21]:

$$B(D) \equiv (D+1)^{-1} - (2D(D+1))^{-\frac{1}{2}}, \quad (5.8)$$

which one can adapt for the Hilbert-Schmidt distance. Using Eq. (5.7), we have that

the lower bound for the Hilbert-Schmidt distance is:

$$D_{\text{HS}}(\hat{\rho}_{\text{Haar}}^{(k)}, \hat{\rho}_T^{(k)}) \geq \frac{4B^2}{D}, \quad (5.9)$$

for time-independent Hamiltonian dynamics.

Another quantity to probe CHSE is the *ensemble entropy*, recently introduced in Ref. [188]. The key idea is to directly compare the states sampled by time evolution to states sampled from the Haar distribution. An ensemble is a collection of pure quantum states with associated probabilities:

$$\mathcal{E} = \{p_j, |\Psi_j\rangle\}, \quad (5.10)$$

where  $j$  can be a discrete value (corresponding to, for example, different measurement outcomes) or a continuous one (such as time). In the case of discretised probability distributions we will use index  $j$  to refer to the different states, but in case of continuous variables we will use  $\Psi$  itself as the variable and write the ensemble as:  $\mathcal{E} = \{P(\Psi), |\Psi\rangle\}$

We will denote the Haar ensemble as  $\mathcal{E}_{\text{Haar}} = \{P^H(\Psi), |\Psi\rangle\}$ , which will be the reference ensemble to compute the ensemble entropy. In the case of a discretised version of the Haar ensemble, we will use the notation  $\mathcal{E}_{\text{Haar}} = \{p_j^H, |\Psi_j\rangle\}$ . The ensemble generated by some aperiodic time evolution, also called the temporal ensemble, will be denoted as  $\mathcal{E}_T = \{P^T(\Psi), |\Psi\rangle\}$  for the continuous version, and as  $\mathcal{E}_T = \{p_j^T, |\Psi_j^T\rangle\}$  for discretised one. The ensemble entropy for this generated temporal ensemble, is then defined as the negative of the Kullback-Leibler divergence:

$$\text{Ent}(\mathcal{E}_T | \mathcal{E}_{\text{Haar}}) = - \int d\Psi P^T(\Psi) \log_2 \left( \frac{P^T(\Psi)}{P^H(\Psi)} \right). \quad (5.11)$$

This quantity directly compares the two ensembles, and is exactly 0 if and only if they are identical. If that holds, it also immediately follows that the Hilbert-Schmidt distance between the moments constructed from these two ensembles,  $\hat{\rho}_{\text{Haar}}^{(k)}$  and  $\hat{\rho}_T^{(k)}$ , will also vanish for any  $k$ .

In practice, when numerically evaluating this measure we are using a finite set of states generated by the time evolution,  $\{|\psi(t)\rangle\}_{t=0}^{T-1}$ , which demands a discretisation of Eq. (5.11), replacing the integral with a discrete summation over a finite set of states and associated probabilities, which we call discretised ensemble entropy (DEE):

$$\text{DEE}(\mathcal{E}_T|\mathcal{E}_{\text{Haar}}) = - \sum_{j=1}^M p_j^T \log_2 \left( \frac{p_j^T}{p_j^H} \right), \quad (5.12)$$

which is effectively an estimator for  $\text{Ent}(\mathcal{E}_T|\mathcal{E}_{\text{Haar}})$ . Further details about how we obtained the discretisation of the Haar ensemble and the numerical calculation can be found in Appendix P.

Finally, before moving on to the first numerical demonstration, we need to present how to actually achieve CHSE in some quantum systems. Obviously it cannot be observed in systems with time-independent Hamiltonians, nor in periodically-driven systems, due to the existence of eigenstates. On the other hand, if we are driving the system at discrete time steps with  $\hat{U}_t$ , and we draw each of these unitaries from the Haar ensemble, the system will, trivially, exhibit CHSE. A very surprising result is that this behaviour can be achieved with  $\hat{U}_t$  sampled solely from a set of two unitaries,  $\{\hat{U}_A, \hat{U}_B\}$ , which are universal [21]. It is interesting to note that almost any pair of randomly sampled unitaries,  $\hat{U}_A, \hat{U}_B \in U(D)$ , will be universal, in the sense that they can be used to construct any other  $\hat{U}_C \in U(D)$  [189]. Then CHSE can be achieved by applying the chosen unitaries according to a “coin-flip” sequence: at each time  $t$  we apply either  $\hat{U}_A$  or  $\hat{U}_B$ , each with probability  $p = 0.5$ . Alternatively, they can be applied in a sequence that is deterministic but also aperiodic, for example through the Fibonacci word  $W_j$ , which is defined recursively [190]. We set two initial words and then construct longer words using a simple stacking rule:

$$W_0 = 1, \quad W_1 = 0; \quad W_{j+1} = W_j W_{j-1}, \quad (5.13)$$

where, in the last equation,  $W_j W_{j-1}$  should be interpreted as appending the two symbols. In theory, this rule can be repeated infinitely many times to give  $W_\infty$ . Using  $W_\infty$  we then implement the discrete-time evolution as a sequence of unitaries  $\hat{U}^{(A)}$  or

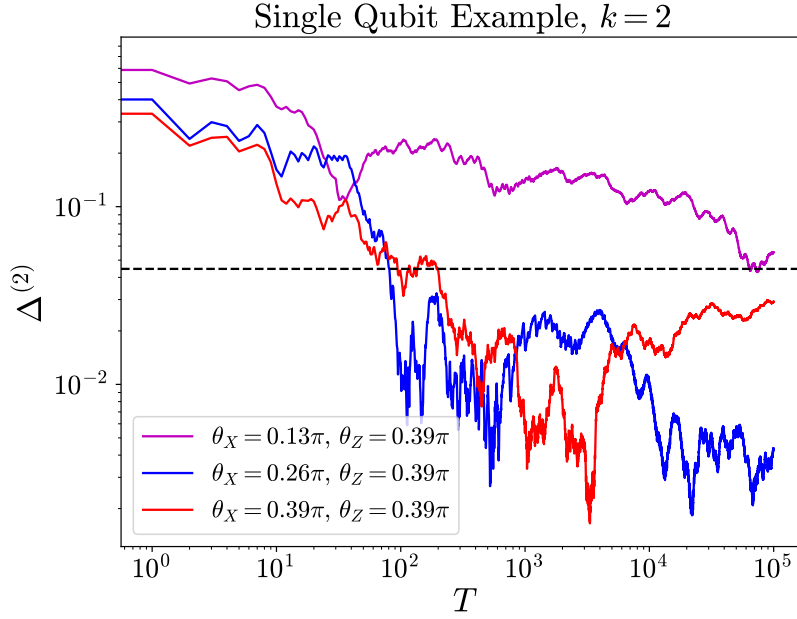


Figure 5.1: Trace distance between the second moments of the Haar ensemble and the temporal ensemble for a single qubit, with the initial state set to  $|\psi(0)\rangle = |0\rangle$ , reproduced based on [21]. We have used three pairs of generators:  $\{(\theta_X = 0.13\pi, \theta_Z = 0.39\pi), (\theta_X = 0.26\pi, \theta_Z = 0.39\pi), (\theta_X = 0.39\pi, \theta_Z = 0.39\pi)\}$ , with the two unitaries  $\hat{U}_X = \exp[-i\theta_X \hat{\sigma}_X]$ ,  $\hat{U}_Z = \exp[-i\theta_Z \hat{\sigma}_Z]$  applied aperiodically, according to the Fibonacci word as described in the main text. The dashed line represents the lower bound for time independent Hamiltonian dynamics. We observe convergence to the Haar ensemble for order  $k = 2$  for each of generator pairs. Adapted from Ref. [21], © (2024) American Physical Society

$\hat{\mathcal{U}}^{(B)}$ . At a given time  $t$  we apply  $\hat{\mathcal{U}}^{(A)}$  if the  $t$ -th symbol in  $W_\infty$  is 0, and  $\hat{\mathcal{U}}^{(B)}$  if it is 1. In practice, computing all symbols of  $W_\infty$  is infeasible, therefore we compute them only up to some desired evolution time  $T$ . It is easy to see that the length of the  $M$ -th word is exactly the  $M$ -th Fibonacci number  $F_M$ :  $|W_M| = F_M$ . For large values of  $M$  the length of  $W_M$  can therefore be approximated by the well known equation [191]:  $|W_M| = F_M \approx \frac{\varphi^M}{\sqrt{5}}$ ,  $\varphi = \frac{1+\sqrt{5}}{2}$ . Hence, in order to compute the dynamics up to time  $T$ , we need to generate the Fibonacci word up to order  $M \approx \log(T) + \frac{1}{2} \log(5) - \log(\varphi)$ .

## 5.2 Warm-up example: Single Qubit

Since the CHSE definition of ergodicity is not tied to local observables, as in the case of ETH, but rather depends on the exploration of the Hilbert space, it can already be observed in the dynamics of a single qubit. Constructing two unitaries:  $\hat{U}_A = \hat{R}_x(\theta_x) = \exp[-i\theta_x \hat{\sigma}_x]$  and  $\hat{U}_B = \hat{R}_z(\theta_z) = \exp[-i\theta_z \hat{\sigma}_z]$ , and layering them

according to the Fibonacci word, we can observe signatures of CHSE for a single qubit, presented in Fig. 5.1. Note that we have used three different pairs of the rotation angles  $\{\theta_x, \theta_z\}$ , and we compute the distance of the  $k = 2$  moment with respect to the Haar ensemble. We numerically observe convergence for all three pairs of generators, showing that aperiodic driving can give rise to behaviour that cannot be observed in the case of Hamiltonian or Floquet dynamics.

We note that while there are fluctuations in the distances with respect to the Haar measure for finite times, as evident in Fig. 5.1, these are expected to decay to zero in the infinite time limit, with detailed analytic arguments presented in Ref. [21]. The main properties required to guarantee the decay to zero are the computational completeness of the used drives,  $\{\hat{U}_A, \hat{U}_B\}$  [189] and the aperiodicity of the drive. The computational completeness guarantees that in principle any unitary operation  $\hat{U}_C$  can be constructed by using the set  $\{\hat{U}_A, \hat{U}_B\}$ . In addition, the aperiodicity of the drive ensures that, in fact, any unitary operation (apart from a zero measure set) will be achieved during the time evolution. All of these operations are also equally likely to appear during the dynamics. Therefore, in the long time limit, the distance between different moments of the Haar ensemble and temporal ensemble are expected to decay to zero.

### 5.3 CHSE in Quantum Circuit Models

Having observed CHSE for a single qubit, we now move on to a many-body setting, in order to investigate the potential effects of the presence of scars and fragmentation on CHSE. Let us consider a chain of  $N$  qudits, labelled by  $i \in \{0, 1, 2, \dots, N-1\}$ , each with local Hilbert space dimension  $d$ . The chain of qudits evolves dynamically by a “brickwork” quantum circuit, made up of layers of 2-local nearest-neighbour unitaries. We choose our brickwork circuit so that each timestep is completely specified by the two 2-local operators  $\hat{U}_e = \exp(i\hat{H}_e)$  and  $\hat{U}_o = \exp(i\hat{H}_o)$ , where  $\hat{U}_e$  makes up the even layer  $\hat{\mathbb{U}}_e = \bigotimes^{N/2} \hat{U}_e$ , and  $\hat{U}_o$  makes up the odd one  $\hat{\mathbb{U}}_o = \hat{\mathbb{I}}[\bigotimes^{N/2-1} \hat{U}_o] \otimes \hat{\mathbb{I}}$ . The system

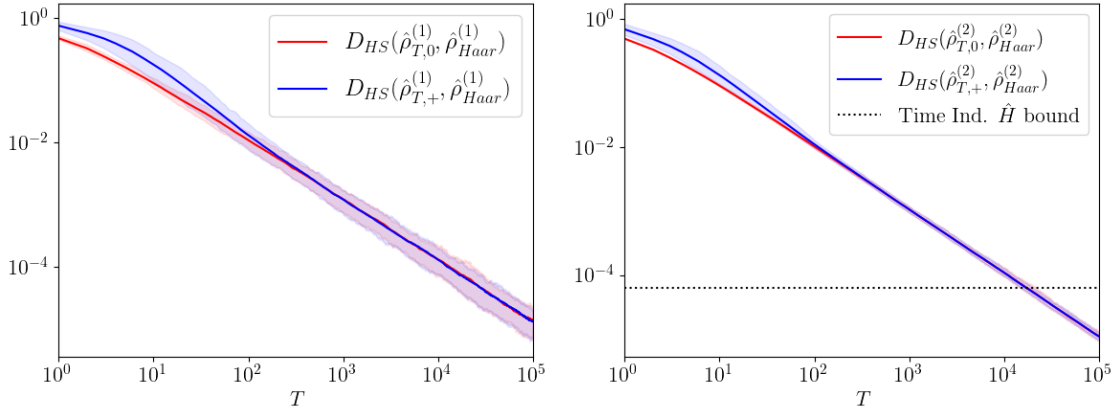


Figure 5.2: Hilbert-Schmidt distance between the Haar moments and the temporal ensemble moments for  $k = 1$  (LEFT) and  $k = 2$  (RIGHT). The results are averaged over 100 different instances of  $\{\hat{U}^{(A)}, \hat{U}^{(B)}\}$ , with the lines corresponding to the average distances, and the shaded areas represent the 10th-90th percentile range. The red plot corresponds to the initial state  $|\psi(0)\rangle = |0\rangle^{\otimes N}$ , and the blue one to  $|\psi(0)\rangle = |+\rangle^{\otimes N}$ . The dotted line in the lower plot for  $k = 2$  represents the lower bound for time independent Hamiltonians, and is surpassed for both initial states.

is then propagated forward one step in time by:

$$\hat{\mathcal{U}} = \hat{\mathcal{U}}_o \hat{\mathcal{U}}_e. \quad (5.14)$$

As already noted for the single qubit example, to generate CHSE, our dynamics cannot be periodic in time. Therefore we introduce two different unitary propagators:  $\hat{\mathcal{U}}^{(A)}$  [specified by the local unitary operators  $\hat{U}_e^{(A)}, \hat{U}_o^{(A)}$ ] and  $\hat{\mathcal{U}}^{(B)}$  [specified by the local operators  $\hat{U}_e^{(B)}, \hat{U}_o^{(B)}$ ]. We then apply these two different unitaries in the aperiodic sequence determined by the Fibonacci word  $W_M$ , as defined in Eq.(5.13). A schematic for the used circuit is presented in Fig. 5.2

We then implement the discrete-time evolution as a sequence of unitaries  $\hat{\mathcal{U}}^{(A)}$  or  $\hat{\mathcal{U}}^{(B)}$ : at a given time  $t$  we apply  $\hat{\mathcal{U}}^{(A)}$  if the  $t$ -th symbol in  $W_\infty$  is 0, and  $\hat{\mathcal{U}}^{(B)}$  if it is 1. The resulting unitary then has the form:

$$\hat{\mathcal{U}}(t) = \dots \hat{\mathcal{U}}^{(B)} \hat{\mathcal{U}}^{(A)} \hat{\mathcal{U}}^{(B)} \hat{\mathcal{U}}^{(A)} \hat{\mathcal{U}}^{(A)} \hat{\mathcal{U}}^{(B)} \hat{\mathcal{U}}^{(A)}. \quad (5.15)$$

This aperiodic sequence of gates will be used throughout this chapter. For the system to exhibit scars, fragmentation, or contain some symmetry, we impose specific constraints on the local bricks  $\hat{U}_e^{(A)}, \hat{U}_o^{(A)}, \hat{U}_e^{(B)}$ , and  $\hat{U}_o^{(B)}$ .

We first numerically probe generic 2-local unitaries, using  $N = 4$  qubits ( $d = 2$ ) with open boundary conditions. The dynamics are defined by the four local unitaries  $\{\hat{U}_e^{(A)}, \hat{U}_o^{(A)}, \hat{U}_e^{(B)}, \hat{U}_o^{(B)}\}$ . Each of these is created using a random Hermitian generator  $\hat{H}_{o/e}^{(A/B)}$ , where the real and imaginary parts of the matrix elements are sampled uniformly from the interval  $[0, 1)$ , and then constructing the unitary as  $\hat{U}_{o/e}^{(A/B)} = \exp(-i\hat{H}_{o/e}^{(A/B)})$ . We sample over 100 different instances of the circuit, using different sets of local unitaries for each of them.

We then compute the Hilbert-Schmidt distances to the Haar moments for  $k = 1, 2$ , using two different initial states:  $|\psi(0)\rangle = |0\rangle^{\otimes N}$  and  $|\psi(0)\rangle = |+\rangle^{\otimes N}$ . At each time-step we compute the first and second moments of the temporal ensemble,  $\hat{\rho}_{T,0/+}^{1/2}$ , where the second subscript denotes the corresponding initial state.

The results are presented in Fig. 5.2. As a first check, for  $k = 1$  we observe that the ensembles generated from both the  $|\psi(0)\rangle = |0\rangle^{\otimes N}$  and the  $|\psi(0)\rangle = |+\rangle^{\otimes N}$  initial states converge to the first moment of the Haar ensemble. These results for  $k = 1$  are not particularly surprising, since any Floquet driving without energy conservation will “heat up” any initial state to the infinite temperature state  $\hat{\rho} = \frac{1}{D}\hat{\mathbb{I}}$ , which is equal to the first Haar moment:  $\hat{\rho}_{\text{Haar}}^{(1)} = \int d\phi |\phi\rangle \langle\phi| = \frac{1}{D}\hat{\mathbb{I}}$ . To numerically investigate CHSE we need to probe  $\hat{\rho}_T^{(k)}$  and  $\hat{\rho}_{\text{Haar}}^{(k)}$  for  $k > 1$ .

Comparing the  $k = 2$  moments of the temporal and Haar ensembles in Fig. 5.2, we again see strong evidence for CHSE. Both initial states give very similar long-time behaviour, overlapping almost perfectly for times  $T > 10^2$ . Interestingly, they display even smaller variation between the different instances of the model than for the  $k = 1$  moment, with all of the results in the 10th-90th percentile range strongly concentrated around the mean values. It is clear that the distances to the Haar moment go below the bound for time-independent Hamiltonians in Eq. (5.9). We also computed the discretised ensemble entropy for this model, presented in section 5.8, where we compare it to models with conserved quantities.

We can also see that this should hold using a Trotterization based argument. Suppose we have a quantum system with  $L$  sites, with a  $d$ -dimensional qudit at each site, resulting in a  $D$ -dimensional Hilbert space  $\mathcal{H}$ , with  $D = d^L$ . Now we can apply one

of two  $k$ -local Hamiltonians at a time, either  $\hat{H}_A$  or  $\hat{H}_B$ , which we can write as:

$$\hat{H}_X = \sum_{j=0}^L \hat{h}_{X,j}, \quad (5.16)$$

where  $X = A, B$ , and  $\hat{h}_{X,j}$  are local operators centered at site  $j$ . We also assume that we can apply them for some predetermined, arbitrary time  $\tau_{A/B}$ , which will result in the unitary  $\hat{U}_X = e^{-i\tau_X \hat{H}_X}$ . The set  $\{\hat{U}_A, \hat{U}_B\}$  will be universal (i.e. we can construct any unitary  $\hat{U}_C \in U(D)$  up to arbitrary precision  $\epsilon$  by applying the two universal gates in some sequence,  $\|\prod_{i=\{A,B\}} \hat{U}_i - \hat{U}_C\| \leq \epsilon$ , where  $\|\dots\|$  denotes the trace norm), provided that [189]:

1. The Hamiltonians  $\hat{H}_A$  and  $\hat{H}_B$  do not share any eigenstates and/or symmetries.
2. Eigenphases of  $\hat{U}_X$  are irrational multiples of  $\pi$ .

We assume that the first condition is satisfied, as generic Hamiltonians (with arbitrary terms  $\hat{h}_{X,j}$ ) will have different eigenstates and will not contain any symmetries. The second condition can be guaranteed, by using a suitable choice of the evolution times  $\tau_X$ .

For each of the two Hamiltonians, we can label the eigenenergies as  $E_j, j = 1, 2, \dots, D$ . Then, the eigenphases of the corresponding unitary will be  $\phi_j = \tau E_j$ . Now, the eigenenergies could be either rational or irrational numbers, but in either case there will exist a time  $\tau$  which will make all of the  $\phi_j$  irrational multiples of  $\pi$ . Hence, we can make the unitaries  $\hat{U}_A, \hat{U}_B$  universal by a suitably choosing the evolution times  $\tau_A$  and  $\tau_B$ .

Now, building upon the previous argument, we can construct a similar argument for the universality of unitaries corresponding to one odd and one even layer in a brickwork layer, as described in Eq. (5.14). Suppose we have a 2-local Hamiltonian,  $\hat{H}$ , acting on an open chain of  $L$  qudits. All of the terms in the Hamiltonian act only on nearest neighbors, so we can decompose it as  $\hat{H} = \sum_{i=0,2,4,\dots}^{L-2} \hat{g}_{i,i+1}^e + \sum_{i=1,3,\dots}^{L-3} \hat{g}_{i,i+1}^o$ . Now, we let the system evolve for some time  $\tau = \eta\delta$ , where  $\eta$  is an arbitrary number of order  $\mathcal{O}(1)$  and  $\delta$  is a parameter which can be arbitrarily small, and we will set it to be of the form  $10^{-a}, a \in \mathbb{Z}^+$ . This will result in the unitary  $\hat{U} = e^{-i\tau\hat{H}} = e^{-i\eta\delta\hat{H}}$ . By

the argument in the paragraph above we can always set  $\eta$  such that the eigenphases of  $\hat{U}$  will be irrational multiples of  $\pi$ .

Writing the terms in the exponent as:  $\eta\delta\hat{g}_{i,i+1}^{o/e} = \tau\hat{h}_{i,i+1}^{o/e}$ , we can identify the  $\hat{\mathbb{U}}_t$  in Eq. (5.14) as a trotterization of some 2-local Hamiltonian, with a single timestep. Hence, the two unitaries are identical up to Trotter error:  $\hat{U} = \hat{\mathbb{U}} + \mathcal{O}(\eta^2\delta^2)$ . Therefore we can make the error arbitrarily small, by tuning the  $\delta$  parameter.

## 5.4 Introducing Hilbert Subspace Ergodicity

CHSE was originally introduced for generic aperiodic drives without any conserved quantities. However, in the presence of (local or nonlocal) conserved quantities the Hilbert space may decouple into several Krylov subspaces  $\mathcal{K}_\alpha$ . Because of this decoupling, it will be impossible to exhibit CHSE in the full Hilbert space. However, the system may still “explore” the subspaces with equal probability, which we term complete Hilbert subspace ergodicity (CHSSE).

Consider some finite,  $D$ -dimensional Hilbert space. The dynamics of this system is governed either by some time dependent Hamiltonian  $\hat{H}(t)$ , with  $t \in \mathbb{R}$ , or a sequence of unitary operations  $\hat{\mathbb{U}}_t$ , with  $t \in \mathbb{N}$ . The Hamiltonian or sequence of unitaries splits into  $\mathcal{N}_\mathcal{K}$  dynamically disconnected Krylov subspaces  $\mathcal{K}_\alpha$ , but is otherwise completely generic. A system will display CHSSE provided that, starting in any initial state in some Krylov subspace,  $|\psi(t=0)\rangle \in \mathcal{K}_\alpha$ , the  $k$ -th moment of the temporal ensemble,  $\hat{\rho}_T^{(k)}$ , will converge to the  $k$ -th moment of the Haar ensemble in the  $\mathcal{K}_\alpha$  subspace, defined as:

$$\begin{aligned} \hat{\rho}_{\mathcal{K}_\alpha\text{-Haar}}^{(k)} &= \int_{|\phi\rangle \in \mathcal{K}_\alpha} d\phi (|\phi\rangle\langle\phi|)^{\otimes k} \\ &= \frac{\sum_{\pi \in \mathcal{S}_k} \sum_{\{\beta_i\}} |\beta_1, \dots, \beta_k\rangle\langle\beta_{\pi(1)}, \dots, \beta_{\pi(k)}|}{D_\alpha(D_\alpha + 1) \dots (D_\alpha + k - 1)}, \end{aligned} \quad (5.17)$$

where  $\pi$  are permutation operators belonging to the permutation group  $\mathcal{S}_k$ ,  $D_\alpha = \dim(\mathcal{K}_\alpha)$  and  $\beta_i$  label different basis states in the Krylov subspace. Note that in this case  $d\phi$  is the unique unitary-invariant measure in the  $\mathcal{K}_\alpha$  subspace. For time-

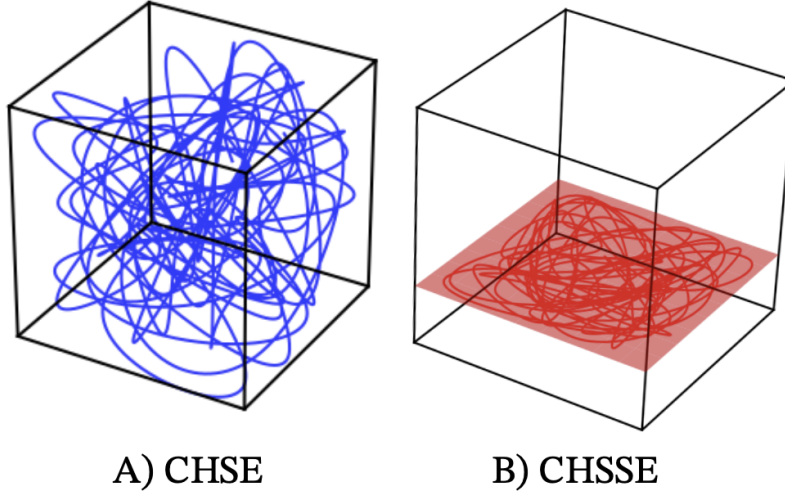


Figure 5.3: Schematic representation for complete Hilbert space ergodicity (A) vs. complete Hilbert space ergodicity in a subspace (B). In the former, the system explores all of the available  $D$ -dimensional Hilbert space, represented by a three-dimensional box. In the latter however, the system can only explore a  $B$ -dimensional subspace of the full Hilbert space, represented by a two-dimensional horizontal slice. In the long-time limit, the system represented in panel (A) will explore all regions within the box with equal probability, whereas the system in panel (B) will explore just the shaded surface.

independent Hamiltonian dynamics, due to the constraint of energy conservation (and the commutant algebra set), we again have a lower bound on the trace distance (and consequently also Hilbert-Schmidt distance) between the  $k$ 'th moments of the temporal ensemble and the Haar ensemble in the decoupled subspaces. By the same argument as in Ref. [21], the bound on the trace distance between the  $k \geq 2$  moments of the temporal ensemble generated by some initial state  $|\psi_\alpha\rangle \in \mathcal{K}_\alpha$  and  $\hat{\rho}_{\mathcal{K}_\alpha - \text{Haar}}^{(k)}$  will be  $B(D_\alpha)$ .

## 5.5 Presence of a single QMBS state

To investigate the effect of ETH-violation on a model which is otherwise CHSE, we first embed QMBS in the brickwork circuit model described in the previous section. To do this we employ the projector embedding method again [87], modify the two-qudit gates,  $\hat{U}_{e/o}^{A/B}$ , so that they instead are of the form:

$$\hat{U}_{e/o}^{(A/B)} = \exp\{i\hat{P}\hat{H}_{e/o}^{(A/B)}\hat{P}\}, \quad (5.18)$$

where  $\hat{P}$  is a chosen two-qudit projectors.

We label the corresponding projector on the full Hilbert space as  $\hat{\mathbb{P}}_{n,n+1} \equiv \hat{\mathbb{I}}_{0,n-1} \otimes \hat{P} \otimes \hat{\mathbb{I}}_{n+2,N-1}$ , where  $\hat{\mathbb{I}}_{i,j}$  is the identity acting on all qudits in the range  $i, i+1, \dots, j-1, j$ . Recall that the set of target states is:

$$\mathcal{T} = \{|\psi_{S_i}\rangle : \hat{\mathbb{P}}_{n,n+1} |\psi_{S_i}\rangle = 0, \forall n\}. \quad (5.19)$$

This protocol of embedding QMBS in a circuit model effectively creates dynamically disconnected subspaces. In case the model supports  $L$  scar states, the full Hilbert space,  $\mathcal{H}_F$ , will consist of  $L+1$  disconnected subspaces, where  $L$  of those will be scar subspaces containing a single state,  $\mathcal{H}_{S_i} = \{|\psi_{S_i}\rangle \langle \psi_{S_i}|\}$ , such that:

$$\mathcal{H}_F = \mathcal{H}_N \oplus \bigoplus_{i=1}^L \mathcal{H}_{S_i} = \mathcal{H}_N \oplus \mathcal{H}_S, \quad (5.20)$$

where we have collected all the scar states into  $\mathcal{H}_S$  for brevity, and the  $\mathcal{H}_N$  contains all of the nonscar states. Note that in this specific circuit construction, all the QMBS are not interacting between themselves, but one could easily construct non-product QMBS by using another layer of unitaries acting within the  $\mathcal{H}_S$  subspace.

This embedding process will not only ensure that the scar states in  $\mathcal{H}_S$  do not thermalize (in any framework, neither CHSE nor ETH), but will impact the evolution of states in  $\mathcal{H}_N$  as well. Since the two subspaces are disconnected by construction, the  $k$ -th moment of the temporal ensemble,  $\hat{\rho}_T^{(k)}$ , generated by the dynamics starting from any initial state  $|\phi_0\rangle \in \mathcal{H}_N$  will not be converging to  $\hat{\rho}_{\text{Haar}}^{(k)}$  in the full Hilbert space. However, since the used unitaries  $\hat{\mathbb{U}}^{(\mathbb{A}/\mathbb{B})}$  are quite generic, we expect it to converge to the Haar averaged state in the nonscar subspace, exhibiting CHSSE within it, as described in the previous section.

To demonstrate the effect of embedding scars into the system, we again consider a simple setup of  $N = 4$  qubits. For the construction of local unitaries, we again generate four different Hermitian operators for each of the 100 different instances, in the same manner as in Section 5.3. However, the local unitaries are modified, as described in Eq. (5.18). We again use:  $\hat{P} = \hat{\mathbb{I}} - |00\rangle \langle 00|$ , which ensures that any of

the 2-local unitaries leaves the  $|00\rangle$  state invariant. The use of this projector leads to a single scar,  $|S_1\rangle = |0\rangle^{\otimes N}$ , which is invariant under the action of both  $\hat{U}^{(A)}$  and  $\hat{U}^{(B)}$ , whereas any other orthogonal state will undergo highly nontrivial dynamics.

To verify this, we perform the time evolution and obtain the temporal ensemble with two different initial states, the scar state  $|S_1\rangle = |0\rangle^{\otimes N}$  and a state in the nonscar subspace  $|\psi_{\text{NS}}\rangle = |1\rangle^{\otimes N}$ . For the former we compute the distance for the  $k = 1, 2$  moments with respect to the Haar moments for the full Hilbert space. For the nonscar initial state we compute the same metrics, but we also compute the distance with respect to the the first and second moments of the Haar ensemble in the nonscar subspace, denoted as  $\hat{\rho}_{\text{NS-Haar}}^{(k)}$ . The results are displayed in Fig. 5.4.

We can clearly see the difference in the dynamics for an initial scar state and nonscar state. Since the  $|0\rangle^{\otimes N}$  scar is not affected by any of the unitaries in the sequence, the two moments for the temporal ensemble ( $\hat{\rho}_S^{(1)} = |0\rangle^{\otimes N} \langle 0|^{\otimes N}$ ,  $\hat{\rho}_S^{(2)} = [|0\rangle^{\otimes N} \langle 0|^{\otimes N}]^{\otimes 2}$ ), as well as the distances  $D_{\text{HS}}(\hat{\rho}_T^{(k)}, \hat{\rho}_{\text{Haar}}^{(k)})$ , with  $k = 1, 2$ , will be constant, as demonstrated.

For the nonscar state  $|1\rangle^{\otimes N}$  we see that the generated first and second moments of the temporal ensemble,  $\hat{\rho}_T^{k=1,2}$ , are clearly getting closer to the respective moments of the Haar ensemble. However, due to the separation of the scar subspace, this initial state will be constrained to exploring only the nonscar subspace. This becomes evident when comparing it to the Haar ensembles for the two moments,  $\hat{\rho}_{\text{Haar}}^{(k=1,2)}$ , as well as the to the nonscar subspace constrained Haar ensemble moments  $\hat{\rho}_{\text{NS-Haar}}^{(k=1,2)}$ . Note that the former is constructed by using Eq. (5.4), and for the latter we use Eq. (5.17) with  $\mathcal{K}_\alpha = \mathcal{H} \setminus \mathcal{H}_S$ , where  $\mathcal{H}_S = \text{span}(\{|0\rangle^{\otimes N}\})$ .

When computing the distance between the generated temporal ensemble moments,  $\hat{\rho}_T^{(k)}$ , and the full space Haar moments,  $\hat{\rho}_H^{(k)}$ , we observe that these are bounded by  $D_{\text{HS}}(\hat{\rho}_{\text{Haar}}^{(k)}, \hat{\rho}_{\text{NS-Haar}}^{(k)})$ , with  $k = 1, 2$ . These bounds can be computed exactly (see Appendix N):

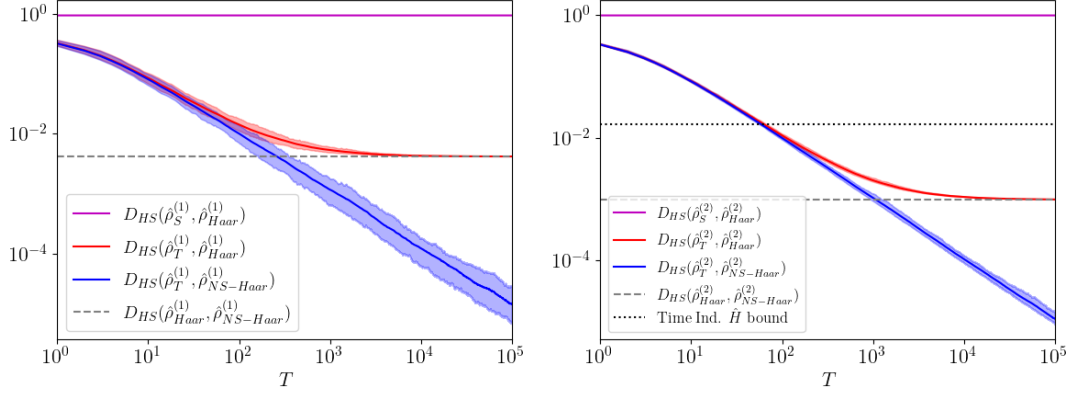


Figure 5.4: Effect of embedding a single QMBS into a brickwork model. The results are obtained by using 100 different instances of  $\{\hat{U}^{(A)}, \hat{U}^{(B)}\}$ . Left: Distance between the first moments of the temporal ensembles and the Haar ensemble. Right: Distance between the second moments of the temporal ensembles and the Haar ensemble. The violet line represents the results for the initial scar state  $|S_1\rangle = |0\rangle^{\otimes N}$ . Since it is invariant under the applied unitaries, the generated moments  $(\hat{\rho}_S^{(k=1,2)})$  are constant, and so is their distance to the respective Haar moments. The red line represents the average distance between the moments of the temporal ensemble  $(\hat{\rho}_T^{(k=1,2)})$  and Haar ensemble  $(\hat{\rho}_{Haar}^{(k=1,2)})$  for the  $|1\rangle^{\otimes N}$  initial state, and the shaded region represents the 10th-90th percentile region. We observe that for both moments, this distance is bounded from below, which is expected due to the separation of the scar and nonscar subspaces. To demonstrate this, we construct the first and second moments of the Haar ensemble constrained to the nonscar subspace,  $(\hat{\rho}_{NS-Haar}^{(k=1,2)})$ , by using Eq. (5.17) with subspace  $\mathcal{K}_{\text{nonscar}} = \mathcal{H} \setminus \mathcal{H}_S$ . The dashed lines represent the distances between the full Hilbert space  $(\hat{\rho}_{Haar}^{(k)})$  and nonscar subspace ensembles  $(\hat{\rho}_{NS-Haar}^{(k)})$  for the first and second moments ( $k = 1, 2$ ), and indicate the lower bounds achievable for a nonscar initial state. Note that these were computed using Eqs. (5.21) and (5.22), with the dimensions set to  $D = 16$ ,  $D_\alpha = 15$ . This is further evidenced by the distance between  $\hat{\rho}_T^{(k)}$  and  $\hat{\rho}_{NS-Haar}^{(k)}$  for first and second moments ( $k = 1, 2$ ), shown in the blue. The line is the average over 100 iterations and the shaded areas represent the 10th-90th percentile. This distance becomes arbitrarily small in the long-time limit for both  $k = 1$  and  $k = 2$ , indicating that the models display CHSSE in the nonscar subspace.

$$D_{HS}(\hat{\rho}_{\text{Haar}}^{(1)}, \hat{\rho}_{\mathcal{K}_\alpha - \text{Haar}}^{(1)}) = \frac{D - D_\alpha}{D_\alpha D}, \quad (5.21)$$

$$D_{HS}(\hat{\rho}_{\text{Haar}}^{(2)}, \hat{\rho}_{\mathcal{K}_\alpha - \text{Haar}}^{(2)}) = \frac{2(D^2 + D - D_\alpha^2 - D_\alpha)}{D_\alpha(D_\alpha + 1)D(D + 1)}, \quad (5.22)$$

where  $D = \dim(\mathcal{H})$ ,  $D_\alpha = \dim(\mathcal{K}_\alpha)$ . For this particular model, the  $k = 1$  bound is  $\frac{1}{D(D-1)}$ , and for  $k = 2$  it is  $\frac{4}{(D+1)D(D-1)}$ . Since  $D = 2^N$ , both of these bounds are exponentially small in system size  $N$  (further details are in Appendix N). When comparing the generated temporal ensembles to the subspace Haar ensembles directly, via  $D_{HS}(\hat{\rho}_T^{(k)}, \hat{\rho}_{\text{NS-Haar}}^{(k)})$ , we see that they become arbitrarily small, and effectively reproduce the behavior observed for a CHSE model, as shown in Fig. 5.2. We also compute the corresponding DEE for these results, which are presented and discussed in section 5.8.

The presence of CHSSE in the nonscar subspace also directly implies that the constructed states are forming  $t$ -designs within that subspace [21, 192]. However, as demonstrated by the numerical results, the convergence rates to form  $t$ -designs are rather slow. In this section we have focused on a single QMBS state, but this can be easily generalised to other cases, presented in Appendix O.

## 5.6 Effect of Hilbert Space Fragmentation

Another known mechanism of ETH-violation is HSF. Here we modify the model presented in Section 5.3 such as to observe *strong* Hilbert space fragmentation. In the distinction between weak and strong violation of ETH discussed in the introduction, this mechanism corresponds to strong ETH-violation, due to exponentially many vanishingly small subspaces.

A known model that displays strong Hilbert space fragmentation for local Hilbert space dimension  $d \geq 3$  is the so-called pair-flip model [95], which was originally studied as a Hamiltonian model and presented in Subsection 2.2.3, but can also be implemented as a gate model. For the latter, the 2-local unitaries have the form:

$$\hat{U}^{\text{PF}} = \sum_{a,b} U_{ab}^{\text{PF}} |aa\rangle \langle bb| + \sum_{a \neq b} e^{i\phi_{ab}} |ab\rangle \langle ab|, \quad (5.23)$$

where  $U_{ab}^{\text{PF}}$  are elements of a Haar random unitary drawn from  $U(d)$  and  $\phi_{ab}$  are random phases drawn uniformly from  $[0, 2\pi)$ . Effectively, this model has the constraint that two neighbouring sites can change their state if and only if they are in the same state, as seen in the first term. The second term implements an arbitrary phase on pairs of states where two sites differ, to break any undesirable symmetries.

The condition that states are allowed to flip only if one of their neighbours is in the same state imposes a strong constraint on what parts of the Hilbert space are available to any initial state. Suppose we have a chain of  $N$  qudits with local Hilbert space dimension  $d$ . If the chain is initialised in a product state where none of the neighbouring sites are in the same state, it cannot transition to any other product state, but can only acquire a time dependent phase, and the dynamics becomes trivial. The number of such states is  $d(d-1)^{N-1}$ , i.e., there are  $d(d-1)^{N-1}$  subspaces of dimension 1. Counting the total number of subspaces and their respective dimensions is highly nontrivial and we refer the reader to Refs. [96, 7, 95]. It can be shown that the number of disconnected subspaces in the pair-flip model with  $d > 2$  is:

$$\mathcal{N}_{\mathcal{K}}(N, d) = \frac{(d-1)^{N+1} - 1}{d-2}, \quad (5.24)$$

which is exponential in system size  $N$ , but still exponentially smaller than the total Hilbert space dimension  $D = d^N$ . Importantly, it can also be shown that the size of the largest subspace for even number of sites  $N$  is:

$$|\mathcal{K}_0(N, d)| = d^N \left(1 + \frac{1}{2} \sum_{j=1}^{N/2} d^{-2j} \binom{1/2}{j} (-1)^j (2\sqrt{d-1})^{2j}\right), \quad (5.25)$$

which is an exponentially vanishing fraction of the total Hilbert space, i.e.

$\lim_{N \rightarrow \infty} (|\mathcal{K}_0(N, d)|/d^N) = 0$ . Since the ratio of the dimension of the largest Krylov subsector versus the dimension of the full Hilbert space tends to 0, this is a form of

*strong* Hilbert space fragmentation.

As seen in the discussion about a scarred system in Sec. 5.5, the scar subspace remained stationary, while a state in the nonscar subspace was displaying CHSSE, exploring all of the available Hilbert space. This gives us good intuition on the consequences of fragmentation. In the case of fragmented models, set up in a way analogous to universal models with CHSE, we expect to see CHSSE for each of the initial states in their Krylov subspaces. To verify this we perform numerics analogous to the ones performed for models in Sections 5.3 and 5.5.

We consider a system of  $N = 4$  qutrits,  $d = 3$ . For these parameters, due to the constraints of the pair-flip model [96], we expect 24 frozen states, six distinct 7-dimensional subspaces, and one 15-dimensional subspace. The system undergoes dynamics defined by applying two global unitaries,  $\hat{U}^{(A)}$  and  $\hat{U}^{(B)}$ . These are defined as a combination of an even and odd layers of 2-local unitaries as defined in Eq. (5.14). All of the individual local unitaries have the form as displayed in Eq. (5.23), but with different values for the free parameters  $\{U_{ab}^{\text{PF}}, \phi_{ab}\}$  for each of  $\{\hat{U}_e^{(A)}, \hat{U}_o^{(A)}, \hat{U}_e^{(B)}, \hat{U}_o^{(B)}\}$  in the circuit.

We first construct the temporal ensembles for all 81 distinct initial computational basis states, and compare the first and second moments to the Haar ensemble, as shown in Fig. 5.5. Analogously to the QMBS state, the frozen states lead to a temporal ensemble for which all moments stay at a distance of  $\mathcal{O}(1)$  with respect to the Haar ensemble. However, the rest of the computational basis states exhibit CHSSE in their respective subspaces, as demonstrated by their convergence to the bounds.

Note that for the largest subspace in this HSF model, the  $k = 1$  bound reduces to  $\frac{2}{D_\alpha}$ , and the  $k = 2$  bound to  $\frac{2}{D_\alpha(D_\alpha+1)}$ , in the large  $N$  limit (see Appendix N). Hence, both of these are decreasing with the system size  $N$ , but at a slower rate than the bounds for the QMBS model.

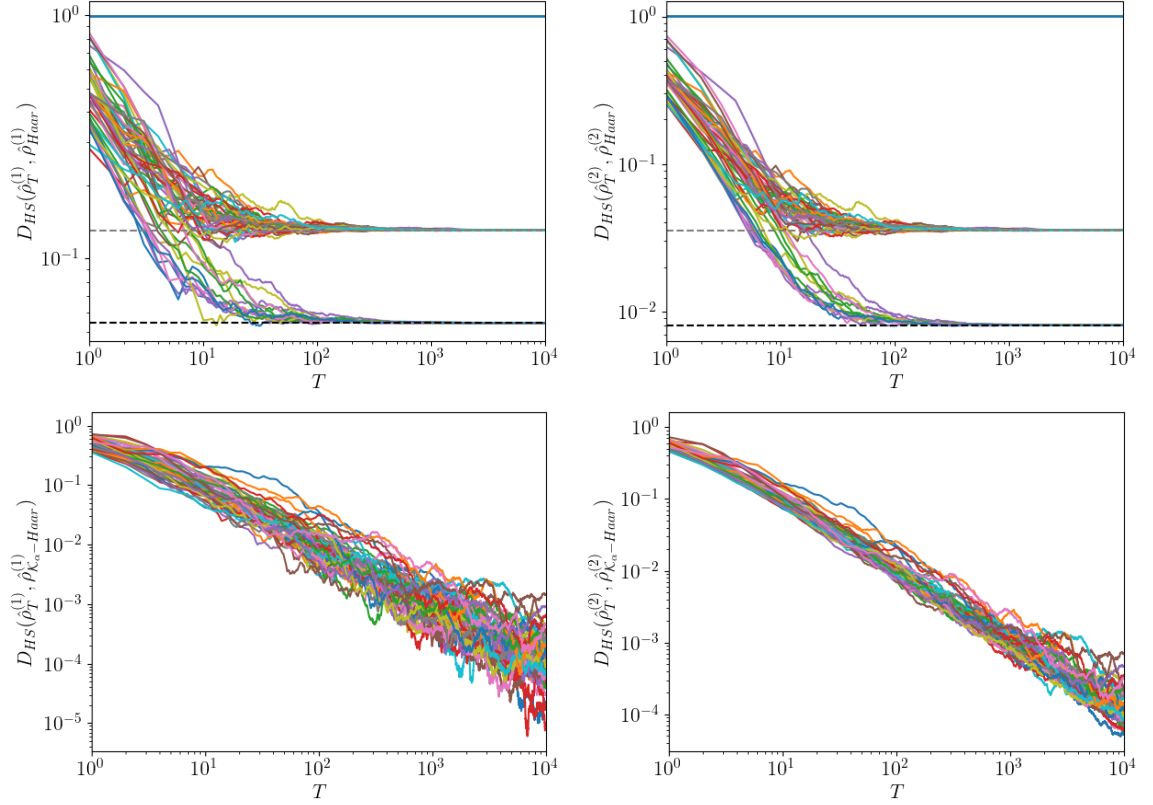


Figure 5.5: Top: Hilbert-Schmidt distance between the first (left) and second (right) moments of the Haar ensemble, and the temporal ensemble generated by the Fibonacci layered pair-flip model, with  $N = 4$ ,  $d = 3$ . We test all possible initial computational basis states to generate the temporal ensembles with a single instance of  $\{\hat{U}^{(A)}, \hat{U}^{(B)}\}$ . For the frozen states, the distance to the first Haar moment stays constant, at around  $\mathcal{O}(1)$ . The 42 states belonging to 7-dimensional subspaces decay and eventually saturate at the distance between the full Hilbert space Haar moments, and the Haar moments in a 7-dimensional subspace (gray dashed line). The states belonging to the largest, 15-dimensional Krylov subspace, however, saturate at a lower bound (black dashed line), due to the larger subspace. Both of these bounds are computed using Eqs. (5.21) and (5.22), with  $D = 81$ ,  $D_\alpha = 15$  (black) and  $D_\alpha = 7$  (gray). Bottom: Hilbert-Schmidt distance the temporal ensemble and the subspace-restricted Haar ensemble for the first (left) and second (right) moments. We test all initial computational basis states, apart from the frozen ones. All of the initial states are converging to the  $\hat{\rho}_{K_\alpha}^{(k)}$  in their respective subspaces, thus displaying CHSSE in them.

## 5.7 Effect of Regular Symmetries

In the ETH framework, the presence of a few local symmetries in a model is generally not classified as an ETH-violation mechanism, as we still observe thermal behaviour within the symmetry subsectors. However, to complete the picture, we also demonstrate the effect of symmetries on an otherwise CHSE model. We again use the 2-local unitaries as defined in Eq. (5.23), but in a system with  $d = 2$ . As any operator can only flip neighbouring sites from  $|00\rangle$  to  $|11\rangle$  (and vice versa), one can deduce that in this case the model will have a global  $U(1)$  symmetry, corresponding to the staggered magnetisation,  $\hat{M}_S = \sum_{j=1}^N (-1)^j |\alpha\rangle_j \langle \alpha|_j$ . Therefore, for system size  $N$ , we will get a splitting into  $N + 1$  subspaces, with dimensions  $\binom{N}{j}$ , where  $j = 0, 1, \dots, N$  labels the different subspaces. We perform similar numerical simulations as for the models in sections 5.5 and 5.6, with  $N = 4$  and up to  $T = 10^4$ . The results are displayed in Fig. 5.6. This effect can also be observed in the DEE as a function of  $T$  for a state belonging to the largest subspace, as shown in Fig. 5.7. The DEE saturates to a bound set by the size of the largest subspace in the long  $T$  limit.

Similarly to the fragmented case, we observe almost perfect convergence to the Haar moments in symmetry subspaces. This indicates that starting in any initial state belonging to a symmetry subsector, we will form  $t$  designs within that subspace for sufficiently long times.

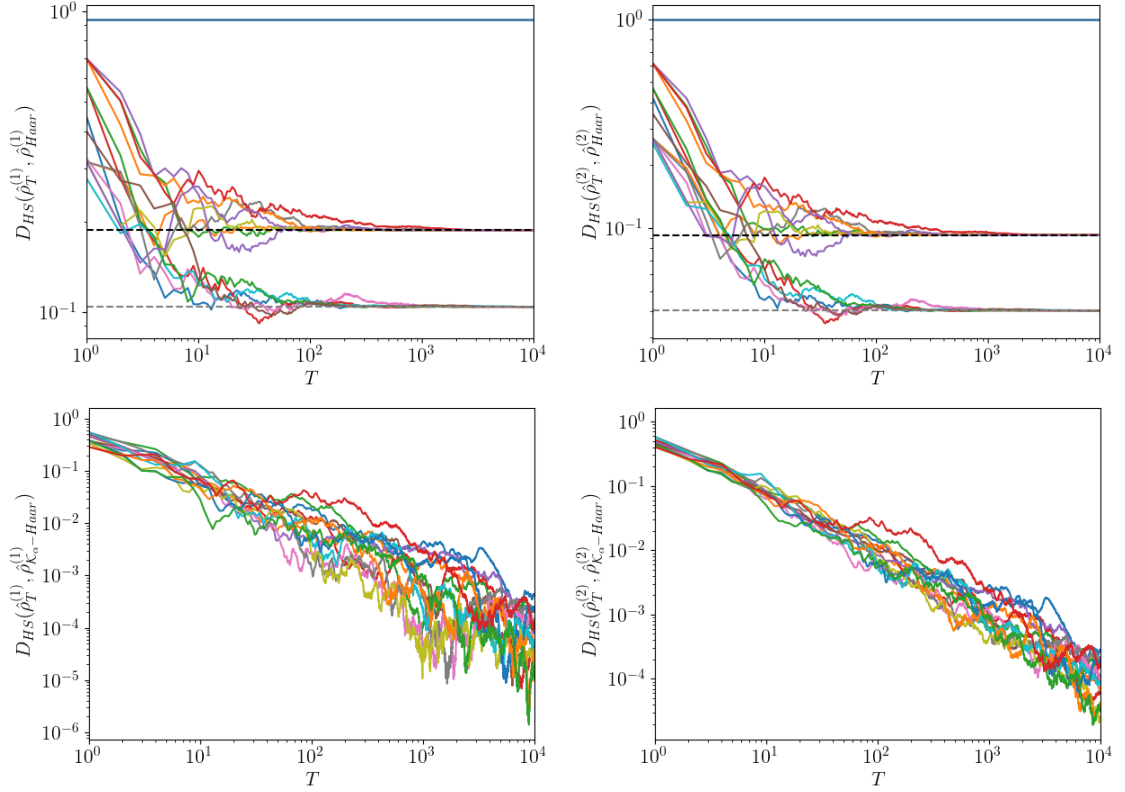


Figure 5.6: Hilbert-Schmidt distance between the first (top) and second (bottom) moments of the Haar ensemble, and the temporal ensemble generated by the Fibonacci layered pair-flip model, with  $N = 4$ ,  $d = 2$ , for a single instance of  $\{\hat{\mathcal{U}}^{(A)}, \hat{\mathcal{U}}^{(B)}\}$ . We test all possible initial product states to generate the temporal ensembles. For the two invariant states, the distance to the first Haar moment stays constant, at  $\mathcal{O}(1)$ . The eight states belonging to four-dimensional subspaces decay and eventually saturate at the distance between the full Hilbert space Haar moments, and the Haar moments in a four-dimensional subspace (black dashed line). The states belonging to the largest, six-dimensional subspace saturate at a lower bound (grey dashed line). Both of these bounds are computed using Eq. (5.21) and Eq. (5.22), with  $D = 16$ ,  $D_\alpha = 4$  (black) and  $D_\alpha = 6$  (gray). Hilbert-Schmidt distance the temporal ensemble and the subspace-restricted Haar ensemble for the first (top) and second (bottom) moments. We test all possible initial computational basis states, apart from the frozen ones. All of the initial states are converging to the  $\hat{\rho}_{\mathcal{K}_\alpha - \text{Haar}}^{(k)}$  in their respective subspaces, thus displaying CHSSE in them.

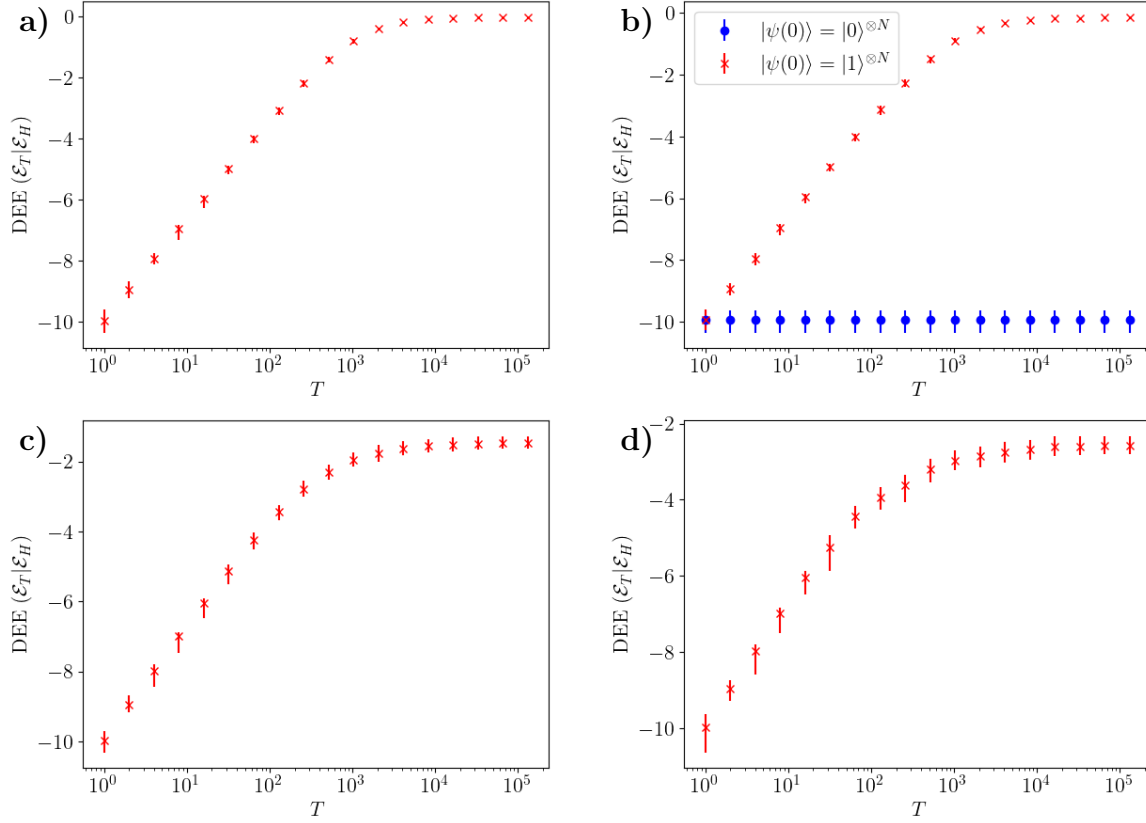


Figure 5.7: Comparison of discretised ensemble entropy (DEE) for the models studied. In all panels, DEE is computed using 100 sets of reference states  $|\psi_j\rangle$  with  $M \approx 1000$  states each; markers indicate the mean and lines or error bars show the full spread. **a)** Generic brickwork circuit initialised in  $|0\rangle^{\otimes N}$ , showing convergence of DEE to zero as the number of generated states  $T$  increases, aligned with expectations for a CHSE model. **b)** Model with a single embedded scar, evaluated for the scar state  $|0\rangle^{\otimes N}$  (blue) and a nonscar state  $|1\rangle^{\otimes N}$  (red). The scar state remains near the minimum entropy  $-\log_2 M$ , while the nonscar state saturates at  $\approx -0.1$ . **c)** Results from circuit with a pair-flip unitary, initialised in  $|0\rangle^{\otimes N}$  belonging to the largest fragmented subspace. The DEE saturates at  $\approx -2$ . **d)** Model with  $U(1)$  symmetry (due to conserved staggered magnetisation), initialised in the largest symmetry subspace. The DEE is again saturating at  $\approx -2$ . Note that the saturation values for all models are aligned with the estimated values in Appendix P.

## 5.8 Comparison of DEE

All of the presented models can also be characterised in terms of the discretised ensemble entropy, as defined in 5.12. The results are presented in Fig. 5.7, with the details of how they were computed in the appendix P.

## 5.9 Concluding remarks

Complete Hilbert space ergodicity has extended the notion of ergodicity from classical to certain quantum systems undergoing aperiodic dynamics. It can thus be seen as complementary to the ETH, which has been conceptually crucial for our understanding of thermalisation in Hamiltonian and Floquet systems.

In this work, we presented the effects of ergodicity breaking mechanisms on systems that otherwise display CHSE. In both scarred and fragmented models, the presence of nonlocal conserved quantities leads to a decoupling between subspaces of the full Hilbert space. Thus, different initial states are constrained to their respective subspaces, and we cannot observe CHSE in the full Hilbert space, but we do observe CHSE in the subspaces. The presented work opens up further avenues and questions for future research. We observe that one can construct Haar ensembles in some designated subspaces. This protocol could be adapted to construct them in decoherence-free subspaces, akin to the prescription described in Ref. [193]. The constructed  $t$ -designs could have practical applications in fault-tolerant quantum computers. Unfortunately, we observe that the temporal ensembles approximate Haar ensembles only at late times. Hence, it remains an important open problem to investigate the convergence rates for the different  $k$ -moments.



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### Conclusion

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In this thesis, we have investigated mechanisms of ergodicity breaking in quantum discrete-time models, motivated by recent advances in digital quantum simulation platforms and the analytical tractability of certain classes of circuit models.

We have addressed the analytical aspect in Chapter 3, where we have demonstrated that weak ergodicity breaking is possible in a provably thermalising and chaotic model, by embedding either a constant number or exponentially many QMBS states. We highlighted its properties via finite-size numerical simulations, but the results are also valid for arbitrary system sizes.

The second major finding presented in the thesis, discussed in Chapter 4, is a connection between gapless excitations, which can be viewed as “asymptotic ground states” and asymptotic QMBS states in a set of models. By focusing on a particular one-dimensional 2-local model, we managed to find the exact form of the AQMBS states, which occur only when an exact QMBS state becomes the  $N$ -qubit W state. Using the found closed-form expression for AQMBS states, we presented an efficient protocol to prepare them on digital quantum simulators with all-to-all connectivity. Using current quantum hardware, we probed the dynamics starting from the AQMBS states, which displayed slower thermalisation at greater system sizes. We also used a trivial product QMBS and a non-scar state to contrast the behaviour of AQMBS states.

These experimental results are significant from two aspects. Firstly, they present the first detection of anomalous thermalisation due to AQMBS. Secondly, in this case a physical phenomenon was first observed on digital quantum simulators, highlighting their versatility in probing new physics.

This opens up both immediate and more speculative avenues for future research. While we have focused on the Floquet version, one could also use quantum computers to probe continuous-time dynamics. One could also implement more sophisticated measurement schemes to extract quantities other than the total magnetisation, such as the fidelity or the Rényi-2 entropy [138, 139]. The major obstacle we have encountered was financial in nature, as these require executing many different circuit instances. The presented arguments also suggest that AQMBS exist in similar models in higher dimensions [182], which should be investigated in more detail in future work. These may provide a particularly challenging benchmark for both digital quantum simulators and classical numerical techniques.

In Chapter 5, we have presented the concept of CHSE and demonstrated its presence in aperiodically structured circuit models. The effects of QMBS, regular symmetries, and HSF were demonstrated both through the distance between the generated temporal ensemble and the Haar ensemble for  $k = 1, 2$  moments, as well as by directly estimating the relative ensemble entropy of the two distributions. One question raised by the presented results is whether a model can have initial states that lead to CHSE and others that do not. I.e., whether probing a single initial state is sufficient to characterise it as obeying CHSE or not. Another natural direction for future work is to identify classes of drives that converge to the Haar ensemble more rapidly.

In summary, we have investigated different ergodicity breaking mechanisms, including QMBS, AQMBS, and HSF in a variety of toy models. The main contributions of this thesis are the demonstration that QMBS can exist in a provably thermalising, chaotic model and the first experimental detection of asymptotic scars.

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## Appendices

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### A Properties of Dual-Unitary Gates

In this section we provide proofs for some properties of DU gates. The most important one is that  $\hat{U}^{\text{DU},1}$  and  $\hat{U}^{\text{DU},2}$  [Eqs. (3.7) and (3.8) in the main text] are DU. Note that this was already shown in Ref. [156], but we provide it for the sake of completeness.

**Claim A.1.** *Let  $\hat{U}_A$  and  $\hat{U}_B$  be two-qudit unitary operators, related by:*

$$\hat{U}_B = (\hat{u}_+ \otimes \hat{u}_-) \hat{U}_A (\hat{v}_- \otimes \hat{v}_+), \quad (\text{A.1})$$

where  $\hat{u}_\pm, \hat{v}_\pm$  are arbitrary single qudit unitaries. Then the dual of  $\hat{U}_B$  is  $\tilde{U}_B = (\hat{v}_+^T \otimes \hat{u}_-) \tilde{U}_A (\hat{v}_- \otimes \hat{u}_+^T)$ , where  $\tilde{U}_A$  is the dual of  $\hat{U}_A$ .

*Proof.* First we write  $\hat{U}_A = \sum_{i,j,k,l=0}^{d-1} [U_A]_{ij}^{kl} |k\rangle \langle i| \otimes |l\rangle \langle j|$ . Then its dual is:

$$\tilde{U}_A = \sum_{i,j,k,l=0}^{d-1} [U_A]_{ij}^{kl} |j\rangle \langle i| \otimes |l\rangle \langle k|, \quad (\text{A.2})$$

as defined by Eq. (3.1). Substituting Eq. (A.2) into Eq. (A.1):

$$\hat{U}_B = \sum_{i,j,k,l=0}^{d-1} [U_A]_{ij}^{kl} (\hat{u}_+ |k\rangle \langle i| \hat{v}_-) \otimes (\hat{u}_- |l\rangle \langle j| \hat{v}_+). \quad (\text{A.3})$$

Explicitly writing out the local unitaries as  $\hat{u}_+ = \sum_{i',k'} [u_+]_{i'}^{k'} |k'\rangle \langle i'|$  and  $\hat{v}_+ = \sum_{j',l'} [v_+]_{j'}^{l'} |l'\rangle \langle j'|$ , and substituting them into Eq. A.3:

$$\hat{U}_B = \sum_{i,j,k,l} \sum_{k',j'} [U_A]_{ij}^{kl} [u_+]_k^{k'} [v_+]_{j'}^j (|k\rangle \langle i| \hat{v}_-) \otimes (\hat{u}_- |l\rangle \langle j|). \quad (\text{A.4})$$

Then the dual of  $\hat{U}_B$  is obtained by exchanging the first ket and the last bra:

$$\begin{aligned}
\tilde{U}_B &= \sum_{i,j,k,l} \sum_{k',j'} [U_A]_{ij}^{kl} [u_+]_k^{k'} [v_+]_{j'}^j (|j\rangle \langle i| \hat{v}_-) \otimes (\hat{u}_- |l\rangle \langle k|) \\
&= \sum_{i,j,k,l} [U_A]_{ij}^{kl} \left( \sum_{j'} [v_+]_{j'}^j |j\rangle \right) \langle i| \hat{v}_- \otimes \hat{u}_- |l\rangle \left( \sum_{k'} [u_+]_k^{k'} \langle k| \right) \\
&= (\hat{v}_+^T \otimes \hat{u}_-) \left\{ \sum_{i,j,k,l=0}^{d-1} [U_A]_{ij}^{kl} |j\rangle \langle i| \otimes |l\rangle \langle k| \right\} (\hat{v}_- \otimes \hat{u}_+^T) \\
&= (\hat{v}_+^T \otimes \hat{u}_-) \tilde{U}_A (\hat{v}_- \otimes \hat{u}_+^T), \tag{A.5}
\end{aligned}$$

as stated.  $\square$

**Corollary A.2.** *If  $\hat{U}_A$  is dual-unitary then so is  $\hat{U}_B = (\hat{u}_+ \otimes \hat{u}_-) \hat{U}_A (\hat{v}_- \otimes \hat{v}_+)$ .*

*Proof.* The transpose of any unitary operator is unitary. Hence, if  $\tilde{U}_A$  is unitary, so must be  $\tilde{U}_B = (\hat{v}_+^T \otimes \hat{u}_-) \tilde{U}_A (\hat{v}_- \otimes \hat{u}_+^T)$ .  $\square$

**Claim A.3.** *If  $\hat{U}$  is DU then  $\hat{S}\hat{U}\hat{S}$  is as well. Its dual is  $\tilde{U}^T$ .*

*Proof.* First we write  $\hat{U} = \sum_{i,j,k,l=0}^{d-1} U_{ij}^{kl} |k\rangle \langle i| \otimes |l\rangle \langle j|$  and its dual as  $\tilde{U} = \sum_{i,j,k,l=0}^{d-1} U_{ij}^{kl} |j\rangle \langle i| \otimes |l\rangle \langle k|$ . Then we have:

$$\hat{S}\hat{U}\hat{S} = \sum_{ijkl} U_{ij}^{kl} |l\rangle \langle j| \otimes |k\rangle \langle i|. \tag{A.6}$$

Its dual is obtained by exchanging the first ket and the last bra:

$$\widetilde{\hat{S}\hat{U}\hat{S}} = \sum_{ijkl} U_{ij}^{kl} |i\rangle \langle j| \otimes |k\rangle \langle l| = \tilde{U}^T. \tag{A.7}$$

Since the transpose of a unitary operator is unitary,  $\widetilde{\hat{S}\hat{U}\hat{S}} = \hat{U}^T$  is also unitary.  $\square$

**Claim A.4.** *The unitary operator  $\hat{U} = \hat{S}\hat{V}$  is self-dual,  $\tilde{U} = \hat{U} = \hat{S}\hat{V}$ .*

*Proof.* We write out the swap operator and the entangling gate explicitly as  $\hat{S} = \sum_{i,j=0}^{d-1} |i\rangle \langle j| \otimes |j\rangle \langle i|$  and  $\hat{V} = \exp\{i \sum_{j=0}^{d-1} \hat{h}^{(j)} \otimes |j\rangle \langle j|\}$ . This gives us:

$$\hat{U} = \hat{S}\hat{V} = \sum_{i,j} |i\rangle \langle j| e^{i\hat{h}^{(i)}} \otimes |j\rangle \langle i|. \tag{A.8}$$

Then its dual is obtained by exchanging the first ket and the last bra:

$$\tilde{U} = \sum_{i,j} |i\rangle \langle j| e^{i\hat{h}^{(i)}} \otimes |j\rangle \langle i| = \hat{U}. \quad \square \tag{A.9}$$

**Corollary A.5.** *The unitary operator  $\hat{U}^{DU,1} = (\hat{u}_+ \otimes \hat{u}_-) \hat{S}\hat{V} (\hat{v}_- \otimes \hat{v}_+)$  is DU and its dual is  $\tilde{U}^{DU,1} = (\hat{v}_+^T \otimes \hat{u}_-) \hat{S}\hat{V} (\hat{v}_- \otimes \hat{u}_+^T)$ .*

*Proof.* This follows from claims A.1 and A.4.

**Corollary A.6.** *The unitary operator  $\hat{U}^{DU,2} = \hat{S}\hat{U}^{DU,1}\hat{S}$  is dual-unitary. Its dual is  $(\hat{U}^{DU,1})^T = (\hat{v}_-^T \otimes \hat{u}_+^T)\hat{V}^T\hat{S}(\hat{v}_+ \otimes \hat{u}_-^T)$ .*

*Proof.* This follows from corollary A.5 and claim A.3.

**Claim A.7.** *The parameterisation in Eq. (3.7) includes all DU gates for  $d = 2$ .*

*Proof.* For  $d = 2$ , any DU gate can be written in the form [18, 154]:

$$\hat{U}^{DU} = e^{i\phi}(\hat{u}^+ \otimes \hat{u}^-)\hat{S}(\hat{C}_z)^\alpha(\hat{v}^- \otimes \hat{v}^+), \quad (\text{A.10})$$

The  $\hat{C}_z$  operator is the controlled-Z gate, which can be written as a diagonal matrix in the computational basis:  $\hat{C}_z = \text{diag}(1, 1, 1, -1)$ . Hence, we can rewrite  $(\hat{C}_z)^\alpha$  as:

$$(\hat{C}_z)^\alpha = \exp[\alpha \ln(\hat{C}_z)] = \exp[\text{diag}(0, 0, 0, \alpha\pi i)] = \exp[\text{diag}(0, \alpha\pi i) \otimes |1\rangle\langle 1|]. \quad (\text{A.11})$$

Therefore it is easy to see that the complete parameterisation from (A.10) is included in the form in (3.7), with  $\hat{h}_{i,j}^{(0)} = 0$  and  $\hat{h}^{(1)} = \text{diag}(0, \alpha\pi i)$ .

## B Non-ergodic cases of DU circuits

In the main text, Section 3.4, we have chosen the even and odd layer unitaries to be from different parameterisations:  $\hat{U}_e \in \hat{U}^{DU,1}$ ,  $\hat{U}_o \in \hat{U}^{DU,2}$ . Recall that the two parameterisations, defined in Eqs. (3.7), (3.8), are related by an exchange of the control and target qudits in the entangling controlled unitary:

$$\hat{U}^{DU,1} = (\hat{u}^+ \otimes \hat{u}^-) \hat{S} \left[ \sum_{j=0}^{d-1} |j\rangle\langle j| \otimes \exp[i\hat{h}^{(j)}] \right] (\hat{v}^- \otimes \hat{v}^+), \quad (\text{B.1})$$

$$\hat{U}^{DU,2} = (\hat{u}^+ \otimes \hat{u}^-) \hat{S} \left[ \sum_{j=0}^{d-1} \exp[i\hat{h}^{(j)}] \otimes |j\rangle\langle j| \right] (\hat{v}^- \otimes \hat{v}^+). \quad (\text{B.2})$$

In the main text we stated that choosing  $\hat{U}_e \in \hat{U}^{DU,1}$ ,  $\hat{U}_o \in \hat{U}^{DU,2}$  ensures the full ergodicity of the models, demonstrated by the eigenvalues of  $\mathcal{M}_\pm$  maps defined in Eq. (3.22). In this section, we will present the cases of models which are not fully ergodic and display chiral behaviour:  $\mathcal{M}_+$  and  $\mathcal{M}_-$  maps indicate different ergodicity classes. For this purposes, we will use either  $\hat{U}_e = \hat{U}_o \in \hat{U}^{DU,1}$  or  $\hat{U}_e = \hat{U}_o \in \hat{U}^{DU,2}$ . Since gates in the even and odd layers are set to be the same, we will construct the  $\mathcal{M}_\pm$  maps according to Eq. (3.10). We also impose the constraints required for the models to contain QMBS states, as written in Eqs. (C1), (C2), (C3). We will consider the

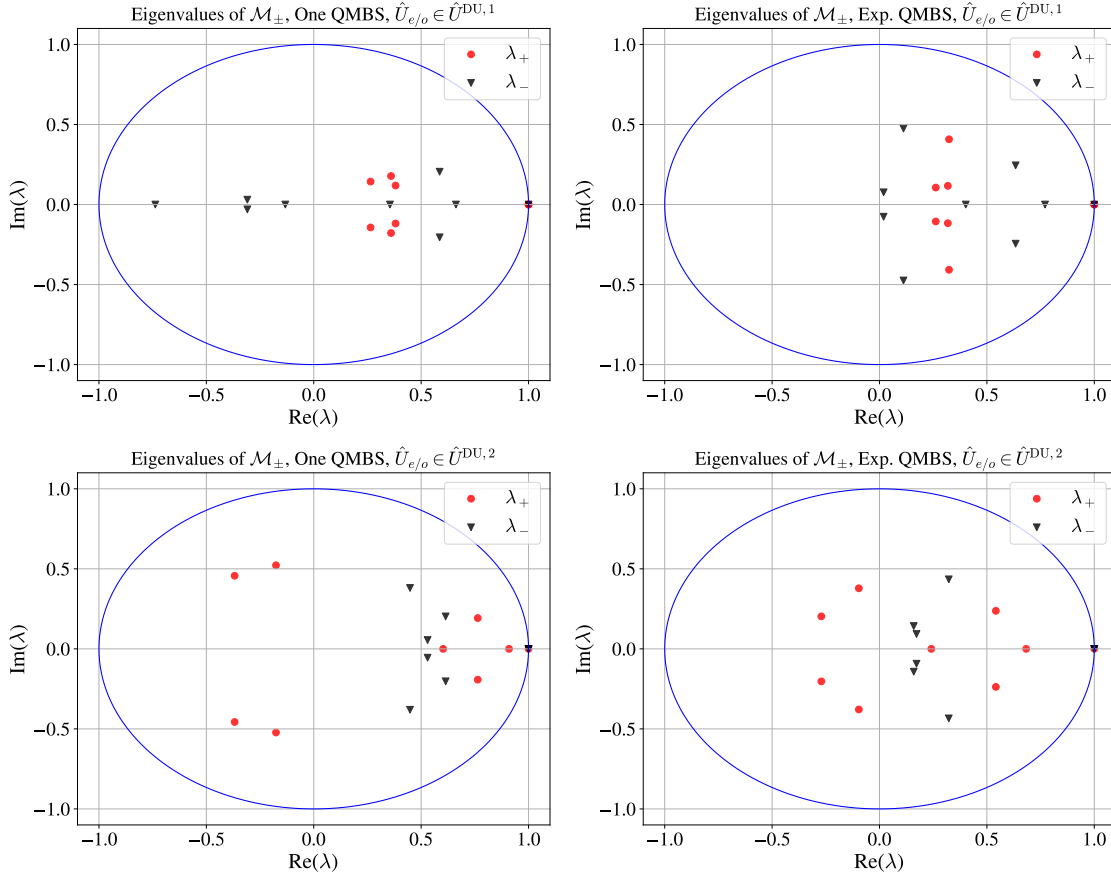


Figure B2: Eigenvalues of  $\mathcal{M}_{\pm}$  maps for circuits with  $\hat{U}_{e/o} \in \hat{U}^{\text{DU},1}$  or  $\hat{U}_{e/o} \in \hat{U}^{\text{DU},2}$ . TOP: Eigenvalues of  $\mathcal{M}_{\pm}$  for  $\hat{U}_e = \hat{U}_o \in \hat{U}^{\text{DU},1}$ , with either a single (left) or exponentially many embedded QMBS (right). In both cases, we observe that the models appear ergodic according to  $\mathcal{M}_{-}$  (only one eigenvalue of  $\lambda_{-} = 1$ ), but non-ergodic according to  $\mathcal{M}_{+}$  (multiple eigenvalues of  $\lambda_{+} = 1$ ). BOTTOM: Eigenvalues of  $\mathcal{M}_{\pm}$  for  $\hat{U}_e = \hat{U}_o \in \hat{U}^{\text{DU},2}$ , with either a single (left) or exponentially many embedded QMBS (right). In contrast to the case  $\hat{U}_e = \hat{U}_o \in \hat{U}^{\text{DU},1}$ , we observe that the models appear ergodic according to  $\mathcal{M}_{+}$  (one eigenvalue of  $\lambda_{+} = 1$ ), but non-ergodic according to  $\mathcal{M}_{-}$  (multiple eigenvalues of  $\lambda_{-} = 1$ ).

two cases of projectors which embed either a single or exponentially many scar states, written in Eqs. (3.17) and (3.20). We set the local Hilbert space dimension to  $d = 3$ . The eigenvalues of the resulting  $\mathcal{M}_{\pm}$  maps are presented in Fig. B2.

We see that setting  $\hat{U}_e = \hat{U}_o \in \hat{U}^{\text{DU},1}$  will lead to ergodic behaviour indicated by the eigenvalues of  $\mathcal{M}_{-}$ , for models with either a single or exponentially many scar states. There are  $d^2 - 1 = 8$  eigenvalues of magnitude  $|\lambda_{-}| < 1$ , indicating decaying correlations and consequently ergodic behaviour along the  $t = -x$  “light-cone”. There is exactly one eigenvalue of  $\lambda_{-} = 1$ , corresponding to the trivial case of the identity operator.

However, the eigenvalues of  $\mathcal{M}_{+}$  in this case indicate *non-ergodic* behaviour, due to non-decaying correlation along the  $x = t$  “light-cone”. There are only six eigenvalues satisfying  $|\lambda_{+}| < 1$  and three eigenvalues of  $\lambda_{+} = 1$ , which are overlapping in the plots in Fig. B2. This indicates that there are two operators that are acted upon trivially

by  $\mathcal{M}_+$ , in addition to the identity  $\hat{\mathbb{I}}$ .

In contrast, setting  $\hat{U}_e = \hat{U}_o \in \hat{U}^{\text{DU},2}$ , we obtain the exact opposite behaviour, for models with either a single or exponentially many QMBS states.  $\mathcal{M}_+$  has  $d^2 - 1 = 8$  eigenvalues with  $|\lambda_+| < 1$ , indicating ergodicity, while  $\mathcal{M}_-$  has only six eigenvalues with  $|\lambda_-| < 1$  and three eigenvalues of  $\lambda_- = 1$ .

This chiral behaviour in the ergodic indicators of  $\mathcal{M}_\pm$  arises due to the structure of the dual-unitary gates and the constraints imposed by the projection operators. Due to the entangling gates acting trivially on the control qubit and the constraints of projectors, there will be stationary operators  $\hat{o}$  for  $\hat{\mathcal{M}}_+$  when the “left” qudit is the control (in  $\hat{U}^{\text{DU},1}$ ), or for  $\hat{\mathcal{M}}_-$  when the “right” qudit is the control (in  $\hat{U}^{\text{DU},2}$ ).

More specifically, any diagonal operator  $\hat{o} = \text{diag}(o_{11}, o_{22}, \dots, o_{dd})$  will be invariant under  $\mathcal{M}_+$  for  $\hat{U}_{e/o} \in \hat{U}^{\text{DU},1}$  and under  $\mathcal{M}_-$  for  $\hat{U}_{e/o} \in \hat{U}^{\text{DU},2}$ . Hence, there will be a total of  $d$  independent basis operators, including the identity, which do not decay along either  $x = \pm t$ , depending on the choice of local DU gates.

Based on the results presented in this section, we have decided to set  $\hat{U}_e \in \hat{U}^{\text{DU},1}$ ,  $\hat{U}_o \in \hat{U}^{\text{DU},2}$  in the main text.  $\hat{U}_e \in \hat{U}^{\text{DU},1}$  ensures the ergodicity for  $\mathcal{M}_-$  and  $\hat{U}_o \in \hat{U}^{\text{DU},2}$  ensures the ergodicity for  $\mathcal{M}_+$ .

## C Reparameterisation of local unitaries

In the main text we rewrite the unitary:

$$\hat{U}_{n,n+1} = \exp[i\hat{H}_{n,n+1}^{(g)}(\hat{h})], \quad (\text{C.1})$$

in the following form:

$$\hat{U}(g, \hat{h}) = \hat{V}^{(g)\dagger}(|0\rangle\langle 0| \otimes \hat{\mathbb{I}} + |1\rangle\langle 1| \otimes e^{i\hat{h}})\hat{V}^{(g)}. \quad (\text{C.2})$$

To see this equivalence, recall the local generators have the form:  $\hat{H}_{n,n+1}^{(g)}(h) = h_{00}|\psi^{(g)}\rangle\langle\psi^{(g)}| + h_{11}|11\rangle\langle 11| + h_{01}|\psi^{(g)}\rangle\langle 11| + h_{01}^*|11\rangle\langle\psi^{(g)}|$ . We now introduce a unitary  $\hat{V}^{(g)}$ , which corresponds to the following basis change:

$$\hat{V}^{(g)}|00\rangle = |00\rangle, \quad \hat{V}^{(g)}|\psi_\perp^{(g)}\rangle = |01\rangle, \quad \hat{V}^{(g)}|\psi^{(g)}\rangle = |10\rangle, \quad \hat{V}^{(g)}|11\rangle = |11\rangle. \quad (\text{C.3})$$

Using the above to transform the generator, we obtain, in the computational basis:

$$\hat{V}^{(g)}\hat{H}^{(g)}\hat{V}^{(g)\dagger} = \begin{pmatrix} \hat{o}_2 & \hat{o}_2 \\ \hat{o}_2 & \hat{h} \end{pmatrix}, \quad (\text{C.4})$$

where  $\hat{0}_2$  are 2 by 2 zero matrices. Using this transformed generator, we can now write:

$$\hat{U}(g, \hat{h}) = \hat{V}^{(g)\dagger} e^{i\hat{V}^{(g)} \hat{H}^{(g)} \hat{V}^{(g)\dagger}} \hat{V}^{(g)} = \hat{V}^{(g)\dagger} \begin{pmatrix} \hat{\mathbb{I}}_2 & \hat{0}_2 \\ \hat{0}_2 & e^{i\hat{h}} \end{pmatrix} \hat{V}^{(g)} \quad (\text{C.5})$$

$$= \hat{V}^{(g)\dagger} [ |0\rangle \langle 0| \otimes \hat{\mathbb{I}} + |1\rangle \langle 1| \otimes e^{i\hat{h}} ] \hat{V}^{(g)}, \quad (\text{C.6})$$

as stated in the main text.

## D Demonstrating that the boundary QMBS is an eigenstate of the considered Hamiltonian and Circuit Models

For brevity, we will omit the  $g$ -dependence in the Hamiltonian and the states. We will write the Hamiltonian as:

$$\hat{\mathbb{H}} = \sum_{n=0}^{N-2} \hat{H}_{n,n+1}, \quad \hat{H}_{n,n+1} = a_{n,n+1} |\psi\rangle \langle \psi| + c_{n,n+1} |11\rangle \langle 11| + b_{n,n+1} |\psi\rangle \langle 11| + b_{n,n+1}^* |11\rangle \langle \psi|.$$

Recall the two states:

$$|\psi_{\perp}\rangle = \frac{1}{\sqrt{1+|g|^2}} (|01\rangle + g|10\rangle), \quad |\psi\rangle = \frac{1}{\sqrt{1+|g|^2}} (g^*|01\rangle - |10\rangle).$$

Now, to write the Hamiltonian in the computational basis, we use:

$$\begin{aligned} |\psi\rangle \langle \psi| &= \frac{1}{1+|g|^2} (|g|^2 |01\rangle \langle 01| + |10\rangle \langle 10| - g^* |01\rangle \langle 10| - g |10\rangle \langle 01|) \\ |11\rangle \langle \psi| &= \frac{1}{\sqrt{1+|g|^2}} (g |11\rangle \langle 01| - |11\rangle \langle 10|) \\ |\psi\rangle \langle 11| &= \frac{1}{\sqrt{1+|g|^2}} (g^* |01\rangle \langle 11| - |10\rangle \langle 11|) \end{aligned}$$

This gives:

$$\begin{aligned} \hat{H}_{n,n+1} &= c_{n,n+1} |11\rangle \langle 11| + \frac{a_{n,n+1}|g|^2}{1+|g|^2} |01\rangle \langle 01| + \frac{a_{n,n+1}}{1+|g|^2} |10\rangle \langle 10| \\ &\quad - \frac{a_{n,n+1} g^*}{1+|g|^2} |01\rangle \langle 10| - \frac{a_{n,n+1} g}{1+|g|^2} |10\rangle \langle 01| + \frac{b_{n,n+1}^* g}{\sqrt{1+|g|^2}} |11\rangle \langle 01| \\ &\quad - \frac{b_{n,n+1}^*}{\sqrt{1+|g|^2}} |11\rangle \langle 10| + \frac{b_{n,n+1} g^*}{\sqrt{1+|g|^2}} |10\rangle \langle 11| - \frac{b_{n,n+1}}{\sqrt{1+|g|^2}} |10\rangle \langle 11| \end{aligned}$$

As an ansatz for a zero energy eigenstate we use:

$$|S\rangle = \sum_{n=0}^{N-1} s_n |e_n\rangle, |e_n\rangle = \hat{X}_n |0\rangle^{\otimes N} = |0\dots 01_n 0\dots 0\rangle$$

The Hamiltonian acts on this state to give:

$$\begin{aligned} H|S\rangle &= \sum_{m=0}^{N-2} \hat{P}_{m,m+1} \hat{h}_{m,m+1} \hat{P}_{m,m+1} \sum_{n=0}^{N-1} s_n |e_n\rangle = \\ &= \sum_{n=0}^{N-2} s_n \hat{P}_{n,n+1} \hat{h}_{n,n+1} \hat{P}_{n,n+1} |e_n\rangle + \sum_{n=1}^{N-1} s_n \hat{P}_{n-1,n} \hat{h}_{n-1,n} \hat{P}_{n-1,n} |e_n\rangle \\ &= \sum_{n=0}^{N-2} s_n \left[ \frac{a_{n,n+1}}{1+|g|^2} |e_n\rangle - \frac{c_{n,n+1}^*}{\sqrt{1+|g|^2}} |e_{n,n+1}\rangle - \frac{g^* a_{n,n+1}}{1+|g|^2} |e_{n+1}\rangle \right] \\ &+ \sum_{n=1}^{N-1} \left[ \frac{|g|^2 a_{n-1,n}}{1+|g|^2} |e_n\rangle + \frac{g c_{n-1,n}^*}{\sqrt{1+|g|^2}} |e_{n-1,n}\rangle - \frac{g a_{n-1,n}}{1+|g|^2} |e_{n-1}\rangle \right] \\ &= \sum_{n=0}^{N-2} \frac{c_{n,n+1}^* (g s_{n+1} - s_n)}{\sqrt{1+|g|^2}} |e_{n,n+1}\rangle + \\ &\sum_{n=0}^{N-2} \frac{a_{n,n+1} (-g s_{n+1} + s_n)}{\sqrt{1+|g|^2}} |e_n\rangle + \sum_{n=1}^{N-1} \frac{a_{n-1,n} g^* (g s_n - s_{n-1})}{1+|g|^2} |e_n\rangle \end{aligned}$$

Hence, if the amplitudes obey  $s_n = g s_{n+1}$  for  $n = 1, \dots, N-1$ , then all terms in round brackets vanish and we have  $\hat{H}|S\rangle = 0$ , demonstrating that it is a zero energy eigenstate. Clearly this is satisfied for  $s_n = g^{-n}$ :

$$|S\rangle = \frac{1}{\mathcal{N}} \sum_{n=1}^{N-1} g^{-n} |e_n\rangle = |\mathcal{B}^{(g)}\rangle,$$

where  $\mathcal{N} = \sum_n |g^{-n}|^2$  is the appropriate normalisation constant. Now that we have obtained the structure of the edge QMBS, we can also verify it is also an eigenstate of the Floquet unitary. To see this, consider the action of any local term present in the Floquet operator, acting on the QMBS state:

$$\begin{aligned} e^{-i\hat{H}_{m,m+1}} |\mathcal{B}^{(g)}\rangle &= e^{-i\hat{H}_{m,m+1}} \frac{1}{\mathcal{N}} \sum_{n=0}^{N-1} g^{-n} |e_n\rangle = \\ &\frac{1}{\mathcal{N}} \left[ \sum_{n=0, n \neq m, m+1}^{N-1} g^{-n} |e_n\rangle \right] + \frac{1}{\mathcal{N}} e^{-i\hat{H}_{m,m+1}} (g^{-m} |e_m\rangle + g^{-m-1} |e_{m+1}\rangle). \end{aligned}$$

We now note that  $(g^{-m} |e_m\rangle + g^{-m-1} |e_{m+1}\rangle) = \frac{g^{-m-1}}{\sqrt{1+|g|^2}} |0\rangle^{\otimes (n-1)} \otimes |\psi_{\perp}^{(g)}\rangle \otimes |0\rangle^{\otimes (N-n-1)}$ . Since the projector  $\hat{P}^{(g)}$  annihilates  $|\psi_{\perp}^{(g)}\rangle$ , this state will stay invariant under the action of the unitary:  $e^{-i\hat{H}_{m,m+1}} (g^{-m} |e_m\rangle + g^{-m-1} |e_{m+1}\rangle) = g^{-m} |e_m\rangle + g^{-m-1} |e_{m+1}\rangle$ .

Therefore, the whole state  $|\mathcal{B}^{(g)}\rangle$  will be invariant as well:

$$\begin{aligned} \frac{1}{\mathcal{N}} \left[ \sum_{n=0, n \neq m, m+1}^{N-1} g^{-n} |e_n\rangle \right] + \frac{1}{\mathcal{N}} e^{-i\hat{H}_{m,m+1}} (g^{-m} |e_m\rangle + g^{-m-1} |e_{m+1}\rangle) = \\ \frac{1}{\mathcal{N}} \left[ \sum_{n=0, n \neq m, m+1}^{N-1} g^{-n} |e_n\rangle \right] + \frac{1}{\mathcal{N}} (g^{-m} |e_m\rangle + g^{-m-1} |e_{m+1}\rangle) = |\mathcal{B}^{(g)}\rangle. \end{aligned}$$

And as such it will be an eigenstate of the Floquet operator.

## E Proof for existence of asymptotic QMBS

In this section, we want to demonstrate the connection between *gapless* excitations (i.e., asymptotic ground states) and *asymptotic* QMBS, which was claimed in Section 4.2. In order to do this, we will first clearly define the setup and assumptions. We will then formally define the central claim, and show that it holds for the considered class of models.

Consider a system of  $N$  qudits on a  $D$ -dimensional lattice  $\Lambda$ , with sites labelled  $n \in \{0, 1, \dots, N-1\}$ . For each site  $n$  we are given a *local projector*

$$\{\hat{P}_{X_n}\}_{n=0}^{N-1},$$

where  $\hat{P}_{X_n}$  acts non-trivially only on a finite neighbourhood  $X_n$  including site  $n$ . We define two local Hamiltonians. First, the *frustration-free*, non-negative Hamiltonian

$$\hat{\mathbb{H}}_+ = \sum_n \hat{P}_{X_n}, \quad (\text{E.1})$$

which satisfies  $\hat{\mathbb{H}}_+ \geq 0$  since each  $\hat{P}_{X_n} \geq 0$ . The *frustration-free* property means that there is a finite set of states  $\{|\mathcal{G}\rangle\}$ , which are ground states both of the Hamiltonian and each individual projector:  $\hat{\mathbb{H}}_+ |\mathcal{G}\rangle = 0$ ,  $\hat{P}_{X_n} |\mathcal{G}\rangle = 0 \forall n$ .

Second, we define the more general Hamiltonian:

$$\hat{\mathbb{H}} = \sum_n \hat{P}_{X_n} \hat{H}_{X_n} \hat{P}_{X_n}, \quad (\text{E.2})$$

where  $\hat{H}_{X_n}$  is a bounded Hermitian operator with support on  $X_n$ . Since  $\hat{H}_{X_n}$  are not necessarily positive semi-definite, neither is  $\hat{\mathbb{H}}$ . But, since  $\hat{P}_{X_n} |\mathcal{G}\rangle = 0$ , the states  $|\mathcal{G}\rangle$  will be zero energy eigenstates of  $\hat{\mathbb{H}}$ , and therefore form scars in case they are somewhere in the middle of the spectrum [87].  $\hat{\mathbb{H}}$  reduces to  $\hat{\mathbb{H}}_+$  if  $\hat{H}_{X_n} = \hat{\mathbb{I}}$  for all  $n$ . Now, just as the ground states  $\{|\mathcal{G}\rangle\}$  become QMBS as we change  $\hat{\mathbb{I}} \rightarrow \hat{H}_{X_n}$ , we want to show that *gapless* excitations (asymptotic ground states) become *asymptotic* QMBS. More formally, assuming that  $\hat{\mathbb{H}}_+ |\mathcal{A}\rangle \xrightarrow{N \rightarrow \infty} 0$ , this leads to  $\hat{\mathbb{H}} |\mathcal{A}\rangle \xrightarrow{N \rightarrow \infty} 0$ .

This will be the main aim of Theorem E.3. Its proof has two main steps:

1. First we show that if  $\hat{\mathbb{H}}_+ |\mathcal{A}\rangle \rightarrow 0$ , then not only does the *sum* of local penalties (encoded by the projectors) vanish, but also certain composite quantities built out of the same projectors. This is shown in Lemmas E.1 and E.2.
2. Then we expand  $\|\hat{\mathbb{H}}_+ |\mathcal{A}\rangle\|^2$  into a sum over pairs of terms. We separate this sum into disjoint cases. Each of these is bounded by one of the positive quantities from Lemmas E.1 and E.2, hence all contributions vanish.

**Lemma E.1.** *If  $\hat{\mathbb{H}}_+ |\mathcal{A}\rangle \xrightarrow{N \rightarrow \infty} 0$  then:  $\langle \mathcal{A} | \hat{\mathbb{H}}'_+ | \mathcal{A} \rangle \xrightarrow{N \rightarrow \infty} 0$  where  $\hat{\mathbb{H}}'_+ \equiv \sum_n \lambda_n \hat{P}_{X_n}$  for any finite couplings  $0 \leq \lambda_n < \infty$ .*

*Proof* Since the operators  $\hat{P}_{X_n}$  and the numbers  $\lambda_n$  are non-negative, we have:

$$\langle \mathcal{A} | \hat{\mathbb{H}}'_+ | \mathcal{A} \rangle = \sum_n \lambda_n \langle \mathcal{A} | \hat{P}_{X_n} | \mathcal{A} \rangle \leq \lambda_{\max} \sum_n \langle \mathcal{A} | \hat{P}_{X_n} | \mathcal{A} \rangle = \lambda_{\max} \langle \mathcal{A} | \hat{\mathbb{H}}_+ | \mathcal{A} \rangle, \quad (\text{E.3})$$

where  $\lambda_{\max} = \max_n \{\lambda_n\}$ . Since  $\hat{\mathbb{H}}_+ |\mathcal{A}\rangle \xrightarrow{N \rightarrow \infty} 0$  we have  $\langle \mathcal{A} | \hat{\mathbb{H}}_+ | \mathcal{A} \rangle \xrightarrow{N \rightarrow \infty} 0$ , and therefore  $\lambda_{\max} \langle \mathcal{A} | \hat{\mathbb{H}}_+ | \mathcal{A} \rangle \xrightarrow{N \rightarrow \infty} 0$ , which implies that  $\langle \mathcal{A} | \hat{\mathbb{H}}'_+ | \mathcal{A} \rangle \xrightarrow{N \rightarrow \infty} 0$ .  $\square$

**Lemma E.2.** *If  $\hat{\mathbb{H}}_+ |\mathcal{A}\rangle \xrightarrow{N \rightarrow \infty} 0$  then:*

1.  $\langle \mathcal{A} | \hat{\mathbb{J}}_+ | \mathcal{A} \rangle \xrightarrow{N \rightarrow \infty} 0$  where  $\hat{\mathbb{J}}_+ \equiv \sum_{\substack{n \neq n' \\ X_n \cap X_{n'} = \emptyset}} \hat{P}_{X_n} \hat{P}_{X_{n'}}$  is non-negative and Hermitian, and the sum is over all non-overlapping pairs of regions,  $X_n \cap X_{n'} = \emptyset$ .
2.  $\langle \mathcal{A} | \hat{\mathbb{F}}_+ | \mathcal{A} \rangle \xrightarrow{N \rightarrow \infty} 0$  where  $\hat{\mathbb{F}}_+ \equiv \sum_{\substack{n < n' \\ X_n \cap X_{n'} \neq \emptyset}} \sum_{\alpha, \alpha'} \left( |\pi_{X_n}^\alpha\rangle \langle \pi_{X_n}^\alpha| + |\pi_{X_{n'}}^{\alpha'}\rangle \langle \pi_{X_{n'}}^{\alpha'}| \right)^2$  is non-negative and Hermitian, and the sum is over all overlapping but non-equal pairs of regions.  $\hat{P}_{X_n} = \sum_{\alpha=0}^{r_n-1} |\pi_{X_n}^\alpha\rangle \langle \pi_{X_n}^\alpha|$  is a partial spectral decomposition of the local projector and  $1 \leq r_n < \infty$  is its rank. Note that we have omitted an implicit identity  $\hat{\mathbb{I}}_{\bar{X}_n}$  acting on all qudits in the complement  $\bar{X}_n$  of  $X_n$ .

*Proof* Since  $\hat{\mathbb{H}}_+ |\mathcal{A}\rangle \xrightarrow{N \rightarrow \infty} 0$ , its norm vanishes in the thermodynamic limit as well:

$$\|\hat{\mathbb{H}}_+ |\mathcal{A}\rangle\|^2 \equiv \langle \mathcal{A} | \hat{\mathbb{H}}_+ \hat{\mathbb{H}}_+ | \mathcal{A} \rangle = \sum_{n, n'} \langle \mathcal{A} | \hat{P}_{X_n} \hat{P}_{X_{n'}} | \mathcal{A} \rangle \xrightarrow{N \rightarrow \infty} 0. \quad (\text{E.4})$$

We now split the sum  $\sum_{n, n'}$  into three groups of terms:  $n = n'$ ,  $n \neq n'$  and  $X_n \cap X_{n'} =$

$\emptyset$ , and  $n \neq n'$  and  $X_n \cap X_{n'} \neq \emptyset$ . This gives:

$$\begin{aligned}
\|\hat{\mathbb{H}}_+ |\mathcal{A}\rangle\|^2 &= \sum_{n,n'} \langle \mathcal{A} | \hat{P}_{X_n} \hat{P}_{X_{n'}} | \mathcal{A} \rangle = \sum_n \langle \mathcal{A} | \hat{P}_{X_n} | \mathcal{A} \rangle \\
&+ \sum_{\substack{n \neq n' \\ X_n \cap X_{n'} = \emptyset}} \langle \mathcal{A} | \hat{P}_{X_n} \hat{P}_{X_{n'}} | \mathcal{A} \rangle + \sum_{\substack{n \neq n' \\ X_n \cap X_{n'} \neq \emptyset}} \langle \mathcal{A} | \hat{P}_{X_n} \hat{P}_{X_{n'}} | \mathcal{A} \rangle \\
&= \langle \mathcal{A} | \hat{\mathbb{H}}_+ | \mathcal{A} \rangle + \langle \mathcal{A} | \hat{\mathbb{J}}_+ | \mathcal{A} \rangle + \sum_{\substack{n < n' \\ X_n \cap X_{n'} \neq \emptyset}} \langle \mathcal{A} | (\hat{P}_{X_n} \hat{P}_{X_{n'}} + \hat{P}_{X_{n'}} \hat{P}_{X_n}) | \mathcal{A} \rangle \\
&\xrightarrow{N \rightarrow \infty} 0,
\end{aligned} \tag{E.5}$$

where we have used  $\sum_n \langle \mathcal{A} | \hat{P}_{X_n} \hat{P}_{X_n} | \mathcal{A} \rangle = \sum_n \langle \mathcal{A} | \hat{P}_{X_n} | \mathcal{A} \rangle = \langle \mathcal{A} | \hat{\mathbb{H}}_+ | \mathcal{A} \rangle$ , and the definition of  $\hat{\mathbb{J}}_+$  given in the statement of the lemma. In the last group of terms, we write the projectors in their partial spectral decomposition, which gives:

$$\begin{aligned}
\|\hat{\mathbb{H}}_+ |\mathcal{A}\rangle\|^2 &= \langle \mathcal{A} | \hat{\mathbb{H}}_+ | \mathcal{A} \rangle + \langle \mathcal{A} | \hat{\mathbb{J}}_+ | \mathcal{A} \rangle \\
&+ \sum_{\substack{n < n' \\ X_n \cap X_{n'} \neq \emptyset}} \sum_{\alpha, \alpha'} \langle \mathcal{A} | \left( |\pi_{X_n}^\alpha\rangle \langle \pi_{X_n}^\alpha| \cdot |\pi_{X_{n'}}^{\alpha'}\rangle \langle \pi_{X_{n'}}^{\alpha'}| \right. \\
&\quad \left. + |\pi_{X_{n'}}^{\alpha'}\rangle \langle \pi_{X_{n'}}^{\alpha'}| \cdot |\pi_{X_n}^\alpha\rangle \langle \pi_{X_n}^\alpha| \right) | \mathcal{A} \rangle \xrightarrow{N \rightarrow \infty} 0
\end{aligned} \tag{E.6}$$

We now rewrite the terms in brackets:

$$\begin{aligned}
&|\pi_{X_n}^\alpha\rangle \langle \pi_{X_n}^\alpha| \cdot |\pi_{X_{n'}}^{\alpha'}\rangle \langle \pi_{X_{n'}}^{\alpha'}| + |\pi_{X_{n'}}^{\alpha'}\rangle \langle \pi_{X_{n'}}^{\alpha'}| \cdot |\pi_{X_n}^\alpha\rangle \langle \pi_{X_n}^\alpha| \\
&= \left( |\pi_{X_n}^\alpha\rangle \langle \pi_{X_n}^\alpha| + |\pi_{X_{n'}}^{\alpha'}\rangle \langle \pi_{X_{n'}}^{\alpha'}| \right)^2 - |\pi_{X_{n'}}^{\alpha'}\rangle \langle \pi_{X_{n'}}^{\alpha'}| - |\pi_{X_n}^\alpha\rangle \langle \pi_{X_n}^\alpha|.
\end{aligned} \tag{E.7}$$

Substituting back into Eq. E.6 gives:

$$\|\hat{\mathbb{H}}_+ |\mathcal{A}\rangle\|^2 = \langle \mathcal{A} | \hat{\mathbb{H}}_+ | \mathcal{A} \rangle + \langle \mathcal{A} | \hat{\mathbb{J}}_+ | \mathcal{A} \rangle + \langle \mathcal{A} | \hat{\mathbb{F}}_+ | \mathcal{A} \rangle - \langle \mathcal{A} | \hat{\mathbb{H}}'_+ | \mathcal{A} \rangle \xrightarrow{N \rightarrow \infty} 0, \tag{E.8}$$

where we have used the definition of  $\hat{\mathbb{F}}_+$  given in the statement of the lemma and:

$$\hat{\mathbb{H}}'_+ = \sum_{\substack{n < n' \\ X_n \cap X_{n'} \neq \emptyset}} \sum_{\alpha, \alpha'} (|\pi_{X_{n'}}^{\alpha'}\rangle \langle \pi_{X_{n'}}^{\alpha'}| + |\pi_{X_n}^\alpha\rangle \langle \pi_{X_n}^\alpha|) = \sum_{\substack{n \neq n' \\ X_n \cap X_{n'} \neq \emptyset}} \sum_{\alpha, \alpha'} |\pi_{X_n}^\alpha\rangle \langle \pi_{X_n}^\alpha| = \sum_n r_n c_n \hat{P}_{X_n}, \tag{E.9}$$

which is in the form of  $\hat{\mathbb{H}}'_+$  in lemma E.1. Note that  $c_n$  ( $0 \leq c_n < \infty$ ) is obtained by summing over the  $X_{n'} \neq n$  that overlap with  $X_n$  and  $r_n$  due to summing over  $\alpha'$ .

Now, since we have that  $\langle \mathcal{A} | \hat{\mathbb{H}}_+ | \mathcal{A} \rangle \xrightarrow{N \rightarrow \infty} 0$  by assumption,  $\langle \mathcal{A} | \hat{\mathbb{H}}'_+ | \mathcal{A} \rangle \xrightarrow{N \rightarrow \infty} 0$  by lemma E.1 and also that  $\hat{\mathbb{F}}_+$  and  $\hat{\mathbb{J}}_+$  are both non-negative Hermitian operators, the only way to have  $\|\hat{\mathbb{H}}_+ |\mathcal{A}\rangle\|^2 \xrightarrow{N \rightarrow \infty} 0$  in Eq. E.8 is if we have:

$$\langle \mathcal{A} | \hat{\mathbb{F}}_+ | \mathcal{A} \rangle \xrightarrow{N \rightarrow \infty} 0 \quad \text{and} \quad \langle \mathcal{A} | \hat{\mathbb{J}}_+ | \mathcal{A} \rangle \xrightarrow{N \rightarrow \infty} 0, \tag{E.10}$$

as required.  $\square$

**Theorem E.3.** *If  $\hat{\mathbb{H}}_+ = \sum_n \hat{P}_{X_n}$  and  $\hat{\mathbb{H}}_+ |\mathcal{A}\rangle \xrightarrow{N \rightarrow \infty} 0$  then the states  $|\mathcal{A}\rangle$  are asymptotic QMBS of  $\hat{\mathbb{H}} = \sum_n \hat{P}_{X_n} \hat{H}_{X_n} \hat{P}_{X_n}$ , with  $\hat{\mathbb{H}} |\mathcal{A}\rangle \xrightarrow{N \rightarrow \infty} 0$ .*

*Proof.* We will show that the norm of  $\hat{\mathbb{H}} |\mathcal{A}\rangle$  vanishes in the thermodynamic limit. First we expand the squared norm:

$$\|\hat{\mathbb{H}} |\mathcal{A}\rangle\|^2 = \langle \mathcal{A} | \hat{\mathbb{H}} \hat{\mathbb{H}} | \mathcal{A} \rangle = \sum_{n,n'} \langle \mathcal{A} | \hat{P}_{X_n} \hat{H}_{X_n} \hat{P}_{X_n} \hat{P}_{X_{n'}} \hat{H}_{X_{n'}} \hat{P}_{X_{n'}} | \mathcal{A} \rangle. \quad (\text{E.11})$$

Since  $\hat{P}_{X_n} \hat{H}_{X_n} \hat{P}_{X_n}$  and  $\hat{P}_{X_{n'}}$  mutually commute, they have a common eigenbasis. We wrote the projectors as  $\hat{P}_{X_n} = \sum_{\alpha=0}^{r_n-1} |\pi_{X_n}^\alpha\rangle \langle \pi_{X_n}^\alpha|$ , and now we similarly expand  $\hat{P}_{X_n} \hat{H}_{X_n} \hat{P}_{X_n} = \sum_{\alpha=0}^{r_n-1} \eta_{X_n}^\alpha |\pi_{X_n}^\alpha\rangle \langle \pi_{X_n}^\alpha|$ , with  $\eta_{X_n}^\alpha \in \mathbb{R}$ . Substituting into Eq. (E.11):

$$\|\hat{\mathbb{H}} |\mathcal{A}\rangle\|^2 = \sum_{n,n'} \sum_{\alpha,\alpha'} \eta_{X_n}^\alpha \eta_{X_{n'}}^{\alpha'} \langle \mathcal{A} | (|\pi_{X_n}^\alpha\rangle \langle \pi_{X_n}^\alpha|) (|\pi_{X_{n'}}^{\alpha'}\rangle \langle \pi_{X_{n'}}^{\alpha'}|) | \mathcal{A} \rangle. \quad (\text{E.12})$$

Now we break up the sum  $\sum_{n,n'}$  into three groups of terms, as also done in Eq. (E.5):

$$\begin{aligned} \|\hat{\mathbb{H}} |\mathcal{A}\rangle\|^2 &= \sum_n \sum_\alpha (\eta_{X_n}^\alpha)^2 \langle \mathcal{A} | (|\pi_{X_n}^\alpha\rangle \langle \pi_{X_n}^\alpha| \otimes) | \mathcal{A} \rangle \\ &+ \sum_{\substack{n \neq n' \\ X_n \cap X_{n'} = \emptyset}} \sum_{\alpha,\alpha'} \eta_{X_n}^\alpha \eta_{X_{n'}}^{\alpha'} \langle \mathcal{A} | (|\pi_{X_n}^\alpha\rangle \langle \pi_{X_n}^\alpha| \cdot |\pi_{X_{n'}}^{\alpha'}\rangle \langle \pi_{X_{n'}}^{\alpha'}|) | \mathcal{A} \rangle \\ &+ \sum_{\substack{n < n' \\ X_n \cap X_{n'} \neq \emptyset}} \sum_{\alpha,\alpha'} \eta_{X_n}^\alpha \eta_{X_{n'}}^{\alpha'} \langle \mathcal{A} | (|\pi_{X_n}^\alpha\rangle \langle \pi_{X_n}^\alpha| \cdot |\pi_{X_{n'}}^{\alpha'}\rangle \langle \pi_{X_{n'}}^{\alpha'}| \\ &+ |\pi_{X_{n'}}^{\alpha'}\rangle \langle \pi_{X_{n'}}^{\alpha'}| \cdot |\pi_{X_n}^\alpha\rangle \langle \pi_{X_n}^\alpha|) | \mathcal{A} \rangle. \end{aligned} \quad (\text{E.13})$$

We now rewrite the term in round brackets in the last line using Eq. E.7, to obtain:

$$\left\| \hat{\mathbb{H}} |\mathcal{A}\rangle \right\|^2 = \sum_n \sum_\alpha (\eta_{X_n}^\alpha)^2 \langle \mathcal{A} | (|\pi_{X_n}^\alpha\rangle \langle \pi_{X_n}^\alpha|) | \mathcal{A} \rangle \quad (\text{E.14})$$

$$+ \sum_{\substack{n \neq n' \\ X_n \cap X_{n'} = \emptyset}} \sum_{\alpha,\alpha'} \eta_{X_n}^\alpha \eta_{X_{n'}}^{\alpha'} \langle \mathcal{A} | (|\pi_{X_n}^\alpha\rangle \langle \pi_{X_n}^\alpha| \cdot |\pi_{X_{n'}}^{\alpha'}\rangle \langle \pi_{X_{n'}}^{\alpha'}|) | \mathcal{A} \rangle \quad (\text{E.15})$$

$$+ \sum_{\substack{n < n' \\ X_n \cap X_{n'} \neq \emptyset}} \sum_{\alpha,\alpha'} \eta_{X_n}^\alpha \eta_{X_{n'}}^{\alpha'} \langle \mathcal{A} | (|\pi_{X_n}^\alpha\rangle \langle \pi_{X_n}^\alpha| + |\pi_{X_{n'}}^{\alpha'}\rangle \langle \pi_{X_{n'}}^{\alpha'}|)^2 | \mathcal{A} \rangle \quad (\text{E.16})$$

$$- \sum_{\substack{n \neq n' \\ X_n \cap X_{n'} \neq \emptyset}} \sum_{\alpha,\alpha'} \eta_{X_n}^\alpha \eta_{X_{n'}}^{\alpha'} \langle \mathcal{A} | (|\pi_{X_n}^\alpha\rangle \langle \pi_{X_n}^\alpha|) | \mathcal{A} \rangle = \textcolor{red}{A} + \textcolor{blue}{B} + \textcolor{red}{C} + D, \quad (\text{E.17})$$

where we have grouped terms together into  $A$ ,  $B$ ,  $C$ , and  $D$  as indicated. Using the triangle inequality:

$$\|\hat{\mathbb{H}}|\mathcal{A}\rangle\|^2 \leq |A| + |B| + |C| + |D|. \quad (\text{E.18})$$

To show that  $\|\hat{\mathbb{H}}|\mathcal{A}\rangle\|^2 \xrightarrow{N \rightarrow \infty} 0$  we can show that each term on the right hand side vanishes for  $N \rightarrow \infty$ . The absolute value of the first term is:

$$\begin{aligned} |A| &= \left| \sum_{n,\alpha} (\eta_{X_n}^\alpha)^2 \langle \mathcal{A} | (|\pi_{X_n}^\alpha\rangle \langle \pi_{X_n}^\alpha|) | \mathcal{A} \rangle \right| \\ &\leq |\eta|_{\max}^2 \sum_{n,\alpha} \langle \mathcal{A} | (|\pi_{X_n}^\alpha\rangle \langle \pi_{X_n}^\alpha|) | \mathcal{A} \rangle = |\eta|_{\max}^2 \langle \mathcal{A} | \hat{\mathbb{H}}_+ | \mathcal{A} \rangle \end{aligned} \quad (\text{E.19})$$

where  $|\eta|_{\max} \equiv \max_{n,\alpha} \{|\eta_n^\alpha|\}$ . Since we are working with the assumption that  $\langle \mathcal{A} | \hat{\mathbb{H}}_+ | \mathcal{A} \rangle \xrightarrow{N \rightarrow \infty} 0$ , the right hand side of Eq. E.19 vanishes and we have  $|A| \xrightarrow{N \rightarrow \infty} 0$ .

The absolute value of the second term is, by the triangle inequality, bounded by:

$$\begin{aligned} |B| &= \left| \sum_{\substack{n \neq n' \\ X_n \cap X_{n'} = \emptyset}} \sum_{\alpha, \alpha'} \eta_{X_n}^\alpha \eta_{X_{n'}}^{\alpha'} \langle \mathcal{A} | (|\pi_{X_n}^\alpha\rangle \langle \pi_{X_n}^\alpha| \cdot |\pi_{X_{n'}}^{\alpha'}\rangle \langle \pi_{X_{n'}}^{\alpha'}|) | \mathcal{A} \rangle \right| \\ &\leq \sum_{\substack{n \neq n' \\ X_n \cap X_{n'} = \emptyset}} \sum_{\alpha, \alpha'} |\eta_{X_n}^\alpha| |\eta_{X_{n'}}^{\alpha'}| \langle \mathcal{A} | (|\pi_{X_n}^\alpha\rangle \langle \pi_{X_n}^\alpha| \cdot |\pi_{X_{n'}}^{\alpha'}\rangle \langle \pi_{X_{n'}}^{\alpha'}|) | \mathcal{A} \rangle \\ &\leq |\eta|_{\max}^2 \sum_{\substack{n \neq n' \\ X_n \cap X_{n'} = \emptyset}} \sum_{\alpha, \alpha'} \langle \mathcal{A} | (|\pi_{X_n}^\alpha\rangle \langle \pi_{X_n}^\alpha| \cdot |\pi_{X_{n'}}^{\alpha'}\rangle \langle \pi_{X_{n'}}^{\alpha'}|) | \mathcal{A} \rangle \\ &= |\eta|_{\max}^2 \sum_{\substack{n \neq n' \\ X_n \cap X_{n'} = \emptyset}} \langle \mathcal{A} | \hat{P}_{X_n} \hat{P}_{X_{n'}} | \mathcal{A} \rangle = |\eta|_{\max}^2 \langle \mathcal{A} | \hat{\mathbb{J}}_+ | \mathcal{A} \rangle. \end{aligned} \quad (\text{E.20})$$

Since  $\langle \mathcal{A} | \hat{\mathbb{J}}_+ | \mathcal{A} \rangle \xrightarrow{N \rightarrow \infty} 0$ , by lemma E.2, the last line vanishes and so  $|B| \xrightarrow{N \rightarrow \infty} 0$ .

Using the triangle inequality again,  $|C|$  is bounded by:

$$\begin{aligned} |C| &= \left| \sum_{\substack{n < n' \\ X_n \cap X_{n'} \neq \emptyset}} \sum_{\alpha, \alpha'} \eta_{X_n}^\alpha \eta_{X_{n'}}^{\alpha'} \langle \mathcal{A} | (|\pi_{X_n}^\alpha\rangle \langle \pi_{X_n}^\alpha| + |\pi_{X_{n'}}^{\alpha'}\rangle \langle \pi_{X_{n'}}^{\alpha'}|)^2 | \mathcal{A} \rangle \right| \\ &\leq \sum_{\substack{n < n' \\ X_n \cap X_{n'} \neq \emptyset}} \sum_{\alpha, \alpha'} |\eta_{X_n}^\alpha| |\eta_{X_{n'}}^{\alpha'}| \langle \mathcal{A} | (|\pi_{X_n}^\alpha\rangle \langle \pi_{X_n}^\alpha| + |\pi_{X_{n'}}^{\alpha'}\rangle \langle \pi_{X_{n'}}^{\alpha'}|)^2 | \mathcal{A} \rangle \\ &\leq |\eta|_{\max}^2 \sum_{\substack{n < n' \\ X_n \cap X_{n'} \neq \emptyset}} \sum_{\alpha, \alpha'} \langle \mathcal{A} | (|\pi_{X_n}^\alpha\rangle \langle \pi_{X_n}^\alpha| + |\pi_{X_{n'}}^{\alpha'}\rangle \langle \pi_{X_{n'}}^{\alpha'}|)^2 | \mathcal{A} \rangle \\ &= |\eta|_{\max}^2 \langle \mathcal{A} | \hat{\mathbb{F}}_+ | \mathcal{A} \rangle. \end{aligned} \quad (\text{E.21})$$

Since we know that  $\langle \mathcal{A} | \hat{\mathbb{F}}_+ | \mathcal{A} \rangle \xrightarrow{N \rightarrow \infty} 0$ , by lemma E.2, the last line vanishes, hence

$|C| \xrightarrow{N \rightarrow \infty} 0$ . Using the triangle inequality, just one last time,  $|D|$  is bounded by:

$$\begin{aligned}
|D| &= \left| \sum_n \sum_\alpha \eta_{X_n}^\alpha \left( \sum_{\alpha'} \sum_{\substack{n' \neq n \\ X_n \cap X_{n'} \neq \emptyset}} \eta_{X_{n'}}^{\alpha'} \right) \langle \mathcal{A} | (|\pi_{X_n}^\alpha\rangle \langle \pi_{X_n}^\alpha|) | \mathcal{A} \rangle \right| \\
&\leq \sum_n \sum_\alpha |\eta_{X_n}^\alpha| \left| \left( \sum_{\alpha'} \sum_{\substack{n' \neq n \\ X_n \cap X_{n'} \neq \emptyset}} \eta_{X_{n'}}^{\alpha'} \right) \right| \langle \mathcal{A} | (|\pi_{X_n}^\alpha\rangle \langle \pi_{X_n}^\alpha|) | \mathcal{A} \rangle \\
&\leq |\eta|_{\max}^2 \sum_{n,\alpha} r_{\max} c_n \langle \mathcal{A} | (|\pi_{X_n}^\alpha\rangle \langle \pi_{X_n}^\alpha|) | \mathcal{A} \rangle = |\eta|_{\max}^2 \langle \mathcal{A} | \hat{\mathbb{H}}'_+ | \mathcal{A} \rangle. \quad (\text{E.22})
\end{aligned}$$

Since  $\langle \mathcal{A} | \hat{\mathbb{H}}'_+ | \mathcal{A} \rangle \xrightarrow{N \rightarrow \infty} 0$ , by lemma E.2, the last line vanishes and we have  $|D| \xrightarrow{N \rightarrow \infty} 0$ . Now, combining everything:  $\|\hat{\mathbb{H}} | \mathcal{A} \rangle\|^2 \leq |A| + |B| + |C| + |D| \xrightarrow{N \rightarrow \infty} 0$ .  $\square$

Similarly, we can also show the presence of AQMBS states in circuit or Floquet models. More precisely, we can formulate it as the following theorem:

**Theorem E.4.** *If  $\hat{\mathbb{H}}_+ = \sum_n \hat{P}_{X_n}$  and  $\hat{\mathbb{H}}_+ | \mathcal{A} \rangle \xrightarrow{N \rightarrow \infty} 0$  then we have we have  $\hat{\mathbb{U}} | \mathcal{A} \rangle \xrightarrow{N \rightarrow \infty} | \mathcal{A} \rangle$  for any finite-depth circuit  $\hat{\mathbb{U}}$  composed of local unitary gates of the form  $\hat{U}_{X_n} = \exp(i \hat{P}_{X_n} \hat{h}_{X_n} \hat{P}_{X_n})$ , where  $\hat{h}_{X_n}$  are arbitrary Hermitian operators which act non-trivially only on the qudits in  $X_n$ .*

*Proof.* Let  $\hat{\mathbb{U}}$  be a circuit of depth  $d$ , which means that it can be expressed as a product of  $d$  layers:

$$\hat{\mathbb{U}} = \hat{\mathbb{U}}^{(d)} \hat{\mathbb{U}}^{(d-1)} \dots \hat{\mathbb{U}}^{(2)} \hat{\mathbb{U}}^{(1)}, \quad (\text{E.23})$$

where the  $\ell$ 'th layer is a product of *commuting* local unitary gates:

$$\hat{\mathbb{U}}^{(\ell)} = \prod_{n \in L^{(\ell)}} \exp(i \hat{P}_{X_n} \hat{h}_{X_n} \hat{P}_{X_n}) = \exp \left( i \sum_{n \in L^{(\ell)}} \hat{P}_{X_n} \hat{h}_{X_n} \hat{P}_{X_n} \right) = \exp \left( i \hat{\mathbb{H}}^{(\ell)} \right). \quad (\text{E.24})$$

Here we have defined the Hamiltonian  $\hat{\mathbb{H}}^{(\ell)} \equiv \sum_{n \in L^{(\ell)}} \hat{P}_{X_n} \hat{h}_{X_n} \hat{P}_{X_n}$  which generates the unitary  $\hat{\mathbb{U}}^{(\ell)}$ , and  $L^{(\ell)}$  is the subset of indices for the commuting gates in the  $\ell$ 'th layer (i.e.,  $X_n \cap X_{n'} = \emptyset$  for any  $n, n' \in L^{(\ell)}$ ).

Now consider the non-negative Hamiltonian:

$$\hat{\mathbb{H}}_+^{(\ell)} \equiv \sum_{n \in L^{(\ell)}} \hat{P}_{X_n}. \quad (\text{E.25})$$

Since we have assumed that  $\hat{\mathbb{H}}_+ | \mathcal{A} \rangle \xrightarrow{N \rightarrow \infty} 0$ , we can use Lemma E.1 and Lemma E.2 to see that:

$$\langle \mathcal{A} | \hat{\mathbb{H}}_+^{(\ell)} \hat{\mathbb{H}}_+^{(\ell)} | \mathcal{A} \rangle = \langle \mathcal{A} | \left( \sum_{n \in L^{(\ell)}} \hat{P}_{X_n} \right) | \mathcal{A} \rangle + \langle \mathcal{A} | \left( \sum_{n \neq n' \in L^{(\ell)}} \hat{P}_{X_n} \hat{P}_{X_{n'}} \right) | \mathcal{A} \rangle \quad (\text{E.26})$$

$$\xrightarrow{N \rightarrow \infty} 0, \quad (\text{E.27})$$

where the first term vanishes due to Lemma E.1 and the second term vanishes due to part 1 of Lemma E.2. Since  $\hat{\mathbb{H}}_+^{(\ell)} | \mathcal{A} \rangle \xrightarrow{N \rightarrow \infty} 0$ , by Theorem E.3 we also have  $\hat{\mathbb{H}}^{(\ell)} | \mathcal{A} \rangle \xrightarrow{N \rightarrow \infty} 0$ . This means that:

$$e^{i\hat{\mathbb{H}}^{(\ell)}} | \mathcal{A} \rangle = \sqrt{1 - |\epsilon_N^{(\ell)}|^2} | \mathcal{A} \rangle + \epsilon_N^{(\ell)} | \psi_N^{(\ell)} \rangle, \quad (\text{E.28})$$

where  $\epsilon_N^{(\ell)} \xrightarrow{N \rightarrow \infty} 0$ . So we have:

$$\hat{\mathbb{U}} | \mathcal{A} \rangle = e^{i\hat{\mathbb{H}}^{(d)}} e^{i\hat{\mathbb{H}}^{(d-1)}} \dots e^{i\hat{\mathbb{H}}^{(1)}} | \mathcal{A} \rangle \quad (\text{E.29})$$

$$\begin{aligned} &= \sqrt{1 - |\epsilon_N^{(d)}|^2} \sqrt{1 - |\epsilon_N^{(d-1)}|^2} \dots \sqrt{1 - |\epsilon_N^{(1)}|^2} | \mathcal{A} \rangle \\ &\quad + \sum_{\ell=1}^d \epsilon_N^{(\ell)} \sqrt{1 - |\epsilon_N^{(\ell-1)}|^2} \sqrt{1 - |\epsilon_N^{(\ell-2)}|^2} \\ &\quad \dots \sqrt{1 - |\epsilon_N^{(1)}|^2} e^{i\hat{\mathbb{H}}^{(d)}} e^{i\hat{\mathbb{H}}^{(d-1)}} \dots e^{i\hat{\mathbb{H}}^{(\ell+1)}} | \psi_N^{(\ell)} \rangle, \end{aligned} \quad (\text{E.30})$$

where to get from Eq. E.29 to Eq. E.30 we repeatedly use Eq. E.28. Now, if the depth  $d$  of the circuit is finite, only the first term of Eq. E.30 remains in the thermodynamic limit, and we have:

$$\hat{\mathbb{U}} | \mathcal{A} \rangle \xrightarrow{N \rightarrow \infty} | \mathcal{A} \rangle, \quad (\text{E.31})$$

as we wished to show.  $\square$

## F Correlation length for Boundary QMBS

In this section we will explicitly compute the correlation length for the state  $|\mathcal{B}^{(g)}\rangle$ . In order to compute them, we will use local  $\hat{X}_i$  operators, although  $\hat{Y}_i$  could have been chosen as well. We will consider the Pauli  $X$  operators acting on sites  $i$  and  $i + r$ , with a distance of  $r$  in between. Recalling the form of the boundary state as:

$|\mathcal{B}^{(g)}\rangle = \frac{1}{\mathcal{N}} \sum_{n=0}^{N-1} g^{-n} \hat{X}_n |0\rangle^{\otimes N}$  and using  $q = g^{-1}$  for brevity, we can see that:

$$\hat{X}_i |\mathcal{B}^{(g)}\rangle = \frac{1}{\mathcal{N}} \left[ q^i |0\rangle^{\otimes N} + \sum_{n=0, n \neq i}^{N-1} q^n \hat{X}_n |0\rangle^{\otimes N} \right] \quad (\text{F.1})$$

$$\hat{X}_{i+r} |\mathcal{B}^{(g)}\rangle = \frac{1}{\mathcal{N}} \left[ q^{i+r} |0\rangle^{\otimes N} + \sum_{n=0, n \neq i+r}^{N-1} q^n \hat{X}_n |0\rangle^{\otimes N} \right] \quad (\text{F.2})$$

$$\hat{X}_{i+r} \hat{X}_i |\mathcal{B}^{(g)}\rangle = \frac{1}{\mathcal{N}} \left[ (q^{i+r} + q^i) |0\rangle^{\otimes N} + \sum_{n=0, n \neq i, i+r}^{N-1} q^n \hat{X}_n |0\rangle^{\otimes N} \right]. \quad (\text{F.3})$$

Using the above, and the notation  $\langle \bullet \rangle \equiv \langle \mathcal{B}^{(g)} | \bullet | \mathcal{B}^{(g)} \rangle$  it immediately follows that:

$$\langle \hat{X}_i \rangle = \frac{\mathcal{N}^2 - |q|^{2i}}{\mathcal{N}^2}, \quad \langle \hat{X}_{i+r} \rangle = \frac{\mathcal{N}^2 - |q|^{2(i+r)}}{\mathcal{N}^2}, \quad \langle \hat{X}_{i+r} \hat{X}_i \rangle = \frac{\mathcal{N}^2 - |q|^{2i} - |q|^{2(i+r)}}{\mathcal{N}^2}. \quad (\text{F.4})$$

Hence, the associated correlation function is:

$$C_{i,i+r} = \langle \hat{X}_{i+r} \hat{X}_i \rangle - \langle \hat{X}_{i+r} \rangle \langle \hat{X}_i \rangle = -\frac{|q|^{(4i+2r)}}{\mathcal{N}^4} = -\frac{|g|^{-4i}}{\mathcal{N}^4} |g|^{-2r} = A_i |g|^{-2r}, \quad (\text{F.5})$$

where  $A_i$  is a site dependent constant. Now expressing the correlation function as:  $C_{i,i+r} = A_i e^{-r/\xi}$ , we can identify  $\xi = \frac{1}{2 \ln |g|}$ . Note that this agrees with the localisation length in the main text up to a factor of 2, which arises because here we used the amplitudes squared (due to the expectation values) and in the main text only the amplitudes in the state.

## G AQMBS in Momentum Space

For clarity, we reiterate the form of AQMBS states from the main text, Eq. (4.19):

$$\langle j | \mathcal{A}_k \rangle = [\mathcal{A}_k]_j = \cos\left[\left(N - j - \frac{1}{2}\right) \frac{k\pi}{N}\right], \quad (\text{G.1})$$

which is effectively a representation in discrete real space, in the  $|j\rangle \equiv \hat{X}_j |0\rangle^{\otimes N}$  basis. In this section we demonstrate that the even-labelled asymptotic QMBS have a particularly convenient representation in momentum space. First, we will relabel the states:  $\mathcal{A}_k \rightarrow \mathcal{A}_m$ , in order to use  $k$  for the momentum basis. The latter is a quantum Fourier transform of the discrete real space basis, and we will use the convention:

$$\langle k | \mathcal{A}_m \rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \langle x | \mathcal{A}_m \rangle e^{-\frac{2\pi i}{N} jk}. \quad (\text{G.2})$$

We first note that for the regular QMBS  $|\mathcal{B}^{(g=1)}\rangle$ , we have  $\langle j | \mathcal{B}^{(g=1)} \rangle = \frac{1}{\sqrt{N}}$ , and therefore we obtain a simple representation in momentum space:

$$\langle k | \mathcal{B}^{(g=1)} \rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \frac{1}{\sqrt{N}} e^{-\frac{2\pi i}{N} j k} = \frac{1}{N} \sum_{j=0}^{N-1} e^{-\frac{2\pi i}{N} j k} = \delta_{0,k}, \quad (\text{G.3})$$

where  $\delta_{0,k}$  is the Kronecker delta. Similarly, for even  $m(= 2l)$  AQMBS, we have:

$$\langle k | \mathcal{A}_m \rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \cos[(N-j-\frac{1}{2})\frac{2l\pi}{N}] e^{-\frac{2\pi i}{N} j k} \quad (\text{G.4})$$

$$= \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \frac{1}{2} [e^{i((N-j-\frac{1}{2})\frac{2l\pi}{N})} + e^{-i((N-j-\frac{1}{2})\frac{2l\pi}{N})}] e^{-\frac{2\pi i}{N} j k} \quad (\text{G.5})$$

$$= \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \frac{1}{2} [e^{i((-j-\frac{1}{2})\frac{2l\pi}{N})} + e^{i((j+\frac{1}{2})\frac{2l\pi}{N})}] e^{-\frac{2\pi i}{N} j k} \quad (\text{G.6})$$

$$= \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \frac{1}{2} [e^{i((-lj-\frac{l}{2}-jk)\frac{2\pi}{N})} + e^{i((lj+\frac{l}{2}-jk)\frac{2l\pi}{N})}] \quad (\text{G.7})$$

$$= \frac{e^{-\frac{l}{2}}}{2\sqrt{N}} \sum_{j=0}^{N-1} [e^{-i((k-N+l)\frac{2\pi j}{N})}] + \frac{e^{+\frac{l}{2}}}{2\sqrt{N}} \sum_{j=0}^{N-1} [e^{-i((k-l)\frac{2\pi j}{N})}] \quad (\text{G.8})$$

$$= \frac{1}{2\sqrt{N}} [e^{-il/2} \delta_{k,N-l} + e^{+il/2} \delta_{k,l}], \quad (\text{G.9})$$

up to a normalisation factor. The even-labelled asymptotic QMBS states are thus equal superpositions of opposite momentum states. Note that for odd-labelled  $|\mathcal{A}_m\rangle$  we do not obtain such a convenient representation, in this labelling convention.

## H Symmetry of the Model at $g = 1$

The translationally invariant Hamiltonian (or Floquet) model possesses an additional symmetry described by  $\hat{\Pi}\hat{R}$ , where the two individual operators are:

$$\begin{aligned} \hat{R} : |i_0, i_1, \dots, i_{N-2}, i_{N-1}\rangle &\rightarrow |i_{N-1}, i_{N-2}, \dots, i_1, i_0\rangle, \\ \hat{\Pi} &= e^{i\pi \sum_j |1\rangle\langle 1|_j}, \end{aligned}$$

where  $i_m \in \{0, 1\}$  denotes the computational basis state at site  $m$ .  $\hat{R}$  is the standard reflection operator, flipping the chain through its midpoint. The operator  $\hat{\Pi}$  acts exactly as the identity for computational basis states with an even number of qubits in the  $|1\rangle$  state, while an odd number of qubits in  $|1\rangle$  leads to an extra  $-1$  factor. We can thus identify  $\hat{\Pi}\hat{R}$  as a reflection transformation, adjusted by the total magnetisation of the transformed computational basis states.

We note that this symmetry appears for  $g = 1$  and at  $g = -1$  the system is invariant

under solely  $\hat{R}$ . In order to demonstrate it, we will consider a Hamiltonian of the form as described in Eq. 4.3:  $\hat{\mathbb{H}}^{(g)} = \sum_{n=0}^{N-2} \hat{H}_{n,n+1}^{(g)}(h_{n+1})$ , with:

$$\hat{H}^{(g)}(h) = h_{00} |\psi^{(g)}\rangle \langle \psi^{(g)}| + h_{11} |11\rangle \langle 11| + h_{01} |\psi^{(g)}\rangle \langle 11| + h_{01}^* |11\rangle \langle \psi^{(g)}|.$$

We note that the reflection operator  $\hat{R}$  swaps the qubits at sites  $j$  and  $N - j - 1$ . Writing out the full Hamiltonian as:

$$\hat{\mathbb{H}} = \sum_{j=0}^{N-2} \{h_{00} |\psi^{(g)}\rangle \langle \psi^{(g)}| + h_{11} |11\rangle \langle 11| + h_{01} |\psi^{(g)}\rangle \langle 11| + h_{01}^* |11\rangle \langle \psi^{(g)}|\}_{j,j+1},$$

and acting on it with the reflection operator we obtain:

$$\begin{aligned} \hat{R} \hat{\mathbb{H}} \hat{R}^\dagger &= \sum_{j=0}^{N-2} \{h_{00} |\psi^{(g)}\rangle \langle \psi^{(g)}| + h_{11} |11\rangle \langle 11| + \\ &h_{01} |\psi^{(g)}\rangle \langle 11| + h_{01}^* |11\rangle \langle \psi^{(g)}|\}_{N-j-1, N-j-2} = \\ &\sum_{j=0}^{N-2} \{h_{00} \hat{S} |\psi^{(g)}\rangle \langle \psi^{(g)}| \hat{S}^\dagger + h_{11} \hat{S} |11\rangle \langle 11| \hat{S}^\dagger + \\ &h_{01} \hat{S} |\psi^{(g)}\rangle \langle 11| \hat{S}^\dagger + h_{01}^* \hat{S} |11\rangle \langle \psi^{(g)}| \hat{S}^\dagger\}_{N-j-2, N-j-1}, \end{aligned} \quad (\text{H.1})$$

where  $\hat{S}$  denotes the 2-qubit swap operator. We also note that:

$$\begin{aligned} \hat{S} |\psi^{(g)}\rangle &= \frac{1}{\sqrt{1+|g|^2}} \hat{S}(g^* |01\rangle - |10\rangle) = \frac{1}{\sqrt{1+|g|^2}} (g^* |10\rangle - |01\rangle) = \\ &-g^* \frac{1}{\sqrt{1+|g|^2}} ((g^*)^{-1} |01\rangle - |10\rangle) = -g^* \sqrt{\frac{1+|g^{-1}|^2}{1+|g|^2}} |\psi^{(g^{-1})}\rangle, \end{aligned} \quad (\text{H.2})$$

and that  $\hat{S} |00\rangle = |00\rangle$ ,  $\hat{S} |11\rangle = |11\rangle$ . Using these and relabelling  $N - j - 2 = j'$ :

$$\begin{aligned} \hat{R} \hat{\mathbb{H}} \hat{R}^\dagger &= \sum_{j'=0}^{N-2} \{h_{00} |\psi^{(g^{-1})}\rangle \langle \psi^{(g^{-1})}| + h_{11} |11\rangle \langle 11| \\ &-g^* h_{01} \sqrt{\frac{1+|g^{-1}|^2}{1+|g|^2}} |\psi^{(g^{-1})}\rangle \langle 11| - g h_{01}^* \sqrt{\frac{1+|g^{-1}|^2}{1+|g|^2}} |11\rangle \langle \psi^{(g^{-1})}|\}_{j', j'+1}. \end{aligned} \quad (\text{H.3})$$

In the case that  $g$  has unit modulus ( $g = e^{i\phi}$ ), it further reduces to:

$$\begin{aligned} \hat{R} \hat{\mathbb{H}} \hat{R}^\dagger &= \sum_{j'=0}^{N-2} \{h_{00} |\psi^{(g^{-1})}\rangle \langle \psi^{(g^{-1})}| + h_{11} |11\rangle \langle 11| \\ &-e^{-i\phi} h_{01} |\psi^{(g^{-1})}\rangle \langle 11| - e^{i\phi} h_{01}^* |11\rangle \langle \psi^{(g^{-1})}|\}_{j', j'+1}. \end{aligned} \quad (\text{H.4})$$

For  $\phi = \pi$  we have  $g = g^{-1} = -1$  and thus obtain:

$$\begin{aligned} \hat{R} \hat{\mathbb{H}} \hat{R}^\dagger &= \sum_{j'=0}^{N-2} \{h_{00} |\psi^{(g^{-1})}\rangle \langle \psi^{(g^{-1})}| + h_{11} |11\rangle \langle 11| + \\ &\quad h_{01} |\psi^{(g)}\rangle \langle 11| + h_{01}^* |11\rangle \langle \psi^{(g)}|\}_{j', j'+1} = \hat{\mathbb{H}}, \end{aligned} \quad (\text{H.5})$$

showing that  $\hat{\mathbb{H}}$  is invariant under a reflection about the midpoint for  $g = -1$ . However, in the case that  $\phi = 0$ , we have  $g = g^{-1} = 1$ , leading to:

$$\begin{aligned} \hat{R} \hat{\mathbb{H}} \hat{R}^\dagger &= \sum_{j'=0}^{N-2} \{h_{00} |\psi^{(g^{-1})}\rangle \langle \psi^{(g^{-1})}| + h_{11} |11\rangle \langle 11| \\ &\quad - h_{01} |\psi^{(g)}\rangle \langle 11| - h_{01}^* |11\rangle \langle \psi^{(g)}|\}_{j', j'+1}. \end{aligned} \quad (\text{H.6})$$

At  $g = 1$  the system is therefore not invariant under the pure reflection. We introduce a transformation  $\hat{\Pi} = e^{i\pi \sum_j |1\rangle \langle 1|_j}$ , which acts on the relevant states as:

$$\begin{aligned} \hat{\Pi} |11\rangle \langle 11|_{j,j+1} \hat{\Pi}^\dagger &= |11\rangle \langle 11|_{j,j+1}, \\ \hat{\Pi} |\psi^{(g)}\rangle \langle \psi^{(g)}|_{j,j+1} \hat{\Pi}^\dagger &= |\psi^{(g)}\rangle \langle \psi^{(g)}|_{j,j+1}, \\ \hat{\Pi} |\psi^{(g)}\rangle \langle 11|_{j,j+1} \hat{\Pi}^\dagger &= -|\psi^{(g)}\rangle \langle 11|_{j,j+1}, \\ \hat{\Pi} |11\rangle \langle \psi^{(g)}|_{j,j+1} \hat{\Pi}^\dagger &= -|11\rangle \langle \psi^{(g)}|_{j,j+1}. \end{aligned} \quad (\text{H.7})$$

Now it is evident that at  $g = 1$  we have:

$$(\hat{\Pi} \hat{R}) \hat{\mathbb{H}} (\hat{\Pi} \hat{R})^\dagger = \hat{\mathbb{H}}, \quad (\text{H.8})$$

showing that at  $g = 1$ , the Hamiltonian is invariant under the combined action of the parity and reflection transformations. Note that, using the same arguments, one can show that the Floquet unitary  $\hat{\mathbb{U}}_F$  has the same symmetries at  $g = \pm 1$ , as it consists of the exponentiated local generators.

## I Models with exponential mid-spectrum degeneracy at the critical point

For particular choices of  $\{\hat{h}_e, \hat{h}_o\}$ , the Hamiltonian (Floquet) model will have an exponential degeneracy at  $E_\alpha(\varphi_\alpha) = 0$ , at the critical point  $g = 1$ . The number of states at  $E_\alpha = 0$  is  $2^{N/2}$ , which was observed for any instance where the local generators have the form  $\hat{h} = \alpha_x \hat{X} + \alpha_y \hat{Y}$ . One particular example of such a model is with  $\hat{h}_e = \hat{X} + 2\hat{Y}$ ,  $\hat{h}_o = 2\hat{X} + \frac{\pi}{2}\hat{Y}$ . The plots of the bipartite entanglement entropy versus the energy of the eigenstates for different system sizes are displayed in Fig. 10.

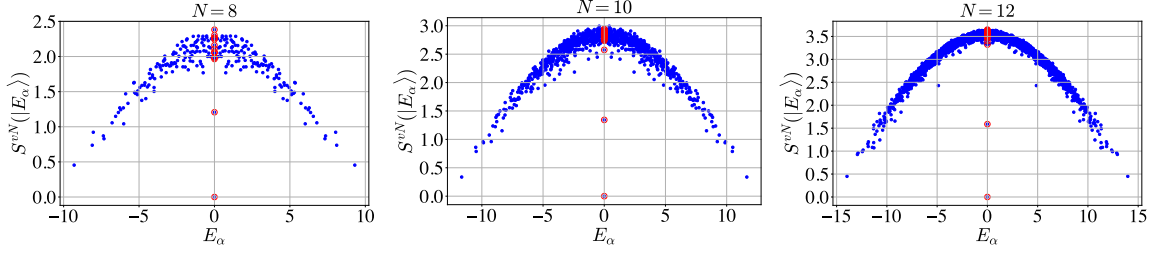


Figure I0: Bipartite entanglement entropy,  $S^{vN}$ , versus the energy  $E_\alpha$ , for the energy eigenstates  $|E_\alpha\rangle$ . We are working with the Hamiltonian model with alternating chains, where  $\hat{h}_e = \hat{X} + 2\hat{Y}$ ,  $\hat{h}_o = 2\hat{X} + \frac{\pi}{2}\hat{Y}$ . The zero energy modes are marked with red circles. We use exact diagonalisation, for system sizes  $N = 8, 10, 12$ .

## J Transition to kinetically constrained models

We note that for  $g = 0$  and  $g \rightarrow \infty$ , the considered Hamiltonian and circuit models reduce to the kinetically constrained quantum East/West models. Recalling the generic form of the used local interactions  $\hat{H}_{n,n+1}^{(g)} = \hat{P}_{n,n+1}^{(g)} \hat{h}_{n,n+1} \hat{P}_{n,n+1}^{(g)}$ , we can write  $\hat{H}_{n,n+1}^{(g)}$  as:

$$\hat{H}_{n,n+1}^{(g)} = a |\psi^{(g)}\rangle \langle \psi^{(g)}| + b |11\rangle \langle 11| + c |\psi^{(g)}\rangle \langle 11| + c^* |11\rangle \langle \psi^{(g)}|,$$

$$|\psi^{(g)}\rangle = \frac{1}{\sqrt{1+|g|^2}} (g^* |01\rangle - |10\rangle).$$

For  $g = 0$ , we obtain  $|\psi^{(g=0)}\rangle = -|10\rangle$ , such that the local interactions reduce to:  $\hat{H}_{n,n+1}^{(g=0)} = a |10\rangle \langle 10| + b |11\rangle \langle 11| - c |10\rangle \langle 11| - c^* |11\rangle \langle 10| = |1\rangle \langle 1| \otimes \hat{h}_{\text{East}}$ , with the single qubit operator  $\hat{h}_{\text{East}}$  determined by  $a, b, c$ . The model composed solely of such interactions will be an instance of so-called quantum East models, possessing kinetic constraints which lead to anomalous transport and ergodicity behaviour [194, 195].

Similarly, for  $g \rightarrow \infty$ , we obtain  $|\psi^{(g \rightarrow \infty)}\rangle = |01\rangle$ , leading to  $\hat{H}_{n,n+1}^{(g \rightarrow \infty)} = a |01\rangle \langle 01| + b |11\rangle \langle 11| + c |01\rangle \langle 11| + c^* |01\rangle \langle 10| = \hat{h}_{\text{West}} \otimes |1\rangle \langle 1|$ . A model composed only of these types of interactions will be an instance of quantum West models, also dictated by strong kinetic constraints [196].

## K Ion Trap Processor Details and Experiment execution

The device specifications of the Quantinuum H1-1 device are available online [179]. This quantum processing unit is based on trapped ions, and supports simulations on

up to 20 qubits. The native gate set is based two parameterised single qubit gates:

$$U_{1q}(\theta, \phi) = \exp[-i\frac{\theta}{2}(\cos(\phi)\hat{X} + \sin(\phi)\hat{Y})], \quad (\text{K.1})$$

$$R_z(\lambda) = \exp[-i\frac{\lambda}{2}\hat{Z}], \quad (\text{K.2})$$

and one entangling 2-qubit  $ZZ$  gate:

$$ZZ(\theta) = \exp[-i\frac{\theta}{2}\hat{Z} \otimes \hat{Z}]. \quad (\text{K.3})$$

The device supports all-to-all connectivity, allowing the execution of up to five 2-qubit gates in parallel.

The typical single qubit and 2-qubit gate error rates are  $1 \times 10^{-5}$  and  $2 \times 10^{-5}$ , respectively. The typical state-preparation and measurement (SPAM) errors are  $3 \times 10^{-5}$ . For relatively shallow circuits, the SPAM errors are the dominant sources of error, while for greater circuit depth the 2-qubit gates infidelities become the prohibitive factor.

In order to execute the model specific 2-qubit generators set to  $\hat{h}_e = \hat{X} + 2\hat{Y}$ ,  $\hat{h}_o = 3\hat{X} + \frac{\pi}{2}\hat{Y}$ , we use the gate decomposition depicted in Fig. 4.11, with the gate decomposition parameters shown in Table 1. When executing the job on the device, the single qubit operations from the different unitary blocks are converted to the native gate-set basis. The compilation was performed using pytket, version 2.7.0. Default optimisation passes were enabled when submitting jobs to the device, including single-qubit gate merging, but no depth-reducing two-qubit re-synthesis was applied. There is also a small compilation error that arises due to this decomposition, as well as the finite precision of rotation angles. All measurements were performed in the computational basis and no readout error mitigation was applied to the obtained data.

Whenever performing simulations for  $N = 8$  and  $N = 12$  (for any of the states  $\{|\mathcal{V}\rangle, |1\rangle^{\otimes N}, |\mathcal{A}_1\rangle\}$ ), we run them in parallel, as the system allows for the simultaneous usage of up to 20 qubits.

By default, the qubit register is always initialised in  $|0\rangle^{\otimes N}$ . Hence, when performing simulations for the QMBS state  $|0\rangle^{\otimes N}$ , we do not have to perform any state preparation prior to the time evolution. The probed non-scar state  $|1\rangle^{\otimes N}$  can be prepared in a straightforward way, by applying the  $\hat{X}$  (which is equivalent, up to a global phase, to  $U_{1q}(\theta = \pi, \phi = 0)$ ) operation on all of the qubits. Similarly, in order to prepare  $|L\rangle = |1\rangle \otimes |0\rangle^{\otimes N-1}$ , we apply  $\hat{X}$  on the first qubit in the chain.

Finally, in order to minimise the costs associated with performing the experiment, we have reused some data points used at  $T = 0$ . Since the circuit constructions are the same, we reused data for the following experimental data points:

1. For the  $|0\rangle^{\otimes N}$  state the data for  $N = 20$  is the same as already used for  $N = 8$  and 12 (which were run in parallel), at  $T = 0$

2. For the  $|1\rangle^{\otimes N}$  state the data for  $N = 20$  is the same as already used for  $N = 8$  and 12 (which were run in parallel), at  $T = 0$
3. For the  $|1\rangle \otimes |0\rangle^{\otimes N-1}$  state we reused the data for all  $g$  values at  $T = 0$ .

## L Angle parameters for implemented time evolution

| $g$                  | 0.5                      | $\frac{2}{3}$            | 1.0                     |
|----------------------|--------------------------|--------------------------|-------------------------|
| $\theta_1, \theta_2$ | 1.2726, 0.82774          | 1.4509, 1.0444           | 1.3782, 1.3782          |
| $\vec{\alpha}_1$     | 3.6052, $\pi/2$ , 1.0727 | 3.6052, $\pi/2$ , 4.1838 | 1.1528, 1.1584, 2.0236  |
| $\vec{\alpha}_2$     | 5.1758, 0.7434, $\pi/2$  | 5.1758, 0.8508, $3\pi/2$ | 4.2948, 1.1584, 5.1648  |
| $\vec{\alpha}_3$     | 0, $\pi/2$ , $\pi/2$     | 0, $\pi/2$ , $\pi/2$     | 0, $\pi/2$ , $\pi/2$    |
| $\vec{\alpha}_4$     | 0, $\pi/2$ , $\pi/2$     | 0, $\pi/2$ , $\pi/2$     | 0, $\pi/2$ , $\pi/2$    |
| $\vec{\alpha}_5$     | 0, 1.0727, 4.2488        | 0, 4.1833, 4.2488        | 2.6888, 1.1584, 5.1308  |
| $\vec{\alpha}_6$     | 0.8274, $\pi/2$ , 5.8198 | 2.4216, $\pi/2$ , 2.6779 | 0.45276, 1.9832, 5.1304 |
| $\theta_3, \theta_4$ | 0.95678, 0.91982         | 1.1983, 1.166            | 1.5559, 1.5559          |
| $\vec{\beta}_1$      | 1.0884, $\pi/2$ , 1.4341 | 1.0884, $\pi/2$ , 1.424  | 5.1088, 1.4377, 1.6084  |
| $\vec{\beta}_2$      | 2.6592, 1.3024, $\pi/2$  | 2.6592, 1.3526, $\pi/2$  | 1.766, 1.4622, 4.8458   |
| $\vec{\beta}_3$      | 0, $\pi/2$ , $\pi/2$     | 0, $\pi/2$ , $\pi/2$     | 0, $\pi/2$ , $\pi/2$    |
| $\vec{\beta}_4$      | 0, $\pi/2$ , $\pi/2$     | 0, $\pi/2$ , $\pi/2$     | 0, $\pi/2$ , $\pi/2$    |
| $\vec{\beta}_5$      | 0, 1.4341, 0.4823        | 0, 1.424, 0.48235        | 3.0077, 1.4622, 1.3756  |
| $\vec{\beta}_6$      | 0.2683, $\pi/2$          | 0.2182, $\pi/2$ , 2.0531 | 0.10959, 1.7039, 1.1745 |

| $g$                  | 1.5                       | 2.0                      |
|----------------------|---------------------------|--------------------------|
| $\theta_1, \theta_2$ | 1.4509, 1.0444            | 1.2726, 0.8277           |
| $\vec{\alpha}_1$     | 2.0344, 0.8508, $3\pi/2$  | 2.0344, 0.7434, $3\pi/2$ |
| $\vec{\alpha}_2$     | 0.46365, $\pi/2$ , 4.1833 | 0.4636, $\pi/2$ , 4.2143 |
| $\vec{\alpha}_3$     | 0, $\pi/2$ , $\pi/2$      | 0, $\pi/2$ , $\pi/2$     |
| $\vec{\alpha}_4$     | 0, $\pi/2$ , $\pi/2$      | 0, $\pi/2$ , $\pi/2$     |
| $\vec{\alpha}_5$     | 2.4216, $\pi/2$ , 5.8198  | 2.3142, $\pi/2$ , 5.8198 |
| $\vec{\alpha}_6$     | 0, 4.1838, 1.1071         | 0, 4.2138, 1.1071        |
| $\theta_3, \theta_4$ | 1.1983, 1.166             | 0.9568, 0.9198           |
| $\vec{\beta}_1$      | 5.8008, 1.3526, $\pi/2$   | 5.8008, 1.3024, $3\pi/2$ |
| $\vec{\beta}_2$      | 4.23, $\pi/2$ , 1.424     | 4.2298, $\pi/2$ , 4.5758 |
| $\vec{\beta}_3$      | 0, $\pi/2$ , $\pi/2$      | 0, $\pi/2$ , $\pi/2$     |
| $\vec{\beta}_4$      | 0, $\pi/2$ , $\pi/2$      | 0, $\pi/2$ , $\pi/2$     |
| $\vec{\beta}_5$      | 0.2182, $\pi/2$ , 5.1947  | 2.8732, $\pi/2$ , 2.0531 |
| $\vec{\beta}_6$      | 0, 1.424, 3.6239          | 0, 4.5757, 3.6239        |

Table 1: Gate parameters for the decomposition depicted in Fig. 4.11 b, c, for the values of  $g$  shown.

## M AQMBS State Preparation Details

We presented the asymptotic QMBS state preparation circuit structure for  $N = 8$  in the main text, Fig. 4.11. For  $N = 2^k$ ,  $k \in \mathbb{Z}_+$ ,  $k > 3$ , we can recursively construct the state preparation circuits from the  $N = 8$  case. For each subsequent  $k$ , we perform  $X$  on the first qubit, and append the entangling gates from the  $k - 1$  state preparation construction to the first  $2^{k-1}$  qubits, as well as the last  $2^{k-1}$  qubits separately. This can already be seen in the  $N = 8$  case: The  $\gamma_2, \gamma_4, \gamma_5$  gates are precisely the entangling operations required for  $N = 4$ , plus the  $X$  gate at the start.

In the case where  $N \neq 2^k$ , we can use an adapted version of the state preparation circuit for  $N' = 2^{\lceil \log_2 N \rceil}$ . This will result in a circuit depth  $\lceil \log_2 N \rceil$ , using  $N - 1$  entangling gates. For clarity, we will write the 2-local gates used for the state preparation circuits as:

$$W_{q1, q2}(\gamma_i) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\gamma_i) & -\sin(\gamma_i) & 0 \\ 0 & \sin(\gamma_i) & \cos(\gamma_i) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (\text{M.1})$$

where the q1, q2 indices denote the labels of the two qubits in the register acted on by  $W_{q1, q2}(\gamma_i)$ . Using this convention, the state preparation protocols in our work can be written out concisely, displayed below. Note that we separate the gates from different layers using “;”.

- $N = 8$ :  $X_0, W_{0,4}(\frac{\pi}{2}); W_{0,2}(0.429), W_{4,6}(1.141); W_{0,1}(0.703), W_{2,3}(0.337), W_{4,5}(1.233), W_{6,7}(0.867), Z_i$  for  $i \in \mathbb{Z}, 4 \leq i \leq 7$
- $N = 12$ :  $X_0, W_{0,6}(\frac{\pi}{2}); W_{0,3}(0.436), W_{6,9}(1.135); W_{0,2}(0.523), W_{3,5}(0.180), W_{6,8}(0.985), W_{9,11}(0.683); W_{0,1}(0.750), W_{3,4}(0.561), W_{6,7}(1.242), W_{9,10}(0.861), Z_i$  for  $i \in \mathbb{Z}, 6 \leq i \leq 11$
- $N = 16$ :  $X_0, W_{0,8}(\frac{\pi}{2}); W_{0,4}(0.438), W_{8,12}(1.133); W_{0,2}(0.704), W_{4,6}(0.370), W_{8,10}(1.201), W_{12,14}(0.867); W_{0,1}(0.766), W_{2,3}(0.720), W_{4,5}(0.639), W_{6,7}(0.326), W_{8,9}(1.245), W_{10,11}(0.931), W_{12,13}(0.851), W_{14,15}(0.805), Z_i$  for  $i \in \mathbb{Z}, 8 \leq i \leq 15$
- $N = 20$ :  $X_0, W_{0,10}(\frac{\pi}{2}); W_{0,5}(0.439), W_{10,15}(1.132); W_{0,3}(0.600), W_{5,8}(0.262), W_{10,13}(1.071), W_{15,18}(0.759); W_{0,2}(0.586), W_{3,4}(0.728), W_{5,7}(0.430), W_{8,9}(0.324), W_{10,12}(0.999), W_{13,14}(0.893), W_{15,17}(0.680), W_{18,19}(0.798); W_{0,1}(0.773), W_{5,6}(0.677), W_{10,11}(1.247), W_{15,16}(0.843), Z_i$  for  $i \in \mathbb{Z}, 10 \leq i \leq 19$

## N Distances between full Hilbert space and subspace Haar moments

First we will compute the result in Eq. (5.21), the Hilbert-Schmidt distance between  $\hat{\rho}_{\text{Haar}}^{(k=1)}$  and  $\hat{\rho}_{\mathcal{K}_\alpha - \text{Haar}}^{(k=1)}$ , using Eqs. (5.4) and (5.17). For  $k = 1$  the moments have a particularly simple form, proportional to the identity, with the subspace Haar moment constrained to its subspace:

$$\hat{\rho}_{\text{Haar}}^{(k=1)} = \frac{1}{D} \hat{\mathbb{I}}_D, \quad \hat{\rho}_{\mathcal{K}_\alpha - \text{Haar}}^{(k=1)} = \frac{1}{D_\alpha} (\hat{\mathbb{I}}_{D_\alpha} + \hat{\mathbb{O}}_{D-D_\alpha}), \quad (\text{N.1})$$

where  $D = \dim(\mathcal{H})$ ,  $D_\alpha = \dim(\mathcal{K}_\alpha)$ ,  $\hat{\mathbb{I}}_M$  is the  $M$  by  $M$  identity and  $\hat{\mathbb{O}}_M$  is the  $M$  by  $M$  zero matrix. The difference between the two is then:

$$\hat{\delta}^{(1)} = \hat{\rho}_{\text{Haar}}^{(k=1)} - \hat{\rho}_{\mathcal{K}_\alpha - \text{Haar}}^{(k=1)} = \frac{1}{D} \hat{\mathbb{I}}_{D-D_\alpha} \oplus \left( \frac{1}{D} - \frac{1}{D_\alpha} \right) \hat{\mathbb{I}}_{D_\alpha},$$

which is diagonal and Hermitian. Therefore:

$$(\hat{\delta}^{(1)})^\dagger \hat{\delta}^{(1)} = \hat{\delta}^{(1)} \hat{\delta}^{(1)} = \frac{1}{D^2} \hat{\mathbb{I}}_{D-D_\alpha} \oplus \left( \frac{1}{D} - \frac{1}{D_\alpha} \right)^2 \hat{\mathbb{I}}_{D_\alpha}.$$

Finally, using the obtained expression we get that:

$$\begin{aligned} D_{HS}(\hat{\rho}_{\text{Haar}}^{(k=1)}, \hat{\rho}_{\mathcal{K}_\alpha - \text{Haar}}^{(k=1)}) &= \text{Tr}[(\hat{\delta}^{(1)})^\dagger \hat{\delta}^{(1)}] = \text{Tr}\left[\frac{1}{D^2} \hat{\mathbb{I}}_{D-D_\alpha} \oplus \left( \frac{1}{D} - \frac{1}{D_\alpha} \right)^2 \hat{\mathbb{I}}_{D_\alpha}\right] \\ &= \frac{D - D_\alpha}{D^2} + D_\alpha \left( \frac{1}{D} - \frac{1}{D_\alpha} \right)^2 = \frac{1}{D_\alpha} - \frac{1}{D} = \frac{D - D_\alpha}{D_\alpha D}. \end{aligned} \quad (\text{N.2})$$

Note that for a QMBS model with a single scar state we have that  $D_\alpha = D - 1$  and therefore the bound reduces to  $\frac{1}{D(D+1)}$ . Since the total Hilbert space dimension is  $d^N$ , this bound is exponentially vanishing with system size. However, it is important to note that the notions of CHSE and CHSSE are not well-defined in the thermodynamic limit. This is due to the fact that the Haar measure, and consequently its moments, are not defined for an infinitely dimensional Hilbert space.

For our HSF model, the bound is also decreasing with system size, but at a much lower rate. First we note that the full Hilbert space dimension is growing more quickly with system size than the dimension of the largest subspace [due to Eq. (5.25)] and  $1/D_\alpha \gg 1/D$ . Hence, for large  $N$ , the bound reduces to  $1/D_\alpha = 1/|\mathcal{K}_0|$  for the largest subspace.

Performing the same calculation for  $k = 2$  is slightly more involved. In this case the moments are proportional to the sum of the identity and a generalised swap operator acting on two  $D_{(\alpha)}$ -dimensional Hilbert spaces for the full space (subspace) Haar measures:

$$\hat{\rho}_{\text{Haar}}^{(k=2)} = \frac{\hat{\mathbb{I}}_{D^2} + \sum_{\beta_1, \beta_2=1}^D |\beta_1, \beta_2\rangle \langle \beta_2, \beta_1|}{D(D+1)}, \quad (\text{N.3})$$

$$\hat{\rho}_{\mathcal{K}_\alpha - \text{Haar}}^{(k=2)} = \frac{\hat{\mathbb{I}}_{D_\alpha^2} + \sum_{\beta_1, \beta_2=1}^{D_\alpha} |\beta_1, \beta_2\rangle \langle \beta_2, \beta_1|}{D_\alpha(D_\alpha + 1)}, \quad (\text{N.4})$$

where we have omitted the  $\hat{\mathbb{O}}$  part in  $\hat{\rho}_{\mathcal{K}_\alpha - \text{Haar}}^{(k=2)}$  for brevity, and without loss of generality ordered the  $\beta_{1,2}$  indices such that  $|\beta_i\rangle \in \mathcal{K}_\alpha$ , if  $\beta_i \leq D_\alpha$  and  $|\beta_i\rangle \notin \mathcal{K}_\alpha$ , if  $\beta_i > D_\alpha$ . To simplify the calculation, we introduce:  $x = \frac{1}{D(D+1)}$  and  $y = \frac{1}{D(D+1)} - \frac{1}{D_\alpha(D_\alpha+1)}$ . First we note that the second contribution to the moments is the generalised swap operator acting on two  $D_\alpha$  dimensional qudits and as such has a diagonal part where  $\beta_1 = \beta_2$  and an off-diagonal part for  $\beta_1 \neq \beta_2$ . Second, when taking the difference between the two moments, the subspace Haar moment will only affect the elements where both  $\beta_1$  and  $\beta_2$  belong to  $\mathcal{K}_\alpha$ . Using these, we obtain:

$$\begin{aligned} \hat{\delta}^{(2)} &= \hat{\rho}_{\text{Haar}}^{(k=2)} - \hat{\rho}_{\mathcal{K}_\alpha - \text{Haar}}^{(k=2)} \\ &= x \sum_{\substack{\beta_1 \notin \mathcal{K}_\alpha \\ \text{or } \beta_2 \notin \mathcal{K}_\alpha}} |\beta_1, \beta_2\rangle \langle \beta_1, \beta_2| + x \sum_{\beta_1 \notin \mathcal{K}_\alpha} |\beta_1, \beta_1\rangle \langle \beta_1, \beta_1| + x \sum_{\substack{\beta_1 \notin \mathcal{K}_\alpha \\ \text{or } \beta_2 \notin \mathcal{K}_\alpha, \\ \beta_1 \neq \beta_2}} |\beta_1, \beta_2\rangle \langle \beta_2, \beta_1| \\ &\quad + y \sum_{\beta_{1,2} \in \mathcal{K}_\alpha} |\beta_1, \beta_2\rangle \langle \beta_1, \beta_2| + y \sum_{\beta_1 \in \mathcal{K}_\alpha} |\beta_1, \beta_1\rangle \langle \beta_1, \beta_1| + y \sum_{\substack{\beta_{1,2} \in \mathcal{K}_\alpha \\ \beta_1 \neq \beta_2}} |\beta_1, \beta_2\rangle \langle \beta_2, \beta_1|, \end{aligned}$$

which is Hermitian. Now:

$$\begin{aligned} (\hat{\delta}^{(2)})^\dagger \hat{\delta}^{(2)} &= \hat{\delta}^{(2)} \hat{\delta}^{(2)} \\ &= x^2 \sum_{\substack{\beta_1 \notin \mathcal{K}_\alpha \\ \text{or } \beta_2 \notin \mathcal{K}_\alpha}} |\beta_1, \beta_2\rangle \langle \beta_1, \beta_2| + 3x^2 \sum_{\beta_1 \notin \mathcal{K}_\alpha} |\beta_1, \beta_1\rangle \langle \beta_1, \beta_1| + x^2 \sum_{\substack{\beta_1 \notin \mathcal{K}_\alpha \\ \text{or } \beta_2 \notin \mathcal{K}_\alpha, \\ \beta_1 \neq \beta_2}} |\beta_1, \beta_2\rangle \langle \beta_2, \beta_1| \\ &\quad + y^2 \sum_{\beta_{1,2} \in \mathcal{K}_\alpha} |\beta_1, \beta_2\rangle \langle \beta_1, \beta_2| + 3y^2 \sum_{\beta_1 \in \mathcal{K}_\alpha} |\beta_1, \beta_1\rangle \langle \beta_1, \beta_1| + y^2 \sum_{\substack{\beta_{1,2} \in \mathcal{K}_\alpha \\ \beta_1 \neq \beta_2}} |\beta_1, \beta_2\rangle \langle \beta_2, \beta_1|, \\ &\quad + \text{traceless terms.} \end{aligned}$$

Finally, we can compute the distance:

$$\begin{aligned} D_{HS}(\hat{\rho}_{\text{Haar}}^{(k=2)}, \hat{\rho}_{\mathcal{K}_\alpha - \text{Haar}}^{(k=2)}) &= \text{Tr}[(\hat{\delta}^{(2)})^\dagger \hat{\delta}^{(2)}] = x^2(D^2 - D_\alpha^2) + 3x^2(D - D_\alpha) \\ &\quad + x^2(D^2 - D_\alpha^2 - D + D_\alpha) + y^2 D_\alpha^2 + 3y^2 D_\alpha + y^2 D_\alpha(D_\alpha - 1) \\ &= \frac{2}{D_\alpha(D_\alpha + 1)} - \frac{2}{D(D + 1)} = \frac{2D_\alpha(D_\alpha + 1) - 2D(D + 1)}{D_\alpha(D_\alpha + 1)D(D + 1)}. \quad (\text{N.5}) \end{aligned}$$

As displayed in Eq. (5.22). Similarly to the  $k = 1$  bound, we can see that this bound

will decrease with system size for both the QMBS and HSF models. For the former,  $D_\alpha = D - 1$  in the case of a single scar, and therefore the  $k = 2$  bound reduces to  $\frac{4}{(D+1)D(D-1)}$ . Since  $D = d^N$ , this bound is exponentially vanishing in system size. For the HSF model, however, this bound reduces to  $\approx \frac{2}{D_\alpha(D_\alpha+1)}$  in the large  $N$  limit where  $D \gg D_\alpha$ . Hence, this bound is also decreasing with system size  $N$ , but at a much slower rate than the bound for the QMBS model.

## O Multiple Scars in CHSE models

In the main text we have focused on the example of a single scar, but the results can be easily generalised to different numbers of scars. For this purpose, we here consider a model with  $N = 4$  qutrits ( $d = 3$ ). We again perform simulations with the brickwork model, where the 2-local unitaries have the form presented in Eq. (5.18). We use four different projectors:

$$\begin{aligned}\hat{P}_1 &= \hat{\mathbb{I}} - |00\rangle\langle 00|, \hat{P}_2 = \hat{\mathbb{I}} - |00\rangle\langle 00| - |11\rangle\langle 11| \\ \hat{P}_{\text{exp}} &= \hat{\mathbb{I}} - |00\rangle\langle 00| - |11\rangle\langle 11| - |01\rangle\langle 01| - |10\rangle\langle 10| \\ \hat{P}_{\text{lin}} &= \hat{\mathbb{I}} - |00\rangle\langle 00| - |11\rangle\langle 11| - |\Phi_-\rangle\langle \Phi_-| \\ |\Phi_-\rangle &\equiv |+-\rangle - |-+\rangle, |\pm\rangle = |0\rangle \pm |1\rangle.\end{aligned}\tag{O.1}$$

Now, using  $\hat{P}_1$  in the embedding procedure will result in a single scar,  $\hat{P}_2$  in two scars,  $\hat{P}_{\text{exp}}$  in  $2^N$  scars and  $\hat{P}_{\text{lin}}$  will give a linear number,  $N + 1$ , of scars. We perform numerical simulations for all four cases, for times up to  $10^4$  and compare the first moments to the Haar ensemble. The results are presented in Fig. O1.

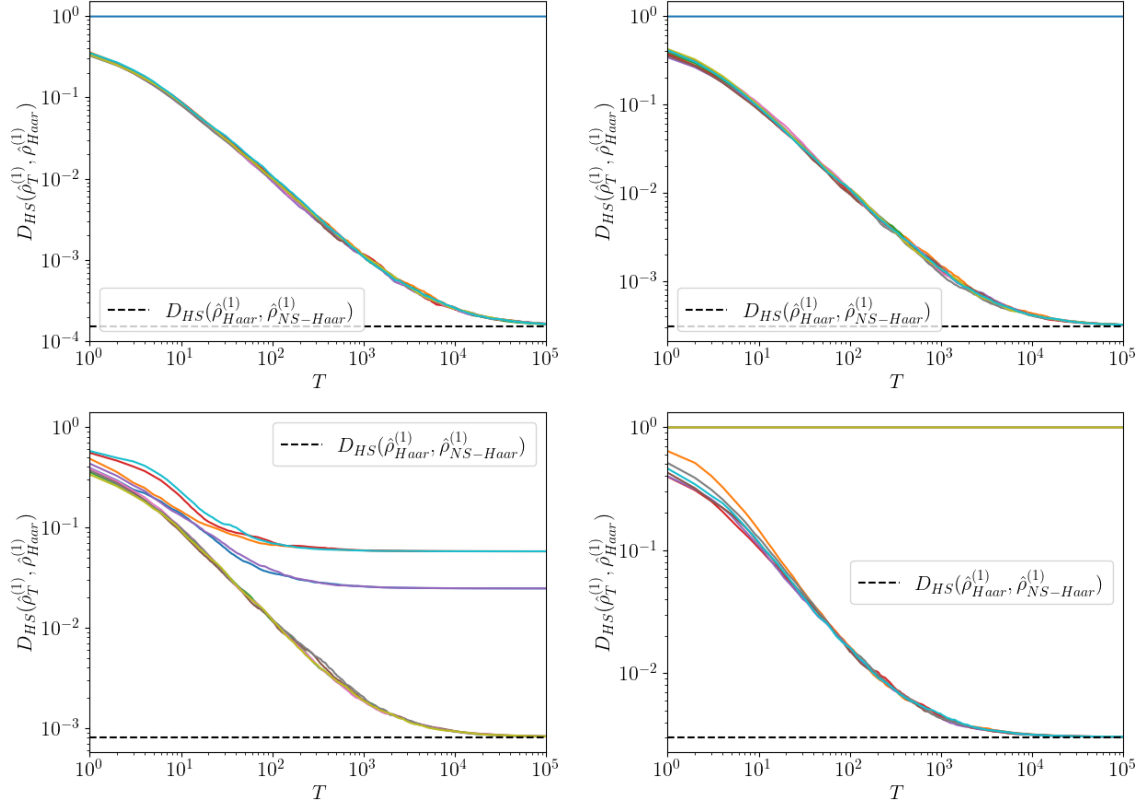


Figure O1: Results comparing the distances between the Haar ensemble and temporal ensemble, with  $k = 1$ , for the local unitaries of the form  $\hat{U} = \exp[i\hat{P}\hat{H}\hat{P}]$ . We employ four different projectors  $\hat{P}_1$ ,  $\hat{P}_2$ ,  $\hat{P}_{lin}$  and  $\hat{P}_{exp}$ , as defined in Eq. (O.1). The results for the different projectors are displayed in that respective order, top to bottom. Note that increasing the number of embedded scars also leads to a decrease in the available space to explore for the other states. Therefore, a greater number of QMBS directly leads to lower bound on the distance, which can be observed from the results. Note that in the linear scar case, the QMBS states are not computational basis states, therefore they are affected by the actions of the local unitaries.

## P Numerical procedure for computing Discretised Ensemble Entropy

In practice, we will end up with a finite set of  $T$  states in the temporal ensemble:  $|\Psi_0^T\rangle, |\Psi_1^T\rangle, |\Psi_2^T\rangle, \dots, |\Psi_{T-1}^T\rangle$ , from which we want to compute an estimator for the discretised ensemble entropy as written in Eq. (5.11). One potential approach would be to exactly separate the Hilbert space into different sections, which are occupying the same “volume,” but this would be quite difficult for arbitrarily large Hilbert space dimension. Therefore, we perform a randomised sampling procedure to estimate the probabilities,  $\{p_j^T, p_j^H\}$  corresponding to the temporal and Haar ensembles:

1. Pick  $T$ ,  $M$  and  $\epsilon$ , with  $\epsilon \in [0, 1)$ .  $T$  and  $M$  should be chosen such that  $T \gg M$ , to minimize the fluctuations in probabilities.
2. Sample  $T$  states generated by the time evolution of chosen aperiodic drive,  $\{|\Psi_i^T\rangle\}$ , as well as  $T$  states from the Haar ensemble,  $\{|\Psi_j^H\rangle\}$ .
3. Sample  $M$  states from Haar ensemble,  $\{|\Phi_j^R\rangle\}$ , which will form a set of reference states used to estimate the probabilities.
4. Remove states from the second set if their overlap is large with any other state, i.e. if  $|\langle\Phi_j|\Phi_k\rangle| > 1 - \epsilon$ , resulting in  $M'$  states. This “filtering” step is performed to ensure numerical stability.
5. Now, the filtered second set will form a reference set for the two sets of states  $\{|\Psi_i^T\rangle\}$  and  $\{|\Psi_i^H\rangle\}$ . We will “bin” the states from these two sets into  $M'$  bins, depending on which state they have the largest overlap with. That is, we will assign the  $i$ -th state state from the temporal (Haar) ensemble to the bin  $j$  which maximises the overlap  $|\langle\Psi_i^{T(H)}|\Phi_j\rangle|$ , giving two sets of counts,  $n_j^T$  and  $n_j^H$ .
6. This will give us two sets of probabilities for the temporal and discretised haar ensembles as:  $p_j^T = n_j^T / \sum_{j=1}^{M'} n_j^T$  and  $p_j^H = n_j^H / \sum_{j=1}^{M'} n_j^H$ . Using these in Eq. (5.12) we obtain the discretised ensemble entropy estimate.

Note that the DEE in Eq. (5.12) is a good proxy for the ensemble entropy in Eq. (5.11), but there are some minor distinctions. Both quantities are upper bounded by 0, but the lower bounds differ. The latter is not bounded from below and can in principle tend to  $-\infty$ , whereas the former has a finite lower bound. The lower bound can be reached in the case of a scar, which remains unchanged throughout the time evolution and thus always has maximum overlap with the same reference state,  $|\Phi_{k'}\rangle$ , for some

$k'$ . This gives the minimum attainable value for DEE:

$$\begin{aligned} & \sum_{j=1}^{M'} \delta_{j,k'} \log_2 \left( \frac{\delta_{j,k'}}{p_j^H} \right) \\ \approx & \sum_{j=1}^{M'} \delta_{j,k'} \log_2 \left( \frac{\delta_{j,k'}}{1/M'} \right) = -\log_2(M'), \end{aligned} \quad (\text{P.1})$$

where we used the approximation that all the probabilities associated with the reference ensemble are equally likely. Note that the same argument can be extended to the case where we are sampling Haar states within a larger subspace, as for example in the case of symmetries or fragmentation. In case the initial state is constrained to a  $B$ -dimensional subspace, the dynamics will generate states in the same  $\frac{B}{D}M = K$  bins. Hence, in the large  $T$  limit the DEE will converge to approximately:

$$\sum_{j=1}^K \frac{1}{K} \log_2 \left( \frac{M}{K} \right), \quad (\text{P.2})$$

which agrees with the large  $T$  limit in Fig. 5.7 for the fragmented and  $U(1)$  symmetric models.

Note that, for a finite set of bins  $M'$ , one can construct dynamics which are not really Hilbert space ergodic, but are just cyclically orbiting between some states belonging to each of the  $M'$  regions of Hilbert space. Therefore, we also perform averaging over different choices of the reference states  $\{|\Psi_j^R\rangle\}$ . In case the estimates of DEE for all different reference sets agree, one can infer that the dynamics indeed exhibits Hilbert (sub-)space ergodicity.