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An Analysis of the Distribution of Wealth in Ireland

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Précis: Lyons' (1972) data on the distribution of Irish wealth are considered and the appropriateness of the lognormal and Pareto models for this data are assessed using relatively sophisticated statistical methods. Allowance is made for the 65 per cent of individuals whose estate size is unrecorded. New estimates of total wealth and of the distribution of wealth in Ireland are obtained. Part of the analysis reported here can be regarded as formalising the procedure adopted by Chesher and McMahon (1977) when considering the same data. The lognormal distribution performs surprisingly well, but the Pareto distribution provides a poor model for the Irish data.

I INTRODUCTION

Chesher and McMahon (1977) in this journal reported an attempt, using elementary graphical methods, to estimate the wealth holdings of individuals for which no record of estate size is given in Lyons' (1972) data concerning the distribution of wealth in Ireland in 1966. The empirical distribution function derived from Lyons' data was plotted on logarithmic-normal probability graph paper for wealth holdings in excess of £2,000. The points obtained were observed to lie approximately along a straight line and, by extrapolating this line to levels of wealth less than £2,000, new frequencies for lower wealth classes were obtained. Using these new frequencies, total wealth in Ireland was estimated as £2,639m – 11 per cent higher than Lyons' estimate.

If the true wealth distribution is well approximated by a lognormal distribution and if the majority of estates whose size is unrecorded are of value less than £2,000, then the procedure used by Chesher and McMahon provides a quick but rough estimate of total wealth holdings. Unfortunately, the sub-

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jective nature of the procedure does not allow probability statements to be made concerning the accuracy of the estimates obtained.

This paper has three purposes. The first is to assess the appropriateness of the lognormal distribution as a model for the Irish wealth distribution in 1966. The second is to formalise the procedure used by Chesher and McMahon and to provide estimates of total wealth with probability statements concerning their accuracy. The third is to investigate the appropriateness of the Pareto distribution as a model for the Irish wealth distribution and to examine the sensitivity of estimates of total wealth to the choice of model for the wealth distribution. Incidentally, limited attention will be given to the manner in which estates are omitted from the estate duty statistics from which Lyons' data derive. Estimation proceeds via the method of maximum likelihood. This is an efficient procedure to use with the large sample available here, but it is one rather rarely used in practice when estimating income and wealth distributions. Thus the analysis reported here may be interesting from a methodological viewpoint.

The lognormal and Pareto distributions are chosen as candidate models for the Irish wealth distribution because they can be derived as limiting distributions of wealth in relatively simple stochastic models of wealth accumulation (Atkinson and Harrison, 1978, Chapter 8, provide a review; see also Pestieau and Possen, 1979, and Sargan, 1957), because they are frequently used as models for wealth distributions, and because of their mathematical tractability.

The omission of individuals from estate duty data is a very serious problem both in Ireland, where in 1966 some 65 per cent of individuals are omitted, and in other countries. For example, in the United Kingdom, over half the estates in any one year are missing from the estate duty statistics (Atkinson and Harrison, p. 35). Omissions on this scale seriously affect the estimation of wealth distributions and wealth distributions are important determinants of government policy. It is worthwhile, therefore, to examine the available data closely to try to determine the nature of the wealth distribution and the extent of wealth holdings and this paper attempts to make a contribution in this direction.

In the next section, the lognormal and Pareto distributions are examined as models for the distribution of wealth in Ireland in 1966 over the population of individuals with recorded estate size. In Section III the individuals with unrecorded estate size are introduced, the models reassessed and new estimates of total wealth holdings are presented. In Section IV the analysis of Section III is extended by imposing a naive model on the conditional probability of estate size being recorded. Section V contains concluding remarks.

II THE DISTRIBUTION OF WEALTH OVER INDIVIDUALS WITH RECORDED WEALTH

Two candidates are considered for the distribution of wealth: the two parameter lognormal distribution with probability density function

$$p_1(w) = (\sigma\sqrt{2\pi}w)^{-1} \exp(-(\log w-\mu)^2/2\sigma^2) \quad w > 0, \sigma > 0, -\infty < \mu < \infty \tag{1}$$

and the two-parameter Pareto distribution with probability density function

$$p_2(w) = \beta A^\beta w^{-(\beta+1)} \qquad w > A > 0, \beta > 0. \tag{2}$$

The lognormal probability density function is unimodal with mode, median and mean located at, respectively, $w = \exp(\mu-\sigma^2)$, $\exp(\mu)$, and $\exp(\mu+\sigma^2/2)$. It has zero order contact at both extremes of its range. The Pareto probability density function is J-shaped with mode, median and mean located, respectively, at $w = A$, $A2^{1/\beta}$ and $A\beta/(\beta-1)$. If $\beta < 1$, the mean is undefined. The jth moment about zero of w is $A\beta/(\beta-j)$, $(\beta>j)$ when w has the Pareto distribution and is $\exp(j\mu+j^2\sigma^2/2)$ when w is lognormally distributed (see Johnson and Kotz, 1979).

It has been suggested that the Pareto distribution provides a reasonably good model for the wealth of rich individuals, but is an unsatisfactory model for the wealth of poor individuals, while the reverse has been said to be true of the lognormal distribution. It is interesting to see if these comments apply to the data in Lyons concerning the Irish wealth distribution in 1966. The data in Lyons (1972) are used here rather than that in Lyons (1975). Though the latter may give a more accurate picture of the frequency distribution of wealth holdings since it is constructed using a more sophisticated mortality multiplier method, the former provides a finer grouping of the data. Fine grouping is important if relatively precise estimates of the parameters of alternative models are to be obtained and if alternative models are to be compared. In fact, for wealth in excess of £5,000 both sets of data seem very similar.

Lyons' data (Table 6.2) provide estimated frequencies of individuals holding wealth in 26 wealth classes. It is reproduced here in Table 2. Denoting the observed frequency in class i (i = 1, 2, . . . , 26) by N_i, the vector $(N_1, N_2, \ldots, N_{26})$ by $\underline{N}$ and the lower and upper delimiters of class i by X_i and X_{i+1}, respectively, the log-likelihood for the 603,972 observations on recorded wealth holdings has the form

$$\log L(\Omega|\underline{N}) = f(\underline{N}) + \sum_{i=1}^{26} N_i \log P_i(\Omega) \tag{3}$$

where $X_{27} = \infty$ and in the case of the lognormal distribution $X_o = 0$, while for the Pareto distribution $X_o = A$. Here $P_i(\Omega)$ is the probability that a single observation from the wealth distribution lies in class i, Ω denotes the parameters of the model under consideration and $f(\underline{N})$ is a function which has zero derivatives with respect to Ω.

For the lognormal model, $P_i(\Omega)$ is given by

$$P_i(\Omega) = \Phi\left(\frac{\log X_{i+1} - \mu}{\sigma}\right) - \Phi\left(\frac{\log X_i - \mu}{\sigma}\right) \tag{4}$$

where $\Omega = (\mu, \sigma)$ and $\Phi(z) = \int_{-\infty}^{z} \Phi(z)dz$ is the standard normal distribution function. For the Pareto model, $P_i(\Omega)$ is given by

$$P_i(\Omega) = A^{\beta}(X_i^{-\beta} - X_{i+1}^{-\beta}) \tag{5}$$

where $\Omega = (A, \beta)$.

The multinomial log-likelihood function in Equation 3 is used under the assumption that $\underline{N}$ arises from grouping a simple random sample of size $N = \sum_{i=1}^{26} N_i = 603{,}972$ individuals' wealth holdings into the 26 classes shown in Table 2 and that the class delimiters are either non-stochastic or, if stochastic, are independent of $\underline{N}$. The use of this log-likelihood function can be questioned. Lyons' data are the end-product of a wealth distribution construction using a mortality multiplier method. Since the mortality multipliers are stochastic and, more importantly, since wealth/age classes are differentially represented, the function in Equation 3 may not be entirely appropriate. However, there is no information available to allow corrections to be made for these features of the data. It is desirable to apply a "rule of thumb" correction to allow for the fact that Lyons' data derive from knowledge of the value of only 26,165 estates. Since the estimations reported in this paper use the data in Lyons (Table 6.2) as it stands, asymptotic standard errors are multiplied up by $(603{,}972/26{,}165)^{1/2} = 4.8$ and χ^2 goodness of fit statistics divided through by $(603{,}972/26{,}165) = 23.1$. Without these corrections the accuracy of the estimates would be seriously overstated. Even these adjustments may not be large enough for it is unlikely that the Revenue Commissioners record the value of estates without error. Turning to the statistical properties of the estimates, for the lognormal distribution the P_i satisfy the regularity conditions for multi-nomial maximum likelihood estimation (see Rao, 1973, p. 360) and the maximum likelihood estimators are consistent, asymptotically efficient and asymptotically normally distributed. The regularity conditions require that $\partial P_i/\partial \Omega_j$ be continuous at the true parameter point, Ω_o, though this can be replaced by the weaker condition that the P_i be totally differentiable at Ω_o. Unfortunately, in general neither of these conditions hold for the Pareto distribution. This can be seen by considering $P_1=P[w$ lies in the least wealth class$]$. If X_2 is the upper delimiter of this class, then $P_1=0$ for $A>X_2$, $P_1=1-A^{\beta}X_2^{-\beta}$ for $A<X_2$ and $\partial P_1/\partial A$ is discontinuous at $A=X_2$. Conditional on $A<X_2$, the regularity conditions hold and the consistency and asymptotic normality of the MLEs follow. Inferences concerning the MLEs can, therefore, be made conditional on $A<X_2$ and this is what is done here.

Maximum likelihood estimates[1] of the parameters of the lognormal and Pareto models are presented in Table 1 together with their asymptotic standard errors, values of the maximum log-likelihood and χ^2 goodness of fit statistics. The frequencies predicted by the models are shown in Table 2.

Table 1: *Maximum likelihood estimates of parameters of lognormal and Pareto distributions using only data on estates with non-zero recorded wealth*

Model	*Parameter*	*Estimate*	*Log-likelihood* x 10^{-2}*	χ^2	*Mean wealth*	*Total wealth*
	μ	6.780 (0.0104)				
Lognormal			– 45.665	396	£3,211 (48.99)	£1,940m (29.58)
	σ	1.609 (0.0077)				
	A	83.38 (0.350)				
Pareto			–667.60	5,413	–	–
	β	0.447 (0.0029)				

Figures in parentheses are asymptotic standard errors.
**Not* corrected for deficient sample size – all other summary statistics *are* corrected; see Section II.

$$\chi^2 = \sum_{\text{all classes}} \left\{ \frac{(\text{observed frequency} - \text{predicted frequency})^2}{\text{predicted frequency}} \right\}$$

Even a cursory glance at Table 1 reveals that the lognormal distribution is far superior to the Pareto distribution as a model for the Irish wealth distribution and this is borne out by the χ^2 statistics and the maximum log-likelihoods which are both reduced by 93 per cent by moving from the Pareto model to the lognormal model. It would be possible to test the lognormal distribution versus the Pareto distribution as alternative composite

1. Maximum likelihood estimates reported here and later were obtained using a quasi-Newton algorithm implemented by Gill and Murray (1972) and based on an algorithm given by Gill, Murray and Pitfield (1972). Convergence was extremely rapid and the solutions reported were reached from a variety of start points.

non-nested hypotheses using the method outlined in Cox (1961 and 1962). However, in the face of the evidence in Tables 1 and 2 this hardly seems necessary. As Cox (1962, p. 407), discussing his method, comments, "We assume that in applications the sample size is large enough to ensure the usual expansions of maximum likelihood theory are good approximations, but not so large that H_f and H_g (the two alternative hypotheses under consideration) can be distinguished with negligible probability of error. (If the probabilities of error were negligible, there would hardly be need for a formal statistical test!)" The sample size used here is very large and a formal statistical test does indeed seem redundant.

Table 2: *Observed frequencies of net capital, Ireland, 1966, and predicted frequencies under the lognormal and Pareto hypotheses-models estimated using only data on estates with non-zero recorded wealth*

Class	*Net capital (£000)*	*Observed frequency*	*Predicted frequencies*	
			Lognormal model	*Pareto model*
1	Under 0.1	47,120	53,344	47,120
2	0.1 – 1.0	286,078	267,823	357,839
3	1.0 – 2.0	98,444	98,667	53,009
4	2.0 – 5.0	80,301	99,514	49,055
5	5.0 – 6.0	15,289	14,314	7,585
6	6.0 – 7.0	10,574	10,688	5,948
7	7.0 – 8.0	7,122	8,254	4,832
8	8.0 – 10.0	13,980	11,837	7,458
9	10.0 – 12.5	10,491	9,603	6,750
10	12.5 – 15.0	6,223	6,378	5,037
11	15.0 – 17.5	5,672	4,481	3,950
12	17.5 – 20.0	4,146	3,292	3,208
13	20.0 – 25.0	5,230	4,437	4,952
14	25.0 – 30.0	2,881	2,791	3,695
15	30.0 – 35.0	2,351	1,876	2,898
16	35.0 – 40.0	1,157	1,325	2,354
17	40.0 – 45.0	745	971	1,963
18	45.0 – 50.0	1,813	733	1,670
19	50.0 – 60.0	1,218	1,014	2,711
20	60.0 – 75.0	821	893	3,031
21	75.0 – 100.0	1,005	745	3,487
22	100.0 – 150.0	452	562	4,212
23	150.0 – 200.0	67	199	2,558
24	200.0 – 250.0	80	90	1,770
25	250.0 – 400.0	157	92	3,197
26	400.0 and over	50	43	13,681
	Total	603,972	603,972	603,970*

*Total differs from observed total because of rounding. Source of Columns 2 and 3: Lyons, Table 6.2.

The fit of the lognormal distribution is quite good in the lower tail and unexpectedly good in the upper tail. In the highest wealth class the predicted frequency is 43 and the observed frequency is 50 individuals. The fit of the Pareto distribution is very poor in the upper tail, 13,681 individuals being predicted in the highest wealth class or 274 times the observed frequency. Indeed, overall the fit of the Pareto distribution is poor, except in the lowest class. Here predicted and observed frequencies are identical, but this is a result which will always be obtained when fitting a Pareto distribution to grouped data by multinomial maximum likelihood.[2] The sparsely populated upper classes carry little weight in the maximum likelihood procedure and it is perhaps not surprising that the Pareto distribution's fit is poor here. It should be noted that even though the upper tail is sparsely populated, the lognormal distribution manages to represent it quite faithfully.

The estimates of the parameters of the lognormal distribution are well determined and imply estimates of mean and total wealth of £3,211 and £1,940m, respectively, with asymptotic standard errors[3] of £48.99 and £30m, respectively. An asymptotic 95 per cent confidence interval for total wealth under the lognormal hypothesis is thus (£1,881m, £1,999m). Lyons reports estimates of mean and total wealth of £3,511 and £2,121m, respectively. His estimates differ from ours because he (sensibly in the context of his study) makes no assumption concerning the functional form of the wealth distribution and, therefore, uses subjectively chosen mid-points for wealth classes when calculating mean and total wealth. Median and modal wealth are estimated here as £66 and £880, respectively, and the Gini coefficient is estimated as 0.745 with an asymptotic 95 per cent confidence interval (.726, .766).

The estimates of the parameters of the Pareto distribution are well determined, but the estimator of β is very low at 0.447. Mean wealth is not defined for $\beta < 1$ and so no inferences concerning total wealth can be made

2. The log-likelihood for the Pareto model can be written using (3) and (5) as

$$\log L(\Omega|\underline{N}) = f(\underline{N}) + N^*\beta \log A + N_1 \log (1 - A^{\beta} X_2^{-\beta}) + g(\beta)$$

where $\frac{dg(\beta)}{dA} = 0$ and $N^* + N_1 = N = \sum_{i=1}^{26} N_i$.

$$\frac{\partial \log L(\Omega|\underline{N})}{\partial A} = 0 \text{ implies } \hat{A} = \left(\frac{N^*}{N}\right)^{1/\hat{\beta}} X_2$$

is the maximum-likelihood estimator of A. The predicted frequency in the first class is $NP_1(\hat{\Omega})$ and substituting $\hat{A}, \hat{\beta}$ into $P_1(\Omega)$ gives $NP_1(\hat{\Omega}) = N_1$.

3. Asymptotic standard errors for mean and total wealth are calculated by noting that estimated mean and total wealth are both well-behaved functions of consistent and asymptotically normally distributed maximum likelihood estimates. A Taylor series expansion of these functions up to first order terms provides the required expressions.

in this model. The conclusion to be drawn from the results of this section is that overall the fit of the lognormal distribution is far superior to the fit of the Pareto distribution and that the lognormal distribution models fit both the upper and lower tail of the Irish wealth distribution quite faithfully. To be sure, the χ^2 statistic for the lognormal distribution is large, but the very large sample size being employed here is at least partly responsible for this. It is possible that the poor fit of the Pareto distribution is due to the omission of estates from the estate duty data and this is one of the matters considered in the next section.

III THE WEALTH DISTRIBUTION INCORPORATING OMITTED INDIVIDUALS

Attention is now turned to the 1,120,278 individuals reported by Lyons as having estates of unrecorded value. It is assumed that the larger an estate is, the more likely it is that its value will be recorded, and further that for estates the size of which exceeds some threshold w*, estate size is recorded with probability one. Estates the value of which is less than w* may or may not have their value recorded. Choice of any finite value for w* necessarily entails some degree of approximation since it is possible that a very large estate could be unrecorded. The method to be employed in this section may be expected to perform well if such events are rare.

If w* is in the interval (X_i, X_{i+1}), the frequencies $N_1, N_2, \ldots, N_i$ understate the true frequencies in classes 1, 2, . . . , i to an unknown extent and the likelihood function (Equation 3) must be modified. If w* is known to lie in an interval no higher than interval i, then the frequency in the class (X_i, X_{i+1}) is accurately represented by $N_1{}^* = N_0 + N_1 + \ldots + N_i$ where N_0 is the known number of estates of unrecorded value. The log-likelihood function for $N^i = (N_1{}^*, N_{i+1}, \ldots, N_{26})$ is

$$\log L_i(\Omega \,|\underline{N}^i) = f^*(\underline{N}^i) + N_1{}^*\log \sum_{j=i}^{i} P_i(\Omega) + \sum_{j=i+1}^{26} N_i \log P_i(\Omega) \qquad (6)$$

With known i (6) can be maximised with respect to Ω to obtain maximum likelihood estimates of Ω.

It is difficult to choose a value for i. One might expect X_{i+1} to be at least equal to the dutiable threshold (£5,000 net dutiable value in Ireland in 1966). Choosing too low a value of i would result in Equation 6 being misspecified. Yet if too high a value were chosen, Equation 6 would still be correctly specified, though estimation would be inefficient since information would have been discarded. It would be convenient if i could be regarded as a parameter of the model and Equation 6 maximised with respect to Ω and i. Unfortunately, since the sample space changes as i alters, this approach is not feasible and, as will be seen, $L_i(\Omega|\underline{N}^1)$ is a monotonic non-increasing function of i. An alternative would be to specify a discrete distribution for i and write a new log-likelihood function

$$\log L'(\Omega|\underline{N}) = \log\left\{\sum_{j=1}^{26} L_j(\Omega|\underline{N}^j)P[i-j]\right\} \tag{7}$$

where P[i=j] is the point probability that i=j, j=1, 2, . . . , 26. Because of the difficulty of specifying a distribution for i, this approach is not pursued.

In what follows a pragmatic approach is adopted. It is assumed that X_{i+1} is most unlikely to exceed £30,000 (i.e., i is unlikely to exceed 14) and the log-likelihood function in Equation 6 is maximised for i=1, 2, . . . , 14 for the lognormal model and the Pareto model in turn. The results are presented in Table 3. In this table the ith row gives the result of fitting by maximum likelihood a lognormal model and a Pareto model to the Irish wealth distribution after merging the lowest i classes and adding to their combined frequencies the number of individuals whose estates are recorded by Lyons as unvalued. Parameter estimates and asymptotic standard errors, values of maximum log-likelihood and χ^2 goodness of fit statistics are reported. In addition, maximum likelihood estimates of mean and total wealth are reported (where possible)[4] with their asymptotic standard errors. In the column labelled "missing wealth" the difference between the estimated total wealth and Lyons' estimate of total wealth is given.

Consider first the log-normal model. For i equal to 1, 2 or 3, the fit of the model is poor, but for i greater than 3, the fit becomes quite good and remains good thereafter. As noted above, the maximum log-likelihoods cannot be compared for different values of i, but to enable some comparisons to be drawn between results obtained using different values of i, the χ^2 statistic has been divided by the number of intervals used in each estimation, minus the number of parameters estimated. This measure stabilises once i exceeds 7, attaining its minimum at this value though there is rather little change in the measure once i exceeds 3. The data suggests that most omitted estates have a value less than £8,000. The high value of the adjusted χ^2 statistic for i=1 suggests that it is most unlikely that the unrecorded estates belong in the lowest wealth class. For $i > 7$ the estimates of μ and σ change rather slowly – a further indication that the majority of omitted estates lie in classes 1 to 7.

The best results seem to be obtained when i = 7 and the individuals with unrecorded wealth are assigned to the lowest seven classes. On this basis mean wealth is estimated as £1,525 with an asymptotic standard error of £138.2 and total wealth is estimated as £2,629m with an asymptotic stan-

4. In the case of the Pareto distribution, only when $\hat{\beta} > 1$.

Table 3: *Maximum likelihood estimates of parameters of lognormal and Pareto distributions incorporating information on the number of estates of unrecorded value under alternative assumptions concerning the maximum size of unvalued estates*

	LOGNORMAL MODEL: Estimates								PARETO MODEL: Estimates								
i	$\hat{\mu}$	$\hat{\sigma}$	*Mean wealth (£)*	*Total wealth (£m)*	*Missing wealth (£m)*	*Log-likelihood x 10-2*	χ^2	$\frac{\chi^2}{\text{no. of classes}-2}$	$\hat{A}$	$\hat{B}$	*Mean wealth (£)*	*Total wealth (£m)*	*Missing wealth (£m)*	*Log-likelihood x 10-2*	χ^2	$\frac{\chi^2}{\text{no. of classes}-2}$	*Upper delimit of wealth class (£000)*
1	3.115	3.295	5,353	9,230	7,109	283.8	2,250	93.8	7.97	0.447	–	–	–	667.6	5,404	225.2	0.1
	(0.020)	(0.018)	(257.5)	(444.0)					(0.163)	(0.003)							
2	4.503	2.379	1,600	2,758	637	42.85	342	14.9	94.2	0.784	–	–	–	122.3	925	40.2	1.0
	(0.027)	(0.015)	(95.0)	(163.8)					(2.31)	(0.007)							
3	5.323	2.329	1,535	2,647	526	40.54	328	14.9	148.8	0.886	–	–	–	92.53	721	32.8	2.0
	(0.047)	(0.027)	(165.6)	(285.5)					(4.79)	(0.010)							
4	5.364	1.982	1,461	2,519	398	14.81	125	6.0	415.5	1.178	2,750	4,742	2,621	30.55	244	11.6	5.0
	(0.046)	(0.027)	(142.9)	(246.4)					(12.9)	(0.014)	(116.3)	(200.4)					
5	5.452	1.963	1,466	2,528	407	14.51	123	6.2	470.3	1.222	2,586	4,458	2,337	27.72	223	11.2	6.0
	(0.039)	(0.021)	(115.5)	(199.2)					(23.3)	(0.022)	(97.10)	(167.4)					
6	5.617	1.923	1,482	2,556	435	13.40	114	6.0	545.8	1.278	2,510	4,327	2,206	23.74	194	10.2	7.0
	(0.049)	(0.026)	(146.1)	(251.9)					(21.2)	(0.019)	(56.81)	(97.96)					
7	5.633	1.851	1,525	2,629	508	9.521	85	4.7	669.2	1.360	2,528	4,358	2,237	15.88	138	7.7	8.0
	(0.048)	(0.023)	(138.2)	(238.3)					(34.4)	(0.028)	(39.33)	(67.82)					
8	5.738	1.844	1,530	2,637	516	9.448	85	5.0	759.8	1.414	2,595	4,423	2,352	14.51	129	7.6	10.0
	(0.041)	(0.021)	(119.6)	(206.2)					(45.2)	(0.031)	(43.32)	(74.69)					
9	5.893	1.800	1,570	2,708	587	8.941	82	5.1	924.1	1.501	2,768	4,773	2,652	12.18	112	7.0	12.5
	(0.094)	(0.044)	(270.2)	(465.9)					(54.0)	(0.030)	(70.71)	(121.95)					
10	5.880	1.739	1,645	2,835	714	8.005	74	4.9	1,141	1.604	3,043	5,245	3,124	9.52	92	6.1	15.0
	(0.104)	(0.047)	(302.9)	(522.3)					(81.6)	(0.041)	(95.38)	(164.5)					
11	5.841	1.744	1,638	2,824	703	7.948	74	5.3	1,232	1.640	3,154	5,439	3,318	9.22	90	6.4	17.5
	(0.071)	(0.030)	(167.2)	(288.3)					(108.9)	(0.053)	(128.38)	(221.36)					
12	5.847	1.756	1,620	2,792	671	7.870	74	5.7	1,288	1.663	3,231	5,571	3,450	9.08	89	6.8	20.0
	(0.325)	(0.125)	(883.3)	(1523.0)					(138.5)	(0.056)	(181.9)	(313.6)					
13	5.856	1.753	1,625	2,801	680	7.818	74	6.2	1,449	1.722	3,457	5,960	3,839	8.78	85	7.1	25.0
	(0.198)	(0.049)	(147.6)	(254.5)					(150.38)	(0.056)	(213.2)	(367.6)					
14	6.054	1.684	1,758	3,030	909	7.465	69	6.3	1,837	1.847	4,006	6,907	4,786	7.84	74	6.7	30.0
	(0.053)	(0.021)	(770.5)	(1328.5)					(639.5)	(0.398)	(390.0)	(672.4)					

Notes

(a) Figures in parentheses are asymptotic standard errors conditional on 1.

(b) Missing wealth – total wealth – £2,121m (Lyons' (1972) estimate of total wealth).

(c) Log-likelihoods are not corrected for deficient sample size.

(d) i is the number of classes merged in estimation.

dard error [5] of £238.3. The Gini coefficient is now estimated as 0.809 with an asymptotic 95 per cent confidence interval (.780, .839).

The fit of the lognormal model with i=7 can be assessed by comparing the predicted and observed frequencies which are presented in Table 4. As can be seen, the fit is quite good overall and, in particular; the fit in the sparsely populated higher classes is good.

Table 4: *Predicted and observed frequencies, lognormal model incorporating individuals with unvalued estates after merging of classes 1 to 7*

Class	*Net capital (£000)*	*Observed frequency*	*Predicted frequency*
1	Under 8.0	1,665,206	1,665,133
2	8.0 – 10.0	13,980	14,148
3	10.0 – 12.5	10,491	11,198
4	12.5 – 15.0	6,723	7,301
5	15.0 – 17.5	5,672	5,072
6	17.5 – 20.0	4,146	3,690
7	20.0 – 25.0	5,230	4,937
8	25.0 – 30.0	2,881	3,088
9	30.0 – 35.0	2,351	2,073
10	35.0 – 40.0	1,157	1,465
11	40.0 – 45.0	745	1,077
12	45.0 – 50.0	1,813	816
13	50.0 – 60.0	1,218	1,136
14	60.0 – 75.0	821	1,012
15	75.0 – 100.0	1,005	861
16	100.0 – 150.0	457	674
17	150.0 – 200.0	67	251
18	200.0 – 250.0	80	119
19	250.0 – 400.0	157	126
20	400.0 and over	50	72
*Total**		1,724,250	1,724,249

*Totals differ because of rounding errors.

Assuming that individuals with unvalued estates had zero wealth, Lyons estimated total wealth as £2,121m. The procedure adopted here raises this estimate by £508m. In fact, the effect of including individuals with unrecorded estates is larger than this, for applying the lognormal distribution to individuals whose estate size was recorded gave an estimate of total wealth of £1,940m (assuming zero wealth holding for the remaining individuals) – £181m lower than Lyons' estimate. Thus, maintaining the lognormal hypothesis and incorporating individuals with unvalued estates raises the maxi-

5. Strictly speaking, these asymptotic standard errors are conditional on i and are not valid once i is chosen by reference to the data.

mum likelihood estimate of total wealth by £689m, just over 35 per cent. It should be noted that the asymptotic standard errors on total wealth are large; this is an unfortunate consequence of the relatively coarse grouping of the data.

In Chesher and McMahon total wealth was estimated using graphical methods at £2,639m. The procedure adopted here formalises Chesher and McMahon's method and produces very similar results. The analysis reported here allows the accuracy of the estimates of total wealth to be assessed.

The Pareto model is now considered briefly. The fit of the Pareto model is less good than that of the lognormal model for $i < 15$. For $i \geqslant 16$ the maximum log-likelihood obtained under the Pareto model is higher than that under the lognormal model. Thus, for wealth in excess of £40,000 it seems that the Pareto model fits better than the lognormal model when individuals with unvalued estates are included.[6]

The estimates of A and β increase rapidly with i and are quite well determined. However, for $i > 2$ the estimates of A (the minimum level of wealth) exceeds £100 (significant at the 5 per cent level). This runs counter to the evidence in Lyons that 47,120 estates had a value of less than £100 and this, together with the very poor fit of the model, leads us to reject the Pareto distribution as a model for the Irish wealth data. For $i > 3$, $\hat{\beta}$ exceeds one and maximum likelihood estimates of total wealth can be obtained. These are presented in Table 3 and can be seen to be nearly twice as high as those obtained under the lognormal hypothesis. The shortcomings of the Pareto distribution suggest that these estimates are unreliable and they are not considered further. One conclusion to be drawn is that the estimate of total wealth is quite sensitive to changes in the assumption concerning the form of the wealth distribution.

The procedure of merging classes adopted in this section involves discarding information on the frequencies of estates which are valued in the lower wealth classes. In the next section the feasibility of using this information is considered.

IV EFFICIENT ESTIMATION OF THE WEALTH DISTRIBUTION INCORPORATING INFORMATION ON OMITTED INDIVIDUALS

Without modelling the Revenue Commission's decision procedure for selecting estates to examine, there is little that can be done to utilise the available frequencies in the lower wealth classes while allowing for the

6. When comparing the fit of alternative models to the upper tails of empirical frequency distributions, one should strictly speaking fit *truncated* lognormal and Pareto distributions to the data. However, in the case of the Pareto distribution such a procedure yields identical results to those reported in Table 3 because of the manner in which the parameter "A" bounds the least wealth class. In the case of the lognormal distribution, fitting truncated and merged lognormal distributions yield similar results in large samples when the lognormal model provides a current specification.

estates whose size is unrecorded. It is not the purpose of this paper to present a model of the Revenue Commission's decision, but the difficulties involved in utilising the frequencies in the lower wealth classes can be seen as follows.

If this information is to be employed, it is necessary to introduce the probability of valuing an estate which, if observed, would be classified into group j. Let these probabilities be denoted r_j, j=1, 2, . . . , 26. The likelihood function for Ω and $r_1, r_2, \ldots, r_{26}$ is then

$$L(\Omega, r_1, r_2, \ldots, r_{26}|\underline{N}^*) = \frac{(N+N_0)!}{N_0!N_1!\ldots N_{26}!}\left\{\prod_{j=1}^{26} P_j(\Omega)(1-r_j)\right\}^{N_0} \prod_{j=1}^{26} (P_j(\Omega)r_j)^{N_j} \qquad (8)$$

where $N^* = (N_0\underline{N})$.

Under the lognormal hypothesis (the Pareto hypothesis is not considered here) this model requires 28 parameters to be estimated from data grouped into only 26 classes and the parameters are clearly not identifiable. A parametric model is needed for the r_js and in this section one simple model is considered. The hypothesis is maintained that for classes 1, 2, . . . , i, r_j is constant and equal to r and that for classes i+1, i+2, . . . , 26, r_j = 1. Thus it is maintained that estates in classes i and below can be omitted from the estate duty statistics and that the probability that an estate in these classes has its size unrecorded is independent of estate size. This hypothesis is something of a "straw man", but it is of interest to investigate it as a preliminary to modelling of the Revenue Commission's selection procedure. Under this hypothesis the log-likelihood function for $\underline{N}^*$ is

$$\log L(\Omega, r|\underline{N}^*) = g(\underline{N}^*) + N_0 \log \sum_{j=1}^{i} P_j(\Omega) + \sum_{j=1}^{26} N_j \log P_j(\Omega) + \left(\sum_{j=1}^{i} N_j\right) \log r + N_0 \log(1-r) \qquad (9)$$

where the $P_j(\Omega)$ are given by (4). There are now only three parameters to be estimated and it is straightforward to show that the maximum likelihood estimator of r is $\hat{r} = \sum_{j=1}^{i} N_j/(N_0 + \sum_{j=1}^{i} N_j)$. Maximum likelihood estimates of μ, σ, A and β under this hypothesis are presented in Table 5 whose content is similar to Table 3.

As expected, this model for the r_js does not perform well. With all omitted individuals assigned to the lowest class (i.e., with i=1), the model involving the r_js degenerates to the merged class model with i=1 obtained previously in Section III and identical results obtain. As i increases and

Table 5: *Maximum likelihood estimates of parameters of lognormal and Pareto distributions incorporating information on number of estates of unrecorded value under alternative assumptions concerning maximum size of estates with unrecorded value assuming constant probability of omission of estates from estate duty data in classes i and below*

	LOGNORMAL MODEL							PARETO MODEL				
i	$\hat{\mu}$	$\hat{\sigma}$	Mean wealth (£)	Total wealth (£m)	Log-likelihood x 10^2	χ^2	χ^2 / no. of classes−2	$\hat{A}$	$\hat{\beta}$	Log-likelihood x 10^2	χ^2	χ^2 / no. of classes−2
1	3.155 (0.020)	3.295 (0.018)	5353 (257.5)	9230 (444.0)	283.8	2250	93.8	7.97 (0.14)	0.447 (0.003)	667.6	5404	225.2
2	5.585 (0.008)	1.616 (0.008)	984 (9.61)	1696 (16.57)	787.3	6879	286.6	84.7 (0.32)	0.756 (0.003)	124.5**	906	37.8
3	6.037 (0.009)	1.439 (0.006)	1178 (8.99)	2031 (15.50)	559.2	5355	223.1	85.81 (0.30)	0.699 (0.003)	619.8	6866	286.0
4	6.418 (0.009)	1.390 (0.006)	1.611 (13.2)	2778 (22.77)	123.6	1265	52.7	86.05 (0.30)	0.640 (0.003)	907.7	8982	374.3
5	6.470 (0.009)	1.397 (0.006)	1711 (13.3)	2950 (22.86)	113.0	1115	46.5	86.04 (0.30)	0.630 (0.003)	983.5	9722	405.1
6	6.508 (0.009)	1.404 (0.006)	1798 (15.3)	3100 (26.45)	96.05	929	38.7	86.01 (0.30)	0.622 (0.003)	1025.9	15021	625.9
7	6.538 (0.009)	1.413 (0.006)	1875 (16.7)	3233 (28.73)	71.66	704	29.3	85.99 (0.30)	0.616 (0.003)	1043.5	10043	418.5
8	6.582 (0.009)	1.429 (0.006)	2004 (18.6)	3455 (32.10)	78.58	751	31.3	85.94 (0.30)	0.606 (0.003)	1103.3	10488	437.0
9	6.620 (0.009)	1.447 (0.006)	2135 (20.8)	3681 (35.82)	73.86	687	28.6	85.88 (0.30)	0.596 (0.003)	1132.9	10539	439.1
10	6.646 (0.009)	1.462 (0.006)	2241 (22.10)	3864 (38.10)	67.14	614	25.6	85.82 (0.30)	0.589 (0.003)	1142.6	10430	434.6
11	6.665 (0.009)	1.475 (0.006)	2329 (20.8)	4016 (35.88)	71.80	650	27.1	85.78 (0.30)	0.583 (0.003)	1154.4	10386	432.8
12	6.680 (0.009)	1.486 (0.006)	2404 (25.7)	4145 (44.33)	74.38	664	27.7	85.74 (0.30)	0.578 (0.003)	1158.9	10294	428.9
13	6.705 (0.010)	1.505 (0.006)	2534 (28.3)	4369 (48.87)	72.80	631	26.3	85.66 (0.30)	0.570 (0.003)	1153.2	10034	418.1
14	6.716 (0.010)	1.519 (0.006)	2618 (30.4)	4514 (52.4)	66.44	563	23.5	85.60 (0.31)	0.564 (0.003)	1138.0	9763	406.8
15	6.727 (0.010)	1.531 (0.006)	2693 (32.1)	4643 (55.31)	65.89	549	22.9	85.54 (0.31)	0.559 (0.003)	1127.0	9556	398.1
16	6.735 (0.010)	1.540 (0.007)	2754 (33.6)	4749 (58.0)	59.76	494	20.6	85.50 (0.31)	0.555 (0.003)	1110.1	9351	389.6
17	6.742 (0.010)	1.548 (0.007)	2805 (35.0)	4837 (60.38)	53.90	447	18.6	85.45 (0.31)	0.551 (0.003)	1093.6	9172	382.2
18	6.747 (0.010)	1.554 (0.007)	2847 (36.2)	4909 (62.36)	62.46	556	23.2	85.42 (0.31)	0.548 (0.003)	1092.0	9087	378.6
19	6.754 (0.010)	1.564 (0.007)	2914 (38.1)	5024 (65.77)	50.31	527	22.0	85.54 (0.31)	0.534 (0.003)	1071.1	8844	368.5
20	6.761 (0.010)	1.575 (0.007)	2986 (40.6)	5149 (69.96)	55.33	480	20.0	85.27 (0.31)	0.538 (0.003)	1041.9	8558	356.6
21	6.768 (0.010)	1.586 (0.007)	3059 (43.0)	5274 (74.17)	54.06	462	19.3	85.17 (0.31)	0.531 (0.003)	1008.7	8228	342.8
22	6.774 (0.010)	1.597 (0.007)	3131 (45.8)	5399 (78.93)	49.52	423	17.6	85.03 (0.32)	0.522 (0.003)	960.2	7818	325.8
23	6.776 (0.010)	1.602 (0.008)	3163 (47.1)	5454 (81.23)	46.77	406	16.9	84.93 (0.32)	0.516 (0.003)	928.1	7566	315.3
24	6.777 (0.010)	1.604 (0.008)	3179 (47.9)	5481 (82.54)	45.99	400	16.7	84.49 (0.32)	0.512 (0.003)	906.1	7390	307.9
25	6.779 (0.010)	1.607 (0.008)	3199 (49.25)	5515 (84.92)	46.00	399	16.6	84.71 (0.32)	0.504 (0.003)	865.9	7065	294.4
26	6.780 (0.010)	1.609 (0.008)	3211 (48.99)	5535 (84.47)	45.67*	396	16.5	83.38 (0.35)	0.447 (0.003)	667.6	5404	225.2

Notes:

*Maximum log-likelihood under log normal hypothesis.

**Maximum log-likelihood under Pareto hypothesis.

Figures in parentheses are asymptotic standard errors.

Log-likelihood are not corrected for deficient sample size.

omitted individuals are assigned to more and more classes, $\hat{\mu}$ increases for all i, $\hat{\sigma}$ increases for $i > 3$ and for $i > 3$ the asymptotic standard errors of $\hat{\mu}$ and $\hat{\sigma}$ are virtually constant at around 0.010 and 0.007, respectively. When i=26 and omitted individuals are assigned to all classes, the results obtained in Section II with the simple lognormal hypothesis where omitted individuals are excluded altogether are obtained (except, of course, that total wealth is increased by the addition of omitted individuals) and it is at this value of i that the log-likelihood achieves its maximum. Thus, the maximum likelihood estimate of i is $\hat{i} = 26$ and if the hypothesis is maintained that the probability of an estate being omitted from the estate duty statistics is constant in all classes in which omission can occur, then, given the data, it is most likely that the omitted individuals are a simple random sample from the *complete* wealth distribution. In fact, of course, this must be regarded as a most unlikely state of affairs and we are led to doubt the hypothesis that the probability that estate size be unrecorded is constant over all classes in which omission can occur. Consequently, little weight should be attached to the estimates of mean and total wealth reported in this section.

Omission of estates is an important feature of the available data. The evidence of this section supports the commonsense view that the probability of an estate being omitted does depend on the size of the estate. More complex models of the process whereby estates are selected for valuation may well result in improved estimates of the true wealth distribution and of total wealth. This is the subject of further research by the author (see Chesher, 1977, for some preliminary results).

V CONCLUSIONS

In this paper Lyons' data concerning the distribution of wealth in Ireland have been examined in order to assess the appropriateness of the lognormal distribution and the Pareto distribution as models for the Irish wealth distribution in 1966, and in order to produce statistical estimates of the size of total wealth holdings in Ireland at that date.

The lognormal distribution outperforms the Pareto distribution when the 65 per cent of individuals with unrecorded wealth are introduced into the analysis and when they are not. One surprising result is the ability of the lognormal distribution, when fitted to all the data on recorded wealth, to model the wealth of the richest individuals. When the alternative models are fitted to the upper tails of the observed distribution, the Pareto distribution outperforms the lognormal distribution only for individuals whose wealth exceeds £40,000; these are the richest 0.4% of all individuals and the richest 1.1% of individuals with recorded wealth.

Adopting the lognormal distribution as an approximate model for the Irish wealth distribution and using data on individuals whose estate size is recorded yields an estimate of mean wealth of £3,211 with an asymptotic standard error of £48.99 and an estimate of total wealth of £1,940m with an asymptotic standard error of £30m. Lyons' estimate of total wealth of £2,121m differs from our estimate of £1,940m because he did not impose a parametric model on the distribution and accordingly used subjective class mid-points in his computations. Assuming a maximum size of omitted estate of £8,000, but imposing no further structure on the manner in which estates are selected for examination, leads to an estimate of mean wealth of £1,525 with an asymptotic standard error of £138.2 and an estimate of total wealth of £2,629m with an asymptotic standard error of £238.3m.

In Chesher and McMahon very similar estimates were produced using graphical methods which may be regarded as approximating the maximum likelihood procedure used here. There total wealth was estimated as £2,639m. An advantage of the analysis of this paper is that the accuracy of the estimates produced can be assessed.

The procedure used to obtain these estimates involves discarding information concerning frequencies in some of the lower wealth classes. To utilise this information the Revenue Commission's selection procedure for choosing estates to be examined has to be modelled. A very simple model in which all estates below some maximum size are subject to equal probability of selection was fitted to the data, but, as expected, gave poor results. To improve the estimates of the Irish wealth distribution and of total wealth it is necessary to model the Revenue Commission's decision procedure. Unfortunately, there is a limit to the amount of information that can be wrung from the hard-won data presented by Lyons. It is difficult to estimate complex models using grouped data when much of the information which allows one model to be distinguished from another is contained in the lower wealth classes where densities are highest, but grouping is generally coarsest. The comment of Harrison (1977) to the effect that there is a great need for better data concerning wealth distributions is apt.

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