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Physical phase space of lattice Yang-Mills theory and the moduli space of flat connections on a Riemann surface

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Abstract

It is shown that the physical phase space of γ-deformed Hamiltonian lattice Yang-Mills theory, which was recently proposed in refs.[1,2], coincides as a Poisson manifold with the moduli space of flat connections on a Riemann surface with $(L-V+1)$ handles and therefore with the physical phase space of the corresponding $(2+1)$-dimensional Chern-Simons model, where L and V are correspondingly a total number of links and vertices of the lattice. The deformation parameter γ is identified with $\frac{2\pi}{k}$ and k is an integer entering the Chern-Simons action.

1 Introduction

It is well-known that there are two closely-related ways to introduce lattice regularization of gauge models. In the approach of Wilson [3] one discretizes all space-time and, thus, replaces the Yang-Mills theory by some statistical mechanics model. In the Hamiltonian approach of Kogut and Susskind [4] one considers the theory in the Minkowskian

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space-time and discretizes only space directions remaining the time continuous. Then one should place on each link of the lattice some phase space and attach to each vertex lattice Gauss-law constraints which are first-class constraints and generate gauge transformations. Thus in the Hamiltonian approach the continuous Yang-Mills theory is replaced by some classical mechanics model with first-class constraints. In fact it is not difficult to show that these two approaches are equivalent if one chooses the cotangent bundle of a Lie group as the phase space placed on a link. However one can consider not only the cotangent bundle and in this case the Hamiltonian approach will lead to results which can not be derived from the Wilson formulation.

In refs.[1,2] I have proposed Hamiltonian lattice gauge models based on the assignment of a Heisenberg double D_+^γ [5, 6, 7, 8, 9, 10] of a Lie group to each link. The Heisenberg double D_+^γ depends on one complex parameter γ and the cotangent bundle of a Lie group can be regarded as a limiting case of D_+^γ when γ goes to zero. Quantization of the Heisenberg double leads to an algebra which contains as subalgebras the quantized universal enveloping algebra $U_q(\mathcal{G})$ and the algebra of functions on the quantum group $Fun_q(G)$ [8, 9, 11], and q is related to γ by means of the following formula: $q = \mathrm{e}^{i\hbar\gamma}$. Only the case of imaginary γ (real q) was considered in refs.[1,2]. In the present paper we find a proper generalization to the case of real γ (q lying on the unit circle). This case seems to be of the most importance in quantum theory since for q a root of unity the Heisenberg double has just a finite number of irreducible finite-dimensional representations and thus the Hilbert space is finite-dimensional too. It permits to develop the weak-coupling expansion for the Hamiltonian lattice Yang-Mills theory which differs from the standard perturbation theory.

The simplest way to get such a generalization seems to be to study the structure of the physical phase space of the usual Hamiltonian lattice gauge models on graphs, and then to deform the physical phase space. We show that the gauge invariance of a gauge model on an arbitrary lattice (or a graph) can be used to reduce the graph to a standard graph with one vertex and $g = L-V+1$ loops (tadpoles), where L and V are a total number of links and vertices of the original graph. The Gauss-law constraints attached to the only vertex of the graph generate residual gauge transformations on the reduced phase space, which is just the direct product of cotangent bundles over all tadpoles. Generalization of gauge models on standard graphs can be obtained in the same way as was done in ref.[2] by replacing the cotangent bundle by the Heisenberg double and the residual Gauss-law constraints by first-class constraints generating the well-known dressing transformations [6, 9] . We note that the Poisson algebra obtained coincides after some transformation [12, 13] with the Poisson algebra introduced by Fock and Rosly [14] to describe the Poisson structure of the moduli space of flat $SL(N)$ connections on a Riemann surface with $g = L-V+1$ handles and find a new antiautomorphism of the Poisson algebra which permits to single out the moduli space of flat $SU(N)$ connections. Thus the γ-deformed Hamiltonian lattice Yang-Mills theory and (2+1)-dimensional Chern-Simons theory with $k = \frac{2\pi}{\gamma}$ have the same physical phase space. Due to the well-known result of Witten [15, 16] the Hilbert space of the (2+1)-dimensional Chern-Simons theory and, therefore, of the lattice Yang-Mills theory is finite-dimensional and coincides with the space of conformal blocks of the WZNW model.

The plan of the paper is as follows. In the second section we consider gauge models on arbitrary graphs and the procedure of reduction to a standard graph. In the third section we firstly remind some simple results from the theory of the Heisenberg double.

Then the deformation of the physical phase space of the Hamiltonian lattice Yang-Mills theory is described and the relation to the moduli space of flat connections and to the (2+1)-dimensional Chern-Simons theory is pointed out. In Conclusion we discuss unsolved problems and perspectives.

2 Gauge models on graphs

In this section we firstly consider gauge models on arbitrary graphs (regular hyper-cubic lattice, triangulation of a surface, simplicial complexes and so on) and then we show that any gauge model on a graph can be reduced to a gauge model on a standard graph. Any graph is described by a set of vertices and a set of links. Each link is thought of as either a path connecting two vertices or a closed path with a marked vertex (tadpole). Two vertices can be connected by any finite number of links. Such a graph is certainly just an arbitrary connected Feynman diagram.

Let us now consider some vicinity of a vertex v which does not contain other vertices and closed paths. Let us denote the paths which go from the vertex v by $l_1(v)$,...,$l_{N_v}(v)$. We call such a path as a vertex path. N_v is a common number of the paths and if there is no closed path for the vertex v then N_v coincides with the number of links going from v to some other vertices of the lattice. With each vertex path $l_i(v)$ one associates a field taking values in the cotangent bundle T^*G of a Lie group. This field can be described by a group-valued matrix $U(l_i(v))$, and an algebra-valued matrix $E(l_i(v))$ with the standard Poisson structure

$$\begin{aligned} \{U^1, U^2\} &= 0 \\ \{E^1, E^2\} &= \frac{1}{2}[E^1 - E^2, C] \\ \{E^1, U^2\} &= CU^2 \end{aligned} \tag{2.1}$$

and fields corresponding to different paths have vanishing Poisson brackets.
In eq.(2.1) we use the standard notations from the theory of quantum groups [17, 18]: for any matrix A acting in some space V one can construct two matrices $A^1 = A \otimes id$ and $A^2 = id \otimes A$ acting in the space $V \otimes V$, the matrix C is the tensor Casimir operator of the Lie algebra $\mathcal{G}$ of the group G: $C = -\eta_{ab}\lambda^a \otimes \lambda^b$ and η_{ab} is the Killing tensor and λ^a form a basis of $\mathcal{G}$.

One can see from eq.(2.1) that the field E should be identified with the right-invariant momentum generating left gauge transformations of the field U. It is useful to introduce a different parametrization of T^*G by means of the left-invariant momentum $\widetilde{E} = -U^{-1}EU$ and of the matrix $\widetilde{U} = U^{-1}$. One can easily check that the fields $\widetilde{U}$ and $\widetilde{E}$ have the same Poisson structure (2.1), momenta E and $\widetilde{E}$ have vanishing Poisson bracket and we shall need the following expression for the bracket of $\widetilde{E}$ and U

$$\{\widetilde{E}^1, U^2\} = -U^2 C \tag{2.2}$$

Let us now attach to the vertex v the following Gauss-law constraints

$$G(v) = \sum_{i=1}^{N_v} E(l_i(v)) = 0 \tag{2.3}$$

These constraints form the Poisson-Lie algebra

$$\{G^1, G^2\} = \frac{1}{2}[G^1 - G^2, C] \tag{2.4}$$

and generate the following gauge transformations of the fields $U(l_i(v))$ and $E(l_i(v))$

$$\begin{aligned} U(l_i(v)) &\rightarrow g(v)U(l_i(v)) \\ E(l_i(v)) &\rightarrow g(v)E(l_i(v))g^{-1}(v) \end{aligned} \tag{2.5}$$

Repeating the same procedure for all of the vertices one gets the phase space which is the direct product of cotangent bundles over all of the vertex paths and a set of the Gauss-law constraints attached to the vertices. The Gauss-law constraints corresponding to different vertices have vanishing Poisson brackets. Taking into account that for each link there are two vertex paths one sees that one has placed on each link two different cotangent bundles. However one can identify these bundles using the fact that the fields U and E and $\widetilde{U}$ and $\widetilde{E}$ have the same Poisson structure. Thus one can impose on the fields attached to one link the following constraints

$$\begin{aligned} \widetilde{U}(1) &= U^{-1}(1) = U(2) \\ \widetilde{E}(1) &= -U^{-1}(1)E(1)U(1) = E(2) \end{aligned} \tag{2.6}$$

So the fields $U(1)$, $E(1)$ and $U(2)$, $E(2)$ are just different coordinates on the same cotangent bundle. The final phase space of the model is thus the direct product of cotangent bundles over all links: $\prod_{links} T^*G$.

Let us note that due to the constraints (2.6) the field $U(l(v_1, v_2))$ corresponding to a link $l(v_1, v_2)$ which connects vertices v_1 and v_2 is transformed by the Gauss-law constraints $G(v_1)$ and $G(v_2)$ as follows

$$U(l(v_1, v_2)) \rightarrow g(v_1)U(l(v_1, v_2))g^{-1}(v_2) \tag{2.7}$$

This is the usual transformation law in lattice Yang-Mills theory. However the field $U(l(v))$ corresponding to a tadpole $l(v)$ attached to a vertex v is transformed by means of conjugations

$$U(l(v)) \rightarrow g(v)U(l(v))g^{-1}(v) \tag{2.8}$$

We see from eq.(2.8) that one can not eliminate the field $U(l(v))$ by means of a gauge transformation.

Observables which are invariant with respect to the gauge transformations (2.7) and (2.8) can be constructed in a standard way. If the graph under consideration is a regular hyper-cubic lattice one gets the usual lattice Yang-Mills model with the following Hamiltonian (which is certainly not unique)

$$H = -\frac{e^2}{2}a^{2-d}\sum_{links} \mathrm{tr}\, E^2(l) - \frac{a^{d-4}}{8e^2}\sum_{plaquettes}(W(\Box) + W^*(\Box)) \tag{2.9}$$

Here the summation is taken over all links and over all plaquettes, d is a dimension of space, e is a coupling constant, a is a lattice length and $W(\Box)$ is the usual Wilson term.

Let us now introduce the notion of the gauge equivalence of two graphs. Two graphs are called gauge equivalent if the corresponding gauge models have the same physical phase space. Let us remind that the physical phase space can be obtained by imposing some gauge conditions and then by solving the Gauss-law constraints . We shall show that any graph is equivalent to a standard graph with one vertex and $g = L - V + 1$ links (all links are tadpoles), where L and V are correspondingly a total number of links and vertices of the original graph.

To prove the statement let us consider some link l connecting two different vertices v_1 and v_2. There are two vertex paths $l(v_1)$ and $l(v_2)$ corresponding to the link l. In what follows we denote the vertex path $l(v_1)$ as l and $l(v_2)$ as l^{-1}. The constraints (2.6) imply $U(l^{-1}) = U^{-1}(l)$ and $E(l^{-1}) = -U^{-1}(l)E(l)U(l)$. Using the gauge invariance under the transformation (2.7) one can impose the gauge condition $U(l) = 1$. Then one has to express the corresponding momentum $E(l)$ through the remaining variables of the phase space. The field $E(l)$ enters two Gauss-law constraints $G(v_1)$ and $G(v_2)$ as follows

$$G(v_1) = E(l) + \sum_{\text{paths}}' E(l_i(v_1)) = 0 \tag{2.10}$$

$$\begin{aligned} G(v_2) &= E(l^{-1}) + \sum_{\text{paths}}' E(l_i(v_2)) = \\ &= -E(l) + \sum_{\text{paths}}' E(l_i(v_2)) = 0 \end{aligned} \tag{2.11}$$

where the summation in eqs.(2.10) and (2.11) goes over all vertex paths excepting $l(v_1)$ and $l(v_2)$ correspondingly.
One can find the field $E(l)$ from eq.(2.10) and inserting the solution into eq.(2.11) one gets instead of two constraints $G(v_1)$ and $G(v_2)$ a residual constraint

$$G(v_1, v_2) = \sum_{\text{paths}}' E(l_i(v_1)) + \sum_{\text{paths}}' E(l_i(v_2)) = 0 \tag{2.12}$$

Now it remains to note that the same Gauss-law constraints correspond to a graph which is obtained from the original graph by shrinking the link l and thus by identifying the vertices v_1 and v_2. Proceeding in the same way one finally gets the standard graph with one vertex and one residual constraint which has the following form

$$G = \sum_{i=1}^{g} E(i) + \tilde{E}(i) = \sum_{i=1}^{g} E(i) - U^{-1}(i)E(i)U(i) = 0 \tag{2.13}$$

where $g = L - V + 1$ is the number of links of the standard graph.
It is not difficult to check that the residual gauge transformations generated by this constraint are the simultaneous conjugations

$$U(i) \to gU(i)g^{-1}, \qquad E(i) \to gE(i)g^{-1} \tag{2.14}$$

It is clear that the gauge fixing just described corresponds to a choice of a maximal tree on a graph. The physical phase space can be now obtained as a factor space of the space, which is the result of the solution of the constraint (2.13), over the action (2.14) of the residual gauge group. This phase space is not a manifold because the gauge group action is not free. This fact seems to be closely related to the well-known Gribov ambiguity.

The reduction procedure just described can be used to calculate the reduced Hamiltonian. In particular it is possible to show that the magnetic part of the Hamiltonian of the (2+1)-dimensional lattice Yang-Mills theory defined on a square lattice with free boundary conditions can be reduced to the following form

$$H_m = \frac{1}{e^2a^2}\sum_{i=1}^{g} tr(U_i^{-1} + U_i) \tag{2.15}$$

and g is equal to the number of plaquettes in this case.

Unfortunately the spectrum of this Hamiltonian is continuous and one can not use it to develop the weak coupling expansion. However let us suppose that we have a way to compactify the physical phase space. It is known that quantization of a compact phase space leads to a finite-dimensional Hilbert space and, therefore, any operator acting in the space has a discrete spectrum and one can easily apply standard perturbation theory. The compactification can be achieved by replacing the cotangent bundles by the Heisenberg doubles D_+^γ and will be discussed in the next section. Let us finally note that in the (2+1)-dimensional case there is another and, may be, more attractive possibility to develop the weak coupling expansion. Taking into account that the electric part of the reduced Hamiltonian contains a term which is proportional to

$$H_e^0 = e^2\sum_{i=1}^{g} trE_i^2 \tag{2.16}$$

one can use the sum of the Hamiltonian H_m and H_e^0 as the first approximation. For $SU(2)$ group the Hamiltonian $H^0 = H_m + H_e^0$ describes an exactly-solvable model and one may hope to calculate exactly its spectrum. It is worthwhile to note that the same Hamiltonian describes the superfluid B-phase of ^{3}He. It would be very interesting to find an integrable generalization of the Hamiltonian H^0 for the $SU(N)$ group.

3 Deformation of the physical phase space

In this section we firstly remind some simple results from the theory of the Heisenberg double (for detailed discussion see refs.[5, 6, 7, 8, 9, 10]). Then we deform the physical phase space of gauge models on a graph by replacing the cotangent bundle and the residual Gauss-law constraint (2.13) by the Heisenberg double and by some deformed constraint correspondingly. After that we show that the deformed phase space coincides as a Poisson manifold with the moduli space of flat connections on a Riemann surface with $g = L-V+1$ handles.

Let G be a matrix algebraic group and $D = G\times G$. For definiteness we consider the case of the $SL(N)$ group. Almost all elements $(x,y)\in D$ can be presented in two equivalent forms as follows

$$\begin{aligned}(x,y) &= (U,U)^{-1}(L_+,L_-) = (U^{-1}L_+,U^{-1}L_-)\\ &= (\widetilde{L}_+,\widetilde{L}_-)^{-1}(\widetilde{U},\widetilde{U}) = (\widetilde{L}_+^{-1}\widetilde{U},\widetilde{L}_-^{-1}\widetilde{U})\end{aligned} \tag{3.17}$$

where $U,\widetilde{U}\in G$, the matrices $L_+,\widetilde{L}_+$ and $L_-,\widetilde{L}_-$ are upper- and lower-triangular, their diagonal parts $l_+,\widetilde{l}_+$ and $l_-,\widetilde{l}_-$ being inverse to each other: $l_+l_- = \widetilde{l}_+\widetilde{l}_- = 1$.

Let all of the matrices be in the fundamental representation V of the group G ($N \times N$ matrices for the $SL(N)$ group). Then the algebra of functions on the group D is generated by the matrix elements x_{ij} and y_{ij}. The matrices $L_\pm$ and U or $\widetilde{L}_\pm$ and $\widetilde{U}$ can be considered as almost everywhere regular functions of x and y. Therefore, the matrix elements $L_{\pm ij}$ and U_{ij} (or $\widetilde{L}_{\pm ij}$ and $\widetilde{U}_{ij}$) define another system of generators of the algebra $FunD$. We define the Poisson structure on the group D in terms of the generators $L_\pm$ and U as follows [8, 9]

$$\{U^1, U^2\} = \gamma[r_\pm, U^1U^2] \tag{3.18}$$

$$\begin{aligned}
\{L_+^1, L_+^2\} &= \gamma[r_\pm, L_+^1 L_+^2] \\
\{L_-^1, L_-^2\} &= \gamma[r_\pm, L_-^1 L_-^2] \\
\{L_+^1, L_-^2\} &= \gamma[r_+, L_+^1 L_-^2]
\end{aligned} \tag{3.19}$$

$$\begin{aligned}
\{L_+^1, U^2\} &= \gamma r_+ L_+^1 U^2 \\
\{L_-^1, U^2\} &= \gamma r_- L_-^1 U^2
\end{aligned} \tag{3.20}$$

Here γ is an arbitrary complex parameter, $r_\pm$ are classical r-matrices which satisfy the classical Yang-Baxter equation and the following relations

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0 \tag{3.21}$$

$$r_- = -Pr_+P, \qquad r_+ - r_- = C \tag{3.22}$$

where P is a permutation in the tensor product $V \otimes V$ ($Pa \otimes b = b \otimes a$). For the $SL(N)$ group the solution of eqs.(3.21-3.22) looks as follows

$$\begin{aligned}
r_+ &= \sum_{i=1}^{N-1} h_i \otimes h_i + 2\sum_{i<j}^{N} e_{ij} \otimes e_{ji} \\
&= -\frac{1}{N} I + \sum_{i=1}^{N} e_{ii} \otimes e_{ii} + 2\sum_{i<j}^{N} e_{ij} \otimes e_{ji}
\end{aligned} \tag{3.23}$$

where $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$ and h_i form an orthonomal basis of the Cartan subalgebra of the $SL(N)$ group: $\sqrt{i(i+1)}h_i = \sum_{k=1}^{i} e_{kk} - ie_{i+1,i+1}$.
In eq.(3.21) using the matrix $r = \sum_a r_1(a) \otimes r_2(a)$ acting in the space $V \otimes V$ one constructs matrices $r^{12} = \sum_a r_1(a) \otimes r_2(a) \otimes id$, $r^{13} = \sum_a r_1(a) \otimes id \otimes r_2(a)$ and $r^{23} = \sum_a id \otimes r_1(a) \otimes r_2(a)$ acting in the space $V \otimes V \otimes V$.

The group D endowed with the Poisson structure (3.18-3.20) is called the Heisenberg double D_+^γ of the group G. It is not difficult to show that the matrices $\widetilde{L}_\pm$ and $\widetilde{U}$ have the same Poisson structure (3.18-3.20) and we shall need the Poisson brackets of $L_\pm$, U and $\widetilde{L}_\pm$, $\widetilde{U}$ [11]

$$\begin{aligned}
\{L_\alpha^1, \widetilde{L}_\beta^2\} &= 0 \qquad for \quad any \quad \alpha, \beta = +, - \\
\{\widetilde{L}_\pm^1, U^2\} &= -\gamma \widetilde{L}_\pm^1 U^2 r_\pm \\
\{L_\pm^1, \widetilde{U}^2\} &= -\gamma L_\pm^1 \widetilde{U}^2 r_\pm \\
\{U^1, \widetilde{U}^2\} &= 0
\end{aligned} \tag{3.24}$$

The cotangent bundle of the group G can be considered as a limiting case of the Heisenberg double. Namely, in the limit $\gamma \to 0$ and $L_\pm \to 1 + \gamma E_\pm$, $E = E_+ - E_-$ the Poisson structure of the Heisenberg double coincides with the canonical Poisson structure of the cotangent bundle T^*G.

Now we are ready to discuss the deformation of gauge models on graphs. The case of imaginary γ was considered in ref.[2] for gauge models on arbitrary graphs. The real γ case is more complicated and it seems to be possible to get a proper deformation only for gauge models on a standard graph. However it does not mean any loss of information because as was shown in preceding section gauge models on arbitrary graphs are equivalent to gauge models on standard graphs.

We begin with the Heisenberg double of the complex $SL(N)$ group and discuss the equation which singles out the real $SU(N)$ form later on. So let us place on each link of a standard graph with g links a Heisenberg double. The phase space is thus the direct product of Heisenberg doubles over all links $\prod_{links} D_+^\gamma$. Then one should replace the Gauss-law constraints (2.13) by some first-class constraints which reduce to the form (2.13) in the limit $\gamma \to 0$. We use the following constraints

$$\begin{aligned} G_\pm &= \widetilde{L}_\pm(1)L_\pm(1)\widetilde{L}_\pm(2)L_\pm(2)\cdots\widetilde{L}_\pm(g)L_\pm(g) = \\ &= G_\pm(1)G_\pm(2)\cdots G_\pm(g) = 1 \end{aligned} \tag{3.25}$$

and we introduced a natural notation $G_\pm(i) \equiv \widetilde{L}_\pm(i)L_\pm(i)$. It is not difficult to verify that these constraints have the following Poisson brackets

$$\begin{aligned} \{G_+^1, G_+^2\} &= \gamma[r_\pm, G_+^1 G_+^2] \\ \{G_-^1, G_-^2\} &= \gamma[r_\pm, G_-^1 G_-^2] \\ \{G_+^1, G_-^2\} &= \gamma[r_+, G_+^1 G_-^2] \end{aligned} \tag{3.26}$$

These Poisson brackets vanish on the constraints surface $G_\pm = 1$ and therefore they are first-class constraints. It is useful to consider instead of two constraints G_+ and G_- one constraint $G = G_-^{-1}G_+ = 1$. This constraint satisfies the following quadratic Poisson algebra

$$\frac{1}{\gamma}\{G^1, G^2\} = G^1 r_+ G^2 - G^2 G^1 r_+ - r_- G^2 G^1 + G^2 r_- G^1 \tag{3.27}$$

In the limit $\gamma \to 0$, $L_\pm \to 1 + \gamma E_\pm$ it reduces to the usual Gauss-law constraint (2.13).

So we have defined the deformed phase space and Gauss-law constraints and now one should just remember that the same phase space and constraints recently appeared in [12, 13] where the relation between the Heisenberg double and the symplectic structure of the moduli space of flat connections on a Riemann surface was studied. Namely it was shown in [12, 13] that there is such a change of variables that the Poisson structure in terms of the new variables coincides with the Poisson structure which was introduced by Fock and Rosly [14] to describe the moduli space. For reader's convinience we present here the corresponding formulas in our notations.

So let us consider the following change of variables [12, 13]

$$\begin{aligned} &A_i = W_i^{-1} L_-^{-1}(i) U(i) L_+(i) W_i \\ &B_i = W_i^{-1} L_-^{-1}(i) L_+(i) W_i \\ &W_i = G_+(i+1)\cdots G_+(g), \quad W_g = 1 \end{aligned} \tag{3.28}$$

The Gauss-law constraint $G = G_-^{-1}G_+ = 1$ can be expressed through the new fields A_i and B_i as follows

$$\begin{aligned} G^{-1} = G_+^{-1}G_- &= A_1^{-1}B_1A_1B_1^{-1}A_2^{-1}B_2A_2B_2^{-1}\cdots A_g^{-1}B_gA_gB_g^{-1} = \\ &= \prod_{i=1}^{g} A_i^{-1}B_iA_iB_i^{-1} = 1 \end{aligned} \tag{3.29}$$

Eq.(3.29) is the defining relation for the holonomies A_i and B_i of a flat connection along the cycles a_i and b_i of a Riemann surface with g handles. Using the Poisson structure of the Heisenberg double one can easily calculate the Poisson structure of the fields A_i and B_i

$$\begin{aligned} & i = 1,\cdots,g \\ \frac{1}{\gamma}\{A_i^1, A_i^2\} &= A_i^1r_+A_i^2 - A_i^2A_i^1r_+ - r_-A_i^2A_i^1 + A_i^2r_-A_i^1 \\ \frac{1}{\gamma}\{B_i^1, B_i^2\} &= B_i^1r_+B_i^2 - B_i^2B_i^1r_+ - r_-B_i^2B_i^1 + B_i^2r_-B_i^1 \\ \frac{1}{\gamma}\{A_i^1, B_i^2\} &= A_i^1r_+B_i^2 - B_i^2A_i^1r_+ - r_+B_i^2A_i^1 + B_i^2r_-A_i^1 \\ & i < j \\ \frac{1}{\gamma}\{A_i^1, A_j^2\} &= A_i^1r_+A_j^2 - A_j^2A_i^1r_+ - r_+A_j^2A_i^1 + A_j^2r_+A_i^1 \\ \frac{1}{\gamma}\{A_i^1, B_j^2\} &= A_i^1r_+B_j^2 - B_j^2A_i^1r_+ - r_+B_j^2A_i^1 + B_j^2r_+A_i^1 \\ \frac{1}{\gamma}\{B_i^1, B_j^2\} &= B_i^1r_+B_j^2 - B_j^2B_i^1r_+ - r_+B_j^2B_i^1 + B_j^2r_+B_i^1 \\ \frac{1}{\gamma}\{B_i^1, A_j^2\} &= B_i^1r_+A_j^2 - A_j^2B_i^1r_+ - r_+A_j^2B_i^1 + A_j^2r_+B_i^1 \end{aligned} \tag{3.30}$$

The Poisson structure (3.30) coincides with the structure which was introduced in [14] for the description of the moduli space of flat $SL(N)$ connections on a Riemann surface with g handles. The Gauss-law constraint (3.29) generates the following gauge transformations

$$A_i \to gA_ig^{-1}, \quad B_i \to gB_ig^{-1} \tag{3.31}$$

where the gauge parameters g depend on the constraint G.
From the point of view of the theory of Poisson-Lie groups one should regard the gauge parameters g as belonging to a Poisson-Lie group. Then eq.(3.31) defines an action of the Poisson-Lie group on the Poisson algebra (3.30) by means of so-called dressing transformations [6, 9].

We have considered up to now only the case of complex $SL(N)$ group. However for physical applications one has to single out the $SU(N)$ real form. It can be done by means of the following condition which seems to be unknown before

$$A_i^* = G_+A_i^{-1}G_+^{-1}, \quad B_i^* = G_+B_i^{-1}G_+^{-1} \tag{3.32}$$

Here A^* is a matrix hermitian-conjugated to A.

It is of no problem to check that this condition is compatible with the Poisson structure (3.30) and with the Gauss-law constraint (3.21). The corresponding anti-automorphism

$$\rho(A_i) = G_+ A_i^{-1} G_+^{-1}, \quad \rho(B_i) = G_+ B_i^{-1} G_+^{-1} \tag{3.33}$$

is not an anti-involution of the Poisson algebra (3.30).
It would be interesting to compare this anti-automorphism with the involution introduced in [19] to quantize the moduli space. Let us note that on the constraints surface $G_\pm = 1$ the condition (3.32) is the standard involution which singles out the $SU(N)$ group.

So we have shown that the γ-deformed physical phase space of a gauge model on a graph coincides with the moduli space of flat connections on a Riemann surface and is compact for the $SU(N)$ group. It is well-known [15] that the same moduli space is a physical phase space of the (2+1)-dimensional Chern-Simons theory, the parameter γ being identified with $\frac{2\pi}{k}$. The Chern-Simons parameter k is required to be integer for the $SU(N)$ group. Due to this relation all correlation functions of the lattice Yang-Mills theory may be expressed through nonlocal correlation functions of the Chern-Simons theory. Quantization of the physical phase space leads to a finite-dimensional Hilbert space which can be identified with the space of conformal blocks of the WZNW model [15, 16] .

4 Conclusion

In this paper the structure of the physical phase space of gauge models on graphs was studied. Any graph was shown to be gauge equivalent to a standard graph and the reduction procedure to the standard graph was described.

The deformation of gauge models on standard graphs based on the assignment of a Heisenberg double to each link was discussed. The physical phase space of the deformed $SL(N)$ gauge model was proved to coincide with the moduli space of flat $SL(N)$ connections on a Riemann surface and an equation (3.35) which singles out the moduli space of flat $SU(N)$ connections was found. As is well-known the same moduli space is the physical phase space of the (2+1)-dimensional Chern-Simons model. By this reason all correlation functions of deformed gauge models can be expressed through nonlocal correlation functions of the Chern-Simons model.

In quantum theory the physical Hilbert space of the Chern-Simons model is known [15] to be finite-dimensional and can be identified with the space of conformal blocks of the WZNW model. It would be interesting by using the relation to the WZNW model to reformulate the eigenvalue problem for the reduced Yang-Mills Hamiltonian in terms of conformal field theory.

The finite-dimensionality of the Hilbert space gives also a possibility to develop a weak-coupling expansion which differs from the asymptotic expansion of the standard perturbation theory. The main contribution in the weak-coupling expansion is given by the magnetic part H_m of the Yang-Mills Hamiltonian. Although the form of the Hamiltonian H_m depends on the space dimension, it is obvious that any Hamiltonian H_m describes an integrable system and, moreover, all of them belongs to the same integrable hierarchy. So, the first step in the weak-coupling expansion is to solve the corresponding integrable systems. It seems to be possible to carry out at least in two space dimensions due to the factorization (2.15) of H_m.

It is worthwhile to note that such a deformation of lattice Yang-Mills theory leads to some kind of duality between magnetic and electric fields. The A and B variables come in the algebra (3.30) on the same footing and play the role of magnetic and electric fields correspondingly. This duality seems to generalize the well-known Kramers-Wigner duality and implies that there may exist a relation between the strong- and weak-coupling expansions.

In this paper we considered only the pure Yang-Mills theory without matter fields. It would be very interesting to include fermions in the consideration.

The Poisson algebra (3.30) can be easily quantized and one gets the quadratic algebra which was introduced in [19] to quantize the Chern-Simons theory. The classical r-matrices $r_\pm$ are to be replaced by the R-matrices $R_\pm(q) = 1 + i\hbar\gamma r_\pm + \cdots$, where $q = e^{i\hbar\gamma}$. It is of no problem to check that in quantum theory the Gauss-law constraints are first-class constraints and one can construct quantum Hamiltonians which commute with the Gauss-law constraints . However the representation theory of the quantized algebra is at present unknown.

Let us note that q has a nonpolynomial dependence on the Planck constant $\hbar$ and thus already "tree" correlation functions of the models will have a nonpolynomial dependence on $\hbar$ as well. It seems to be an indication that correlation functions of the models correspond to a summation over infinitely-many number of the usual Feynman diagrams. It is not excluded that the parameter γ plays the role of an infrared cut-off. Due to the fact that there is the additional parameter γ for the models one may expect that these models have more rich phase structure than the usual lattice gauge theory.

Let us finally notice that q-deformed lattice gauge theory was considered in refs.[14, 20, 19, 21] in connection with the Chern-Simons theory.

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