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The centre of the graph and moduli algebras at roots of 1.

S.A.Frolov*

Section Physik, Munich University
Theresienstr.37, 80333 Munich, Germany [†]

Abstract

The structure of the centres $\mathcal{Z}(\mathcal{L}_g)$ and $\mathcal{Z}(\mathcal{M}_g)$ of the graph algebra $\mathcal{L}_g(sl_2)$ and the moduli algebra $\mathcal{M}_g(sl_2)$ is studied at roots of 1. It is shown that $\mathcal{Z}(\mathcal{L}_g)$ can be endowed with the structure of the Poisson graph algebra. The elements of $Spec(\mathcal{Z}(\mathcal{M}_g))$ are shown to satisfy the defining relation for the holonomies of a flat connection along the cycles of a Riemann surface. The irreducible representations of the graph algebra are constructed.

1 Introduction

The Poisson structure of the moduli space of flat connections on a Riemann surface with g handles can be described by means of a quadratic Poisson algebra, which was introduced by Fock and Rosly [1] and here will be called the Poisson graph algebra. Let us remind the definition of the algebra [1].

Let G be a matrix algebraic group and $D_g = G^{\times 2g}$. An arbitrary element d of D_g is parametrized by matrices A_i and B_i as $d = (A_1, B_1, \dots, A_g, B_g) \in D_g$. Let all of the

*Alexander von Humboldt fellow

[†]Permanent address: Steklov Mathematical Institute, Vavilov st.42, GSP-1, 117966 Moscow, RUSSIA

matrices be in the fundamental representation of the group G . Then the algebra of functions on D_g is generated by the matrix elements $(A_i)_{mn}$ and $(B_i)_{mn}$.

Definition 1. The Poisson graph algebra $\mathcal{PL}_g$ is an algebra of regular functions on D_g with the following Poisson structure

$$\begin{aligned}
& i = 1, \dots, g \\
& \frac{k}{2\pi} \{A_i^1, A_i^2\} = A_i^1 r_+ A_i^2 - A_i^2 A_i^1 r_+ - r_- A_i^2 A_i^1 + A_i^2 r_- A_i^1 \\
& \frac{k}{2\pi} \{B_i^1, B_i^2\} = B_i^1 r_+ B_i^2 - B_i^2 B_i^1 r_+ - r_- B_i^2 B_i^1 + B_i^2 r_- B_i^1 \\
& \frac{k}{2\pi} \{A_i^1, B_i^2\} = A_i^1 r_+ B_i^2 - B_i^2 A_i^1 r_+ - r_+ B_i^2 A_i^1 + B_i^2 r_- A_i^1 \\
& i < j \\
& \frac{k}{2\pi} \{A_i^1, A_j^2\} = A_i^1 r_+ A_j^2 - A_j^2 A_i^1 r_+ - r_+ A_j^2 A_i^1 + A_j^2 r_+ A_i^1 \\
& \frac{k}{2\pi} \{A_i^1, B_j^2\} = A_i^1 r_+ B_j^2 - B_j^2 A_i^1 r_+ - r_+ B_j^2 A_i^1 + B_j^2 r_+ A_i^1 \\
& \frac{k}{2\pi} \{B_i^1, B_j^2\} = B_i^1 r_+ B_j^2 - B_j^2 B_i^1 r_+ - r_+ B_j^2 B_i^1 + B_j^2 r_+ B_i^1 \\
& \frac{k}{2\pi} \{B_i^1, A_j^2\} = B_i^1 r_+ A_j^2 - A_j^2 B_i^1 r_+ - r_+ A_j^2 B_i^1 + A_j^2 r_+ B_i^1
\end{aligned} \tag{1.1}$$

Here k is an arbitrary complex parameter, $r_{\pm}$ are classical r -matrices which satisfy the classical Yang-Baxter equation and the following relations

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0 \tag{1.2}$$

$$r_- = -Pr_+P, \quad r_+ - r_- = C \tag{1.3}$$

where P is a permutation in the tensor product $V \otimes V$ ($Pa \otimes b = b \otimes a$).

In eq.(1.1-1.3) we use the standard notations from the theory of quantum groups [2, 3]: for any matrix A acting in some space V one can construct two matrices $A^1 = A \otimes id$ and $A^2 = id \otimes A$ acting in the space $V \otimes V$ and for any matrix $r = \sum_a r_1(a) \otimes r_2(a)$ acting in the space $V \otimes V$ one constructs matrices $r^{12} = \sum_a r_1(a) \otimes r_2(a) \otimes id$, $r^{13} = \sum_a r_1(a) \otimes id \otimes r_2(a)$ and $r^{23} = \sum_a id \otimes r_1(a) \otimes r_2(a)$ acting in the space $V \otimes V \otimes V$. The matrix C is the tensor Casimir operator of the Lie algebra $\mathcal{G}$ of the group G : $C = -\eta_{ab} \lambda^a \otimes \lambda^b$, η_{ab} is the Killing tensor and λ^a form a basis of $\mathcal{G}$.

Let us now identify the matrices A_i and B_i with holonomies of a flat connection along the cycles a_i and b_i of a Riemann surface with g handles. Then A_i and B_i should satisfy the following defining relations

$$M = B_g A_g^{-1} B_g^{-1} A_g \cdots B_1 A_1^{-1} B_1^{-1} A_1 = 1 \tag{1.4}$$

These relations can be regarded as first-class constraints imposed on the variables of D_g . The gauge transformations generated by these constraints are just the simultaneous conjugations of A_i and B_i

$$A_i \rightarrow h A_i h^{-1}, \quad B_i \rightarrow h B_i h^{-1}$$

Let us now consider two Poisson subalgebras of $\mathcal{PL}_g$. The first subalgebra $\mathcal{I}$ consists of all of the functions vanishing on the constraints surface:

$$\mathcal{I} = \{f \in \mathcal{PL}_g : f|_{M=1} = 0\}$$

and the second subalgebra $\mathcal{F}_{\mathcal{I}}$ is the maximal subalgebra of $\mathcal{P}\mathcal{L}_g$ such that the subalgebra $\mathcal{I}$ is a Poisson ideal of $\mathcal{F}_{\mathcal{I}}$:

$$\mathcal{F}_{\mathcal{I}} = \{f \in \mathcal{P}\mathcal{L}_g : \{f, h\} \in \mathcal{I} \quad \forall h \in \mathcal{I}\}.$$

In particular it is not difficult to check that any function f which is invariant with respect to simultaneous conjugations of A_i and B_i

$$f(hA_1h^{-1}, \dots, hB_g h^{-1}) = f(A_1, \dots, B_g)$$

belongs to $\mathcal{F}_{\mathcal{I}}$.

Definition 2. The Poisson algebra of functions on the moduli space of flat connections on a Riemann surface with g handles or the Poisson moduli algebra $\mathcal{P}\mathcal{M}_g$ is defined as a quotient of the algebra $\mathcal{F}_{\mathcal{I}}$ over the ideal $\mathcal{I}$

$$\mathcal{P}\mathcal{M}_g = \mathcal{F}_{\mathcal{I}}/\mathcal{I}.$$

It was shown by Fock and Rosly that the algebra $\mathcal{P}\mathcal{M}_g$ coincides with the canonical Poisson algebra of functions on the moduli space defined by the Atiyah-Bott symplectic structure.

Quantization of the Poisson graph algebra leads to an associative algebra which was introduced in [4] (see also [5]). In the present paper we study the structure of the centre of the quantized graph and moduli algebras for the simplest case of $SL(2)$ group and, in what follows, present definitions and results only for this case. Our definition of the moduli algebra differs from the definition given in [4], where the truncated case was considered, and can be regarded as the standard one from quantum theory of constraints systems.

The plan of the paper is as follows. In the second section we introduce the graph and moduli algebras. Then we describe an extension of the graph algebra and the isomorphism between the extended graph algebra $\mathcal{L}_g^*$ and the tensor product of g copies of $\mathcal{L}_1^*$ [6, 7]. In the third section we study the centre $\mathcal{Z}(\mathcal{L}_1)$ of $\mathcal{L}_1$ and prove that $\mathcal{Z}(\mathcal{L}_1)$ is isomorphic to $\mathcal{P}\mathcal{L}_1$. In the forth section, using the isomorphism mentioned in the second section, we generalize the results obtained in the third section to $\mathcal{L}_g$ and show that the elements of $Spec(\mathcal{Z}(\mathcal{M}_g(sl_2)))$ satisfy the defining relation (1.4). In the fifth section the irreducible representations of $\mathcal{L}_g$ are constructed. In Conclusion we discuss unsolved problems.

2 Graph and moduli algebras

Definition 3. The graph algebra $\mathcal{L}_g(sl_2)$ is an associative algebra with unit element, generated by matrix elements of $A_i, B_i \in End C^2 \otimes \mathcal{L}_g(sl_2)$, $i = 1, \dots, g$ and $(A_i)_{11}^{-1}$, $(B_i)_{11}^{-1}$, M_{11}^{-1} which are subject to the following relations

$$\begin{aligned} i = 1, \dots, g \\ A_i^1 R_+ A_i^2 R_+^{-1} = R_- A_i^2 R_-^{-1} A_i^1, \quad B_i^1 R_+ B_i^2 R_+^{-1} = R_- B_i^2 R_-^{-1} B_i^1 \\ A_i^1 R_+ B_i^2 R_+^{-1} = R_+ B_i^2 R_-^{-1} A_i^1 \end{aligned} \tag{2.5}$$

$$\begin{aligned}
& i < j \\
& A_i^1 R_+ A_j^2 R_+^{-1} = R_+ A_j^2 R_+^{-1} A_i^1, \quad A_i^1 R_+ B_j^2 R_+^{-1} = R_+ B_j^2 R_+^{-1} A_i^1 \\
& B_i^1 R_+ B_j^2 R_+^{-1} = R_+ B_j^2 R_+^{-1} B_i^1, \quad B_i^1 R_+ A_j^2 R_+^{-1} = R_+ A_j^2 R_+^{-1} B_i^1 \\
& (A_i)_{11} (A_i)_{11}^{-1} = (A_i)_{11}^{-1} (A_i)_{11} = (B_i)_{11} (B_i)_{11}^{-1} = (B_i)_{11}^{-1} (B_i)_{11} = 1 \\
& M_{11}^{-1} (q^{-3g} B_g A_g^{-1} B_g^{-1} A_g \cdots B_1 A_1^{-1} B_1^{-1} A_1)_{11} = \\
& (q^{-3g} B_g A_g^{-1} B_g^{-1} A_g \cdots B_1 A_1^{-1} B_1^{-1} A_1)_{11} M_{11}^{-1} = 1 \\
& \det_q A_i = (A_i)_{11} (A_i)_{22} - q^2 (A_i)_{21} (A_i)_{12} = 1 \\
& \det_q B_i = (B_i)_{11} (B_i)_{22} - q^2 (B_i)_{21} (B_i)_{12} = 1
\end{aligned} \tag{2.6}$$

and we denote by 1 the unit element of any algebra throughout the paper.

Here $R_{\pm}$ -matrices

$$R_+ = q^{-\frac{1}{2}} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \quad R_- = q^{\frac{1}{2}} \begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q^{-1} - q & 1 & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}$$

satisfy the quantum Yang-Baxter equation and the following relations

$$\begin{aligned}
& R^{12} R^{13} R^{23} = R^{23} R^{13} R^{12} \\
& R_+ = P R_-^{-1} P, \quad R_{\pm}(q) = 1 + \frac{2\pi i}{k} \hbar r_{\pm} + O(\hbar^2) \\
& q = \exp\left(\frac{2\pi i}{p} \hbar\right), \quad p = k + 2\hbar
\end{aligned}$$

We have introduced the element M_{11}^{-1} because the Poisson structure (1.1) is degenerate on the surface $M_{11} = 0$ and because, as we will see in the fifth section, if the element M_{11} is invertible then all irreducible representations of the graph algebra are p^{3g} -dimensional.

Using the explicit expressions for $R_{\pm}$ one can write down the commutation relations (2.6) in terms of elements $(A_i)_{mn}$ and $(B_i)_{mn}$. The corresponding formulas are given in Appendix.

It is well known that for fixed index i the algebra generated by matrix elements of A_i (or B_i) is isomorphic to $U_q(sl_2)$ [8] and the standard generators of $U_q(sl_2)$ can be expressed through $(A_i)_{mn}$ as follows

$$K = (A_i)_{11}, \quad X_+ = -\frac{q^{\frac{1}{2}}}{q - q^{-1}} (A_i)_{21}, \quad X_- = -\frac{q^{\frac{1}{2}}}{q - q^{-1}} (A_i)_{11}^{-1} (A_i)_{12}$$

The Casimir element of $U_q(sl_2)$ is equal to

$$c_i = \text{tr}_q A_i = q^{-1} (A_i)_{11} + q (A_i)_{22}$$

However the Casimir elements c_i are not central elements of the graph algebra and, moreover, for generic values of q the graph algebra has a trivial centre.

It is not difficult to show that for fixed index i the algebra generated by A_i and B_i is isomorphic to the quantized algebra of functions on the Heisenberg double of a Lie group [9, 10, 11, 12].

Let us introduce monodromies M_i

$$M_i = q^{-3(g-i+1)} B_g A_g^{-1} B_g^{-1} A_g \cdots B_i A_i^{-1} B_i^{-1} A_i$$

The monodromies satisfy the following commutation relations

$$\begin{aligned} i &\leq j \\ M_i^1 R_+ M_j^2 R_+^{-1} &= R_- M_j^2 R_-^{-1} M_i^1, \quad M_i^1 R_+ A_j^2 R_+^{-1} = R_- A_j^2 R_-^{-1} M_i^1 \\ M_i^1 R_+ B_j^2 R_+^{-1} &= R_- B_j^2 R_-^{-1} M_i^1 \\ i &< j \\ A_i^1 R_+ M_j^2 R_+^{-1} &= R_+ M_j^2 R_+^{-1} A_i^1, \quad B_i^1 R_+ M_j^2 R_+ = R_+ M_j^2 R_+^{-1} B_i^1 \\ \det_q M_i &= 1 \end{aligned} \tag{2.7}$$

Using the Gauss decomposition one can represent the monodromies as follows

$$M_i = m_-^{-1}(i) K_i m_+(i)$$

where $m_{\pm}(i)$ are upper- and lower-triangular matrices with the unity on the diagonals and K_i are diagonal matrices. It follows from relations (2.7) that the matrix elements of K_i form a commutative subalgebra of $\mathcal{L}_g$.

Let $\mathcal{L}_g^*$ be an extension of the graph algebra $\mathcal{L}_g$ by means of the elements $Q_i^{\pm 1} = K_i^{\pm \frac{1}{2}}$. The relations (2.7) and the commutation relations of Q_i are presented in components in Appendix. Then the following proposition is a refinement of some results from [6, 7]:

Proposition 1. The extended graph algebra $\mathcal{L}_g^*$ is isomorphic to the tensor product of g copies of the extended graph algebra $\mathcal{L}_1^*$. The isomorphism is given by means of the following formulas

$$A_i = M_+^{-1}(i+1) \bar{A}_i M_+(i+1), \quad B_i = M_+^{-1}(i+1) \bar{B}_i M_+(i+1) \tag{2.8}$$

where

$$\begin{aligned} M_+(i) &= Q_i m_+(i), \quad M_-(i) = Q_i^{-1} m_-(i) \\ M_{\pm}(i) &= \bar{G}_{\pm}(i) \bar{G}_{\pm}(i+1) \cdots \bar{G}_{\pm}(g) \\ \bar{G}_i &= q^{-3} \bar{B}_i \bar{A}_i^{-1} \bar{B}_i^{-1} \bar{A}_i \end{aligned}$$

Remark 1. The Proposition 1 is valid for arbitrary graph algebra. In the case of $SL(2)$ group the element $(K_i)_{11} = (M_i)_{11} = (\bar{G}_i)_{11} \cdots (\bar{G}_g)_{11}$, all elements $(M_i)_{11}$ and $(\bar{G}_i)_{11}$ are invertible due to invertibility of M_{11} and one can easily show that formulas (2.8) define the isomorphism of $\mathcal{L}_g(sl_2)$ and $\mathcal{L}_1^{\otimes g}$.

There is no natural anti-involution of the graph algebra. However, the following anti-automorphism [13] plays the role of $*$ -operation on $\mathcal{L}_g$:

$$\rho(A_i) = M_+ A_i^{-1} M_+^{-1}, \quad \rho(B_i) = M_+ B_i^{-1} M_+^{-1} \tag{2.9}$$

The square of the anti-automorphism is equal to

$$\rho^2(A_i) = M A_i M^{-1}, \quad \rho^2(B_i) = M B_i M^{-1}$$

It is worthwhile to note that this anti-automorphism acts on the elements M_{ij} as follows

$$\rho(M_{\pm}) = M_{\mp}$$

Let us introduce a set Φ of quantum constraints

$$\Phi_{ij} = M_{ij} - \delta_{ij} \quad (2.10)$$

where M_{ij} are components of $M = M_1$.

Let us consider the left and right ideals of the graph algebra, generated by the set Φ and let $\mathcal{F}_{\mathcal{I}_L}$ ($\mathcal{F}_{\mathcal{I}_R}$) be the maximal subalgebra of $\mathcal{L}_g$ such that $\mathcal{I}_L$ ($\mathcal{I}_R$) is a two-sided ideal of $\mathcal{F}_{\mathcal{I}_L}$ ($\mathcal{F}_{\mathcal{I}_R}$). Let $\mathcal{F}_{\mathcal{I}}$ be the intersection of $\mathcal{F}_{\mathcal{I}_L}$ and $\mathcal{F}_{\mathcal{I}_R}$:

$$\mathcal{F}_{\mathcal{I}} = \{f \in \mathcal{L}_g : \mathcal{I}_L f \subset \mathcal{I}_L \text{ and } f \mathcal{I}_R \subset \mathcal{I}_R\}.$$

It is obvious that $\mathcal{F}_{\mathcal{I}}$ is a subalgebra of $\mathcal{L}_g$ and the elements $\Phi_{ij} \in \mathcal{F}_{\mathcal{I}}$.

Definition 4. Let $\mathcal{I}$ be the intersection of the algebra $\mathcal{F}_{\mathcal{I}}$ with the union of $\mathcal{I}_L$ and $\mathcal{I}_R$. The moduli algebra $\mathcal{M}_g$ is defined as a quotient of the algebra $\mathcal{F}_{\mathcal{I}}$ over the ideal $\mathcal{I}$

$$\mathcal{M}_g = \mathcal{F}_{\mathcal{I}} / \mathcal{I} \quad (2.11)$$

Some comments are in order. It is clear that $\mathcal{I}$ is a two-sided ideal of $\mathcal{F}_{\mathcal{I}}$. It follows from the action of the anti-automorphism ρ on the elements M_{ij} that $\rho(\mathcal{I}_L) = \mathcal{I}_R$ and $\rho(\mathcal{I}_R) = \mathcal{I}_L$ and, hence if $f \in \mathcal{F}_{\mathcal{I}}$ and $i \in \mathcal{I}$ then $\rho(f) \in \mathcal{F}_{\mathcal{I}}$ and $\rho(i) \in \mathcal{I}$. Thus the moduli algebra inherits the anti-automorphism ρ from $\mathcal{L}_g$. Moreover it is possible to show that, being restricted to the moduli algebra, the anti-automorphism becomes an anti-involution of $\mathcal{M}_g$.

Our aim is to describe the structure of the centre $\mathcal{Z}(\mathcal{L}_g)$ of the graph algebra at roots of 1, and in the remainder of the paper we assume that q is a primitive p^{th} root of unity, p being odd. Due to the Proposition 1 we can begin with the study of the centre $\mathcal{Z}(\mathcal{L}_1)$ of the algebra $\mathcal{L}_1$.

3 The centre of the graph algebra $\mathcal{L}_1(sl_2)$

Let a_{ij} , b_{ij} and M_{11}^{-1} be the generators of $\mathcal{L}_1(sl_2)$ with the commutation relations given by formulas (6.23) from the Appendix.

Proposition 2. 1) The centre $\mathcal{Z}(\mathcal{L}_1)$ of $\mathcal{L}_1(sl_2)$ is generated by the elements $a_{11}^{\pm p}$, a_{12}^p , a_{21}^p , $b_{11}^{\pm p}$, b_{12}^p, b_{21}^p and M_{11}^{-p} , subject to the single relation

$$M_{11}^{-p}(\mathcal{B}\mathcal{A}^{-1}\mathcal{B}^{-1}\mathcal{A})_{11} = 1 \quad (3.12)$$

where $\mathcal{A}_{ij} = a_{ij}^p$, $\mathcal{B}_{ij} = b_{ij}^p$ for $i, j \neq 2$ simultaneously and $\det \mathcal{A} = \det \mathcal{B} = 1$.

2) The algebra $\mathcal{L}_1(sl_2)$ is a free $\mathcal{Z}(\mathcal{L}_1)$ -module with basis the set of monomials

$$a_{11}^{r_1} a_{12}^{s_1} a_{21}^{t_1} b_{11}^{r_2} b_{12}^{s_2} b_{21}^{t_2} \quad \text{and} \quad 0 \leq r_i, s_i, t_i \leq p-1.$$

3) The ring $\mathcal{L}_1(sl_2)$ is an integral domain.

Proof. Let us introduce a new set of generators of $\mathcal{L}_1(sl_2)$ by means of the following formulas

$$X_1 = (a_{11}^{-1} a_{12} - b_{11}^{-1} b_{12}) b_{11}^2, \quad X_2 = a_{11} a_{21} b_{11}^{-2}$$

$$X_3 = a_{11}^{-2}b_{12}b_{11}, \quad X_4 = (b_{21}b_{11}^{-1} - a_{21}a_{11}^{-1})a_{11}^2$$

In terms of these generators the relations (6.23) from the Appendix take the form

$$\begin{aligned} a_{11}b_{11} &= qb_{11}a_{11}, & a_{11}X_i &= X_ia_{11}, & b_{11}X_i &= X_ib_{11} \\ X_1X_4 &= q^2X_4X_1, & X_iX_{i+2} &= q^{-2}X_{i+2}X_i, & i &= 1, 2 \\ X_iX_{i+1} &= q^2X_{i+1}X_i + q^2 - 1, & i &= 1, 2, 3 \\ M_{11}^{-1}X_1 &= q^2X_1M_{11}^{-1}, & M_{11}^{-1}X_3 &= q^2X_3M_{11}^{-1} \\ M_{11}^{-1}X_2 &= q^{-2}X_2M_{11}^{-1}, & M_{11}^{-1}X_4 &= q^{-2}X_4M_{11}^{-1} \\ M_{11}^{-1}q^{-2}(1 + X_1X_2 + X_1X_4 + X_3X_4 + X_1X_2X_3X_4) &= 1 \end{aligned} \quad (3.13)$$

It is now of no problem to show that the elements M_{11}^{-p} , $a_{11}^{\pm p}$, $b_{11}^{\pm p}$ and X_i^p generate the centre of $\mathcal{L}_1(sl_2)$. To do this one should use the following lemma, which can be easily proved by induction

Lemma 1. Let elements c , Z and W satisfy the relations $cZ = Zc$, $cW = Wc$, $ZW = q^2WZ + c$, then the following relation is valid

$$Z^mW^n = \sum_{k=0}^m q^{2(m-k)(n-k)} c^k W^{n-k} Z^{m-k} \frac{(n)_q!}{(n-k)_q!} \frac{(m)_q!}{(m-k)_q!(k)_q!}$$

where $m \leq n$, $(n)_q = \frac{1-q^{2n}}{1-q^2}$.

The first part of the Proposition 2 follows now from the simple relations between the elements a_{ij}^p , b_{ij}^p and X_i^p

$$\begin{aligned} X_1^p &= (a_{11}^{-p}a_{12}^p - b_{11}^{-p}b_{12}^p)b_{11}^{2p}, & X_2^p &= a_{11}^p a_{21}^p b_{11}^{-2p} \\ X_3^p &= a_{11}^{-2p}b_{12}^p b_{11}^p, & X_4^p &= (b_{21}^p b_{11}^{-p} - a_{21}^p a_{11}^{-p})a_{11}^{2p} \end{aligned}$$

which can be proved by using the well-known Lemma

Lemma 2. Let elements a and b satisfy the relation $ab = q^2ba$ or $a(a+b) = q^2(a+b)a$ and $q^p = 1$, then

$$(a+b)^p = a^p + b^p$$

The relation (3.12) will be proved later in this section (see Proposition 4).

The second part of the Proposition 2 follows now from the obvious observation that the products $a_{11}^{\pm r_1} a_{12}^{s_1} a_{21}^{t_1} b_{11}^{\pm r_2} b_{12}^{s_2} b_{21}^{t_2} M_{11}^{-v}$, where $r_i, s_i, t_i, v \in N$, are a basis of $\mathcal{L}_1(sl_2)$.

Relations (3.13) show that $\mathcal{L}_1(sl_2)$ is the tensor product of the Weil algebra, generated by a_{11} and b_{11} , and the algebra $\mathcal{X}$ generated by X_i and M_{11}^{-1} and, therefore, to prove that $\mathcal{L}_1(sl_2)$ is an integral domain, it is enough to show that $\mathcal{X}$ is an integral domain. We have to prove that if $fg = 0$ then either $f = 0$ or $g = 0$.

An arbitrary element $f \in \mathcal{X}$ can be presented in the form

$$f = \sum_{i,l=0}^{\infty} \sum_{j,k=0}^{p-1} f_{ijkl}(M_{11}^{-p}, X_2^p, X_3^p) X_1^i X_2^j X_3^k X_4^l$$

It is obvious that if $f \neq 0$, then $X_1 f \neq 0$. Let us show that $fX_1 = 0$ if and only if $f = 0$. It is clear that it is enough to prove the statement only for elements of the form

$$f = \sum f_{ij} X_1^i X_2^j$$

Using the commutation relation for X_1 and X_2 one gets

$$\begin{aligned} fX_1 &= \sum f_{ij} \left(q^{-2j} X_1^{i+1} X_2^j + (q^{-2j} - 1) X_1^i X_2^{j-1} \right) \\ &= \sum \left(f_{ij} q^{-2j} + (q^{-2(j+1)} - 1) f_{i+1, j+1} \right) X_1^{i+1} X_2^j \end{aligned}$$

This expression is equal to zero only if

$$f_{ij} = (q^{2j} - q^{-2}) f_{i+1, j+1}$$

Let $j = kp + j_0$, $0 \leq j_0 \leq p-1$. By simple induction one gets

$$f_{ij} = (q^{2j_0} - q^{-2})(q^{2(j_0+1)} - q^{-2}) \dots (q^{2(j_0+p-j_0-1)} - q^{-2}) f_{i+p-j_0, (k+1)p} = 0$$

Thus if $fX_1 = 0$ then $f = 0$. In the same manner one can show that if $f \neq 0$ then $fX_4 \neq 0$, $X_4f \neq 0$, $f(1 + X_1X_2) \neq 0$, $(1 + X_1X_2)f \neq 0$.

Let us note that for generic values of q this statement is not valid and the ring $\mathcal{X}$ is not an integral domain.

Let us consider elements $\tilde{f} = X_1^{p-1} X_4^{p-1} (1 + X_1X_2)^{p-1} f$ and $\tilde{g} = g X_1^{p-1} X_4^{p-1} (1 + X_1X_2)^{p-1}$. We have just shown that equation $\tilde{f} = 0$ ($\tilde{g} = 0$) is equivalent to equation $f = 0$ ($g = 0$), therefore it is enough to show that if $\tilde{f}\tilde{g} = 0$ then either $\tilde{f} = 0$ or $\tilde{g} = 0$. Elements $\tilde{f}$ and $\tilde{g}$ can be written in the form

$$\begin{aligned} \tilde{f} &= \sum_{j,l=0}^{\infty} \sum_{i,k=0}^{p-1} f_{ijkl} (M_{11}^{-p}, X_2^p, X_3^p) M_{11}^i X_1^j (1 + X_1X_2)^k X_4^l \\ \tilde{g} &= \sum_{j,l=0}^{\infty} \sum_{i,k=0}^{p-1} g_{ijkl} (M_{11}^{-p}, X_2^p, X_3^p) M_{11}^i X_1^j (1 + X_1X_2)^k X_4^l \end{aligned}$$

The product of these elements is equal to

$$\begin{aligned} \tilde{f}\tilde{g} &= \sum_{j_1, j_2, l_1, l_2=0}^{\infty} \sum_{i_1, i_2, k_1, k_2=0}^{p-1} q^{2i_2(j_1-l_1)-2j_2(k_1+l_1)} f_{i_1 j_1 k_1 l_1} g_{i_2 j_2 k_2 l_2} \times \\ &\quad M_{11}^{i_1+i_2} X_1^{j_1+j_2} (1 + X_1X_2)^{k_1+k_2} X_4^{l_1+l_2} \end{aligned}$$

It is now of no problem to show that $\tilde{f}\tilde{g} = 0$ if and only if either $\tilde{f} = 0$ or $\tilde{g} = 0$.

This proves the Proposition.

One can introduce the Poisson structure on the centre of $\mathcal{L}_1(sl_2)$ by means of the following formula (see, for example, [16])

$$\frac{k}{2\pi} \{x, y\} = \lim_{\kappa \rightarrow q} \frac{xy - yx}{1 - \kappa^{p^2}} \quad (3.14)$$

Let $\mathcal{PL}_g^*$ be the extension of $\mathcal{PL}_g$ by means of the element M_{11}^{-1} . Using relations (3.13) and Lemma 1 one can easily calculate the Poisson brackets between the generators of $\mathcal{Z}(\mathcal{L}_1)$ and prove the following proposition

Proposition 3. The centre $\mathcal{Z}(\mathcal{L}_1)$ of $\mathcal{L}_1(sl_2)$ endowed with the Poisson structure (3.14) is isomorphic to the extended Poisson graph algebra $\mathcal{PL}_1^*(sl_2)$ and the isomorphism is given by the formulas

$$\phi(a_{ij}^p) = \alpha_{ij}, \quad \phi(b_{ij}^p) = \beta_{ij},$$

where α_{ij}, β_{ij} are generators of the extended Poisson graph algebra $\mathcal{PL}_1^*(sl_2)$ and $i, j \neq 2$ simultaneously.

Remark 2. Let us note that as a by-product we have proven the well-known theorem that $\mathcal{Z}_0(\mathcal{U}_q(sl_2))$ is isomorphic to $C[SL_2^*]$ (see, e.g. [14, 15, 16]).

To proceed with the study of the centre of the graph algebra $\mathcal{L}_g$ we will need to know how the central elements $\mathcal{M}_{ij} = M_{ij}^p = (BA^{-1}B^{-1}A)_{ij}^p$ are expressed through the elements a_{ij}^p and b_{ij}^p . Another reason to find these expressions is that the ideal of $\mathcal{Z}(\mathcal{L}_g)$, generated by the elements $M_{ij}^p - \delta_{ij}$, belongs to the centre $\mathcal{Z}(\mathcal{I})$.

Proposition 4. The central elements M_{ij}^p are expressed through the generators a_{ij}^p and b_{ij}^p of $\mathcal{Z}(\mathcal{L}_1(sl_2))$ by means of the following formula

$$\mathcal{M} = \mathcal{B}\mathcal{A}^{-1}\mathcal{B}^{-1}\mathcal{A},$$

where $\mathcal{A}_{ij} = a_{ij}^p$, $\mathcal{B}_{ij} = b_{ij}^p$ for $i, j \neq 2$ simultaneously and $\det \mathcal{A} = \det \mathcal{B} = 1$.

Proof. Let us introduce matrices $D = BA^{-1}$ and $C = B^{-1}A$. We firstly show that the matrix $\mathcal{M}$ is expressed through the matrix elements d_{ij}^p, c_{ij}^p as follows

$$\mathcal{M} = \mathcal{D}\mathcal{C}$$

where $\mathcal{D}_{ij} = d_{ij}^p$, $\mathcal{C}_{ij} = c_{ij}^p$ for $i, j \neq 2$ simultaneously.

The matrices D and C have the following commutation relations

$$D^1 R_+ D^2 R_+^{-1} = R_- D^2 R_-^{-1} D^1, \quad C^1 R_+ C^2 R_+^{-1} = R_- C^2 R_-^{-1} C^1$$

$$C^1 R_+ D^2 R_+^{-1} = R_+ D^2 R_+^{-1} C^1, \quad \det_q C = \det_q C = q^3$$

which are rewritten in components in the Appendix.

Using the relations (6.24) and Lemma 2 one gets

$$M_{11}^p = (DC)_{11}^p = d_{11}^p(c_{11} + d_{11}^{-1}d_{12}c_{21})^p = d_{11}^p c_{11}^p + d_{12}^p c_{21}^p = (\mathcal{D}\mathcal{C})_{11}$$

$$\begin{aligned} M_{12}^p &= (DC)_{12}^p = (d_{11}c_{12} + q^2 d_{12}c_{11}^{-1}c_{21}c_{12} + q^3 d_{12}c_{11}^{-1})^p \\ &= (d_{11}c_{12} + q^2 d_{12}c_{11}^{-1}c_{21}c_{12})^p + (d_{12}c_{11}^{-1})^p = (d_{11} + q^2 d_{12}c_{11}^{-1}c_{21})^p c_{12}^p + d_{12}^p c_{11}^{-p} \\ &= d_{11}^p c_{12}^p + d_{12}^p c_{11}^{-p} c_{21}^p c_{12}^p + d_{12}^p c_{11}^{-p} = (\mathcal{D}\mathcal{C})_{12} \end{aligned}$$

$$\begin{aligned} M_{21}^p &= (DC)_{21}^p = (d_{21}c_{11} + q^2 d_{11}^{-1}d_{21}d_{12}c_{21} + q^3 d_{11}^{-1}c_{21})^p \\ &= d_{11}^{-p} c_{21}^p + (d_{21}c_{11} + q^2 d_{11}^{-1}d_{21}d_{12}c_{21})^p = d_{11}^{-p} c_{21}^p + d_{21}^p (c_{11} + d_{11}^{-1}d_{12}c_{21})^p \\ &= d_{11}^{-p} c_{21}^p + d_{21}^p (c_{11}^p + d_{11}^{-p} d_{12}^p c_{21}^p) = (\mathcal{D}\mathcal{C})_{21} \end{aligned}$$

To complete the proof of the Proposition 4 one should show that

$$\mathcal{C} = \mathcal{B}^{-1}\mathcal{A}, \quad \mathcal{D} = \mathcal{B}\mathcal{A}^{-1}$$

It can be done by using the following lemma which can be easily proved with the help of the Lemma 2:

Lemma 3. Let elements c , Z and W satisfy the relations $cZ = Zc$, $cW = Wc$, $ZW - c = q^{-2}(WZ - c)$ and $q^p = 1$ then

$$(ZW - c)^p = Z^p W^p - c^p$$

Then the calculation of c_{ij}^p gives

$$\begin{aligned}
c_{11}^p &= (B^{-1}A)_{11}^p = \left((q^2b_{22} + (1-q^2)b_{11})a_{11} - q^2b_{12}a_{21} \right)^p \\
&= \left(q^2b_{11}^{-1}a_{11} + q^4b_{11}^{-1}b_{21}b_{12}a_{11} + (1-q^2)b_{11}a_{11} - q^2b_{12}a_{21} \right)^p \\
&= \left(b_{11}^{-1}a_{11} + q^2b_{11}^{-1}b_{21}b_{12}a_{11} - qa_{21}b_{12} \right)^p \\
&= b_{11}^{-p}a_{11}^p \left(1 + q^2(b_{21} - a_{21}a_{11}^{-1}b_{11})b_{12} \right)^p \\
&= b_{11}^{-p}a_{11}^p \left(1 + (b_{21} - a_{21}a_{11}^{-1}b_{11})^p b_{12}^p \right) = (\mathcal{B}^{-1}\mathcal{A})_{11}
\end{aligned}$$

$$\begin{aligned}
c_{12}^p &= (B^{-1}A)_{12}^p = (q^2b_{22}a_{12} - qa_{22}b_{12})^p \\
&= (-a_{11}^{-1}b_{12} - q^2a_{11}^{-1}a_{21}a_{12}b_{12} + qb_{22}a_{12})^p \\
&= -a_{11}^{-p}b_{12}^p + (-q^2a_{11}^{-1}a_{21}a_{12}b_{12} + qb_{22}a_{12})^p \\
&= -a_{11}^{-p}b_{12}^p + a_{12}^p(-qa_{11}^{-1}a_{21}b_{12} + b_{22})^p \\
&= -a_{11}^{-p}b_{12}^p + a_{12}^p \left(b_{11}^{-1} + (b_{21}b_{11}^{-1} - a_{21}a_{11}^{-1})b_{12} \right)^p \\
&= -a_{11}^{-p}b_{12}^p + a_{12}^p b_{11}^{-p} \left(1 + (b_{21}b_{11}^{-1} - a_{21}a_{11}^{-1})b_{12}b_{11} \right)^p \\
&= -a_{11}^{-p}b_{12}^p + a_{12}^p b_{11}^{-p} \left(1 + (b_{21} - a_{21}a_{11}^{-1}b_{11})^p b_{12}^p \right) = (\mathcal{B}^{-1}\mathcal{A})_{12}
\end{aligned}$$

$$\begin{aligned}
c_{21}^p &= (B^{-1}A)_{21}^p = (-q^2b_{21}a_{11} + b_{11}a_{21})^p \\
&= a_{11}^p(-q^2b_{21} + b_{11}a_{21}a_{11}^{-1})^p = a_{11}^p(-b_{21}^p + b_{11}^p a_{21}^p a_{11}^{-p}) = (\mathcal{B}^{-1}\mathcal{A})_{21}
\end{aligned}$$

The calculation of the matrix elements $\mathcal{D}_{ij}$ can be done in the same manner.

The matrix elements of M can be expressed through the generators a_{11} , b_{11} and X_i as follows

$$\begin{aligned}
M_{11} &= 1 + X_2X_1 + X_1X_4 + X_3X_4 + X_3X_2X_1X_4 \\
&= q^{-2}(1 + X_1X_2 + X_1X_4 + X_3X_4 + X_1X_2X_3X_4) \\
M_{12} &= -q^{-2}((1 + X_1X_2)X_3 + X_1) + q^{-2}X_1M_{11}b_{11}^{-2} + X_3M_{11}a_{11}^2b_{11}^{-2} \\
M_{21} &= -q^{-2}(X_2(1 + X_3X_4) + X_4) + X_4M_{11}a_{11}^{-2} + q^2X_2M_{11}a_{11}^{-2}b_{11}^2
\end{aligned} \tag{3.15}$$

It follows from the Proposition 4 that

$$\begin{aligned}
M_{11}^p &= 1 + X_1^pX_2^p + X_1^pX_4^p + X_3^pX_4^p + X_1^pX_2^pX_3^pX_4^p \\
M_{12}^p &= -(1 + X_1^pX_2^p)X_3^p - X_1^p + X_1^pM_{11}^pb_{11}^{-2p} + X_3^pM_{11}^pa_{11}^{2p}b_{11}^{-2p} \\
M_{21}^p &= -X_2^p(1 + X_3^pX_4^p) - X_4^p + X_4^pM_{11}^pa_{11}^{-2p} + X_2^pM_{11}^pa_{11}^{-2p}b_{11}^{2p}
\end{aligned} \tag{3.16}$$

One can use eqs.(3.15) to prove the Proposition 4.

We are now ready to discuss the structure of the centre of the graph algebra $\mathcal{L}_g$ and the moduli algebra $\mathcal{M}_g$.

4 The centre of the graph and moduli algebras

Proposition 5. 1) The centre $\mathcal{Z}(\mathcal{L}_g)$ of $\mathcal{L}_g(sl_2)$ is generated by the elements $a_{i11}^{\pm p}$, a_{i12}^p , a_{i21}^p , $b_{i11}^{\pm p}$, b_{i12}^p , b_{i21}^p and M_{11}^{-p} , subject to the single relation

$$M_{11}^{-p}(\mathcal{B}_g\mathcal{A}_g^{-1}\mathcal{B}_g^{-1}\mathcal{A}_g \cdots \mathcal{B}_1\mathcal{A}_1^{-1}\mathcal{B}_1^{-1}\mathcal{A}_1)_{11} = 1$$

2) The algebra $\mathcal{L}_g(sl_2)$ is a free $\mathcal{Z}(\mathcal{L}_1(sl_2))$ -module with basis the set of monomials $\prod_{i=1}^g (a_{i11}^{r_{i1}} a_{i12}^{s_{i1}} a_{i21}^{t_{i1}} b_{i11}^{r_{i2}} b_{i12}^{s_{i2}} b_{i21}^{t_{i2}})$ and $0 \leq r, s, t \leq p-1$.

3) The ring $\mathcal{L}_g(sl_2)$ is an integral domain.

4) The centre $\mathcal{Z}(\mathcal{L}_g)$ endowed with the Poisson structure (3.14) is isomorphic to the extended Poisson graph algebra $\mathcal{PL}_g^*(sl_2)$.

5) The centre $\mathcal{Z}(\mathcal{L}_g)$ is isomorphic to the tensor product of g copies of the extended Poisson graph algebra $\mathcal{PL}_g^*(sl_2)$. The isomorphism is given by means of the following formulas

$$\mathcal{A}_i = \mathcal{M}_+^{-1}(i+1) \bar{\mathcal{A}}_i \mathcal{M}_+(i+1), \quad \mathcal{B}_i = \mathcal{M}_+^{-1}(i+1) \bar{\mathcal{B}}_i \mathcal{M}_+(i+1) \quad (4.17)$$

where

$$\begin{aligned} (\mathcal{A}_i)_{mn} &= (A_i)_{mn}^p, & (\mathcal{B}_i)_{mn} &= (B_i)_{mn}^p \\ (\bar{\mathcal{A}}_i)_{mn} &= (\bar{A}_i)_{mn}^p, & (\bar{\mathcal{B}}_i)_{mn} &= (\bar{B}_i)_{mn}^p \end{aligned}$$

and $m, n \neq 2$ simultaneously,

$$\mathcal{M}_i = \mathcal{M}_-^{-1}(i) \mathcal{M}_+(i) = \mathcal{B}_g \mathcal{A}_g^{-1} \mathcal{B}_g^{-1} \mathcal{A}_g \cdots \mathcal{B}_i \mathcal{A}_i^{-1} \mathcal{B}_i^{-1} \mathcal{A}_i \quad (4.18)$$

$$\begin{aligned} \mathcal{M}_\pm(i) &= \bar{\mathcal{G}}_\pm(i) \bar{\mathcal{G}}_\pm(i+1) \cdots \bar{\mathcal{G}}_\pm(g) \\ \bar{\mathcal{G}}_i &= \bar{\mathcal{G}}_-^{-1}(i) \bar{\mathcal{G}}_+(i) = \bar{\mathcal{B}}_i \bar{\mathcal{A}}_i^{-1} \bar{\mathcal{B}}_i^{-1} \bar{\mathcal{A}}_i \end{aligned} \quad (4.19)$$

and $\bar{A}_i, \bar{B}_i$ are matrices from the Proposition 1.

Proof. It is obvious that Propositions 5.1-5.4 follow from Propositions 1, 2, 3, and 5.5, thus it is enough to prove the Proposition 5.5. The matrix elements of $M_+(i)$ can be expressed through the matrix elements of M_i as follows

$$\begin{aligned} M_+(i)_{11} &= m_{i11}^{\frac{1}{2}}, & M_+(i)_{12} &= m_{i11}^{-\frac{1}{2}} m_{i12}, & M_+(i)_{22} &= m_{i11}^{-\frac{1}{2}}, \\ M_+^{-1}(i)_{11} &= m_{i11}^{-\frac{1}{2}}, & M_+^{-1}(i)_{12} &= -q m_{i11}^{-\frac{1}{2}} m_{i12}, & M_+^{-1}(i)_{22} &= m_{i11}^{\frac{1}{2}} \end{aligned}$$

Using formulas (2.8) one gets

$$\begin{aligned} (A_i)_{11}^p &= (\bar{a}_{i11} - q^2 m_{i+1,12} \bar{a}_{i21})^p = \bar{a}_{i11}^p - m_{i+1,12}^p \bar{a}_{i21}^p = (\widetilde{\mathcal{M}}_+^{-1}(i+1) \bar{\mathcal{A}}_i \widetilde{\mathcal{M}}_+(i+1))_{11} \\ (A_i)_{21}^p &= (m_{i+1,11} \bar{a}_{i21})^p = m_{i+1,11}^p \bar{a}_{i21}^p = (\widetilde{\mathcal{M}}_+^{-1}(i+1) \bar{\mathcal{A}}_i \widetilde{\mathcal{M}}_+(i+1))_{21} \\ (A_i)_{12}^p &= (\bar{a}_{i11} m_{i+1,11}^{-1} m_{i+1,12} + \bar{a}_{i12} m_{i+1,11}^{-1} - \bar{a}_{i21} m_{i+1,11}^{-1} m_{i+1,12}^2 - \bar{a}_{i22} m_{i+1,11}^{-1} m_{i+1,12})^p \\ &= \bar{a}_{i11}^{-p} m_{i+1,11}^{-p} (\bar{a}_{i11}^2 m_{i+1,12} + \bar{a}_{i11} \bar{a}_{i12} - \bar{a}_{i11} \bar{a}_{i21} m_{i+1,12}^2 - m_{i+1,12} - q^2 \bar{a}_{i21} \bar{a}_{i12} m_{i+1,12})^p \\ &= \bar{a}_{i11}^{-p} m_{i+1,11}^{-p} \left((\bar{a}_{i11} - q^2 \bar{a}_{i21} m_{i+1,12}) (\bar{a}_{i11} m_{i+1,12} + \bar{a}_{i12}) - m_{i+1,12} \right)^p \\ &= \bar{a}_{i11}^{-p} m_{i+1,11}^{-p} \left((\bar{a}_{i11} - q^2 \bar{a}_{i21} m_{i+1,12})^p (\bar{a}_{i11} m_{i+1,12} + \bar{a}_{i12})^p - m_{i+1,12}^p \right) \\ &= (\widetilde{\mathcal{M}}_+^{-1}(i+1) \bar{\mathcal{A}}_i \widetilde{\mathcal{M}}_+(i+1))_{12} \end{aligned} \quad (4.20)$$

where $\widetilde{\mathcal{M}}_{i+1,kl} = m_{i+1,kl}^p$.

To get these expressions one should use the commutativity of $m_{i+1,kl}$ and $\bar{a}_{imn}$, the Lemmas 2 and 3 and eq.(4.20) follows from the identification

$$\bar{a}_{i11} - q^2 \bar{a}_{i21} m_{i+1,12} = Z, \quad \bar{a}_{i11} m_{i+1,12} + \bar{a}_{i12} = W, \quad m_{i+1,12} = c$$

One sees that to prove the Proposition 5.4 one should show that

$$\widetilde{\mathcal{M}}_i = \mathcal{M}_i \tag{4.21}$$

The matrix elements of M_i can be expressed through the matrix elements of $\bar{G}_i$ as follows

$$\begin{aligned} m_{i11} &= \bar{g}_{i11} \bar{g}_{i+1,11} \cdots \bar{g}_{g11} \\ m_{i12} &= \sum_{k=i}^g \bar{g}_{i11} \cdots \bar{g}_{k-1,11} \bar{g}_{k12} \\ m_{i21} &= \sum_{k=i}^g \bar{g}_{i11} \cdots \bar{g}_{k-1,11} \bar{g}_{k21} \end{aligned}$$

Using these expressions and the Lemma 2 one immediately gets that

$$\widetilde{\mathcal{M}}_+(i) = \widetilde{\Gamma}_\pm(i) \cdots \widetilde{\Gamma}_\pm(g)$$

where $(\widetilde{\Gamma}_\pm(i))_{mn} = (\bar{\mathcal{G}}_\pm(i))_{mn}^p$.

Thus to complete the proof of the Proposition 5.4 it remains to remember that the equation

$$\widetilde{\Gamma}_i = \widetilde{\Gamma}_-^{-1}(i) \widetilde{\Gamma}_+(i) = \bar{\mathcal{G}}_i = \bar{\mathcal{B}}_i \bar{\mathcal{A}}_i^{-1} \bar{\mathcal{B}}_i^{-1} \bar{\mathcal{A}}_i$$

was proved in Proposition 4.

The Proposition 5 is proved.

Remark 3. Strictly speaking to define the matrices $\mathcal{M}_\pm(i)$, $\bar{\mathcal{G}}_\pm(i)$ one should use the elements $(m_{i11}^{\pm \frac{1}{2}})^p$, $(\bar{g}_{i11}^{\pm \frac{1}{2}})^p$ from the extended graph algebras. However the formulas (4.17) (and (2.8)) do not depend on them due to the obvious relations $(m_{i11}^{\frac{1}{2}})^2 = m_{i11}$, $(\bar{g}_{i11}^{\frac{1}{2}})^2 = \bar{g}_{i11}$ and we use the matrices $\mathcal{M}_\pm$, $\bar{\mathcal{G}}_\pm$ only to simplify the notations and the proof of eq.(4.21).

Remark 4. The method described in the third section to prove the Proposition 4 can be applied to prove eq.(4.18) without using the decomposition (4.19).

Let us proceed with the study of the centre $\mathcal{Z}(\mathcal{M}_g)$ of the moduli algebra.

Proposition 6. The centre $\mathcal{Z}(\mathcal{F}_\mathcal{I})$ coincides with $\mathcal{Z}(\mathcal{L}_g)$ and the centre $\mathcal{Z}(\mathcal{I})$ is the ideal of $\mathcal{Z}(\mathcal{L}_g)$, generated by the elements $M_{ij}^p - \delta_{ij}$.

Proof. It is obvious that $\mathcal{Z}(\mathcal{L}_g) \subset \mathcal{Z}(\mathcal{F}_\mathcal{I})$. As was proved in Proposition 5, $\mathcal{L}_g$ and, hence $\mathcal{F}_\mathcal{I}$ are integral domains and, therefore $\mathcal{Z}(\mathcal{I}) \subset \mathcal{Z}(\mathcal{F}_\mathcal{I})$. Let z belongs to $\mathcal{Z}(\mathcal{F}_\mathcal{I})$. It is not difficult to show that the elements $\bar{a}_{k11} M_{21}^{p-1} M_{12}^{p-1}$ and $\bar{b}_{k11} M_{21}^{p-1} M_{12}^{p-1}$ belong to $\mathcal{F}_\mathcal{I}$. The ring $\mathcal{L}_g$ is an integral domain and, hence the following equations should be valid

$$z \bar{a}_{k11} = \bar{a}_{k11} z, \quad z \bar{b}_{k11} = \bar{b}_{k11} z$$

It follows from these equations that the element z should be of the form

$$z = z((X_k)_i, \bar{a}_{k11}^p, \bar{b}_{k11}^p)$$

. Taking into account that z commutes with M_{12} and M_{21} one gets

$$z(X_k)_i = (X_k)_i z \quad \forall k = 1, \dots, g \quad \text{and} \quad \forall i = 1, 2, 3, 4.$$

Therefore, z belongs to $\mathcal{Z}(\mathcal{L}_g)$ and $\mathcal{Z}(\mathcal{F}_{\mathcal{I}})$ coincides with $\mathcal{Z}(\mathcal{L}_g)$.

It can be easily shown using eqs.(3.15) and (3.16) that any element $z \in \mathcal{Z}(\mathcal{I})$ belongs to the ideal of $\mathcal{Z}(\mathcal{L}_g)$ generated by the elements $M_{ij}^p - \delta_{ij}$.

Let us consider a subalgebra $\mathcal{J}$ of $\mathcal{F}_{\mathcal{I}}$ which consists of the elements such that the commutator of an element $j \in \mathcal{J}$ with any element $f \in \mathcal{F}_{\mathcal{I}}$ belongs to the ideal $\mathcal{I}$:

$$\mathcal{J} = \{j \in \mathcal{F}_{\mathcal{I}} : jf - fj \in \mathcal{I} \quad \forall f \in \mathcal{F}_{\mathcal{I}}\}$$

It is obvious that $\mathcal{Z}(\mathcal{L}_g) \subset \mathcal{J}$ is the centre of $\mathcal{J}$, $\mathcal{I}$ is an ideal of $\mathcal{J}$ and the centre $\mathcal{Z}(\mathcal{M}_g)$ coincides with the factor algebra of $\mathcal{J}$ over $\mathcal{I}$

$$\mathcal{Z}(\mathcal{M}_g) = \mathcal{J}/\mathcal{I}$$

As was shown before the graph algebra $\mathcal{L}_g$ and, therefore, the algebras $\mathcal{F}_{\mathcal{I}}$ and $\mathcal{J}$ are finitely generated over $\mathcal{Z}(\mathcal{L}_g)$. As has been just proved $\mathcal{Z}(\mathcal{I}) \subset \mathcal{Z}(\mathcal{L}_g)$ and, hence the moduli algebra $\mathcal{M}_g = \mathcal{F}_{\mathcal{I}}/\mathcal{I}$ and the centre $\mathcal{Z}(\mathcal{M}_g)$ are finitely generated over $\mathcal{Z}(\mathcal{L}_g)/\mathcal{Z}(\mathcal{I})$. Let us consider $\text{Spec}(\mathcal{Z}(\mathcal{M}_g))$ and $\text{Spec}(\mathcal{Z}(\mathcal{L}_g)/\mathcal{Z}(\mathcal{I}))$, i.e. the set of all algebra homomorphisms from $\mathcal{Z}(\mathcal{M}_g)$ and $\mathcal{Z}(\mathcal{L}_g)/\mathcal{Z}(\mathcal{I})$ to C . It is clear that $\text{Spec}(\mathcal{Z}(\mathcal{L}_g))$ is isomorphic to $C^{4g} \times (C^\times)^{2g}$, i.e. a complex affine space of dimension $6g$ with $2g$ hyperplanes of codimension 1 removed. Then $\text{Spec}(\mathcal{Z}(\mathcal{L}_g)/\mathcal{Z}(\mathcal{I}))$ is a submanifold of $C^{4g} \times (C^\times)^{2g}$ which is singled out by means of eq.(1.4).

5 Irreducible representations of the graph algebra

It is clear that every irreducible $\mathcal{L}_g$ -module V is finite dimensional and the centre $\mathcal{Z}(\mathcal{L}_g)$ acts by scalar operators on V and, therefore, there is a homomorphism $\chi_V \in \text{Spec}(\mathcal{Z}(\mathcal{L}_g)) : \mathcal{Z}(\mathcal{L}_g) \rightarrow C$, the central character of V , such that

$$z.v = \chi_V(z)v$$

for all $z \in \mathcal{Z}(\mathcal{L}_g)$ and $v \in V$.

Let us consider an ideal $\mathcal{I}_\chi$ of $\mathcal{L}_g$ generated by elements $z - \chi_V(z)$, $z \in \mathcal{Z}(\mathcal{L}_g)$. Then every irreducible representation of the algebra $\mathcal{L}_g^\chi = \mathcal{L}_g/\mathcal{I}_\chi$ is an irreducible representation of $\mathcal{L}_g$ with the central character $\chi_V(z)$.

Due to the Proposition 1 it is enough to construct representations of $\mathcal{L}_1$. As was shown in the third section the algebra $\mathcal{L}_1$ is isomorphic to the tensor product of the Weil algebra, generated by a_{11} and b_{11} , and the algebra $\mathcal{X}$ generated by X_i . There is no problem in constructing representations of the Weil algebra and we begin with the discussion of irreducible representations of $\mathcal{X}$.

Proposition 7. The algebra $\mathcal{X}_\chi = \mathcal{X}/\mathcal{I}_\chi$, $\chi(M_{11}^p) \neq 0$ is a simple p^4 -dimensional algebra and, hence, the algebra $\mathcal{L}_1^\chi$ is simple p^6 -dimensional.

Proof. We are going to show that the unity element belongs to an arbitrary (nonzero) ideal $\mathcal{J}$ and, hence, the ideal coincides with $\mathcal{X}_\chi$ and $\mathcal{X}_\chi$ is a simple algebra.

An arbitrary element $f \in \mathcal{X}_\chi$ can be presented in the form

$$\begin{aligned} f &= \sum c_{ijkl} X_1^i (1 + X_1 X_2)^j (1 + X_3 X_4)^k X_4^l + d_{ijkl} X_1^i (1 + X_1 X_2)^j (1 + X_3 X_4)^k X_3^l \\ &+ f_{ijkl} X_2^i (1 + X_1 X_2)^j (1 + X_3 X_4)^k X_4^l + g_{ijkl} X_2^i (1 + X_1 X_2)^j (1 + X_3 X_4)^k X_3^l \end{aligned}$$

where $i, j, k, l = 0, 1, \dots, p-1$.

Let f belong to an ideal $\mathcal{J}$ and let $l_2(l_3)$ be the maximal power of $X_2(X_3)$ in the element f . Then one can easily show that the element $X_1^{l_2} f X_4^{l_3}$, which obviously belongs to $\mathcal{J}$, can be represented in the form

$$X_1^{l_2} f X_4^{l_3} = \sum c_{ijkl} X_1^i (1 + X_1 X_2)^j (1 + X_3 X_4)^k X_4^l$$

with some new coefficients c_{ijkl} .

This element can be rewritten as follows

$$X_1^{l_2} f X_4^{l_3} = \sum c_{\alpha ijk} M_{11}^\alpha (1 + X_1 X_2)^i X_1^j X_4^k + d_{\alpha ijk} M_{11}^\alpha (1 + X_3 X_4)^i X_1^j X_4^k$$

where M_{11} is given by eq.(3.15).

Multiplying $X_1^{l_2} f X_4^{l_3}$ on $(1 + X_1 X_2)^l$, where l is the maximal power of $(1 + X_3 X_4)$ in $X_1^{l_2} f X_4^{l_3}$ one gets that the ideal $\mathcal{J}$ contains an element of the form

$$\sum c_{\alpha ijk} M_{11}^\alpha (1 + X_1 X_2)^i X_1^j X_4^k$$

Let us now suppose that $X_1^p X_4^p \neq 0$. Then the elements X_1 and X_4 are invertible and using the commutation relations of M_{11} , X_1 and X_4 with X_i one can easily show that the element of the form

$$(1 + X_1 X_2)^l \sum c_\alpha M_{11}^\alpha X_1^{p-\alpha} X_4^{p-\alpha}$$

belongs to $\mathcal{J}$.

If $X_2^p = 0$ then the element $(1 + X_1 X_2)$ is invertible and from the commutation relations of $(1 + X_1 X_2)$ with X_i one gets that the ideal $\mathcal{J}$ contains M_{11} and, hence the unity element. If $X_2^p \neq 0$ then using the commutation relations of X_2 with X_i one gets that the element $(1 + X_1 X_2)^l$ belongs to $\mathcal{J}$. Now using the relation

$$(1 + X_1 X_2)(1 + X_3 X_4) = (1 + X_3 X_4)(1 + X_1 X_2) + (q^2 - 1)X_1 X_4$$

one can easily show that $\mathcal{J}$ contains the element $X_1 X_4$ and, therefore, the unity element.

Let us now consider the case $X_1^p = 0$. Then the element $(1 + X_1 X_2)$ is invertible and from the commutation relations of M_{11} and $(1 + X_1 X_2)$ with X_i one can get that the element of the form

$$X_1^{l_1} X_4^{l_4} \sum c_{\alpha i} M_{11}^\alpha (1 + X_1 X_2)^i$$

belongs to $\mathcal{J}$.

Let $X_4^p \neq 0$. Using the commutation relations of X_4 with X_i one gets that $\mathcal{J}$ contains the element of the form

$$X_1^{l_1} \sum c_i (1 + X_1 X_2)^i$$

With the help of X_2 one gets that the element $X_1^{l_1} X_2^{l_2} (1 + X_1 X_2)^l$ and, hence, $X_1^{l_1} X_2^{l_2}$ belongs to $\mathcal{J}$ and using firstly X_1 and then X_2 one sees that the unity element is in $\mathcal{J}$.

If $X_4^p = 0$ then the element $(1 + X_3 X_4)$ is invertible and using this element and X_2 one shows that the element $X_4^{p-1} \sum c_\alpha M_{11}^\alpha$ belongs to $\mathcal{J}$. Using again X_2 one gets that $\mathcal{J}$

contains the element $X_4^{p-1}X_2^l$. With the help of X_1 and then X_3 one gets that the unity element belongs to $\mathcal{J}$. The case $X_4^p = 0$, $X_1 \neq 0$ can be considered in the same manner. The Proposition 7 is proved.

The algebra $\mathcal{X}_\chi$, being simple, is isomorphic to $M_{p^2}(C)$ and, therefore, the following proposition is valid.

Proposition 7. Every irreducible representation of $\mathcal{X}$ is isomorphic to one of the following p^2 -dimensional:

$$\begin{aligned}
1) \quad & X_1\Psi(k, l) = \Psi(k+1, l) \\
& X_2\Psi(k, l) = q^{-2k}(y_2x_1 + 1 - q^{2k})\Psi(k-1, l) + z_2q^{-2(k+l)}\Psi(k, l+1) \\
& X_3\Psi(k, l) = q^{2(k+l)}(y_3x_4 + 1 - q^{-2l})\Psi(k, l-1) + z_3q^{2k}\Psi(k+1, l) \\
& X_4\Psi(k, l) = q^{-2k}\Psi(k, l+1)
\end{aligned} \tag{5.22}$$

Here $\Psi(k, l)$ is a basis of a p^2 -dimensional vector space, $k, l = 0, 1, \dots, p-1$ and we use the following notations

$$\begin{aligned}
\Psi(p, l) &= x_1\Psi(0, l), & x_1\Psi(-1, l) &= \Psi(p-1, l) \\
\Psi(k, p) &= x_4\Psi(k, 0), & x_4\Psi(k, -1) &= \Psi(k, p-1)
\end{aligned}$$

The complex parameters $x_1, x_4, y_2, y_3, z_2, z_3$ should satisfy the following equations

$$\begin{aligned}
(y_2x_1 + 1)(y_3x_4 + 1) &\neq 0, & z_2z_3 &= 0 \\
1 + q^{-2}z_3(y_2x_1 + 1) + q^2z_2(y_3x_4 + 1) &= 0
\end{aligned}$$

The central character of the representation is defined by the formulas

$$\begin{aligned}
\chi(X_1^p) &= x_1, & \chi(X_4^p) &= x_4 \\
\chi(X_2^p) &= \frac{1}{x_1}((y_2x_1 + 1)^p - 1) + z_2^p x_4 \\
\chi(X_3^p) &= \frac{1}{x_4}((y_3x_4 + 1)^p - 1) + z_3^p x_1
\end{aligned}$$

The element M_{11} acts in the representation as follows

$$M_{11}\Psi(k, l) = q^{2(l-k)}(y_2x_1 + 1)(y_3x_4 + 1)\Psi(k, l)$$

$$\begin{aligned}
2) \quad & X_1\Psi(k, l) = \Psi(k+1, l) \\
& X_2\Psi(k, l) = -\Psi(k-1, l) + b_1q^{-2(k+l)}\Psi(k, l+1) + b_2q^{-2(k+2l)}\Psi(k+1, l+2) \\
& X_3\Psi(k, l) = -q^{2k}\Psi(k, l-1) + c_{p-1}q^{2(k+2l)}\Psi(p+k-1, p+l-2) \\
& X_4\Psi(k, l) = q^{-2k}\Psi(k, l+1)
\end{aligned}$$

where

$$\begin{aligned}
x_1x_4b_1b_2c_{p-1} &\neq 0, & 1 + q^{10}x_1x_4b_2c_{p-1} &= 0 \\
\chi(X_1^p) &= x_1, & \chi(X_4^p) &= x_4 \\
\chi(X_2^p) &= (b_1^p + b_2^p x_1x_4)x_4 - x_1^{-1}
\end{aligned}$$

$$\chi(X_3^p) = c_{p-1}^p x_1^{p-1} x_4^{p-2} - x_4^{-1}$$

$$M_{11}\Psi(k, l) = q^{2(l-k+3)} x_1 x_4 b_1 c_{p-1} \Psi(k, l)$$

Now irreducible representations of the graph algebra $\mathcal{L}_1(sl_2)$ can be constructed using, for example, the following representation of the Weil algebra

$$a_{11}\Psi(m) = \Psi(m+1), \quad b_{11}\Psi(m) = \beta_{11}q^{-m}\Psi(m)$$

$$\Psi(m+p) = \alpha_{11}\Psi(m), \quad \chi(a_{11}^p) = \alpha_{11}, \quad \chi(b_{11}^p) = \beta_{11}^p$$

Then the graph algebra acts in the tensor product of the representations of the Weil algebra and the algebra $\mathcal{X}$

$$\begin{aligned} a_{11}\Psi(k, l, m) &= \Psi(k, l, m+1) \\ b_{11}\Psi(k, l, m) &= \beta_{11}q^{-m}\Psi(k, l, m) \\ a_{12}\Psi(k, l, m) &= \beta_{11}^{-2}q^{2m}\Psi(k+1, l, m+1) + \beta_{11}^{-2}z_3q^{2(m+k+1)}\Psi(k+1, l, m+3) \\ &\quad + \beta_{11}^{-2}q^{2(m+k+l+1)}(y_3x_4 + 1 - q^{-2l})\Psi(k, l-1, m+3) \\ a_{21}\Psi(k, l, m) &= \beta_{11}^2q^{2(m-k)}(y_2x_1 + 1 - q^{2k})\Psi(k-1, l, m-1) \\ &\quad + \beta_{11}^2z_2q^{2(m-k-l)}\Psi(k, l+1, m-1) \\ b_{12}\Psi(k, l, m) &= \beta_{11}^{-1}q^{m+2(k+l)}(y_3x_4 + 1 - q^{-2l})\Psi(k, l-1, m+2) \\ &\quad + \beta_{11}^{-1}z_3q^{m+2k}\Psi(k+1, l, m+2) \\ b_{21}\Psi(k, l, m) &= \beta_{11}^3q^{3m-2k+2}(y_2x_1 + 1 - q^{2k})\Psi(k-1, l, m-2) \\ &\quad + \beta_{11}^3q^{3m-2(k+l)+2}\Psi(k, l+1, m-2) \\ &\quad + \beta_{11}q^{m-2k}\Psi(k, l+1, m-2) \end{aligned}$$

and we present formulas only for the representation (5.22) of $\mathcal{X}$.

Using this representation and Proposition 1 one can easily construct all irreducible representations of the graph algebra $\mathcal{L}_g$.

Let us briefly discuss representations of the moduli algebra $\mathcal{M}_g$. In this case one should consider only representations with $\chi(M_{11}^p) = 1$, $\chi(M_{12}^p) = \chi(M_{21}^p) = 0$. Let V_0 be the submodule of an irreducible left $\mathcal{L}_g$ -module V which is annihilated by the elements $\Phi_{ij} = M_{ij} - \delta_{ij}$:

$$V_0 = \{\Psi \in V : \Phi_{ij}\Psi = 0\}$$

and let δV_0 be a submodule of V_0 which consists of the vectors of the following form:

$$\delta V_0 = \{\Psi \in V_0 : \Psi = \Phi_{ij}\chi_{ij} \text{ for some } \chi_{ij} \in V\}$$

Then it is obvious that the moduli algebra acts in the factor module $V_{ph} = V_0/\delta V_0$.

6 Conclusion

In this paper we studied the structure of the centre and irreducible representations of the graph algebra. The next problem to be solved is to construct unitary representations of the graph algebra. The anti-automorphism ρ (2.9) can be used to define unitary representations of $\mathcal{L}_g$. Namely, let V be a left $\mathcal{L}_g$ -module and β be a bilinear form on $V \times V$, such that $\beta(v_2, \lambda v_1) = \lambda \beta(v_2, v_1)$, $\beta(\lambda v_2, v_1) = \lambda^* \beta(v_2, v_1)$, $\lambda \in C$. A representation

is called unitary if $\beta(v_2, f v_1) = \beta(\rho(f) v_2, v_1)$ for all $v_1, v_2 \in V$ and $f \in \mathcal{L}_g$. It is obvious that only representations with central characters, satisfying the equations $\mathcal{A}_i \mathcal{M} = \mathcal{M} \mathcal{A}_i$, $\mathcal{B}_i \mathcal{M} = \mathcal{M} \mathcal{B}_i$, can be unitary.

We didn't study irreducible representations of the moduli algebra, however there are some indications that V_{ph} is an irreducible $\mathcal{M}_g$ -module. One should prove (or disprove) this conjecture and show that the dimension of V_{ph} is given by Verlinde's formula.

It would be interesting to clarify the relation between this approach to quantization of the moduli space and the geometric quantization [17, 18]. It seems that a choice of a point of $Spec(\mathcal{Z}(\mathcal{M}_g))$ corresponds to a choice of a polarization on the moduli space.

It seems that the results obtained in the paper can be generalized to the graph algebras corresponding to arbitrary quantized universal enveloping algebras.

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Appendix

Let matrices A and B satisfy the commutation relations of $\mathcal{L}_1$

$$A^1 R_+ A^2 R_+^{-1} = R_- A^2 R_-^{-1} A^1, \quad B^1 R_+ B^2 R_+^{-1} = R_- B^2 R_-^{-1} B^1$$

$$A^1 R_+ B^2 R_+^{-1} = R_+ B^2 R_-^{-1} A^1$$

$$\det_q A = a_{11} a_{22} - q^2 a_{21} a_{12} = \lambda_a, \quad \det_q B = b_{11} b_{22} - q^2 b_{21} b_{12} = \lambda_b$$

These relations can be rewritten in components as follows

$$\begin{aligned} a_{11} a_{12} &= q^{-2} a_{12} a_{11}, & a_{11} a_{21} &= q^2 a_{21} a_{11}, & a_{11} a_{22} &= a_{22} a_{11} \\ [a_{12}, a_{21}] &= -(1 - q^{-2}) a_{11} (a_{11} - a_{22}) \Leftrightarrow a_{12} a_{21} = q^2 a_{21} a_{12} + (1 - q^{-2}) (\lambda_a - a_{11}^2) \\ [a_{12}, a_{22}] &= -(1 - q^{-2}) a_{11} a_{12}, & [a_{21}, a_{22}] &= (1 - q^{-2}) a_{21} a_{11} \end{aligned}$$

and the same relations for b_{ij} .

$$\begin{aligned} a_{11} b_{11} &= q b_{11} a_{11}, & a_{11} b_{12} &= q^{-1} b_{12} a_{11}, & a_{11} b_{21} &= q b_{21} a_{11} + (q - q^{-1}) b_{11} a_{21} \\ a_{11} b_{22} &= q^{-1} b_{22} a_{11} + q^{-1} (q - q^{-1})^2 b_{11} a_{11} + (q - q^{-1}) b_{12} a_{21} \end{aligned}$$

$$\begin{aligned} a_{12} b_{11} &= q b_{11} a_{12} + (q - q^{-1}) b_{12} a_{11}, & a_{12} b_{12} &= q b_{12} a_{12} \\ a_{12} b_{21} &= q^{-1} b_{21} a_{12} + q^{-1} (q - q^{-1})^2 b_{12} a_{21} \\ &\quad + q^{-2} (q - q^{-1}) (b_{22} a_{11} + b_{11} a_{22} + (q^{-2} - 2) b_{11} a_{11}) \\ a_{12} b_{22} &= q^{-1} b_{22} a_{12} + q^{-1} (q - q^{-1})^2 b_{11} a_{12} + (q - q^{-1}) b_{12} a_{22} - q^{-2} (q - q^{-1}) b_{12} a_{11} \end{aligned}$$

$$\begin{aligned} a_{21} b_{11} &= q^{-1} b_{11} a_{21}, & a_{21} b_{12} &= q^{-1} b_{12} a_{21} + q^{-2} (q - q^{-1}) b_{11} a_{11}, & a_{21} b_{21} &= q b_{21} a_{21} \\ a_{21} b_{22} &= q b_{22} a_{21} + (q - q^{-1}) b_{21} a_{11} \end{aligned}$$

$$\begin{aligned}
a_{22}b_{11} &= q^{-1}b_{11}a_{22} + q^{-1}(q - q^{-1})^2b_{11}a_{11} + (q - q^{-1})b_{12}a_{21} \\
a_{22}b_{22} &= qb_{22}a_{22} - q^{-3}(q - q^{-1})^2b_{11}a_{11} + (q - q^{-1})b_{21}a_{12} - q^{-2}(q - q^{-1})b_{12}a_{21} \\
a_{22}b_{21} &= q^{-1}b_{21}a_{22} + q^{-1}(q - q^{-1})^2b_{21}a_{11} + (q - q^{-1})b_{22}a_{21} - q^{-2}(q - q^{-1})b_{11}a_{21} \\
a_{22}b_{12} &= qb_{12}a_{22} + (q - q^{-1})b_{11}a_{12}
\end{aligned} \tag{6.23}$$

Let matrices C and D satisfy the following relations

$$C^1 R_+ D^2 R_+^{-1} = R_+ D^2 R_+^{-1} C^1$$

The relations look in components as follows

$$\begin{aligned}
c_{11}d_{11} &= d_{11}c_{11} - q(q - q^{-1})d_{12}c_{21}, & c_{11}d_{21} &= d_{21}c_{11} + q(q - q^{-1})(d_{11} - d_{22})c_{21} \\
c_{11}d_{12} &= d_{12}c_{11}, & c_{11}d_{22} &= d_{22}c_{11} + q^{-1}(q - q^{-1})d_{12}c_{21} \\
c_{12}d_{11} &= d_{11}c_{12} + q(q - q^{-1})d_{12}(c_{11} - c_{22}), & c_{12}d_{12} &= q^2d_{12}c_{12} \\
c_{12}d_{21} + q^{-1}(q - q^{-1})c_{11}(d_{11} - d_{22}) &= q^{-2}d_{21}c_{12} + q^{-1}(q - q^{-1})(d_{11} - d_{22})c_{22} \\
c_{12}d_{22} &= d_{22}c_{12} - q^{-1}(q - q^{-1})d_{12}(c_{11} - c_{22}) \\
c_{21}d_{11} &= d_{11}c_{21}, & c_{21}d_{12} &= q^{-2}d_{12}c_{21}, & c_{21}d_{21} &= q^2d_{21}c_{21}, & c_{21}d_{22} &= d_{22}c_{21} \\
c_{22}d_{11} &= d_{11}c_{22} + q^{-1}(q - q^{-1})d_{12}c_{21}, & c_{22}d_{22} &= d_{22}c_{22} - q^{-3}(q - q^{-1})d_{12}c_{21} \\
c_{22}d_{12} &= d_{12}c_{22}, & c_{22}d_{21} &= d_{21}c_{22} - q^{-1}(q - q^{-1})(d_{11} - d_{22})c_{21}
\end{aligned} \tag{6.24}$$

The nontrivial commutation relations of the elements c_{ij} with $d_{11}^{\frac{1}{2}}$, which are used to define $\mathcal{L}_g^*$, are given by the formulas

$$\begin{aligned}
c_{11}d_{11}^{\frac{1}{2}} &= d_{11}^{\frac{1}{2}}c_{11} + (1 - q)d_{11}^{-\frac{1}{2}}d_{12}c_{21} \\
c_{12}d_{11}^{\frac{1}{2}} &= d_{11}^{\frac{1}{2}}c_{12} - (1 - q)d_{11}^{-\frac{1}{2}}d_{12}(c_{11} - c_{22}) + q^{-3}(1 - q)^2d_{11}^{-\frac{3}{2}}d_{12}^2c_{21}
\end{aligned}$$

Let matrices A and M have the commutation relations

$$M^1 R_+ A^2 R_+^{-1} = R_+ A^2 R_+^{-1} M^1$$

The relations look in components as follows

$$\begin{aligned}
a_{11}m_{11} &= m_{11}a_{11}, & a_{11}m_{12} &= m_{12}a_{11} - q(q - q^{-1})m_{11}a_{12} \\
a_{11}m_{21} &= m_{21}a_{11} + q^{-1}(q - q^{-1})m_{11}a_{21} \\
a_{11}m_{22} &= m_{22}a_{11} + q(q - q^{-1})(m_{12}a_{21} - m_{21}a_{12}) - (q - q^{-1})^2m_{11}(a_{22} - a_{11}) \\
a_{12}m_{11} &= q^2m_{11}a_{12}, & a_{12}m_{12} &= m_{12}a_{12} \\
a_{12}m_{21} &= m_{21}a_{12} - q^{-1}(q - q^{-1})m_{11}(a_{11} - a_{22}) \\
a_{12}m_{22} &= q^{-2}m_{22}a_{12} - q^{-1}(q - q^{-1})m_{12}(a_{11} - a_{22}) + q^{-1}(q - q^{-1})(q^2 - q^{-2})m_{11}a_{12} \\
a_{21}m_{11} &= q^{-2}m_{11}a_{21}, & a_{21}m_{12} &= m_{12}a_{21} + q^{-1}(q - q^{-1})m_{11}(a_{11} - a_{22}) \\
a_{21}m_{21} &= m_{21}a_{21}, & a_{21}m_{22} &= q^2m_{22}a_{21} + q(q - q^{-1})m_{21}(a_{11} - a_{22})
\end{aligned}$$

$$\begin{aligned}
a_{22}m_{11} &= m_{11}a_{22}, & a_{22}m_{12} &= m_{12}a_{22} + q^{-1}(q - q^{-1})m_{11}a_{12} \\
a_{22}m_{21} &= m_{21}a_{22} - q^{-3}(q - q^{-1})m_{11}a_{21} \\
a_{22}m_{22} &= m_{22}a_{22} - q^{-1}(q - q^{-1})(m_{12}a_{21} - m_{21}a_{12}) \\
&\quad + q^{-2}(q - q^{-1})^2 m_{11}(a_{22} - a_{11})
\end{aligned} \tag{6.25}$$

The nontrivial commutation relations of the elements m_{ij} with $a_{11}^{\frac{1}{2}}$, which are used to define $\mathcal{L}_g^*$, are given by the formulas

$$\begin{aligned}
a_{11}^{\frac{1}{2}}m_{12} &= m_{12}a_{11}^{\frac{1}{2}} + (1 - q)m_{11}a_{11}^{-\frac{1}{2}}a_{12} \\
a_{11}^{\frac{1}{2}}m_{21} &= m_{21}a_{11}^{\frac{1}{2}} + (1 - q^{-1})m_{11}a_{11}^{-\frac{1}{2}}a_{21}
\end{aligned}$$

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