

TRANSIENT STABILITY OF
POWER GENERATING STATIONS
SYNCHRONOUSLY CONNECTED TO
LOW INERTIA ELECTRICAL POWER SYSTEMS



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Abstract

The ever-present requirement to decarbonize energy generation, consequently the impetus to increase energy levels from sustainable sources means that wind turbines and solar photovoltaic installations have become major contributors to the energy pool. All studies of isolated grids (e.g. Ireland) indicate that increase of these sources weakens the ability of the frequency in the transmission and distribution system to remain stable after transient disturbances because of the consequent decreased inertia in the electrical power system. The behaviour of the grid and in particular the grid frequency is manifested as an increase of the Rate of Change of Frequency (ROCOF) which may give rise to oscillations in the entire transmission system. Therefore, the goal of maximizing the renewable energy level on a transmission system *when the grid inertia is low* without compromising the safety and integrity of the existing generator assets should be attained. Hence a rigorous approach permitting quantitative understanding of low inertia grids based on dynamical models is developed. These are torsional nonlinear oscillators which may oscillate about a temporary equilibrium orientation and so may describe the response of the grid and the grid connected assets to transient frequency changes. They are inspired by previous applications in statistical mechanics, and comprise two bodies rotating about a fixed axis connected by a nonlinear spring subject to external torques. This stimulus causes a change in the angular velocity of both masses resulting in an oscillatory motion of the entire coupled system. The models are verified by comparison with actual measurements in existing power generating stations so highlighting limitations to the operational range of synchronous generators. Thus one may identify the minimum system inertia and so the limit to the quantity of grid connected non-synchronous generation. The governing nonlinear equations of motion are solved by adapting techniques similar to those developed for stochastic differential equations (v. *The Langevin Equation*, 4th edition, W.T. Coffey and Yu.P. Kalmykov, World Scientific, 2017). The results allow simple analytic description of systems with high wind electricity generation or other asynchronous renewable energy sources and so support the overall goal of decarbonizing electricity generation.

For my father

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1 INTRODUCTION

In a changing electricity market invariably demanding an increase in the energy generated by sustainable sources, wind turbines and solar photovoltaic installations will from now on represent major contributors to the electrical energy generating portfolio. However, such non carbonizing sources [1] have a direct bearing on the frequency stability within the transmission and distribution network because they do not contribute any inertia to it, implying that the rate at which grid frequency changes in the event of a loss of load or generation [2] is of critical concern in the context of these grid connected sources and loads. The current limit for the *Rate of Change of Frequency* (ROCOF) is 0.5 Hz/s which is empirically arrived at by the transmission system operator, based *both* on the recommendations of all major turbine and generator manufacturers and on everyday experience. In the light of the newly available sustainable energy generating sources, the effect of an increase of ROCOF to 1 Hz/s on the conventional larger fossil fuel fired turbo and hydro-generators run by the Irish electricity utility ESB (Electricity Supply Board) and connected to the relatively small and isolated grid of the island of Ireland must be investigated. Now, *renewable* generation sources, contrary to conventional *synchronously* grid-connected turbo generators of gas, oil or even coal fired power stations, are predominantly *asynchronously* connected to the grid. Thus, on increasing such generation the *stability of the entire electricity network is reduced* implying that the existing conventional power generation units are also affected. This phenomenon is manifested by an increase in electrical power oscillations of conventional units and ensuing tripping of some larger units occurring during transient events at times of high wind generation. Furthermore, other transient events like a *sudden change in the generation to load balance* following a trip of a generating unit, disconnection of a large load or during a system separation event typically manifest themselves as a grid frequency drop, rise or even oscillation. Besides affecting the frequency of the grid, such Rate of Change of Frequency (ROCOF) events may have adverse effects on the mechanical integrity of various generator components, ultimately leading to fatigue damage of these due to frequent exposure to torsional oscillations. *Historically*, most dynamical models used to describe generators connected to large grid systems, and in consequence their equations of motion, assume that a grid has *infinite inertia* [3]. However, some recent measurements cannot be explained by such a simplified formalism as generator-grid feedback is ignored. Hence, new dynamical methods must be developed taking account of the *finite inertia* of a grid with *high renewable penetration*. Here, to quantify the influence of potentially severe ROCOF events

on both the operation and the integrity of conventional generating units, entirely new dynamical models (in this context) involving torsional anharmonic oscillation about a temporary equilibrium orientation (rather than the fixed one hitherto assumed) and so automatically incorporating finite grid inertia (loosely known as itinerant oscillators because the centre of oscillation is moving) will be used to calculate the ensuing electrical and mechanical response of the generator sets. These will be verified by experimental observations available at actual power generating stations within the island of Ireland.

1.1 THEORY AND PRACTICE

Initially localised, rapidly changing time varying disturbances when suddenly applied to a transmission system (e.g., trip of a generator or disconnection of a part of the distribution network) invariably cause a change in frequency in the *overall* transmission and distribution system. This event occurs because the ensuing transient responses affect the electromagnetic torque in the air gap of grid connected synchronous generators, inducing *inter alia* a rotor angle oscillation resulting in mechanical torsional oscillations in the shaft line of the turbo or hydro-generator and turbine accompanied by electrical power output oscillations. Now, much research has focussed on disturbances due to three-phase faults [4] or network switching events in *large* grid systems [5] [6] (e.g., continental Europe or north America) which typically have *sufficient inertia* to limit ROCOF [7] to values not considered a risk to the generating plant. Therefore, to initially quantify the influence of *potentially stronger* ROCOF events in *isolated* grids on the operation and the integrity of a turbo [8] or hydro-set, one must precisely define the ROCOF and so determine the ensuing electrical and mechanical response of the sets in order to assess the consequences for individual components. During a typical system perturbation, a sudden *imbalance* occurs between the *generated electrical power* P_g and the *electrical power consumption* P_l , which can be determined by [9], here shown as the change of the kinetic energy E_{kin} of the grid with $E_{kin} = \frac{1}{2} J_{grid} \omega_{grid}^2$ with J_{grid} being the inertia of the grid and $\omega_{grid}(t)$ its angular velocity:

$$P_g - P_l = \frac{d\left(\frac{1}{2} J_{grid} \omega_{grid}^2\right)}{dt} \quad (1.1)$$

Also, by defining the inertia time constant H_{gen} for a turbine-generator-shaft as the ratio of its kinetic energy to the apparent load of the unit we have with J_{gen} being the inertia of a

generator, $\omega_{grid,0}$ is the nominal system frequency and S_{gen} the apparent load of the generator we have;

$$H_{gen} = \frac{\frac{1}{2} J_{gen} \frac{\omega_{grid,0}^2}{p^2}}{S_{gen}} \quad (1.2)$$

Since all the turbo-generators used in Ireland have only one pole pair, $p=1$, Eq.(1.2) can be simplified to:

$$H_{gen} = \frac{J_{gen} \omega_{grid,0}^2}{2S_{gen}} \quad (1.3)$$

assuming that the generators follow the same grid frequency $\omega_{grid,0} = \omega_{grid}$:

$$H_{gen} = \frac{J_{gen} \omega_{grid}^2}{2S_{gen}} \quad (1.4)$$

Converting Eq.(1.1) to per unit values by dividing with the system apparent load S_{grid} and using Eq.(1.2) with $H_{grid} S_{grid} = \sum_i H_{geni} S_{geni}$ we have after the differentiation:

$$\bar{P}_g - \bar{P}_l = 2 \sum_i \frac{H_{geni} S_{geni}}{S_{grid}} \bar{\omega}_{grid} \frac{d\bar{\omega}_{grid}}{dt} \quad (1.5)$$

and

$$\bar{P}_g - \bar{P}_l = 2H_{grid} \bar{\omega}_{grid} \frac{d\bar{\omega}_{grid}}{dt} \quad (1.6)$$

Assuming that, the unitized grid velocity $\bar{\omega}_{grid} \approx 1$ and with $\omega = 2\pi f$, we have the rate of change of frequency as:

$$\boxed{\frac{d\bar{f}_{grid}}{dt} = \frac{\bar{P}_g - \bar{P}_l}{2H_{grid}}} \quad (1.7)$$

Past experience in the island of Ireland, and of the Irish grid in particular, shows the rate of change of frequency following Eq.(1.7) accurately and that the graph is smooth.

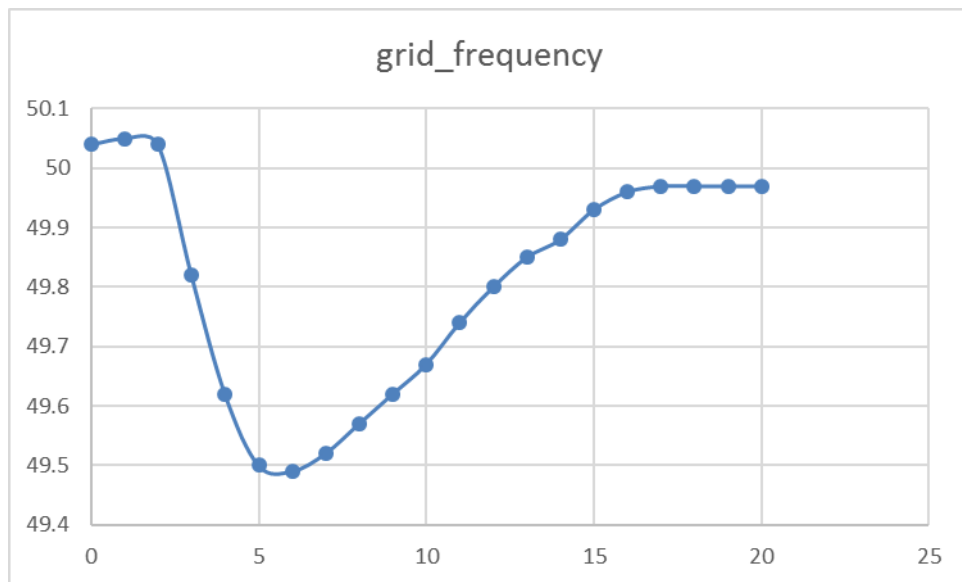


Figure 1-1: Frequency Drop with ROCOF of 0.17Hz/s
 Vertical Axis: Grid Frequency in Hz
 Horizontal Axis: Time in seconds

However, an event experienced in a large gas fired generating station (see Figure 1-2) during the spring of 2013 (see Figure 1-3) triggered an interest in understanding the interaction between a generating station and the grid because it did not follow this formalism but instead showed an interaction between the electrical power output of the generating unit and the grid frequency.



Figure 1-2: 410MW Combined Cycle Gas Turbine, Dublin Bay Power

Furthermore, the power oscillations should have been decreasing but they increased. This event constitutes the starting point of the research described in this thesis.

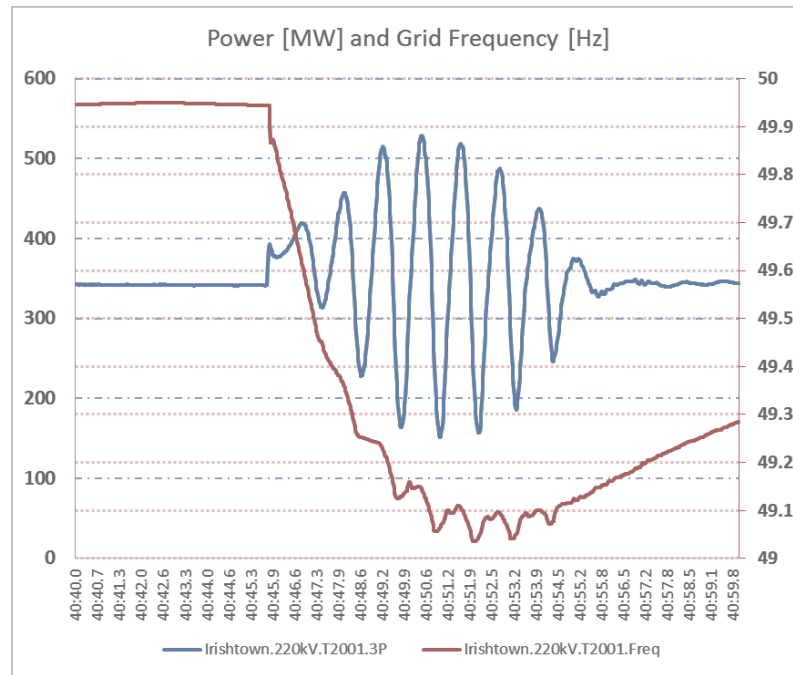


Figure 1-3 Historical event, Power output and Grid frequency

Left scale: Power (blue graph), Right: Grid frequency (red graph)

Output of a 400MW synchronous generator as a response of a real ROCOF event, real time in [min:sec].

Historically, such perturbation of a generator would have been described by models describing the motion of a sphere oscillating in a bowl (see [10, p. 459] or [11, p. 942]).

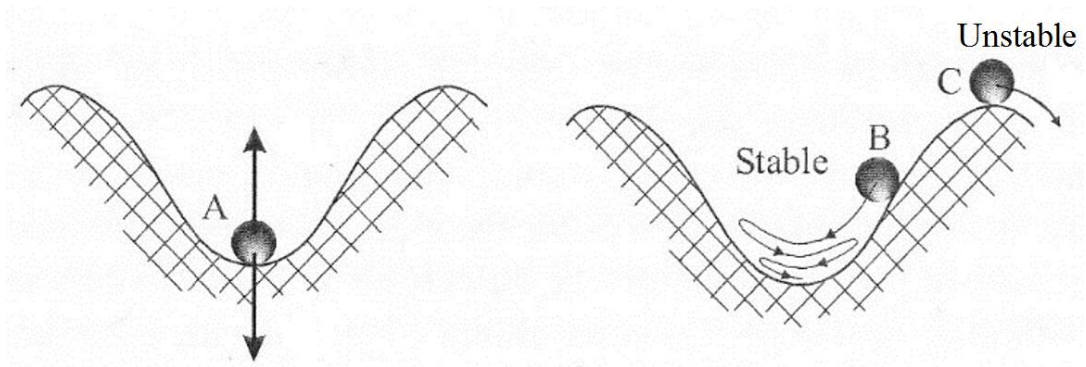


Figure 1-4: Generator states, A steady state, B perturbed, picture copied from: [10]

However, this simple mechanical analogue would neither allow the description of an increasing power output oscillation, nor an oscillating frequency. It would *need* to describe, besides the motion of the spherical particle in a potential well, *also* the motion of the well itself.

Intuitively one could just take the simplified illustration of balancing generation and consumption of electrical power.

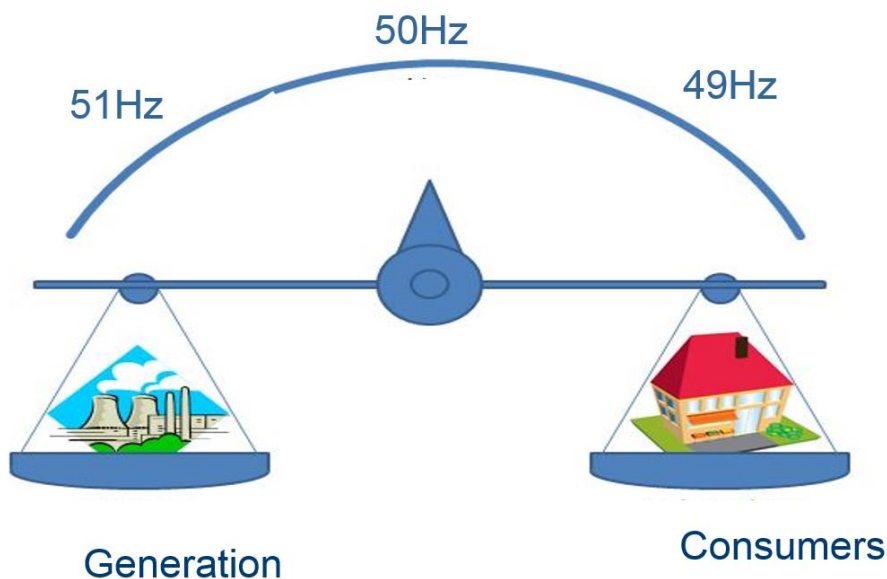


Figure 1-5: Equilibrium of Frequency

The above figure (Figure 1-1) illustrates that when the generation and consumption of electricity are equal then the frequency remains at 50Hz, in case of the US it would be 60Hz. If the weighing scale is a real scale then it becomes obvious that the movement of the scale and the speed the scale would leave its equilibrium does not depend only of the difference between the two sides in weight but depends also on the absolute mass of each side, for example if the scale would have 1 kg on one side and 2 kg on the other, it would move faster than having on one side 1000 kg and on the other 1001 kg. This phenomenon and further analysis of published works on this subject (e.g., Ref. [5]) will facilitate a better understanding and offering a critique of the new definition of ROCOF, introduced to allow the operation of the Irish Transmission System with lower inertia, for the Irish Grid Code, viz., “the generator must remain synchronised to the transmission system for a rate of change of frequency of up to 1 Hz per second averaged over 500 milliseconds” [12]. As this coarse-grained definition has potentially serious implications for plant integrity, the assessment of risk to generating units shall be determined by suitably adapting methods of stochastic dynamics (which are essentially similar to the itinerant oscillator model for both translational and for fixed axis rotational Brownian motion [13] [14] [15] [16] [17] [18] [19] [20] [21] of a rigid charged rotator in a cage of rotating neighbours) to analyse the expected frequency traces, reinforced via *MATLAB*[®] models to create such signals. Furthermore, to determine the impact of *different* ROCOF values on *specific* machines, calculations will be undertaken using simplified mathematical models of turbines and

generators subjected to sudden disturbances. This will complement assessment of the risks to the power generating equipment based on both *MATLAB*[®]-*SIMPOWERSYSTEMS*[®] model studies and analytical dynamics models.

1.2 ROCOF DEFINITION

The starting point of our analysis will be the proposed new definition of ROCOF for the Irish Grid Code stating that the generator must *remain synchronised to the Transmission System for a Rate of Change of Frequency of up to 1 Hz per second averaged over 500 milliseconds* [22]. However, such a *coarse-grained* definition has potentially serious implications for the assessment of risk to generating units as indicated below where we graphically illustrate the different values of ROCOF which can be calculated from one and the same actually occurring frequency transient, depending on the time window chosen for Δt . Further, in this thesis, in order to facilitate analysis of the impact of different ROCOF values on *specific* machines, simulations and calculations will be undertaken using simplified mathematical models of turbines and generators subjected to various disturbances. This procedure will allow an assessment of the risks to the power generating equipment which will be outlined based on the results.

1.3 FINE AND COARSE-GRAINED DEFINITIONS OF ROCOF

Two definitions of ROCOF are commonly used. Fundamentally an **Average Change** is defined in terms of infinitesimals Δf , Δt of frequency and time as proposed in the Irish Grid Code:

$$ROCOF = \frac{\Delta f}{\Delta t} \quad (1.8)$$

whereas an **Instantaneous Change** is defined via the time derivative of the frequency:

$$ROCOF = \frac{df}{dt} \quad (1.9)$$

as shown below, in Figure 1-6.

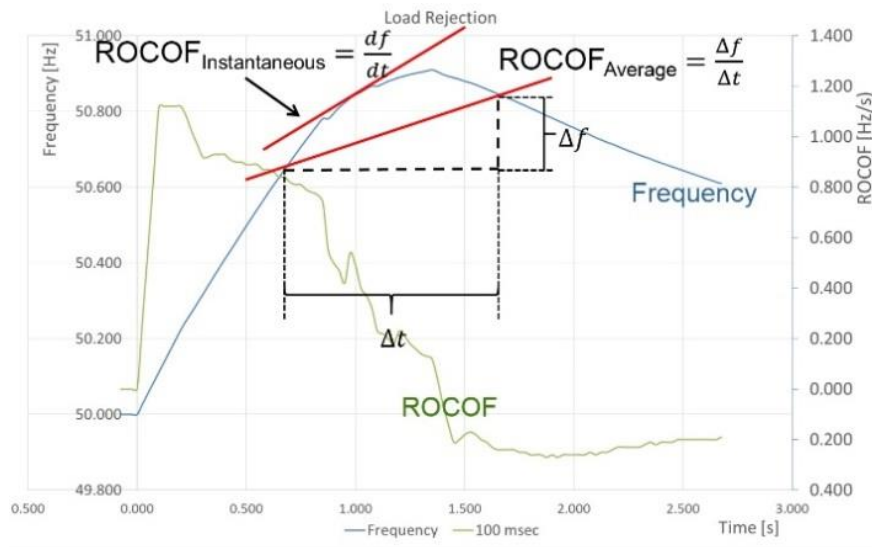


Figure 1-6 Sample Plot of a frequency-rise event with ROCOF definitions

Now, by taking a sample frequency trace and plotting the *average* ROCOF traces with values of $\Delta t=50\text{ms}$, 100ms and 500ms , marked differences become evident (see also [23]). It appears that the *fine grained* ROCOF plots with $\Delta t=50\text{ms}$ and 100ms are almost identical whereas in contrast the *coarse grained* ROCOF plot with $\Delta t=500\text{ms}$ differs substantially from these.

For example, considering the maximum ROCOF value in the plots, the fine-grained plots show a value 15% higher than the coarse-grained ones as shown in Figure 1-7.

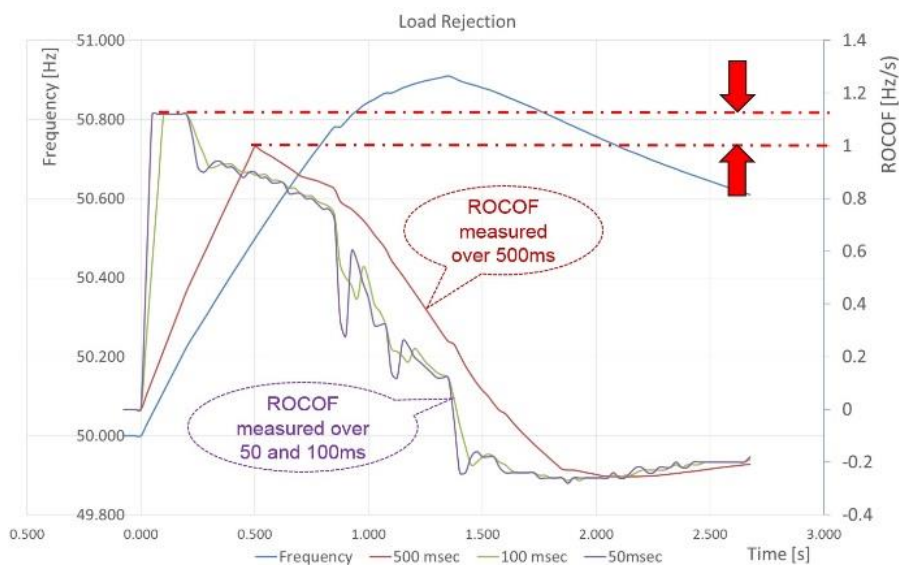


Figure 1-7 Increase of ROCOF with larger time window.
ROCOF plots with $\Delta t=50\text{ms}$, $\Delta t=100\text{ms}$ and $\Delta t=500\text{ms}$

Furthermore, on analysing a sample ROCOF event, the duration (with a value of e.g. $\geq 0.8\text{Hz/s}$) with the fine-grained plot shows an increase of around 30% over the coarse-grained definition as shown in Figure 1-8.

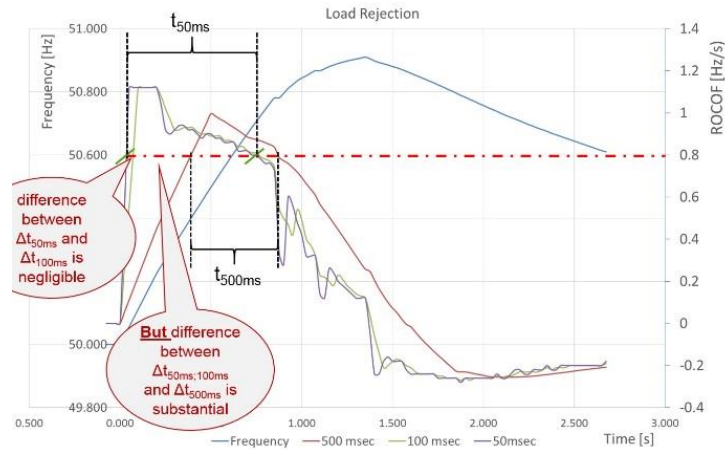


Figure 1-8 Frequency rise trace with examples of ROCOF with varying Δt
ROCOF plots with $\Delta t=50\text{ms}$, $\Delta t=100\text{ms}$ and $\Delta t=500\text{ms}$

Clearly, the plots indicate that an inadequate definition may underestimate the duration and magnitude of the ROCOF event and thus exacerbate the risks to connected machinery. By using a (possibly) inappropriate average rate, the disturbance will be underestimated and consequently appear much smaller than which the generating plant will *actually experience* in practice. 90 events in the period from October 2010 till May 2013 [24] were analysed and Figure 1-9 shows the maximum ROCOF values experienced during these events when measured over a 100 and 500ms window.

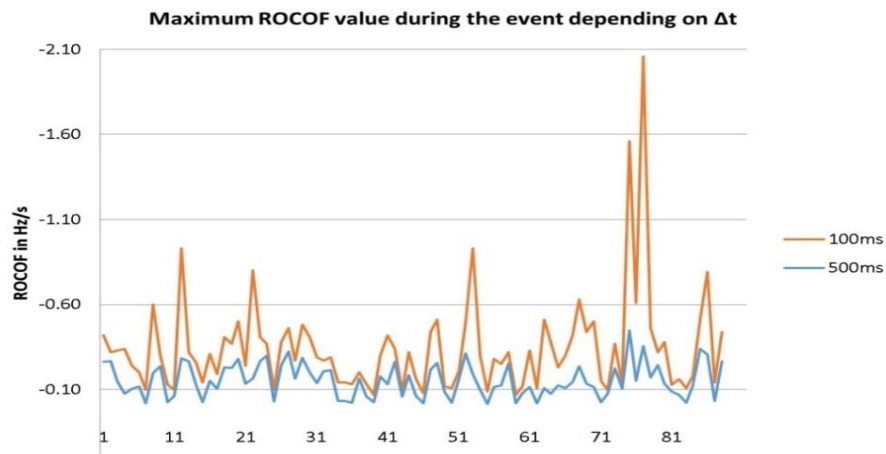


Figure 1-9 Illustration of maximum ROCOF
90 events between 2010 and 2013 were analyzed with $\Delta t=100\text{ms}$ and $\Delta t=500\text{ms}$

The consequences of an inadequate definition are potentially very serious for a turbo-set because its stability and overall integrity depends both on the maximum value [25] of ROCOF and the duration of higher ROCOF values as well known in the industry. An inappropriate definition will also lead to consequences when specifying ROCOF capabilities to manufacturers to deliver compliant equipment or to assess existing equipment for compliance. This is particularly topical in relation to the ongoing drafting of new network codes under the European Union's 3rd Energy package [26] to enhance market legislation. The implications for the turbo-set also depend on several other factors ultimately involving ROCOF. Paramount amongst these is the electromagnetic torque in the air gap of the generator. This air gap torque is in equilibrium with the mechanical torque applied by the turbine and it also depends on the stator current as well as the rotor current. Now, especially at low loads and with the generator leading, (operation in under-excited mode) the air gap torque reduces to values such that if a sufficiently large ROCOF event occurs, the generator may exhibit *large oscillations* [27] or even a *pole slip*.

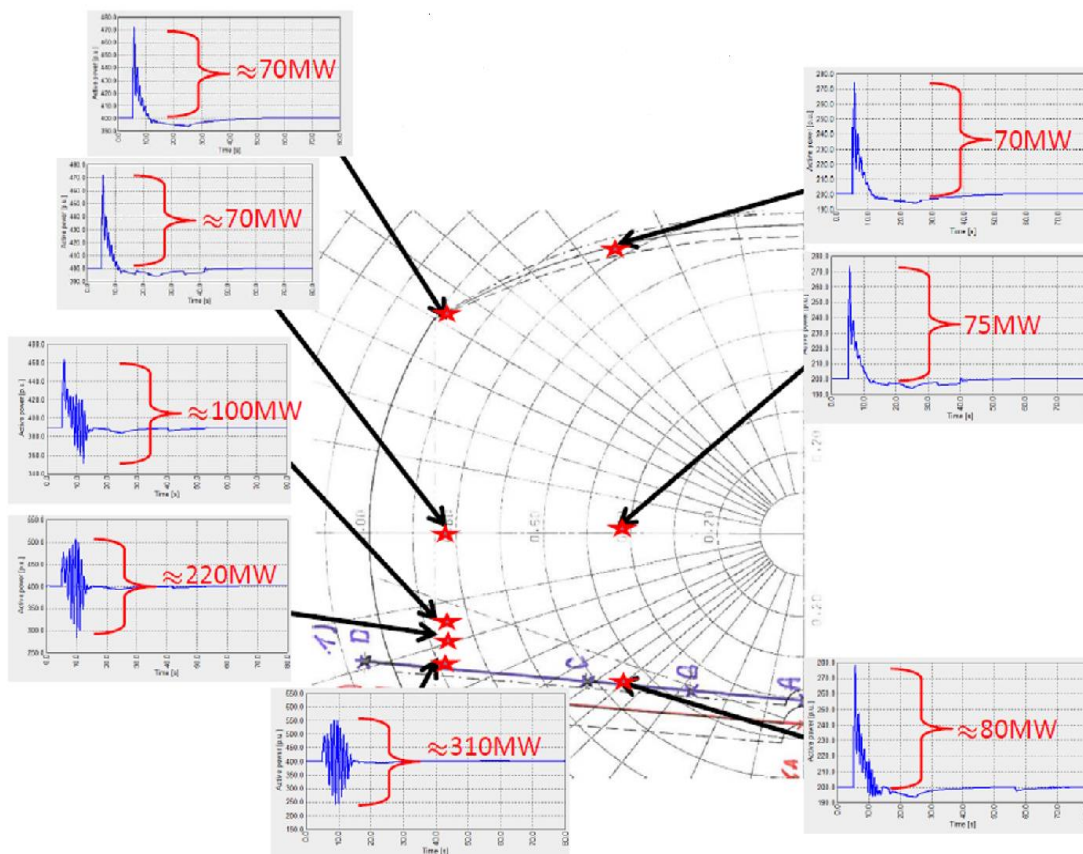


Figure 1-10, Power oscillations after frequency event

1.4 TECHNICAL RISKS TO A POWER GENERATING STATION

1.4.1 Controllers and Operational Issues

Operational events such as those experienced during a frequency change [28] in the grid require a very fast response from plant control systems [29] [30]. Demands on the control system will be exacerbated when ROCOF is larger because the rapid change of the grid frequency also directly influences the underlying process. Clearly various power generation technologies are subjected to different operational phenomena during rapid transients such as those experienced during ROCOF events. Examples of these occur in combined cycle and open cycle gas turbine technology, where the electrical power output of the plant reduces with the angular velocity of the gas turbine rotor and hence the electrical frequency. Here the volumetric flow through the compressor reduces as the rotor slows down leading to an increase in turbine inlet temperature and changes in the air to fuel ratio in the combustion system. These have the potential for undesirable operational events such as turbine over temperature or flame blow out. Other examples of potential unwanted operational phenomena include hydraulic transients in the run of the river or pumped storage power stations, associated with changes in flow. Therefore, undesirable pressure oscillations may result, leading to pressure rises and consequential breach of design limits of mechanical components giving rise to potential rupture of metallic penstocks or mechanical failures of the turbine and its subcomponents. Particularly plants with longer penstocks or tail race tunnels are more susceptible to such events [31, p. 544]. For all plant, high ROCOF events cause rapid reduction in the rotational speed of synchronous plant. Thus, as the shaft slows down, kinetic energy is ultimately translated into electrical energy via the turbine and generator shaft, causing a higher short-term peak in electrical output power. The consequences of situations such those illustrated above is that *significant* additional demands are placed on plant control systems during events with high ROCOF values. Plant controllers must be designed so, as to accommodate an increased governor reaction and, also to compensate for high speed transient effects within the process itself. Failure to adequately control such extreme events can lead to automatic plant protective actions and to cascade tripping of generation plant. Thus, speed and load controllers as well as voltage controllers of all rotating plant must be reviewed to determine if they can respond to a stronger ROCOF event. Further implications of an extreme ROCOF event are as follows:

1.4.2 Auxiliary Plant

It is not known if auxiliary plant [32] containing high inertia motors such as boiler feed pumps, mills, fans or gas compressors can respond in a manner such that they continue to operate during a pronounced ROCOF event. Operational behaviour of these devices will be affected by both changes in voltage and in frequency [33]. Each individual item of plant must be analysed to determine if any implications due to a powerful ROCOF event exist. High speed bus transfers will also need to be reviewed where applicable.

1.4.3 Protection Devices

Generator Protection [34]: The introduction of so called ROCOF relays should also be investigated. In small island grids both ROCOF and phase shift relays are used to determine potentially high ROCOF values possibly impacting on the integrity of individual components within the turbine and generator. However, the currently available ROCOF relays in the market are not built to the quality standards normally specified for larger generation units. This is true, particularly in relation to redundancy and failure rates of which no experience exists within generation utilities. Regarding existing generator protection functions [35] [36] there is a need to re-calculate and recalibrate the settings, such as the pole slip protection function.

Turbine Protection: Some turbine over-speed protection systems have an embedded logic that anticipates over-speed based on rate of speed change. It is unclear without further study whether increasing ROCOF limits will cause these protection systems to inadvertently trip the unit. A rapid fall in the electrical power output due to power swings caused by ROCOF could falsely be interpreted as due to the opening of a remote breaker. In particular, the load controller of the turbine closes its associated control valves during such events and additional generation loss would therefore be experienced.

Excitation Systems: In the ESB power generation fleet, various technologies are used to control the voltage of generators. In particular mechanical rolling sector voltage regulators as well as numerical multichannel voltage regulators are used. Irrespective of the basic differences in the design, the behaviour of the various automatic voltage regulators (AVR) may be significantly different. For example, for historical reasons, some units lack any over and under-excitation limiters. Others may have different types of power system stabilizer circuits. In addition, the tuning of these devices may be different *which would never be a reason for concern at relatively low ROCOF values*. Therefore, it follows that the dynamical behaviour of a power system stabiliser and more generally the generator and

the entire turbo-set will need to be evaluated to determine the response during a ROCOF event at the higher limit value.

1.4.4 Mechanical Integrity

A typical steam or gas turbine-generator rotor system contains a number of relatively stiff regions [37]. The main bodies of the turbine rotors support the turbine wheels and blades as well as the part of the generator rotor length containing the winding axial slots. These relatively stiff rotor regions are usually connected by, comparatively speaking, flexible shafts. Now the turbo-generator generates electrical power when a force is applied by the driving medium acting on the turbine blades. This in turn impacts a torque to the turbine generator shaft which is resisted by the electromagnetic air-gap torque in the generator. These applied torques cause a static torsional displacement [38] [39] along the entire length of the turbine and generator shaft so that one end is twisted in the opposite direction relative to the other. However, during a ROCOF event the electromagnetic torque in the air gap of the generator will suddenly alter, whereas the mechanical torque applied to the turbine shaft via the forces acting on the turbine blades [40] cannot alter so rapidly. This *torque delay* phenomenon is due both to the delay in reaction of the control systems and the higher inertia of the turbine/compressor rotors as well as the steam or water characteristics of the process. This *variation* in torque causes a transient change of the torsional displacement, with consequent additional twisting of the shaft line [41, p. 224] connecting turbine and generator. Furthermore, in certain cases, the turbine-generator shaft system will *torsionally oscillate* following a transient event, giving rise to *alternating twisting and untwisting motions* of the turbine-generator shafts. However, due to the relatively light mechanical damping of turbine-generator shaft systems [42], we may ultimately have tens of seconds of torsional oscillation following high ROCOF events. In addition, multiple events may over a prolonged period lead to *fatigue damage of components* [43] [44] [45]. Thus, detailed analysis will be required to determine whether any impact on the integrity of the various components exists. An initial assessment of components at risk in gas and steam turbine-generator rotor systems could be couched as follows:

Couplings: In addition to the above-mentioned failure mechanisms, coupling bolts may either shear or otherwise accumulate fatigue damage due to high levels of transient torques during a ROCOF event. For example, if a coupling is shrunk onto a shaft it may slip during an extreme ROCOF event resulting in fretting damage which can significantly reduce the fatigue life of the coupling and shaft. Cracks can also be initiated and grow at these

locations with subsequent propagation due to either normal or abnormal loading during other transient events.

Rotors and Shafts: Generally, the main body regions of an individual rotor have significantly larger diameter sections than the rotor extensions at each end. On turbine-generators these shaft extensions often contain seals and bearing journals with *abrupt* changes of diameter possibly terminating in integral or shrunk-on couplings. These shaft extensions are *very flexible* relative to the main body regions. Therefore, the *torsional stiffness characteristics* of a turbine-generator are dominated by the relatively low stiffness values of the spans between the main rotor bodies. Thus, during a ROCOF event, these rotor extensions will tend to twist relative to the much stiffer main body regions. Hence, changes of section or keyways will act as stress raisers and initiation of cracking in these locations may be consequently observed [46].

Turbine Blades: In addition to shaft torsional oscillations, the last few stages of the blades of a low-pressure steam turbine rotor can, if excited, execute coupled vibrations resulting in significant motion of the low-pressure rotor elements at the axial location where the blades are attached.

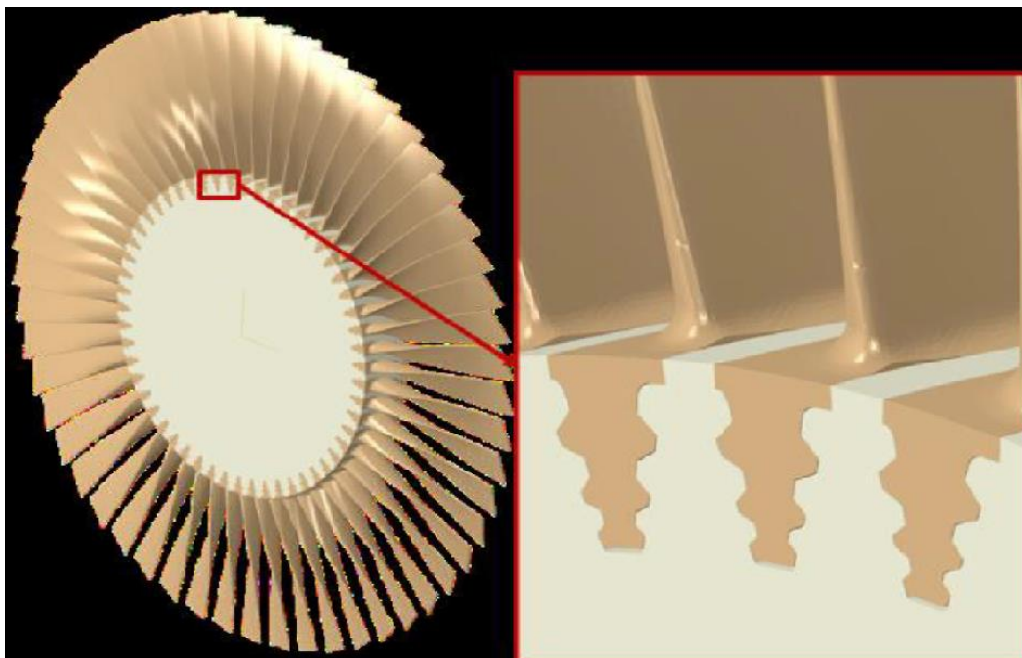


Figure 1-11: CAD models of the last stage blades [47]

In particular, if the natural frequency of the low pressure (LP) shaft system mode has a frequency sufficiently close to the natural frequency of a blade, significant vibration

coupling can occur. Thus, the shafts and blades will vibrate in unison. Hence the stresses at the blade root or in blade attachment features to the wheels can become large, possibly resulting in fatigue cracking of the blades at these locations.

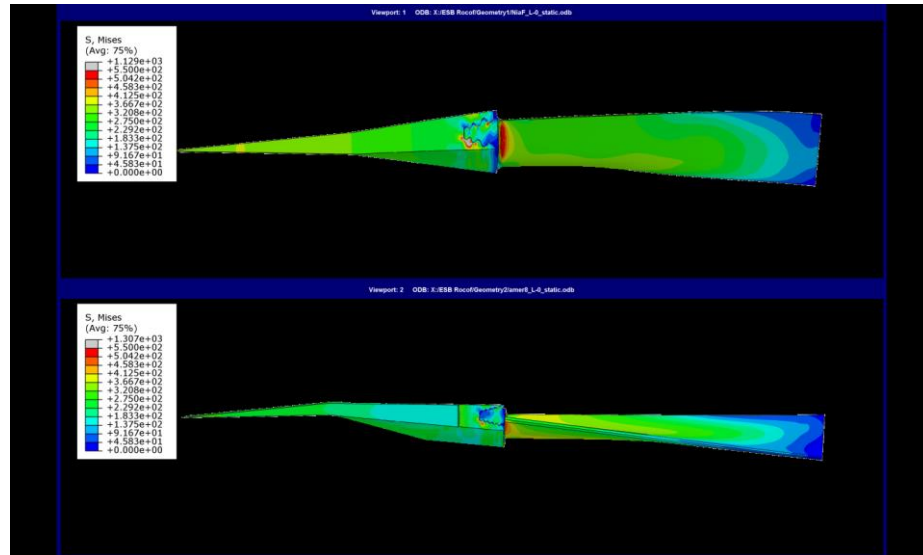


Figure 1-12: Stress distribution at nominal speed [47]

Therefore, more information concerning the nature of the ROCOF torque input waveform is needed in order to assess the risk of blade damage due to such coupled vibrations.

Retaining Rings: The retaining rings on turbo-generators have a *shrink fit surface* at the inner diameter of the in-board end which is tightly shrunk onto a matching circumferential surface at the end of the generator rotor body. Relative motion may occur between the retaining ring and the rotor during a ROCOF event possibly resulting in scoring of the shrink fit surfaces of the retaining rings.

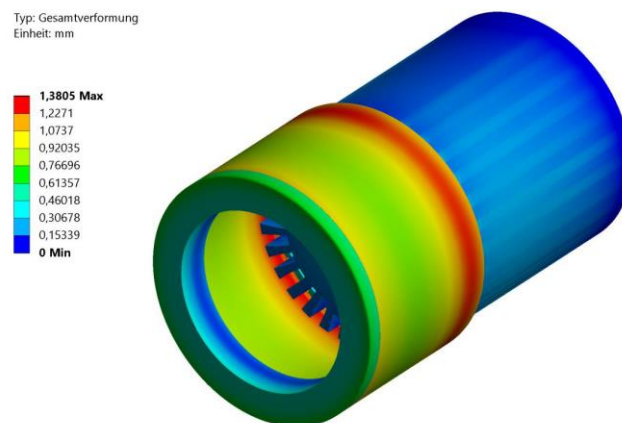


Figure 1-13, Deformation of retaining ring at nominal speed [47]

Since the shrink fit locations of the retaining rings are highly stressed, the resultant scoring of the surface may act as a stress raiser, thus cracks may again initiate and grow in this location.

End Windings: There may be a risk of damage to end windings due to stator-current increases within the ROCOF context. Changes to the air gap torque and in general changes to the electromagnetic field in the end winding area result in an increase of forces on the generator stator end-windings. As the end-winding basket is the weakest area of the generator stator these forces and their implications will need to be reviewed.

Gas Turbine Issues: Modern gas turbine compressors are susceptible to damage during disturbance events. For example, there may be a risk of rubbing of compressor blades during a ROCOF event. Also, the implications of a ROCOF event on the *interval between overhauls* would need to be determined. These issues should be investigated in conjunction with the manufacturers for each type of gas turbine.

1.4.5 Analysis of Potential Consequences:

The potential consequences of the issues as described above can be broadly divided into two categories.

The first category may be described as *operational* consequences, whereby a unit either fails to deliver the required response during a severe ROCOF event, or indeed trips, leading to a further loss of electrical power generation in the system with the potential to initiate further cascade tripping events, leading to load shedding, system islanding or an entire system black-out.

The second category can be broadly described as *mechanical integrity* issues. Due to mechanisms such as those already described, ROCOF events also have the potential to result in reduced component life, decreased overhaul intervals, increased inspection requirements, or in the worst case, catastrophic failures. These can happen as a consequence of either a *single* ROCOF event or *cumulative* damage caused by a series of such events. Potential also exists for consequential machinery damage, forced plant outages and injury to personnel.

The likelihood of such events cannot be determined without further analysis; this is also a function of the nature of the ROCOF events and the regularity of their occurrence. The usage of a coarse-grained definition in the ROCOF context exposes the conventional generating units to increased mechanical and electrical stresses. Though adequate modelling is required to understand the interaction between the generator and the turbine, it is also of crucial importance to include in the modelling the interaction between generating plant and the grid, especially in a small isolated grid as in the island of Ireland. As a further step, various electrical measurements should be made in order to determine the viability and adequacy of the models used, viz the computer models and the mathematical models.

1.5 FURTHER STEPS AND LAYOUT OF THE THESIS

The thesis is structured as follows, first starting with modelling a generating station in *MATLAB*[®] whereby the transient behavior of the unit connected to an infinite grid [48] (See Figure 1-14) is modelled. This allows one to identify specific modelling requirements, optimize modelling techniques and is the fastest way to gain an understanding of how a generating unit reacts to a frequency event as historical events are also accessible.



Figure 1-14: Single Machine Infinite Grid representation

As second step the generator is modelled in a multi-machine environment embedded in a near reality transmission system model (see Figure 1-15). Hence, all the load is lumped and the generators are individually connected to the transmission system. The model which represents a near to reality setup will allow one to reproduce transient disturbances as experienced in an actual ROCOF event.

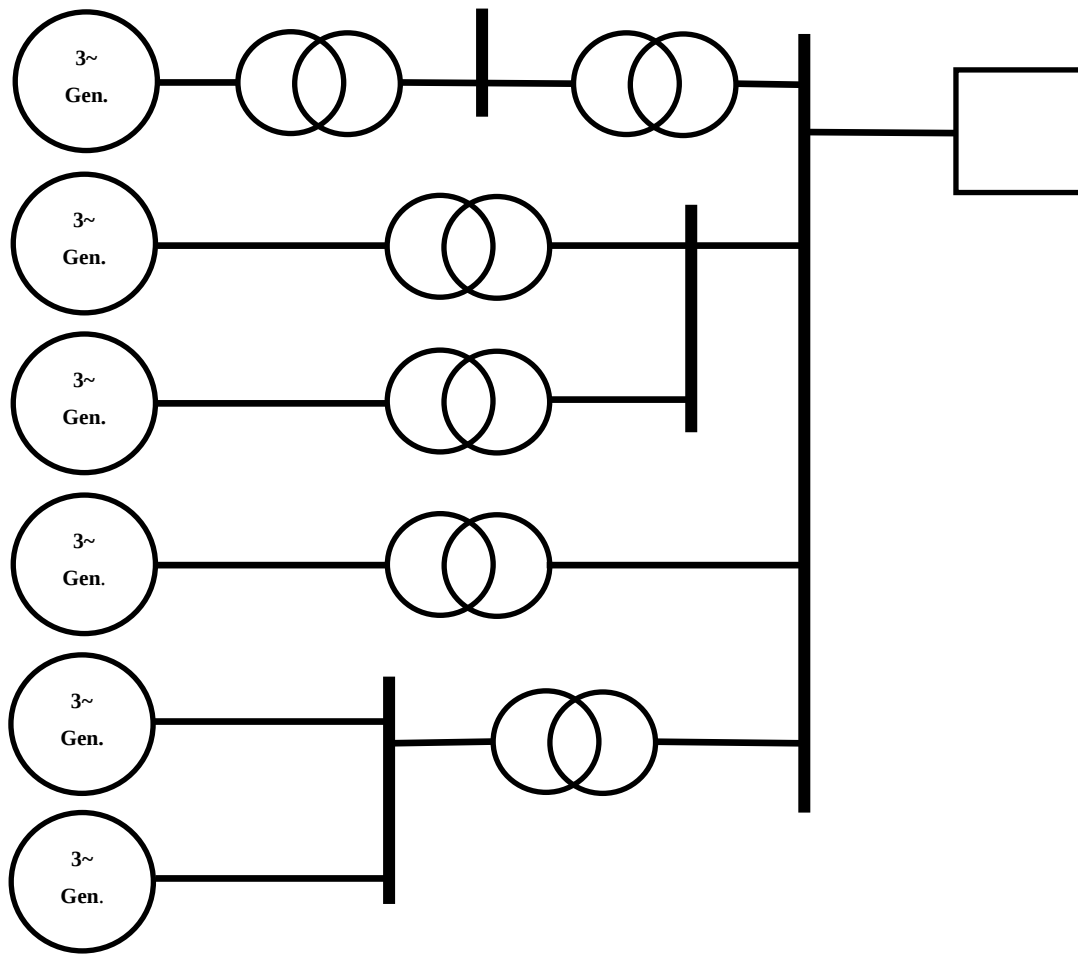


Figure 1-15: Typical transmission system layout

Unfortunately, such modelling does not yield any intuitive understanding of the oscillatory behavior of a generating station, particularly the interaction between station and grid. To allow this, the multi-machine system is reduced to a *two-machine* model consisting of the generating unit under review and with all the other generating stations lumped into *one* single very large station. This will bolster the attempt to identify the parameter set of a lumped generator. Such a two-machine model can then be described by a simple mechanical analogue allowing the development of an analytic dynamics solution approximately describing *how* generating units interact with a *finite* grid. This solution will facilitate an understanding of how the units react to transient disturbances as seen during ROCOF events. Having a mathematical formalism describing these events allows predictability of the behaviour of a unit and also allows for the development of active rather than re-active controls. In the model described in Figure 1-16 the influence of transformers and transmission lines was assumed negligent and therefore these are not modelled. If the generator unit is very far from the grid as it is the case with assets connected to a long

transmission line, this simplification is not valid as the inductance of the line introduces delays in response to disturbances.

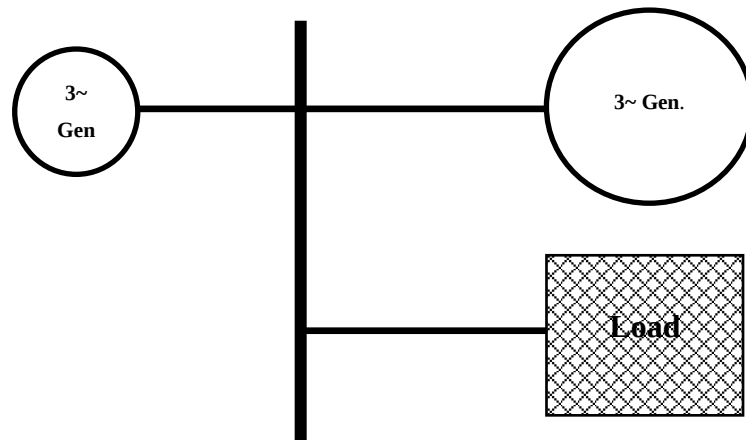


Figure 1-16: Reduced Two Machine model with lumped Generation and lumped Load

The analytic solution of this two-body phenomenon will be accomplished using *MATLAB*[®] and various analytic techniques. Finally, the operation of a power system stabilizer circuit will be analyzed, again using both *MATLAB*[®] and analytic tools.

2 MODELLING OF GRID WITH MATWORKS SIMPOWERSYSTEM

2.1 MODELLING WITH EXISTING SIMPOWERSYSTEM® COMPONENTS

The aim of this simulation is to determine the effects of various ROCOF rates, particularly these due to ramp-like transient changes of the grid frequency on the electrical power output, the load angle, the rotor speed and the mechanical torques of the turbo-set shaft line. This treatment consists in modelling a turbo-set consisting of a generator, two low pressure turbines (LP), one intermediate pressure (IP) and one high pressure turbine (HP) including an automatic voltage regulator and power system stabilizer with *SimPowerSystems*® components in *MATLAB/SIMULINK*® and imposing a frequency change on the system. The system line diagram is illustrated in Figure 2-1.

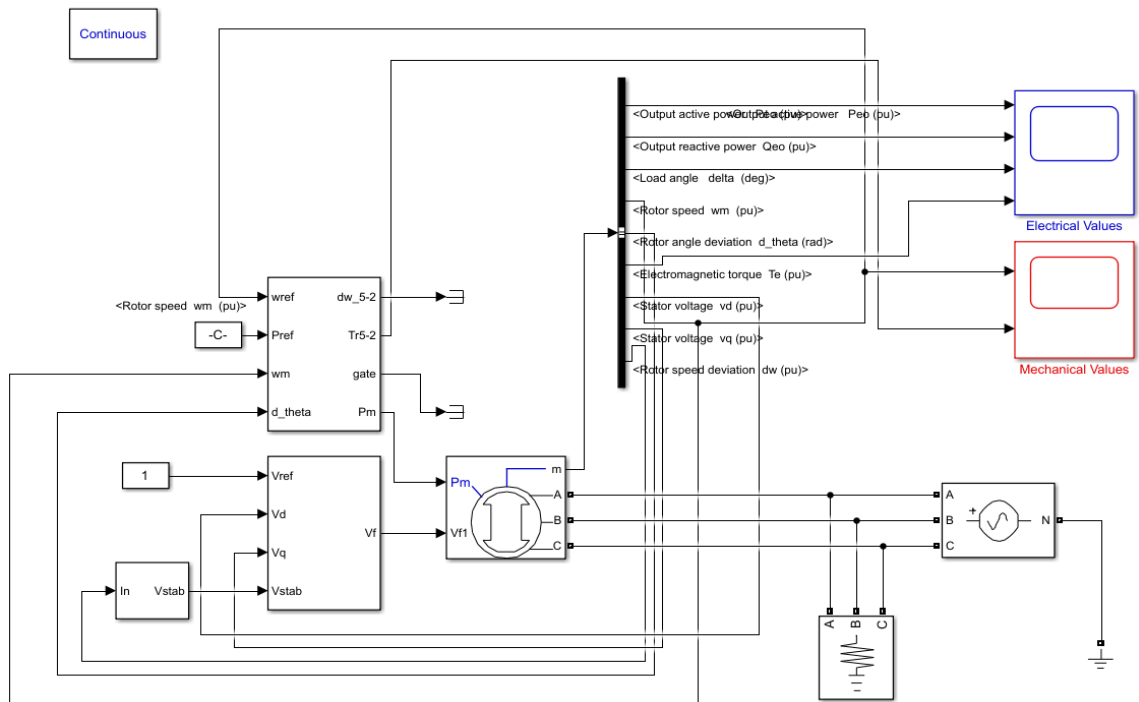


Figure 2-1: Generator model with MATLAB/SimPowerSystems

A grid frequency change of $df/dt = -0.25\text{Hz/s}$ and $t = 4\text{s}$ duration was imposed on the generator and the results were compared with those of disturbances with $(-0.5\text{Hz/s} \mid 2\text{s})$, $(-1\text{Hz/s} \mid 1\text{s})$ and $(-2\text{Hz/s} \mid 0.5\text{s})$ respectively. The results shown in Figure 2-2 illustrate that both the electrical and mechanical stresses on the turbines and the generator increase as

the ROCOF value is increased. The power oscillations of these units are purely due to Synchronous Inertial Response of the unit as the frequency response was switched off.

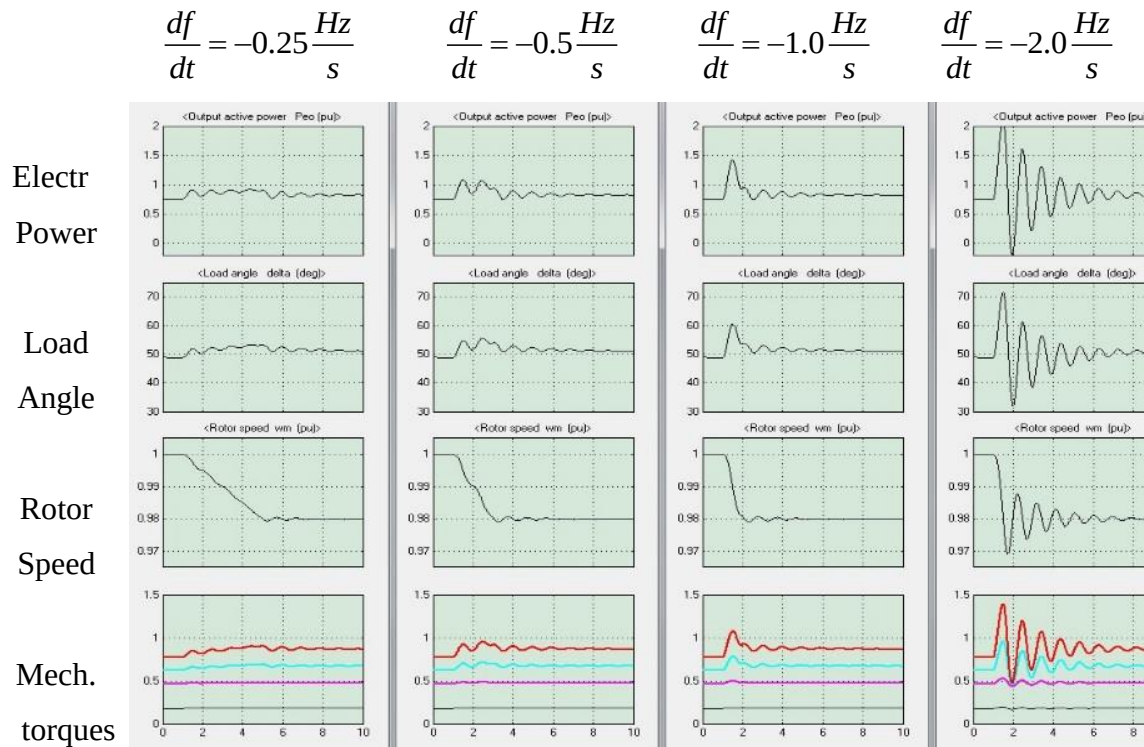


Figure 2-2 Response to a ramp like frequency drop
Active power [pu], rotor angle [deg], rotor speed [pu] and mechanical torques [pu] for various ROCOF values

By increasing the ROCOF values the output of the generator may rise beyond its nominal value. This may trigger overcurrent devices which would open the generator circuit breaker with consequential further loss of generation within the transmission system. The extreme load oscillations may also result in control circuits wrongly interpreting this response as a load rejection, leading to further implications for plant.

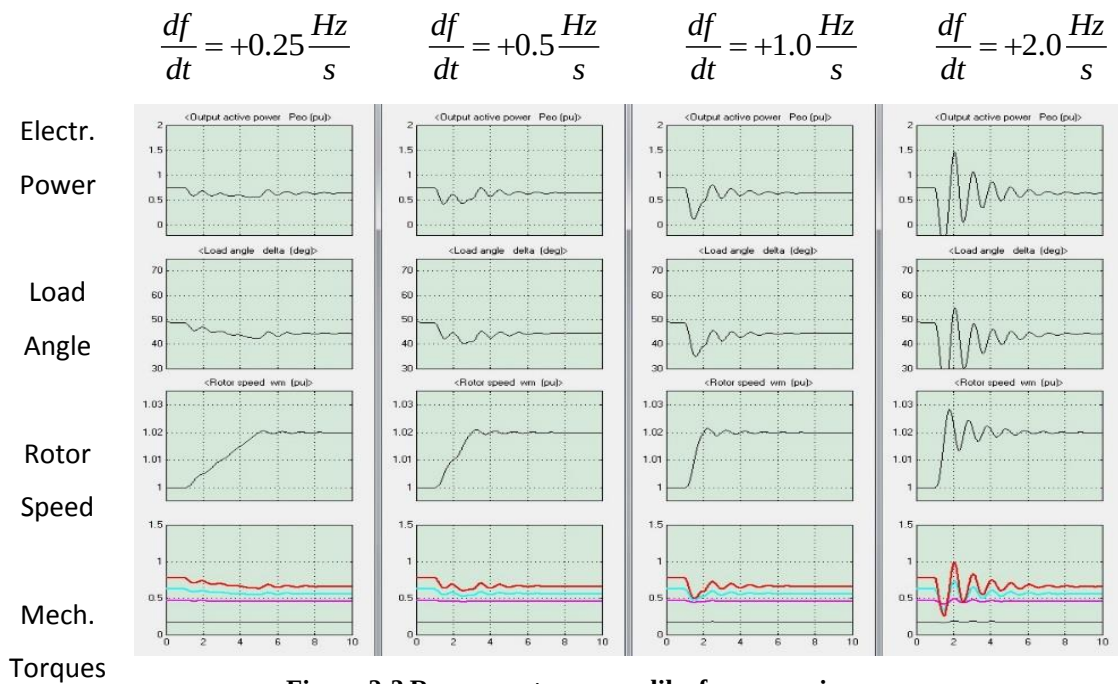


Figure 2-3 Responses to a ramp like frequency increase
Active power [pu], rotor angle [deg], rotor speed [pu] and mechanical torques [pu] for ROCOF of +0.25Hz/s; +0.5Hz/s; +1Hz/s and +2Hz/s

The effect of a positive ROCOF was also examined, occurring for example when a large load is lost, as may happen when the DC interconnector to the UK trips during export. The graphs in Figure 2-3 show the increase in stresses resulting from such a load shedding event when modelled.

Clearly for both negative and positive ROCOF the model-results highlight possible risks to the turbo-set which can be associated with four underlying effects:

- large changes in electrical power output possibly affecting controllers such as excitation systems, power system stabilizers, turbine controller and unit controllers
- torque and angle oscillations affecting the mechanical integrity of the turbo-set
- large changes in MW input from the prime mover due to governor response (see also Figure 2-26)
- inherent response of a unit to falling frequency such as gas compressor performance or inertial energy delivery

The risks could also be broadly categorized as *operational risks* covering the responses of automatic control systems and protection circuits on the one hand and on the other hand *risks to the integrity* of the mechanical and electrical components used in the generation of electricity.

However, in reality the grid frequency does not change in a ramp like manner. Therefore, for a better understanding of the behaviour of a generating unit modelling of a frequency trace like that usually seen in real time measurements, is needed.

2.2 RESPONSE OF A GENERATOR TO REALISTIC FREQUENCY TRACES

As part of delivering a Secure Sustainable Electricity System (DS3) by the Irish transmission system operator *EirGrid*, the demand to increase sustainable energy sources influences the maximum Rate of Change of Frequency (ROCOF). During periods of increased electricity generation from wind turbine generators the ROCOF rate rises. Therefore, we must now analyze how conventional power generating stations such as fossil fuel powered stations and hydro power stations (both using synchronous generators) respond to such transient phenomena. These transient events which will be further analyzed here result from a change in the balance of energy generation to energy consumption as seen when a generating plant suddenly trips or the load in the grid suddenly changes.

The Irish grid operator *EirGrid* has defined four rather extreme frequency traces shown in Figure 2-4. In particular, the first case is a frequency rise event, usually seen when a very large consumer is suddenly removed. This event, in the context of the Irish grid, could happen when either a large distribution area is disconnected or a DC Interconnector between Ireland and Great Britain opens during times when electricity is exported. The second case arises when during a high frequency event the interconnector suddenly reverses. Similarly, a frequency drop event with a subsequent rapid rise might occur when during a generator trip (with the interconnector exporting) the interconnector reverses and thus, drastically increases the inflow of power. The frequency drop scenario is the most common and happens usually when a large generator trips.

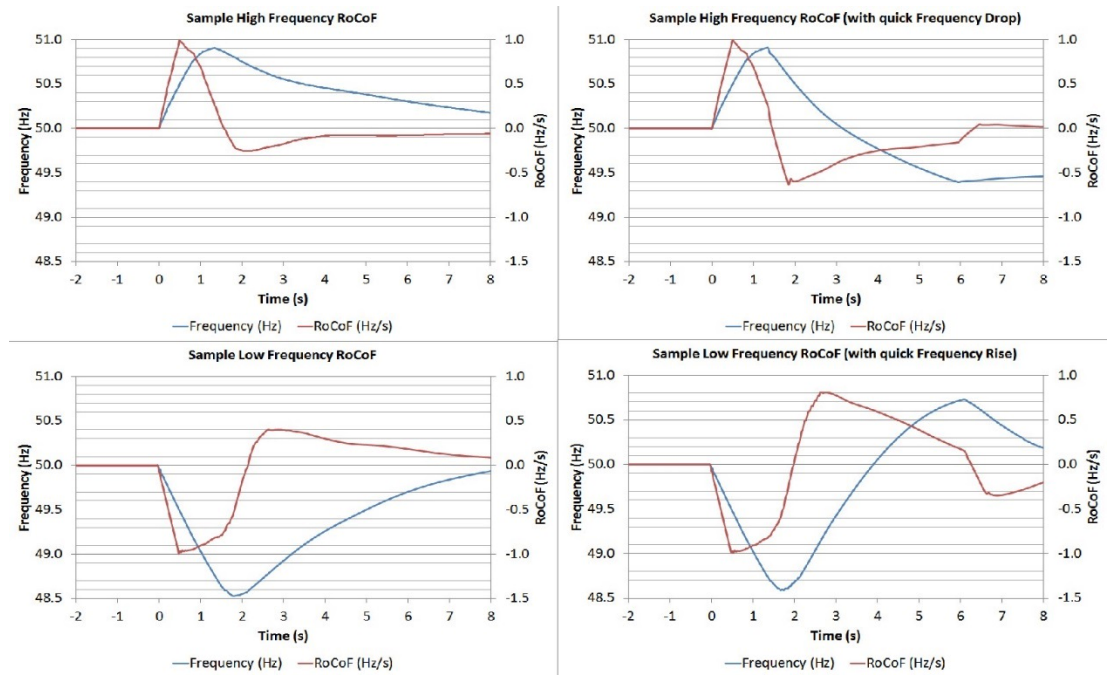


Figure 2-4 Predefined frequency disturbances
Blue graphs: Grid frequency; Red graphs: ROCOF

To assess the implications for power generating units a model consisting of a setup representing an infinite grid with a variable frequency source and adjustable grid impedance is required.

2.3 MODELLING AN INFINITE GRID

The program chosen to create such a model was *MATLAB*[®] and the embedded graphical programming environment *SIMULINK*[®]. The library enabling the building of blocks specifically yielding acceptable representation of a power generating station including turbine, turbine-controller, synchronous generators, voltage regulators and other components is *SimPowerSystems*[®]. As *SIMULINK*[®] does not have a variable 3 phase frequency source which can reproduce predefined or even prerecorded frequency traces such as those shown above, such a source was developed. By assuming a constant voltage-level a model replaying the four *extreme* frequency traces (see Figure 2-4) was designed using certain fundamental blocks from the *SimPowerSystems*[®] library, (see Figure 2-5):

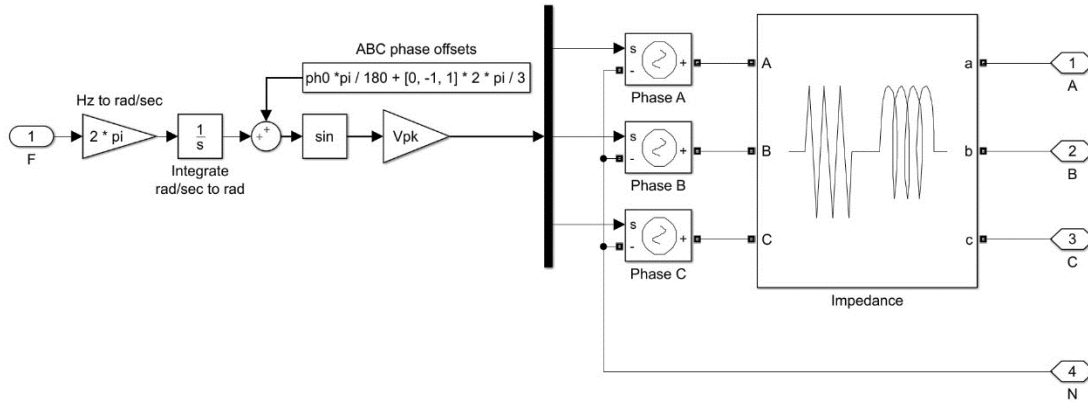


Figure 2-5 Signal Builder, frequency block for simulations

This variable frequency source, here called *signal builder*, was embedded in the model and connected to a generator as seen in Figure 2-6.

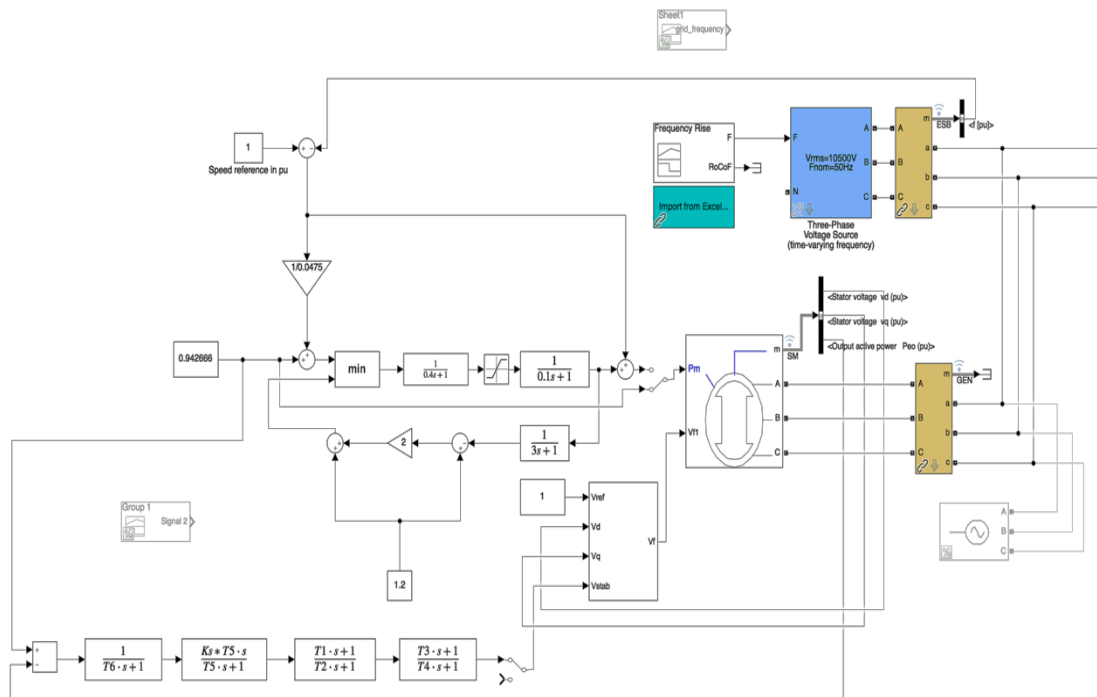


Figure 2-6 Generator model with constant mechanical torque

A particular power generating station with an open cycle gas turbine connected to a turbo-generator was chosen in order to validate the model. The specific parameters of this power generating station used are listed in Table 2-1

Table 2-1 Generator Parameters (Aghada CT11)

Parameter	Description	Value
S_N [VA]	Apparent Power	$100.9 \cdot 10^6$
U_N [V]	Nominal Voltage, phase-Phase	$10.5 \cdot 10^3$
T_{do}' [s]	Transient direct axis time constant	5.437
T_{do}'' [s]	Subtransient direct axis time constant	0.04
T_{qo}' [s]	Transient quadrature axis time constant	3.7
T_{qo}'' [s]	Subtransient quadrature axis time constant	0.04
x_d [pu]	Direct Axis Synchronous Reactance	1.761
x_d' [pu]	Direct Axis Transient Synchronous Reactance	0.237
x_d'' [pu]	Direct Axis Subtransient Synchronous Reactance	0.147
x_q [pu]	Axis Synchronous Reactance	1.65
x_q' [pu]	Axis Transient Synchronous Reactance	0.375
x_q'' [pu]	Axis Subtransient Synchronous Reactance	0.161
x_l [pu]	Leakage Reactance	0.135
H [s]	Inertia constant	4.5

2.3.1 Model Validation

For the purpose of validating the model an actual system perturbation is used whereby measurements of the system frequency and the actual electrical power are obtained as seen in Figure 2-7. This figure is assembled by Eirgrid and shows how the electrical power output of the generating unit changes as a reaction to a frequency dip in the system.



Figure 2-7 Actual system perturbation event

Blue graph: System frequency [Hz]; Red graph: Unit output [MW];
Black graph: expected power output [MW]

During this particular event, at about 90 seconds after the frequency perturbation, a command was given, increasing the load to 70MW with a rate of 5MW/min. This explains why the expected response deviates from the actual measured power output of the unit. To achieve similar behavior, a ramp was introduced into the model. Furthermore, to gain an understanding of how a generator responds without the influence of a turbine controller and the embedded frequency influence function [49] [12, p. CC13], a bypass switch was introduced and the ensuing behavior is illustrated in Figure 2-8

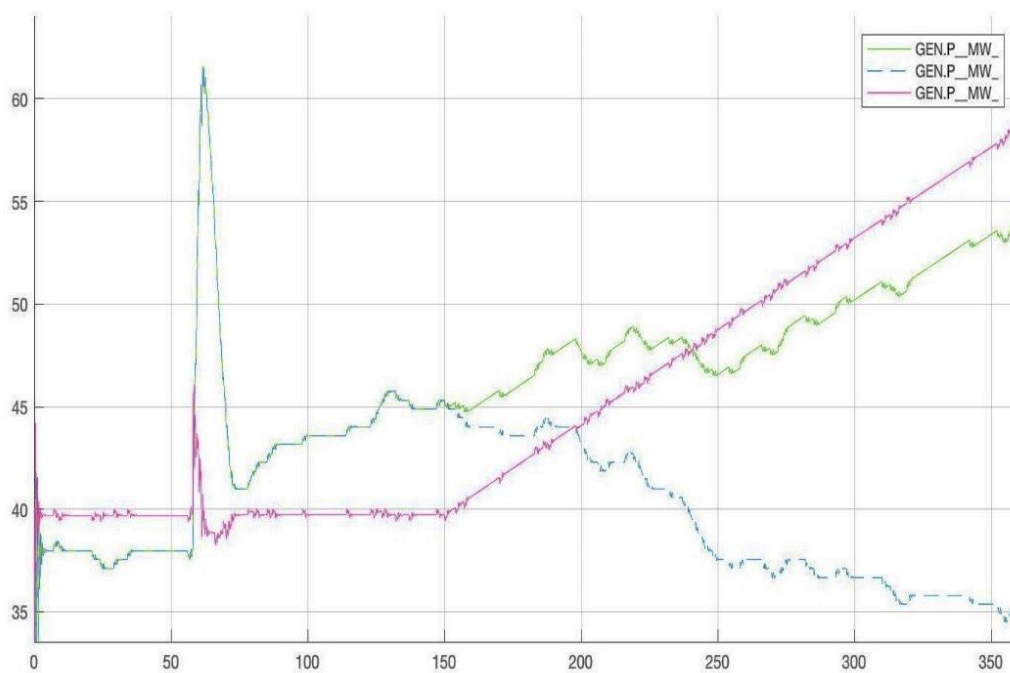


Figure 2-8 Modelled power output response

Red graph: pure generator response, Turbine controller bypassed

Blue dashed graph: Power output without power increase command modelled

Green graph: fully modelled response

Each synchronous generator on the system reacts to such events and a Synchronous Inertial Response (SIR) is experienced (spike on red graph at around 60 s in Figure 2-8).

2.3.2 Reaction to predefined frequency traces

To understand which part of the oscillation can be attributed to *Synchronous Inertial Response* (SIR) (which follows from a sudden change of the rotor angle) and which is due to *governor response* (in particular the Primary Operating Reserve, POR, see also Table 3-1) an analysis is undertaken at the four extreme operating points of a generator as shown in the generator capability chart (see Figure 2-9):

- I. Maximum Active Power and maximum lagging Reactive Power (90MW, +87MVar)
- II. Maximum Active Power and maximum leading Reactive Power (90MW, -36MVar)
- III. Minimum Active Power and maximum lagging Reactive Power (15MW, +124MVar)
- IV. Minimum Active Power and minimum leading Reactive Power (15MW, -47MVar)

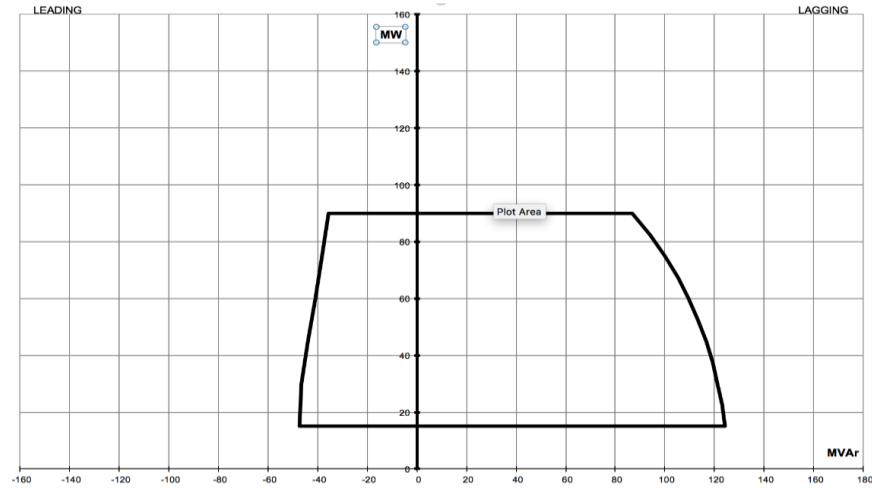


Figure 2-9 Generator Capability Chart [50]

2.3.3 Unit response at various operating points

To visualize the behaviour of the generating station following the various disturbances the most common ROCOF scenarios (see Figure 2-4) were modelled at the extreme operating points as defined above. In the analysed cases the generator was operated with the turbine at maximum load and minimum load and the generator operated *maximum overexcited (full lagging)* and *maximum underexcited (full leading)*. The *SIMULINK®* model used is shown in Figure 2-6.

Table 2-2 Simulation Runs at various operating points

	Frequency Rise		Frequency Rise, fast drop		Frequency Drop	Frequency Drop, fast rise
	P [MW]	Q [MVar]	Simulation run	Simulation run	Simulation run	Simulation run
Case I	90	87	1, Figure 2-10	5, Figure 2-14	9, Figure 2-18	13, Figure 2-22
Case II	90	-36	2, Figure 2-11	6, Figure 2-15	10, Figure 2-19	14, Figure 2-23
Case III	15	124	3, Figure 2-12	7, Figure 2-16	11, Figure 2-20	15, Figure 2-24
Case IV	15	-47	4, Figure 2-13	8, Figure 2-17	12, Figure 2-21	16, Figure 2-25

The figures below show (for the simulation runs 1-16) the response of the generator power output to a predefined transient event as described in the cases I-IV given by the Irish TSO *EirGrid* to all transmission connected generating stations. Simulation runs 1-4 show the response to a frequency rise event.

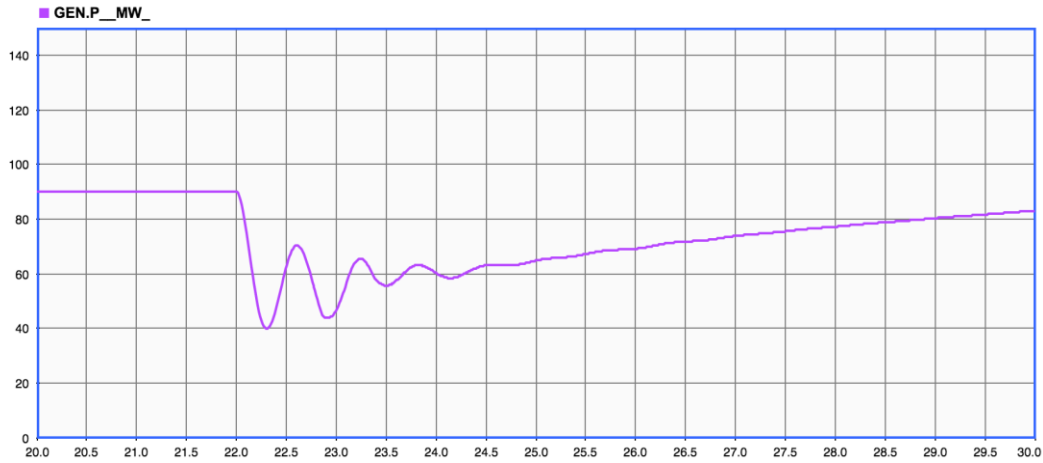


Figure 2-10: Sim-Run 1, Freq. Rise, Max Load – Max Overexcited

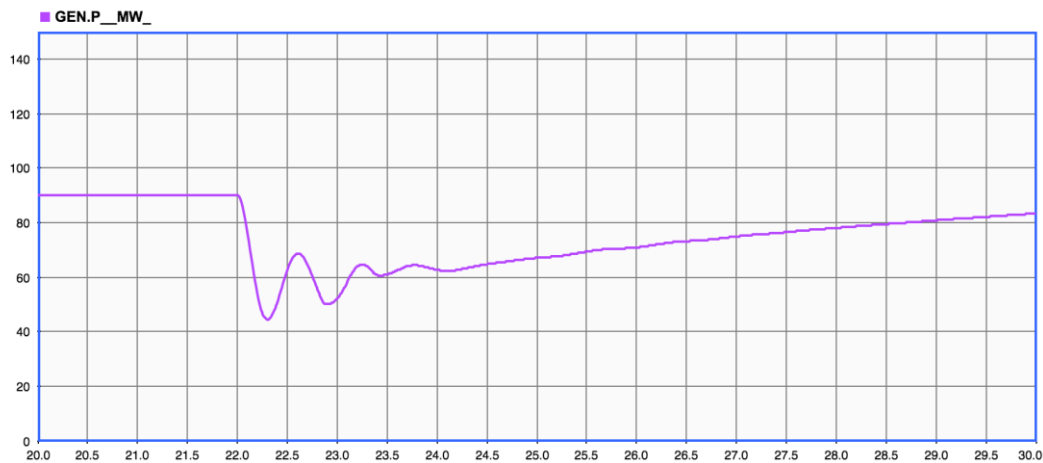


Figure 2-11: Sim-Run 2, Freq. Rise, Max Load – Max Underexcited

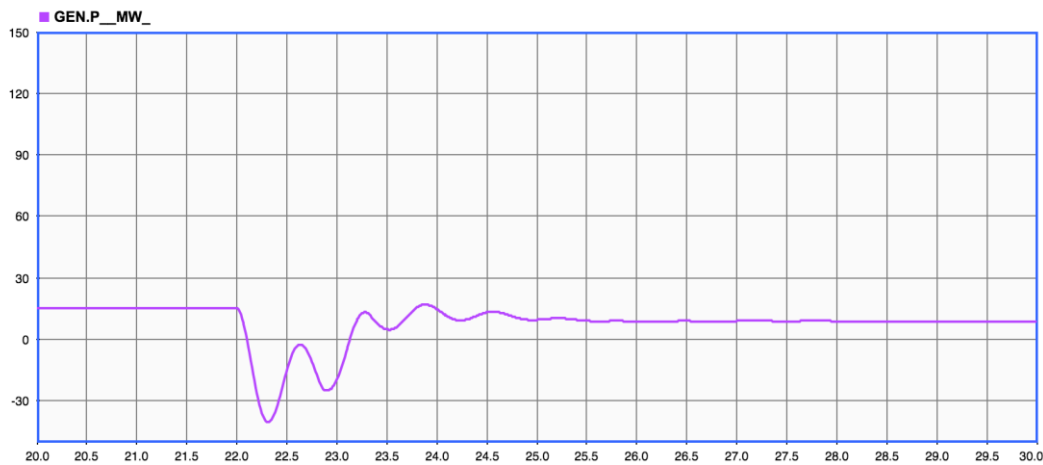


Figure 2-12: Sim-Run 3, Freq. Rise, Min Load – Max Overexcited

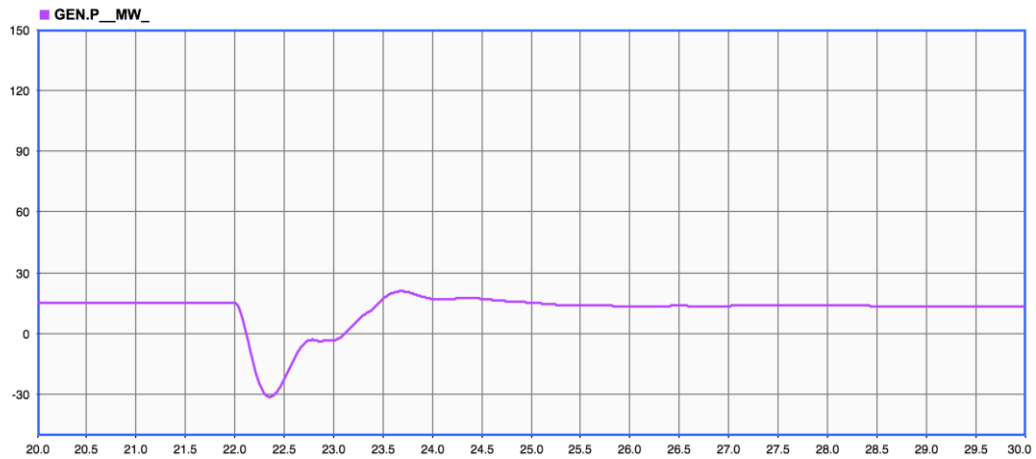


Figure 2-13: Sim-Run 4, Freq. Rise, Min Load – Max Underexcited

As expected the power output drops suddenly when the frequency rises. However, the unit remains stable and does not experience any prolonged oscillations.

If the model of the unit is exposed to frequency rise with subsequent fast drop then the response of the unit also seems uncritical even though an alarm would be triggered indicating that the units went into reverse power, however without any further implications.

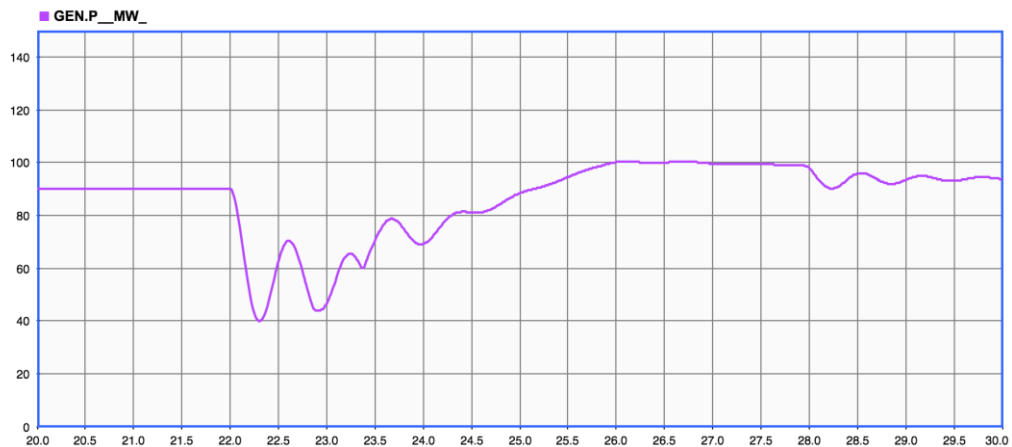


Figure 2-14: Sim-Run 5, Freq. Rise-Fast Drop, Max Load – Max Overexcited

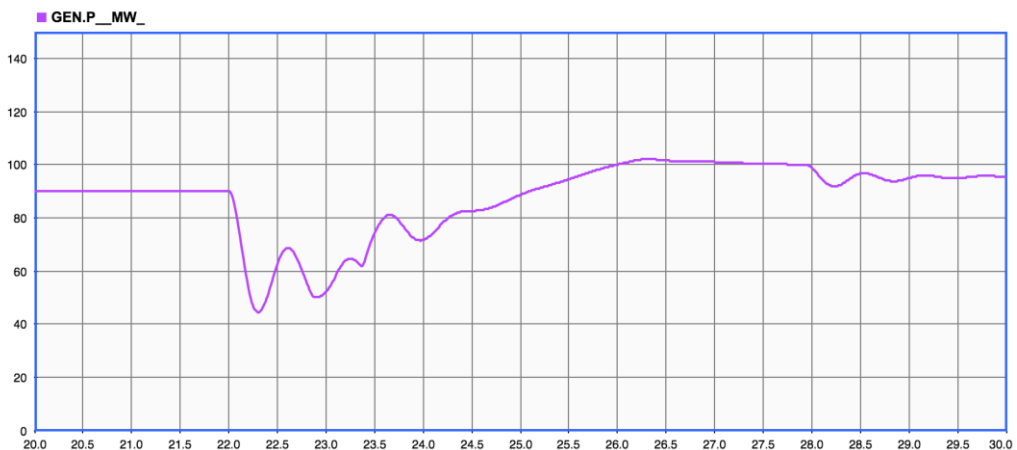


Figure 2-15: Sim-Run 6, Freq. Rise-Fast Drop, Max Load–Max Underexcited

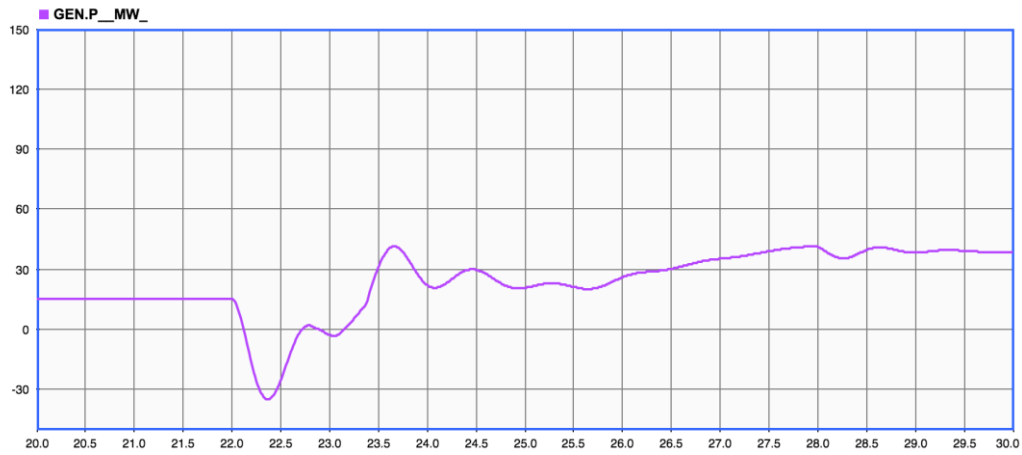


Figure 2-16: Sim-Run 7, Freq. Rise-Fast Drop, Max Load-Max Overexcited

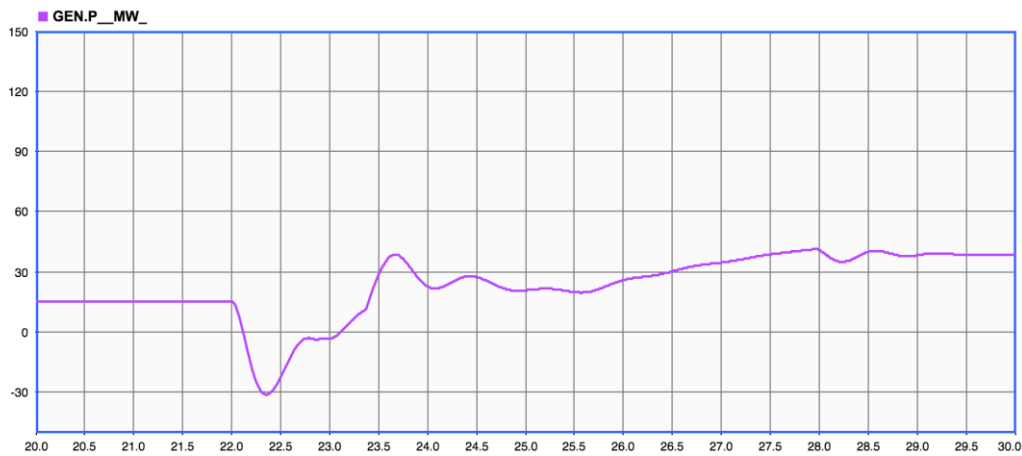


Figure 2-17: Sim-Run 8, Freq. Rise-Fast Drop, Max Load-Max Underexcited

When a unit is exposed to a frequency drop event the oscillatory behaviour of the generated electrical power is also not critical as protection settings would not be exceeded. The power oscillations are illustrated below and are depicting the simulation runs 9-12 as per Table 2-2.

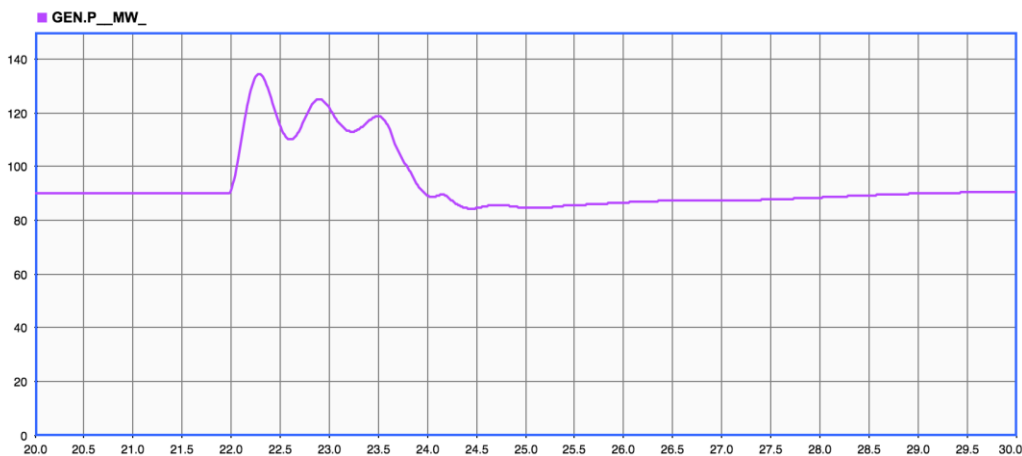


Figure 2-18: Sim-Run 9, Freq. Drop, Max Load-Max Overexcited

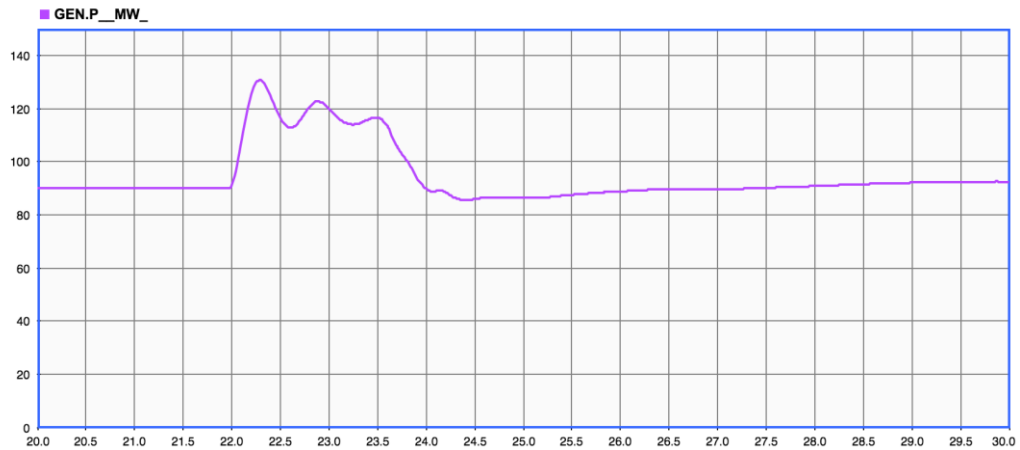


Figure 2-19: Sim-Run 10, Freq. Drop, Max Load–Max Underexcited

At minimum load the oscillatory behaviour changes compared to the full load examples show initially a similar increase of the generated power, however after a few seconds the influence of the turbine controller which increases the power output is observed.

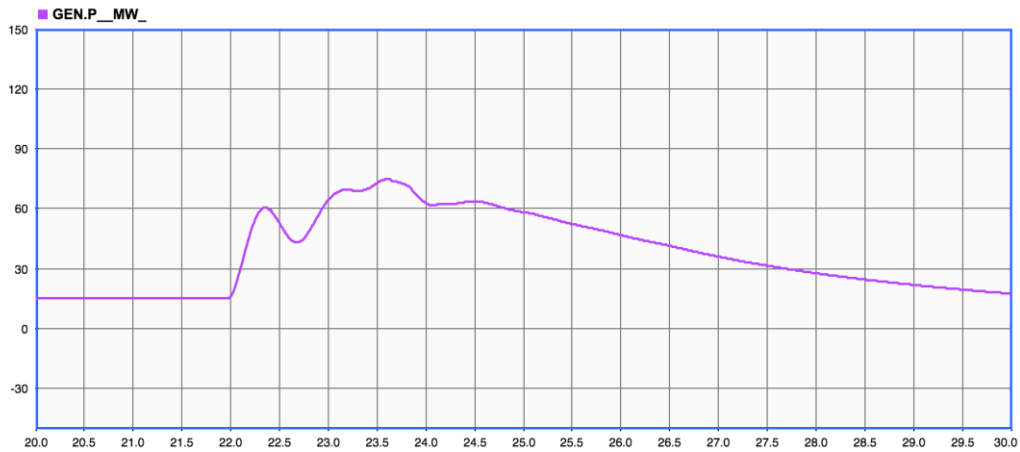


Figure 2-20: Sim-Run 11, Freq. Drop, Min Load–Max Overexcited

Again, as seen also at full load the oscillation seemed to be more damped with the generator in lagging power factor (see above Figure 2-20)

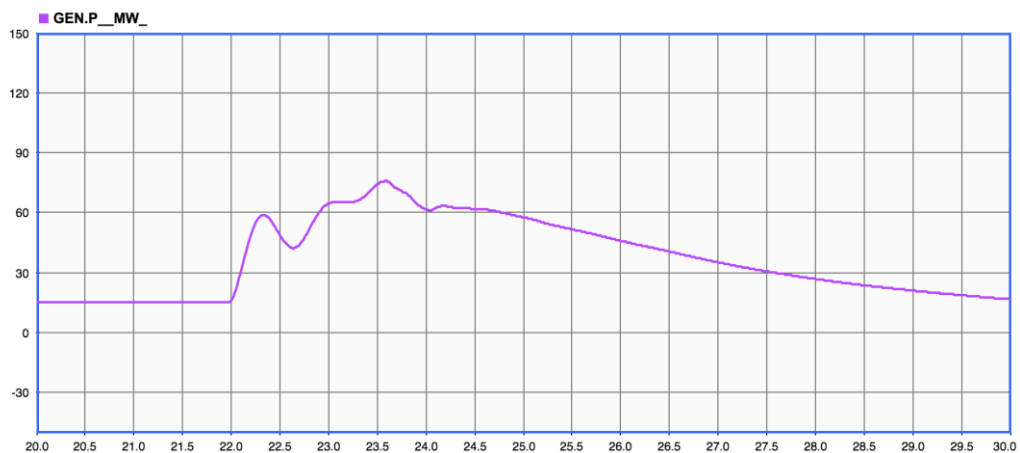


Figure 2-21: Sim-Run 12, Freq. Drop, Min Load–Max Underexcited

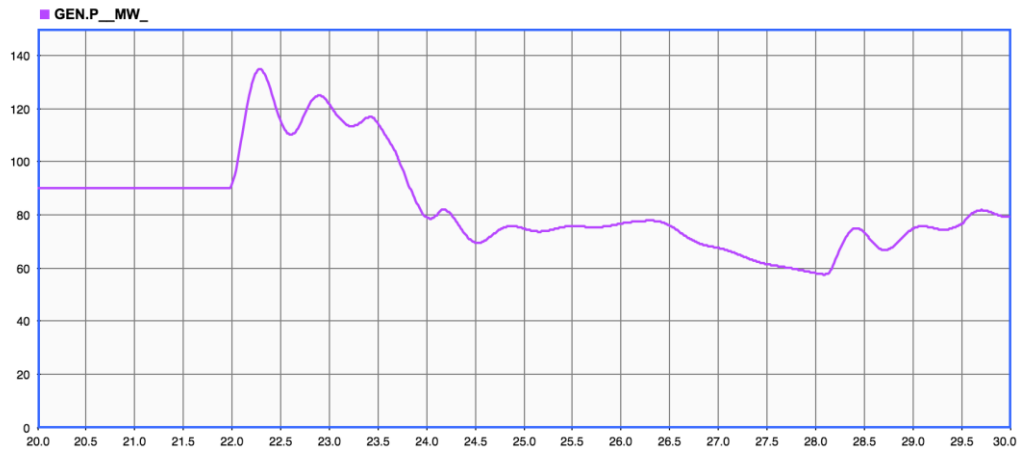


Figure 2-22: Sim-Run 13, Freq. drop – fast rise, Max Load–Max Overexcited

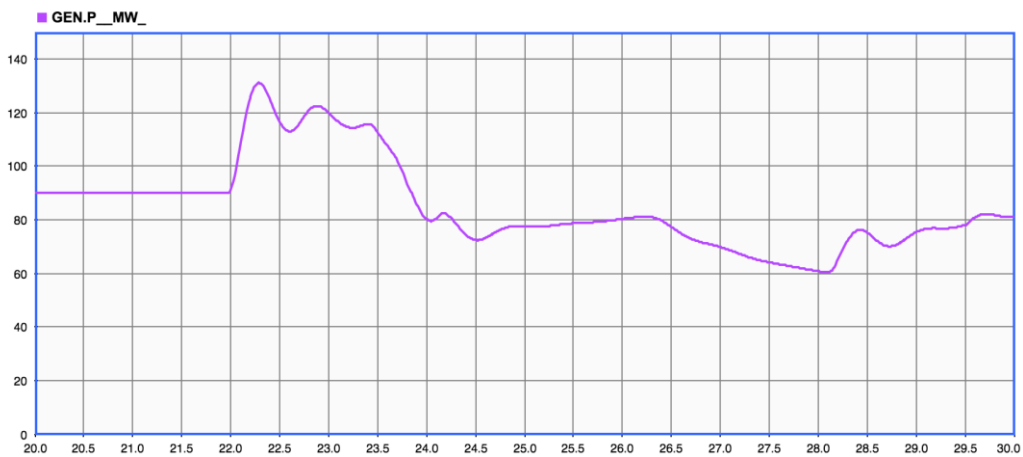


Figure 2-23: Sim-Run 14, Freq. drop – fast rise, Max Load–Max Underexcited

If the unit is on low loads as can be seen in simulation runs 15 and 16 the primary operating reserve becomes more evident. This is due to the influence of the turbine load controller trying to increase the power output in order to mitigate against the reduced frequency in the system

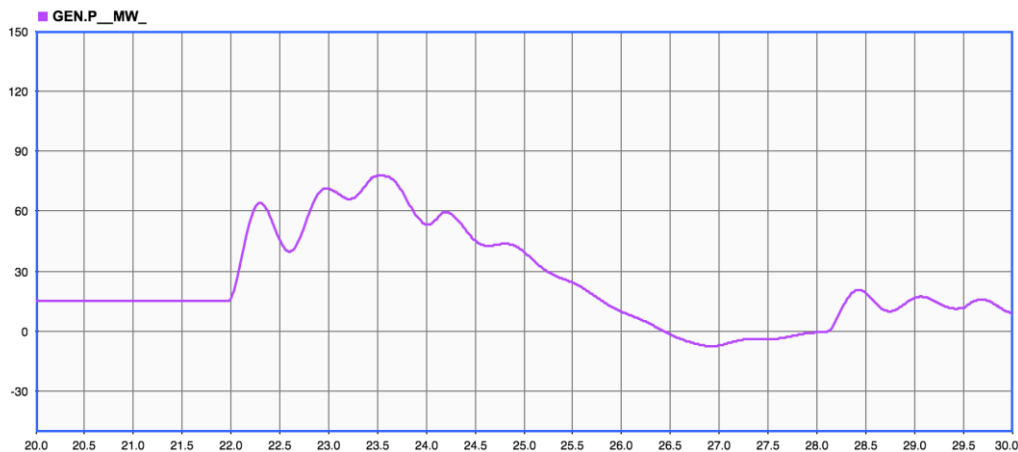


Figure 2-24: Sim-Run 15, Freq. drop – fast rise, Min Load–Max Overexcited

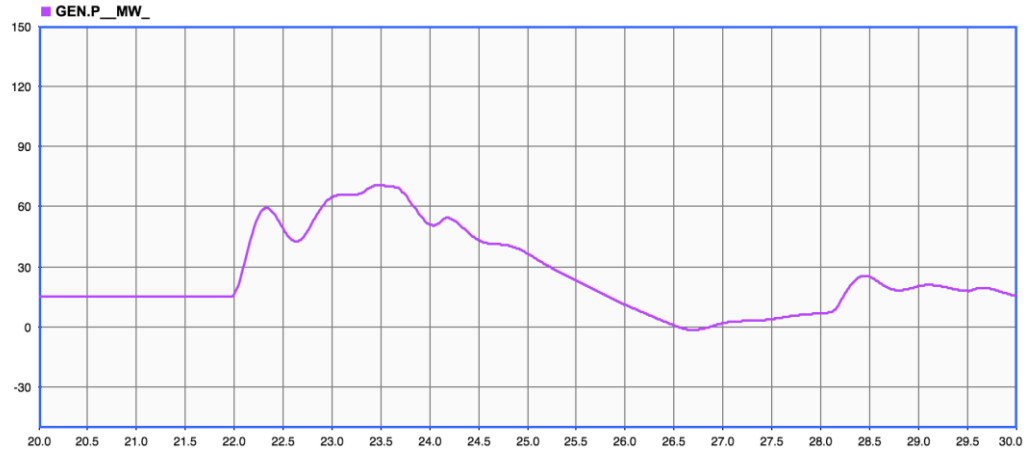


Figure 2-25: Sim-Run 16, Freq. drop – fast rise, Min Load–Max Overexcited

As the previous figures do not allow easy evaluation of some effects, the results were combined into a 3D plot with the electromagnetic torque and the time on x and y axis and the power output on the z axis. As the most common transient fault in the system is a frequency drop, the response of a turbine generator was repeated for various values of the electromagnetic torque varying from 20 to 100% of its nominal torque. When analysing Figure 2-26 the Synchronous Inertial Response (SIR) of the unit appears identical for all power output scenarios.

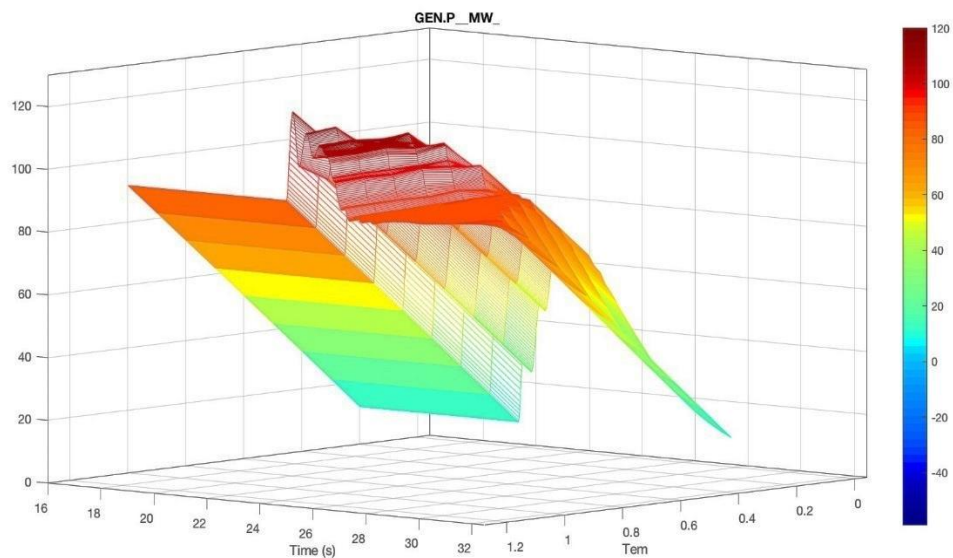


Figure 2-26: unit exposed to frequency drop at various load levels
Same frequency drop, varying electromagnetic torque from 0.2 to 1 pu

However, in Figure 2-26 this finding is still only an assumption, prompted to remove the influence of the turbine, particularly the influence of the load-controller. The response of the load controller to a frequency drop is to increase the output power depending on its actual load and an adjusted droop.

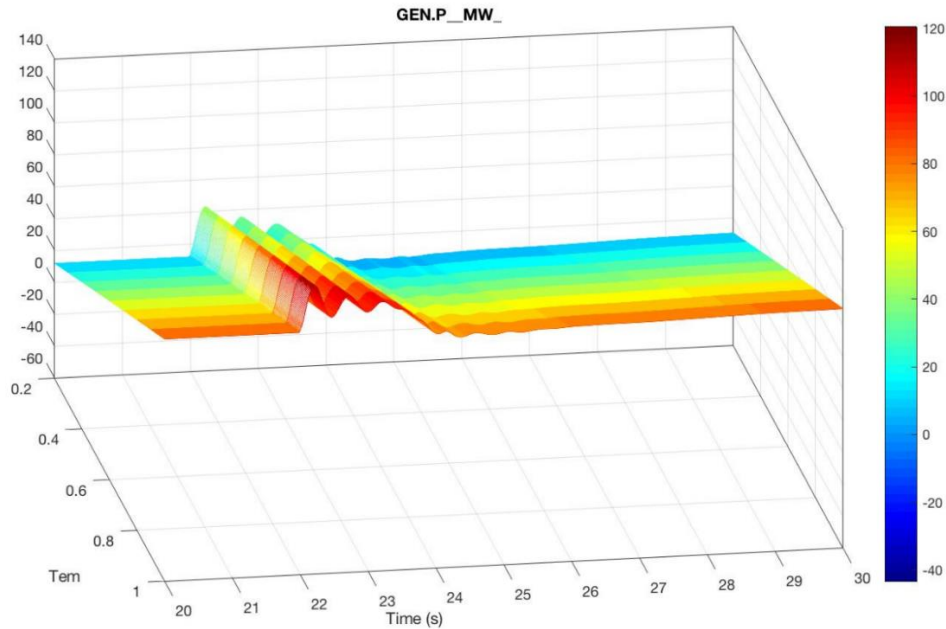


Figure 2-27: Synchronous Inertial Response
Same frequency drop, varying electromagnetic torque from 0.2 to 1 pu

In contrast in Figure 2-27 the independence between SIR and the power output at each operating point is obvious for the frequency trace studied.

2.3.4 Synchronous Inertial Effect in an Infinite Grid

This effect is enhanced as the inertia of the generator-turbine shaft increases. Here the inertia (which for this machine is $H=4.5\text{s}$) was changed from 1 to 12 to visualize the effect.

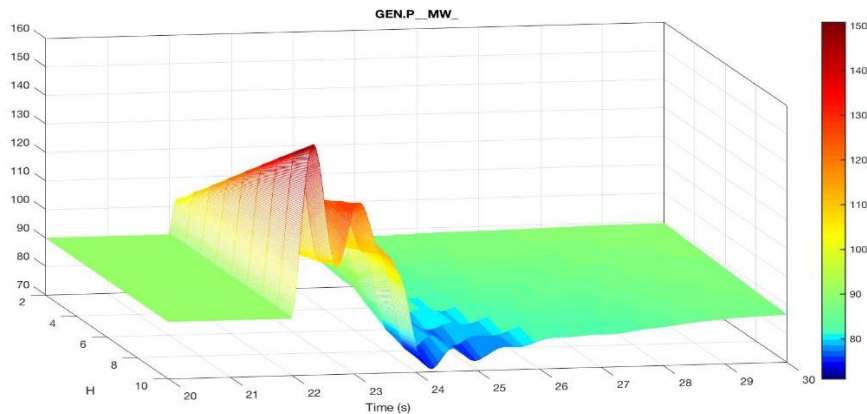


Figure 2-28: Influence of the Turboshaft Inertia H on the Synchronous Inertial Response

To visualize the relationship between the damping or decay of the oscillation to the electromagnetic torque in the airgap, in particular the influence of the reactive power component on the oscillation an analysis of a large steam turbine generating set was undertaken, here Moneypoint generating station. In this case the influence of the *turbine controller* and the *power system stabilizer* embedded in the *automatic voltage regulator* may be discounted as they are switched off.

Table 2-3: Legend for Figure 2-29

Test / Graph	Power [p.u.]	Operating Point
1 - red	0.85, max load	Full leading
2 - green	0.85, max load	Unity
3 - blue	0.85, max load	Full lagging
4 - light green	0.28, min load	Full leading
5 - orange	0.28, min load	Unity
6 - light red	0.28, min load	Full lagging



Figure 2-29: Influence of operating point to damping of the electrical power output

Obviously from the graphs in Figure 2-29 of the simulations based on Table 2-3, the damping is enhanced when the stator and rotor currents increase, meaning that a greater electro-magnetic torque exists in the air gap of the generator.

At full load the mechanical torques for the turbine-generator shaft are calculated. The simulation results are shown in Figure 2-30 for a four-stage turbine like Moneypoint Generating Station consisting of a turbo-shaft with one generator connected to two low-pressure turbines, one intermediate-pressure Turbine and one high-pressure turbine.

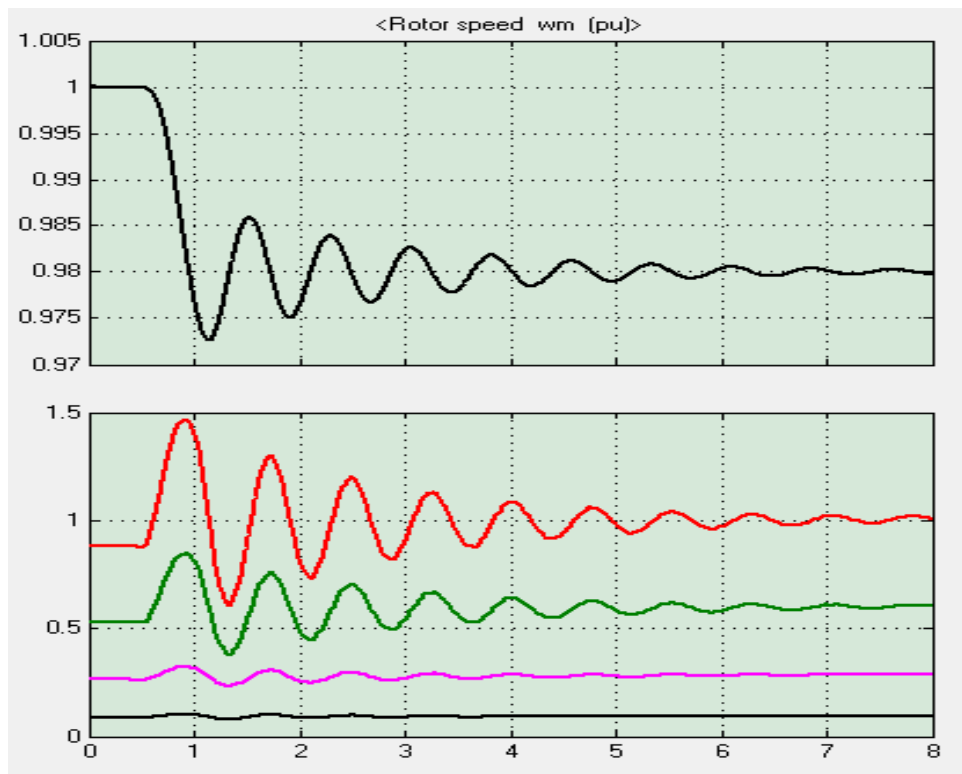


Figure 2-30: Rotor speed and mechanical torques

top, black graph: rotor speed in per unit values
below, red: Torque between LP2 and Generator
below, green: Torque between LP1 and LP2
below, violet: Torque between IP and LP2
below, black: Torque between HP and IP

The individual torques are added up and so the greatest torque appears at the coupling to the generator. Based on these torques the stresses on the turbine-generator shaft can be calculated. Definitely, at the level of torque indicated in Figure 2-30 the turbine should not experience any fatigue cracks. However, if cracks due to other failure mechanisms preexist then such increased torque above the nominal rating may introduce further crack propagation.

2.4 MODELLING A FINITE GRID WITH A MULTI-MACHINE-MODEL

Here, a very simplified grid was created with *MATLAB*[®] to model a transient event occurring when a large generator trips. Using a *finite* system facilitates understanding of how a generator reacts in a *low inertia* system

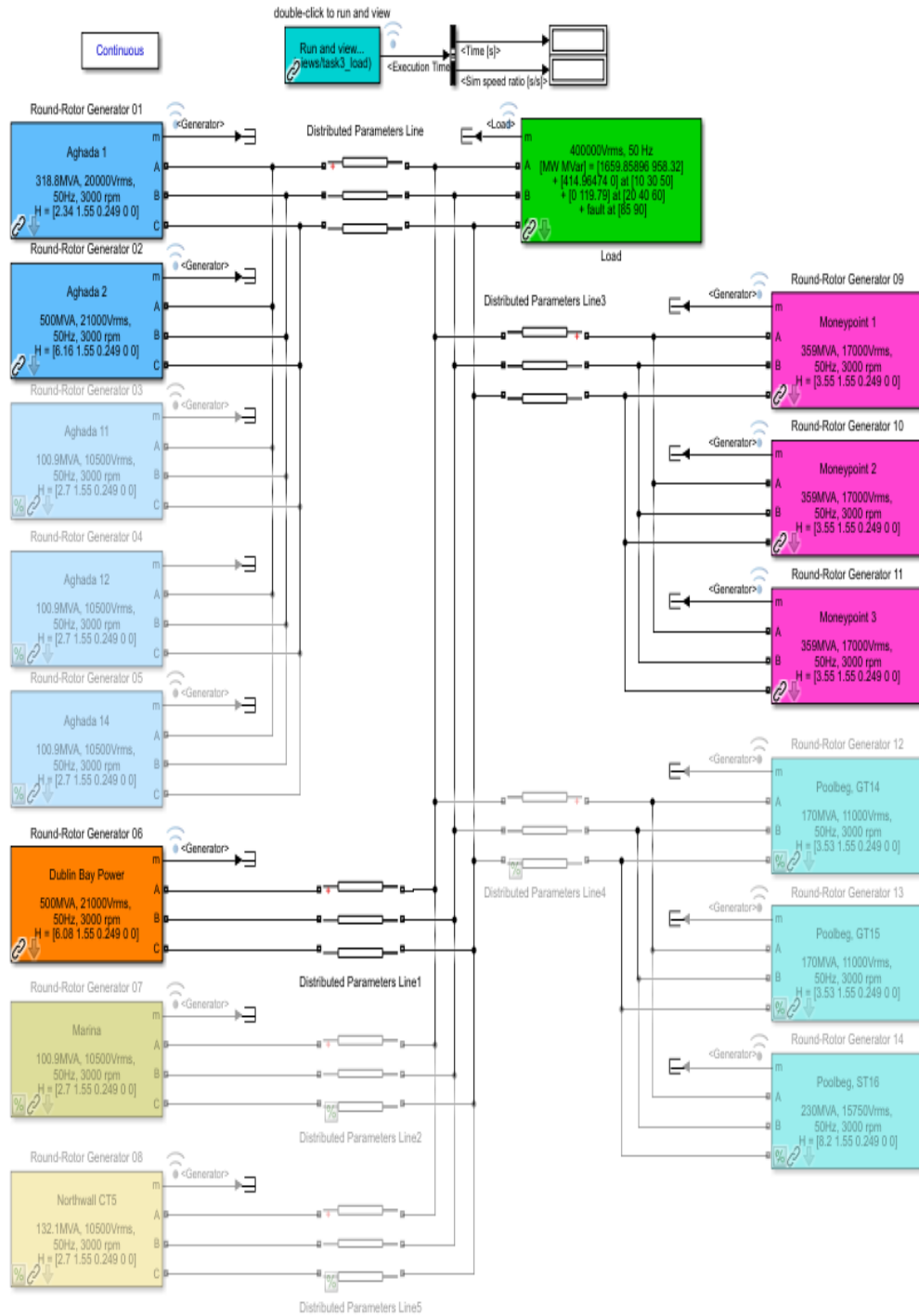


Figure 2-31: Finite grid model with generators from the ESB fleet
(This is a partial representation of the grid)

Now introducing various disturbances corresponding to suddenly increasing and decreasing the load in the system, we get:



Figure 2-32: Frequency plot of a system experiencing sudden load changes

The figure depicts a system experiencing sudden increases of load (at 30s, 40s, 60s and 70s) and sudden decreases of load (at 40s, 50s and 80s)

If the generators are analyzed in detail, one can see that they do not oscillate in identical fashion instead slightly diverting from each other

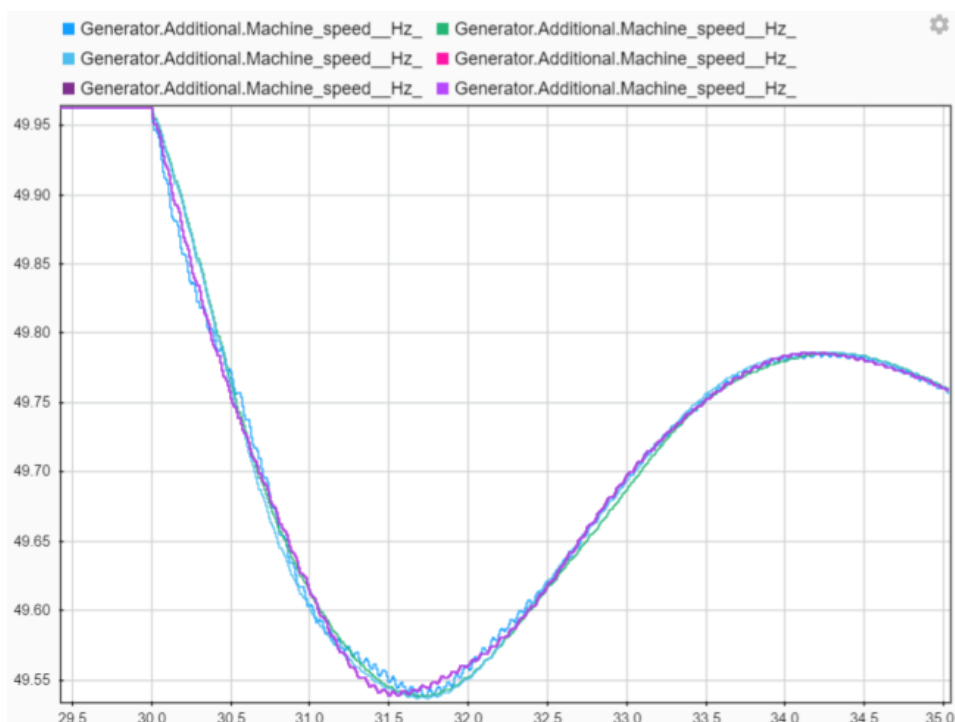


Figure 2-33: Individual rotor speed oscillations

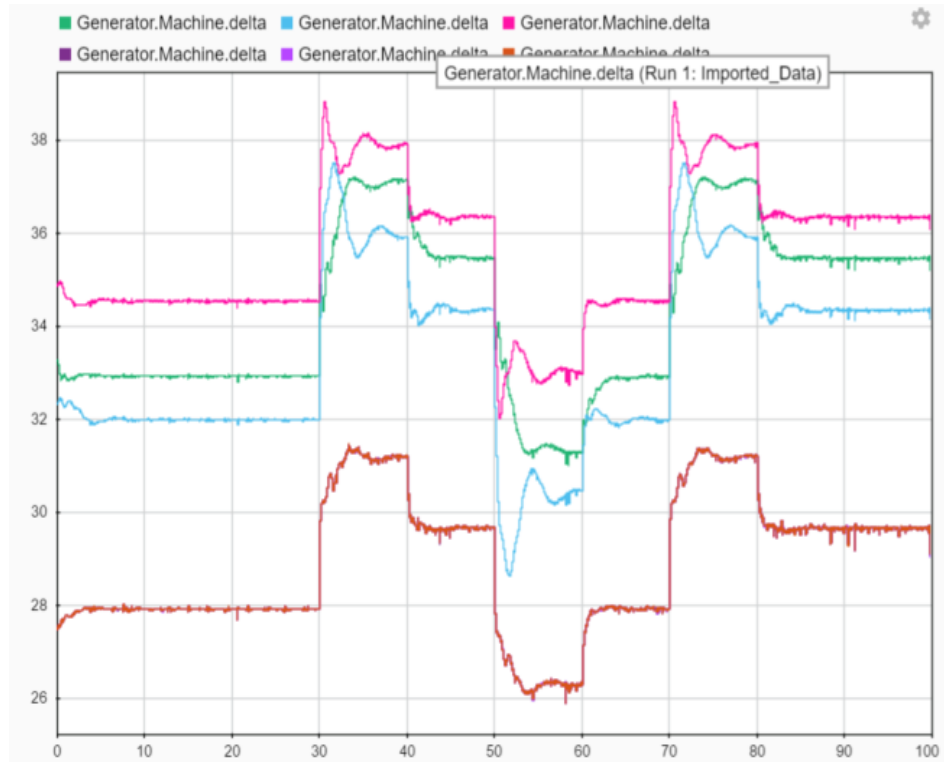


Figure 2-34: Rotor angles of individual generators

However, the rotor angles of the generators significantly deviate from each other. Now, to reduce the multi machine model shown in Figure 2-31 we depict below the frequency trace of the multi machine model that the lumped machine model shall replicate

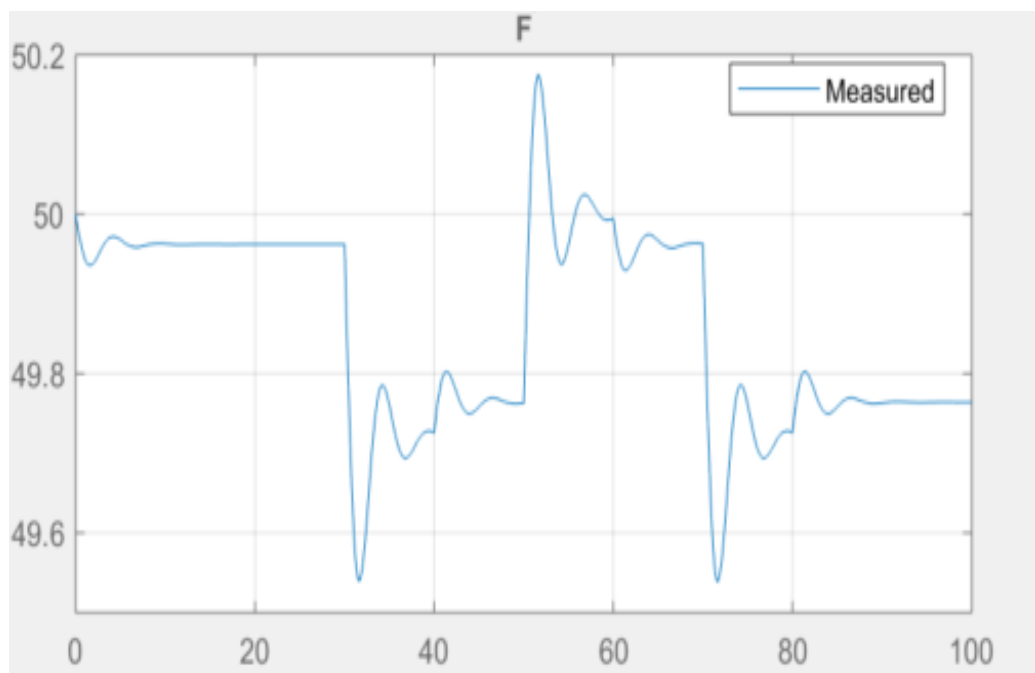


Figure 2-35 desired frequency behaviour of the lumped grid

After a few iterations the oscillatory behaviour of the lumped generator with all the machine parameters estimated, is obtained

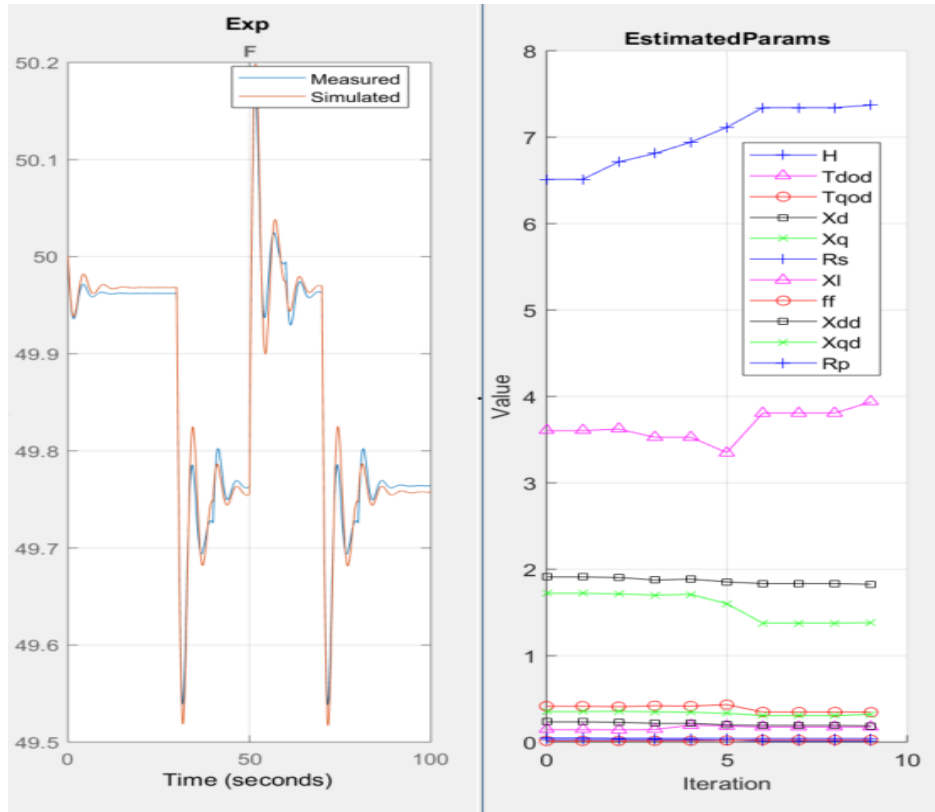


Figure 2-36: Estimated machine parameters of a lumped generator

The estimation technique used by *MATLAB*[®] therefore yields parameter values which in turn may be used to define an accurate model of a lumped generator thereby allowing one to use analytic methods to describe accurately a two-machine system. However, such estimations might only be adequate in so far as it is assumed that the parameters do not alter with changing fault situations as these are determined for only a specific fault and so may lead to them becoming time-varying in certain instances.

2.5 MODELLING A FINITE GRID WITH A REDUCED TWO-MACHINE-MODEL

A generator within a network consisting of $i+1$ generators may be analyzed by combining the i generators into a *single* machine which is denoted by the abbreviation “grid”. The initial approach is to add all kinetic energies of the generators to be lumped into one. By using the inertial constant H we have

$$H = \frac{\frac{1}{2} J_{gen} \omega^2}{S_n} \quad (2.1)$$

The kinetic energy of a Generator is then:

$$\frac{1}{2} J_{gen} \omega^2 = HS \quad (2.2)$$

The kinetic energy of the lumped Generator which is denoted with *grid* is

$$\frac{1}{2} J_{grid} \omega_{grid}^2 = \frac{1}{2} J_{gen,1} \omega_{gen,1}^2 + \frac{1}{2} J_{gen,2} \omega_{gen,2}^2 + \dots + \frac{1}{2} J_{gen,i} \omega_{gen,i}^2 \quad (2.3)$$

Now, by using the inertial constant H and the apparent load S in order to summate the kinetic energies of each individual generator irrespective of the angular velocity of each generator as for example for large 2 and 4 pole round rotor turbo-generators and multi-pole salient pole hydro-generators, we have

$$H_{grid} = \sum_i \frac{H_{gen,i} S_{gen,i}}{S_{grid}} \quad (2.4)$$

where

$$S_{grid} = \sum_i S_{gen,i} \quad (2.5)$$

so that

$$H_{grid} = \sum_i \frac{H_{gen,i} S_{gen,i}}{\sum_i S_{gen,i}} \quad (2.6)$$

To calculate the electrical parameters, we can either graphically estimate them [51, p. 33] from a plot resulting from a three-phase short circuit in a multimachine system or they can be analytically calculated [52].

Furthermore, to obtain the characteristics of a lumped generator we sum [52] all individual AC generator currents

$$I_{grid,AC}(t) = \sum_i I_{gen,i,AC}(t) \quad (2.7)$$

Moreover, just as with the AC component in Eq.(2.7) we may write for the DC component of the current during a short circuit event.

$$I_{grid,DC}(t) = \sum_i I_{gen,i,DC}(t) \quad (2.8)$$

Now, with U_{LN} being the line to ground voltage, we can also write an expression for the *steady state short-circuit current*

$$I_k = \frac{U_{LN}}{X_d} \quad (2.9)$$

In like manner using the *transient synchronous reactance* X_d' we have the *transient short circuit current*

$$I'_k = \frac{U_{LN}}{X'_d} . \quad (2.10)$$

Furthermore, we have for the *sub-transient short-circuit current* component

$$I''_k = \frac{U_{LN}}{X''_d} . \quad (2.11)$$

Now, at any given time t the short circuit current of a given generator is

$$i_k = I_{gen,AC}(t) + I_{gen,DC}(t) \quad (2.12)$$

and the AC component of a generator short circuit current is [51]

$$I_{gen,AC}(t) = \sqrt{2} \left[(I''_k - I'_k) e^{-t/T_d''} + (I'_k - I_k) e^{-t/T_d'} + I_k \right] \sin(\omega t - \varphi_k) \quad (2.13)$$

while the DC component of that current is

$$I_{gen,DC}(t) = \sqrt{2} I''_k e^{-t/T_g} \sin(\varphi_k) \quad (2.14)$$

Therefore, at any given time, the short circuit current of a generator is:

$$i_k = \sqrt{2} \left[(I''_k - I'_k) e^{-t/T_d''} + (I'_k - I_k) e^{-t/T_d'} + I_k \right] \sin(\omega t - \varphi_k) + \sqrt{2} I''_k e^{-t/T_g} \sin(\varphi_k) \quad (2.15)$$

As the *generator stator resistance* is very small compared to the *reactance* we can assume $\varphi_k = 90^\circ$ or $\varphi_k = \frac{\pi}{2}$. Then, we have $\sin \varphi_k = 1$ and $\sin(\omega t - \varphi_k) = \cos(\omega t)$, therefore the foregoing equation becomes:

$$i_k = \sqrt{2} \left[(I_k'' - I_k') e^{-t/T_d''} + (I_k' - I_k) e^{-t/T_d'} + I_k \right] \cos \omega t + \sqrt{2} I_k'' e^{-t/T_g} \quad (2.16)$$

meaning that for any generator in the system we have

$$i_{k,i} = \sqrt{2} \left[(I_{k,i}'' - I_{k,i}') e^{-t/T_{d,i}''} + (I_{k,i}' - I_{k,i}) e^{-t/T_{d,i}'} + I_{k,i} \right] \cos \omega t + \sqrt{2} I_{k,i}'' e^{-t/T_{g,i}} \quad (2.17)$$

Hence, the short circuit in a system with n generators becomes:

$$i_{k,grid} = \sum_i^n i_{k,i} \quad (2.18)$$

and so, with Eq.(2.17)

$$\begin{aligned} i_{k,grid} = & \sqrt{2} \left[(I_{k,1}'' - I_{k,1}') e^{-t/T_{d,1}''} + (I_{k,1}' - I_{k,1}) e^{-t/T_{d,1}'} + I_{k,1} \right] \cos \omega t + \sqrt{2} I_{k,1}'' e^{-t/T_{g,1}} \\ & + \sqrt{2} \left[(I_{k,2}'' - I_{k,2}') e^{-t/T_{d,2}''} + (I_{k,2}' - I_{k,2}) e^{-t/T_{d,2}'} + I_{k,2} \right] \cos \omega t + \sqrt{2} I_{k,2}'' e^{-t/T_{g,2}} + \dots \\ & + \sqrt{2} \left[(I_{k,n}'' - I_{k,n}') e^{-t/T_{d,n}''} + (I_{k,n}' - I_{k,n}) e^{-t/T_{d,n}'} + I_{k,n} \right] \cos \omega t + \sqrt{2} I_{k,n}'' e^{-t/T_{g,n}} \end{aligned} \quad (2.19)$$

Assuming now that the time constants T_d'', T_d' and T_g for all machines are identical, which is a valid assumption as most of the turbo generators in Ireland have similar values of $T_d'' \approx 0.04s, T_d' \approx 5s$. Now with:

$$T_d'' = T_{d,1}'' = T_{d,2}'' = T_{d,n}'' \quad (2.20)$$

and

$$T_d' = T_{d,1}' = T_{d,2}' = T_{d,n}' \quad (2.21)$$

and also

$$T_g = T_{g,1} = T_{g,2} = T_{g,n} , \quad (2.22)$$

As previously shown, this is a reasonable assumption in practice. Now, we have as far as large scale synchronous generators are concerned:

$$\begin{aligned}
i_{k,grid} = & \sqrt{2} \left[(I''_{k,1} - I'_{k,1}) e^{-t/T_d''} + (I'_{k,1} - I_{k,1}) e^{-t/T_d'} + I_{k,1} \right] \cos \omega t + \sqrt{2} I''_{k,1} e^{-t/T_g} + \\
& + \sqrt{2} \left[(I''_{k,2} - I'_{k,2}) e^{-t/T_d''} + (I'_{k,2} - I_{k,2}) e^{-t/T_d'} + I_{k,2} \right] \cos \omega t + \sqrt{2} I''_{k,2} e^{-t/T_g} + \dots \\
& + \sqrt{2} \left[(I''_{k,n} - I'_{k,n}) e^{-t/T_d''} + (I'_{k,n} - I_{k,n}) e^{-t/T_d'} + I_{k,n} \right] \cos \omega t + \sqrt{2} I''_{k,n} e^{-t/T_g}
\end{aligned} \tag{2.23}$$

or

$$\begin{aligned}
i_{k,grid} = & \sqrt{2} \left\{ (I''_{k,1} - I'_{k,1}) + (I''_{k,2} - I'_{k,2}) + \dots + (I''_{k,n} - I'_{k,n}) \right\} e^{-t/T_d''} \cos \omega t + \\
& + \left\{ (I'_{k,1} - I_{k,1}) + (I'_{k,2} - I_{k,2}) + \dots + (I'_{k,n} - I_{k,n}) \right\} e^{-t/T_d'} \cos \omega t + \\
& + \left\{ I_{k,1} + I_{k,2} + \dots + I_{k,n} \right\} \cos \omega t + \sqrt{2} e^{-t/T_g} (I''_{k,1} + I''_{k,2} + \dots + I''_{k,n})
\end{aligned} \tag{2.24}$$

so that:

$$\begin{aligned}
i_{k,grid} = & \sqrt{2} \left\{ (I''_{k,1} + I''_{k,2} + \dots + I''_{k,n}) - (I'_{k,1} + I'_{k,2} + \dots + I'_{k,n}) \right\} e^{-t/T_d''} \cos \omega t + \\
& + \left\{ (I'_{k,1} + I'_{k,2} + \dots + I'_{k,n}) - (I_{k,1} + I_{k,2} + \dots + I_{k,n}) \right\} e^{-t/T_d'} \cos \omega t + \\
& + \left\{ I_{k,1} + I_{k,2} + \dots + I_{k,n} \right\} \cos \omega t + \sqrt{2} e^{-t/T_g} (I''_{k,1} + I''_{k,2} + \dots + I''_{k,n})
\end{aligned} \tag{2.25}$$

Now by defining the short circuit current for the grid as

$$\begin{aligned}
i_{k,grid} = & \sqrt{2} \left\{ \left(\sum_i^n I''_{k,i} \right) - \left(\sum_i^n I'_{k,i} \right) \right\} e^{-t/T_d''} \cos \omega t + \left\{ \left(\sum_i^n I'_{k,i} \right) - \left(\sum_i^n I_{k,i} \right) \right\} e^{-t/T_d'} \cos \omega t + \\
& + \left(\sum_i^n I_{k,i} \right) \cos \omega t + \sqrt{2} e^{-t/T_g} \left(\sum_i^n I''_{k,i} \right)
\end{aligned} \tag{2.26}$$

we can write Eq.(2.25) in a form, similar to Eq.(2.16):

$$\begin{aligned}
i_{k,grid} = & \sqrt{2} \left[\left(\left(\sum_i^n I''_{k,i} \right) - \left(\sum_i^n I'_{k,i} \right) \right) e^{-t/T_d''} + \left(\left(\sum_i^n I'_{k,i} \right) - \left(\sum_i^n I_{k,i} \right) \right) e^{-t/T_d'} + \left(\sum_i^n I_{k,i} \right) \right] \cos \omega t + \\
& + \sqrt{2} \left(\sum_i^n I''_{k,i} \right) e^{-t/T_g}
\end{aligned} \tag{2.27}$$

Furthermore, we can in turn rewrite this equation as

$$i_{k,grid} = \sqrt{2} \left[(I''_{k,grid} - I'_{k,grid}) e^{-t/T_d''} + (I'_{k,grid} - I_{k,grid}) e^{-t/T_d'} + I_{k,grid} \right] \cos \omega t + \sqrt{2} I''_{k,grid} e^{-t/T_g} \tag{2.28}$$

Inserting Eqs.(2.9), (2.10) and (2.11) into Eq.(2.15) we have the short circuit contribution from one generator based on the reactance:

$$i_k = \sqrt{2}U_{LN} \left[\left(\frac{1}{X_d''} - \frac{1}{X_d'} \right) e^{-t/T_d''} + \left(\frac{1}{X_d'} - \frac{1}{X_d} \right) e^{-t/T_d'} + \frac{1}{X_d} \right] \cos \omega t + \sqrt{2} \frac{U_{LN}}{X_d''} e^{-t/T_g} \quad (2.29)$$

Moreover, by inserting Eqs. (2.37), (2.38) and (2.39) into Eq.(2.28). We have for the short circuit current of the lumped generator at any given time t

$$i_{k,grid} = \sqrt{2}U_{LN} \left[\left(\sum_i \frac{1}{X_{d,i}''} - \sum_i \frac{1}{X_{d,i}'} \right) e^{-t/T_d''} + \left(\sum_i \frac{1}{X_{d,i}'} - \sum_i \frac{1}{X_{d,i}} \right) e^{-t/T_d'} + \sum_i \frac{1}{X_{d,i}} \right] \cos \omega t \\ + \sqrt{2}U_{LN} \left(\sum_i \frac{1}{X_{d,i}''} \right) e^{-t/T_g} \quad (2.30)$$

Thus, we have for the synchronous reactance of the lumped generator:

$$I_{k,grid} = \sum_i I_{k,i} \quad (2.31)$$

which can be expressed as:

$$I_{k,grid} = \sum_i \frac{U_{LN,i}}{X_{d,i}} \quad (2.32)$$

Now, assuming a voltage $U_{grid,LN}$ the expression for the *synchronous* reactance of a lumped generator can be written as

$$\frac{U_{LN,grid}}{X_{d,grid}} = \sum_i \frac{U_{LN,i}}{X_{d,i}}. \quad (2.33)$$

Thus we can now write for the *synchronous* reactance of a lumped generator with a terminal line to ground voltage $U_{LN,grid}$

$$X_{d,grid} = \frac{U_{LN,grid}}{\sum_i \frac{U_{LN,i}}{X_{d,i}}} \quad (2.34)$$

Equivalently for the transient synchronous inertia, we have

$$X'_{d,grid} = \frac{U_{LN,grid}}{\sum_i \frac{U_{LN,i}}{X'_{d,i}}} \quad (2.35)$$

In a similar way, we have for the sub-transient synchronous inertia

$$X''_{d,grid} = \frac{U_{LN,grid}}{\sum_i \frac{U_{LN,i}}{X''_{d,i}}} \quad (2.36)$$

For similar line voltages at all generators we can simply write:

$$X_{d,grid} = \frac{1}{\sum_i \frac{1}{X_{d,i}}} \quad (2.37)$$

and in light manner:

$$X'_{d,grid} = \frac{1}{\sum_i \frac{1}{X'_{d,i}}} \quad (2.38)$$

and

$$X''_{d,grid} = \frac{1}{\sum_i \frac{1}{X''_{d,i}}} \quad (2.39)$$

Using these lumped parameters we can model the *limited* two-machine system describing a generator connected to a *finite* grid. Hence, if a frequency drop event similar to that shown in Figure 1-3 is recreated, the resulting load oscillations in the context of the two-machine MATLAB model now exhibits comparable oscillatory behaviour.

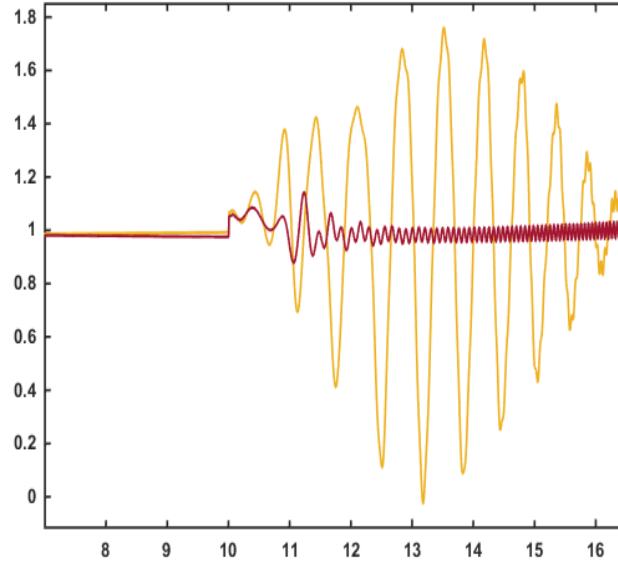


Figure 2-37: generator power output oscillations when connected to a lumped grid

Yellow graph: Generator power oscillations similar to them in Figure 1-3

Red graph: Generator power oscillations when lumped grid PSS controller was optimized

The generator power oscillations depicted in Figure 2-37 show good agreement with the actual oscillation depicted in Figure 1-3. More interestingly, after the attempt to minimise the power oscillation of the specific generator was unsuccessful some parameters in the PSS of the lumped generator were changed thereby achieving optimum damping of the oscillation which is shown in the above figure by the red graph. This plot highlights the interaction between the grid and generator in a finite system.

Now, if the time constants are varying from each other, then a different approach as it is shown below shall be used. For different relaxation functions we have:

$$f_i(t) = A_i e^{-t/T_i} . \quad (2.40)$$

The total signal is the sum of the relaxation functions

$$c(t) = \sum_{i=1}^N f_i(t) = \sum_{i=1}^N A_i e^{-t/T_i} . \quad (2.41)$$

Next, we introduce the definition of the integral relaxation time T_{in} and the effective relaxation time T_{ef}

$$T_{in} = \frac{1}{c(0)} \int_0^{\infty} c(t) dt = \frac{\sum_{i=1}^N A_i T_i}{\sum_{i=1}^N A_i} , \quad (2.42)$$

$$T_{ef} = -\frac{c(0)}{\dot{c}(t)} = \frac{\sum_{i=1}^N A_i}{\sum_{i=1}^N (A_i / T_i)}, \quad (2.43)$$

Now, we can compare the normalized signal $c(t)/c(0)$ with the following relaxation functions

$$c_{in}(t) = e^{-t/T_{in}}, \quad (2.44)$$

$$c_{ef}(t) = e^{-t/T_{ef}} \quad (2.45)$$

The result is shown in Figure 2-38. One can see that $c_{ef}(t)$ gives a good approximation of the real signal at shorter times $t < 1$, while $c_{in}(t)$ gives a good approximation at long times $t > 10$.

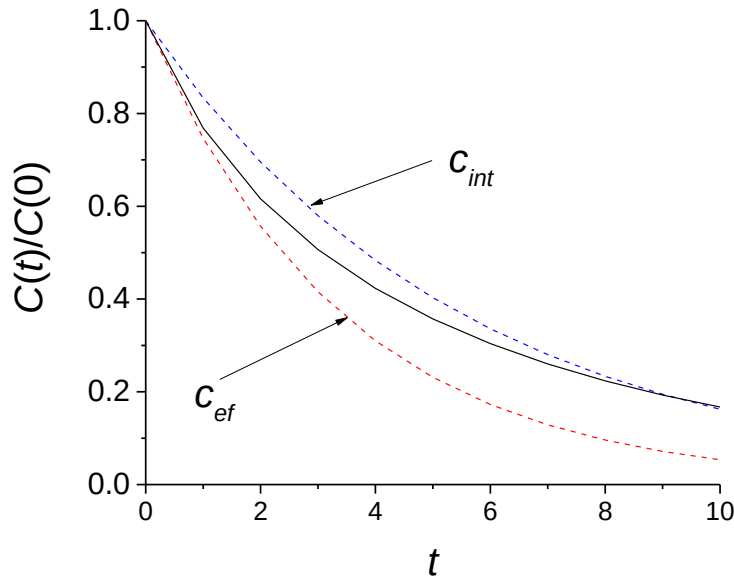


**Figure 2-38: Total signal vs. time for $A_i = 1$ and $T_i = i$ (solid line).
Dashed lines: relaxation functions.**

At this stage it seems that the effective relaxation time is more appropriate for our use. However, further analysis shall be subject of future publications.

3 MATHEMATICAL MODELS

We have already seen that an ever-increasing requirement to integrate greater and greater amounts of electrical energy from renewable sources especially from wind turbines and solar photo-voltaic installations exists. Moreover, recent experience in the island of Ireland demonstrates that this requirement influences the behaviour of conventional generating stations. One observation is the *change* in the electrical power output of synchronous generators following a transient disturbance, especially their oscillatory behaviour accompanied by similar oscillatory behaviour of the grid frequency, both becoming more pronounced with reducing grid inertia. This behaviour *cannot be reproduced with the existing mathematical models* as described by Figure 1-4, indicating that an understanding of the behaviour of synchronous generators, subjected to various disturbances especially in a system with low inertia requires a different modelling technique, here the *itinerant oscillator model* borrowed from statistical mechanics. Thus, two models of a generating station, based on a type of double pendulum described by a system of coupled nonlinear differential equations and suitable for analysis of its stability corresponding to infinite or finite grid inertia, are presented. Formal analytic solutions of the equations of motion of the grid-generator system are given and compared with numerical solutions. In particular, the new finite grid model will allow one to identify limitations to the operational range of the synchronous generators used in conventional power generation and also to identify limits, such as the allowable Rate of Change of Frequency which is currently set to $\pm 0.5\text{Hz/s}$. This, as we have seen, is a major factor in describing the volatility of a grid. Analytical dynamics modelling will also indicate requirements to the total inertia necessary, which is currently provided by conventional power generators only, thus allowing one to maximise the usage of grid connected non-synchronous generators e.g. wind turbines and solar photo-voltaic installations.

3.1 INTRODUCTION

Clearly, the requirement to decarbonize electrical energy production gives rise to an increase of wind turbine and solar photo-voltaic based energy generation. These renewable generation sources in comparison to conventional *synchronously* grid connected turbo generators of gas, oil or even coal fired power stations are as we have seen *asynchronously* connected to the grid. Inadvertently, by increasing the extent of *asynchronously grid connected generation* the stability of the entire electricity network is reduced thereby also

affecting the existing conventional power generation units. Examples are the unwanted increase of electrical power oscillations of conventional power generating units and also consequential tripping of some larger units which occur during transient events at times of high wind generation in the system. In addition, certain transient events like a sudden generation to load imbalance which arises after a trip of a generating unit, a disconnection of a large load or during a system separation event also manifest themselves in a grid frequency drop, rise or even oscillation. We have also seen that the weakening of the grid frequency strength manifests itself via an increase of the Rate of Change of Frequency (ROCOF). Historically, the ROCOF was limited by the manufacturers of conventional generating units to $\pm 0.5\text{Hz/s}$, suggesting that an increase in excess of this value needs to be analysed especially in relation to the manufacturer's specification thereby allowing for a further increase of non-synchronously connected electrical generation sources like wind turbine generators and solar PV installations. Now to enable one to initially quantify the influence of more severe ROCOF events on the operation and the integrity of conventional generating units it is necessary to define an appropriate dynamical model which we can mathematically analyse for the purpose of calculation of the ensuing electrical and mechanical behaviour of the generator sets. Our objective is to assess the consequences for their individual components, e.g. the generator with its electrical windings in the rotor and stator, the turbine with its blades and the coupling to the generator.

The chapter is organized as follows. We first envisage the generator-grid system as a driven damped torsional pendulum (making large oscillations), where we suppose that the grid has infinite inertia. This leads to a torque-balance equation in the form of the equation of motion of the angle δ between the rotor and stator fields which is now no longer *constant*, instead oscillating about a fixed axis because of the transient disturbance. This *infinite* grid inertia rotator model allows one to estimate the energy dependent frequency of the disturbance in terms of an elliptic integral as well as the response to a short impulsive change of torque due to a transient fault (trip of a generator or disconnection of a large consumer). Next, the model is generalized to a *finite* grid inertia (as obtains in an isolated grid like that of the island of Ireland). The generalization leads to two coupled nonlinear rotator-model-like equations for the motion of the generator-grid system whereby the system may oscillate about a temporary equilibrium position, hence, the generic term "itinerant oscillator". These nonlinear equations may be solved using matrix continued fractions, moreover the response of the generator-grid system so obtained following the sudden imposition of a set of impulses is substantially in accordance with an actual ROCOF event. This second model explicitly takes account of the fact that reviewing the

measured data, indicates that the *grid frequency also exhibits an oscillation in phase with the power oscillation, indicating that these two are influencing each other*. Since the grid inertia in the island of Ireland is far from infinite any transient disturbance will induce a rotor angle oscillation which in turn leads to a grid oscillation. Hence, the assumption that is valid for an infinite grid (model 1, see 3.2 below), namely that no feedback of the oscillation experienced by the turbine generator shaft back to the grid exists, leads us to the conclusion that the *single mass mechanical analogue is not useful when applied to grids with small inertia*. Thus, the main focus of our attention will be on the second model.

3.2 A SINGLE MASS ROTATING PENDULUM MODEL: EQUATION OF MOTION (MODEL 1)

The dynamical equations of central importance in power system stability analysis are the torsional equations of motion describing the effect of imbalance between the electromagnetic and mechanical torques of the individual machines. Now during *stable* operation of a synchronous generator, the electromagnetic torque in the airgap between rotor and stator of the generator balances the mechanical torque applied by the turbine rotor to the rotor of the generator, $\tau_{el} = \tau_m$. However, when a disturbance occurs, an accelerating torque which can be either positive or negative acts on the turbo shaft inertia. The turbo shaft comprises the coupled turbine and generator rotor, which during such a disturbance either accelerates or de-accelerates.

The electric currents in the stator windings of a synchronized generator create an electromagnetic field in the air gap of the generator [11]. The strength of the magnetic field $B_s = |\mathbf{B}_s|$ is constant for constant stator currents. Now, since these currents are sinusoidal the field vector $\mathbf{B}_s$ rotates about the generator axis with angular velocity $\omega_0 = 2\pi f_0$, where f_0 is the grid frequency. In a *synchronous* machine, the rotor field vector $\mathbf{B}_r$ rotates with the same angular velocity as the stator field. The magnitude of the rotor field $B_r = |\mathbf{B}_r|$ is constant as the field is induced by a DC current. Thus, the following electromagnetic torque τ_{el} arises in the air gap of the generator, viz.,

$$\tau_{el} = k\mathbf{B}_r \times \mathbf{B}_s, \quad (3.1)$$

where k is a characteristic constant of the machine which depends on its geometry.

Equation (3.1) can be rewritten as:

$$\boldsymbol{\tau}_{el} = k \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ B_R \cos \delta_R & B_R \sin \delta_R & 0 \\ B_S \cos \delta_S & B_S \sin \delta_S & 0 \end{vmatrix} = \mathbf{e}_z k B_R B_S \sin \delta, \quad (3.2)$$

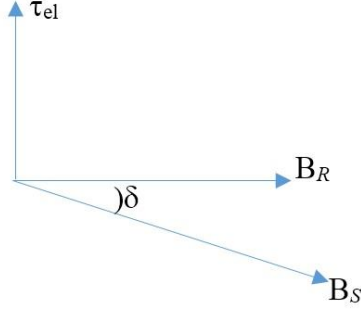


Figure 3-1 The electromagnetic torque
 $\tau_{el} = k B_S B_R \sin \delta$ in the air gap of the generator.

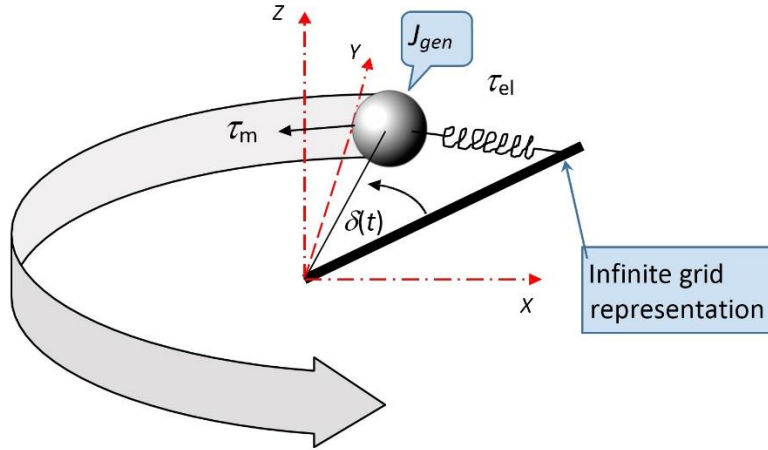


Figure 3-2 Scheme with an infinite inertia grid

where δ_R and δ_S are the angles between the vectors $\mathbf{B}_R$ and $\mathbf{B}_S$ and the x-axis, respectively, and $\delta = \delta_S - \delta_R$ is the angle between $\mathbf{B}_S$ and $\mathbf{B}_R$ (see Figure 3-1 and Figure 3-2). The angle δ is called the rotor angle or in some cases the torque angle. The additional mechanical torque τ_m is transmitted from the turbine to the generator via the (bolted) coupling of the turbine and the generator rotors. Now if an imbalance between the torques τ_{el} and τ_m exists, the net torque $\tau_m - \tau_{el}$ causes a rate of change of angular momentum of the generator. Consequently, the angle $\delta(t)$ between $\mathbf{B}_S$ and $\mathbf{B}_R$ either increases or decreases. The total rotational kinetic energy of the system is $\Gamma = J \dot{\theta}^2 / 2$, where

$\theta(t) = \omega_0 t + \delta(t)$ and J is the combined moment of inertia of generator and turbine while the potential energy is $\Pi = kB_s B_r [1 - \cos(\theta(t) - \omega_0 t)]$. The total energy dissipated due to the damping torque is $D = K_D (\dot{\theta}(t) - \omega_0)^2 / 2$. The damping torque in this case is the torque which is created by the *asynchronous* operation of the generator. As we are analysing the behaviour of a synchronous generator any oscillation of its frequency results in creation of a *so-called* slip which results in a change to the electrical torque in the air gap whereby the power output is increased or decreased. Clearly such a torque only exists when the angular velocity of the generator rotor *differs* from the angular velocity of the grid. All other energy losses, like friction or thermal ones are neglected; hence we have the differential equation for the motion of δ :

$$J\ddot{\delta}(t) + K_D\dot{\delta}(t) + kB_s B_r \sin \delta(t) = \tau_m, \quad (3.3)$$

or

$$\ddot{\delta}(t) + \beta\dot{\delta}(t) + \xi \sin \delta(t) = \bar{\tau}_m, \quad (3.4)$$

where we have the parameter $\beta = K_D / J$ which is a friction parameter, the force parameter $\xi = kB_s B_r / J$, and the applied mechanical torque from the turbine $\bar{\tau}_m = \tau_m / J$. This single-mass rotator model is commonly used to analyze the dynamic response of a synchronous generator in an infinite grid [11] [53] [54]. The mechanical analogue is a *driven damped pendulum*. In accordance with Eq. (3.4), both grid and generator initially rotate with constant rotor angle $\delta_0 = \arcsin[\tau_m / (kB_s B_r)] = \text{const}$ (see Figure 3-2). The bar in Figure 3-2 symbolizing the grid, rotates with the grid frequency and is of *infinite* inertia, meaning that it cannot be influenced by the inertia of a single generator. In the steady-state situation both inertias, comprising that of the generator turbine shaft J_{gen} and the grid J_{grid} rotate in a horizontal x-y-plane about the fixed z-axis with angular velocity ω_0 corresponding to the nominal grid-frequency of 50Hz. However, during *transient* faults on the system, like the trip of a generator or the disconnection of a large load, the grid-frequency changes. Hence the generator will experience an *additional* torque as the grid inertia alters its angular velocity. This disturbance in turn leads to an oscillation of the rotor angle $\delta(t)$. The rotor angle oscillations in turn cause a power output oscillation because the electrical power of a generator is determined via the rotor angle as $P_{el} = P_{\text{max}} \sin \delta(t)$. Hence the rate of change of the grid frequency is of major importance. If a disturbance of the motion takes place the

energy-dependent period T of oscillations of the angle $\delta(t)$ may be estimated using the idealized free motion of a generator when both the damping and driving torques are absent, i.e., $\beta = 0$ and $\tau_m = 0$. The solution of Eq. (3.4) in this case can be obtained as follows [55]. For $\beta = \bar{\tau}_m = 0$, we rearrange Eq. (3.4) as

$$2\dot{\delta}(t)\ddot{\delta}(t) + 2\xi\dot{\delta}(t)\sin\delta(t) = \frac{d}{dt}\left[\dot{\delta}^2(t) - 2\xi\cos\delta(t)\right] = 0. \quad (3.5)$$

Thus, we have from Eq. (3.5)

$$\dot{\delta}^2(t) = 2\xi[\cos\delta(t) - \cos\delta_0] = 4\xi\left[\sin^2\frac{\delta_0}{2} - \sin^2\frac{\delta(t)}{2}\right] \quad (3.6)$$

or

$$\frac{d\delta}{2\sqrt{\sin^2(\delta_0/2) - \sin^2(\delta/2)}} = \sqrt{\xi}dt, \quad (3.7)$$

where δ_0 is a constant of integration. By integrating Eq. (3.7) with the initial condition $\delta(t_0) = 0$, we have

$$\int_0^{\delta} \frac{d(\delta/2)}{\sqrt{m - \sin^2(\delta/2)}} = \int_0^{m^{-1/2}\sin\delta/2} \frac{dt}{\sqrt{(1-mt^2)(1-t^2)}} = \sqrt{\xi}(t-t_0), \quad (3.8)$$

where $m = \sin^2\delta_0/2$, so yielding the solution

$$\sin[\delta(t)/2] = \sqrt{m} \operatorname{sn}\left[\sqrt{\xi}(t-t_0)|m\right], \quad (3.9)$$

Here $\operatorname{sn}(u|m)$ is a Jacobian elliptic function [56] with modulus $m = \sin^2(\delta_0/2)$, δ_0 is the amplitude and t_0 is the initial time at which $\delta = \delta_0$ and $\dot{\delta} = 0$. The exact expression for the period T is given by

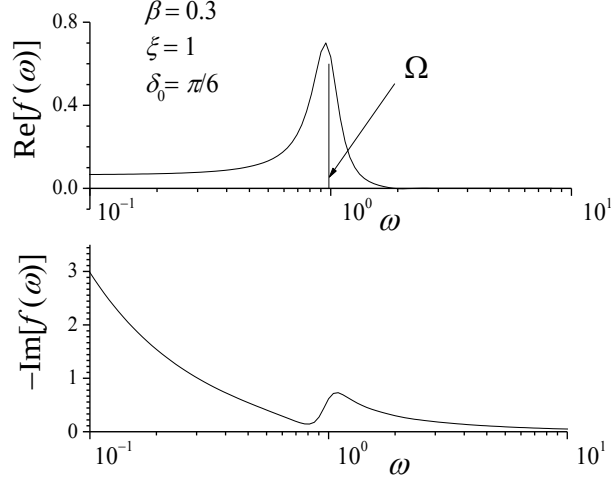


Figure 3-3 Real and imaginary parts of the linear combination
 $f(\omega) = [\tilde{c}_{i-1}(i\omega) - \tilde{c}_{i-1}(i\omega)] / 2i$, calculated by using matrix continued fraction method. The frequency of free motion Ω is given by Eq.(3.11)

$$T = 4 \sqrt{\frac{J}{kB_S B_R}} K(m), \quad (3.10)$$

where $K(m)$ is a complete elliptic integral of the first kind [56]. Thus, the angular frequency of the periodic motion is

$$\Omega = \frac{2\pi}{T} = \frac{\pi \sqrt{kB_S B_R / J}}{2K(m)}. \quad (3.11)$$

In the general case, we suppose that a disturbance of the motion occurs at the instant $t_0 = 0$ thereupon the torque $\bar{\tau}_m$ changes from the initial value $\bar{\tau}_m^0 = \xi \sin \delta_0$ to $\bar{\tau}_m < \bar{\tau}_m^0$. Here $\delta_0 = \delta(0)$ is the initial angle. The value of δ_0 depends on the applied torque of the turbine. The initial value of $\dot{\delta}_0 = \dot{\delta}(0)$ is zero. We have from Eq. (3.4) for the function $c_{nq}(t) = \dot{\delta}^n r^q$ with $r = e^{-i\delta(t)}$

$$\begin{aligned} \frac{d}{dt} c_{nq}(t) &= n \dot{\delta}^{n-1} r^q \ddot{\delta} - iq \dot{\delta}^{n+1} r^q \\ &= -iq c_{n+1q}(t) - n \dot{\delta}^{n-1} r^q [\beta \dot{\delta}(t) + \xi \sin \delta(t) - \bar{\tau}_m]. \end{aligned} \quad (3.12)$$

Here we substitute $\ddot{\delta}(t) = -\beta \dot{\delta}(t) - \xi \sin \delta(t) + \bar{\tau}_m$ from Eq. (3.4). By using $\sin \delta(t) = (r^{-1} - r) / 2i$ in Eq. (3.12), we then have the following set of differential-recurrence relations for $c_{nq}(t)$:

$$\frac{d}{dt}c_{nq}(t) = -\beta nc_{nq}(t) - iq c_{n+1q}(t) - \frac{in}{2} \left[\xi c_{n-1q+1}(t) + 2i\bar{\tau}_m c_{n-1q}(t) - \xi c_{n-1q-1}(t) \right]. \quad (3.13)$$

By introducing infinite column vectors

$$\mathbf{C}_0(t) = 0, \quad \mathbf{C}_n(t) = \begin{pmatrix} \vdots \\ c_{n-1-2}(t) \\ c_{n-1-1}(t) \\ c_{n-10}(t) \\ c_{n-11}(t) \\ c_{n-12}(t) \\ \vdots \end{pmatrix} \quad (n > 0), \quad (3.14)$$

the differential-recurrence equations (3.13) can be transformed into the tri-diagonal matrix form which may be solved explicitly,

$$\frac{d}{dt} \mathbf{C}_n(t) = \mathbf{Q}_n^- \mathbf{C}_{n-1}(t) + \mathbf{Q}_n \mathbf{C}_n(t) + \mathbf{Q}_n^+ \mathbf{C}_{n+1}(t). \quad (3.15)$$

Here the matrices $\mathbf{Q}_n$ and $\mathbf{Q}_n^\pm$ are

$$\mathbf{Q}_n = -\beta(n-1)\mathbf{I}, \quad (3.16)$$

$$\mathbf{Q}_n^+ = -i \begin{pmatrix} \ddots & \ddots & \ddots & & \\ \ddots & -1 & 0 & \ddots & \\ \ddots & 0 & 0 & 0 & \ddots \\ & \ddots & 0 & 1 & \ddots \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \quad (3.17)$$

$$\mathbf{Q}_n^- = -(n-1) \begin{pmatrix} \ddots & \ddots & \ddots & & \\ \ddots & \bar{\tau}_m & i\xi/2 & \ddots & \\ \ddots & -i\xi/2 & \bar{\tau}_m & i\xi/2 & \ddots \\ & \ddots & -i\xi/2 & \bar{\tau}_m & \ddots \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \quad (3.18)$$

and $\mathbf{I}$ is the unit matrix of infinite dimension.

Equation (3.15) may now be formally solved by the matrix continued fraction method [14] [57]. On considering the initial conditions $\delta(0) = \delta_0$ and $\dot{\delta}(0) = 0$ and taking the Laplace

transformation of Eq. (3.15), we have algebraic matrix recurrence relations for the Laplace transforms $\tilde{\mathbf{C}}_n(s)$:

$$-s\tilde{\mathbf{C}}_1(s) + \mathbf{Q}_1^+ \tilde{\mathbf{C}}_2(s) = -\mathbf{C}_1(0), \quad (3.19)$$

$$\mathbf{Q}_n^- \tilde{\mathbf{C}}_{n-1}(s) + (\mathbf{Q}_n - s\mathbf{I}) \tilde{\mathbf{C}}_n(s) + \mathbf{Q}_n^+ \tilde{\mathbf{C}}_{n+1}(s) = 0 \quad (n > 1), \quad (3.20)$$

where

$$\tilde{\mathbf{C}}_n(s) = \int_0^\infty \mathbf{C}_n(t) e^{-st} dt. \quad (3.21)$$

Here we have used

$$\mathbf{C}_1(0) = \begin{pmatrix} \vdots \\ e^{2i\delta_0} \\ e^{i\delta_0} \\ 1 \\ e^{-i\delta_0} \\ e^{-2i\delta_0} \\ \vdots \end{pmatrix}, \quad \mathbf{C}_n(0) = 0, \quad n > 2.$$

We seek the solution of the linear Eqs. (3.19) and (3.20) as [14]

$$\tilde{\mathbf{C}}_n(s) = \mathbf{\Lambda}_n \mathbf{Q}_n^- \tilde{\mathbf{C}}_{n-1}(s) + \mathbf{R}_n, \quad (3.22)$$

where $\mathbf{\Lambda}_n$ is the matrix continued fraction defined by the 2 index recurrence matrix equation

$$\mathbf{\Lambda}_n = \left[s\mathbf{I} - \mathbf{Q}_n - \mathbf{Q}_n^+ \mathbf{\Lambda}_{n+1} \mathbf{Q}_{n+1}^- \right]^{-1}, \quad (3.23)$$

so, allowing one to express $\mathbf{\Lambda}_n$ as an *infinite* matrix continued fraction [57]. Substituting Eq. (3.22) into Eqs. (3.19) and (3.20) yields

$$\left[s\mathbf{I} - \mathbf{Q}_1 - \mathbf{Q}_1^+ \mathbf{\Lambda}_2 \mathbf{Q}_2^- \right] \mathbf{R}_1 = \mathbf{C}_1(0) + \mathbf{Q}_1^+ \mathbf{R}_2, \quad (3.24)$$

$$\left[s\mathbf{I} - \mathbf{Q}_n - \mathbf{Q}_n^+ \mathbf{\Lambda}_{n+1} \mathbf{Q}_{n+1}^- \right] \mathbf{R}_n = \mathbf{Q}_n^+ \mathbf{R}_{n+1} \quad (n > 1). \quad (3.25)$$

Equations (3.24) and (3.25) can be solved exactly for the desired *particular solutions* $\mathbf{R}_n$ in terms of the matrix continued fraction $\mathbf{\Delta}_1$:

$$\mathbf{R}_1 = \mathbf{\Delta}_1 \mathbf{C}_1(0), \quad \mathbf{R}_n = 0 \quad (n > 1). \quad (3.26)$$

Substituting Eq. (3.26) into Eq. (3.22) yields the matrix continued-fraction solution of Eqs. (3.19) and (3.20) for the Laplace transforms $\tilde{\mathbf{C}}_n(s)$ [7]

$$\tilde{\mathbf{C}}_1(s) = \mathbf{\Delta}_1 \mathbf{C}_1(0), \quad (3.27)$$

$$\tilde{\mathbf{C}}_n(s) = \mathbf{\Delta}_n \mathbf{Q}_n^- \tilde{\mathbf{C}}_{n-1}(s) = \mathbf{\Delta}_n \mathbf{Q}_n^- \mathbf{\Delta}_{n-1} \mathbf{Q}_{n-1}^- \dots \mathbf{\Delta}_1 \mathbf{C}_1(0) \quad (n > 1). \quad (3.28)$$

The matrix continued fraction solution we have obtained is very convenient for the purpose of computation (various algorithms for calculating matrix continued fractions are discussed in Refs. [14] and [57]). As far as the calculation of the infinite matrix continued fraction $\mathbf{\Delta}_n$ is concerned, we initially approximated it by some matrix continued fraction of *finite* order (by putting $\mathbf{Q}_{n+1} = 0$ at some $n = K$). At the same time, we confined the dimensions of the matrices $\mathbf{Q}_n$ and $\mathbf{Q}_n^\pm$ to some finite number $2M + 1$. Both K and M depend on the model parameters and must be chosen taking account of the desired degree of accuracy of the calculation. The calculation of the time-dependent functions $c_{nq}(t)$ can then be accomplished using the Fast Fourier Transform (FFT) algorithm as

$$\begin{aligned} c_{nq}(t) &\approx \frac{1}{\pi} \operatorname{Re} \left[\int_0^{\Omega'(N-1)} \tilde{c}_{nq}(i\omega) e^{i\omega t} d\omega \right] \\ &= \frac{\Omega'}{\pi} \operatorname{Re} \left[\sum_{s=1}^N \tilde{c}_{nq}(i\Omega'(s-1)) e^{i\Omega'(s-1)t} \right] \end{aligned} \quad (3.29)$$

or

$$\begin{aligned} c_{nq}(T(r-1)) &= \frac{\Omega'}{\pi} \operatorname{Re} \left[\sum_{s=1}^N \tilde{c}_{nq}(i\Omega'(s-1)) e^{i\Omega'T(s-1)(r-1)} \right] \\ &\quad (r = 1, 2, \dots, N), \end{aligned} \quad (3.30)$$

where

$$\Omega' = \frac{2\pi}{NT}, \quad N = 2^k. \quad (3.31)$$

Having calculated $c_{nq}(t)$, we can evaluate all other functions of interest. For example, we have

$$\sin \delta(t) = \frac{c_{0-1}(t) - c_{01}(t)}{2i} \quad (3.32)$$

and so on. The results of such calculations for the linear combination of the one-sided Fourier transforms $\tilde{c}_{1\pm 1}(i\omega)$ in the frequency domain are shown in Figure 3-3. Equation (3.4) can also be solved numerically using *Mathematica*[®]. A comparison of the numerical and continued fraction solutions of Eq. (3.4) is shown in Figure 3-4.

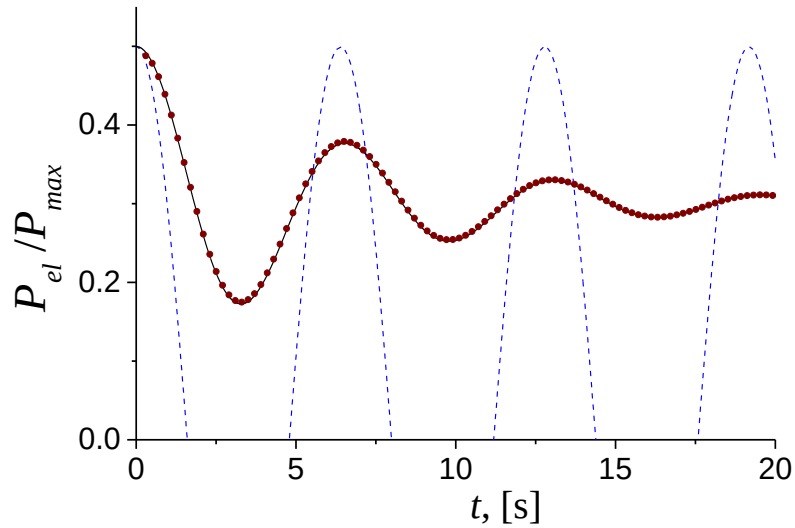


Figure 3-4 Solid line: numerical solution of Eq. (3.4) vs continued fraction solution obtained by using *Mathematica*[®].

Circles: continued fraction solution of Eq. (3.4) in the frequency domain following from the inverse Fourier transform with initial conditions $\delta_0 = \pi / 6$, $\dot{\delta}_0 = 0$ for the following model parameters $\xi = 1$, $\bar{\tau}_m = 0.3$, $\beta = 0.3$. Dashed line: analytical solution for $\beta = 0$, Eq. (3.9) whereby we may estimate the period analytically

Figure 3-5 shows the response of the system to a set of impulses (representing the alternating character of changes of $\bar{\tau}_m$ due to external events). This result can be compared with the measured power output of a single shaft combined cycle 400MW synchronous generator after an actual ROCOF event (see Figure 1-3).

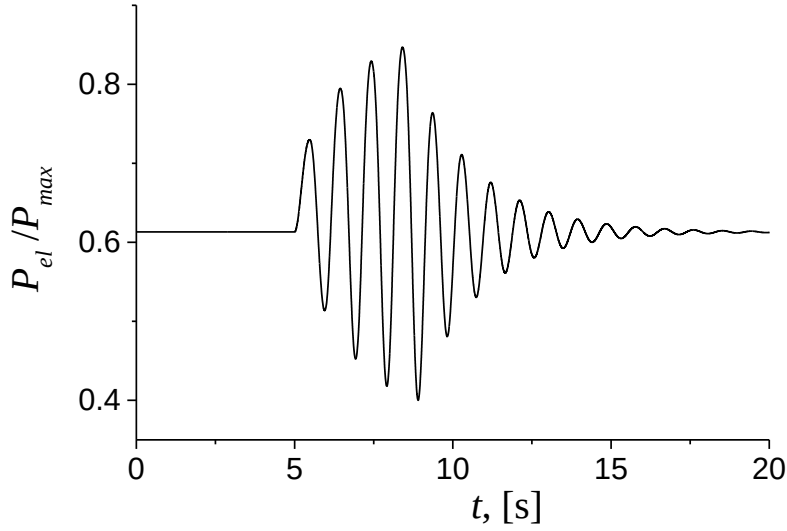


Figure 3-5 Response of the system (Eq. (3.4)) to a set of 4 impulses with torque $\bar{\tau}_m$ as shown in Figure 3-6

for the initial conditions $\delta_0 = 0.66$, $\dot{\delta}_0 = 0$ and the following model parameters $\xi = 60$, $\beta = 1$, $\Delta \bar{\tau}_m = 4$.

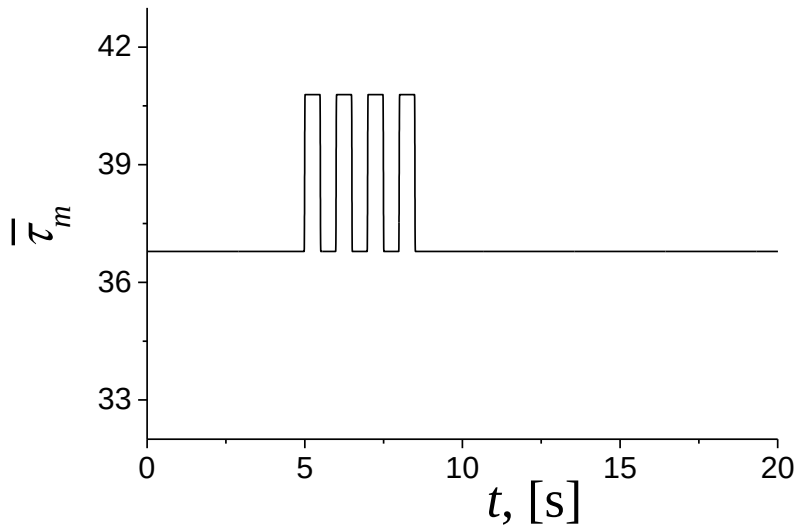


Figure 3-6: Set of impulses of torque $\bar{\tau}_m$

Here, the frequency of the torque pulses $\bar{\tau}_m$ is approximately $f_{\bar{\tau}} \approx 1\text{Hz}$ which is below the resonance frequency of $f_{\bar{\tau}} < \frac{\sqrt{\xi}}{2\pi}$ and has damping ratio $\zeta = \frac{\beta}{2\sqrt{\xi}} = 0.06$.

The power oscillation obtained from the model (Figure 3-5) reproduces qualitatively well the experimentally measured oscillation of the power in Figure 1-3 however it assumes *infinite* grid inertia, i.e. a *fixed* grid frequency. In spite of this, the most striking feature of Figure 1-3 is that the *grid frequency executes an oscillation in phase with the power*

oscillation which also indicates that the two are influencing each other. Furthermore, recent observations indicate that the smaller the inertia in the grid, the more the output power of a generator oscillates. Now the grid frequency experiences such drops about 20-30 times annually (see also Figure 1-9). These are usually due to trips of generating stations or disconnection of a large load (e.g. when an HVDC interconnector trips while exporting electric energy from Ireland to Great Britain). Hence, the ability to calculate exactly how the grid frequency behaves during such disturbances *demands an approach whereby that frequency can be influenced by the transient behaviour of a generating station rather than just the ability to calculate the maximum ROCOF alone*. As seen in the previous analysis, any transient disturbance can induce a rotor angle oscillation resulting in mechanical oscillations of the generator-turbine rotor. Hence, we *have oscillations of the power output of the set leading to a grid oscillation and in turn this grid oscillation again influences how the turbo-set oscillates and so on*. Thus, the assumption that is valid for an *infinite grid*, namely, that *no feedback* of the oscillation experienced by the turbine generator shaft back to the grid exists, leads one to the conclusion that the single-mass model is not useful when applied to grids with *small* inertia so that we need a *two-mass pendulum* model with two degrees of freedom.

3.3 THE TWO-MASS ROTATING PENDULUM (MODEL 2)

This model, where the grid now has *finite* moment of inertia such as obtains in the isolated grid of Ireland (see Figure 3-7), leads by Newton's third law to the following set of two coupled equations of motion for the generator and grid

$$J_{grid} \ddot{\theta}_{grid} + K_D (\dot{\theta}_{grid} - \dot{\theta}_{gen}) + \tau_{el\max} \sin(\theta_{grid} - \theta_{gen}) = \tau_1 = \tau_{grid} - \tau_{el\text{-}remain}, \quad (3.33)$$

$$J_{gen} \ddot{\theta}_{gen} + K_D (\dot{\theta}_{gen} - \dot{\theta}_{grid}) + \tau_{el\max} \sin(\theta_{gen} - \theta_{grid}) = \tau_2 = \tau_{gen}, \quad (3.34)$$

where $\tau_{el\max}$ is the maximum electromagnetic torque in the air gap, τ_2 is the torque executed by the turbine on the generator, τ_1 is the resulting torque executed on the grid (sum of all turbine torques less the torques due to the loads and remaining generators $\tau_{el\text{-}remain}$ on the grid). Here the damping torques characterized by the damping coefficient K_D are those torques, existing only when the rotor angle oscillates.

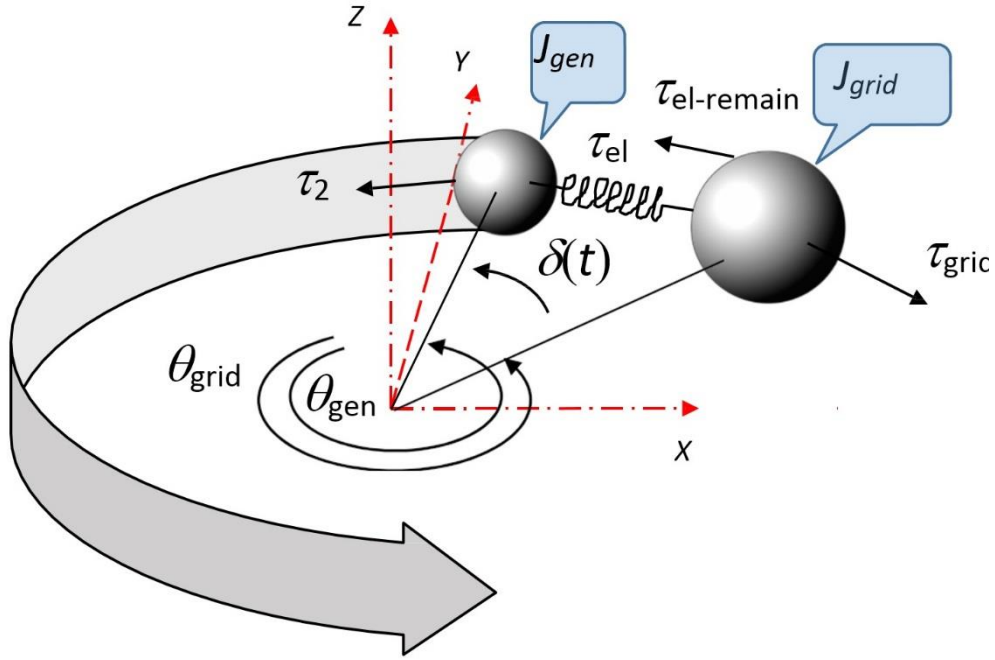


Figure 3-7 Scheme with a finite inertia of the grid

In this case, the turbine controller reacts to a drop of the frequency by increasing the output of the set. Thus, it opens the control valves to allow an increase of the driving medium consequently resulting in an increased power output. All generating units in the grid have such similar control behaviour. We emphasize that Eqs. (3.33) and (3.34) represent a model

which still has limitations, e.g. the energy losses due to resistances in the generators and lines are ignored.

The formal solution of the coupled Eqs. (3.33) and (3.34) is obtained as follows. Equations (3.33) and (3.34) can be rewritten as

$$\ddot{\theta}_i + \beta_i(\dot{\theta}_i - \dot{\theta}_j) + \xi_i \sin(\theta_i - \theta_j) = \bar{\tau}_i \quad (i = 1, 2 \text{ and } j = 2, 1), \quad (3.35)$$

where $\theta_1 = \theta_{grid}$, $\beta_1 = K_D^1 / J_{grid}$, $\xi_1 = \tau_{el\max} / J_{grid}$, $\theta_2 = \theta_{gen}$, $\beta_2 = K_D^2 / J_{gen}$, $\xi_2 = \tau_{el\max} / J_{gen}$, $\bar{\tau}_1 = \tau_1 / J_{grid}$ and $\bar{\tau}_2 = \tau_2 / J_{grid}$.

Now we again introduce functions related to the observables viz

$$c_{n_2 q_2}^{n_1 q_1} = \dot{\theta}_1^{n_1} \dot{\theta}_2^{n_2} r_1^{q_1} r_2^{q_2}, \quad n=0,1,2,\dots, \quad q=0, \pm 1, \pm 2, \dots \quad (3.36)$$

where $r_i = e^{-i\theta_i(t)}$.

Then, noting Eqs. (3.35) and (3.36), we have the multi-term algebraic recurrence equations

$$\begin{aligned} \frac{d}{dt} c_{n_2 q_2}^{n_1 q_1} = & -(\beta_1 n_1 + \beta_2 n_2) c_{n_2 q_2}^{n_1 q_1} + \beta_1 n_1 c_{n_2+1 q_2}^{n_1-1 q_1} + \beta_2 n_2 c_{n_2-1 q_2}^{n_1+1 q_1} - i q_1 c_{n_2 q_2}^{n_1+1 q_1} - i q_2 c_{n_2+1 q_2}^{n_1 q_1} \\ & + \bar{\tau}_1 n_1 c_{n_2 q_2}^{n_1-1 q_1} + \frac{i}{2} \xi_1 n_1 (c_{n_2 q_2+1}^{n_1-1 q_1-1} - c_{n_2 q_2-1}^{n_1-1 q_1+1}) + \bar{\tau}_2 n_2 c_{n_2-1 q_2}^{n_1 q_1} + \frac{i}{2} \xi_2 n_2 (c_{n_2-1 q_2-1}^{n_1 q_1+1} - c_{n_2-1 q_2+1}^{n_1 q_1-1}). \end{aligned} \quad (3.37)$$

Thus, again introducing column vectors $\mathbf{C}_n(t)$ defined as

$$\mathbf{C}_1(t) = (\mathbf{c}_{00}), \quad \mathbf{C}_2(t) = \begin{pmatrix} \mathbf{c}_{10} \\ \mathbf{c}_{01} \end{pmatrix}, \dots, \mathbf{C}_n(t) = \begin{pmatrix} \mathbf{c}_{n-10} \\ \mathbf{c}_{n-21} \\ \vdots \\ \mathbf{c}_{0n-1} \end{pmatrix} \quad (3.38)$$

where the sub-vectors are

$$\mathbf{c}_{n_1 n_2}(t) = \begin{pmatrix} \vdots \\ c_{n_2 2}^{n_1-1}(t) \\ c_{n_2 1}^{n_1 0}(t) \\ c_{n_2 0}^{n_1 1}(t) \\ c_{n_2 -1}^{n_1 2}(t) \\ \vdots \end{pmatrix}, \quad (3.39)$$

it follows that the differential-recurrence Eq. (3.37) can once again be transformed into the matrix three-term recurrence equation (3.15), (which may be formally solved) namely,

$$\frac{d}{dt} \mathbf{C}_n(t) = \mathbf{Q}_n^- \mathbf{C}_{n-1}(t) + \mathbf{Q}_n \mathbf{C}_n(t) + \mathbf{Q}_n^+ \mathbf{C}_{n+1}(t), \quad (3.40)$$

where the matrices $\mathbf{Q}_n$ and $\mathbf{Q}_n^\pm$ are given by

$$\mathbf{Q}_n = -\beta_1 \begin{pmatrix} (n-1)\mathbf{I} & (n-1)\mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & (n-2)\mathbf{I} & (n-2)\mathbf{I} & \ddots & \vdots \\ \mathbf{0} & \ddots & \ddots & \ddots & \mathbf{0} \\ \vdots & \ddots & \mathbf{0} & \mathbf{I} & \mathbf{I} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} - \beta_2 \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{I} & \mathbf{I} & \mathbf{0} & \ddots & \vdots \\ \mathbf{0} & \ddots & \ddots & \ddots & \mathbf{0} \\ \vdots & \ddots & (n-2)\mathbf{I} & (n-2)\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & (n-1)\mathbf{I} & (n-1)\mathbf{I} \end{pmatrix}, \quad (3.41)$$

$$\mathbf{Q}_n^+ = -i \begin{pmatrix} \mathbf{q}^+ & \mathbf{q}_+ & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{q}^+ & \mathbf{q}_+ & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{q}^+ & \mathbf{q}_+ \end{pmatrix}, \quad (3.42)$$

$$\mathbf{Q}_n^- = \begin{pmatrix} (n-1)\mathbf{q}^- & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{q}_- & (n-2)\mathbf{q}^- & \ddots & \vdots \\ \mathbf{0} & 2\mathbf{q}_- & \ddots & \mathbf{0} \\ \vdots & \ddots & \ddots & \mathbf{q}^- \\ \mathbf{0} & \cdots & \mathbf{0} & (n-1)\mathbf{q}_- \end{pmatrix}, \quad (3.43)$$

where

$$\mathbf{q}^+ = \begin{pmatrix} -q_{\max} + 1 & 0 & \cdots & 0 \\ 0 & -q_{\max} + 2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & q_{\max} \end{pmatrix}, \quad \mathbf{q}_+ = \begin{pmatrix} q_{\max} & 0 & \cdots & 0 \\ 0 & q_{\max} - 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -q_{\max} + 1 \end{pmatrix} \quad (3.44)$$

$$\mathbf{q}^- = \begin{pmatrix} \bar{\tau}_1 & -i\xi_1/2 & \cdots & 0 \\ i\xi_1/2 & \bar{\tau}_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -i\xi_1/2 \\ 0 & \cdots & i\xi_1/2 & \bar{\tau}_1 \end{pmatrix}, \quad \mathbf{q}_- = \begin{pmatrix} \bar{\tau}_2 & i\xi_2/2 & \cdots & 0 \\ -i\xi_2/2 & \bar{\tau}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & i\xi_2/2 \\ 0 & \cdots & -i\xi_2/2 & \bar{\tau}_2 \end{pmatrix}, \quad (3.45)$$

and $\mathbf{I}$ is a unit matrix with dimension $2q_{\max}$ (here we have confined the dimensions of the matrices $\mathbf{q}^\pm$ and $\mathbf{q}_\pm$ to some finite number $2q_{\max}$).

Using the Laplace transformation, Eq. (3.40) can be reduced to the matrix recurrence relations for the Laplace transforms $\tilde{\mathbf{C}}_n(s)$, Eqs. (3.19) and (3.20), where

$$\mathbf{C}_1(0) = \begin{pmatrix} \vdots \\ c_{02}^{0-1}(0) \\ c_{01}^{00}(0) \\ c_{00}^{01}(0) \\ c_{0-1}^{02}(0) \\ \vdots \end{pmatrix} \quad (3.46)$$

$c_{0q_2}^{0q_1}(0) = e^{-i(q_1\theta_1(0)+q_2\theta_2(0))}$, and $\mathbf{C}_n(0) = 0$ for $n = 2, 3, \dots$ because $\dot{\theta}_1(0) = \dot{\theta}_2(0) = 0$. Then by invoking the general method for solving three term matrix recurrence equations [14], we have the explicit matrix continued fraction solution, Eq. (3.27). The calculation of the time-dependent functions $c_{n_2 q_2}^{n_1 q_1}(t)$ can be then accomplished using the FFT algorithm.

Finally, the following properties of Eqs. (3.33) and (3.34) are useful in practical applications. First by adding Eqs. (3.33) and (3.34), we have

$$J_{grid} \ddot{\theta}_{grid} + J_{gen} \ddot{\theta}_{gen} = \tau_1 + \tau_2. \quad (3.47)$$

Next writing Eqs. (3.33) and (3.34) in the form of the general Eq. (3.35) and subtracting the second equation from the first equation, we have

$$\ddot{\theta}_1 - \ddot{\theta}_2 + (\beta_1 + \beta_2)(\dot{\theta}_1 - \dot{\theta}_2) + (\xi_1 + \xi_2)\sin(\theta_1 - \theta_2) = \bar{\tau}_1 - \bar{\tau}_2. \quad (3.48)$$

Equation (3.48) is identical in form to Eq. (3.4) of the damped pendulum equation and thus can be solved in a similar way using the matrix continued fraction solution. However, the difference is that *now the grid dynamics are influenced by the generator* in accordance with Eqs. (3.47) and (3.48) thus the grid simply no longer oscillates with a *constant* frequency ω_0 as in a grid with infinite inertia. Therefore we need to integrate Eq. (3.47) twice and combine it with the continued fraction solution of Eq. (3.48) in order to

determine the overall dynamics of the system. For analysis of the generator and grid response to various disturbances we use numerical solutions of Eqs.(3.33) and (3.34) obtained below in *Mathematica*[®]. At this stage, we remark that after a disturbance, all generators on the system increase their mechanical torque, via increasing the load of the turbine. This controller functionality, also called *frequency response* or *operating reserve* [12], is realised in the turbine controllers and is divided into various categories as described in Table 3-1

Table 3-1 Operating reserve provided by units which are synchronised

Description	Time frame after an event
Primary Operating Reserve (POR)	5-15 seconds
Secondary Operating Reserve (SOR)	15-90 seconds
Tertiary Operating Reserve 1 (TOR1)	90 seconds – 5 minutes
Tertiary Operating Reserve 2 (TOR2)	5 minutes – 20 minutes
Replacement Reserve	20 minutes – 4 hours
Substitute Reserve	4 hours – 24 hours

In order to enable one to understand the behaviour of a generator during a transient disturbance, all these torques, emanating from both the controllers and all the other remaining generators in the power system, were combined together and visualised either as a solitary impulse or a set of impulses (caption in Figure 3-5). Thus **Error! Reference source not found.** shows the response of the system (grid frequency $f_{grid} = \dot{\theta}_{grid} / (2\pi)$, generator frequency $f_{gen} = \dot{\theta}_{gen} / (2\pi)$, power output $P_{el} / P_{max} = \sin \delta(t)$ and $ROCOF = \Delta f_{grid} / \Delta t = \Delta \dot{\theta}_{grid} / (2\pi \Delta t)$) to a set of short impulsive changes of τ_1 (which as we saw represents the alternating character of changes of τ_1 due to external events such as oscillations of other generators or influences of control systems) for different values of the inertia ratio $x = J_{grid} / J_{gen}$: $x = 7$ (column (a)), $x = 30$ (column (b)) and $x = 100$ (column (c)). Here $\delta = \theta_{grid} - \theta_{gen}$ and x is a parameter describing how much inertia is inherent in the transmission system. For example, for $x = 7$ the grid has 7 times as much inertia as the generator in question. The initial value of the angle δ_0 allows one to obtain a maximum initial power output which is proportional to $\sin \delta_0$ yet while still remaining in the linear

regime (where $\sin \delta_0 \approx \delta_0$). The recovery of the frequency to its initial (i.e. normal) value is accomplished by adjusting τ_1 following a perturbation. This recovery is in practice achieved by increasing the power output of those generating units still remaining connected to the system. The power output increase is a function which is called *primary and secondary operating reserve*.

The responses of the generator-grid system to a short impulsive change in τ_1 are shown below. The calculations correspond to the initial conditions $\delta_0 = 0.66$, $\dot{\theta}_{gen}(0) = \dot{\theta}_{grid}(0) = 0$ and the following parameters: $J_{gen} = 71 \cdot 10^3 \text{ kgm}^2$, $\beta_{gen} / J_{gen} = 0.42 \text{ s}^{-1}$, $\beta_{grid} / J_{grid} = 2.8 / x \text{ s}^{-1}$, and $\tau_2 = \tau_1 = 1.3 \cdot 10^6 \text{ Nm}$.

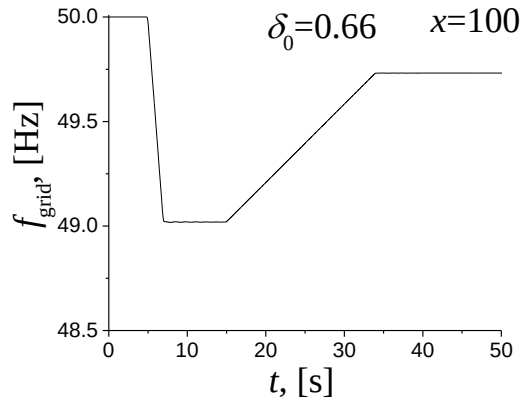


Figure 3-8: Grid frequency plot, 100 similar turbo-sets connected to the grid

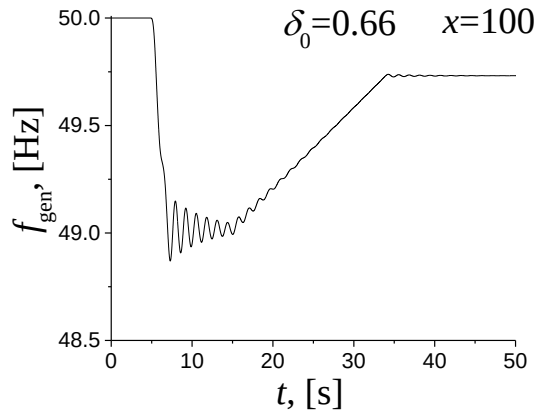


Figure 3-9: Generator frequency plot, 100 similar sets connected to the grid

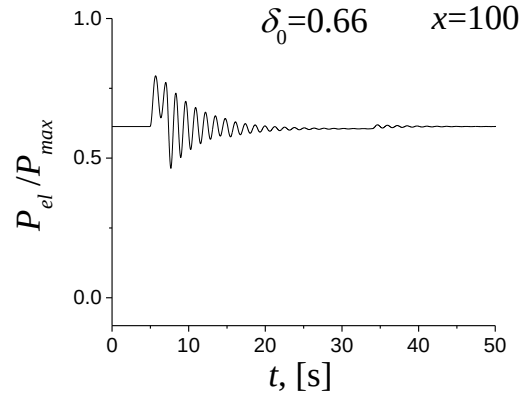


Figure 3-10: Generator Power oscillations, 100 similar sets connected to the grid

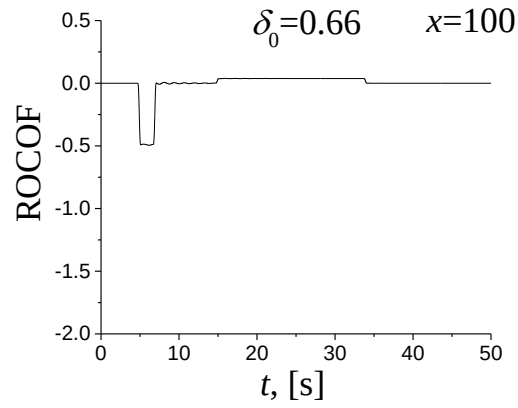


Figure 3-11: ROCOF rate, 100 similar sets connected to the grid

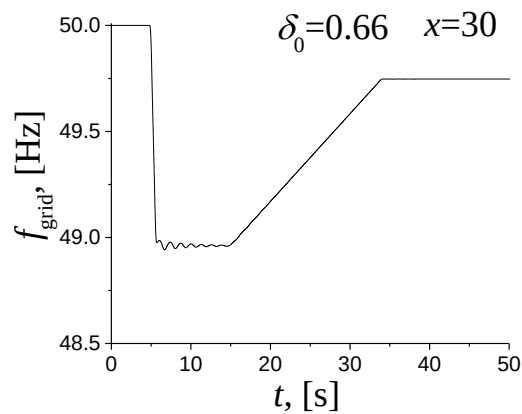


Figure 3-12: Grid frequency plot, 30 similar turbo-sets connected to the grid

In the case of a grid where 30 turbosets are connected, the grid frequency exhibits weak oscillations following the disturbance.

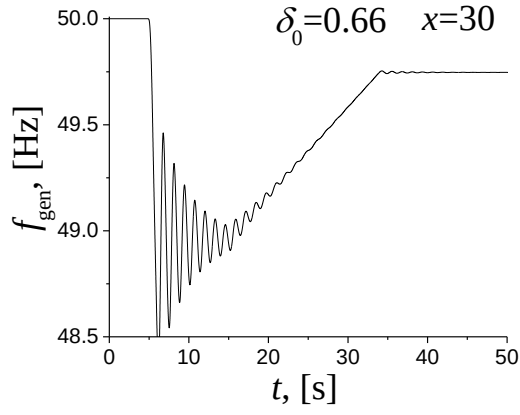


Figure 3-13: Generator frequency plot, 30 similar sets connected to the grid

Moreover, regarding the generator frequency, the oscillations are now increased in amplitude

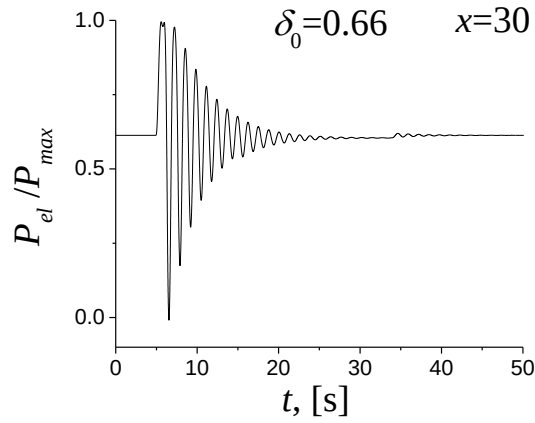


Figure 3-14: Generator Power oscillations, 30 similar sets connected to the grid

In a similar way, the electrical power output also oscillates with a higher amplitude.

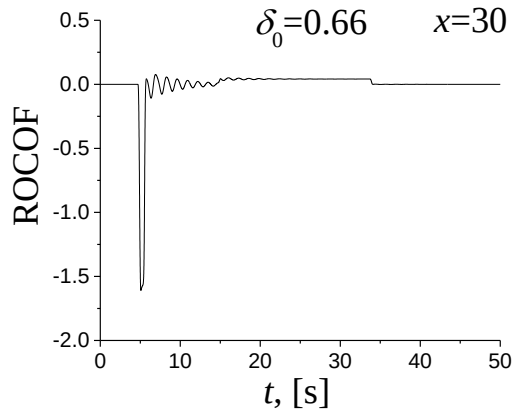


Figure 3-15: ROCOF rate, 30 similar sets connected to the grid

Furthermore, decreasing the inertia by two thirds, increases the ROCOF rate by approximately a factor of 3. This corresponds to those calculations where Eq.(1.7) is used.

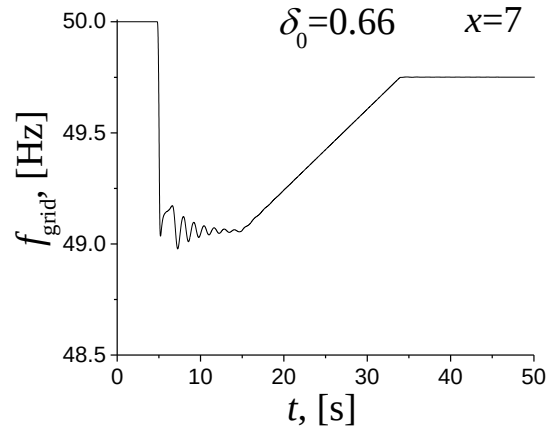


Figure 3-16: Grid frequency plot, 7 similar turbo-sets connected to the grid

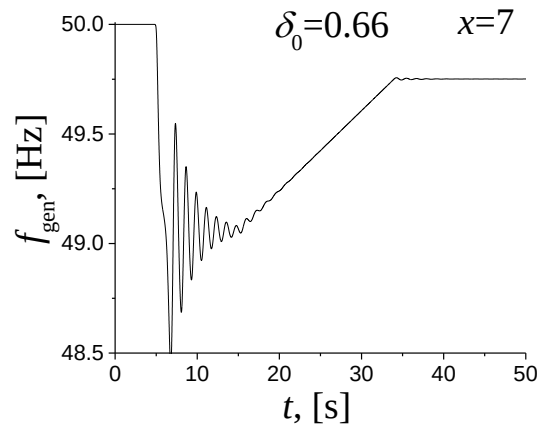


Figure 3-17: Generator frequency plot, 7 similar sets connected to the grid

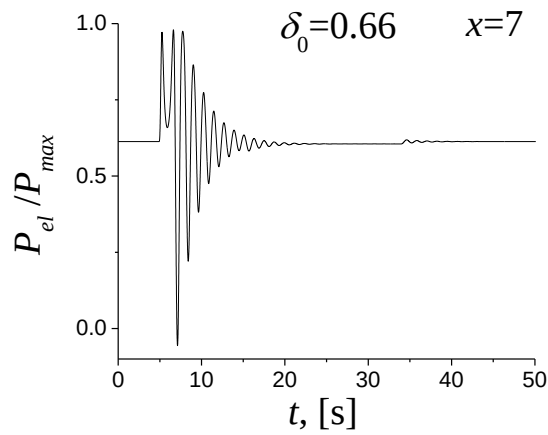


Figure 3-18: Generator Power oscillations, 7 similar sets connected to the grid

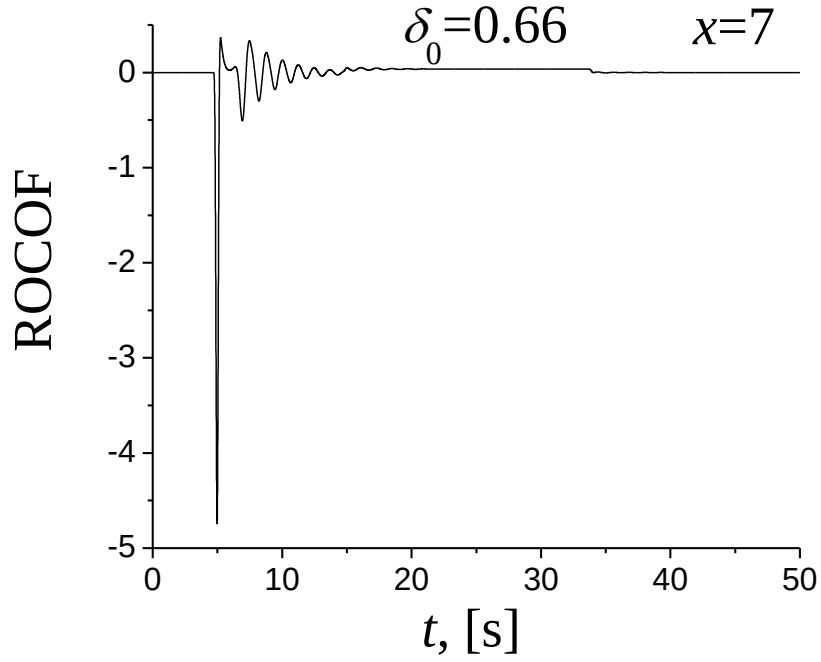


Figure 3-19: ROCOF rate, 7 similar sets connected to the grid

However, the ROCOF rate does not now follow the relationship described by Eq.(1.7). According to Figure 3-11 for $x=100$ the ROCOF rate is $\frac{df}{dt} = 0.5 \frac{\text{Hz}}{\text{s}}$ while for $x=30$ the ROCOF rate is $\frac{df}{dt} = 1.7 \frac{\text{Hz}}{\text{s}}$, which is as predicted by Eq.(1.7). Nevertheless, when we further increase the inertia to a multiple of $x=7$ then the calculated value according to Eq.(1.7) should have been about $\frac{df}{dt} = 7.1 \frac{\text{Hz}}{\text{s}}$ in contrast it is only $\frac{df}{dt} = 4.7 \frac{\text{Hz}}{\text{s}}$. This decreased ROCOF rate may be misleading in reaching an understanding of the ROCOF parameter and its importance as a tool for describing stability in power systems. Therefore, it appears that for very low inertia systems the inertia itself and not the ROCOF rate should be used to describe the stability such power systems. This conclusion is exemplified in Figure 3-20, where the ROCOF rate does not exceed for the given case a rate of $\frac{df}{dt} = 2 \frac{\text{Hz}}{\text{s}}$, yet, the system still falls asynchronous.

The response of the grid (Eqs. (3.33) and (3.34)) to an applied impulse in addition to the permanent torque τ_1 (i.e. a short impulsive change in τ_1) is shown in Figure 3-20.

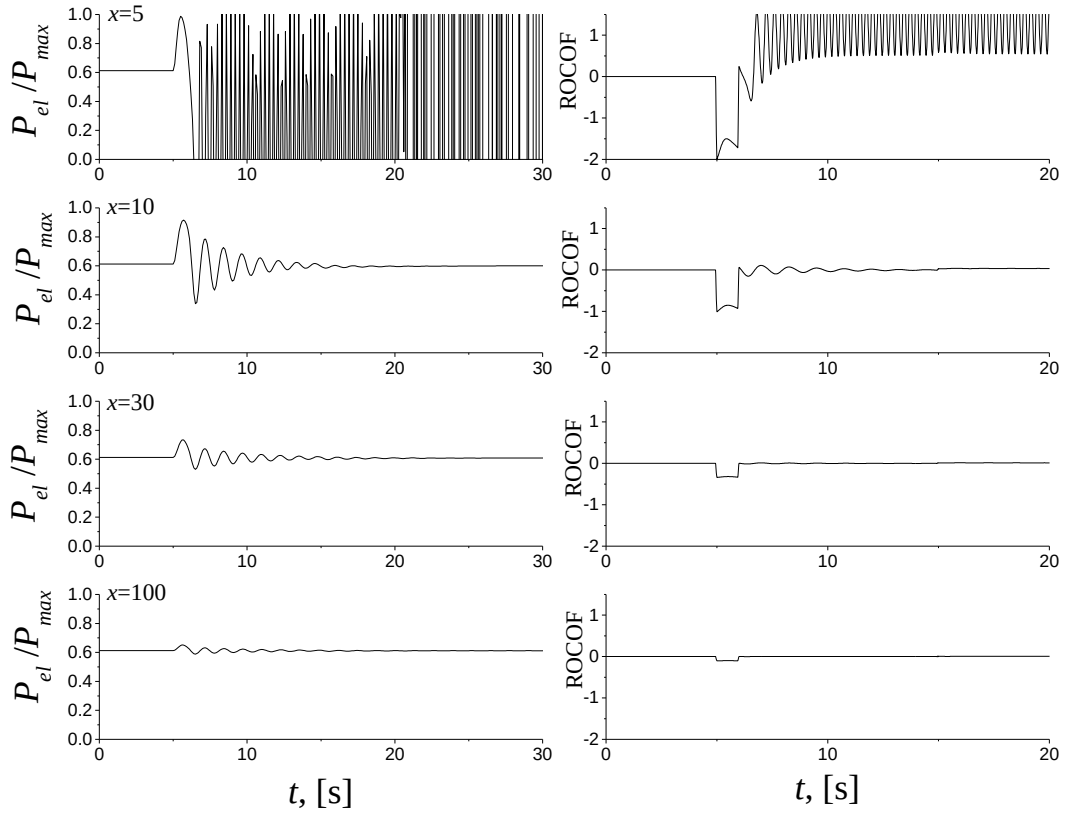


Figure 3-20 Response to torque changes in the grid
The response of the system to the same additional impulse of torque $\Delta\tau_{grid}$ for different values of x .

We emphasized above, that the numerical solution of Eqs. (3.33) and (3.34) indicates that for $x = 5$ the generator experiences a pole slip as evident in Figure 3-20. However, as we can now see, this finding *contradicts* the commonly used formula for ROCOF namely

$$\frac{df}{dt} = \frac{\Delta P}{P} \frac{f_0}{2H_{grid}} \quad (3.49)$$

where ΔP [MW] is the power change per unit machine base, P [MW] is the total power generated and H [s] is the system inertia. Equation (3.49) expressed via the parameters of the model, namely $2HP_N = J_{grid}\omega_0^2$ and $\Delta P \approx \Delta\tau\omega_0$, gives

$$\frac{df}{dt} = \frac{\Delta P}{P} \frac{f_0}{2H} \approx \frac{\Delta P f_0}{2HP_N} = \frac{\Delta\tau\omega_0 f_0}{J_{grid}\omega_0^2} = \frac{\Delta\tau}{2\pi x J_{gen}} \quad (3.50)$$

The comparison of numerical calculations of df/dt vs. $x = J_{grid}/J_{gen}$ with the traditionally used simple Eq. (3.50) is shown in Figure 3-21.

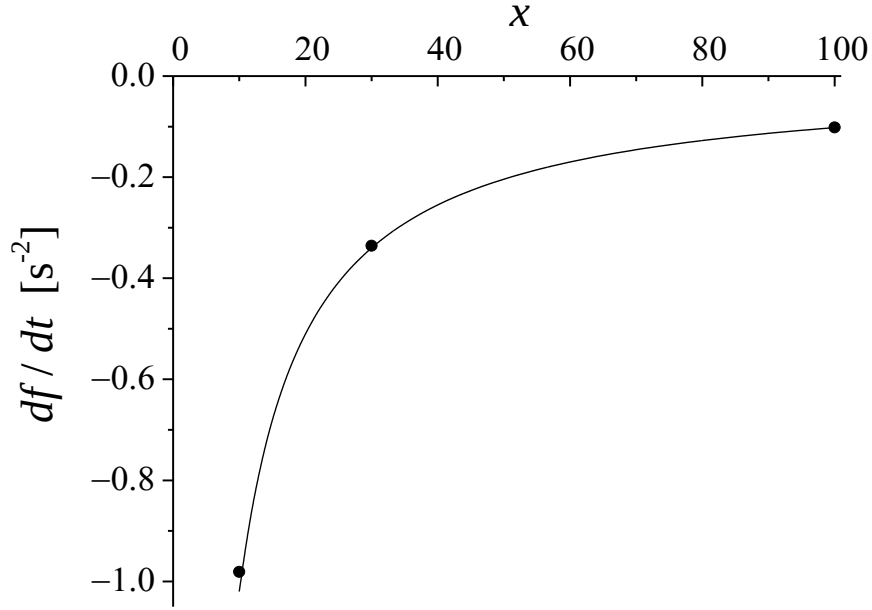


Figure 3-21 df/dt vs. $x = J_{grid} / J_{gen}$
for $J_{gen} = 71 \cdot 10^3 \text{ kg} \cdot \text{m}^2$, $\tau_{gen} = 1.3 \cdot 10^6 \text{ N} \cdot \text{m}$ and $\Delta\tau_{grid} = -3.5\tau_{gen}$.
Solid line: Eq. (3.50), filled circles: solution of Eqs. (3.33) and (3.34).

Clearly good agreement exists between the numerical solution of Eqs. (3.33) and (3.34) and the approximation contained in Eq. (3.50) for $x > 10$. However, for values of x less than 10 one cannot use such an approximation as can be seen in Figure 3-22 and explained above.

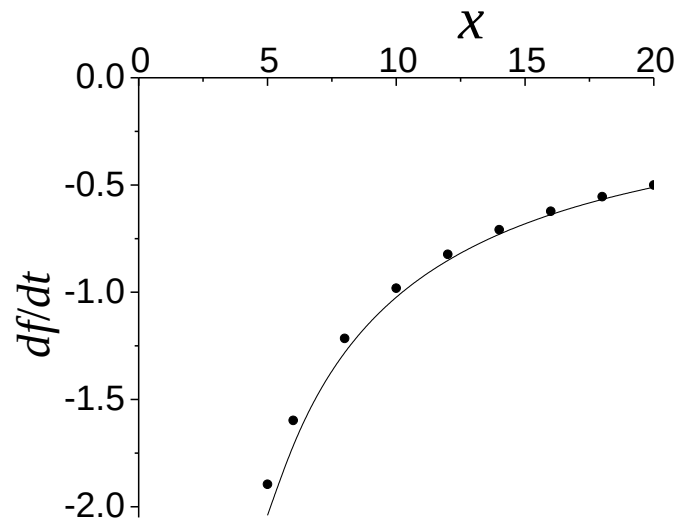


Figure 3-22: df/dt for $x < 20$

At this point if we consider a two-body system like a grid – generator system which interacts to each other and whose dynamics is described by a system of two ordinary coupled differential equations of the second order. The first model is a Kuramoto-like model. In the Kuramoto-like model the generator and the grid are considered rotating objects with moments of inertia J_{gen} and J_{grid} , respectively. The damping torque is proportional to the deviation of the grid/generator frequency from reference frequency Ω . Moreover, the generator and the grid are synchronized by the interaction torque $\tau_{el} \sin(\theta_{gen} - \theta_{grid})$. The resulting system of equations describing the mutual dynamics of two bodies is:

$$J_{gen} \frac{d^2 \theta_{gen}}{dt^2} + K_{gen} \left(\frac{d\theta_{gen}}{dt} - \Omega \right) + \tau_{el} \sin(\theta_{gen} - \theta_{grid}) = \tau_{gen}, \quad (3.51)$$

$$J_{grid} \frac{d^2 \theta_{grid}}{dt^2} + K_{grid} \left(\frac{d\theta_{grid}}{dt} - \Omega \right) + \tau_{el} \sin(\theta_{grid} - \theta_{gen}) = \tau_{grid}. \quad (3.52)$$

The rotating double-pendulum model differs from the Kuramoto like model by the damping torque which is now proportional to the difference between grid and generator frequencies, namely,

$$J_{gen} \frac{d^2 \theta_{gen}}{dt^2} + K \left(\frac{d\theta_{gen}}{dt} - \frac{d\theta_{grid}}{dt} \right) + \tau_{el} \sin(\theta_{gen} - \theta_{grid}) = \tau_{gen}, \quad (3.53)$$

$$J_{grid} \frac{d^2 \theta_{grid}}{dt^2} + K \left(\frac{d\theta_{grid}}{dt} - \frac{d\theta_{gen}}{dt} \right) + \tau_{el} \sin(\theta_{grid} - \theta_{gen}) = \tau_{grid}. \quad (3.54)$$

Both models give identical results only in the case of infinite grid $J_{grid} \rightarrow \infty$, because in that case $\theta_{grid}(t) = \Omega t + \theta_{grid}(0)$ and the solution of both equations is simplified to the solution of the same equation for $\delta(t) = \theta_{gen}(t) - \theta_{grid}(t)$ analogue of a *driven damped pendulum*. We are interested here in the finite grid inertia case, so, we consider first the double pendulum model. By adding Eqs. (3.53) and (3.54), we have

$$J_{grid} \ddot{\theta}_{grid} + J_{gen} \ddot{\theta}_{gen} = \tau_{gen} + \tau_{grid}. \quad (3.55)$$

Next, we use the indexes 1 for grid and 2 for generator and rewrite Eqs. (3.53) and (3.54) as

$$\ddot{\theta}_i + \beta_i (\dot{\theta}_i - \dot{\theta}_j) + \xi_i \sin(\theta_i - \theta_j) = \bar{\tau}_i, \quad i = 1, 2 \quad (3.56)$$

where $\beta_i = K / J_i$, $\xi_i = \tau_{el} / J_i$ and $\bar{\tau}_i = \tau_i / J_i$. Subtracting the second equation from the first equation, we have

$$\ddot{\theta}_1 - \ddot{\theta}_2 + (\beta_1 + \beta_2)(\dot{\theta}_1 - \dot{\theta}_2) + (\xi_1 + \xi_2)\sin(\theta_1 - \theta_2) = \bar{\tau}_1 - \bar{\tau}_2. \quad (3.57)$$

Equation (3.57) is the mechanical analogue of a *driven damped pendulum*, viz.,

$$\ddot{\delta} + \beta\dot{\delta} + \xi\sin\delta = \tau, \quad \delta = \theta_{grid} - \theta_{gen}. \quad (3.58)$$

where $\beta = \beta_1 + \beta_2$, $\xi = \xi_1 + \xi_2$ and $\tau = \bar{\tau}_1 - \bar{\tau}_2$. Thus, it can be solved using matrix continued fraction solution of the damped pendulum equation. However, now the difference is that the grid dynamics are influenced by the generator in accordance with Eq. (3.55) and the grid does not oscillate anymore with a constant frequency ω_0 as in a grid with infinite inertia. However we still need to integrate Eq. (3.55) twice and combine it with the continued fraction solution of Eq. (3.58) in order to determine the overall dynamics of the system. This integration we accomplish for the case $\dot{\theta}_i(0) = \Omega$ and $\tau_{gen} = -\tau_{grid} = const$. So, we have from Eqs. (3.55) and (3.58):

$$J_1\theta_1(t) + J_2\theta_2(t) = (J_1 + J_2)\Omega t + J_1\theta_1(0) + J_2\theta_2(0). \quad (3.59)$$

and

$$\theta_1(t) - \theta_2(t) = \delta(t). \quad (3.60)$$

Equations (3.59) and (3.60) permit us to derive the time evolution of angles $\theta_i(t)$, viz.,

$$\theta_1(t) = \Omega t + \frac{J_1\theta_1(0) + J_2\theta_2(0)}{J_1 + J_2} + \frac{J_2}{J_1 + J_2}\delta(t). \quad (3.61)$$

and

$$\theta_2(t) = \Omega t + \frac{J_1\theta_1(0) + J_2\theta_2(0)}{J_1 + J_2} - \frac{J_1}{J_1 + J_2}\delta(t). \quad (3.62)$$

One can see that $\theta_{grid}(t) = \Omega t + \theta_{grid}(0)$ if $J_{grid} = J_1 \rightarrow \infty$. In the case of the Kuramoto-like model we cannot simplify the solution as in the case of the rotating double pendulum (cage-like) model. We will have to solve system of equations using the matrix continued fraction solution which is by far more complicated than for the damped pendulum equation.

3.4 THE KURAMOTO MODEL, SYNCHRONIZATION TIME OF CONVENTIONAL POWER GENERATING STATIONS

3.4.1 Introduction to the Kuramoto Model

Many dynamical systems in physics, chemistry, biology or engineering represent coupled interacting objects either rotating or oscillating in time. To study the dynamics of such systems, several models have been proposed. The typical model considered here is a phase oscillator network, in particular, Kuramoto oscillators. This class of models characterizes the collective dynamics of a variety of phase oscillator systems ranging from chemical reactions [58] and neural networks [59, 60] to coupled Josephson junctions [61], laser arrays [62], optomechanical systems [63] and mean-field quantum systems [64]. Studies of this model typically concentrate on the synchronization processes occurring in phase oscillator systems [65]. Kuramoto-like dynamics also appear for models describing the frequency dynamics of power grids [65, 66, 67], which is the subject of this chapter. The dynamics of the phases ϕ_j of the rotating synchronous generators and motors, representing power plants, are described by the swing equation

$$J_j \frac{d^2 \phi_j}{dt^2} + K_j \frac{d\phi_j}{dt} + \sum_{l=1}^N \tau_{jl}^{\max} \sin(\phi_j - \phi_l) = \tau_j. \quad (3.63)$$

where J_j is the moment of inertia and K_j is the damping coefficient characterizing the damping torque acting on a synchronous machine, $\tau_{jl}^{\max} = \tau_{lj}^{\max}$ denotes electromagnetic torque (the coupling strength of two machines j and l) and τ_j is the resulting torque on the generator. Each machine rotates with a frequency near to the grid reference frequency $\Omega = 2\pi f_0$ with ($f_0 = 50\text{Hz}$). Thus, the phase of each machine is defined as $\theta_j = \Omega t + \phi_j$.

3.4.2 Dynamics of a machine coupled to a grid

The equation of motion of a generator coupled to a grid can be derived from the set of (3.63). Thus, we separate a particular machine (say $j = N$) from all the others and designate it as a *generator* ($J_{j=N} = J_{\text{gen}}, \phi_{j=N} = \phi_{\text{gen}}, \tau_{j=N} = \tau_{\text{gen}}$). All other machines are regarded as *well synchronised* ($\phi_{j \neq N} = \phi_{\text{grid}}$) and as having the same parameters

$\tau_{j \neq N}^{\max} = \tau_{\text{gen}}^{\max}$. We then have

$$J_{gen} \frac{d^2 \phi_{gen}}{dt^2} + K_{gen} \frac{d\phi_{gen}}{dt} + \sum_{l=1}^{N-1} \tau_{genl}^{\max} \sin(\phi_{gen} - \phi_{grid}) = \tau_{gen}, \quad j = N, \quad (3.64)$$

$$J_j \frac{d^2 \phi_{grid}}{dt^2} + K_j \frac{d\phi_{grid}}{dt} + \tau_{jgen}^{\max} \sin(\phi_{grid} - \phi_{gen}) = \tau_j, \quad j \neq N. \quad (3.65)$$

By summing all equations with $j \neq N$ we have

$$\sum_{j=1}^{N-1} J_j \frac{d^2 \phi_{grid}}{dt^2} + \sum_{j=1}^{N-1} K_j \frac{d\phi_{grid}}{dt} + \sum_{j=1}^{N-1} \tau_{jgen}^{\max} \sin(\phi_{grid} - \phi_{gen}) = \sum_{j=1}^{N-1} \tau_j. \quad (3.66)$$

Equations (3.64) and (3.66) can be rewritten as

$$J_{gen} \frac{d^2 \phi_{gen}}{dt^2} + K_{gen} \frac{d\phi_{gen}}{dt} + \tau_{el} \sin(\phi_{gen} - \phi_{grid}) = \tau_{gen}, \quad (3.67)$$

$$J_{grid} \frac{d^2 \phi_{grid}}{dt^2} + K_{grid} \frac{d\phi_{grid}}{dt} + \tau_{el} \sin(\phi_{grid} - \phi_{gen}) = \tau_{grid}. \quad (3.68)$$

Here $J_{grid} = \sum_{j=1}^{N-1} J_j$, $K_{grid} = \sum_{j=1}^{N-1} K_j$, $\tau_{grid} = \sum_{j=1}^{N-1} \tau_j$ and $\tau_{el} = \sum_{j=1}^{N-1} \tau_{jgen} = \sum_{j=1}^{N-1} \tau_{genj}$.

Next, we make the substitution $\phi_{grid} = \theta_{grid} - \Omega t$ and $\phi_{gen} = \theta_{gen} - \Omega t$ yielding

$$J_{gen} \frac{d^2 \theta_{gen}}{dt^2} + K_{gen} \left(\frac{d\theta_{gen}}{dt} - \Omega \right) + \tau_{el} \sin(\theta_{gen} - \theta_{grid}) = \tau_{gen}, \quad (3.69)$$

$$J_{grid} \frac{d^2 \theta_{grid}}{dt^2} + K_{grid} \left(\frac{d\theta_{grid}}{dt} - \Omega \right) + \tau_{el} \sin(\theta_{grid} - \theta_{gen}) = \tau_{grid}. \quad (3.70)$$

In this Kuramoto-like model [67] the damping torque is proportional to the *deviation* of the grid/generator frequency from the reference frequency Ω . Another cage-like [68, 69] model is considered in [70], where the *damping torque* is proportional to the *difference* between grid and generator frequencies, namely

$$J_{gen} \frac{d^2 \theta_{gen}}{dt^2} + K \left(\frac{d\theta_{gen}}{dt} - \frac{d\theta_{grid}}{dt} \right) + \tau_{el} \sin(\theta_{gen} - \theta_{grid}) = \tau_{gen}, \quad (3.71)$$

and

$$J_{grid} \frac{d^2 \theta_{grid}}{dt^2} + K \left(\frac{d\theta_{grid}}{dt} - \frac{d\theta_{gen}}{dt} \right) + \tau_{el} \sin(\theta_{grid} - \theta_{gen}) = \tau_{grid} . \quad (3.72)$$

Equations (3.71) and (3.72) can both be derived from the dynamical equations for the phases θ_j analogous to Eq.(3.63), namely

$$J_j \frac{d^2 \theta_j}{dt^2} + \sum_{l=1}^N K_{jl} \left(\frac{d\theta_j}{dt} - \frac{d\theta_l}{dt} \right) + \sum_{l=1}^N \tau_{jl}^{\max} \sin(\theta_j - \theta_l) = \tau_j . \quad (3.73)$$

From the above derivation we have

$$J_{gen} \frac{d^2 \theta_{gen}}{dt^2} + \sum_{l=1}^{N-1} K_{genl} \left(\frac{d\theta_{gen}}{dt} - \frac{d\theta_{grid}}{dt} \right) + \sum_{l=1}^{N-1} \tau_{genl}^{\max} \sin(\theta_{gen} - \theta_{grid}) = \tau_{gen} . \quad (3.74)$$

$$\sum_{j=1}^{N-1} J_j \frac{d^2 \theta_{grid}}{dt^2} + \sum_{j=1}^{N-1} K_{jgen} \left(\frac{d\theta_{grid}}{dt} - \frac{d\theta_{gen}}{dt} \right) + \sum_{j=1}^{N-1} \tau_{jgen}^{\max} \sin(\theta_{grid} - \theta_{gen}) = \sum_{j=1}^{N-1} \tau_j . \quad (3.75)$$

Following the substitutions $J_{grid} = \sum_{j=1}^{N-1} J_j$, $\tau_{grid} = \sum_{j=1}^{N-1} \tau_j$, $K = \sum_{l=1}^{N-1} K_{genl} = \sum_{j=1}^{N-1} K_{jgen}$, and

$\tau_{el} = \sum_{l=1}^{N-1} \tau_{genl}^{\max} = \sum_{j=1}^{N-1} \tau_{jgen}^{\max}$ we have Eqs. (3.71) and (3.72). Now, both systems have the same

steady state $\theta^* = (\theta_{gen}^*, \theta_{grid}^*)$ as determined by the algebraic equations

$$\tau_{el} \sin(\theta_{gen}^* - \theta_{grid}^*) = \tau_{gen} , \quad (3.76)$$

and

$$\tau_{el} \sin(\theta_{grid}^* - \theta_{gen}^*) = \tau_{grid} . \quad (3.77)$$

It is necessary for the existence of a steady state that the sum of the generated torque τ_{gen} and the grid torque τ_{grid} is equal to zero in the stable state or $\tau_{gen} + \tau_{grid} = 0$. Ultimately, we have only one equation for the determination of a steady state viz

$$\tau_{el} \sin \delta^* = \tau , \quad (3.78)$$

where $\tau = \tau_{gen} = -\tau_{grid}$ and $\delta^* = \theta_{gen}^* - \theta_{grid}^* = \phi_{gen}^* - \phi_{grid}^*$. Equation (3.78) contains only the difference of phases $\theta_{gen}^* - \theta_{grid}^*$ which is the reason why the steady state is determined only

up to a global phase shift. This phase shift could be chosen in a manner that $\theta_{gen}^* + \theta_{grid}^* = 2\Omega t$ or $\phi_{gen}^* + \phi_{grid}^* = 0$ [58].

3.4.3 Susceptibility (stationary case)

Let $\mathbf{x}^* = (x_1^*, x_2^*, \dots, x_N^*)$ be a steady state of a system in a representation space. Suppose that on applying a small perturbation Δp to one of the system parameters p_k , namely $p'_k = p_k + \Delta p$, the fixed point changes by a certain amount $\mathbf{x}^* \rightarrow \mathbf{x}'^*(\varepsilon)$. Then the system susceptibility due to the parameter p_k is defined as [65]

$$\chi_{p_k \rightarrow j} = \lim_{\Delta p \rightarrow 0} \frac{x'_j(\Delta p) - x_j^*}{\Delta p}, \quad (3.79)$$

In [65] two general cases involving a *small perturbation* of the dynamical system are considered. In the *first case* the electromagnetic torque is perturbed i.e. $\tau'_{el} = \tau_{el} + \Delta \tau_{el}$. Here the *susceptibility* describes the response of a network to changes of the *interaction process*. In the *second scenario* the torque τ itself is perturbed $\tau' = \tau + \Delta \tau$. Here the *susceptibility* describes the response of the *collective* dynamics of a network to changes of *unit magnitude*. In both cases the word *response* means the determination of the response strength. Both perturbations cause the steady-state phase of the network to alter from $\boldsymbol{\theta}^* = (\theta_{gen}^*, \theta_{grid}^*)$ to $\boldsymbol{\theta}'^* = (\theta_{gen}'^*, \theta_{grid}'^*)$.

In the *first case* the new steady state is determined by the algebraic equation

$$\tau_{el} \sin \delta = \tau_{el}^* \sin \delta^* \quad (3.80)$$

In accordance with Eq. (3.80) the new difference of phases is

$$\delta = \arcsin \left[\tau_{el}^* \sin \delta^* / \tau_{el} \right] \quad (3.81)$$

In the following we calculate the response to perturbation of the difference of the phases. Therefore, we are interested in the corresponding increment of δ which is (see also Figure 3-23)

$$d\delta = - \frac{\tau_{el}^* \sin \delta^*}{\tau_{el}^2 \sqrt{1 - (\tau_{el}^* / \tau_{el})^2 \sin^2 \delta^*}} d\tau_{el} \quad (3.82)$$

Now, we have for a small perturbation $\Delta \delta$ of the state $\delta^* = \theta_{gen}^* - \theta_{grid}^*$

$$\Delta\delta = \Delta\phi_{gen} - \Delta\phi_{grid} \approx - \left. \frac{\tau_{el}^* \sin \delta^* \Delta\tau_{el}}{\tau_{el}^2 \sqrt{1 - (\tau_{el}^* / \tau_{el})^2 \sin^2 \delta^*}} \right|_{\tau_{el} = \tau_{el}^*} = - \frac{\Delta\tau_{el}}{\tau_{el}^*} \tan \delta^*, \quad (3.83)$$

where

$$\Delta\phi_{gen} = \phi_{gen}^* - \phi_{gen}, \quad \Delta\phi_{grid} = \phi_{grid}^* - \phi_{grid}. \quad (3.84)$$

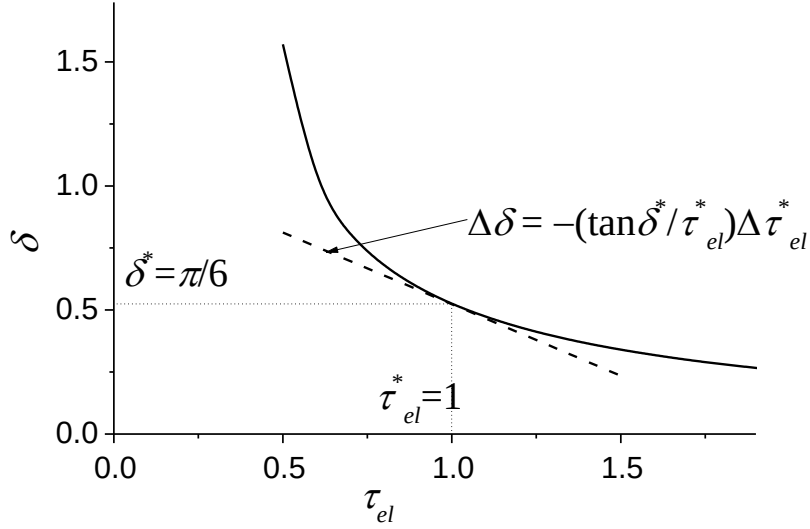


Figure 3-23: Stationary angle δ vs coupling parameter τ_{el}
for a given initial coupling parameter $\tau_{el}^* = 1$ and an initial stationary angle δ^*
Solid line: Eq. (3.81), dashed line: Eq. (3.83)

Equation (3.83) can also be written in the matrix form

$$\mathbf{A}\boldsymbol{\kappa} = -(\Delta\tau_{el} / \tau_{el}^*) \tan \delta^* \mathbf{q}, \quad (3.85)$$

where

$$\boldsymbol{\kappa} = \begin{pmatrix} \Delta\phi_{gen} \\ \Delta\phi_{grid} \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (3.86)$$

The matrix $\mathbf{A}$ is singular and therefore cannot be inverted. However, we can use the pseudoinverse of $\mathbf{A}$, namely $\mathbf{A}^+$. We then have

$$\boldsymbol{\kappa} = -(\Delta\tau_{el} / \tau_{el}^*) \tan \delta^* \mathbf{r}, \quad (3.87)$$

where

$$\mathbf{r} = \mathbf{A}^+ \mathbf{q} = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \quad (3.88)$$

Then the system susceptibility due to a *perturbation* of the electric torque parameter τ_{el} is

$$\chi_{\tau_{el} \rightarrow gen} = \lim_{\varepsilon \rightarrow 0} \frac{\Delta \phi_{gen}}{\Delta \tau_{el}} = -\tan \delta^* / (2\tau_{el}), \quad (3.89)$$

$$\chi_{\tau_{el} \rightarrow grid} = \lim_{\varepsilon \rightarrow 0} \frac{\Delta \phi_{grid}}{\Delta \tau_{el}} = \tan \delta^* / (2\tau_{el}). \quad (3.90)$$

The perturbed electrical torque in the *linear approximation* may also be written in terms of the unperturbed electrical torque viz

$$F' \approx \tau_{el} \sin \delta^* + \Delta \tau_{el} \sin \delta^* + \tau_{el} \cos \delta^* (\Delta \phi_{gen} - \Delta \phi_{grid}) = \tau_{el} \sin \delta^* = F. \quad (3.91)$$

In the *second case* the increment of δ is

$$d\delta = (\sec \delta / \tau_{el}) d\tau, \quad (3.92)$$

so that we have for a *small* perturbation $\Delta \delta$ of the state $\delta^* = \theta_{gen}^* - \theta_{grid}^*$ (see Figure 3-24)

$$\Delta \delta = (\sec \delta^* / \tau_{el}) \Delta \tau. \quad (3.93)$$

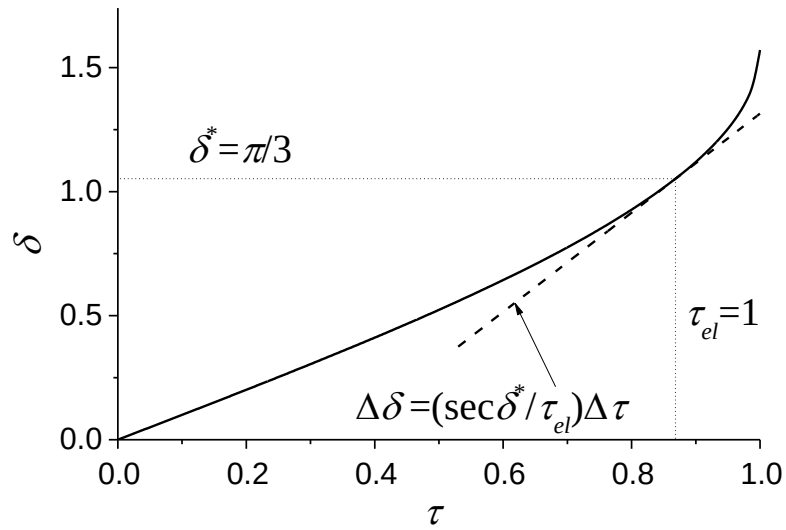


Figure 3-24 Stationary angle δ vs the torque τ
for given coupling parameter $\tau_{el} = 1$ and initial stationary angle $\delta^* = \pi / 3$. Solid line: Eq.(3.78), dashed line: Eq. (3.93)

This equation can again be written in the matrix form

$$\mathbf{A}\mathbf{\kappa} = (\Delta \tau / \tau_{el}) \sec \delta^* \mathbf{q}. \quad (3.94)$$

Solving this equation for the vector $\mathbf{\kappa}$ yields

$$\mathbf{\kappa} = (\Delta\tau / \tau_{el}) \sec \delta^* \mathbf{r}. \quad (3.95)$$

Then the system susceptibility due to variation of the parameter τ is

$$\chi_{\tau \rightarrow gen} = \lim_{\varepsilon \rightarrow 0} \frac{\Delta\phi_{gen}}{\Delta\tau} = \sec \delta^* / (2\tau_{el}) \quad (3.96)$$

and

$$\chi_{\tau \rightarrow grid} = \lim_{\Delta\tau \rightarrow 0} \frac{\Delta\phi_{grid}}{\Delta\tau} = -\sec \delta^* / (2\tau_{el}). \quad (3.97)$$

3.4.4 Order parameter

An order parameter measuring synchronization and phase locking in complex oscillator networks is [71]

$$r(t) = \frac{1}{N} \sum_{i=1}^N e^{i\theta_i(t)}. \quad (3.98)$$

The absolute value of the order parameter is

$$S(t) = |r(t)|^2 = \frac{1}{N^2} \sum_{i,j=1}^N e^{i[\theta_i(t) - \theta_j(t)]} = \frac{1}{N^2} \sum_{i,j=1}^N \cos[\theta_i(t) - \theta_j(t)]. \quad (3.99)$$

A single measure quantifying the phase ordering is then given by the *long-time average* of the absolute value of the order parameter [72] viz

$$S_\infty = \lim_{t_1 \rightarrow \infty} \lim_{t_2 \rightarrow \infty} \frac{1}{t_2 - t_1} \int_{t_1}^{t_1+t_2} f(t) dt = \frac{1}{N^2} \sum_{i,j=1}^N \left\langle \cos[\theta_i(t) - \theta_j(t)] \right\rangle_t =. \quad (3.100)$$

Therefore, order parameters serve to measure *phase coherence* in networks. Moreover, analysis of order parameters following changes in model parameters, describe the transition from a completely *incoherent* state to a state of full *synchronicity*.

For the particular case of our model, equations (3.71) and (3.72), the absolute value of the order parameter simplifies to

$$s(t) = \frac{1}{2} + \frac{1}{2} \cos \delta(t), \quad (3.101)$$

where $\delta = \theta_{gen} - \theta_{grid}$.

3.4.5 Synchronization time (dynamic case)

We assume that the magnitude of a model parameter (say τ_{el} in Eqs. (3.71) and (3.72)) is suddenly altered at time $t = 0$ from τ_{el}^I to τ_{el}^{II} . We are then interested in the relaxation of the system, starting from an equilibrium state I with $\delta(0) = \delta_I$ (Here we are considering a two-body interaction described by Eqs. (3.71)-(3.72) $\sin \delta_I = \tau / \tau_{el}^I$) to another equilibrium state II with $\delta(\infty) = \delta_{II}$ ($\sin \delta_{II} = \tau / \tau_{el}^{II}$). The dynamics of the system, immediately following the alteration of the interaction parameter τ_{el} , may be described using the normalized relaxation function [68]

$$f(t) = \frac{s(t) - s_{II}}{s_I - s_{II}}. \quad (3.102)$$

Suppose now that a given normalized function $q(t)$ has exponential behaviour with relaxation time T so that $q(t) = e^{-t/T}$, then the relaxation time is defined as

$$T = \int_0^\infty q(t) dt, \quad (3.103)$$

By analogy we may introduce the integral relaxation time for the normalized relaxation function $f(t)$ which in general is not exponential [68]

$$T_{\text{int}} = \int_0^\infty f(t) dt = \tilde{f}(0), \quad (3.104)$$

where

$$\tilde{f}(s) = \int_0^\infty f(t) e^{-st} dt \quad (3.105)$$

denotes the Laplace transform. The integral relaxation time characterizes a process in which a system, following changes in a particular system parameter, goes from one stable state to another stable state and could be considered as the *characteristic time* of a *synchronization process*. As usual we have $f(T_{\text{int}}) = e^{-1}$.

Now, if some normalized function $f(t)$ oscillates and decays, we can approximate the curve crossing all positive maxima as the exponential $q(t) = e^{-t/T}$. The time interval between two adjacent maxima is

$$T_{\text{os}} = \frac{2\pi}{\Omega} \quad (3.106)$$

We have for the first maximum at $t = T_{os}$

$$f(T_{os}) = q(T_{os}), \quad (3.107)$$

The integral relaxation time of the function $q(t) = e^{-t/T}$ is then given by Eq. (3.103), we have

$$T_{int} = T = -T_{os} / \log f(T_{os}), \quad (3.108)$$

To find T_{os} we must solve Eqs. (3.71)-(3.72). As usual Eqs.(3.71) and (3.72) can be rewritten as

$$\ddot{\theta}_i + \beta_i(\dot{\theta}_i - \dot{\theta}_j) + \xi_i \sin(\theta_i - \theta_j) = \tau_i, \quad (i = 1, 2 \text{ and } j = 2, 1), \quad (3.109)$$

where $\theta_1 = \theta_{grid}$, $\beta_1 = K / J_{grid}$, $\xi_1 = \tau_{el} / J_{grid}$, $\theta_2 = \theta_{gen}$, $\beta_2 = K / J_{gen}$, $\xi_2 = \tau_{el} / J_{gen}$, $\tau_1 = \tau_{grid} / J_{grid}$ and $\tau_2 = \tau_{gen} / J_{gen}$. Subtracting the second equation from the first one, we have

$$\ddot{\delta} + \beta\dot{\delta} + \xi \sin \delta = \tau_m, \quad (3.110)$$

where $\beta = \beta_1 + \beta_2$, $\xi = \xi_1 + \xi_2$, $\tau_m = \tau_1 - \tau_2$. This single-mass model is commonly used to analyze the dynamic response of a synchronous generator in an infinite grid [73, 74]. The mechanical analogue is a *driven damped pendulum*. In accordance with Eq. (3.110) both grid and generator initially rotate with constant rotor angle $\delta_0 = \arcsin[\tau_m / \xi] = \text{const}$ with angular velocity ω_0 corresponding to the nominal grid-frequency of 50Hz. However, during *transient* faults on the system like the trip of a generator or the disconnection of a large load, the grid-frequency changes. Hence the generator will experience an *additional* torque as the grid inertia alters its angular velocity. This disturbance leads to an oscillation of the rotor angle $\delta(t)$. The rotor angle oscillations in turn result in a power output oscillation because the electrical power of a generator is determined via the rotor angle as $P_{el} = P_{max} \sin \delta(t)$.

The solution of the damped pendulum equation (3.110) in terms of continued fractions is given in [70]. The frequency Ω of oscillations can be estimated as the frequency of free motion [70]

$$\Omega = \frac{\pi \sqrt{\xi}}{2K(m)}, \quad (3.111)$$

where $K(m)$ is a complete elliptic integral of the first kind [75] with modulus $m = \sin^2(\delta_0 / 2)$ and δ_0 is the amplitude. For small amplitude oscillations Eq. (3.110) reduces to the linear differential equation

$$\ddot{\delta} + \beta\dot{\delta} + \xi\delta = \tau_m, \quad (3.112)$$

so that the frequency Ω of small oscillations can be written

$$\Omega = \sqrt{\xi - \beta^2 / 4}. \quad (3.113)$$

Here we are interested in the solution for the relaxation function (3.102). We suppose that a disturbance of the motion occurs at the instant $t_0 = 0$ when the interaction parameter ξ changes from the initial value ξ_I to ξ_{II} . Here $\delta_I = \delta(0)$ is the initial angle. The value of δ_I depends on the applied torque of the turbine viz, $\sin \delta_I = \tau_m / \xi_I$. The initial value of $\dot{\delta}_I = \dot{\delta}(0)$ is zero. To find the normalized relaxation function $f(t)$ we may use a complete set of functions which have finite area under the curves in the time domain, viz

$$a_{nq}(t) = \dot{\delta}^n \sin^q(\delta - \delta_{II}) \quad (3.114)$$

$$b_{nq}(t) = \dot{\delta}^n \sin^q(\delta - \delta_{II}) [1 - \cos(\delta - \delta_{II})] \quad (3.115)$$

The relaxation function (3.102) can then be expressed via $a_{nq}(t)$ and $b_{nq}(t)$ as

$$f(t) = \frac{\cos \delta(t) - \cos \delta_{II}}{\cos \delta_I - \cos \delta_{II}} = \frac{a_{01}(t) \sin \delta_{II} + b_{00}(t) \cos \delta_{II}}{a_{01}(0) \sin \delta_{II} + b_{00}(0) \cos \delta_{II}}, \quad (3.116)$$

Next we derive from Eq. (3.110) a set of differential-recurrence equations for $a_{nq}(t)$ and $b_{nq}(t)$. First, we rewrite Eq. (3.110) as

$$\ddot{\delta} = -\beta\dot{\delta} - \xi_{II} \left\{ \sin(\delta - \delta_{II}) \cos \delta_{II} + [\cos(\delta - \delta_{II}) - 1] \sin \delta_{II} \right\}, \quad (3.117)$$

The time derivatives of the $a_{nq}(t)$ given by Eq. (3.114) are

$$\frac{d}{dt} a_{nq}(t) = n \dot{\delta}^{n-1} \ddot{\delta} \sin^q(\delta - \delta_{II}) + q \dot{\delta}^n \sin^{q-1}(\delta - \delta_{II}) \cos(\delta - \delta_{II}) \quad (3.118)$$

By substituting Eq. (3.117) into Eq. (3.118) we have differential-recurrence relations for the $a_{nq}(t)$

$$\frac{d}{dt} a_{nq} = -\beta n a_{nq} + q a_{n+1q-1} - q b_{n+1q-1} - \xi_{II} n [a_{n-1q+1} \cos \delta_{II} - b_{n-1q} \sin \delta_{II}] \quad (3.119)$$

The time derivatives of the $b_{nq}(t)$ are given by Eq. (3.115) which by analogy leads to the following set of differential-recurrence relations

$$\begin{aligned} \frac{d}{dt} b_{nq} = & -\beta n b_{nq} + (q+1) a_{n+1q+1} - q b_{n+1q-1} \\ & -\xi_{\text{II}} n \left[a_{n-1q+2} \sin \delta_{\text{II}} + b_{n-1q+1} \cos \delta_{\text{II}} - 2b_{n-1q} \sin \delta_{\text{II}} \right] \end{aligned} \quad (3.120)$$

By introducing infinite column vectors

$$\mathbf{C}_0(t) = 0, \quad \mathbf{C}_n(t) = \begin{pmatrix} a_{n-10} \\ b_{n-10} \\ a_{n-11} \\ b_{n-11} \\ \vdots \end{pmatrix} \quad (n > 0), \quad (3.121)$$

the scalar two index differential-recurrence equations (3.119) and (3.120) can then be transformed into the tri-diagonal matrix form

$$\frac{d}{dt} \mathbf{C}_n(t) = \mathbf{Q}_n^- \mathbf{C}_{n-1}(t) + \mathbf{Q}_n \mathbf{C}_n(t) + \mathbf{Q}_n^+ \mathbf{C}_{n+1}(t). \quad (3.122)$$

Here the matrices $\mathbf{Q}_n$ and $\mathbf{Q}_n^\pm$ are

$$\mathbf{Q}_n = -\beta(n-1)\mathbf{I}, \quad (3.123)$$

$$\mathbf{Q}_n^+ = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 1 & -1 & 0 & 0 & 0 & \dots \\ 0 & -1 & 0 & 0 & 2 & \dots \\ 0 & 0 & 2 & -2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (3.124)$$

$$\mathbf{Q}_n^- = -\xi_{\text{II}}(n-1) \begin{pmatrix} 0 & -\sin \delta_{\text{II}} & \cos \delta_{\text{II}} & 0 & 0 & \dots \\ 0 & -2 \sin \delta_{\text{II}} & 0 & \cos \delta_{\text{II}} & \sin \delta_{\text{II}} & \dots \\ 0 & 0 & 0 & -\sin \delta_{\text{II}} & \cos \delta_{\text{II}} & \dots \\ 0 & 0 & 0 & -2 \sin \delta_{\text{II}} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (3.125)$$

and $\mathbf{I}$ is the unit matrix of infinite dimension. Equation (3.122) is again solved by the matrix continued fraction method [68, 76]. On considering the initial conditions $\delta(0) = \delta_I$ and $\dot{\delta}(0) = 0$ and then taking the Laplace transformation of Eq. (3.122),

$$\tilde{\mathbf{C}}_n(s) = \int_0^\infty \mathbf{C}_n(t) e^{-st} dt, \quad (3.126)$$

we have the matrix recurrence relations for the Laplace transforms $\tilde{\mathbf{C}}_n(s)$:

$$-s\tilde{\mathbf{C}}_1(s) + \mathbf{Q}_1^+ \tilde{\mathbf{C}}_2(s) = -\mathbf{C}_1(0), \quad (3.127)$$

$$\mathbf{Q}_n^- \tilde{\mathbf{C}}_{n-1}(s) + (\mathbf{Q}_n - s\mathbf{I}) \tilde{\mathbf{C}}_n(s) + \mathbf{Q}_n^+ \tilde{\mathbf{C}}_{n+1}(s) = 0 \quad (n > 1), \quad (3.128)$$

where

$$\tilde{\mathbf{C}}_n(s) = \int_0^\infty \mathbf{C}_n(t) e^{-st} dt. \quad (3.129)$$

Here we have used

$$\mathbf{C}_1(0) = \begin{pmatrix} 1 \\ 1 - \cos(\delta_I - \delta_{II}) \\ \sin(\delta_I - \delta_{II}) \\ \sin(\delta_I - \delta_{II}) [1 - \cos(\delta_I - \delta_{II})] \\ \sin^2(\delta_I - \delta_{II}) \\ \sin^2(\delta_I - \delta_{II}) [1 - \cos(\delta_I - \delta_{II})] \\ \vdots \end{pmatrix}, \quad \mathbf{C}_n(0) = 0, \quad n > 2.$$

We seek the solution of the linear matrix Eqs. (3.127) and (3.128). At first, we consider a finite number of Eqs. (3.128) (for example, $n_{\max}=5$) by putting $\tilde{\mathbf{C}}_n(s) = 0$ for $n > n_{\max}$. We then have from Eqs. (3.127) and (3.128)

$$-s\tilde{\mathbf{C}}_1(s) + \mathbf{Q}_1^+ \tilde{\mathbf{C}}_2(s) = -\mathbf{C}_1(0), \quad (3.130)$$

$$\mathbf{Q}_2^- \tilde{\mathbf{C}}_1(s) + (\mathbf{Q}_2 - s\mathbf{I}) \tilde{\mathbf{C}}_2(s) + \mathbf{Q}_2^+ \tilde{\mathbf{C}}_3(s) = 0, \quad (3.131)$$

$$\mathbf{Q}_3^- \tilde{\mathbf{C}}_2(s) + (\mathbf{Q}_3 - s\mathbf{I}) \tilde{\mathbf{C}}_3(s) + \mathbf{Q}_3^+ \tilde{\mathbf{C}}_4(s) = 0, \quad (3.132)$$

$$\mathbf{Q}_4^- \tilde{\mathbf{C}}_3(s) + (\mathbf{Q}_4 - s\mathbf{I}) \tilde{\mathbf{C}}_4(s) = 0, \quad (3.133)$$

Next, we express $\tilde{\mathbf{C}}_4(s)$ from Eq. (3.133) as

$$\tilde{\mathbf{C}}_4(s) = \frac{\mathbf{I}}{s\mathbf{I} - \mathbf{Q}_4} \mathbf{Q}_4^- \tilde{\mathbf{C}}_3(s), \quad (3.134)$$

By substituting $\tilde{\mathbf{C}}_4(s)$ into Eq. (3.132) we have an expression for $\tilde{\mathbf{C}}_3(s)$

$$\tilde{\mathbf{C}}_3(s) = \frac{\mathbf{I}}{s\mathbf{I} - \mathbf{Q}_3 - \mathbf{Q}_3^+ \frac{\mathbf{I}}{s\mathbf{I} - \mathbf{Q}_4} \mathbf{Q}_4^-} \mathbf{Q}_3^- \tilde{\mathbf{C}}_2(s), \quad (3.135)$$

By continuing this procedure for all the other equations, we ultimately have

$$\tilde{\mathbf{C}}_1(s) = \frac{\mathbf{I}}{s\mathbf{I} - \mathbf{Q}_1^+ \frac{\mathbf{I}}{s\mathbf{I} - \mathbf{Q}_2 - \mathbf{Q}_2^+ \frac{\mathbf{I}}{s\mathbf{I} - \mathbf{Q}_3 - \mathbf{Q}_3^+ \frac{\mathbf{I}}{s\mathbf{I} - \mathbf{Q}_4} \mathbf{Q}_4^-} \mathbf{Q}_3^-} \mathbf{Q}_2^-} \mathbf{C}_1(0). \quad (3.136)$$

Hence, the matrix continued-fraction solution of Eqs. (3.127) and (3.128) for the Laplace transforms $\tilde{\mathbf{C}}_n(s)$ [7] is

$$\tilde{\mathbf{C}}_1(s) = \Delta_1 \mathbf{C}_1(0), \quad (3.137)$$

where Δ_n is the matrix continued fraction, again defined by the 2 index recurrence equation

$$\Delta_n = \left[s\mathbf{I} - \mathbf{Q}_n - \mathbf{Q}_n^+ \Delta_{n+1} \mathbf{Q}_{n+1}^- \right]^{-1}, \quad (3.138)$$

allowing us to represent Δ_n as an infinite matrix continued fraction [68] as before.

The matrix continued fraction solution we have used is once more very convenient for the purpose of computation (various algorithms for calculating matrix continued fractions are discussed in Refs. [68]). As far as the calculation of the infinite matrix continued fraction Δ_n is concerned, we approximate it by some matrix continued fraction of finite order (by putting $\mathbf{Q}_{n+1} = 0$ at some $n = n_{\max}$). Simultaneously, we confine the dimensions of the matrices $\mathbf{Q}_n$ and $\mathbf{Q}_n^\pm$ to some finite number $2q_{\max}$. Both $n_{\max}$ and $q_{\max}$ depend on the model parameters used and they must be chosen taking account of the desired degree of accuracy of the calculation.

Using a one-sided Fourier transformation the set of differential-recurrence equations for the functions $a_{nq}(t)$ and $b_{nq}(t)$ is then converted to a set of algebraic equations for the one-

sided Fourier transforms $\tilde{a}_{nq}(i\omega)$ and $\tilde{b}_{nq}(i\omega)$ which can be solved via matrix continued fractions. The resulting solution yields an analytical (in terms of matrix continued fractions) result for the one-sided Fourier transform of the normalized relaxation function

$$\tilde{f}(s) = \frac{\tilde{a}_{01}(s) \sin \delta_{II} + \tilde{b}_{00}(s) \cos \delta_{II}}{a_{01}(0) \sin \delta_{II} + b_{00}(0) \cos \delta_{II}}. \quad (3.139)$$

and so, the area under the curve $S = \tilde{f}(0)$.

3.4.6 Results and discussions

Figure 3-25 show the results of the calculations (via the matrix continued fraction method) of the spectrum of the relaxation function $f(t)$ and the area under that function Eq. (3.139). Function $f(t)$ is recovered from its spectrum via inverse Fourier transformation as described in detail in [70].

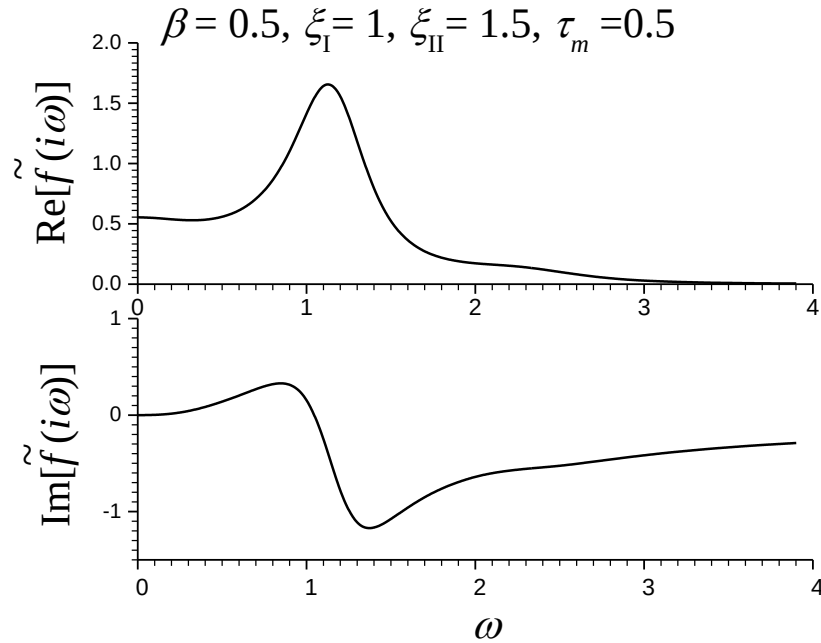


Figure 3-25: Frequency dependence of the real and imaginary parts of one-side Fourier transform $\tilde{f}(i\omega)$ ($s = i\omega$) of relaxation function Eq. (3.139) for damping parameter $\beta = 0.5$, torque parameter $\tau_m = 0.5$ and initial coupling parameter $\xi_I = 1$ and final coupling parameter $\xi_{II} = 1.5$.

The result of the application of the inverse Fourier transformation is presented in Figure 3-26

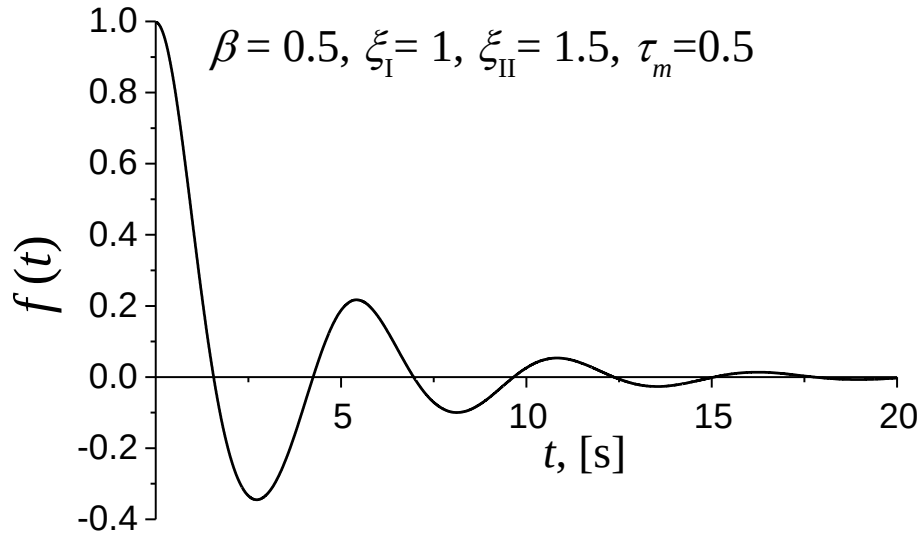


Figure 3-26: Time dependence of the relaxation function
 $f(t)$ Eq. (3.116) for damping parameter $\beta = 0.5$, torque parameter $\tau_m = 0.5$,
initial coupling parameter $\xi_I = 1$ and final coupling parameter $\xi_{II} = 1.5$.

Figure 3-27 shows the results of the calculations of the area under the curve $S = \tilde{f}(0)$

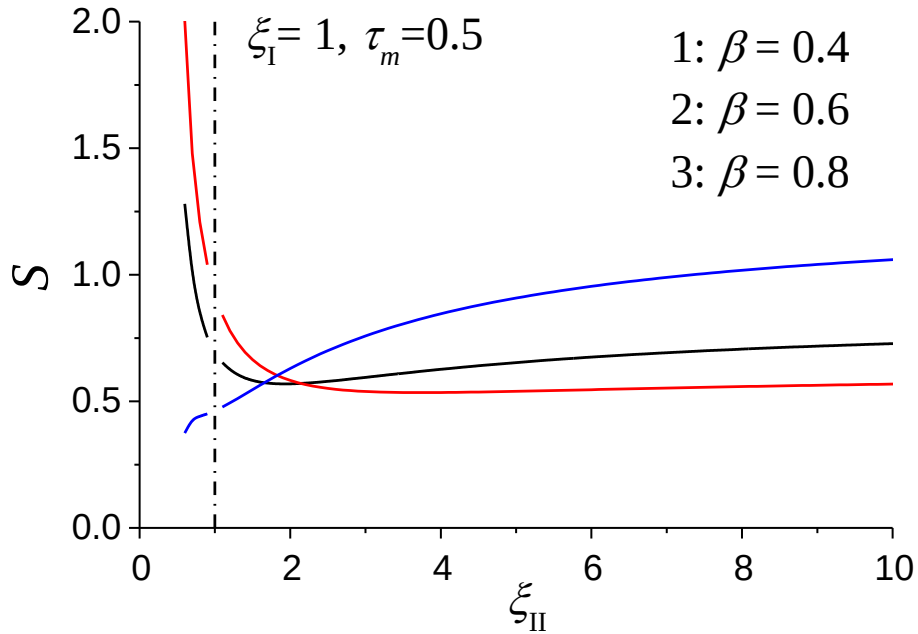


Figure 3-27: Area under the normalized relaxation function
 $S = \tilde{f}(0)$ vs. final coupling parameter ξ_{II} for torque parameter $\tau_m = 0.5$ initial
coupling parameter $\xi_I = 1$ and various damping parameters $\beta = 0.4, 0.6, 0.8$.

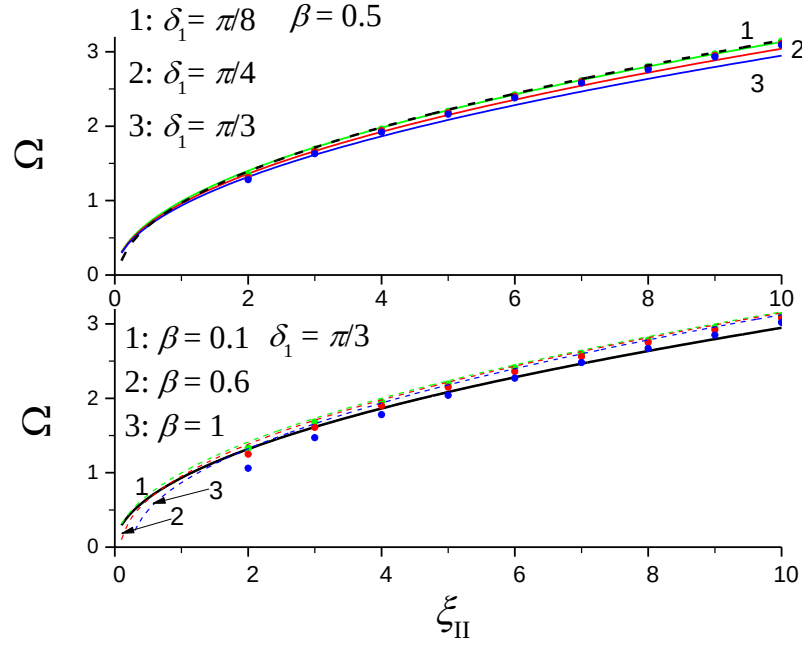


Figure 3-28: Frequency of oscillation of relaxation function $f(t)$ vs. final coupling parameter ξ_{II}

for initial coupling parameter $\xi_I = 1$ and various initial angles (a) $\delta_1 = \pi/8, \pi/4, \pi/3$, and damping parameters (b) $\beta = 0.4, 0.6, 0.8$. Solid line: Eq.(3.111), dashed line: Eq.(3.113) and circles: frequency of the maximum value of $\text{Re}[\tilde{f}(i\omega)]$ (see Figure 3-25).

Figure 3-28 demonstrates the dependence of the frequency of the oscillation of the relaxation function $f(t)$ as a function of the coupling parameter ξ_{II} . Then, Figure 3-29 shows the time dependence of the relaxation function $f(t)$ Eq.(3.116) for an initial coupling parameter $\xi_I = 1$, initial angle $\delta_1 = \pi/3$ and various values of the damping β and final coupling parameters ξ_{II} . Thus, we ultimately obtain the integral relaxation time Eq. (3.108).

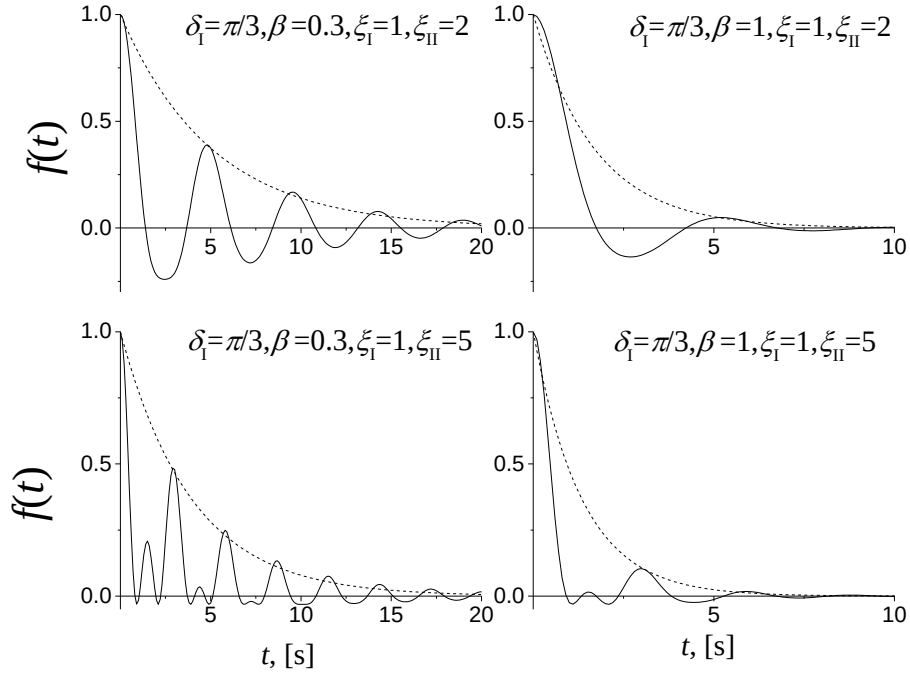


Figure 3-29: Time dependence of the relaxation function $f(t)$ Eq. (3.116)

for initial coupling parameter $\xi_I = 1$ and initial angle $\delta_I = \pi / 3$ and different values of damping parameters β and final coupling parameters ξ_{II} . Dashed lines: exponential approximation of the maxima of $f(t)$.

The result of calculation of T_{int} vs. the final coupling parameter ξ_{II} is shown in Figure 3-30.

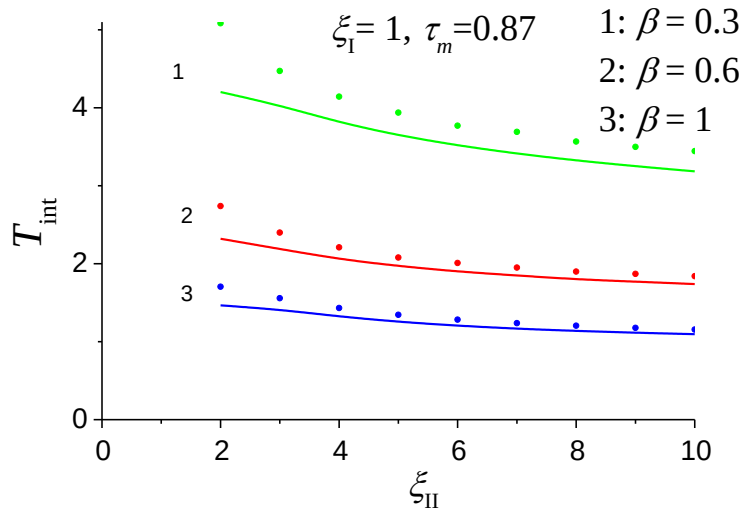


Figure 3-30: Integral relaxation time vs. final coupling parameter ξ_{II}

Eq. (3.108) for torque parameter $\tau_m = 0.87$ initial coupling parameter $\xi_I = 1$ and various damping parameters $\beta = 0.3, 0.6, 1$.

Solid line: Eq. (3.113),

circles: $\Omega = \omega_{\text{max}}$, where ω_{max} is the frequency of the maximum value of $\text{Re}[\tilde{f}(i\omega)]$ (see Figure 3-25).

Figure 3-30 shows the integral relaxation time (which is $T_{\text{int}} = T = -T_{\text{os}} / \log f(T_{\text{os}})$) vs. the final coupling parameter ξ_{II} . The solid line corresponds to $T_{\text{os}} = 2\pi / \Omega = 2\pi / \omega_{\text{so}}$, where $\omega_{\text{so}} = \sqrt{\xi - \beta^2 / 4}$ is the frequency of small oscillations. The circles correspond to $T_{\text{os}} = 2\pi / \Omega = 2\pi / \omega_{\text{max}}$, where ω_{max} is the frequency of the maximum value of $\text{Re}[\tilde{f}(i\omega)]$. The circles are different from the lines because the frequencies ω_{max} and ω_{so} differ (see circles and dashed lines in Figure 3-28). The analytical expression for the line shape $\text{Re}[\tilde{f}(i\omega)]$ is unknown, it can only be calculated via continued fraction method. However, it can be proven that for well-known line shapes, that ω_{max} and ω_{so} differ. For example, for a Lorentz line shape (which is $f_L = \frac{\omega_{\text{so}}^2}{\omega_{\text{so}}^2 - (\omega + i\Delta\omega)^2}$, where $\Delta\omega$ is half-width of the line) $\omega_{\text{max}} = \omega_{\text{so}} \sqrt{1 + \frac{\Delta\omega^2}{\omega_{\text{so}}^2} \pm 2\Delta\omega \sqrt{\omega_{\text{so}}^2 + \Delta\omega^2}}$. Here ω_{max} and ω_{so} coincide only if $\Delta\omega = 0$, which is not the case studied here.

3.5 APPLICATION OF LAGRANGE'S EQUATIONS

3.5.1 Grid with low inertia

The total kinetic energy of the system is $\Gamma = J_{\text{grid}} \dot{\theta}_{\text{grid}}^2 / 2 + J_{\text{gen}} \dot{\theta}_{\text{gen}}^2 / 2$, while the potential energy is $\Pi = kB_S B_R [1 - \cos(\theta_{\text{gen}}(t) - \theta_{\text{grid}}(t))]$. The total dissipated energy due to the damping in a *synchronised* mechanism is $D = K(\dot{\theta}_{\text{gen}}(t) - \dot{\theta}_{\text{grid}}(t))^2 / 2$. All the other energy losses are neglected. If we use the relevant form of Lagrange's equations, viz

$$\frac{d}{dt} \frac{\partial \Gamma}{\partial \dot{\theta}_{\text{gen}}} + \frac{\partial D}{\partial \dot{\theta}_{\text{gen}}} + \frac{\partial \Pi}{\partial \theta_{\text{gen}}} = \tau_{\text{gen}} \quad (3.140)$$

and

$$\frac{d}{dt} \frac{\partial \Gamma}{\partial \dot{\theta}_{\text{grid}}} + \frac{\partial D}{\partial \dot{\theta}_{\text{grid}}} + \frac{\partial \Pi}{\partial \theta_{\text{grid}}} = \tau_{\text{grid}}. \quad (3.141)$$

We then have the nonlinear equations of motion

$$J_{gen} \ddot{\theta}_{gen}(t) + K [\dot{\theta}_{gen}(t) - \dot{\theta}_{grid}(t)] + kB_S B_R \sin [\theta_{gen}(t) - \theta_{grid}(t)] = \tau_{gen} \quad (3.142)$$

and

$$J_{grid} \ddot{\theta}_{grid}(t) + K [\dot{\theta}_{grid}(t) - \dot{\theta}_{gen}(t)] + kB_S B_R \sin [\theta_{grid}(t) - \theta_{gen}(t)] = \tau_{grid} \quad (3.143)$$

3.5.2 Grid with infinite inertia

For infinite grid inertia we have $\theta_{grid}(t) = \omega_0 t$. Then equation (3.4) can be rewritten as

$$J_{gen} \ddot{\theta}_{gen}(t) + K [\dot{\theta}_{gen}(t) - \omega_0] + kB_S B_R \sin [\theta_{gen}(t) - \omega_0 t] = \tau_{gen} \quad (3.144)$$

By again introducing the notation $\delta(t) = \theta_{gen}(t) - \omega_0 t$ we can rewrite Eq. (3.144) as

$$J_{gen} \ddot{\delta}(t) + K \dot{\delta}(t) + kB_S B_R \sin \delta(t) = \tau_{gen} \quad (3.145)$$

This model is again a damped pendulum equation and is commonly used to analyze the dynamic response of a *synchronous* generator in an infinite grid. We already saw that the mechanical analog is a *driven damped pendulum* as shown in Figure 3-7.

3.6 INTRODUCING THE EQUIVALENT ELECTRICAL CIRCUITS

To obtain the equations of motion including the energy stored in the various magnetic fields [77] regarded as constituting a part of an electric circuit, we again use Lagrange's equations. Here K_e is the kinetic energy of the system, V the potential energy and $\frac{\partial P}{\partial \dot{q}_i}$ is the damping torque which now also accounts for the heat losses in the stator and line resistances. Thus, we use Lagrange's equations in the form

$$\frac{d}{dt} \left(\frac{\partial K_e}{\partial \dot{q}_i} \right) - \frac{\partial K_e}{\partial q_i} + \frac{\partial P}{\partial \dot{q}_i} + \frac{\partial V}{\partial q_i} = Q_i \quad (3.146)$$

Now, to analyze a three-phase synchronous generator and to assess the contribution to the swing equations by various electromagnetic variables the electrical phase currents are usually transformed via the *Park transformation* (*dq0*, *direct*, *quadrature* and *zero direction*) [77].

$$\begin{bmatrix} i_d \\ i_q \\ i_0 \end{bmatrix} = \sqrt{\frac{2}{3}} \begin{bmatrix} \cos \vartheta & \cos\left(\vartheta - \frac{2\pi}{3}\right) & \cos\left(\vartheta + \frac{2\pi}{3}\right) \\ -\sin \vartheta & -\sin\left(\vartheta - \frac{2\pi}{3}\right) & -\sin\left(\vartheta + \frac{2\pi}{3}\right) \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} \quad (3.147)$$

The calculations may then be carried out in the rotor rotating reference frame. Now, for synchronous generators, and specifically for the three currents i_d , i_q and i_0 , we can stipulate that a single generator, connected to an infinite grid, has 4 degrees of freedom where we can use three of them to represent the charges associated with the currents and one for the mechanical motion of the generator rotor. Thus, we have the coupled equations

$$\frac{d}{dt} \begin{pmatrix} \frac{\partial K_e}{\partial \dot{q}_1} \\ \frac{\partial K_e}{\partial \dot{q}_2} \\ \frac{\partial K_e}{\partial \dot{q}_3} \\ \frac{\partial K_e}{\partial \dot{q}_4} \end{pmatrix} - \begin{pmatrix} \frac{\partial K_e}{\partial q_1} \\ \frac{\partial K_e}{\partial q_2} \\ \frac{\partial K_e}{\partial q_3} \\ \frac{\partial K_e}{\partial q_4} \end{pmatrix} + \begin{pmatrix} \frac{\partial P}{\partial \dot{q}_1} \\ \frac{\partial P}{\partial \dot{q}_2} \\ \frac{\partial P}{\partial \dot{q}_3} \\ \frac{\partial P}{\partial \dot{q}_4} \end{pmatrix} + \begin{pmatrix} \frac{\partial V}{\partial q_1} \\ \frac{\partial V}{\partial q_2} \\ \frac{\partial V}{\partial q_3} \\ \frac{\partial V}{\partial q_4} \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{pmatrix} \quad (3.148)$$

Consequently, for a system with n generators Lagrange's equations become

$$\frac{d}{dt} \begin{pmatrix} \frac{\partial K_{e1}}{\partial \dot{q}_{11}} \\ \frac{\partial K_{e1}}{\partial \dot{q}_{21}} \\ \frac{\partial K_{e1}}{\partial \dot{q}_{31}} \\ \frac{\partial K_{e1}}{\partial \dot{q}_{41}} \\ \vdots \\ \frac{\partial K_{en}}{\partial \dot{q}_{1n}} \\ \frac{\partial K_{en}}{\partial \dot{q}_{2n}} \\ \frac{\partial K_{en}}{\partial \dot{q}_{3n}} \\ \frac{\partial K_{en}}{\partial \dot{q}_{4n}} \end{pmatrix} - \begin{pmatrix} \frac{\partial K_{e1}}{\partial q_{11}} \\ \frac{\partial K_{e1}}{\partial q_{21}} \\ \frac{\partial K_{e1}}{\partial q_{31}} \\ \frac{\partial K_{e1}}{\partial q_{41}} \\ \vdots \\ \frac{\partial K_{en}}{\partial q_{1n}} \\ \frac{\partial K_{en}}{\partial q_{2n}} \\ \frac{\partial K_{en}}{\partial q_{3n}} \\ \frac{\partial K_{en}}{\partial q_{4n}} \end{pmatrix} + \begin{pmatrix} \frac{\partial P_1}{\partial \dot{q}_{11}} \\ \frac{\partial P_1}{\partial \dot{q}_{21}} \\ \frac{\partial P_1}{\partial \dot{q}_{31}} \\ \frac{\partial P_1}{\partial \dot{q}_{41}} \\ \vdots \\ \frac{\partial P_n}{\partial \dot{q}_{1n}} \\ \frac{\partial P_n}{\partial \dot{q}_{2n}} \\ \frac{\partial P_n}{\partial \dot{q}_{3n}} \\ \frac{\partial P_n}{\partial \dot{q}_{4n}} \end{pmatrix} + \begin{pmatrix} \frac{\partial V_1}{\partial q_{11}} \\ \frac{\partial V_1}{\partial q_{21}} \\ \frac{\partial V_1}{\partial q_{31}} \\ \frac{\partial V_1}{\partial q_{41}} \\ \vdots \\ \frac{\partial V_n}{\partial q_{1n}} \\ \frac{\partial V_n}{\partial q_{2n}} \\ \frac{\partial V_n}{\partial q_{3n}} \\ \frac{\partial V_n}{\partial q_{4n}} \end{pmatrix} = \begin{pmatrix} Q_{11} \\ Q_{21} \\ Q_{31} \\ Q_{41} \\ \vdots \\ Q_{1n} \\ Q_{2n} \\ Q_{3n} \\ Q_{4n} \end{pmatrix} \quad (3.149)$$

Specifically, we have assigned the four degrees of freedom as follows. For the *direct* axis (referred to as the *d* axis)

$$q_1 = \int i_d dt \equiv \frac{i_d}{s} \quad (3.150)$$

$$\dot{q}_1 = i_d \quad (3.151)$$

$$Q_1 = u_d \quad (3.152)$$

In like manner, we can define for the *quadrature* or *q* axis

$$q_2 = \int i_q dt \equiv \frac{i_q}{s} \quad (3.153)$$

$$\dot{q}_2 = i_q \quad (3.154)$$

$$Q_2 = u_q \quad (3.155)$$

Furthermore, for the mechanical system we have

$$q_3 = \mathcal{G}_{gen} \quad (3.156)$$

while its first derivative is

$$\dot{q}_3 = \frac{d\mathcal{G}_{gen}}{dt} = \dot{\mathcal{G}}_{gen} = \omega_{gen} \quad (3.157)$$

Moreover, for the applied torque we have

$$Q_3 = -\tau_{mgen} \quad (3.158)$$

For the fourth degree of freedom we can use the current in the star point of the generator, even though many generators would have a high Ohmic star point. Thus, we have

$$q_4 = \int i_0 dt \equiv \frac{i_0}{s} \quad (3.159)$$

$$\dot{q}_4 = i_0 \quad (3.160)$$

$$Q_4 = 0 \quad (3.161)$$

Now, we take a simplistic approach based on introducing a single inductance in the *q* and the other in the *d* direction. For the electrical energy we have for the energy stored in the inductances

$$K_{ee} = \frac{1}{2} L_d i_d^2 + \frac{1}{2} L_q i_q^2 \quad (3.162)$$

or

$$K_{ee} = \frac{1}{2} L_d \dot{q}_1^2 + \frac{1}{2} L_q \dot{q}_2^2 \quad (3.163)$$

while for the n^{th} generator

$$K_{een} = \frac{1}{2} L_{dn} \dot{q}_{1n}^2 + \frac{1}{2} L_{qn} \dot{q}_{2n}^2 \quad (3.164)$$

also, for the mechanical kinetic energy of a single generator

$$K_{em} = \frac{1}{2} J_{gen} \dot{q}_3^2 \quad (3.165)$$

and for a system with n generators we have

$$K_{emn} = \frac{1}{2} J_{genn} \dot{q}_{3n}^2 \quad (3.166)$$

Thus, the total kinetic energy becomes the *sum* of the electrical and the mechanical energy

$$K_e = K_{ee} + K_{em} \quad (3.167)$$

For the total kinetic energy of a system with n generators we then have

$$K_{en} = \frac{1}{2} L_{dk} \dot{q}_{1k}^2 + \frac{1}{2} L_{qk} \dot{q}_{2k}^2 + \frac{1}{2} J_{kgen} \dot{q}_{3k}^2 \quad (3.168)$$

which is equivalent to

$$K_{en} = \frac{1}{2} L_{dn} i_{dn}^2 + \frac{1}{2} L_{qn} i_{qn}^2 + \frac{1}{2} J_{genn} \omega_n^2 \quad (3.169)$$

Now to write the equations of motion of the various matrix components the partial derivatives of the kinetic energy should be evaluated:

$$\frac{\partial K_{en}}{\partial \dot{q}_{1n}} = \frac{\partial K_{en}}{\partial i_{dn}} = \frac{\partial \left(\frac{1}{2} L_{dn} i_{dn}^2 + \frac{1}{2} L_{qn} i_{qn}^2 + \frac{1}{2} J_{genn} \omega_n^2 \right)}{\partial i_{dn}} \quad (3.170)$$

or

$$\frac{\partial K_{en}}{\partial \dot{q}_{1n}} = \frac{\partial K_{en}}{\partial i_{dn}} = L_{dn} i_{dn} \quad (3.171)$$

$$\frac{\partial K_{en}}{\partial \dot{q}_{2n}} = \frac{\partial K_{en}}{\partial i_{qn}} = \frac{\partial \left(\frac{1}{2} L_{dn} i_{dn}^2 + \frac{1}{2} L_{qn} i_{qn}^2 + \frac{1}{2} J_{genn} \omega_n^2 \right)}{\partial i_{qn}} \quad (3.172)$$

so that

$$\frac{\partial K_{en}}{\partial \dot{q}_{2n}} = \frac{\partial K_{en}}{\partial i_{qn}} = L_{qn} i_{qn} \quad (3.173)$$

For the third degree of freedom we can write:

$$\frac{\partial K_{en}}{\partial \dot{q}_{3n}} = \frac{\partial K_{en}}{\partial \omega_n} = \frac{\partial \left(\frac{1}{2} L_{dn} i_{dn}^2 + \frac{1}{2} L_{qn} i_{qn}^2 + \frac{1}{2} J_{genn} \omega_n^2 \right)}{\partial \omega_n} = J_{genn} \omega_n \quad (3.174)$$

Now equation (3.174) may be more conveniently written by introducing the inductive reactance $X = \omega L$. In this way we can write:

$$\frac{\partial K_{en}}{\partial \dot{q}_{3n}} = \frac{\partial K_{en}}{\partial \omega_n} = \frac{\partial \left(\frac{1}{2} \frac{X_{dn}}{\omega_n} i_{dn}^2 + \frac{1}{2} \frac{X_{qn}}{\omega_n} i_{qn}^2 + \frac{1}{2} J_{genn} \omega_n^2 \right)}{\partial \omega_n} \quad (3.175)$$

so that

$$\frac{\partial K_{en}}{\partial \dot{q}_{3n}} = \frac{\partial K_{en}}{\partial \omega_n} = -\frac{1}{2} \frac{X_{dn}}{\omega_n^2} i_{dn}^2 - \frac{1}{2} \frac{X_{qn}}{\omega_n^2} i_{qn}^2 + J_{genn} \omega_n \quad (3.176)$$

This equation in itself is a very interesting finding as in a stable situation where the angular velocity is constant, the mechanical angular momentum is equal to the electrical one produced due to the current *a fortiori* the electromagnetic field in the airgap of the generator viz

$$-\frac{1}{2} \frac{X_{dn}}{\omega^2} i_{dn}^2 - \frac{1}{2} \frac{X_{qn}}{\omega^2} i_{qn}^2 + J_{genn} \omega_n = 0 \quad (3.177)$$

or

$$\frac{1}{2} \frac{X_{dn}}{\omega^2} i_{dn}^2 + \frac{1}{2} \frac{X_{qn}}{\omega^2} i_{qn}^2 = J_{genn} \omega_n \quad (3.178)$$

We remark that the foregoing equation resembles that found by A. Einstein and W.J.de Haas in treating the momentum of molecular currents which is described in the physics

literature as the Einstein-de Haas effect [78], however this similarity requires further investigation.

During transient events, we would also need to introduce the transient synchronous reactance X'_d and the subtransient synchronous reactance X''_d .

Now, in almost all synchronous generators the star-point is usually grounded through a high-ohmic resistance. Therefore, the fourth degree of freedom is *in this case* neglected. Thereafter, numerical methods can be used to solve (3.149) which will form the basis of future publications.

4 BEHAVIOUR OF POWER SYSTEM STABILIZERS (PSS1A) DUE TO GRID FREQUENCY CHANGES IN LIGHT INERTIA GRIDS

We have seen that during transient events, the absolute requirement to increase the amount of renewable energy sources provided by non-synchronous generation sources such as wind turbines and photo voltaic installations may in practice result in oscillatory response of the power output of grid connected conventional turbo-generators. Hence in this situation the adequacy of a *Power System Stabilizer (PSS)* circuit must be examined. As a particular example, the IEEE PSS1A model used in an automatic voltage regulator of a generator connected to a gas turbine is chosen. In tandem with analytical (s-plane) study of the circuit the behaviour of the PSS [79] via a *MatLab* model created using *SimPowerSystems* is reviewed.

4.1 INTRODUCTION

The purpose of *power system stabilizers* is to reduce torsional oscillations of large turbine generator shafts caused by oscillations in the generated power of grid connected synchronous generators. Such transient behaviour occurs, in particular, after switching operations in the grid resulting in changes of grid impedance and may also manifest itself as inter-area oscillations creating other various problems for the grid. However, recent measurements on generators connected to the relatively isolated transmission system of the island of Ireland exhibit *increased* power oscillations. This problem must be immediately addressed from both an experimental and theoretical point of view in order to understand the implications for both generation assets and the grid itself. Thus, in order to examine the adequacy of power system stabilisers of the commonly used type PSS1A, a model is created in *MatLab* using transfer function blocks from the *Simulink* library, in particular using the corresponding *Simulink embedded linear analysis tool* from which a Bode plot is drawn. This plot is then compared with the corresponding results of the analytic solution obtained via s-plane analysis. Next the output of the PSS1A is used as an input to an Automatic Voltage Regulator (AVR). Likewise, the AVR is modelled within *Simulink* using cascaded transfer function blocks thereby yielding the Bode plots which are once more compared with the analytic solution. Finally, again using *Simulink* we have

also modelled the generator and the corresponding overall transfer function in order to obtain a comprehensive model valid for all parameter values. The block diagram underlying the *Simulink* model is shown in Figure 4-1 PSS1A and the AVR Function blocks are as programmed in *Simulink* and the model used for the analytic review is shown in Figure 4-1.

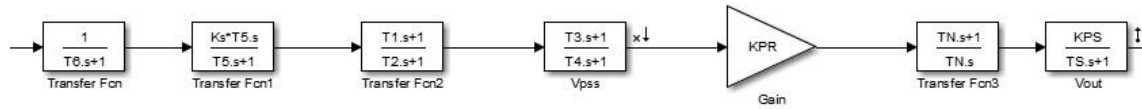


Figure 4-1 PSS1A and the AVR Function blocks are as programmed in *Simulink*

Initially we will analyse first the PSS part of the logic (see Figure 4-2)

$$V_{in} \xrightarrow{\frac{1}{1+sT_6}} V_1 \xrightarrow{K_s \frac{sT_5}{1+sT_5}} V_2 \xrightarrow{\frac{1+sT_1}{1+sT_2}} V_3 \xrightarrow{\frac{1+sT_3}{1+sT_4}} V_{PSS}$$

Figure 4-2 Block diagram of PSS1A model

Usually, when regarded as input signal to the PSS, the accelerating power is written in the form $\vec{\tau}_{accel} = \vec{\tau}_{mech} - \vec{\tau}_{elec}$. Here the mechanical component is the torque applied by the turbine and the electrical component is the electromagnetic torque in the airgap. If both torques are in equilibrium, meaning that the vectors are of *similar magnitude* but of *opposite direction*, the accelerating torque becomes zero and the turboshaft is stable at its nominal speed. In applications in which a PSS1A is used the electrical power alone is usually measured and the dc component is filtered out so that only the oscillating component is used as the input signal V_{in} to the PSS1A circuit, as shown in Figure 4-2. We study in this chapter how the PSS1A reacts to an oscillatory input signal corresponding to oscillation of the accelerating torque. Finally, the electrical power oscillation resulting from a ROCOF (Rate of Change of Frequency) event is studied. As an initial step the PSS outlined in Figure 4-2 was incorporated into a *MATLAB*[®]-*SIMULINK*[®] model representing a *single* machine connected to an *infinite* grid.

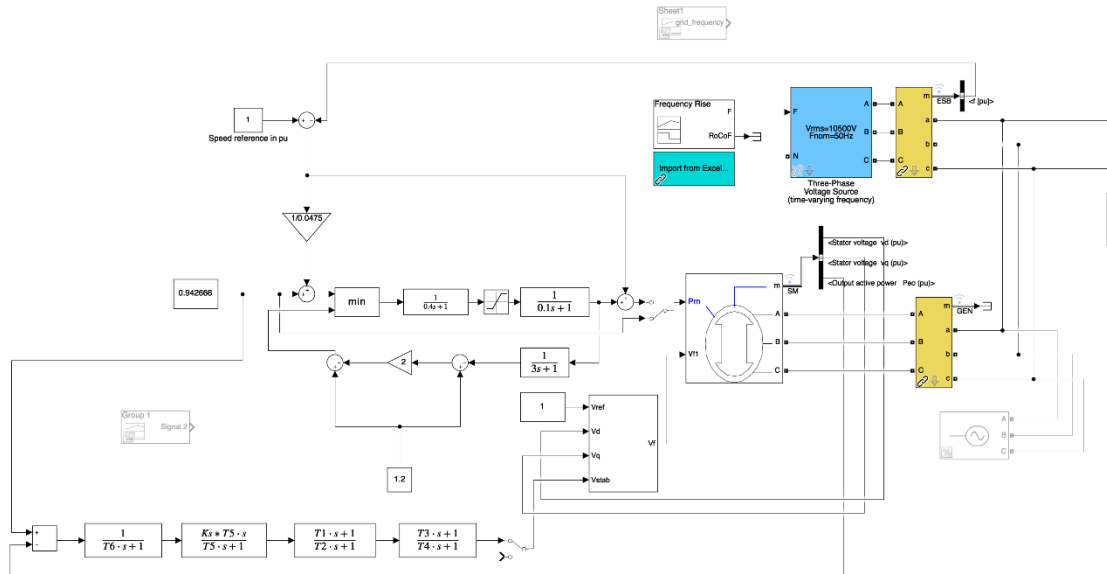


Figure 4-3 Single machine connected to an infinite grid model

The generator parameters were obtained from Table 2-1 and the PSS parameters were obtained from Table 4-1.

The figures below (Figure 4-4) show the response to a transient disturbance and the corresponding response of the generator:

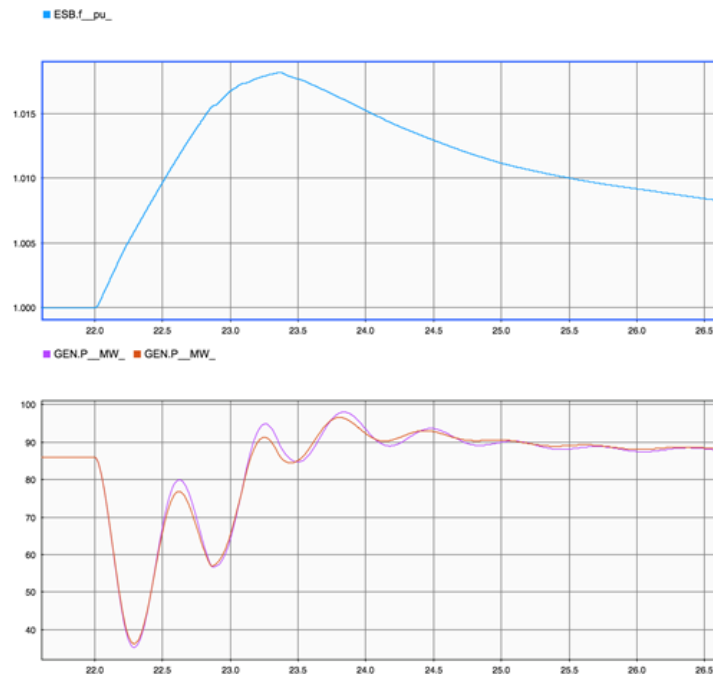


Figure 4-4 Response of a Generator to a frequency rise event and subsequent oscillations

Purple with PSS active, Red with PSS *not* active

As apparent from the oscillations shown above, the PSS has a very small damping effect, indeed almost negligible. Consequently, to gain a better understanding of the influence of this particular type of PSS an analytic approach is undertaken.

4.2 MODELS FOR PSS BEHAVIOR

The PSS model provides an input (V_{PSS}) to the AVR, the purpose being to produce *active damping of the power oscillations* associated with the torque oscillations in the airgap of the synchronous generator. A variety of signals V_{PSS} may be used depending on the particular design of the PSS model. Here as a stabilizing signal V_{PSS} we chose the signal *generated by PSS1A* which is a well-known model. The block diagram of a PSS1A model is shown in Figure 4-1. Here V_{in} is the input signal, while V_1 , V_2 and V_3 are intermediate signals, K_s is a factor of proportionality while the T_i are time constants as given in Table 4-1.

Table 4-1 Parameter PSS1A

Parameter	T_1	T_2	T_3	T_4	T_5	T_6	K_s
Value	0.4 s	1.0 s	0.1 s	0.05 s	2.0 s	0.028 s	0.8

Initially we draw the Bode Plot of the transfer function as outlined in Figure 4-5 using *MATLAB-SIMULINK®*.

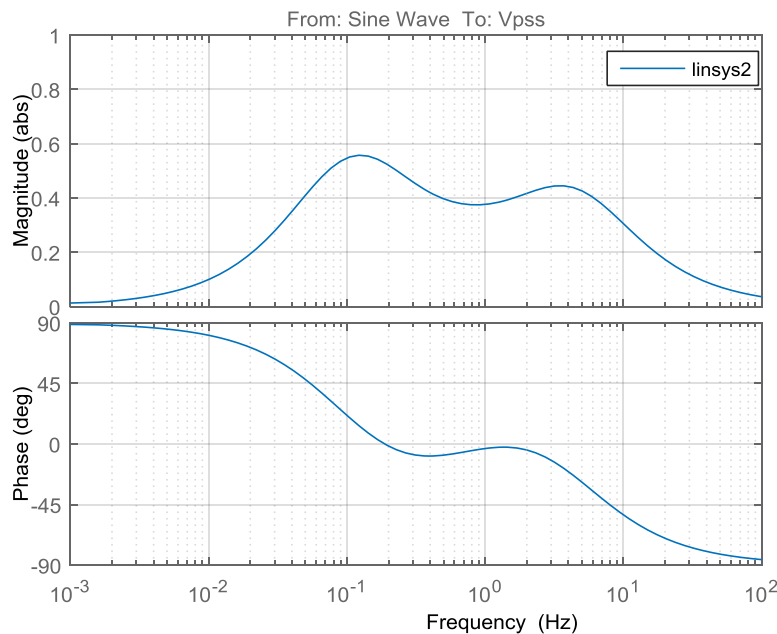


Figure 4-5 Bode diagram of the transfer function obtained with *Simulink*

4.3 FREQUENCY OF THE POWER OUTPUT OSCILLATIONS FOLLOWING DISTURBANCE OF THE GRID

In Ref. [22] and the preceding chapter, we have already described a model of a generating station based on a double pendulum. This model is described by a system of coupled nonlinear differential equations and is suitable for analysis of the generator stability with either infinite or finite grid inertia. In particular we have seen that the two-mass pendulum model (itinerant oscillator) where the grid has *finite* moment of inertia, yields the following set of coupled equations of motion for the generator and grid [22]

$$J_{grid}\ddot{\theta}_{grid} + K_D(\dot{\theta}_{grid} - \dot{\theta}_{gen}) + \tau_{el\max} \sin(\theta_{grid} - \theta_{gen}) = \tau_1, \quad (4.1)$$

$$J_{gen}\ddot{\theta}_{gen} + K_D(\dot{\theta}_{gen} - \dot{\theta}_{grid}) + \tau_{el\max} \sin(\theta_{gen} - \theta_{grid}) = \tau_2, \quad (4.2)$$

Here J_i denotes the relevant moment of inertia, $\tau_{el\max}$ is the maximum electromagnetic torque in the air gap, τ_2 is the torque applied by the turbine to the generator, τ_1 is the resulting torque applied to the grid (sum of all turbine torques less the torques due to the loads and remaining generators $\tau_{el-remain}$ on the grid). Here the damping torques characterized by the damping coefficient K_D , are the torques which act only when the rotor angular velocity *differs* from the grid angular frequency.

By addition of Eqs. (4.1) and (3.34), we have

$$J_{grid}\ddot{\theta}_{grid} + J_{gen}\ddot{\theta}_{gen} = \tau_1 + \tau_2. \quad (4.3)$$

In the stationary regime $\tau_s = \tau_1 + \tau_2 = 0$. This is merely the condition for zero acceleration of the system. Consequently, we can directly integrate Eq. (4.3) yielding the first integral,

$$\omega_{grid}(t) = \omega_{gen}(t) = \omega_0. \quad (4.4)$$

Here $\omega_i(t) = \dot{\theta}_i(t)$ and $\omega_0 = \omega_{grid}(0) = \omega_{gen}(0)$ is the grid frequency in the stationary regime, $f_0 = \omega_0 / 2\pi = 50$ Hz.

However, if the grid load is *perturbed* so that the total torque becomes $\tau'_s = \tau_s + \Delta\tau_s$ then the system will either *accelerate* or *decelerate* depending on the sign of $\Delta\tau_s$. However, we can again integrate Eq. (4.3) yielding the first integral

$$J_{grid}\omega_{grid}(t) + J_{gen}\omega_{gen}(t) = \Delta\tau_s t + (J_{grid} + J_{gen})\omega_0. \quad (4.5)$$

In order to determine the frequencies $\omega_{grid}(t)$ and $\omega_{gen}(t)$, we need an additional equation.

To obtain such an equation we rewrite Eqs. (4.1) and (4.2) as the system

$$\ddot{\theta}_i + \beta_i(\dot{\theta}_i - \dot{\theta}_j) + \xi_i \sin(\theta_i - \theta_j) = \bar{\tau}_i \quad (i = 1, 2 \text{ and } j = 2, 1), \quad (4.6)$$

where $\theta_1 = \theta_{grid}$, $\beta_1 = K_D^1 / J_{grid}$, $\xi_1 = \tau_{el\max} / J_{grid}$, $\theta_2 = \theta_{gen}$, $\beta_2 = K_D^2 / J_{gen}$, $\xi_2 = \tau_{el\max} / J_{gen}$, $\bar{\tau}_1 = \tau_1 / J_{grid}$ and $\bar{\tau}_2 = \tau_2 / J_{grid}$. Next by subtracting the second equation from the first equation and introducing the phase difference $\delta(t) = \theta_1(t) - \theta_2(t)$, we have

$$\ddot{\delta}(t) + (\beta_1 + \beta_2) \dot{\delta}(t) + (\xi_1 + \xi_2) \sin \delta(t) = \tau_r, \quad (4.7)$$

where $\tau_r = \bar{\tau}_1 - \bar{\tau}_2$. This is simply the single-mass version of the model, which is commonly used to analyze the dynamic response of a *synchronous generator* in an *infinite grid* [11] [53]. The mechanical analogue as illustrated in Figure 3-7 is a *driven damped pendulum*. Now the differential equation (3.4) or (4.7) is essentially the same as that describing the Josephson current in a superconducting junction. In particular, the angle $\delta(t)$ of the pendulum described by Eq.(4.7) corresponds to the phase difference $\phi(t)$ of the two superconductors in a Josephson junction. The corresponding equation [14] for $\phi(t)$ is

$$\ddot{\phi}(t) + \frac{1}{CR} \dot{\phi}(t) + \frac{2eI_c}{\hbar C} \sin \phi(t) = \frac{2eI}{\hbar C}. \quad (4.8)$$

where C and R are respectively the capacitance and the resistance of the junction, I_c and I are respectively, the supercurrent and dc bias current amplitudes, e is the electron charge, and $\hbar$ is Planck's reduced constant.

Equations (4.4) and (3.4) indicate that both the grid and the generator initially rotate with constant angular frequency ω_0 and rotor angle $\delta_0 = \arcsin[\tau_r / (\xi_1 + \xi_2)]$. Now we have shown previously and also in [14] [22] how to estimate the frequency of oscillations of the angle $\delta(t)$ via the idealized free motion of a pendulum when the damping $\beta_1 + \beta_2$ and torque τ_r are both absent, i.e., $\beta_1 + \beta_2 = 0$ and $\tau_r = 0$. Here the angular frequency of the periodic motion is [22]

$$\Omega = \frac{\pi \sqrt{\xi_1 + \xi_2}}{2K(m)}, \quad (4.9)$$

where $K(m)$ is a complete elliptic integral of the first kind [56] and

$$\sin[\delta(t)/2] = \sqrt{m} \operatorname{sn}\left[\sqrt{\xi_1 + \xi_2} t | m\right], \quad (4.10)$$

in which $\operatorname{sn}(u|m)$ is a Jacobian elliptic function [56] with modulus $m = \sin^2(\delta_0/2)$.

Thus, we have for $\dot{\delta}(t)$

$$\dot{\delta}(t) = \omega_{grid}(t) - \omega_{gen}(t) = 2\sqrt{m(\xi_1 + \xi_2)} \operatorname{cn}\left[\sqrt{\xi_1 + \xi_2} t | m\right], \quad (4.11)$$

where $\operatorname{cn}(u|m)$ is again a Jacobian elliptic function [56]. By combining Eqs. (4.5) and (4.11) it is clear that both $\omega_{grid}(t)$ and $\omega_{gen}(t)$ contain a *permanent term* with the unperturbed grid frequency ω_0 , a relaxation term $\Delta\tau_s t$ and an *oscillatory term* which varies like $\operatorname{cn}\left[\sqrt{\xi_1 + \xi_2} t | m\right]$. The frequency of this oscillation is given by Eq. (3.11). Now, for *small* oscillations we may rewrite Eq. (3.4) as

$$\ddot{\delta}(t) + (\beta_1 + \beta_2) \dot{\delta}(t) + (\xi_1 + \xi_2) \delta(t) = \tau_r. \quad (4.12)$$

Clearly the frequency of the damped harmonic motion described by Eq. (4.12) and its period are both independent of the amplitude and are given by

$$\Omega_{os} = \sqrt{\xi_1 + \xi_2}, \quad T_{os} = 2\pi / \sqrt{\xi_1 + \xi_2}. \quad (4.13)$$

However, by expanding the Jacobian elliptic function $K(m)$ in Eq(3.11) in the Fourier series [56]

$$K(m) = \frac{\pi}{2} \left(1 + \sum_{n=1}^{\infty} \left(\frac{(2n-1)!!}{(2n)!!} \right)^2 m^n \right), \quad (4.14)$$

we have the power series expansion of the frequency of the *power output oscillations*

$$\Omega = \Omega_{os} \left(1 - \frac{\delta_0^2}{8} + \frac{17\delta_0^4}{768} + O(\delta_0^6) \right) \quad (4.15)$$

exemplifying how the frequency now depends on the amplitude of the initial angle δ_0 .

Finally, the lightly damped response of the oscillator Eq. (4.12) is given by

$$\delta(t) = e^{-(\beta_1 + \beta_2)(t-t_0)/2} \delta_0 \cos[\Omega_1(t-t_0)], \quad (4.16)$$

where $\Omega_1 = \Omega_{os} \sqrt{1 - (\beta_1 + \beta_2)^2 / 4\Omega_{os}^2}$. Here the factor $\sqrt{1 - (\beta_1 + \beta_2)^2 / 4\Omega_{os}^2}$ represents the correction to the frequency of the undamped motion. However, precise numerical calculations indicate that Eq. (3.11) must also be corrected to account for the damping $\beta_1 + \beta_2$. Nevertheless, using the same correction factor as given above for the frequency of nonlinear oscillations Ω yields good agreement with numerical calculations of the frequency of the power output oscillations (see Figure 4-6 and Figure 4-7), namely

$$\Omega_p = \Omega \sqrt{1 - (\beta_1 + \beta_2)^2 / 4\Omega^2} . \quad (4.17)$$

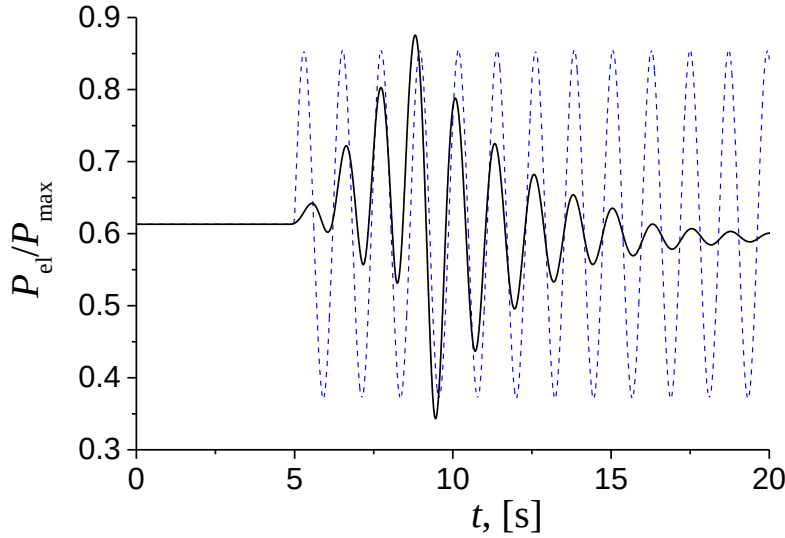


Figure 4-6 Solid line shows the response of the system (Eqs. (4.1) and (3.34)) to a set of four impulses of torque $\bar{\tau}_m$

for initial conditions $\delta_0 = 0.66$, $\dot{\delta}_0 = 0$ with model parameters $\xi = 60$, $\beta = 1$, $\bar{\tau}_m = 37$, $\Delta \bar{\tau}_m = 4$. The dashed line is an indicative representation gained via an oscillation of angular velocity (frequency) Ω_p given by Eq.(4.17)

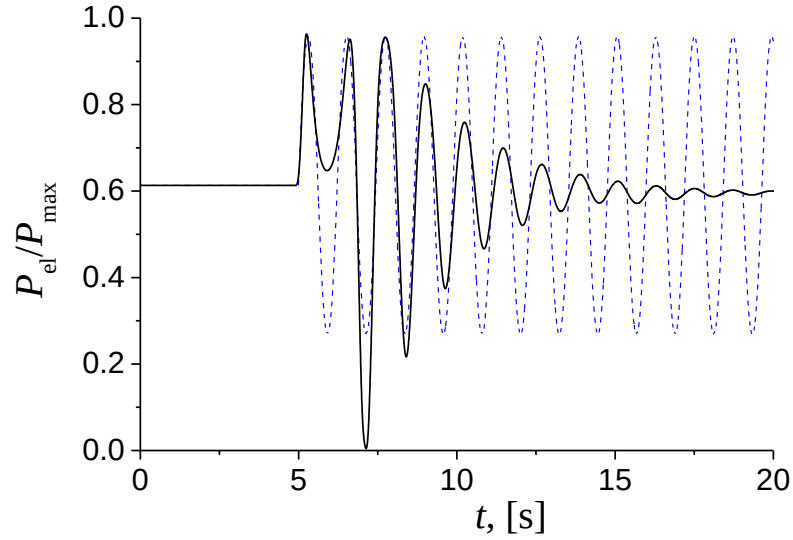


Figure 4-7 Response of the system (Eqs. (4.1) and (3.34)) to a single impulse of torque $\bar{\tau}_m$ for initial conditions $\delta_0 = 0.66, \dot{\delta}_0 = 0$.

4.4 SOLUTION FOR THE TRANSFER FUNCTION

To study the transfer function analytically we select an input signal which in general corresponds to an actual signal of the generator and which can also yield the stabilizing signal V_{PSS} in closed form. Thus, the signal may conveniently be taken as the damped oscillation

$$V_{in} = Ae^{-\lambda t} \sin \omega t. \quad (4.18)$$

Here the frequency is estimated from the generator-grid model which we have previously described. In accordance with the first block in Figure 4-2 the Laplace transform [80] of the signal V_1 is

$$\tilde{V}_1(s) = \frac{1}{1 + sT_6} \tilde{V}_{in}(s), \quad (4.19)$$

where

$$\tilde{V}_{in}(s) = \frac{A\omega}{\omega^2 + (\lambda + s)^2}. \quad (4.20)$$

On taking the inverse transform of Eq. (4.19) we then have with the initial condition $V_1(0) = 0$

$$T_6 \frac{dV_1(t)}{dt} + V_1(t) = V_{in}(t) = Ae^{-\lambda t} \sin \omega t \quad (4.21)$$

which has the complete solution

$$V_1(t) = a_1 e^{-\lambda t} \sin \omega t + b_1 e^{-\lambda t} \cos \omega t + c_1 e^{-t/T_6} \quad (4.22)$$

where

$$a_1 = \frac{A(1 - \lambda T_6)}{(1 - \lambda T_6)^2 + (\omega T_6)^2}, \quad b_1 = -\frac{A\omega T_6}{(1 - \lambda T_6)^2 + (\omega T_6)^2}, \quad c_1 = \frac{A\omega T_6}{(1 - \lambda T_6)^2 + (\omega T_6)^2}. \quad (4.23)$$

Comparison of the input signal V_{in} and the intermediate signal V_1 is shown in Figure 4-8 for $A=1$, $\lambda=0.3$, $\omega=5.2$. The behavior is shown both, in the time as well as in the frequency domain.

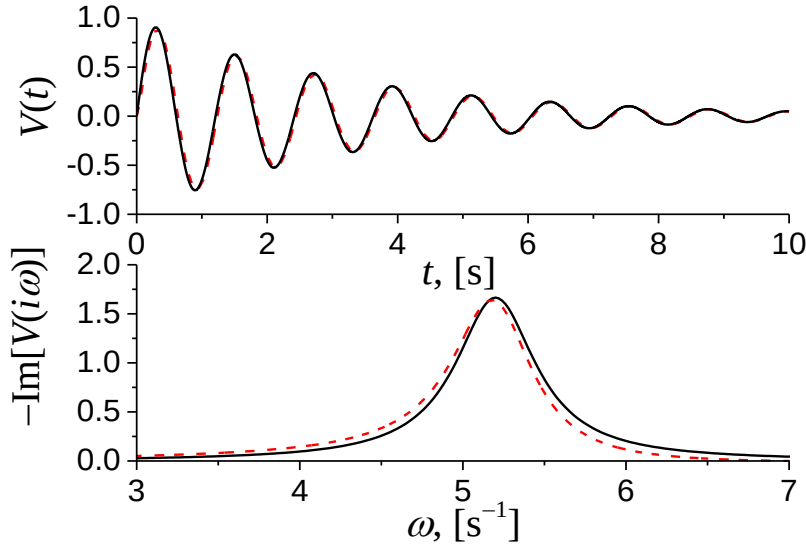


Figure 4-8 Signals $V_{in}(t)$ (solid lines) and $V_1(t)$ (dashed lines) for $A=1$, $\lambda=0.3$, $\omega=5.2$

The second block in Figure 4-2 yields

$$\tilde{V}_2(s) = K_s \frac{sT_5}{1 + sT_5} \tilde{V}_1(s). \quad (4.24)$$

Again by inverse transformation of Eq. (4.24) we have with the initial condition $V_2(0) = 0$

$$\begin{aligned} T_5 \frac{dV_2(t)}{dt} + V_2(t) &= K_s T_5 \frac{dV_1(t)}{dt} \\ &= a'_1 e^{-\lambda t} \sin \omega t + b'_1 e^{-\lambda t} \cos \omega t + c'_1 e^{-t/T_6} \end{aligned} \quad (4.25)$$

where

$$a'_1 = -K_s T_5 (a_1 \lambda + b_1 \omega), \quad b'_1 = K_s T_5 (a_1 \omega - b_1 \lambda), \quad c'_1 = -K_s T_5 c_1 / T_6. \quad (4.26)$$

Thus, we have

$$V_2(t) = a_2 e^{-\lambda t} \sin \omega t + b_2 e^{-\lambda t} \cos \omega t + c_2 e^{-t/T_6} + d_2 e^{-t/T_5}, \quad (4.27)$$

where

$$a_2 = \frac{(1 - \lambda T_5) a'_1 + \omega T_5 b'_1}{(1 - \lambda T_5)^2 + (\omega T_5)^2}, \quad b_2 = \frac{(1 - \lambda T_5) b'_1 - \omega T_5 a'_1}{(1 - \lambda T_5)^2 + (\omega T_5)^2}, \quad (4.28)$$

$$c_2 = -\frac{T_6 c'_1}{T_5 - T_6}, \quad d_2 = \frac{\omega T_5 a'_1 - (1 - \lambda T_5) b'_1}{(1 - \lambda T_5)^2 + (\omega T_5)^2} + \frac{T_6 c'_1}{T_5 - T_6}. \quad (4.29)$$

The input V_{in} and intermediate signals V_2 are compared in Figure 4-9 Signals $V_{in}(t)$ (solid lines) and $V_2(t)$ (dashed lines)..

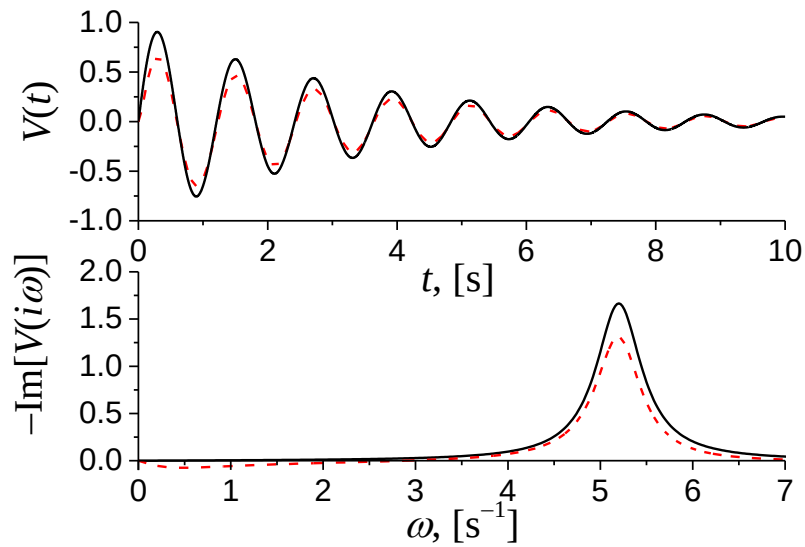


Figure 4-9 Signals $V_{in}(t)$ (solid lines) and $V_2(t)$ (dashed lines).

Parameters are the same as in Figure 4-8 Signals $V_{in}(t)$ (solid lines) and $V_1(t)$ (dashed lines) for $A=1$, $\lambda = 0.3$, $\omega = 5.2$.

The third block in Figure 4-1 yields the Laplace transform of the intermediate signal $V_3(t)$ viz

$$\tilde{V}_3(s) = \frac{1+sT_1}{1+sT_2} \tilde{V}_2(s) \quad (4.30)$$

or in the time domain with ($V_3(0) = 0$)

$$\begin{aligned} T_2 \frac{dV_3(t)}{dt} + V_3(t) &= T_1 \frac{dV_2(t)}{dt} + V_2(t) \\ &= a'_2 e^{-\lambda t} \sin \omega t + b'_2 e^{-\lambda t} \cos \omega t + c'_2 e^{-t/T_6} + d'_2 e^{-t/T_5} \end{aligned} \quad (4.31)$$

where

$$a'_2 = (1 - \lambda T_1) a_2 - \omega T_1 b_2, \quad b'_2 = \omega T_1 a_2 + (1 - \lambda T_1) b_2 \quad (4.32)$$

$$c'_2 = (1 - T_1 / T_6) c_2, \quad d'_2 = (1 - T_1 / T_5) d_2 \quad (4.33)$$

leading to the solution for the initial condition $V_3(0) = 0$

$$V_3(t) = a_3 e^{-\lambda t} \sin(\omega t - \phi) + b_3 e^{-\lambda t} \cos(\omega t - \phi) + c_3 e^{-t/T_6} + d_3 e^{-t/T_5} + e_3 e^{-t/T_2}. \quad (4.34)$$

where

$$a_3 = \frac{(1 - \lambda T_2) a'_2 + \omega T_2 b'_2}{(1 - \lambda T_2)^2 + (\omega T_2)^2}, \quad b_3 = \frac{(1 - \lambda T_2) b'_2 - \omega T_2 a'_2}{(1 - \lambda T_2)^2 + (\omega T_2)^2} \quad (4.35)$$

$$c_3 = -\frac{T_6 c'_2}{T_2 - T_6}, \quad d_3 = -\frac{T_5 d'_2}{T_2 - T_5}, \quad (4.36)$$

$$e_3 = \frac{\omega T_2 a'_2 - (1 - \lambda T_2) b'_2}{(1 - \lambda T_2)^2 + (\omega T_2)^2} + \frac{T_6 c'_2}{T_2 - T_6} + \frac{T_5 d'_2}{T_2 - T_5}. \quad (4.37)$$

Again, the comparison of the input signal V_{in} and the intermediate signal V_3 is shown in Figure 4-10.

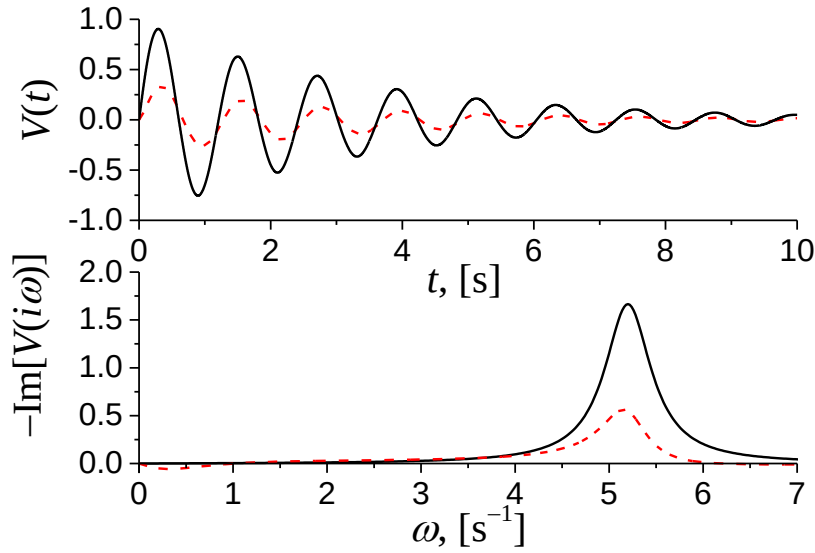


Figure 4-10 Signals $V_{in}(t)$ (solid lines) and $V_3(t)$ (dashed lines).

Parameters are the same as in Figure 4-5 Bode diagram of the transfer function obtained with *Simulink*.

Finally, the last block in Figure 4-2 yields for the transform of the intermediate signal $V_3(t)$

$$\tilde{V}_{PSS}(s) = \frac{1+sT_3}{1+sT_4} \tilde{V}_3(s) \quad (4.38)$$

or in the time domain ($V_3(0) = 0$)

$$\begin{aligned} T_4 \frac{dV_{PSS}(t)}{dt} + V_{PSS}(t) &= T_3 \frac{dV_3(t)}{dt} + V_3(t) \\ &= a'_3 e^{-\lambda t} \sin \omega t + b'_3 e^{-\lambda t} \cos \omega t + c'_3 e^{-t/T_6} + d'_3 e^{-t/T_5} + e'_3 e^{-t/T_2} \end{aligned} \quad (4.39)$$

where

$$a'_3 = (1 - \lambda T_3) a_3 - \omega T_3 b_3, \quad b'_3 = \omega T_3 a_3 + (1 - \lambda T_3) b_3, \quad (4.40)$$

$$c'_3 = (1 - T_3 / T_6) c_3, \quad d'_3 = (1 - T_3 / T_5) d_3, \quad e'_3 = (1 - T_3 / T_2) e_3 \quad (4.41)$$

so that with initial condition $V_{PSS}(0) = 0$ we have

$$V_{PSS}(t) = a_4 e^{-\lambda t} \sin \omega t + b_4 e^{-\lambda t} \cos \omega t + c_4 e^{-t/T_6} + d_4 e^{-t/T_5} + e_4 e^{-t/T_2} + f_4 e^{-t/T_4}. \quad (4.42)$$

where

$$a_4 = \frac{(1 - \lambda T_4) a'_3 + \omega T_4 b'_3}{(1 - \lambda T_4)^2 + (\omega T_4)^2}, \quad b_4 = \frac{(1 - \lambda T_4) b'_3 - \omega T_4 a'_3}{(1 - \lambda T_4)^2 + (\omega T_4)^2}, \quad (4.43)$$

$$c_4 = -\frac{T_6 c'_3}{T_4 - T_6}, \quad d_4 = -\frac{T_5 d'_3}{T_4 - T_5}, \quad e_4 = -\frac{T_2 e'_3}{T_4 - T_2}, \quad (4.44)$$

$$f_4 = \frac{\omega T_4 a'_2 - (1 - \lambda T_4) b'_2}{(1 - \lambda T_4)^2 + (\omega T_4)^2} + \frac{T_6 c'_3}{T_4 - T_6} + \frac{T_5 d'_3}{T_4 - T_5} + \frac{T_2 e'_3}{T_4 - T_2}. \quad (4.45)$$

The input V_{in} and intermediate signal V_{PSS} are compared in Figure 4-11.

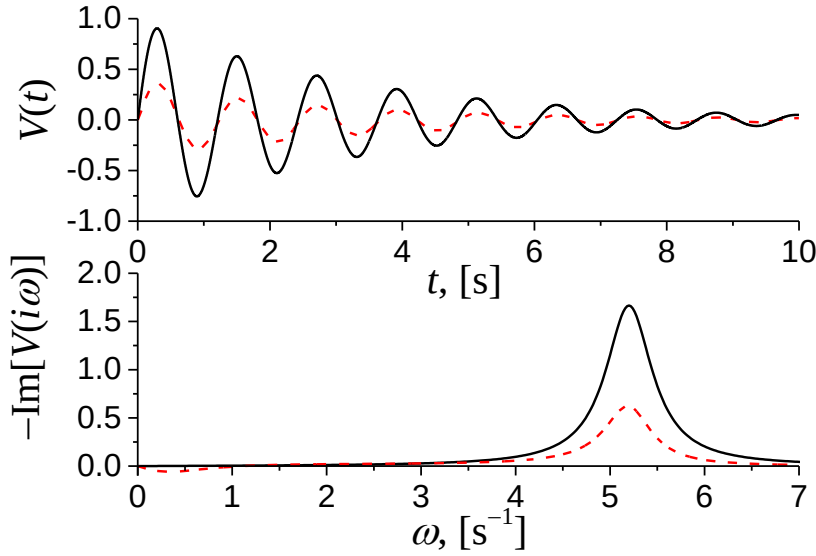


Figure 4-11 Signals $V_{in}(t)$ (solid lines) and $V_{PSS}(t)$ (dashed lines).

The parameters are the same as in Figure 4-2.

Now by introducing a column vector $\mathbf{V}_{in}$ and a row vector $\mathbf{V}_{PSS}$ which are defined as follows

$$\mathbf{V}_{in} = \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \begin{pmatrix} A \\ 0 \end{pmatrix}. \quad (4.46)$$

and

$$\mathbf{V}_{PSS} = (e^{-\lambda t} \sin \omega t, \quad e^{-\lambda t} \cos \omega t, \quad e^{-t/T_6}, \quad e^{-t/T_5}, \quad e^{-t/T_2}, \quad e^{-t/T_4}), \quad (4.47)$$

the stabilizing signal V_{PSS} can be succinctly written as the matrix product:

$$V_{PSS} = \mathbf{V}_{PSS} \prod_{i=1}^4 \mathbf{S}_i \mathbf{D}_i \mathbf{V}_{in} \quad (4.48)$$

where the matrices $\mathbf{S}_i$ and $\mathbf{D}_i$ correspond to the i^{th} block and the various matrices in Eq.(4.48) are defined by

$$\mathbf{D}_1 = \mathbf{I}_2, \quad \mathbf{D}_4 = \mathbf{I}_5 - T_3 \begin{pmatrix} \lambda & \omega & 0 & 0 & 0 \\ -\omega & \lambda & 0 & 0 & 0 \\ 0 & 0 & T_6^{-1} & 0 & 0 \\ 0 & 0 & 0 & T_5^{-1} & 0 \\ 0 & 0 & 0 & 0 & T_2^{-1} \end{pmatrix} \quad (4.49)$$

$$\mathbf{D}_2 = -K_S T_5 \begin{pmatrix} \lambda & \omega & 0 \\ -\omega & \lambda & 0 \\ 0 & 0 & T_6^{-1} \end{pmatrix}, \quad \mathbf{D}_3 = \mathbf{I}_4 - T_1 \begin{pmatrix} \lambda & \omega & 0 & 0 \\ -\omega & \lambda & 0 & 0 \\ 0 & 0 & T_6^{-1} & 0 \\ 0 & 0 & 0 & T_5^{-1} \end{pmatrix}. \quad (4.50)$$

$$\mathbf{S}_1 = \begin{pmatrix} \frac{1 - \lambda T_6}{(1 - \lambda T_6)^2 + (\omega T_6)^2} & \frac{\omega T_6}{(1 - \lambda T_6)^2 + (\omega T_6)^2} \\ \frac{-\omega T_6}{(1 - \lambda T_6)^2 + (\omega T_6)^2} & \frac{1 - \lambda T_6}{(1 - \lambda T_6)^2 + (\omega T_6)^2} \\ \frac{\omega T_6}{(1 - \lambda T_6)^2 + (\omega T_6)^2} & \frac{-(1 - \lambda T_6)}{(1 - \lambda T_6)^2 + (\omega T_6)^2} \end{pmatrix}, \quad (4.51)$$

$$\mathbf{S}_2 = \begin{pmatrix} \frac{1 - \lambda T_5}{(1 - \lambda T_5)^2 + (\omega T_5)^2} & \frac{\omega T_5}{(1 - \lambda T_5)^2 + (\omega T_5)^2} & 0 \\ \frac{-\omega T_5}{(1 - \lambda T_5)^2 + (\omega T_5)^2} & \frac{1 - \lambda T_5}{(1 - \lambda T_5)^2 + (\omega T_5)^2} & 0 \\ 0 & 0 & -\frac{T_6}{T_5 - T_6} \\ \frac{\omega T_5}{(1 - \lambda T_5)^2 + (\omega T_5)^2} & \frac{-(1 - \lambda T_5)}{(1 - \lambda T_5)^2 + (\omega T_5)^2} & \frac{T_6}{T_5 - T_6} \end{pmatrix}, \quad (4.52)$$

$$\mathbf{S}_3 = \begin{pmatrix} \frac{1-\lambda T_2}{(1-\lambda T_2)^2 + (\omega T_2)^2} & \frac{\omega T_2}{(1-\lambda T_2)^2 + (\omega T_2)^2} & 0 & 0 \\ \frac{-\omega T_2}{(1-\lambda T_2)^2 + (\omega T_2)^2} & \frac{1-\lambda T_2}{(1-\lambda T_2)^2 + (\omega T_2)^2} & 0 & 0 \\ 0 & 0 & -\frac{T_6}{T_2-T_6} & 0 \\ 0 & 0 & 0 & -\frac{T_5}{T_2-T_6} \\ \frac{\omega T_2}{(1-\lambda T_2)^2 + (\omega T_2)^2} & \frac{-(1-\lambda T_2)}{(1-\lambda T_2)^2 + (\omega T_2)^2} & \frac{T_6}{T_2-T_6} & \frac{T_5}{T_2-T_6} \end{pmatrix}, \quad (4.53)$$

$$\mathbf{S}_4 = \begin{pmatrix} \frac{1-\lambda T_4}{(1-\lambda T_4)^2 + (\omega T_4)^2} & \frac{\omega T_4}{(1-\lambda T_4)^2 + (\omega T_4)^2} & 0 & 0 & 0 \\ \frac{-\omega T_4}{(1-\lambda T_4)^2 + (\omega T_4)^2} & \frac{1-\lambda T_4}{(1-\lambda T_4)^2 + (\omega T_4)^2} & 0 & 0 & 0 \\ 0 & 0 & -\frac{T_6}{T_4-T_6} & 0 & 0 \\ 0 & 0 & 0 & -\frac{T_5}{T_4-T_6} & 0 \\ 0 & 0 & 0 & 0 & -\frac{T_5}{T_4-T_6} \\ \frac{\omega T_4}{(1-\lambda T_4)^2 + (\omega T_4)^2} & \frac{-(1-\lambda T_4)}{(1-\lambda T_4)^2 + (\omega T_4)^2} & \frac{T_6}{T_4-T_6} & \frac{T_5}{T_4-T_6} & \frac{T_5}{T_4-T_6} \end{pmatrix}. \quad (4.54)$$

Here $\mathbf{I}_n$ is the unit matrix $n \times n$.

4.5 COMPARISON OF THE VARIOUS BLOCKS

The consequences of sequentially introducing the transfer functions of the individual blocks are shown in Figure 4-2 using the results of the nonlinear analysis already available in Chapter 3. The comparison of the response of the double pendulum system (Eqs.(4.1)-(3.34)) *before* (input signal V_{in}) and *after* passing through the various blocks of the PSS (signals V_i) is shown in Figure 4-12. The calculation is accomplished numerically using direct and inverse Fourier transformation as described in [22].

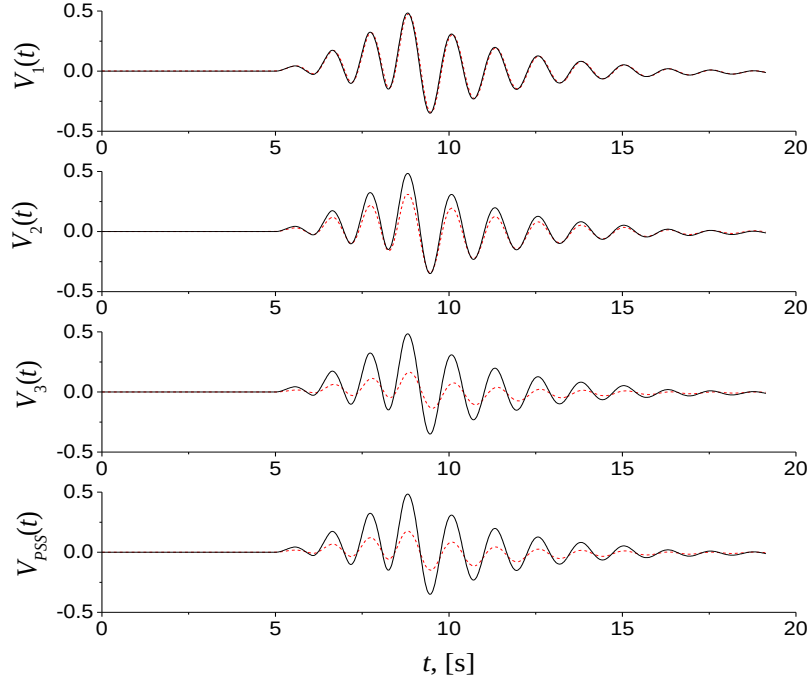


Figure 4-12 Response of the two mass pendulum model

as defined by Eqs. (4.1)-(3.34), to a set of impulses of torque $\bar{\tau}_m$ for initial conditions $\delta_0 = 0.66$, $\dot{\delta}_0 = 0$ and the following model parameters $\xi = 60$, $\beta = 1$, $\bar{\tau}_m = 37$, and $\Delta\bar{\tau}_m = 4$. Signals $V_m(t) = (\omega / \omega_0) \sin[\Delta\delta(t)]$ with $\Delta\delta(t) = \theta_{gen}(t) - \theta_{grid}(t) - \delta_0$ (solid lines) and $V_i(t)$ (dashed lines) corresponds to the block diagram of the PSS1a model (Figure 4-2 Block diagram of PSS1A model).

Furthermore, the analytically calculated Bode diagram is identical to that obtained by using modelling techniques in *SIMULINK*[®] (see Figure 4-5)

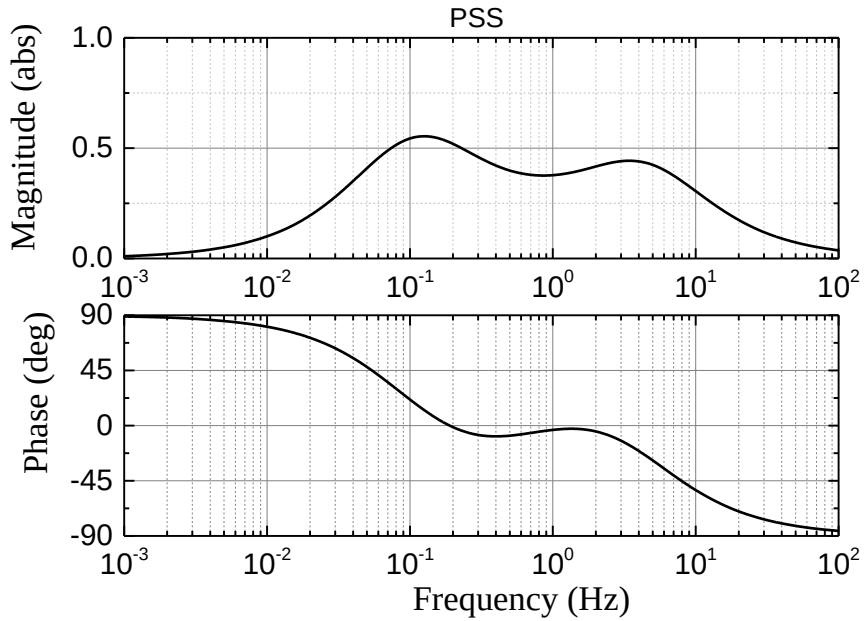


Figure 4-13: Analytically obtained Bode Diagram of the PSS1A model

4.6 INTRODUCTION OF THE AVR FUNCTION

The output of the PSS is now used as input to the AVR which is represented by the block diagram shown in the figure below.

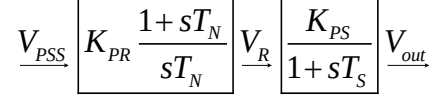


Figure 4-14 Block diagram of a static excitation diagram

Here $T_N = 2$ s is the integration time of the regulator, $T_S = 1.8$ ms is the time constant of the bridge, K_{PR} and K_{PS} are constants of proportionality. For simplicity, we set $K_{PR} = K_{PS} = 1$. As before we form the signal $V_R(t)$ corresponding to the first block in Figure 4-14, namely

$$\tilde{V}_R(s) = \frac{1 + sT_N}{sT_N} \tilde{V}_{PSS}(s) \quad (4.55)$$

The corresponding differential equation with *zero initial conditions* is

$$\begin{aligned} T_N \frac{dV_R(t)}{dt} &= T_N \frac{dV_{PSS}(t)}{dt} + V_{PSS}(t) \\ &= a'_4 e^{-\lambda t} \sin \omega t + b'_4 e^{-\lambda t} \cos \omega t + c'_4 e^{-t/T_6} + d'_4 e^{-t/T_5} + e'_4 e^{-t/T_2} + f'_4 e^{-t/T_4}, \end{aligned} \quad (4.56)$$

where

$$a'_4 = (1 - \lambda T_N) a_4 - \omega T_N b_4, \quad b'_4 = \omega T_N a_4 + (1 - \lambda T_N) b_4, \quad (4.57)$$

$$c'_4 = (1 - T_3 / T_6) c_4, \quad d'_4 = (1 - T_3 / T_5) d_4, \quad e'_4 = (1 - T_3 / T_2) e_4, \quad f'_4 = (1 - T_3 / T_4) f_4 \quad (4.58)$$

with solution subject to the initial condition $V_{PSS}(0) = 0$:

$$\begin{aligned} V_R(t) &= a_R e^{-\lambda t} \sin \omega t + b_R e^{-\lambda t} \cos \omega t \\ &+ c_R e^{-t/T_6} + d_R e^{-t/T_5} + e_R e^{-t/T_2} + f_R e^{-t/T_4} + s_R \end{aligned} \quad (4.59)$$

where

$$a_R = \omega b'_4 - \lambda a'_4, \quad b_R = -\omega a'_4 - \lambda b'_4 \quad (4.60)$$

$$c_R = -T_6 c'_4 / T_N, \quad d_R = -T_5 d'_4 / T_N, \quad e_R = -T_2 e'_4 / T_N, \quad (4.61)$$

$$f_R = -T_2 f'_4 / T_N, \quad s_R = -b_R - c_R - d_R - e_R - f_R \quad (4.62)$$

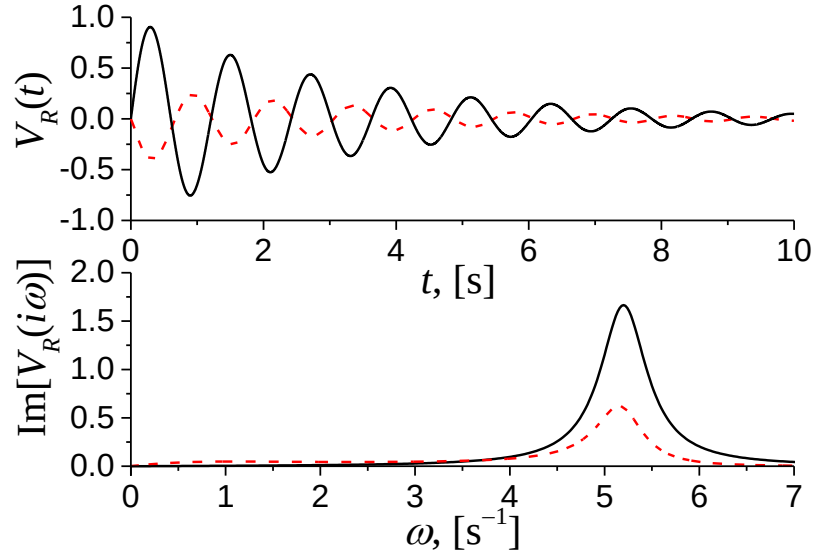


Figure 4-15 Signals $V_{in}(t)$ (solid lines) and $V_R(t)$ (dashed lines).
The parameters are the same as in Figure 4-2

Then we have for the output signal(4.63)

$$\tilde{V}_{out}(s) = \frac{1}{1 + sT_s} \tilde{V}_R(s) \quad (4.64)$$

and

$$T_s \frac{dV_{out}(t)}{dt} + V_{out}(t) = V_R(t) \quad (4.65)$$

where

$$V_R(t) = a_R e^{-\lambda t} \sin \omega t + b_R e^{-\lambda t} \cos \omega t + c_R e^{-t/T_6} + d_R e^{-t/T_5} + e_R e^{-t/T_2} + f_R e^{-t/T_4} + s_R. \quad (4.66)$$

Hence with $V_{out}(0) = 0$, we have

$$V_{out}(t) = a_{out} e^{-\lambda t} \sin \omega t + b_{out} e^{-\lambda t} \cos \omega t + c_{out} e^{-t/T_6} + d_{out} e^{-t/T_5} + e_{out} e^{-t/T_2} + f_{out} e^{-t/T_4} + g_{out} e^{-t/T_s} + s_R \quad (4.67)$$

where

$$a_{out} = \frac{(1 - \lambda T_S) a_R + \omega T_S b_R}{(1 - \lambda T_S)^2 + (\omega T_S)^2}, b_{out} = \frac{(1 - \lambda T_S) b_R - \omega T_S a_R}{(1 - \lambda T_S)^2 + (\omega T_S)^2}, \quad (4.68)$$

$$c_{out} = \frac{T_6 c_R}{T_6 - T_S}, \quad d_{out} = \frac{T_5 d_R}{T_5 - T_S}, \quad e_{out} = \frac{T_2 e_R}{T_2 - T_S}, \quad f_{out} = \frac{T_4 f_R}{T_4 - T_S}, \quad (4.69)$$

$$g_{out} = \frac{\omega T_S a_R + (\lambda^2 T_S^2 + \omega^2 T_S^2 - \lambda T_S) b_R}{(1 - \lambda T_S)^2 + (\omega T_S)^2} + \frac{T_S c_R}{T_S - T_6} + \frac{T_S d_R}{T_S - T_5} + \frac{T_S e_R}{T_S - T_2} + \frac{T_S f_R}{T_S - T_4}. \quad (4.70)$$

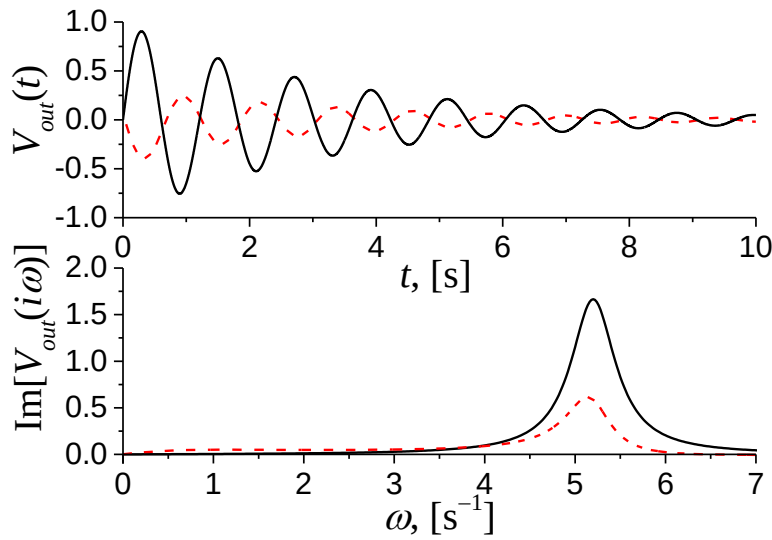


Figure 4-16: Signals $V_{in}(t)$ (solid lines) and $V_{out}(t)$ (dashed lines).
The parameters are the same as in Figure 4-2

The analytic solution (see Figure 4-17 below) and modelling with *SIMULINK*[®] (see Figure 4-4) show that the influence of the power system stabilizer is not adequate as the gain around the 1 Hz area seems to low.

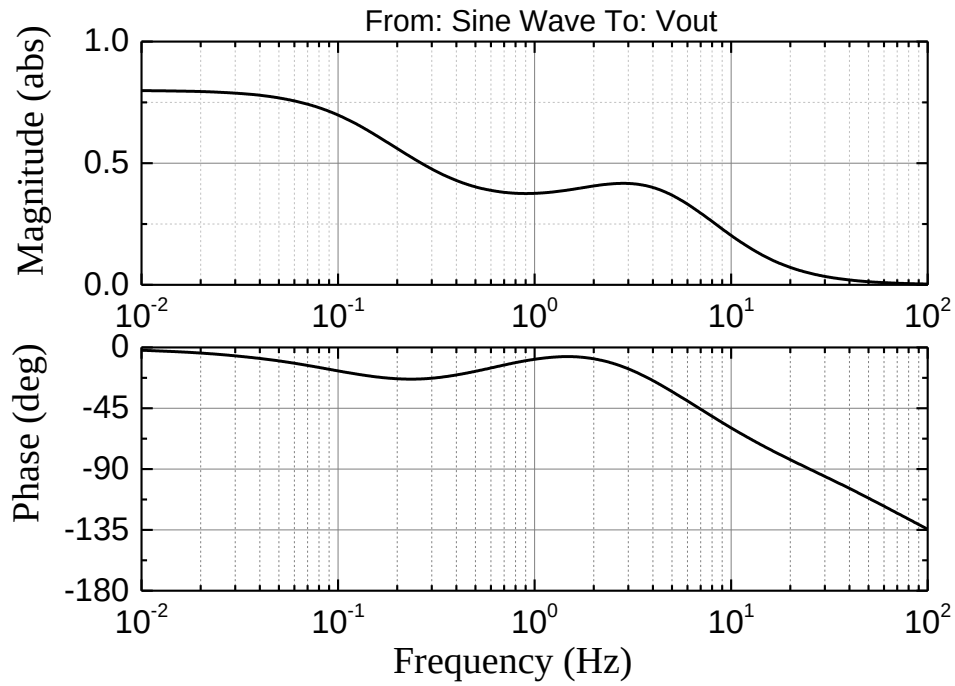


Figure 4-17 Bode Plot of PSS1A and AVR

4.7 ABOUT THE USABILITY OF PSS1A CONTROLLERS

We have seen for this specific PSS1A (with the actual parameters embedded in a power generating station), that when subjected to an oscillatory signal it *actively dampens a power oscillation*. Such oscillations occur when a generator is exposed to a ROCOF event. In practice, the accelerating torque becomes positive when the voltage at the terminal is raised, as the AVR increases the excitation current, in order to raise the electromagnetic torque [81, p. 334] in the airgap, thereby *forcing active damping of the torque oscillations and in doing so, dampens the power oscillations*. This set up in a finite grid would in consequence *actively dampen any grid frequency oscillations* which may have been *induced by the generator power output oscillations*. However, in the cases studied, particularly during a ROCOF event, one should discern *if* for the given PSS1A a change of parameters would yield better results *or* whether a different type of PSS like the PSS2B or PSS4B could be more appropriate as these systems use, beside the power oscillation as an input, also the actual values of the generator frequency, the generator current and also the generator voltage.

5 FINDINGS AND CONCLUSIONS

I have discussed dynamical models for the transient stability of conventional power generating stations connected to low inertia grid systems such as obtain in the island of Ireland and other isolated grids. These models are based both on *MATLAB* © simulations and on analytic solutions of the nonlinear pendulum-like equations which describe the motion of the coupled generator-grid system. The procedure may be summarized as follows:

The treatment initially involved computer simulations of the generator-grid system using modelling with *MATLAB*®, in particular the *SIMULINK*® graphical interface. Next, the equations of motion of the generator-grid system for the transient and steady state responses due to a stimulus, occurring as a result of a ROCOF event, were solved analytically. The analytic solutions are obtained by initially regarding the generator-grid system as connected to an infinite grid, leading to a nonlinear pendulum-like equation for the motion of the system. This equation may be numerically solved using matrix continued fraction methods already familiar in stochastic nonlinear oscillators. The model, of course, does not take into account the finite inertia of isolated grids pertaining to the Irish situation. Therefore, one needs a new model which incorporates torsional oscillation about a moving equilibrium position and, in so doing, may represent the feedback from the generator to an isolated grid, that is *finite* grid inertia.

The two degree of freedom model (itinerant oscillator) has the advantage that it also contains, as a special case, the infinite grid model and may be solved in a similar way using matrix continued fractions. The model also yields approximate expressions for the frequency of large oscillations using analytical techniques familiar in the analysis of pendulums making energy dependent large oscillations.

I emphasise that all the analytical models are based on the *Ansatz* of *lumping* all the generation fleet (except the particular or tagged one under discussion) into a single generator.

The parameters of the lumped generator-grid system were determined both analytically and empirically. I remark that the lumped generator-grid *Ansatz* allows the subsequent calculation of the dynamics of synchronous grid-connected generators as a function of the *lumped* parameters of the grid. In this way it facilitates a solution using *MATLAB* modelling as well as also facilitating the development of an analytic dynamics solution.

The solutions so obtained now provide the predictability required in order to assess how various units will react to transient disturbances if sudden torque changes are introduced by perturbations such as those imposed by ROCOF events.

The observed oscillations, which lead *inter alia* to further oscillations of the grid and vice versa, can now be quantified. Therefore, I believe that the analysis yields an accurate representation of actual events such as the one illustrated in Figure 1-3. Consequently, the model can provide a description of both the power station and grid behaviour during such frequency events. Moreover it provides analytical support for, as well as simplifying the *MATLAB* modelling. In this way my approach lends support to efforts to maximise the amount of renewable energy produced by *non-synchronously* connected generation plant such as wind turbines and photovoltaic installations. These are of the upmost importance in the context of the decarbonisation of electricity generation.

One major finding of this research is that the analytical models which are commonly used and yield simple algebraic formulae for ROCOF cannot describe frequency changes in low inertia systems as exemplified by Figure 3-22.

In short, I have described a viable mathematical model where the grid itself is postulated as a rigid rotator representing a grid with decreased inertia as obtains when it is operated *with a large amount of non-synchronous penetration*. This situation occurs in the island of Ireland and other isolated grids particularly during times of strong penetration by (predominantly) wind turbines and photo-voltaic installations. Clearly, the results highlight the potential risks which may occur if a component of the generator-grid system (called a unit) is exposed to a disturbance in a system with high wind penetration and low inertia. In contrast, in traditional large inertia systems, such as the European or North American grid, the ensuing oscillation due to a ROCOF event is quickly extinguished, while this is not true of *isolated* grids where the damping may be very light. Hence in systems *with small inertia* the stresses on any particular unit may also increase. For example, take a turbine generator shaft. Here although the oscillatory mechanical torques may not reach a level so high that one expects to see immediate crack development, nevertheless, the magnitude of the oscillations experienced due to ROCOF events may introduce additional crack-propagation where a crack already exists.

As far as technical risks are concerned, these can be broadly associated with three driving processes:

- Large changes in electrical power output affecting controllers such as excitation systems, power system stabilizers, turbine and unit controllers
- Large changes in the electromagnetic torque introducing critical mechanical oscillations affecting the mechanical integrity of the generator, the turbine rotor and its components
- Inherent response of a unit to falling frequency such as gas compressor performance or inertial energy delivery, possibly triggering a protection disconnecting the unit from the grid leading to cascade events, and *a fortiori* leading to a blackout of the entire electricity network.

In summary, the main conclusion of this thesis is that, by using the solutions as set out in the equations describing the rotating double pendulum or itinerant oscillator model, one may define *safe operational limits for synchronously connected generators in a small inertia system* by simply calculating the minimum amount of inertia required within the system as shown in Figure 3-20. In this way, one can gain a better understanding of how such low inertia systems react during transient disturbances. One specific finding is that the present commonly used definition of ROCOF, particularly the simplistic way of calculating it, is inadequate for low inertia electrical power systems and that the present use of ROCOF to describe weak grids needs further discussion.

I have shown that we are able to take the rotating double pendulum model described in this research ([22] and Appendix A) and adapt it to describe molecular movement ([13] and Appendix B). In essence, the model used *is again* a nonlinear itinerant oscillator model. However, it now represents a cage of neighbouring molecules which carries in its interior a electric dipole directly analogous to the generator rotor and the magnetic field in the bore of a generator. The cage model is familiar in statistical mechanics where it has been used in an *approximate form* (namely that the cage is much more massive than the dipole) to describe the microwave and far infrared absorption of polar dielectric fluids. The methods described in this thesis allow one to solve this model exactly for any inertia ratios so providing a complete description of the dynamics of the model.

As far as future work is concerned actual torque measurements would provide data for the numerical quantification and assessment of the severity of electrical faults experienced during transient events as seen during ROCOF events. These measurements would also allow superior assessment vs lifetime implications for the mechanical and electrical components of the generator, associated turbines and their auxiliary systems. The measurement of the actual mechanical torques will also facilitate verification of the models, complement the desktop analysis and will enable adjustment of them so that future theoretical reviews of the components can yield more accurate results.

Finally, this work primarily supports the objective of maximizing the amount of renewable energy obtained from non-synchronous generating sources as currently exist in the island of Ireland namely wind turbines and solar photo voltaic installations, thus facilitating the absolute requirement to decarbonize electricity generation.

6 APPENDIX A



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Models for the transient stability of conventional power generating stations connected to low inertia systems*

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Abstract. An ever-increasing requirement to integrate greater amounts of electrical energy from renewable sources especially from wind turbines and solar photo-voltaic installations exists and recent experience in the island of Ireland demonstrates that this requirement influences the behaviour of conventional generating stations. One observation is the change in the electrical power output of synchronous generators following a transient disturbance especially their oscillatory behaviour accompanied by similar oscillatory behaviour of the grid frequency, both becoming more pronounced with reducing grid inertia. This behaviour cannot be reproduced with existing mathematical models indicating that an understanding of the behaviour of synchronous generators, subjected to various disturbances especially in a system with low inertia requires a new modelling technique. Thus two models of a generating station based on a double pendulum described by a system of coupled nonlinear differential equations and suitable for analysis of its stability corresponding to infinite or finite grid inertia are presented. Formal analytic solutions of the equations of motion are given and compared with numerical solutions. In particular the new finite grid model will allow one to identify limitations to the operational range of the synchronous generators used in conventional power generation and also to identify limits, such as the allowable Rate of Change of Frequency which is currently set to ± 0.5 Hz/s and is a major factor in describing the volatility of a grid as well as identifying requirements to the total inertia necessary, which is currently provided by conventional power generators only, thus allowing one to maximise the usage of grid connected non-synchronous generators, *e.g.*, wind turbines and solar photo-voltaic installations.

1 Introduction

The requirement to decarbonize electrical energy production is leading to an increase of wind turbine and solar photo-voltaic based energy generation. These renewable generation sources in comparison to conventional *synchronously* grid connected turbo generators of gas, oil or even coal fired power stations are *asynchronously* connected to the grid. Inadvertently, by increasing the extent of asynchronously grid connected generation the stability of the entire electricity network is reduced thereby also affecting the existing conventional power generation units. Examples are the increase of electrical power oscillations of conventional power generating units and also consequential tripping of some larger units which occur during transient events at times of high wind generation in the system. In addition certain transient events like a sudden generation to load imbalance which arises after a trip of a generating unit, a disconnection of a large load or during a system separation event manifest themselves in a grid frequency drop, rise or even oscillation. The weakening of the grid frequency strength manifests itself via an increase of the Rate of Change of Frequency (ROCOF). Historically, the ROCOF was limited by the manufacturers of conventional generating units to ± 0.5 Hz/s suggesting that an increase in excess of this value needs to be analysed especially in relation to the manufacturer's specification thereby allowing for a further increase of non-synchronously connected electrical generation sources like wind turbine generators and solar PV installations. Now to enable one to initially quantify the influence of more severe ROCOF events on the

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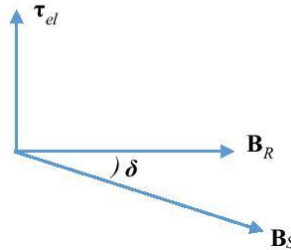


Fig. 1. The electromagnetic torque $\tau_{el} = k B_S B_R \sin \delta$ in the air gap of the generator.

operation and the integrity of conventional generating units it is necessary to define an appropriate dynamical model which we can mathematically analyse for the purpose of calculation of the ensuing electrical and mechanical behaviour of the generator sets. Our objective being to assess the consequences for their individual components, *e.g.* the generator with its electrical windings in the rotor and stator, the turbine with its blades and the coupling to the generator.

The paper is organized as follows. We first envisage the generator-grid system as a driven damped pendulum (making large oscillations), where we suppose that the grid has infinite inertia. This leads to a torque-balance equation in the form of the equation of motion of the angle δ between the rotor and stator fields which is now no longer constant instead oscillating because of the transient disturbance. This infinite grid inertia model allows one to estimate the energy dependent frequency of the disturbance in terms of an elliptic integral as well as the response to a short impulsive change of torque due to a transient fault (trip of a generator or disconnection of a large consumer). Next, the model is generalized to a finite grid inertia (such as obtains in an isolated grid like that of the island of Ireland). The generalization leads to two coupled nonlinear pendulum-like equations for the motion of the generator-grid system whereby the system may oscillate about an equilibrium position. These equations may be solved using matrix continued fractions and the response of the generator-grid system so obtained following the sudden imposition of a set of impulses is substantially in accordance with an actual ROCOF event. This second model explicitly takes account of the fact that when reviewing the measured data, it is clear that the grid frequency also exhibits a slight oscillation in phase with the power oscillation, indicating that these two are influencing each other. Since the grid inertia in the island of Ireland is not infinite any transient disturbance will induce a rotor angle oscillation which in turn leads to a grid oscillation. Hence, the assumption that is valid for an infinite grid (model 1), namely that no feedback of the oscillation experienced by the turbine generator shaft back to the grid exists, leads us to the conclusion that the single-mass mechanical analogue is not useful when applied to grids with small inertia. Thus the main focus of our attention will be on the second model.

2 A single-mass model: equation of motion (model 1)

The dynamical equations of central importance in power system stability analysis are the rotational inertial equations of motion describing the effect of imbalance between the electromagnetic torque and the mechanical torque of the individual machines. Now during stable operation of a synchronous generator, the electromagnetic torque in the airgap between rotor and stator of the generator is equal to the mechanical torque applied by the turbine rotor to the rotor of the generator, $\tau_{el} = \tau_m$. However when a disturbance occurs, an accelerating torque which can be either positive or negative acts on the turbo shaft inertia. The turbo shaft comprises the coupled turbine and generator rotor, which during such a disturbance either accelerates or de-accelerates.

The electric currents in the stator windings of a synchronized generator create an electromagnetic field in the air gap of the generator [1]. The strength of the magnetic field $B_S = |\mathbf{B}_S|$ is constant for constant stator currents. Now, since these currents are sinusoidal the field vector $\mathbf{B}_S$ rotates about the generator axis with angular velocity $\omega_0 = 2\pi f_0$, where f_0 is the grid frequency. In a synchronous machine the rotor field vector $\mathbf{B}_R$ rotates with the same angular velocity as the stator field. The magnitude of the rotor field $B_R = |\mathbf{B}_R|$ is constant as the field is induced by a DC current. Thus, the following electromagnetic torque τ_{el} arises in the air gap of the generator, *viz.*,

$$\tau_{el} = k \mathbf{B}_R \times \mathbf{B}_S, \quad (1)$$

where k is a characteristic constant of the machine which depends on its geometry. Equation (1) can be rewritten as (see fig. 1)

$$\tau_{el} = k \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ B_R \cos \delta_R & B_R \sin \delta_R & 0 \\ B_S \cos \delta_S & B_S \sin \delta_S & 0 \end{vmatrix} = \mathbf{e}_z k B_R B_S \sin \delta, \quad (2)$$

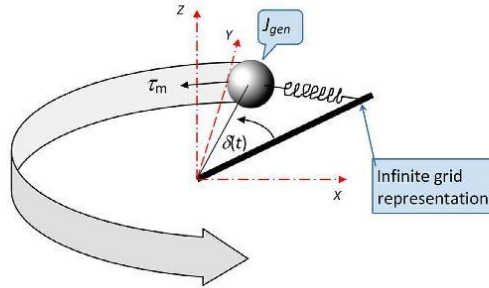


Fig. 2. Scheme with an infinite inertia of the grid.

where δ_R and δ_S are the angles between the vectors $\mathbf{B}_R$ and $\mathbf{B}_S$ and the X -axis, respectively, and $\delta = \delta_S - \delta_R$ is the angle between $\mathbf{B}_S$ and $\mathbf{B}_R$ (see fig. 1). The angle δ is called the rotor angle or in some cases the torque angle. The additional mechanical torque τ_m is transmitted from the turbine to the generator through the coupling of the turbine rotor and the generator rotor. Now if an imbalance between the torques τ_{el} and τ_m exists, the net torque $\tau_m - \tau_{el}$ causes a rate of change of angular momentum of the generator. Consequently, the angle $\delta(t)$ between $\mathbf{B}_S$ and $\mathbf{B}_R$ either increases or decreases. The total rotational kinetic energy of the system is $I = J\dot{\theta}^2/2$, where $\theta(t) = \omega_0 t + \delta(t)$ and J is the combined moment of inertia of generator and turbine. The potential energy of the system is $\Pi = kB_S B_R [1 - \cos(\theta(t) - \omega_0 t)]$. The total energy dissipated due to the damping torque is $D = K_D(\dot{\theta}(t) - \omega_0)^2/2$. The damping torque in this case is the torque which is created by the asynchronous operation of the generator. As we are analysing the behaviour of a synchronous generator an oscillation of its frequency results in creation of a so called slip which when negative creates a mechanical torque to the shaft, when positive it creates an additional electrical torque whereby the power output is increased. Hence it only exists when the angular velocity of the generator rotor *differs* from the angular velocity of the grid. All other energy losses, like friction or thermal losses are neglected hence we have the differential equation for δ :

$$J\ddot{\delta}(t) + K_D\dot{\delta}(t) + kB_S B_R \sin \delta(t) = \tau_m,$$

or

$$\ddot{\delta}(t) + \beta\dot{\delta}(t) + \xi \sin \delta(t) = \bar{\tau}_m, \quad (3)$$

where $\beta = K_D/J$, $\xi = kB_S B_R/J$, and $\bar{\tau}_m = \tau_m/J$. This single-mass model is commonly used to analyze the dynamic response of a synchronous generator in an infinite grid [1–3]. The mechanical analogue is a *driven damped pendulum*. In accordance with eq. (3) both grid and generator initially rotate with constant rotor angle $\delta_0 = \arcsin[\tau_m/(kB_S B_R)] = \text{const.}$ (see fig. 2). The bar in fig. 2 symbolizing the grid, moves according to the grid frequency and is of *infinite* inertia, meaning that it cannot be influenced by the inertia of a single generator. In the steady-state situation both inertias, comprising the generator turbine shaft J_{gen} and the grid J_{grid} rotate in a plane horizontally about the z -axis with angular velocity ω_0 corresponding to the nominal grid-frequency of 50 Hz. However, during *transient* faults on the system like the trip of a generator or the disconnection of a large load, the grid-frequency changes. Hence the generator will experience an *additional* torque as the grid inertia alters its angular velocity. This disturbance leads to an oscillation of the rotor angle $\delta(t)$. The rotor angle oscillations result in a power output oscillation because the electrical power of a generator is determined via the rotor angle as $P_{el} = P_{\max} \sin \delta(t)$. Hence the rate of change of the grid frequency is of major importance.

If a disturbance of the motion takes place the period T of oscillations of the angle $\delta(t)$ may be estimated by considering the idealized free motion of a generator when the damping and torque τ_m are absent, *i.e.*, $\beta = 0$ and $\tau_m = 0$. The solution of eq. (3) in this case is given by (see appendix A) [4]

$$\sin [\delta(t)/2] = \sqrt{m} \operatorname{sn} \left[\sqrt{\xi}(t - t_0) \mid m \right], \quad (4)$$

where $\operatorname{sn}(u \mid m)$ is a Jacobian elliptic function [5] with modulus $m = \sin^2(\delta_0/2)$, δ_0 is the amplitude and t_0 is the initial time at which $\delta = \delta_0$ and $\dot{\delta} = 0$. The exact expression for the period T is given by

$$T = 4\sqrt{\frac{J}{kB_S B_R}} K(m), \quad (5)$$

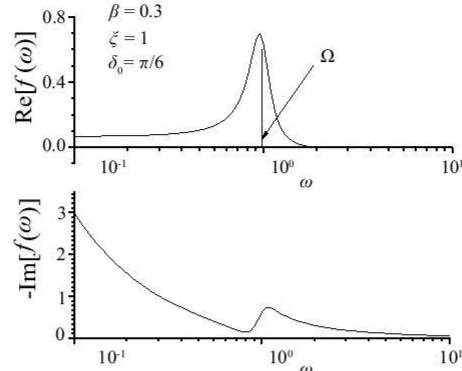


Fig. 3 Real and imaginary parts of the linear combination $f(\omega) = [\tilde{c}_{1-1}(i\omega) - \tilde{c}_{1-1}(i\omega)]/2i$ calculated by using the matrix continued fraction method (see appendix B). The frequency of free motion Ω is given by eq. (6).

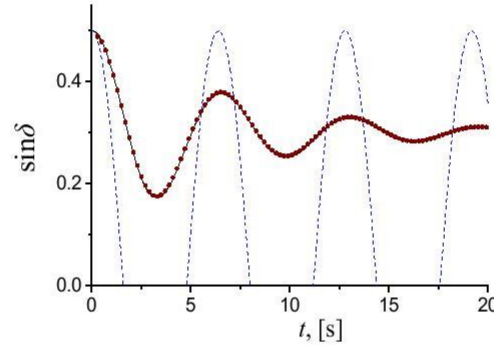


Fig. 4 Solid line: numerical solution of eq. (3) obtained by using **Mathematica**[®]. Circles: continued fraction solution of eq. (3) in the frequency domain following from the inverse Fourier transform (see appendix B) with initial conditions $\delta_0 = \pi/6$, $\dot{\delta}_0 = 0$ for the following model parameters $\xi = 1$, $\bar{\tau}_m = 0.3$, $\beta = 0.3$. Dashed line: analytical solution for $\beta = 0$, eq. (4) whereby we may estimate the period analytically.

where $K(m)$ is a complete elliptic integral of the first kind [5]. Thus the angular frequency of the periodic motion is

$$\Omega = \frac{2\pi}{T} = \frac{\pi\sqrt{kB_S B_R/J}}{2K(m)}. \quad (6)$$

The solution of the damped pendulum equation (3) in terms of continued fractions is given in appendix B [6]. We suppose that a disturbance of the motion occurs at the instant $t_0 = 0$ when the torque $\bar{\tau}_m$ changes from the initial value $\bar{\tau}_m^0 = \xi \sin \delta_0$ to $\bar{\tau}_m < \bar{\tau}_m^0$. Here $\delta_0 = \delta(0)$ is the initial angle. The value of δ_0 depends on the applied torque of the turbine. The initial value of $\dot{\delta}_0 = \dot{\delta}(0)$ is zero. By deriving from eq. (3) a set of differential-recurrence equations for the functions $c_{nq}(t) = \delta^n r^q$, we can solve this set for the one-sided Fourier transforms $\tilde{c}_{nq}(i\omega)$ of $c_{nq}(t)$ via matrix continued fractions. Next, using the fast Fourier transform algorithm allows us to recover the function $c_{nq}(t)$. Having calculated $c_{nq}(t)$, we can evaluate all other functions of interest. For example, we have

$$\sin \delta(t) = \frac{c_{0-1}(t) - c_{01}(t)}{2i} \quad (7)$$

and so on. The results of such calculations for the linear combination of the one-sided Fourier transforms $\tilde{c}_{1\pm 1}(i\omega)$ in the frequency domain are shown in fig. 3.

Equation (3) can also be solved numerically using **Mathematica**[®]. A comparison of the numerical and continued fraction solutions of eq. (3) is shown in fig. 4. Figure 5 shows the response of the system to a set of impulses (representing the alternating character of changes of $\bar{\tau}_m$ due to external events). This result can be compared with the measured power output of a single shaft combined cycle 400 MW synchronous generator after a real ROCOF event [1] (see fig. 6).

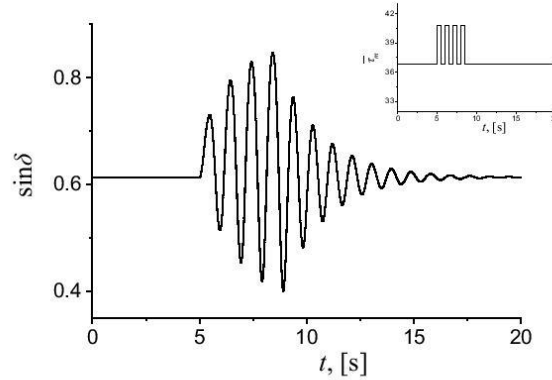


Fig. 5. Response of the system (eq. (3)) to a set of impulses of torque $\bar{\tau}_m$ for the initial conditions $\delta_0 = 0.66$, $\dot{\delta}_0 = 0$ and the following model parameters $\xi = 60$, $\beta = 1$.

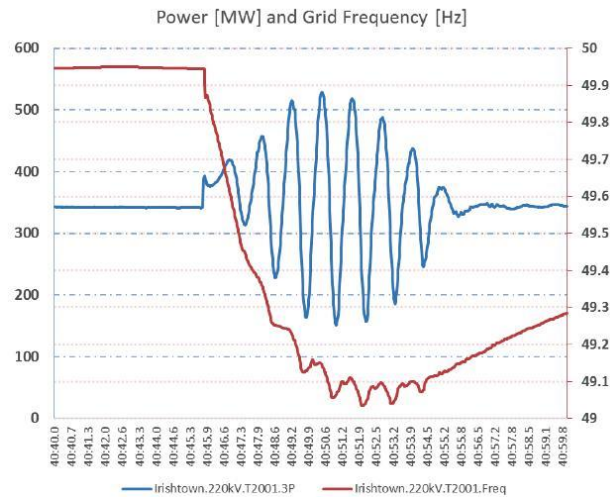


Fig. 6. Power output (blue, left scale) and grid frequency (red, right scale) of a 400 MW synchronous generator as a response of a real ROCOF event, real time in [min:sec].

The power oscillation obtained from the model (fig. 5) reproduces qualitatively well the experimentally measured oscillation of the power in fig. 6. However this model assumes infinite grid inertia, *i.e.* a fixed grid frequency. The most important feature of fig. 6 is that the grid frequency executes an oscillation in phase with the power oscillation indicating that they influence each other. Furthermore, recent observations indicate that the smaller the inertia in the grid, the more the output power of a generator oscillates. Now the grid frequency experiences such drops about 20–30 times annually. These are usually due to trips of generating stations or disconnection of a large load (which, for example, can happen when an HVDC interconnector trips while exporting electric energy from Ireland to Great Britain). Hence the ability to calculate exactly how the frequency behaves during such disturbances demands an approach whereby that frequency can be influenced by the transient behaviour of a generating station rather than just the ability to calculate the maximum ROCOF alone. As seen in the previous analysis, any transient disturbance can induce a rotor angle oscillation resulting in mechanical oscillations of the generator-turbine rotor. Hence, we have oscillations of the power output of the set leading to a grid oscillation and in turn this grid oscillation again influences how the turbo-set oscillates and so on. Thus the assumption that is valid for an *infinite grid*, namely, that no feedback of the oscillation experienced by the turbine generator shaft back to the grid exists, leads one to the conclusion that the single-mass model is not useful when applied to grids with *small* inertia so that we need a two-mass pendulum model with two degrees of freedom.

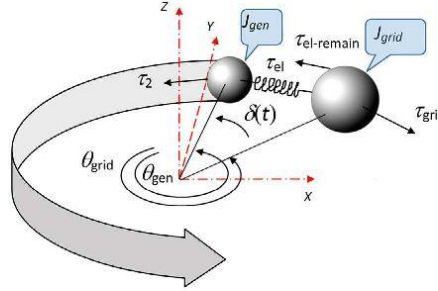


Fig. 7. Scheme with a finite inertia of the grid.

3 The two-mass pendulum (model 2)

This model, where the grid now has *finite* moment of inertia such as obtains in the isolated grid of Ireland (see fig. 7), leads to the following set of two coupled equations of motion for the generator and grid:

$$J_{grid}\ddot{\theta}_{grid} + K_D(\dot{\theta}_{grid} - \dot{\theta}_{gen}) + \tau_{el\max} \sin(\theta_{grid} - \theta_{gen}) = \tau_1 = \tau_{grid} - \tau_{el-remain}, \quad (8)$$

$$J_{gen}\ddot{\theta}_{gen} + K_D(\dot{\theta}_{gen} - \dot{\theta}_{grid}) + \tau_{el\max} \sin(\theta_{gen} - \theta_{grid}) = \tau_2 = \tau_{gen}, \quad (9)$$

where $\tau_{el\max}$ is the maximum electromagnetic torque in the air gap, τ_2 is the torque applied by the turbine to the generator, τ_1 is the resulting torque applied to the grid (sum of all turbine torques less the torques due to the loads and remaining generators $\tau_{el-remain}$ on the grid). Here the damping torques, characterized by the damping coefficient K_D , are those torques, which are present only when the rotor angle oscillates. In this case, the turbine controller reacts to a drop of the frequency by increasing the output of the set. It opens the valves to allow an increase of the power output. All generating units in the grid have this control behaviour. We emphasize that eqs. (8) and (9) represent a model which still has limitations, *e.g.* the energy losses due to resistances in the generators and lines are ignored.

The formal solution of eqs. (8) and (9) is given in appendix C. For the purpose of analysis of the generator and grid response to various disturbances we use below numerical solutions of eqs. (8) and (9) obtained in *Mathematica*®. At this stage one needs to point out that after a disturbance, all generators in the system increase their mechanical torque, by means of increasing the load of the turbine. This behaviour is programmed in the turbine controllers and is called primary (POR), secondary (SOR) and tertiary operating reserve (TOR). To enable one to understand the behaviour of a generator during a transient disturbance, all these torques from controllers and all the remaining generators in the power system were combined and visualised as an impulse or a set of impulses (caption in fig. 5). Figure 8 shows the response of the system (grid frequency $f_{grid} = \dot{\theta}_{grid}/(2\pi)$, generator frequency $f_{gen} = \dot{\theta}_{gen}/(2\pi)$, power output $P_{el}/P_{\max} = \sin \delta(t)$ and ROCOF $= \Delta f_{grid}/\Delta t = \Delta \dot{\theta}_{grid}/(2\pi \Delta t)$) to a set of short impulsive changes of τ_1 (as mentioned earlier representing the alternating character of changes of τ_1 due to external events such as oscillations of other generators) for different values of the inertia ratio $x = J_{grid}/J_{gen}$: $x = 7$ (column (a)), $x = 30$ (column (b)) and $x = 100$ (column (c)). Here $\delta = \theta_{grid} - \theta_{gen}$ and x is a parameter describing how much inertia lies within the transmission system. For example, for $x = 7$, the grid has 7 times as much inertia as the generator in question. The initial value of the angle δ_0 allows one to obtain a maximum initial power output which is proportional to $\sin \delta_0$ and yet allows it to still remain in the linear regime (where $\sin \delta_0 \approx \delta_0$). The recovery of the frequency to its initial (*i.e.* normal) value is accomplished by adjusting τ_1 following a perturbation. This recovery is in practice achieved by increasing the power output of those generating units remaining connected to the system. The power output increase is a function which is called primary and secondary operating reserve. Clearly, the power output (third row plots in fig. 8) qualitatively corresponds to the measurements seen in reality. Figure 8 also shows the ROCOF (fourth row) corresponding to the first row with $\Delta t = 100$ ms. The response of the system (eqs. (8) and (9)) to an applied impulse in addition to the permanent torque τ_1 (*i.e.* a short impulsive change in τ_1) is shown in fig. 9.

Now it is important to emphasize that the numerical solution of eqs. (8) and (9) indicates that for $x = 5$ the generator would experience a pole slip as seen in fig. 9. This finding is in contradiction with the commonly used formula for ROCOF, namely

$$\frac{df}{dt} = \frac{\Delta P}{P} \frac{f_0}{2H}, \quad (10)$$

where ΔP [MW] is the power change per unit machine base, P [MW] is the total power generated and H [s] is the system inertia. Equation (10) expressed via the parameters of the model, namely $2HP_N = J_{grid}\omega_0^2$ and $\Delta P \approx \Delta \tau \omega_0$,

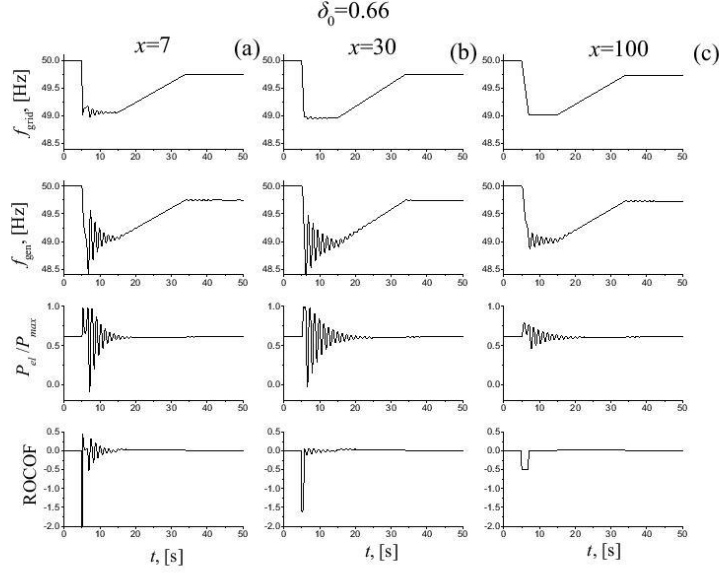


Fig. 8. Response of the generator-grid system to a short impulsive change in τ_1 . The calculations correspond to the initial conditions $\delta_0 = 0.66$, $\theta_{gen}(0) = \theta_{grid}(0) = 0$ and the following parameters: $J_{gen} = 71 \cdot 10^3 \text{ kg} \cdot \text{m}^2$, $\beta_{gen}/J_{gen} = 0.42 \text{ s}^{-1}$, $\beta_{grid}/J_{grid} = 2.8/x \text{ s}^{-1}$, and $\tau_2 = \tau_1 = 1.3 \cdot 10^6 \text{ N} \cdot \text{m}$.

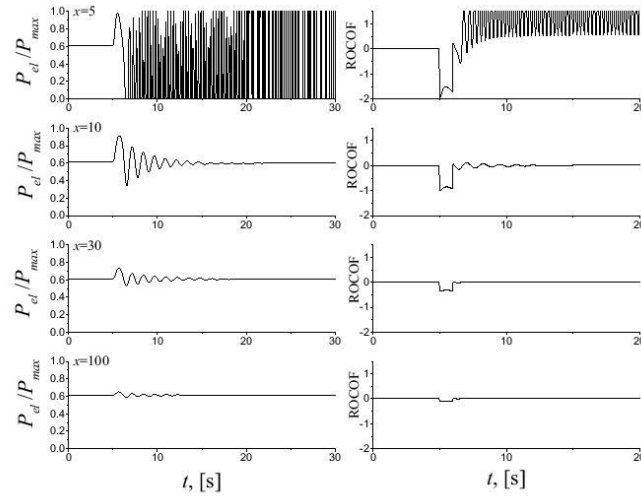


Fig. 9. The response of the system to the same additional impulse of torque $\Delta\tau_{grid}$ for different values of x .

gives

$$\frac{df}{dt} = \frac{\Delta P}{P} \frac{f_0}{2H} \approx \frac{\Delta P f_0}{2HP_N} = \frac{\Delta\tau \omega_0 f_0}{J_{grid} \omega_0^2} = \frac{\Delta\tau}{2\pi x J_{gen}}. \quad (11)$$

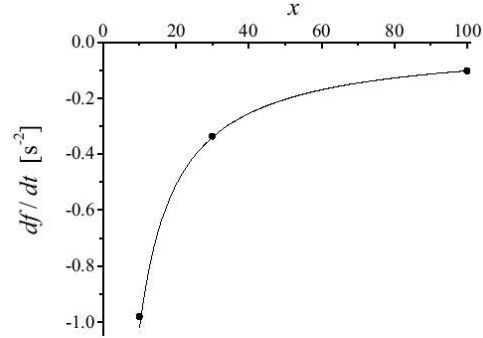


Fig. 10. df/dt vs. $x = J_{\text{grid}}/J_{\text{gen}}$ for $J_{\text{gen}} = 71 \cdot 10^3 \text{ kg} \cdot \text{m}^2$, $\tau_{\text{gen}} = 1.3 \cdot 10^6 \text{ N} \cdot \text{m}$ and $\Delta\tau_{\text{grid}} = -3.5\tau_{\text{gen}}$. Solid line: eq. (11). Filled circles: solution of eqs. (8) and (9).

The comparison of numerical calculations of df/dt vs. $x = J_{\text{grid}}/J_{\text{gen}}$ with the traditionally used eq. (11) is shown in fig. 10. Clearly good agreement exists between the numerical solution of eqs. (8) and (9) and the approximation contained in eq. (11) for $x > 10$. For values of x less than 10 one cannot use such an approximation.

4 Conclusion

We have described a viable mathematical model where the grid is postulated as a rotating mass representing a grid with decreased inertia as is obtained when it is operated *with a large amount of non-synchronous penetration*. This situation occurs in the island of Ireland and other isolated grids during times of strong penetration by (predominantly) wind turbines, etc.

Clearly, the results of the analysis highlight the potential risks when a unit is exposed to a disturbance in a system with high wind penetration and consequential low inertia. Whereas in traditional large inertia systems the oscillation steadily decreases we perceive that in systems with small inertia the stresses on the unit may also increase. Although the mechanical torques may not reach a level where one expects to see immediate crack development in a turbine generator shaft component nevertheless the magnitude of the oscillations experienced could introduce additional crack-propagation where a crack already exists.

The technical risks can be broadly associated with three driving processes:

- a) Large changes in electrical power output affecting controllers such as excitation systems, power system stabilizers, turbine controller and unit controllers.
- b) Large changes in the electromagnetic torque introducing critical mechanical oscillations affecting both the mechanical integrity of the generator and the turbine rotor and its components.
- c) Inherent response of a unit to falling frequency such as gas compressor performance or inertial energy delivery possibly triggering a protection to disconnect the unit from the grid leading to cascade events, possibly even to a blackout of the entire electricity network.

However our major finding is that by using the solutions set out in the equations describing the double pendulum, one may define the safe operational limits for synchronously connected generators, in a small inertia system, thereby gaining a better understanding of how low inertia systems react during transient disturbances and thus maximizing the quantity of renewable energy which may be generated.

We thank Dr W.J. Dowling for a critical reading of the paper and useful comments and suggestions.

Appendix A. Solution of free pendulum equation

For $\beta = \bar{\tau}_m = 0$, we rearrange eq. (3) as

$$2\dot{\delta}(t)\ddot{\delta}(t) + 2\xi\dot{\delta}(t)\sin\delta(t) = \frac{d}{dt} \left[\dot{\delta}^2(t) - 2\xi\cos\delta(t) \right] = 0. \quad (\text{A.1})$$

Thus we have from eq. (A.1)

$$\dot{\delta}^2(t) = 2\xi [\cos \delta(t) - \cos \delta_0] = 4\xi \left[\sin^2 \frac{\delta_0}{2} - \sin^2 \frac{\delta(t)}{2} \right], \quad (\text{A.2})$$

or

$$\frac{d\delta}{2\sqrt{\sin^2(\delta_0/2) - \sin^2(\delta/2)}} = \sqrt{\xi} dt, \quad (\text{A.3})$$

where δ_0 is an integration constant. By integrating eq. (A.3) with the initial condition $\delta(t_0) = 0$, we have

$$\int_0^\delta \frac{d(\delta/2)}{\sqrt{m - \sin^2(\delta/2)}} = \int_0^{m^{-1/2} \sin \delta/2} \frac{dt}{\sqrt{(1 - mt^2)(1 - t^2)}} = \sqrt{\xi}(t - t_0), \quad (\text{A.4})$$

where $m = \sin^2 \delta_0/2$, so yielding eq. (4).

Appendix B. Matrix continued fraction solution of damped pendulum equation

In the general case, we have for the function $c_{nq}(t) = \delta^n r^q$ with $r = e^{-i\delta(t)}$

$$\begin{aligned} \frac{d}{dt} c_{nq}(t) &= n\delta^{n-1} r^q \dot{\delta} - iq\delta^{n+1} r^q \\ &= -iqc_{n+1q}(t) - n\delta^{n-1} r^q [\beta\dot{\delta}(t) + \xi \sin \delta(t) - \bar{\tau}_m]. \end{aligned} \quad (\text{B.1})$$

Here we substitute $\ddot{\delta}(t) = -\beta\dot{\delta}(t) - \xi \sin \delta(t) + \bar{\tau}_m$ from eq. (3). By using $\sin \delta(t) = (r^{-1} - r)/2i$ in eq. (B.1), we have the following differential-recurrence relation for $c_{nq}(t)$:

$$\frac{d}{dt} c_{nq}(t) = -\beta n c_{nq}(t) - iq c_{n+1q}(t) - \frac{in}{2} [\xi c_{n-1q+1}(t) + 2i\bar{\tau}_m c_{n-1q} - \xi c_{n-1q-1}(t)]. \quad (\text{B.2})$$

By introducing infinite column vectors,

$$\mathbf{C}_0(t) = 0, \quad \mathbf{C}_n(t) = \begin{pmatrix} \vdots \\ c_{n-1-2}(t) \\ c_{n-1-1}(t) \\ c_{n-10}(t) \\ c_{n-11}(t) \\ c_{n-12}(t) \\ \vdots \end{pmatrix} \quad (n > 0), \quad (\text{B.3})$$

the differential-recurrence equations (B.2) can be transformed into the tri-diagonal matrix form

$$\frac{d}{dt} \mathbf{C}_n(t) = \mathbf{Q}_n^- \mathbf{C}_{n-1}(t) + \mathbf{Q}_n \mathbf{C}_n(t) + \mathbf{Q}_n^+ \mathbf{C}_{n+1}(t). \quad (\text{B.4})$$

Here the matrices $\mathbf{Q}_n$ and $\mathbf{Q}_n^\pm$ are

$$\mathbf{Q}_n = -\beta(n-1)\mathbf{I}, \quad (\text{B.5})$$

$$\mathbf{Q}_n^+ = -i \begin{pmatrix} \ddots & \ddots & \ddots & & \\ & -1 & 0 & \ddots & \\ & \ddots & 0 & 0 & \ddots \\ & & \ddots & 0 & 1 & \ddots \\ & & & \ddots & \ddots & \ddots \end{pmatrix}, \quad (\text{B.6})$$

$$\mathbf{Q}_n^- = -(n-1) \begin{pmatrix} \ddots & \ddots & \ddots & & \\ & \bar{\tau}_m & i\xi/2 & \ddots & \\ & \ddots & -i\xi/2 & \bar{\tau}_m & i\xi/2 & \ddots \\ & & \ddots & -i\xi/2 & \bar{\tau}_m & \ddots \\ & & & \ddots & \ddots & \ddots \end{pmatrix}, \quad (\text{B.7})$$

and $\mathbf{I}$ is the unit matrix of infinite dimension.

Equation (B.4) is solved by the matrix continued fraction method [6, 7]. On considering the initial conditions $\delta(0) = \delta_0$ and $\dot{\delta}(0) = 0$ and taking the Laplace transform of eq. (B.4), we obtain the matrix recurrence relations for the Laplace transforms $\tilde{\mathbf{C}}_n(s)$:

$$-s\tilde{\mathbf{C}}_1(s) + \mathbf{Q}_1^+\tilde{\mathbf{C}}_2(s) = -\mathbf{C}_1(0), \quad (\text{B.8})$$

$$\mathbf{Q}_n^-\tilde{\mathbf{C}}_{n-1}(s) + (\mathbf{Q}_n - s\mathbf{I})\tilde{\mathbf{C}}_n(s) + \mathbf{Q}_n^+\tilde{\mathbf{C}}_{n+1}(s) = 0 \quad (n > 1), \quad (\text{B.9})$$

where

$$\tilde{\mathbf{C}}_n(s) = \int_0^\infty \mathbf{C}_n(t)e^{-st}dt. \quad (\text{B.10})$$

Here we have noted that

$$\mathbf{C}_1(0) = \begin{pmatrix} \vdots \\ e^{2i\delta_0} \\ e^{i\delta_0} \\ 1 \\ e^{-i\delta_0} \\ e^{-2i\delta_0} \\ \vdots \end{pmatrix}, \quad \mathbf{C}_n(0) = 0, \quad n > 2.$$

We seek the solution of the linear eqs. (B.8) and (B.9) as

$$\tilde{\mathbf{C}}_n(s) = \mathbf{\Delta}_n \mathbf{Q}_n^-\tilde{\mathbf{C}}_{n-1}(s) + \mathbf{R}_n, \quad (\text{B.11})$$

where $\mathbf{\Delta}_n$ is the matrix continued fraction defined by the recurrence equation

$$\mathbf{\Delta}_n = [s\mathbf{I} - \mathbf{Q}_n - \mathbf{Q}_n^+\mathbf{\Delta}_{n+1}\mathbf{Q}_{n+1}^-]^{-1}, \quad (\text{B.12})$$

which allows us to represent $\mathbf{\Delta}_n$ as an infinite matrix continued fraction [7]. Substituting eq. (B.11) into eqs. (B.8) and (B.9) yields

$$[s\mathbf{I} - \mathbf{Q}_1 - \mathbf{Q}_1^+\mathbf{\Delta}_2\mathbf{Q}_2^-]\mathbf{R}_1 = \mathbf{C}_1(0) + \mathbf{Q}_1^+\mathbf{R}_2, \quad (\text{B.13})$$

$$[s\mathbf{I} - \mathbf{Q}_n - \mathbf{Q}_n^+\mathbf{\Delta}_{n+1}\mathbf{Q}_{n+1}^-]\mathbf{R}_n = \mathbf{Q}_n^+\mathbf{R}_{n+1} \quad (n > 1). \quad (\text{B.14})$$

Equations (B.13) and (B.14) can be solved exactly for the desired *particular solutions* $\mathbf{R}_n$ in terms of the matrix continued fraction Δ_1 :

$$\mathbf{R}_1 = \Delta_1 \mathbf{C}_1(0), \quad \mathbf{R}_n = 0 \quad (n > 1). \quad (\text{B.15})$$

Substituting eq. (B.15) into eq. (B.11) yields the matrix continued-fraction solution of eqs. (B.8) and (B.9) for the Laplace transforms $\tilde{\mathbf{C}}_n(s)$ [7]

$$\tilde{\mathbf{C}}_1(s) = \Delta_1 \mathbf{C}_1(0), \quad (\text{B.16})$$

$$\tilde{\mathbf{C}}_n(s) = \Delta_n \mathbf{Q}_n^- \tilde{\mathbf{C}}_{n-1}(s) = \Delta_n \mathbf{Q}_n^- \Delta_{n-1} \mathbf{Q}_{n-1}^- \dots \Delta_1 \mathbf{C}_1(0) \quad (n > 1). \quad (\text{B.17})$$

The matrix continued fraction solution we have obtained is very convenient for the purpose of computation (various algorithms for calculating matrix continued fractions are discussed in refs. [6] and [7]). As far as the calculation of the infinite matrix continued fraction Δ_n is concerned, we approximated it by some matrix continued fraction of finite order (by putting $\mathbf{Q}_{n+1} = 0$ at some $n = K$). At the same time, we confined the dimensions of the matrices $\mathbf{Q}_n$ and $\mathbf{Q}_n^\pm$ to some finite number $2M + 1$. Both K and M depend on the model parameters and must be chosen taking account of the desired degree of accuracy of the calculation. The calculation of the time-dependent functions $c_{nq}(t)$ can be then accomplished using the Fast Fourier Transform (FFT) algorithm as

$$\begin{aligned} c_{nq}(t) &\approx \frac{1}{\pi} \operatorname{Re} \left[\int_0^{\Omega'(N-1)} \tilde{c}_{nq}(i\omega) e^{i\omega t} d\omega \right] \\ &= \frac{\Omega'}{\pi} \operatorname{Re} \left[\sum_{s=1}^N \tilde{c}_{nq}(i\Omega'(s-1)) e^{i\Omega'(s-1)t} \right], \end{aligned} \quad (\text{B.18})$$

or

$$\begin{aligned} c_{nq}(T(r-1)) &= \frac{\Omega'}{\pi} \operatorname{Re} \left[\sum_{s=1}^N \tilde{c}_{nq}(i\Omega'(s-1)) e^{i\Omega'(s-1)T(r-1)} \right] \\ (r &= 1, 2, \dots, N), \end{aligned} \quad (\text{B.19})$$

where

$$\Omega' = \frac{2\pi}{NT}, \quad N = 2^k. \quad (\text{B.20})$$

Appendix C. Matrix continued fraction solution of eqs. (8) and (9)

Equations (8) and (9) can be rewritten as

$$\ddot{\theta}_i + \beta_i(\dot{\theta}_i - \dot{\theta}_j) + \xi_i \sin(\theta_i - \theta_j) = \bar{\tau}_i \quad (i = 1, 2 \text{ and } j = 2, 1), \quad (\text{C.1})$$

where $\theta_1 = \theta_{\text{grid}}$, $\beta_1 = K_D^1/J_{\text{grid}}$, $\xi_1 = \tau_{el \text{ max}}/J_{\text{grid}}$, $\theta_2 = \theta_{\text{gen}}$, $\beta_2 = K_D^2/J_{\text{gen}}$, $\xi_2 = \tau_{el \text{ max}}/J_{\text{gen}}$, $\bar{\tau}_1 = \tau_1/J_{\text{grid}}$ and $\bar{\tau}_2 = \tau_2/J_{\text{grid}}$.

Now we introduce

$$c_{n_1 q_1}^{n_2 q_2} = \dot{\theta}_1^{n_1} \dot{\theta}_2^{n_2} r_1^{q_1} r_2^{q_2}, \quad n = 0, 1, 2, \dots, \quad q = 0, \pm 1, \pm 2, \dots, \quad (\text{C.2})$$

where $r_i = e^{-i\theta_i(t)}$. Then, noting eqs. (C.1) and (C.2), we have

$$\begin{aligned} \frac{d}{dt} c_{n_1 q_1}^{n_2 q_2} &= -(\beta_1 n_1 + \beta_2 n_2) c_{n_2 q_2}^{n_1 q_1} + \beta_1 n_1 c_{n_2+1 q_2}^{n_1-1 q_1} + \beta_2 n_2 c_{n_2-1 q_2}^{n_1+1 q_1} - i q_1 c_{n_2 q_2}^{n_1+1 q_1} - i q_2 c_{n_2+1 q_2}^{n_1 q_1} \\ &\quad + \bar{\tau}_1 n_1 c_{n_2 q_2}^{n_1-1 q_1} + \frac{i}{2} \xi_1 n_1 (c_{n_2 q_2+1}^{n_1-1 q_1-1} - c_{n_2 q_2-1}^{n_1-1 q_1+1}) + \bar{\tau}_2 n_2 c_{n_2-1 q_2}^{n_1 q_1} + \frac{i}{2} \xi_2 n_2 (c_{n_2-1 q_2-1}^{n_1 q_1+1} - c_{n_2-1 q_2+1}^{n_1 q_1-1}). \end{aligned} \quad (\text{C.3})$$

Thus on introducing column vectors $\mathbf{C}_n(t)$ defined as

$$\mathbf{C}_1(t) = (\mathbf{c}_{00}), \quad \mathbf{C}_2(t) = \begin{pmatrix} \mathbf{c}_{10} \\ \mathbf{c}_{01} \end{pmatrix}, \dots, \quad \mathbf{C}_n(t) = \begin{pmatrix} \mathbf{c}_{n-10} \\ \mathbf{c}_{n-21} \\ \vdots \\ \mathbf{c}_{0n-1} \end{pmatrix}, \quad (\text{C.4})$$

where the subvectors are

$$\mathbf{c}_{n_1 n_2}(t) = \begin{pmatrix} \vdots \\ c_{n_2 2}^{n_1-1}(t) \\ c_{n_2 1}^{n_1 0}(t) \\ c_{n_2 0}^{n_1 1}(t) \\ c_{n_2-1}^{n_1 2}(t) \\ \vdots \end{pmatrix}, \quad (\text{C.5})$$

it follows that the differential-recurrence equation (C.3) can be transformed into the matrix three-term recurrence equation (B.4), namely,

$$\frac{d}{dt}\mathbf{C}_n(t) = \mathbf{Q}_n^- \mathbf{C}_{n-1}(t) + \mathbf{Q}_n \mathbf{C}_n(t) + \mathbf{Q}_n^+ \mathbf{C}_{n+1}(t), \quad (\text{C.6})$$

where the matrices $\mathbf{Q}_n$ and $\mathbf{Q}_n^\pm$ are given by

$$\mathbf{Q}_n = -\beta_1 \begin{pmatrix} (n-1)\mathbf{I} & (n-1)\mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & (n-2)\mathbf{I} & (n-2)\mathbf{I} & \ddots & \vdots \\ \mathbf{0} & \ddots & \ddots & \ddots & \mathbf{0} \\ \vdots & \ddots & \mathbf{0} & \mathbf{I} & \mathbf{I} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} - \beta_2 \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{I} & \mathbf{I} & \mathbf{0} & \ddots & \vdots \\ \mathbf{0} & \ddots & \ddots & \ddots & \mathbf{0} \\ \vdots & \ddots & (n-2)\mathbf{I} & (n-2)\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & (n-1)\mathbf{I} & (n-1)\mathbf{I} \end{pmatrix}, \quad (\text{C.7})$$

$$\mathbf{Q}_n^+ = -i \begin{pmatrix} \mathbf{q}^+ & \mathbf{q}_+ & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{q}^+ & \mathbf{q}_+ & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{q}^+ & \mathbf{q}_+ \end{pmatrix}, \quad (\text{C.8})$$

$$\mathbf{Q}_n^- = \begin{pmatrix} (n-1)\mathbf{q}^- & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{q}_- & (n-2)\mathbf{q}^- & \ddots & \vdots \\ \mathbf{0} & 2\mathbf{q}_- & \ddots & \mathbf{0} \\ \vdots & \ddots & \ddots & \mathbf{q}_- \\ \mathbf{0} & \cdots & \mathbf{0} & (n-1)\mathbf{q}_- \end{pmatrix}, \quad (\text{C.9})$$

where

$$\mathbf{q}^+ = \begin{pmatrix} -q_{\max} + 1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & -q_{\max} + 2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & q_{\max} \end{pmatrix}, \quad \mathbf{q}_+ = \begin{pmatrix} q_{\max} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & q_{\max} - 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & -q_{\max} + 1 \end{pmatrix} \quad (\text{C.10})$$

$$\mathbf{q}^- = \begin{pmatrix} \bar{\tau}_1 & -i\xi_1/2 & \cdots & \mathbf{0} \\ i\xi_1/2 & \bar{\tau}_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -i\xi_1/2 \\ \mathbf{0} & \cdots & i\xi_1/2 & \bar{\tau}_1 \end{pmatrix}, \quad \mathbf{q}_- = \begin{pmatrix} \bar{\tau}_2 & i\xi_2/2 & \cdots & \mathbf{0} \\ -i\xi_2/2 & \bar{\tau}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & i\xi_2/2 \\ \mathbf{0} & \cdots & -i\xi_2/2 & \bar{\tau}_2 \end{pmatrix}, \quad (\text{C.11})$$

and $\mathbf{I}$ is a unit matrix with dimension $2q_{\max}$ (here we have confined the dimensions of the matrices $\mathbf{q}^\pm$ and $\mathbf{q}_\pm$ to some finite number $2q_{\max}$).

Using the Laplace transformation, eq. (C.6) can be reduced to the matrix recurrence relations for the Laplace transforms $\tilde{\mathbf{C}}_n(s)$, eqs. (B.8) and (B.9), where

$$\mathbf{C}_1(0) = \begin{pmatrix} \vdots \\ c_{02}^{0-1}(0) \\ c_{01}^{00}(0) \\ c_{00}^{01}(0) \\ c_{0-1}^{02}(0) \\ \vdots \end{pmatrix}, \quad (\text{C.12})$$

$c_{0q_2}^{0q_1}(0) = e^{-i(q_1\theta_1(0)+q_2\theta_2(0))}$, and $\mathbf{C}_n(0) = 0$ for $n = 2, 3, \dots$, because $\dot{\theta}_1(0) = \dot{\theta}_2(0) = 0$. Then by invoking the general method for solving the matrix recurrence equations [6], we have the matrix continued fraction solution, eq. (B.16). The calculation of the time-dependent functions $c_{n_2q_2}^{n_1q_1}(t)$ can be then accomplished using the FFT algorithm.

Finally, the following properties of eqs. (8) and (9) can be useful in practical applications. First by adding eqs. (8) and (9), we have

$$J_{\text{grid}}\ddot{\theta}_{\text{grid}} + J_{\text{gen}}\ddot{\theta}_{\text{gen}} = \tau_1 + \tau_2. \quad (\text{C.13})$$

Next writing eqs. (8) and (9) in the form of eq. (C.1) and subtracting the second equation from the first equation, we have

$$\ddot{\theta}_1 - \ddot{\theta}_2 + (\beta_1 + \beta_2)(\dot{\theta}_1 - \dot{\theta}_2) + (\xi_1 + \xi_2)\sin(\theta_1 - \theta_2) = \bar{\tau}_1 - \bar{\tau}_2. \quad (\text{C.14})$$

Equation (C.14) is identical in form to eq. (3) thus it can be solved in a similar way using matrix continued fraction solution of the damped pendulum equation (see appendix B). However, the difference is that now the grid dynamics are influenced by the generator in accordance with eq. (C.13) and the grid does not oscillate anymore with a constant frequency ω_0 as in a grid with infinite inertia. However we still need to integrate eq. (C.13) twice and combine it with the continued fraction solution of eq. (C.14) in order to determine the overall dynamics of the system.

References

1. P. Kundur, *Power system stability and control*, in *The EPRI Power System Engineering Series* (McGraw Hill Inc., NY, 1994).
2. M. Zarifakis, W.T. Coffey, *Impact of elevated Rate of Change of Frequency (RoCoF) on conventional power generation plant in the Island of Ireland*, VGB PowerTech (8/2014).
3. M. Zarifakis, W.T. Coffey, *Transient Stability of conventional power generating stations during times of high wind penetration*, in *Arbeitskreis Energie, Vortraege auf der DPG Fruehjahrstagung in Berlin 2015* (9/2015).
4. E.T. Whittaker, G.N. Watson, *A Course of Modern Analysis*, 4th edition (Cambridge University Press, London, 1927).
5. M. Abramowitz, I.A. Stegun (Editors), *Handbook of Mathematical Functions* (Dover, New York, 1972).
6. W.T. Coffey, Yu.P. Kalmykov, *The Langevin Equation*, 4th edition (World Scientific, Singapore, 2017).
7. H. Risken, *The Fokker-Planck Equation*, 2nd edition (Springer, Berlin, 1989).

7 APPENDIX B

Cage model of polar fluids: Finite cage inertia generalization

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Cage model of polar fluids: Finite cage inertia generalization

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The *itinerant oscillator model* describing rotation of a dipole about a fixed axis inside a cage formed by its surrounding polar molecules is revisited in the context of modeling the dielectric relaxation of a polar fluid via the Langevin equation. The dynamical properties of the model are studied by averaging the Langevin equations describing the complex orientational dynamics of two bodies (molecule-cage) over their realizations in phase space so that the problem reduces to solving a system of three index linear differential-recurrence relations for the statistical moments. These are then solved in the frequency domain using matrix continued fractions. The linear dielectric response is then evaluated for extensive ranges of damping, dipole moment ratio, and cage-dipole inertia ratio and along with the usual inertia corrected microwave Debye absorption gives rise to significant far-infrared absorption with a comb-like structure of harmonic peaks. The model may be also regarded as an extension of Budó's [J. Chem. Phys. **17**, 686 (1949)] treatment of molecules containing rotating polar groups to include inertial effects. Published by AIP Publishing. [<http://dx.doi.org/10.1063/1.4993791>]

I. INTRODUCTION

The cage model or itinerant oscillator^{1–6} introduced by Hill^{7,8} and Sears⁹ is a schematic representation of the dynamical behavior of a molecule in a fluid embodying the fact that a typical molecule of the fluid may vibrate about a temporary equilibrium position. Sears⁹ used a translational version of the model to evaluate the velocity correlation function for liquid argon, while Hill^{7,8} used a two-dimensional rotational version to explain relaxation (Debye) and far-infrared (FIR) absorption spectrum of dipolar fluids. Hill considered a specific mechanism whereby at any instant an individual polar molecule may be regarded as confined to a temporary equilibrium position in the potential well created by its cage of polar nearest neighbors, where the potential energy surface may in general have several minima. The molecule is considered to librate in this cage, i.e., execute torsional oscillations about temporary equilibrium positions. The Brownian motion of the cage gives rise to the Debye absorption,^{10,11} while the librational motion of the molecule gives rise to the resonance absorption in the far infrared region. Using an approximate analysis of this model based on the Smoluchowski equation¹⁰ for the rotational diffusion of an assembly of non-interacting dipolar molecules, Hill demonstrated that the frequency of the FIR absorption peak is inversely proportional to the square root of the moment of inertia of a molecule. Hill's treatment based on the small oscillation (harmonic potential) approximation was further expanded upon, using the Langevin equation of the theory of the Brownian motion, in a series of papers by Wyllie^{12,13} and by Coffey *et al.*^{14–17} The angular

velocity and orientational correlation functions were explicitly evaluated in the small oscillation approximation.^{5,6,10} These results are summarized in Refs. 1 and 2. All these analyses, however, invariably rely on a small oscillation approximation because no reliable method of treating the finite oscillations of a pendulum when the Brownian torques are included had existed. Thus, some of the most important nonlinear aspects of the relaxation processes were omitted, an example being the dependence of the frequency of oscillation of the dipole on the amplitude of the oscillation. A preliminary attempt to include nonlinear or anharmonic effects (cosine potential) has been made in Refs. 17 and 18. The equations of motion of the model when the restriction to fixed axis rotation is relaxed were given by Coffey.¹⁸ Recently it became possible to treat the Brownian motion in a potential other than a parabolic one in a general fashion using matrix continued fractions.¹⁹ The solution was illustrated¹⁹ by considering the Brownian motion of a rotator in a cosine potential with a *fixed* equilibrium position.

The various applications to chemical physics are discussed extensively in Refs. 20–24. Yet another application of the model is relaxation of ferrofluids²⁵ (colloidal suspensions of single-domain ferromagnetic particles). A three-dimensional rotational version excluding inertial effects, termed an “egg model,” has been used by Shliomis and Stepanov,²⁶ to simultaneously explain the Brownian and Néel relaxation in ferrofluids,²⁵ which are due to the rotational diffusion of the particles and random reorientations of the magnetization inside the particles, respectively. The analysis in the context of ferrofluids yields similar equations of motion

to those given by Damle *et al.*²⁷ and by van Kampen²⁸ for the translational itinerant oscillator. This formulation has the advantage that, in the noninertial limit, the equations of motion automatically decouple into those of the molecule and its surroundings (cage). Such a decoupling of the exact equations of motion is also possible in the inertial case if we assume a massive cage. These results are summarized in Refs. 6 and 10. Moreover in Ref. 6, it was also shown how to extend the model to the fractional time random walk process where the characteristic waiting time between encounters of molecules is divergent.

The disadvantage of all these analyses, however, is that due to the inherent mathematical difficulties arising from the intrinsically nonlinear nature of the governing Langevin equations, they invariably rely on the assumption that the inertia of the molecule is much smaller than the inertia of the cage. The assumption allows one to decouple the equations of motion into the equation of motion of the cage itself which is just that of a free inertial rigid Brownian rotator and the motion of the encaged dipole or tagged molecule alone which is that of an inertial rigid Brownian rotator in a sinusoidal potential. Hence all the correlation functions of the model may be given in a closed form. However, in using this *ansatz*, some of the most important aspects of the relaxation processes are unavoidably omitted, for example, the perturbation of the cage rotation by the molecule. As a result, the relaxation modes of the system separate into those of the tagged molecule and those of its surroundings, irrespective of the form of the interaction potential between the cage and the tagged molecule, which in general is not physical.

It is the purpose of this paper to illustrate using linear response theory (which relates the ac response to the step-off response following the removal of a constant field $\mathbf{F}$) how the itinerant oscillator model may be solved exactly for the first time for the case of *finite inertia of the cage* as compared to that of the molecule. In this case, the rotational motion of the cage is no longer independent and is undoubtedly influenced by the probe. The treatment of a cage with finite inertia may be justified as follows. In solid bodies (e.g., ferromagnetic particles), the internal potentials are due to the crystalline structure of the bodies and in consequence are rather stable.¹⁰ This is not true of fluids and dense liquids where the angular displacement of the surroundings is comparable with the displacement of a probe. Hence it is reasonable to conceive of the cage as undergoing drift or diffusive motion as it is intrinsic to the model. If the number of molecules surrounding the probe is not very large, it also seems reasonable to use finite cage inertia. Then the rotation of the cage can no longer be simplified by treating it as merely rotational Brownian motion of a heavy object since it must now be described in a more complicated manner as rotation *perturbed by the probe molecule*.

We shall describe the cage model mainly in the context of the theory of dielectric relaxation. By averaging the Langevin equations describing the complex dynamics of two bodies (molecule-cage) over its realizations, the problem is first reduced to solving a system of linear differential-recurrence relations for the statistical moments. It is then shown how these three-term differential-recurrence relations may be formally solved for the Fourier transform of the after-effect function

via matrix continued fractions, thereby allowing one to exactly analyze the susceptibility spectrum $\chi(\omega)$ of the model regardless of the inertia ratio and dipole-cage interaction strength. In this form, the model may be regarded as an extension of Budó's²⁹ treatment of molecules containing rotating polar groups to include inertial effects, thereby automatically leading to far infrared resonance (THz) absorption as well as the accompanying microwave Debye absorption so that both absorptions may be linked together in the same model.

II. SOLUTION OF THE LANGEVIN EQUATIONS FOR A DIPOLE INTERACTING WITH ITS SURROUNDINGS

A typical member of a system of dipolar molecules (specified by an angle ϕ_1) and cage (specified by an angle ϕ_2) rotating about a fixed axis in an external uniform dc applied field $\mathbf{F}$ (which is needed in order to use linear response theory) is in general governed by a Hamiltonian H given by

$$H = \frac{1}{2}I_1\dot{\phi}_1^2 + \frac{1}{2}I_2\dot{\phi}_2^2 - \mu_1 F \cos \phi_1 - \mu_2 F \cos \phi_2 - \mu_1 R \cos(\phi_1 - \phi_2), \quad (1)$$

where $F = |\mathbf{F}|$, μ_1 and I_1 are the dipole moment and the moment of inertia of the molecule, respectively, and μ_2 and I_2 are the dipole moment and the moment of inertia of the surroundings (i.e., the cage), respectively. The final term in Eq. (1) represents the dipole-cage interaction, where R is the reaction field of the surroundings¹⁰ which represents their influence on the probe molecule. Here we shall use the quasi-stationary approximation for the reaction field $R(t) = R$. The equation of motion of the encaged polar molecule in the external uniform dc field and the field of the rotating cage is in accordance with that of van Kampen,³⁰

$$I_1\ddot{\phi}_1(t) = \zeta_1(\dot{\phi}_2(t) - \dot{\phi}_1(t)) - \mu_1 R \sin(\phi_1(t) - \phi_2(t)) - \mu_1 F \sin \phi_1(t) + \lambda_1(t). \quad (2)$$

The terms $\zeta_1(\dot{\phi}_2(t) - \dot{\phi}_1(t))$ and $\lambda_1(t)$ represent the dissipative and fluctuating Brownian torques acting on the molecule due to the heat bath. The equation of motion of the cage in the presence of the dc field $\mathbf{F}$ and the field of the polar molecule is on applying Newton's third law³⁰

$$I_2\ddot{\phi}_2(t) = \zeta_1(\dot{\phi}_1(t) - \dot{\phi}_2(t)) - \zeta_2\dot{\phi}_2(t) - \mu_1 R \sin(\phi_2(t) - \phi_1(t)) - \mu_2 F \sin \phi_2(t) - \lambda_1(t) + \lambda_2(t). \quad (3)$$

The terms $-\zeta_2\dot{\phi}_2(t)$ and $\lambda_2(t)$ are again the dissipative and the stochastic torques on the surroundings of the cage which are generated by the heat bath in which the cage is embedded. In Eqs. (2) and (3), the white noise torques $\lambda_i(t)$ are centered Gaussian random variables with correlation functions

$$\overline{\lambda_i(t)\lambda_i(t')} = 2kT\zeta_i\delta(t-t'), \quad (4)$$

where kT is the thermal energy (k is Boltzmann's constant).

In order to use linear response theory, we consider the special case where a small external dc field $\mathbf{F}$, which has been applied to the system in the infinite past, is suddenly switched off at time $t = 0$. In our study, we are interested in the relaxation of the system consisting of the cage and embedded dipole,

starting from an initial equilibrium state at $t = 0$ with the Maxwell-Boltzmann distribution function $W_F(\phi_1, \phi_2, \dot{\phi}_1, \dot{\phi}_2)$ given by

$$W_F(\phi_1, \phi_2, \dot{\phi}_1, \dot{\phi}_2) = Z_F^{-1} e^{-\eta_1^2 \dot{\phi}_1^2 - \eta_2^2 \dot{\phi}_2^2 + \xi_1 \cos \phi_1 + \xi_2 \cos \phi_2 + \xi_{\text{int}} \cos(\phi_1 - \phi_2)} \quad (5)$$

to another equilibrium state as $t \rightarrow \infty$ with the new Maxwell-Boltzmann distribution function

$$W_0(\phi_1, \phi_2, \dot{\phi}_1, \dot{\phi}_2) = W_{F=0}(\phi_1, \phi_2, \dot{\phi}_1, \dot{\phi}_2) = Z_0^{-1} e^{-\eta_1^2 \dot{\phi}_1^2 - \eta_2^2 \dot{\phi}_2^2 + \xi_{\text{int}} \cos(\phi_1 - \phi_2)}. \quad (6)$$

Here Z_F is the partition function defined as

$$Z_F = \int_0^{2\pi} \int_0^{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\eta_1^2 \dot{\phi}_1^2 - \eta_2^2 \dot{\phi}_2^2 + \xi_1 \cos \phi_1 + \xi_2 \cos \phi_2 + \xi_{\text{int}} \cos(\phi_1 - \phi_2)} \times d\phi_1 d\phi_2 d\dot{\phi}_1 d\dot{\phi}_2,$$

where $\eta_1 = \sqrt{I_1/(2kT)}$ and $\eta_2 = \sqrt{I_2/(2kT)}$ are characteristic time constants, $\xi_1 = \mu_1 F/(kT)$ and $\xi_2 = \alpha \xi_1 = \mu_2 F/(kT)$ are the field-dipole and field-cage interaction parameters, $\alpha = \mu_2/\mu_1$ is the ratio of dipole moments, and $\xi_{\text{int}} = \mu_1 R/(kT)$ is the dipole-cage interaction parameter. The dynamics of the cage-dipole system immediately following the removal of the external dc field may be described using the normalized relaxation function

$$C(t) = \frac{\varphi(t) + \psi(t)}{\varphi(0) + \psi(0)}, \quad t > 0, \quad (7)$$

where we have defined the individual response functions

$$\varphi(t) = \mu_1 (\langle \cos \phi_1 \rangle(t) - \langle \cos \phi_1 \rangle_0), \quad (8)$$

$$\psi(t) = \mu_2 (\langle \cos \phi_2 \rangle(t) - \langle \cos \phi_2 \rangle_0). \quad (9)$$

Here the angular brackets $\langle A \rangle(t)$ are the time-dependent ensemble averages associated with an observable A and the brackets $\langle A \rangle_0$ designate the *equilibrium* ensemble averages, namely,

$$\langle A \rangle_0 = \int_0^{2\pi} \int_0^{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(\phi_1, \phi_2, \dot{\phi}_1, \dot{\phi}_2) W_0(\phi_1, \phi_2, \dot{\phi}_1, \dot{\phi}_2) \times d\phi_1 d\phi_2 d\dot{\phi}_1 d\dot{\phi}_2. \quad (10)$$

In the linear approximation in the applied dc field, $\mu_i F/(kT) \ll 1$, the initial conditions for the functions $\varphi(t)$ and $\psi(t)$ are

$$\varphi(0) \approx \frac{F}{kT} \left[\mu_1^2 (\langle \cos^2 \phi_1 \rangle_0 - \langle \cos \phi_1 \rangle_0^2) + \mu_1 \mu_2 \times (\langle \cos \phi_1 \cos \phi_2 \rangle_0 - \langle \cos \phi_1 \rangle_0 \langle \cos \phi_2 \rangle_0) \right], \quad (11)$$

$$\psi(0) \approx \frac{F}{kT} \left[\mu_2^2 (\langle \cos^2 \phi_2 \rangle_0 - \langle \cos \phi_2 \rangle_0^2) + \mu_1 \mu_2 \times (\langle \cos \phi_1 \cos \phi_2 \rangle_0 - \langle \cos \phi_1 \rangle_0 \langle \cos \phi_2 \rangle_0) \right]. \quad (12)$$

Thus, the individual response function $\varphi(t)$ now refers to the embedded dipole while $\psi(t)$ refers to that of the surroundings. Since we have assumed in order to use linear response theory that the initial amplitude F of the external uniform dc field is small ($\mu_i F/(kT) \ll 1$), the complex susceptibility $\chi(\omega) = \chi'(\omega) - i\chi''(\omega)$ and so the ac response of such an assembly can then be determined from the linear response theory formula^{31,32}

$$\frac{\chi(\omega)}{\chi} = 1 - i\omega \int_0^{\infty} C(t) e^{-i\omega t} dt, \quad (13)$$

where $\chi = \chi'(0) = [\varphi(0) + \psi(0)] F^{-1}$ is the static susceptibility and the normalized relaxation function $C(t)$ is given by Eqs. (7)–(9).

III. MATRIX CONTINUED FRACTION SOLUTION FOR THE RESPONSE FUNCTION

The one-sided Fourier transform of the normalized relaxation function $C(t)$ is calculated as follows. As shown in [Appendix A](#), the Langevin equations, Eqs. (2) and (3), can be transformed into differential-recurrence equations for the statistical moments $f_{m_2}^{m_1 q_1}(t)$ pertaining to the observables, which are given by

$$f_{m_2}^{m_1 q_1}(t) = \langle H_{m_1}(\eta_1 \dot{\phi}_1) H_{m_2}(\eta_2 \dot{\phi}_2) e^{-iq_1 \phi_1} e^{-iq_2 \phi_2} \rangle(t), \quad m_i = 0, 1, 2, \dots, \quad q_i = 0, \pm 1, \pm 2, \dots, \quad (14)$$

where $H_m(z)$ are the Hermite polynomials. The ensuing calculation advantage is that (in this step-off case) the indices q_1 and q_2 are no longer independent of each other; thus, we may introduce the simplified moments $c_{m_1 m_2}^q(t) = f_{m_2}^{m_1 q}(t)$ associated with the expectation values of observables. Finally taking account of the general form of the differential-recurrence equations for the functions $f_{m_2}^{m_1 q_1}(t)$ from [Appendix A](#), Eq. (A6), we have the corresponding differential-recurrence equations for the moments $c_{m_1 m_2}^q(t)$ (where the external field parameters ξ_1 and ξ_2 are now zero), viz.,

$$\begin{aligned} \eta \frac{d}{dt} c_{m_1 m_2}^q &= - \left(\gamma_1 \beta_1 m_1 + \frac{\gamma_2^2}{\gamma_1} \beta_1 m_2 + \gamma_2 \beta_2 m_2 \right) c_{m_1 m_2}^q + \gamma_2 \beta_1 m_2 c_{m_1+1 m_2-1}^q + \gamma_2 \beta_1 m_1 c_{m_1-1 m_2+1}^q + 2\gamma_2 \beta_1 m_1 m_2 c_{m_1-1 m_2-1}^q \\ &\quad - 2 \frac{\gamma_2^2}{\gamma_1} \beta_1 m_2 (m_2 - 1) c_{m_1 m_2-2}^q + \frac{i\xi_{\text{int}}}{2} \left[\gamma_1 m_1 (c_{m_1-1 m_2}^{q-1} - c_{m_1-1 m_2}^{q+1}) + \gamma_2 m_2 (c_{m_1 m_2-1}^{q+1} - c_{m_1 m_2-1}^{q-1}) \right] \\ &\quad - \frac{i}{2} \gamma_1 q (c_{m_1+1 m_2}^q + 2m_1 c_{m_1-1 m_2}^q) + \frac{i}{2} \gamma_2 (q-1) (c_{m_1 m_2+1}^q + 2m_2 c_{m_1 m_2-1}^q). \end{aligned} \quad (15)$$

Here $\eta^{-1} = \eta_1^{-1} + \eta_2^{-1}$, $\gamma_i = \eta/\eta_i$ is an inertia ratio, and all other notations are given in Appendix A. The set of Eq. (15) represents three index linear differential-recurrence equations. The solution of this system of equations for $c_{m_1 m_2}^q(t) = f_{m_2 - q + 1}^{m_1 q}(t)$ is far simpler than those for $f_{m_2 q_2}^{m_1 q_1}(t)$ which comprise four index differential-recurrence equations and which are obtained when $F \neq 0$.

Proceeding, we first introduce the supercolumn vector $\mathbf{C}_n(t)$, which is defined as follows:

$$\mathbf{C}_n(t) = \begin{pmatrix} \mathbf{c}_{0n}(t) \\ \mathbf{c}_{1n-1}(t) \\ \vdots \\ \mathbf{c}_{n0}(t) \end{pmatrix}, \quad \mathbf{c}_{m_1 m_2}(t) = \begin{pmatrix} \vdots \\ c_{m_1 m_2}^{-1}(t) \\ c_{m_1 m_2}^0(t) \\ c_{m_1 m_2}^1(t) \\ \vdots \end{pmatrix}, \quad n \geq 0. \quad (16)$$

Thus the multi-term differential-recurrence equation recurrence Eq. (15) for the expectation values of the observables can be transformed into a supermatrix four-term differential-recurrence equation of the form

$$\eta \frac{d}{dt} \mathbf{C}_n(t) = \mathbf{Q}_n^- \mathbf{C}_{n-2}(t) + \mathbf{Q}_n^- \mathbf{C}_{n-1}(t) + \mathbf{Q}_n \mathbf{C}_n(t) + \mathbf{Q}_n^+ \mathbf{C}_{n+1}(t), \quad (17)$$

where the supermatrices $\mathbf{Q}_n^-$, $\mathbf{Q}_n$, and $\mathbf{Q}_n^+$ are given in Appendix B. By taking the Laplace transform of Eq. (17), we have the algebraic four-term recurrence equation in the frequency domain

$$\eta s \tilde{\mathbf{C}}_n(s) - \delta_{n0} \eta \mathbf{C}_0(0) = \mathbf{Q}_n^- \tilde{\mathbf{C}}_{n-2}(s) + \mathbf{Q}_n^- \tilde{\mathbf{C}}_{n-1}(s) + \mathbf{Q}_n \tilde{\mathbf{C}}_n(s) + \mathbf{Q}_n^+ \tilde{\mathbf{C}}_{n+1}(s), \quad (18)$$

where

$$\tilde{\mathbf{C}}_n(s) = \int_0^\infty \mathbf{C}_n(t) e^{-st} dt. \quad (19)$$

Here we have used the initial condition $\mathbf{C}_n(0) = 0$ for $n > 0$ because $\langle H_n e^{-iq\phi_i} \rangle = 0$ for $n > 0$ as demanded by the initial Maxwell-Boltzmann distribution. The initial value vector $\mathbf{C}_0(0)$ is given in Appendix B. Next by invoking the general continued-fraction method for solving matrix recurrence relations,^{10,33} we have the formal solution for the spectrum $\tilde{\mathbf{C}}_0(s)$ which contains the elements of interest to us as defined in Eq. (22), viz.,

$$\tilde{\mathbf{C}}_0(s) = \eta \Delta_0(s) \mathbf{C}_0(0), \quad (20)$$

where the matrix $\Delta_n(s)$ is to be calculated from its matrix continued-fraction definition, viz.,

$$\Delta_n(s) = \left[\eta s \mathbf{I} - \mathbf{Q}_n - \mathbf{Q}_n^+ \Delta_{n+1}(s) (\mathbf{Q}_{n+1}^- + \mathbf{Q}_{n+1}^+ \Delta_{n+2}(s) \mathbf{Q}_{n+2}^-) \right]^{-1}. \quad (21)$$

Having determined the column vector $\tilde{\mathbf{C}}_0(s)$, we then have the spectrum of the relaxation function $C(t)$ given by Eq. (7),

$$\begin{aligned} \tilde{C}(i\omega) &= \frac{\tilde{\varphi}(i\omega) + \tilde{\psi}(i\omega)}{\varphi(0) + \psi(0)} \\ &= \frac{\tilde{c}_{00}^1(i\omega) + \tilde{c}_{00}^{1*}(-i\omega) + \alpha [\tilde{c}_{00}^0(i\omega) + \tilde{c}_{00}^{0*}(-i\omega)]}{c_{00}^1(0) + c_{00}^{1*}(0) + \alpha [c_{00}^0(0) + c_{00}^{0*}(0)]}, \end{aligned} \quad (22)$$

where the asterisks denote the complex conjugate and we have noted that the response functions of the dipole $\varphi(t)$ and cage $\psi(t)$ are given in terms of the elements of the column vector $\mathbf{C}_0(t)$, Eq. (16), as

$$\begin{aligned} \varphi(t) &= \frac{\mu_1}{2} (c_{00}^1(t) + c_{00}^{1*}(t)), \\ \psi(t) &= \frac{\mu_2}{2} (c_{00}^0(t) + c_{00}^{0*}(t)), \end{aligned} \quad (23)$$

while the corresponding spectra in the frequency domain are

$$\begin{aligned} \tilde{\varphi}(i\omega) &= \frac{\mu_1}{2} [\tilde{c}_{00}^1(i\omega) + \tilde{c}_{00}^{1*}(-i\omega)], \\ \tilde{\psi}(i\omega) &= \frac{\mu_2}{2} [\tilde{c}_{00}^0(i\omega) + \tilde{c}_{00}^{0*}(-i\omega)]. \end{aligned} \quad (24)$$

These equations, on calculating the matrix continued-fraction Eq. (21), will yield an exact numerical solution of our problem.

IV. APPROXIMATE EXPRESSIONS FOR THE COMPLEX SUSCEPTIBILITY

For purposes of calculation, it is also useful to have analytical solutions for particular limiting cases, thereby yielding some insight into the underlying physics as well as serving as a benchmark for the matrix continued fraction solution given above. First by adding Eqs. (2) and (3), we have for $F = 0$,

$$I_2 \ddot{\phi}_2(t) + I_1 \dot{\phi}_1(t) = -\zeta_2 \dot{\phi}_2(t) + \lambda_2(t). \quad (25)$$

Next on dividing Eq. (2) by I_1 and Eq. (3) by I_2 and subtracting the second equation from the first, we have

$$\begin{aligned} \ddot{\theta}(t) + \frac{\zeta_1}{I_1} \left(1 + \frac{I_1}{I_2}\right) \dot{\theta}(t) - \frac{I_1 \zeta_2}{I_2} \dot{\phi}_2(t) + \frac{\mu_1 R}{I_1} \left(1 + \frac{I_1}{I_2}\right) \sin \theta(t) \\ = \lambda_1(t) + \frac{I_1}{I_2} [\lambda_1(t) - \lambda_2(t)], \end{aligned} \quad (26)$$

where $\theta(t) = \phi_1(t) - \phi_2(t)$. Clearly, Eqs. (25) and (26) decouple in the limit $I_1/I_2 \ll 1$ and so lead to the itinerant oscillator equations³⁴

$$I_2 \ddot{\phi}_2(t) = -\zeta_2 \dot{\phi}_2(t) + \lambda_2(t), \quad (27)$$

$$I_1 \ddot{\theta}(t) = -\zeta_1 \dot{\theta}(t) - \mu_1 R \sin \theta(t) + \lambda_1(t). \quad (28)$$

The equations of motion have now been reduced to the Langevin equation of a free Brownian rotator and that of a dipole in a cosine potential. Furthermore, for infinite cage inertia $I_1 \ll I_2$, the cage (surroundings) autocorrelation function is^{6,10,34} available in the closed form,

$$C_s(t) = \frac{\psi(t)}{\psi(0)} = \frac{\langle \cos \phi_2 \rangle(t) - \langle \cos \phi_2 \rangle_0}{\langle \cos \phi_2 \rangle(0) - \langle \cos \phi_2 \rangle_0} \approx \frac{\langle \cos \phi_2(0) \cos \phi_2(t) \rangle}{\langle \cos \phi_2(0) \cos \phi_2(0) \rangle} = \exp \left\{ -\frac{1}{2\beta_2} \left[\frac{t}{\eta_2} + \frac{1}{\beta_2} \left(e^{-\frac{\beta_2 t}{\eta_2}} - 1 \right) \right] \right\}. \quad (29)$$

On expanding the double exponential, we have

$$C_s(t) = e^{\gamma_2} \sum_{n=0}^{\infty} \frac{1}{(-2\beta_2^2)^n n!} e^{-(1+2\beta_2^2 n) \frac{t}{2\eta_2\beta_2}}. \quad (30)$$

The Laplace transform of Eq. (30) is ($s = i\omega$)

$$\tilde{C}_s(i\omega) = \frac{2\eta\beta_2}{\gamma_2} e^{(2\beta_2^2)^{-1}} \times \sum_{n=0}^{\infty} \frac{1}{(-2\beta_2^2)^n n! (1 + 2\beta_2^2 n + 2i\omega\eta\beta_2/\gamma_2)}. \quad (31)$$

Equation (31) is useful for calculating the susceptibility of the surroundings via the linear response formula, Eq. (13), which in this case is

$$\frac{\chi_s(\omega)}{\chi_s(0)} = 1 - i\omega\tilde{C}_s(i\omega). \quad (32)$$

Now the dipole autocorrelation function $\varphi(t)$ has been examined in detail^{10,15,35} in the small oscillation (itinerant oscillator) approximation. Invariably $\varphi(t)$ gives rise to a pronounced FIR absorption peak in the frequency domain. The characteristic frequency ω_0 of this FIR peak is given by

$$\omega_0 = \frac{\gamma_1}{\eta} \sqrt{\frac{\xi_{\text{int}}}{2}}, \quad (33)$$

and the complete expression for the overall correlation function $C(t)$ as defined by Eq. (7) is given in the *small oscillation* and *large cage approximation* by the product

$$C(t) = C_s(t)C_{\mu_1}(t) = C_s(t) \exp \left\{ -\frac{1}{\xi_{\text{int}}} \left[1 - \exp \left(-\frac{\beta_1\gamma_1 t}{2\eta} \right) \right] \times \left(\cos(\omega_1 t) + \frac{\beta_1}{2\omega_1} \sin(\omega_1 t) \right) \right\}, \quad (34)$$

where the damped natural frequency of dipole librations ω_1 is given by the equation

$$\omega_1 = \frac{\gamma_1}{2\eta} \sqrt{2\xi_{\text{int}} - \beta_1^2}. \quad (35)$$

The term in curly braces in Eq. (34) obviously represents an infinite set of decaying harmonic oscillations giving rise in the frequency domain to a comb-like harmonic peak structure at FIR frequencies.

Furthermore, if the cage inertial parameter as defined in Ref. 4 is sufficiently small ($\beta_2^{-2} = I_2 kT/\zeta_2^2 \leq 0.1$) and the dipole kinetic energy satisfies the condition $I_1 \omega_0^2 \ll kT$, the exact complex susceptibility for small oscillations, using Eqs. (13) and (34), may be closely approximated by the simple closed expression^{10,34,35}

$$\frac{\chi(\omega)}{\chi} = \frac{1}{1 + i\eta\omega\tau_D} \left[\frac{2\beta_2^2}{2\beta_2^2 + i\eta\omega\tau_D} + \frac{i\eta\omega\tau_D}{\xi_{\text{int}} - 2(\eta\omega/\gamma_1)^2 + 2i\beta_1\eta\omega/\gamma_1} \right], \quad (36)$$

where $\tau_D = 2\beta_2/\gamma_2$ is the Debye relaxation time of the cage. The form of Eq. (36) was originally given in Ref. 14 for the single friction itinerant oscillator model (see also Ref. 1). The first (or non-resonant) term in Eq. (36) stemming from $C_s(t)$ is essentially the well-known Rocard equation³⁶ of the inertia

corrected Debye theory of dielectric relaxation of noninteracting dipoles, and so due to the heavily damped cage motion, the second lightly damped harmonic oscillator term $C_{\mu_1}(t)$ represents in our simplified picture the high frequency FIR effects due to the cage-embedded dipole interaction. If the series expansion of the exact small oscillation solution Eq. (34) leading directly to the approximate Eq. (36) is continued, one can clearly see the emergence of the FIR peak structure at harmonics of the fundamental frequency given by Eq. (35).

V. RESULTS AND DISCUSSION

We exhibit the results of numerical analysis of the susceptibility $\chi(\omega)$ of the model in Figs. 1–8. In Figs. 1 and 2, we show $\chi(\omega)$ versus ω for various values of the dipole-cage interaction parameter ξ_{int} for a relatively lightly damped cage and dipole. Clearly the real part of the susceptibility $\chi'(\omega)$ does not differ appreciably from that of the cage alone even for strong dipole-cage interaction. However, this is not so for the imaginary part of the susceptibility $\chi''(\omega)$ where even for moderate dipole-cage coupling (curve 1), we see a pronounced FIR shoulder. The shoulder gradually evolves into a distinct peak structure as the interaction increases with a steady progress of

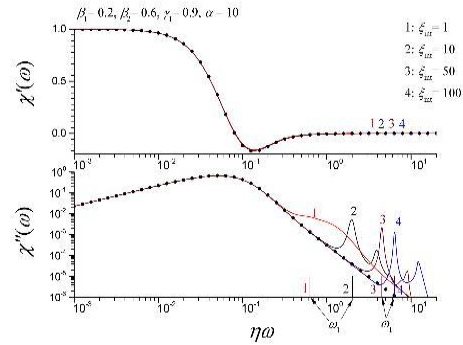


FIG. 1. Real and imaginary parts of susceptibility: numerically exact solution from Eq. (13) vs ω for various values of the dipole-cage interaction parameter ξ_{int} . Circles: pure cage susceptibility Eq. (32).

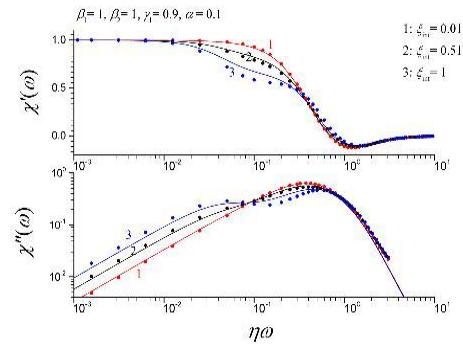


FIG. 2. Real and imaginary parts of susceptibility: numerically exact solution from Eq. (13) vs ω for various values of the dipole-cage interaction parameter ξ_{int} . Circles: result yielded by the closed form small oscillation solution for the correlation function Eq. (34).

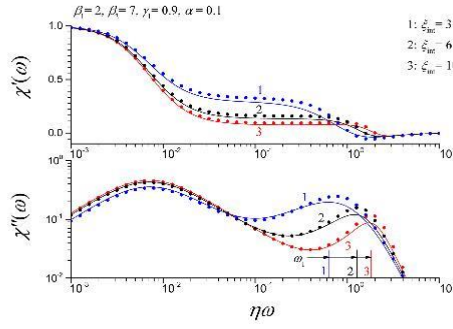


FIG. 3. Real and imaginary parts of susceptibility: numerically exact solution from Eq. (13) vs ω for various values of the dipole-cage interaction parameter ξ_{int} . Circles: approximate small oscillation solution Eq. (36).

the fundamental frequency of the peak structure to higher and higher frequencies as ξ_{int} increases which is to be expected in view of the damped natural frequency Eq. (35). Figure 2 which pertains to relatively strong damping of both dipole and cage shows that the closed form small oscillation solution, Eq. (34), provides a reasonable approximation to the exact solution in this case, the accuracy of which decreases as the dipole-cage coupling strength increases. Here the damping on both entities is so strong that the distinct peak structure does not appear at all, the only manifestation of the high frequency behavior being the more rapid high frequency fall-off of $\chi''(\omega)$ in comparison to that of the cage alone as depicted in Fig. 1. Figure 3 for a lightly damped dipole shows how the approximate small oscillation solution Eq. (36) provides a good approximation to the exact matrix continued fraction solution. Furthermore for the parameter values chosen, Eq. (36) provides a better and better approximation to the exact continued fraction solution as the interaction strength is increased together with a progressive shift of the fundamental frequency to higher and higher values as the interaction strength is increased again in accordance with Eqs. (33) and (35). Figure 4 shows that the effect of the dipole moment ratio as compared to the pure cage susceptibility is profound with the FIR response effects becoming more and more pronounced as the cage dipole moment increases.

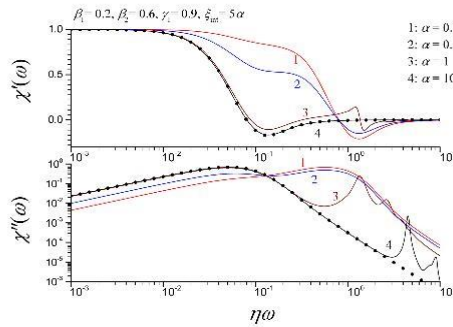


FIG. 4. Real and imaginary parts of susceptibility: numerically exact solution from Eq. (13) vs ω for various values of the dipole moment ratio parameter α . Circles: cage susceptibility Eq. (32).

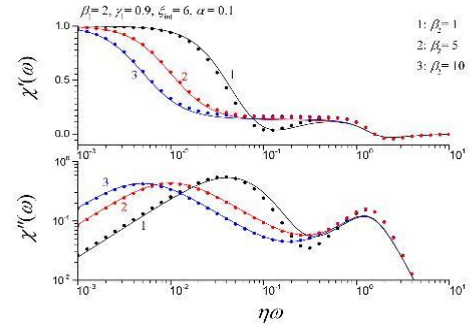


FIG. 5. Real and imaginary parts of susceptibility: numerically exact solution from Eq. (13) vs ω for various values of the cage friction parameter β_2 . Circles: approximate small oscillation solution Eq. (36).

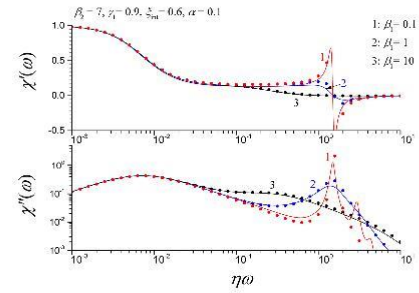


FIG. 6. Real and imaginary parts of susceptibility: numerically exact solution from Eq. (13) vs ω for various values of the dipole friction parameter β_1 . Circles: approximate small oscillation solution Eq. (36). Exact solution displays the characteristic comb-like structure.

Figure 5 shows the complex susceptibility for various values of the cage friction parameter β_2 for given dipole-cage interaction. It is clear that in this case of a relatively highly damped dipole that the simple single resonance formula given by Eq. (36) provides a reasonable description of the high frequency behavior. In contrast, Fig. 6 shows that for small damping of the dipole, Eq. (36) fails as the high frequency peak structure

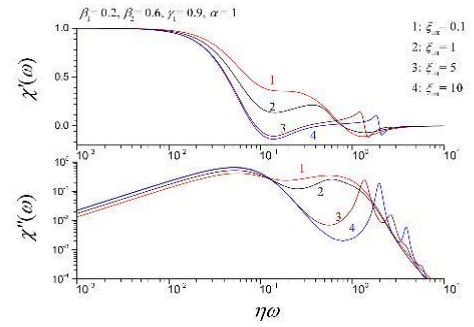


FIG. 7. Real and imaginary parts of susceptibility: numerically exact solution from Eq. (13) vs ω for various values of the dipole-cage interaction parameter ξ_{int} .

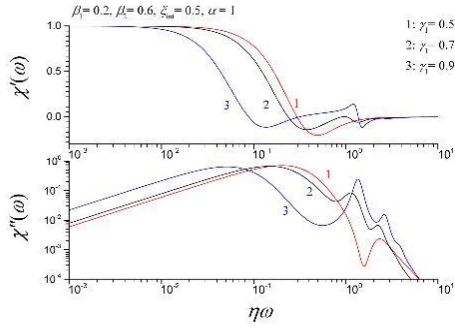


FIG. 8. Real and imaginary parts of susceptibility: numerically exact solution from Eq. (13) vs ω for various values of the dipole inertia parameter γ_1 .

predicted by the exact solution cannot be reproduced at all. Figure 7 shows that as the dipole-cage interaction parameter ξ_{int} is increased, the FIR resonance absorption is greatly increased at the expense of the mid-frequency absorption (curve 1). Figure 8 shows that the FIR absorption becomes more and more pronounced as the dipole inertia parameter increases again as one would intuitively expect.

VI. CONCLUSION

The caging phenomenon is a common hallmark of diffusion in both dense liquids³⁷ and granular systems.³⁸ As far as applications to dielectric relaxation are concerned, the cage model or itinerant oscillator is a simple mechanical system which seamlessly links the microwave and FIR absorption and can explain their role, namely, a caged-dipole effect (i.e., the FIR absorption peaks) and a rotational diffusion effect at low frequencies (i.e., Debye-like peak). In essence, the model represents the diffusion of a dipole molecule in a periodic cosine potential under the influence of a thermal bath. The periodic potential, representing the dipole-cage interaction, is not fixed but moves since the cage itself undergoes rotational Brownian motion. The model, which was originally used in translational guise to describe diffusion in cold liquids,^{10,39} has

also recently found applications in the description of microrheology in living matter⁴⁰ and mesorheological granular experiments.³⁸ Also in a much modified form, it has been used to examine the stability of large synchronous generators connected to low inertia systems where predominantly renewable energy sources like wind turbine generators or photovoltaic installations are connected.⁴¹

Here we have restricted ourselves to linear response theory which nevertheless is able to reproduce some noticeable phenomena observed in a liquid. Previously for the purpose of mathematical simplification, the cage was considered as being much heavier in comparison to the embedded dipolar molecule, which allowed one to separate the equations of motion of the cage and molecule. Here, in contrast we have given a rigorous solution of the problem in terms of matrix continued fractions by taking into account the finite inertia of the cage as well as large oscillations of the dipole in the cosine potential. For the purposes of physical interpretation, some limiting cases have been recovered, namely, the case of a massive cage and small oscillations of the dipole allowing one to assess the accuracy of these solutions.

The assumption of a cage with finite inertia and the exact matrix continued fraction solution is the main advance on the previous treatments.¹⁰ The exact solution which we have presented here permits one to analyse the situation when the cage diffusion is no longer independent and is simultaneously influenced by the probe allowing one to reproduce previous results which have been obtained for heavy cages. Thus, it also allows one to examine the range of validity of the existing approximations. This influence is reflected in the susceptibility spectrum $\chi(\omega)$ of the model. The calculations may with suitable adaptation be used to describe ferrofluids. Here relaxation over the internal magnetocrystalline anisotropy barriers inside a ferrofluid particle occurs along with Debye relaxation of the fluid particle itself.

APPENDIX A: RECURRENCE RELATIONS FOR THE FUNCTIONS $f_{m_2 q_2}^{m_1 q_1}(t)$

The time derivative of the function $f_{m_2 q_2}^{m_1 q_1}(t)$ associated with the observables is

$$\frac{d}{dt} f_{m_2 q_2}^{m_1 q_1}(t) = \left(\left[\dot{H}_{m_1} - \frac{iq_1}{2\eta_1} (H_{m_1+1} + 2m_1 H_{m_1-1}) \right] H_{m_2} e^{-iq_1 \phi_1} e^{-iq_2 \phi_2} + \left[\dot{H}_{m_2} - \frac{iq_2}{2\eta_2} (H_{m_2+1} + 2m_2 H_{m_2-1}) \right] H_{m_1} e^{-iq_1 \phi_1} e^{-iq_2 \phi_2} \right). \quad (\text{A1})$$

The terms involving derivatives like $\dot{H}_{m_1} H_{m_2} e^{-iq_1 \phi_1} e^{-iq_2 \phi_2}$ and $H_{m_1} \dot{H}_{m_2} e^{-iq_1 \phi_1} e^{-iq_2 \phi_2}$ occurring on the right hand side of Eq. (A1) can be written as

$$\begin{aligned} \dot{H}_{m_1} H_{m_2} e^{-iq_1 \phi_1} e^{-iq_2 \phi_2} &= 2\eta_1 \ddot{\phi}_1 m_1 H_{m_1-1} H_{m_2} e^{-iq_1 \phi_1} e^{-iq_2 \phi_2} = -m_1 H_{m_1-1} H_{m_2} e^{-iq_1 \phi_1} e^{-iq_2 \phi_2} \left(2\beta_1 (\phi_1 - \phi_2) \right. \\ &\quad \left. + \frac{\xi_{\text{int}}}{\eta_1} \sin(\phi_1 - \phi_2) + \frac{\xi_1}{\eta_1} \sin \phi_1 - \frac{\lambda_1}{\eta_1 kT} \right), \end{aligned} \quad (\text{A2})$$

where

$$\beta_i = \frac{\xi_i \eta_i}{I_i} = \frac{\xi_i}{2\eta_i kT}. \quad (\text{A3})$$

Thus averaging¹⁰ of the term $\dot{H}_{m_1} H_{m_2} e^{-iq_1\phi_1} e^{-iq_2\phi_2}$ leads to

$$\begin{aligned} \langle \dot{H}_{m_1} H_{m_2} e^{-iq_1\phi_1} e^{-iq_2\phi_2} \rangle &= -\frac{\beta_1}{\eta_1} m_1 \langle H_{m_1} H_{m_2} e^{-iq_1\phi_1} e^{-iq_2\phi_2} \rangle + \frac{\beta_1}{\eta_2} m_1 \langle H_{m_1-1} (H_{m_2+1} + 2m_2 H_{m_2-1}) e^{-iq_1\phi_1} e^{-iq_2\phi_2} \rangle \\ &\quad + \frac{i\xi_{\text{int}}}{2\eta_1} m_1 \langle H_{m_1-1} H_{m_2} (e^{-i(q_1-1)\phi_1} e^{-i(q_2+1)\phi_2} - e^{-i(q_1+1)\phi_1} e^{-i(q_2-1)\phi_2}) \rangle \\ &\quad + \frac{i\xi_1}{2\eta_1} m_1 \langle H_{m_1-1} H_{m_2} (e^{-i(q_1-1)\phi_1} - e^{-i(q_1+1)\phi_1}) e^{-iq_2\phi_2} \rangle. \end{aligned} \quad (\text{A4})$$

In a similar manner, averaging¹⁰ of the term $H_{m_1} \dot{H}_{m_2} e^{-iq_1\phi_1} e^{-iq_2\phi_2}$ yields

$$\begin{aligned} \langle H_{m_1} \dot{H}_{m_2} e^{-iq_1\phi_1} e^{-iq_2\phi_2} \rangle &= \frac{\beta_1}{\eta_2} m_2 \langle H_{m_1+1} H_{m_2-1} e^{-iq_1\phi_1} e^{-iq_2\phi_2} \rangle - \frac{\beta_2}{\eta_2} m_2 \langle H_{m_1} H_{m_2} e^{-iq_1\phi_1} e^{-iq_2\phi_2} \rangle - \frac{\eta_1 \beta_1}{\eta_2^2} m_2 \langle H_{m_1} (H_{m_2} \\ &\quad + 2(m_2 - 1) H_{m_2-2}) e^{-iq_1\phi_1} e^{-iq_2\phi_2} \rangle + \frac{i\xi_{\text{int}}}{2\eta_2} m_2 \langle H_{m_1} H_{m_2-1} (e^{-i(q_1+1)\phi_1} e^{-i(q_2-1)\phi_2} \\ &\quad - e^{-i(q_1-1)\phi_1} e^{-i(q_2+1)\phi_2}) \rangle + \frac{i\xi_2}{2\eta_2} m_2 \langle H_{m_1} H_{m_2-1} e^{-iq_1\phi_1} (e^{-i(q_2-1)\phi_2} - e^{-i(q_2+1)\phi_2}) \rangle. \end{aligned} \quad (\text{A5})$$

Finally, by utilizing Eqs. (A4) and (A5), we have the differential-recurrence equations for the statistical moments $f_{m_2 q_2}^{m_1 q_1}(t)$, Eq. (14), viz.,

$$\begin{aligned} \eta \frac{d}{dt} f_{m_2 q_2}^{m_1 q_1} &= - \left(\gamma_1 \beta_1 m_1 + \frac{\gamma_2^2}{\gamma_1} \beta_1 m_2 + \gamma_2 \beta_2 m_2 \right) f_{m_2 q_2}^{m_1 q_1} + \gamma_2 \beta_1 m_2 f_{m_2-1 q_2}^{m_1+1 q_1} + \gamma_2 \beta_1 m_1 f_{m_2+1 q_2}^{m_1-1 q_1} + 2\gamma_2 \beta_1 m_1 m_2 f_{m_2-1 q_2}^{m_1-1 q_1} \\ &\quad - 2\frac{\gamma_2^2}{\gamma_1} \beta_1 m_2 (m_2 - 1) f_{m_2-2 q_2}^{m_1 q_1} + \frac{i\xi_{\text{int}}}{2} \left[\gamma_1 m_1 (f_{m_2 q_2+1}^{m_1-1 q_1-1} - f_{m_2 q_2-1}^{m_1-1 q_1+1}) + \gamma_2 m_2 (f_{m_2-1 q_2-1}^{m_1 q_1+1} - f_{m_2-1 q_2+1}^{m_1 q_1-1}) \right] \\ &\quad - \frac{i}{2} \left[\gamma_1 q_1 (f_{m_2 q_2}^{m_1+1 q_1} + 2m_1 f_{m_2 q_2}^{m_1-1 q_1}) + \gamma_2 q_2 (f_{m_2+1 q_2}^{m_1 q_1} + 2m_2 f_{m_2-1 q_2}^{m_1 q_1}) \right] + \frac{i}{2} \left[\gamma_1 \xi_1 m_1 (f_{m_2 q_2}^{m_1-1 q_1-1} - f_{m_2 q_2}^{m_1-1 q_1+1}) \right. \\ &\quad \left. + \gamma_2 \xi_2 m_2 (f_{m_2-1 q_2-1}^{m_1 q_1} - f_{m_2-1 q_2+1}^{m_1 q_1}) \right]. \end{aligned} \quad (\text{A6})$$

APPENDIX B: MATRICES $\mathbf{Q}_n^-$, $\mathbf{Q}_n$, and $\mathbf{Q}_n^+$ AND INITIAL VALUE VECTOR $\mathbf{C}_n(0)$

The supermatrices $\mathbf{Q}_n^-$, $\mathbf{Q}_n$, and $\mathbf{Q}_n^+$ in Eq. (17) are

$$\mathbf{Q}_n = \begin{pmatrix} \mathbf{q}_{0n} & \mathbf{p}_n & 0 & \cdots & 0 \\ \mathbf{p}_1 & \mathbf{q}_{1n-1} & \mathbf{p}_{n-1} & \ddots & \vdots \\ 0 & \mathbf{p}_2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \mathbf{q}_{n-11} & \mathbf{p}_1 \\ 0 & \cdots & 0 & \mathbf{p}_n & \mathbf{q}_{n0} \end{pmatrix}, \quad (\text{B1})$$

$$\mathbf{Q}_n^+ = \frac{i}{2} \begin{pmatrix} \gamma_2(\mathbf{r} - \mathbf{I}) & -\gamma_1 \mathbf{r} & 0 & \cdots & 0 \\ 0 & \gamma_2(\mathbf{r} - \mathbf{I}) - \gamma_1 \mathbf{r} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \gamma_2(\mathbf{r} - \mathbf{I}) - \gamma_1 \mathbf{r} \end{pmatrix}, \quad (\text{B2})$$

$$\mathbf{Q}_n^- = \frac{i}{2} \begin{pmatrix} \gamma_2 \mathbf{w}_n & 0 & \cdots & 0 \\ -\gamma_1 \mathbf{s}_1 & \gamma_2 \mathbf{w}_{n-1} & \ddots & \vdots \\ 0 & -\gamma_1 \mathbf{s}_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \gamma_2 \mathbf{w}_1 \\ 0 & \cdots & 0 & -\gamma_1 \mathbf{s}_n \end{pmatrix}, \quad (\text{B3})$$

$$\mathbf{Q}_n^- = \begin{pmatrix} \mathbf{u}_n & 0 & \cdots \\ \mathbf{v}_{1n-1} & \ddots & \ddots \\ 0 & \ddots & \mathbf{u}_2 \\ \vdots & \ddots & \mathbf{v}_{n-11} \\ 0 & \cdots & 0 \end{pmatrix}, \quad (\text{B4})$$

while the submatrices have the form

$$\mathbf{q}_{n_1 n_2} = - \left(\gamma_1 \beta_1 n_1 + \frac{\gamma_2^2}{\gamma_1} \beta_1 n_2 + \gamma_2 \beta_2 n_2 \right) \mathbf{I}, \quad (\text{B5})$$

$$\mathbf{p}_n = \gamma_2 \beta_1 n \mathbf{I}, \quad (\text{B6})$$

$$\mathbf{u}_n = -2\frac{\gamma_2^2}{\gamma_1} \beta_1 n (n-1) \mathbf{I}, \quad (\text{B7})$$

$$\mathbf{v}_{n_1 n_2} = 2\gamma_2 \beta_1 n_1 n_2 \mathbf{I}, \quad (\text{B8})$$

$$\mathbf{r} = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots \\ \ddots & -1 & 0 & 0 \\ \ddots & 0 & 0 & 0 \\ \ddots & 0 & 0 & 1 \\ \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \quad (\text{B9})$$

$$\mathbf{s}_n = n \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & -4 & \xi_{\text{int}} & 0 & \ddots & \ddots \\ \ddots & -\xi_{\text{int}} & -2 & \xi_{\text{int}} & 0 & \ddots \\ \ddots & 0 & -\xi_{\text{int}} & 0 & \xi_{\text{int}} & 0 \\ \ddots & \ddots & 0 & -\xi_{\text{int}} & 2 & \xi_{\text{int}} \\ \ddots & \ddots & \ddots & 0 & -\xi_{\text{int}} & 4 \end{pmatrix}, \quad (\text{B10})$$

$$\mathbf{w}_n = n \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & -6 & \xi_{\text{int}} & 0 & \ddots & \ddots \\ \ddots & -\xi_{\text{int}} & -4 & \xi_{\text{int}} & 0 & \ddots \\ \ddots & 0 & -\xi_{\text{int}} & -2 & \xi_{\text{int}} & 0 \\ \ddots & \ddots & 0 & -\xi_{\text{int}} & 0 & \xi_{\text{int}} \\ \ddots & \ddots & \ddots & 0 & -\xi_{\text{int}} & 2 \end{pmatrix}. \quad (\text{B11})$$

The initial value vector in Eq. (18) is calculated in the following manner:

$$\mathbf{C}_0(0) = \begin{pmatrix} \vdots \\ c_{00}^{-1}(0) \\ c_{00}^0(0) \\ c_{00}^1(0) \\ \vdots \end{pmatrix}, \quad (\text{B12})$$

where the elements are given by

$$\begin{aligned} c_{00}^q(0) &= \langle e^{-iq\phi_1 + i(q-1)\phi_2} \rangle(0) - \langle e^{-iq\phi_1 + i(q-1)\phi_2} \rangle_0 \\ &= Z^{-1} \int_0^{2\pi} \int_0^{2\pi} e^{\xi_1 \cos \phi_1 + \xi_2 \cos \phi_2 + \xi_{\text{int}} \cos(\phi_1 - \phi_2)} d\phi_1 d\phi_2 \\ &\quad - Z_0^{-1} \int_0^{2\pi} \int_0^{2\pi} e^{\xi_{\text{int}} \cos(\phi_1 - \phi_2)} d\phi_1 d\phi_2. \end{aligned} \quad (\text{B13})$$

Next using the series expansion of Eq. (B13) with respect to ξ , we have $c_{00}^q(0)$ in terms of the modified Bessel functions,

$$\begin{aligned} c_{00}^q(0) &\approx \xi_1 \langle (\cos \phi_1 + \alpha \cos \phi_2) (e^{-iq\phi_1 + i(q-1)\phi_2}) \rangle_0 \\ &= \frac{I_{q-1}(\xi_{\text{int}}) + \alpha I_q(\xi_{\text{int}})}{2I_0(\xi_{\text{int}})}, \end{aligned} \quad (\text{B14})$$

where $I_q(z)$ are the modified Bessel functions of the first kind and we have also used the fact that $\langle \cos \phi_1 + \alpha \cos \phi_2 \rangle_0 = 0$.

- ¹M. W. Evans, G. J. Evans, W. T. Coffey, and P. Grigolini, *Molecular Dynamics* (Wiley, New York, 1982).
- ²W. T. Coffey, M. W. Evans, and P. Grigolini, *Molecular Diffusion and Spectra* (Wiley, New York, 1984).
- ³E. P. Gross, *J. Chem. Phys.* **23**, 1415 (1955).
- ⁴R. A. Sack, *Proc. Phys. Soc., Sect. B* **70**, 414 (1957).
- ⁵W. T. Coffey, G. P. Johari, Yu. P. Kalmykov, and S. V. Titov, *J. Phys.: Condens. Matter* **15**, 2961 (2003).
- ⁶W. T. Coffey, Yu. P. Kalmykov, and S. V. Titov, *Adv. Chem. Phys.* **126**, 131 (2003).
- ⁷N. E. Hill, *Proc. Phys. Soc.* **82**, 723 (1963).
- ⁸N. E. Hill, W. E. Vaughan, A. H. Price, and M. Davies, *Dielectric Properties and Molecular Behavior* (D. Van Nostrand Co., London, 1969).
- ⁹V. F. Sears, *Proc. Phys. Soc.* **86**, 953 (1965).
- ¹⁰W. T. Coffey and Yu. P. Kalmykov, *The Langevin Equation*, 4th ed. (World Scientific, Singapore, 2017).
- ¹¹P. Debye, *Polar Molecules* (Chemical Catalog, New York 1929), reprinted by Dover Publications, New York, 1954.
- ¹²G. Wyllie, *J. Phys. C* **4**, 564 (1971).
- ¹³G. A. P. Wyllie, in *Dielectric and Related Molecular Processes*, Specialist Periodical Reports, Senior Reporter, M. Davies (The Chemical Society, London, 1972), Vol. 1.
- ¹⁴J. H. Calderwood and W. T. Coffey, *Proc. R. Soc. London A* **365**, 269 (1977).
- ¹⁵W. T. Coffey, P. M. Corcoran, and M. W. Evans, *Proc. R. Soc. A* **410**, 61 (1987).
- ¹⁶W. T. Coffey, P. M. Corcoran, and J. K. Vij, *Proc. R. Soc. London A* **414**, 339 (1987).
- ¹⁷W. T. Coffey, P. M. Corcoran, and J. K. Vij, *Proc. R. Soc. A* **425**, 169 (1989).
- ¹⁸W. T. Coffey, *J. Chem. Phys.* **107**, 4960 (1997).
- ¹⁹W. T. Coffey, Yu. P. Kalmykov, and S. V. Titov, *J. Chem. Phys.* **115**, 9895 (2001).
- ²⁰P. L. Nordio and A. Polemino, *Chem. Phys.* **180**, 109 (1994).
- ²¹A. Polemino and J. H. Freed, *Adv. Chem. Phys.* **83**, 89 (1993).
- ²²G. J. Moro, P. L. Nordio, M. Moro, and A. Polemino, *J. Chem. Phys.* **101**, 693 (1994).
- ²³A. Polemino and M. Moro, *J. Chem. Phys.* **101**, 703 (1994).
- ²⁴M. C. C. Ribeiro and P. S. Santon, *J. Mol. Liq.* **68**, 33 (1996).
- ²⁵M. I. Shliomis, *Sov. Phys. Usp.* **17**, 153 (1974).
- ²⁶M. I. Shliomis and V. I. Stepanov, in *Relaxation Phenomena in Condensed Matter*, Advances in Chemical Physics Vol. 87, edited by W. T. Coffey (Wiley, New York, 1994), p. 1.
- ²⁷P. S. Damle, A. Sjölander, and K. S. Singwi, *Phys. Rev.* **165**, 277 (1968).
- ²⁸N. G. van Kampen, *Stochastic Processes in Physics and Chemistry*, 2nd ed. (North-Holland, Amsterdam, 1992).
- ²⁹A. Budó, *J. Chem. Phys.* **17**, 686 (1949).
- ³⁰N. G. van Kampen, *Stochastic Processes in Physics and Chemistry*, 3rd ed. (Elsevier, North-Holland, 2007).
- ³¹B. K. P. Scaife, *Principles of Dielectrics* (Oxford University Press, London, 1989), revised edition 1998.
- ³²B. K. P. Scaife, *Complex Permittivity* (The English Universities Press, London, 1971).
- ³³H. Risken, *The Fokker-Planck Equation*, 2nd ed. (Springer, Berlin, 1989).
- ³⁴W. T. Coffey and M. E. Walsh, *J. Chem. Phys.* **106**, 7625 (1997).
- ³⁵M. E. Walsh and P. M. Déjardin, *J. Phys. B: At. Mol. Opt. Phys.* **32**, 2057 (1999).
- ³⁶M. I. Shliomis and V. I. Stepanov, *Adv. Chem. Phys.* **87**, 1 (1994).
- ³⁷A. Cavagna, *Phys. Rep.* **476**, 51 (2009).
- ³⁸A. Lasanta and A. Puglisi, *J. Chem. Phys.* **143**, 064511 (2015).
- ³⁹H. D. Vollmer, *Z. Phys. B: Condens. Matter* **33**, 103 (1979).
- ⁴⁰E. Fodor, K. Kanazawa, H. Hayakawa, P. Visco, and F. van Wijland, *Phys. Rev. E* **90**, 042724 (2014).
- ⁴¹M. Zarifakis, W. T. Coffey, Yu. P. Kalmykov, and S. V. Titov, *Eur. Phys. J. Plus* **132**, 289 (2017).

8 REFERENCES

- [1] F. Gonzalez-Longatt, E. Chikuni and E. Rashayi, "Effects of the Synthetic Inertia from Wind Power on the Total System Inertia after a Frequency Disturbance," in *IEEE Industrial Conference on Industrial Technology*, Cape Town, 2013.
- [2] R. Ruedenberg, "The Frequencies of Natural Power Oscillations in Interconnected Generating and Distribution Systems," *AIEE Transactions*, vol. 62, pp. 791-803, 1943.
- [3] T. Keve and H. Roeloffzen, *Baustein Elektrische Maschine*, Stuttgart, Berlin, Koeln, Mainz,: Verlag Berliner Union, Verlag W. Kohlhammer, 1978.
- [4] J. S. Joyce, T. Kulig and D. Lambrecht, "Torsional Fatigue of Turbine-Generator shafts caused by different electrical system faults and switching operations," *IEEE Transactions on Power Apparatus and Systems*, Vols. PAS-97, no. 5, pp. 1965-1973, September/October 1978.
- [5] J. Machowski, J. Bialek and J. Bumby, *Power System Dynamics, Stability and Control*, New York: John Wiley & Sons, 2008.
- [6] D. Manik, M. Rohden, H. Ronellenfitch, X. Zhang, S. Hallerberg, D. Witthaut and M. Timme, *Physical Review E* 95, vol. 012319, 2017.
- [7] G. Ivkovic, "Application of rate of change of frequency constraints for high wind penetration scenarios in a small power system," in *Australian Universities Power Engineering Conference, AUPEC*, Hobart, Australia, 2013.
- [8] EPRI, "Determination of Torsional Fatigue Life of Large Turbine Generator Shafts - EL 3083," EPRI, 1984.
- [9] P. Tielens and D. Van Hertem, "Grid Inertia and Frequency Control in Power Systems with High Penetration of Renewables," *Young Researchers Symposium in Electrical Power Engineering*, edition 6, 16 April 2012.
- [10] E. Mircea and M. Shahidehpour, Eds., *Handbook of Electrical Power System Dynamics*, Hoboken, New Jersey: John Wiley & Sons, 2013.
- [11] P. Kundur, *Power System Stability and Control*, New York, NY: McGraw-Hill Education - Europe, 2005.

- [12] EirGrid, “EirGrid Grid Code Version 6.0,” 22 July 2015. [Online]. Available: <http://www.eirgridgroup.com/site-files/library/EirGrid/GridCodeVersion6.pdf>. [Accessed 2015].
- [13] W. T. Coffey, M. Zarifakis, Y. P. Kalmykov, S. V. Titov, W. J. Dowling and A. S. Titov, “Cage model of polar fluids: Finite cage inertia generalization,” *Journal Chemical Physics*, vol. 147, no. 034509, 2017.
- [14] W. T. Coffey and Y. P. Kalmykov, *The Langevin Equation*, 4 ed., Singapore: World Scientific, 2017.
- [15] H. D. Vollmer, “Two particle model for the diffusion of interacting particles in periodic potentials,” *Z. Phys. B: Condens Matter*, vol. 33, p. 103, 1979.
- [16] E. Fodor, K. Kanazawa, H. Hayakawa, P. Visco and F. van Wijland, “Energetics of active fluctuations in living cells,” *Physical Review E*, vol. 90, p. 0427724, 2014.
- [17] A. Lasanta and A. Puglisi, “An itinerant oscillator model with cage inertia for mesorheological granular experiments,” *Journal Chemical Physics*, vol. 143, p. 064511, 2015.
- [18] J. Weizenecker, “The Fokker-Planck Equation for Coupled Brown-Neel-Rotation,” *Phys. Med. Biol.*, in press.
- [19] V. Sears, “The itinerant oscillator model of liquids,” *Proc. Phys. Soc. London*, vol. 86, p. 953, 1965.
- [20] N. G. van Kampen, “Stochastic Processes in Physics and Chemistry,” 3rd ed., Amsterdam, North Holland, 2007, p. 224.
- [21] P. Debye, “Zur Theorie der anomalen Dispersion im Gebiete der langwelligen elektrischen Strahlung,” in *Ber. Der Deutschen Physikalischen Gesellschaft*, vol. 15, New York, Dover, 1913, pp. 777-793.
- [22] M. Zarifakis, W. T. Coffey, Y. P. Kalmykov and S. V. Titov, “Models for the transient stability of conventional power generating stations connected to low inertia systems,” *The European Physical Journal Plus*, vol. 132, no. 289, 2017.
- [23] K. Creighton, M. McClure, J. O'Higgins, T. McCartan and A. Rogers, “Increased Wind Generation in Ireland and Northern Ireland and the Impact on Rate of Change of Frequency,” EirGrid, Dublin.
- [24] EirGrid, “Frequency raw data from system disturbances,” 2015. [Online]. Available:

http://www.eirgrid.com/media/FrequencyTransientAnalysis_HighSpeedFrequencyData.xlsx.

- [25] British Standards, *BS EN 60034, Rotating electrical machines*, British Standards, 2010.
- [26] European Commission, “Market Legislation,” 2017. [Online]. Available: <http://ec.europa.eu/energy/en/topics/markets-and-consumers/market-legislation>. [Accessed 24 09 2017].
- [27] J. Taborda, “Analysis of power oscillations in Dublin Bay Power (internal ESB),” 2014.
- [28] M. Peydayesh and R. Baldick, “The effects of very fast response to frequency fluctuation,” *International Association for Energy Economics*, 2012.
- [29] P. W. Murrill, *Fundamentals of process control theory*, 3rd ed., North Carolina: ISA - The Instrumentation, Systems and Automation Society, 2005.
- [30] H. M. Power and R. J. Simpson, *Introduction to Dynamics and Control*, Maidenhead, Berkshire: McGraw-Hill Book Company Limited, 1978.
- [31] M. Eremia and M. Shahindehpour, Eds., *Handbook of electrical power system dynamics: modelling, stability and control*, Hoboken, New Jersey: John Wiley & Sons, 2013.
- [32] Bagert, Emmerich, Marquard, Schulz, Weber and Wittner, *Elektrischer Eigenbedarf*, 3rd ed., Berlin: VDE Verlag GmbH, 2012.
- [33] H. Lindner, H. Brauer and C. Lehmann, *Taschenbuch der Elektrotechnik und Elektronik*, Leipzig: VEB Fachbuchverlag, 1985.
- [34] D. Reimert, *Protective Relaying for power generation System*, Boca Raton, Florida: Taylor & Francis Group, 2006.
- [35] H. Titze, *Fehler und Fehlerschutz in elektrischen Drehstromanlagen, Die Fehler und ihre Berechnung*, vol. 1, Wien: Springer Verlag, 1951.
- [36] H. Titze, *Fehler und Fehlerschutz in elektrischen Drehstromanlagen*, vol. 2, Wien: Springer Verlag, 1953.
- [37] J. M. Vance, *Rotordynamics of Turbomachinery*, New York, Chichester, Brisbane, Toronto, Singapore: John Wiley & Sons, 1988.
- [38] M. C. Jackson, S. D. Umans, R. D. Dunlop, S. H. Horowitz and A. C. Parikh, “Turbine-Generator Shaft Torques and Fatigue: Part 1 - Simulation Methods

and Fatigue Analysis,” *IEEE Transactions on Power Apparatus and Systems*, Vols. PAS-98, no. No. 6, December 1979.

- [39] EPRI, “Torsional Interaction between Electrical Network Phenomena and Turbine-Generator Shafts: Plant Vulnerability,” *EPRI Technical Report 1013460*, November 2006.
- [40] T. J. Hammons, “Stressing of large turbine-generators at shaft couplings and LP turbine final-stage blade roots following clearance of grid system faults and faulty synchronisation,” *IEEE Transactions on Power System Apparatus and Systems*, Vols. PAS-99, no. 4, pp. 1652-1662, July 1980.
- [41] G. Klempner and I. Kerszenbaum, *Handbook of Large Turbo-Generator Operation and Maintenance*, 2nd ed., Hoboken, New Jersey: John Wiley & Sons, 2008.
- [42] K. V. Frolov, Y. L. Izrailev, N. A. Makhutov, E. M. Morozov and V. Z. Parton, *Thermal stresses and strength of turbines, Calculation and Design*, 2nd ed., R. B. Hetnarski, Ed., New York, Washington, Philadelphia, London: Hemisphere Publishing Corporation, 1991.
- [43] S. Aboreshaid and S. Al-Dhalaan, “Stochastic Evaluation of Turbine-Generator shaft-fatigue due to system faults and faulty synchronization,” in *Power Engineering Society Winter Meeting*, Singapore, 2000.
- [44] R. D. Dunlop, S. H. Horowitz, A. C. Parikh, M. C. Jackson and S. D. Umans, “Turbine-Generator Shaft Torques and Fatigue: Part II - Impact of System Disturbances and High Speed Reclosure,” *IEEE Transactions on Power Apparatus and Systems*, Vols. PAS-98, no. No. 6, pp. 2308-2328, November/December 1979.
- [45] M. C. Jackson and S. D. Umans, “Turbine-Generator Shaft Torques and Fatigue: Part III - Refinements to Fatigue Model and Test Results,” *IEEE Transactions on Power Apparatus and Systems*, Vols. PAS-99, no. 3, pp. 1259-1268, May/June 1980.
- [46] EPRI, “A Methodology for Predicting Fatigue Life of Turbine Generator Shafts - EL 4333,” EPRI, 1985.
- [47] N. Kohtz, A. Bagaviev, S. Exnovski, J. Folchert, F. Hellmann, R. Hoffmann, A. Karpunow, A. Keskin, C. Kreisher and S. Kulig, “Assessment of equipment

- integrity, Phase 2 of ROCOF Electrical Stability Studies,” TUV Rheinland Industrie Service, Cologne, 2017.
- [48] R. Ruedenberg, *Elektrische Schaltvorgaenge*, 5th ed., H. Dorsch and P. Jacottet, Eds., Berlin, Heidelberg, New York: Springer Verlag, 1974.
 - [49] The European Commission, “Access to European Union Law,” 2 August 2017. [Online]. Available: http://eur-lex.europa.eu/legal-content/EN/TXT/?uri=uriserv:OJ.L_.2017.220.01.0001.01.ENG&toc=OJ:L:2017:220:TOC. [Accessed 5 September 2017].
 - [50] Alsthom, *Maintenance Manual for Turbo-Generator (Aghada 11, 12, 14)*, Belfort: Alsthom, 1985.
 - [51] R. Roeper, *Kurzschlussstroeme in Drehstromnetzen*, 6th ed., Berlin, Muenchen: Siemens Aktiengesellschaft, 1984.
 - [52] BSi, “BS IEC 61363-1 - Electrical installations of ships and mobile and fixed offshore units,” *British Standards*, 1998.
 - [53] M. Zarifakis and W. T. Coffey, “Impact of elevated rate of change of frequency (ROCOF) on conventional power generation plant in the island of Ireland,” *VGB POWERTECH, International Journal for Electricity and Heat Generation*, vol. 8, pp. 47-53, 8 2014.
 - [54] M. Zarifakis and W. T. Coffey, “Transient stability of conventional power generating stations during times of high wind penetration,” in *Vortraege auf der DPG Fruehjahrstagung in Berlin 2015*, Berlin, 2015.
 - [55] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, 4th ed., London: Cambridge University Press, 1927.
 - [56] M. Abramowitz and I. A. Stegun, Eds., *Handbook of Mathematical Functions*, New York: Dover Publications, 1972.
 - [57] H. Risken, *The Fokker-Planck Equation*, 2nd Edition ed., Berlin: Springer, 1989.
 - [58] Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence*, Berlin: Springer, 1984.
 - [59] H. Sompolinsky, D. Golomb, and D. Kleinfeld, “Global processing of visual stimuli in a neural network of coupled oscillators,” *Proc. Natl. Acad. Sci. U.S.A*, vol. 87, p. 7200, 1990.

- [60] C. Kirst, M. Timme, and D. Battaglia, “Dynamic information routing in complex networks,” *Nat. Commun.*, vol. 7, p. 11061, 2016.
- [61] K. Wiesenfeld, P. Colet, and S. H. Strogatz, “Synchronization Transitions in a Disordered Josephson Series Array,” *Phys. Rev. Lett.*, vol. 76, p. 404, 1996.
- [62] A. G. Vladimirov, G. Kozireff, and P. Mandel, “Synchronization of weakly stable oscillators and semiconductor laser arrays,” *Europhys. Lett.*, vol. 61, no. 5, p. 613, 2003.
- [63] G. Heinrich, M. Ludwig, J. Qian, B. Kubala, and F. Marquardt, “Collective Dynamics in Optomechanical Arrays,” *Phys. Rev. Lett.*, vol. 107, p. 043603, 2011.
- [64] D. Witthaut and M. Timme, “Kuramoto dynamics in Hamiltonian systems,” *Phys. Rev. E*, vol. 90, p. 032917, 2014.
- [65] D. Manik, M. Rohden, H. Ronellenfitsch, X. Zhang, S. Hallerberg, D. Witthaut, and M. Timme, “Network susceptibilities: Theory and applications,” *Phys. Rev. E*, vol. 95, p. 012319, 2017.
- [66] J. Machowski, J. Bialek, and J. Bumby, *Power System Dynamics, Stability, and Control*, New York: John Wiley & Sons, 2008.
- [67] A. H. N. a. N. F. P. G. Filatrella, “Analysis of a power grid using a Kuramoto-like model,” *Eur. Phys. J. B*, vol. 61, p. 485, 2008.
- [68] W. T. Coffey, Yu. P. Kalmykov, *The Langevin Equation*, 4th ed., Singapore: World Scientific, 2017.
- [69] W. T. Coffey, M. Zarifakis, Y. P. Kalmykov, S. V. Titov, W. J. Dowling and A. S. Titov, “Cage model of polar fluids: Finite cage inertia generalization,” *J.Chem Phys.*, vol. 147, p. 034509, 2017.
- [70] M. Zarifakis, W. T. Coffey, Y. P. Kalmykov and S. V. Titov, “Models for the transient stability of conventional power generating stations connected to low inertia systems,” *The European Physical Journal Plus*, vol. 132, p. 289, 2017.
- [71] M. Schröder, M. Timme, and D. Witthaut, “A universal order parameter for synchrony in networks of limit cycle oscillators,” *Chaos*, vol. 27, p. 073119, 2017.
- [72] M. Rohden, A. Sorge, D. Witthaut, and M. Timme, “Impact of network topology on synchrony of oscillatory power grids,” *Chaos*, vol. 24, p. 013123, 2014.

- [73] P. Kundur, Power system stability and control The EPRI Power System Engineering Series, New York: McGraw Hill Inc., 1994.
- [74] M. Zarifakis, W.T. Coffey, "Impact of elevated Rate of Change of Frequency (RoCoF) on conventional power generation plant in the Island of Ireland," *VGB PowerTech*, p. 8, 2014.
- [75] M. Abramowitz and I. A. Stegun (Eds.), Handbook of Mathematical Functions, New York: Dover, 1972.
- [76] H. Risken, The Fokker-Planck Equation, 2nd ed., Berlin: Springer, 1989.
- [77] IEEE, *IEEE Guide for Synchronous Generators Modelling Practices and Applications in Power System Stability Analyses*, New York: The Institute of Electrical and Electronics Engineers Inc., 2007.
- [78] A. Einstein and W. J. de Haas, "Experimenteller Nachweis der Ampereschen Molekularstroeme," *Deutsche Physikalische Gesellschaft, Verhandlungen*, vol. 17, pp. 152-170, 1915.
- [79] P. Deverill, A. Bennett, N. Healey, R. Hyde, P. Rottier and P. Deverill, "Enhanced Control of Generators within a Marine Electrical Power System," in *7th International Naval Engineering Conference*, 2004.
- [80] J. J. DiStefano III, A. R. Stubberud and I. J. Williams, Theory and Problems of Feedback and Control Systems, New York: McGraw-Hill Book Company, 1967.
- [81] G. Mueller, Elektrische Maschinen - Aufbau und Wirkungsweise, 6th ed., Berlin: VEB Verlag Technik, 1985.
- [82] Eirgrid, "Modification proposal Form (MPID 229)," 04 09 2012. [Online]. Available: <http://www.eirgridgroup.com/site-files/library/EirGrid/MPID-229-RATE-OF-CHANGE-OF-FREQUENCY.PDF>.
- [83] M. Schroeder, M. Timme and D. Witthaut, *Chaos* 27, 073119, 2017.
- [84] L. Balmer and C. P. Lewis, Solution of Problems in Control Engineering, Bath: Pitman Paperbacks, 1970.
- [85] S. Backhaus and M. Chertkov, "Getting a grip on the electrical grid," *Physics Today*, vol. 42, 2013.
- [86] G. Filatrella, H. Nielsen and N. Pedersen, *European Physical Journal B*, vol. 61, no. 485, 2008.

