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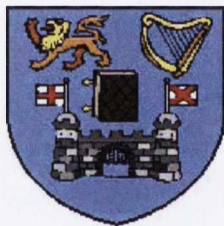
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Yang-Mills Instantons on the Taub-NUT Space and Supersymmetric $N=2$ Gauge Theories with Impurities

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A thesis submitted to the University of Dublin, Trinity College
in Partial Fulfillment of the Requirements for the Degree of
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Summary

We write a formula for arbitrary charge calorons, instantons on $\mathbb{R}^3 \times S^1$, in terms of the Green's function of the Laplacian defined for the Nahm Transform, thus generalising the formula for the charge one caloron derived by Kraan and van Baal in [1]. The Laplacian is constructed from Nahm data. The usual approach to the Nahm Transform involves an integration over the interval on which the Nahm data are defined. By using Green's functions we avoid this integration and our formula is straightforward to use.

Using the same approach, we derive a formula for an $SU(2)$ instanton on the Taub-NUT space. Here, the Laplacian is constructed from Bow data that solve the Nahm Equations in the interior of the interval. The Bow data includes bifundamental data at the end-points of the interval.

We write the Lagrangian for the low-energy effective field theory on the D3-brane in a Chalmers-Hanany-Witten configuration of intersecting D3-, D5- and NS5-branes [2] [3], by adding bifundamental fields to the Lagrangian written in [4]. The low-energy theory on the D3-branes is described by $N = 2$ Super-Yang-Mills gauge theory with codimension one defects. The supersymmetric vacuum conditions for the gauge theory give the Bow data for an instanton on the Taub-NUT space.

We write an explicit expression for a charge one $SU(2)$ instanton on the Taub-NUT space, in terms of the Green's function values at jumping points and end-points of the Nahm interval. For the charge one instanton we find the Green's function explicitly.

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Chapter 1

Introduction

1.1 Overview

There is a close correspondence between monopoles and a system of equations called the Nahm Equations on the one hand, and quantum gauge theories that arise from String Theory D-branes on the other. This correspondence takes the form of an identification of the space of vacua of the gauge theory with the moduli space of fields that satisfy the Nahm Equations or (for a monopole) the Bogomolny Equations [5] [6]. Our aim is to write a formula for an instanton in a curved background called the Taub-NUT space, in terms of fields that solve the Nahm Equations *and* that parametrize the space of vacua of a gauge theory arising from a particular configuration of D-branes, which we will specify. To do this, we will need to use the Nahm Transform.

The Nahm Transform was first introduced by Nahm in 1980 in [7] as a generalization of the important Atiyah, Drinfel'd, Hitchin and Manin (ADHM) construction for instantons [8]. It is a map between fields that solve self-duality equations such as the Nahm Equations and the Bogomolny Equations mentioned above. Since it was developed by Nahm, it has been used to construct instantons on various spaces. In this case we study instantons in curved backgrounds called asymptotically locally

flat (ALF) spaces, in particular the Taub-NUT space which is described in Section 1.6.

The Nahm Transform essentially involves three steps. It is first necessary to solve the Nahm Equations, a system of non-linear ODEs. Then we can use the solutions we have found to write a Dirac-type equation called the Weyl equation. Finally we perform an integration on the solutions to the Weyl equation.

Nahm provided the technique to deal with the first two of these steps, but the final step involves performing a straightforward but tedious calculation. Here we use an approach that eliminates the need to do this final step. This approach uses Green's functions of operators defined in the Nahm Transform. We can thus write a formula for an instanton in two steps, avoiding a significant amount of computation in the process. This method was introduced by Corrigan, Fairlie, Templeton and Goddard in [9], for instantons.

In Sections 1.2 and 1.3 we describe Yang-Mills instantons, Nahm's Equations, Monopoles and the Nahm Transform. Then, in Section 1.4, we demonstrate both approaches to the Nahm Transform by finding the Dirac dipole using each method, and by finding the Higgs field of the $U(2)$ monopole using the Green's function method.

It is natural to wonder how the connection to quantum gauge theory arises. In fact, D-branes and instantons are closely related. Both are BPS objects (i.e. they satisfy a mass-charge relation), and as shown in [10] a $D(p-4)$ -brane will, from the point of view of a Dp -brane coincident in the $p-4$ dimensions, behave as an instanton. We can therefore realise given instanton configurations in string theory as D-brane configurations [11]. Since there is a $(p+1)$ -dimensional quantum $U(1)$ gauge theory on the worldvolume of a single Dp -brane, we can see how the connection between instantons and quantum gauge theories arises. In Section 1.5 of this chapter we will discuss gauge theories on D-branes in more detail.

In Chapters 2 and 3 we consider a configuration of intersecting D3- and D5-

branes, and we write a related formula for calorons (instantons on $\mathbb{R}^3 \times S^1$). This configuration was studied in [12] and the action for the gauge theory on the world-volume of the D3-branes was written in $N = 2, d = 3$ superspace in [4]. A formula for a charge one caloron was derived in [1] [13], and here we generalize this formula for charge k .

We expand on the work done in [4] by introducing an NS5-brane to their model, and we write the Lagrangian for the gauge theory on the D3-brane in this configuration. We demonstrate that the classical supersymmetric vacua of this gauge theory correspond to Bow data for an instanton on the Taub-NUT space. In Chapter 4 we derive a formula for charge k instantons in the Taub-NUT space. The NS5-brane and the Taub-NUT space are related by T-duality [14], thus the instanton formula in Chapter 4 is related to the brane configuration in the presence of the NS5-brane. Finally, in Chapter 5, we give an explicit expression for a charge one instanton on the Taub-NUT space. A statement of the results we have achieved and presented in this thesis, along with a brief description of these, is given in Section 1.7.

1.2 Yang-Mills Instantons

1.2.1 The Instanton Equations

A good description of Yang-Mills instantons, which we follow here, is given in [15]. Yang-Mills Theory is a gauge theory whose observable quantities are invariant under the space-time dependent transformations of a non-Abelian gauge group $SU(N)$. The Yang-Mills gauge field $A = A_\mu dx^\mu$ takes values in the Lie algebra of the gauge group, i.e. A_μ is an $N \times N$ Hermitian traceless matrix. The curvature is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu].$$

Physical quantities are invariant under transformations

$$A_\mu \rightarrow g^{-1} A_\mu g + i g^{-1} \partial_\mu g,$$

$$F_{\mu\nu} \rightarrow g^{-1} F_{\mu\nu} g.$$

The pure Yang-Mills action is

$$S = -\frac{1}{2} \int_{\mathbb{R}^4} d^4x \text{Tr} F_{\mu\nu} F^{\mu\nu} = - \int_{\mathbb{R}^4} \text{Tr}(F \wedge *F), \quad (1.1)$$

where $F = F_{\mu\nu} dx^\mu dx^\nu$ and can be written in terms of the one-form A as

$$F = dA - iA \wedge A. \quad (1.2)$$

In the above the star operator is the Hodge star ($*F_{12} = F_{34}$ and cyclic permutations). The covariant derivative is $D_\mu = \partial_\mu - iA_\mu$ or, in terms of forms, the covariant derivative of F is

$$DF = dF - iA \wedge F + iF \wedge A. \quad (1.3)$$

The Bianchi Identity,

$$DF = 0,$$

follows from substituting Equation (1.2) into Equation (1.3). The equations of motion are found by varying the action:

$$D * F = 0. \quad (1.4)$$

So we see that if we find F such that

$$F = \pm * F, \quad (1.5)$$

the equation of motion is solved.

We require the action to be finite, so $F \rightarrow 0$ as $|x| \rightarrow \infty$. We do not need A to be zero in order to satisfy this requirement, it will suffice if A is a gauge transformation of zero, i.e. if

$$A_\mu \rightarrow ig^{-1}\partial_\mu g.$$

So g gives a map from S_∞^3 to $SU(N)$. This map has an integer winding number

$$k = -\frac{1}{8\pi^2} \int_{\mathbb{R}^4} \text{Tr}(F \wedge F). \quad (1.6)$$

This integer k is the second Chern class. It is a topological invariant that counts the number of times the three-sphere at infinity wraps itself around the group, and we shall often refer to it as the instanton charge. Different values of k label different sectors of the space of field configurations.

We can decompose F into its self-dual and anti-self-dual parts, F_+ and F_- respectively,

$$F = F_+ \oplus F_-.$$

Then Equations (1.1) and (1.6) can be rewritten as follows:

$$S = \|F\|^2 = \|F_+\|^2 + \|F_-\|^2, \quad (1.7)$$

$$8\pi^2 k = \|F_+\|^2 - \|F_-\|^2. \quad (1.8)$$

We deduce the following lower bound on the Lagrangian:

$$L \geq 8\pi^2 |k|, \quad (1.9)$$

where the equality holds if and only if $F = \pm *F$. The solutions to

$$F = *F, \quad (1.10)$$

are called instantons and are labelled by their charge k . Solutions to the equation $F = -*F$ are anti-instantons and have charge $-k$. Since we can get to one from the other by a parity transformation, we will concentrate here on instantons. So we see that a Yang-Mills instanton is a gauge field A that solves the self-duality equations $F = *F$ and approaches pure gauge at infinity. Instantons are classified by their topological charge k .

1.2.2 A Brief History of Instanton Solutions

An excellent review of instantons and monopoles is given in [16], while a good up-to-date review of instantons, monopoles and their connection to supersymmetry and string theory is given in [17]. Here we outline the developments in the study of Yang-Mills instantons as they occurred.

In 1975, the charge one $SU(2)$ instanton was written explicitly by Belavin, Polyakov, Schwartz and Tyupkin [18]. It has the following form:

$$A_\mu = \frac{\rho^2(x - X)_\nu}{(x - X)^2((x - X)^2 + \rho^2)} \bar{\eta}_{\mu\nu}^i (g\sigma_i g^{-1}), \quad (1.11)$$

where ρ and X_μ give the scale and position of the instanton respectively, and g is the $SU(2)$ gauge parameter. These eight parameters are the degrees of freedom of the charge one solution. The 't Hooft tensor [19], $\bar{\eta}$, in Equation (1.11) is composed of unit quaternions e_μ :

$$\bar{\eta}_{\mu\nu} = \bar{e}_{[\mu} e_{\nu]} = \bar{e}_\mu e_\nu - \bar{e}_\nu e_\mu. \quad (1.12)$$

We choose the following representation for the quaternions:

$$\begin{aligned} e_0 &= 1_{2 \times 2}, \\ e_j &= -i\sigma_j, \end{aligned} \quad (1.13)$$

where the Pauli matrices, σ_j , are defined as follows:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The 't Hooft tensors $\eta_{\mu\nu}$ and $\bar{\eta}_{\mu\nu}$ have the very useful property that they are self-dual and anti-self-dual, respectively. That is,

$$\eta_{\mu\nu} = \epsilon_{\mu\nu\lambda\rho} \eta_{\lambda\rho} = * \eta_{\mu\nu}, \quad \bar{\eta}_{\mu\nu} = -\epsilon_{\mu\nu\lambda\rho} \bar{\eta}_{\lambda\rho} = - * \bar{\eta}_{\mu\nu}. \quad (1.14)$$

They can be written as a linear combination of the Pauli matrices, as follows:

$$\eta_{\mu\nu} = e_{[\mu} \bar{e}_{\nu]} = \eta_{\mu\nu}^a \sigma_a, \quad (1.15)$$

$$\bar{\eta}_{\mu\nu} = \bar{\eta}_{\mu\nu}^a \sigma_a. \quad (1.16)$$

The solution written in Equation (1.11) is known as the BPST instanton.

The next important step came with the introduction of the 't Hooft ansatz, which is written below:

$$A_\mu = \frac{1}{2} \bar{\eta}_{\mu\nu} \partial_\nu \log \phi, \quad (1.17)$$

where

$$\phi = 1 + \sum_k \frac{\rho^2}{|x - X|^2}, \quad (1.18)$$

and ρ and X are the instanton scale and position, as before. The 't Hooft ansatz allowed the construction of arbitrary charge instantons, but does not give a complete construction of all arbitrary charge solutions. In 1977 the Atiyah-Ward ansatz [20] provided a recursive method of finding solutions to the self-duality equations, which should generate the complete solution. However this approach encountered problems with global singularities, and in 1978 Atiyah, Drinfel'd, Hitchin and Manin [8] found a complete construction for all arbitrary charge instantons on $\mathbb{R}^4$.

We briefly present the details of the ADHM construction here. We start with a quaternionic $(k+1) \times k$ matrix $\Delta(x)$ that is linear in x and satisfies the condition that $\Delta^\dagger \Delta$ is a real invertible $k \times k$ matrix. We must then find a $k+1$ -dimensional column vector Ψ such that

$$\Delta^\dagger \Psi = 0, \quad \text{and} \quad \Psi^\dagger \Psi = 1. \quad (1.19)$$

Finally the charge k instanton solution is found from

$$A_\mu = \Psi^\dagger \partial_\mu \Psi. \quad (1.20)$$

It can be shown that the curvature obtained from A_μ in Equation (1.20) is self-dual, and that the construction gives the complete solution for instantons on $\mathbb{R}^4$ [8].

1.3 Monopoles, Calorons, and the Nahm Transform

1.3.1 Monopoles

A magnetic monopole (Φ, A) is a time-independent, three-dimensional gauge field A and a Higgs field Φ that transform as follows under gauge transformations:

$$\Phi \rightarrow g^{-1} \Phi g, \quad A_j \rightarrow g^{-1} A_j + i g^{-1} \frac{dg}{dx_j}, \quad i = 1, 2, 3.$$

The self-duality equations for a monopole are called the Bogomolny Equations:

$$D\Phi = - * F, \quad (1.21)$$

where $D\Phi = d\Phi - iA \wedge \Phi + i\Phi \wedge A$.

A monopole which solves Equation (1.21) and has finite energy

$$E = \int_{\mathbb{R}^3} (||F||^2 + ||D\Phi||^2) d^3x, \quad (1.22)$$

is called a BPS monopole [5] [6]. Monopoles are classified by their non-Abelian charge:

$$m = \frac{1}{4\pi} \int_{\mathbb{R}^3} \text{tr } F \wedge D\Phi. \quad (1.23)$$

In 1974 the first nonabelian monopoles solution were found separately by 't Hooft [21] and Polyakov [22]. They showed that monopoles are solutions to a Yang-Mills-Higgs field theory with Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + D_\mu \Phi D^\mu \Phi - \frac{\lambda}{4} V(\Phi), \quad (1.24)$$

where Φ is a Higgs field. An exact analytic solution can be found in the case where $\lambda \rightarrow 0$. This is the 't Hooft-Polyakov monopole in the BPS limit [5] [6]:

$$\Phi = \frac{\vec{\sigma} \cdot \vec{r}}{2r^2} (2\lambda r \coth 2\lambda r - 1) \quad (1.25)$$

$$A = \frac{d\vec{x} \cdot (\vec{\sigma} \times \vec{r})}{2r^2} \left(\frac{2\lambda r}{\sinh 2\lambda r} - 1 \right). \quad (1.26)$$

With the appearance of the Nahm Transform in the early 1980s, there was a method of constructing monopole solutions analogous to (in fact, generalising) the ADHM construction for instantons.

1.3.2 The Nahm Transform

The Nahm Transform generalises the ADHM construction to include a construction for monopoles and periodic instantons called calorons [7] [23] [24] [25]. Here we give a brief description of the Nahm Transform.

Nahm data consists of $k \times k$ matrices $T_\mu, \mu = 0, \dots, 3$, where T_0 is a connection

in the x_0 -direction (which is parametrised by s from here on) and $T_i, i = 1, 2, 3$, are scalar fields. The Nahm data have rank k and satisfy the following equations, called the Nahm Equations:

$$\begin{aligned}\frac{dT_1}{ds} - i[T_0, T_1] + i[T_2, T_3] &= 0 \\ \frac{dT_2}{ds} - i[T_0, T_2] + i[T_3, T_1] &= 0 \\ \frac{dT_3}{ds} - i[T_0, T_3] + i[T_1, T_2] &= 0.\end{aligned}\tag{1.27}$$

The Nahm data T_μ are in fact a connection T_0 , and three endomorphisms T_j , on an $SU(k)$ vector bundle E over an interval $\mathcal{I}$ parametrised by s . The inner product on the space of $\mathcal{L}^2$ sections of E is

$$\langle \alpha, \beta \rangle = \int_{\mathcal{I}} \alpha^\dagger \beta ds.\tag{1.28}$$

The Nahm Transform provides a method of obtaining monopoles from Nahm data. It works as follows:

We define a Dirac-type operator from the Nahm data. This we call the Weyl operator, $\mathcal{D}$. It acts on a bundle $S \otimes E$, where S is a rank two Hermitian spinor bundle, and E is the Hermitian bundle described above. We write the Weyl operator here:

$$\mathcal{D} = e_0 \otimes \left(\frac{d}{ds} - iT_0 \right) - i \sum_{j=1}^3 e_j \otimes T_j.\tag{1.29}$$

We use this slash notation throughout this work to indicate multiplication by the Pauli matrices. We choose the same representation for the unit quaternions e_μ as we wrote in Section 1.2.2. The operator dual to $\mathcal{D}$ can be obtained using (1.28). We define the twisted dual Weyl operator:

$$\mathcal{D}_t^\dagger = -e_0 \otimes \left(\frac{d}{ds} - i(T_0 - t_0) \right) - i \sum_{j=1}^3 e_j \otimes (T_j - t_j),\tag{1.30}$$

where $t_\mu = (t_0, t_j) \in \mathbb{R}^4$. To be precise, instead of $T_\mu - t_\mu$, we should write $T_\mu \otimes 1 - 1 \otimes t_\mu$. For the sake of clearer presentation we leave out this extra notation. In all of the calculations we perform we use Abelian ($k=1$) Nahm data and the expression above is correct as it is written. The orthonormal basis of the kernel of the twisted Weyl operator is formed by the columns of Ψ , i.e.

$$\mathcal{D}_t^\dagger \Psi = 0. \quad (1.31)$$

In other words Ψ forms an orthonormal basis for solutions to the Weyl Equation (1.31). Now, once we solve the Weyl Equation for Ψ , we can write an expression for a monopole (Φ, A) using the formulae:

$$A_j = i \int ds \Psi^\dagger \frac{\partial}{\partial t_j} \Psi, \quad (1.32)$$

$$\Phi = \int ds \Psi^\dagger s \Psi. \quad (1.33)$$

The Nahm Transform also works in the opposite direction, i.e. we can get Nahm data from monopole data. Applied twice it returns the original fields. It exchanges rank and charge, so that the rank of the Nahm data will give the charge on the monopole, while we can see from Equations (1.32) and (1.33) that the rank of the monopole is $\dim \text{Ker } \mathcal{D}_t^\dagger$.

It is particularly interesting to write the Laplacian of the Weyl operator:

$$\begin{aligned} \mathcal{D}_t^\dagger \mathcal{D}_t = e_0 \otimes & \left(- \left(\frac{d}{ds} - i(T_0 - t_0) \right)^2 + \sum_i (T_i - t_i)^2 \right) \\ & - i e_i \otimes \sum_i \left(\frac{dT_i}{ds} - i[T_0, T_i] + i[T_j, T_k] \right). \end{aligned} \quad (1.34)$$

The second term is zero due to the Nahm Equations. Therefore the Laplacian is proportional to the identity in spin space, and commutes with sigma matrices. It is also positive definite as we can see from the first term in Equation (4.22), so

$\text{Ker } \mathcal{D}_t^\dagger \mathcal{D}_t = 0$. Thus it is invertible and has a Green's function. The Green's function will also commute with the sigma matrices. These properties can be used to write instanton formulae in terms of the Green's function.

The Nahm transform was generalised further by Braam and van Baal [26], who formulated it as a correspondence between solutions of self-duality equations on dual spaces. As such, the constructions for instantons, monopoles and calorons are specific examples of the Nahm Transform in use. They considered self-dual fields on $\mathbb{R}^4/H$, where H is a translational subgroup under which the physics is invariant [26]. The Nahm transform then provides a correspondence between self-dual fields on $\mathbb{R}^4/H$ and $\mathbb{R}^4/H^*$. We have outlined the Nahm Transform for $H = \mathbb{R}$, i.e. for BPS magnetic monopoles. In this case H^* is $\mathbb{R}^3$ and the correspondence is between monopoles and Nahm data. Also, in these terms the Nahm Transform for calorons is a correspondence between self-dual fields on $\mathbb{R}^4/\mathbb{Z}$ and S^1 . Another example is the ADHM construction, which corresponds to the case where H is a point in $\mathbb{R}^4$.

1.3.3 Boundary Conditions for the Nahm Transform

The Nahm transform allows us to construct a magnetic monopole from Nahm data over an interval on $\mathbb{R}$. Here we consider the boundary conditions on that interval [7] [25] [27] [28]. There are three possibilities. The interval can be the whole of $\mathbb{R}$, it can extend infinitely in either the positive or negative direction, or it can be a finite interval $[p, q]$ with $p, q \in \mathbb{R}$.

We call points such as p and q jumping points. Nahm data T_μ on an interval, away from the jumping points, solve the Nahm Equations (1.27). We may have a number of neighbouring intervals with a jumping point at each boundary. If we have $n + 1$ neighbouring intervals there will be n jumping points that we shall denote by $\lambda_i, i = 1, \dots, n$. There may be jumps in the rank of the Nahm data from one interval to the next, however we shall not consider this case here but will concentrate on the case where the rank of the Nahm data in neighbouring intervals is equal. In such a

situation there will be discontinuities in the Nahm data at the jumping points and we must require that the limits of the Nahm data on either side approaching the discontinuity exist. The Nahm equations in the vicinity of a given jumping point, λ_p become:

$$\frac{d(T_i)_{ab}}{ds} - i[T_0, T_i]_{ab} + i[T_j, T_k]_{ab} = -\frac{1}{2}\delta(s - \lambda_p)Q_{a\alpha}^\dagger(\sigma_i)_{\alpha\beta}Q_{\beta b}, \quad (1.35)$$

and cyclic permutations, where Q is a $2k$ -component vector, α, β are spinor indices that run from 1 to 2, and a, b label the matrix elements of the rank k Nahm data. We shall only consider Abelian Nahm data in this work, in which case the commutators in Equations (1.35) vanish. The discontinuity in the Nahm data at λ_p can then be found by integrating Equations (1.35) for Abelian Nahm data about the jumping point:

$$T_i(\lambda_p+) - T_i(\lambda_p-) = -\frac{1}{2}Q^\dagger\sigma_iQ, \quad i = 1, 2, 3. \quad (1.36)$$

1.3.4 Calorons

If the Nahm interval from the previous section is a circle, then the Nahm transform will produce periodic instantons on $\mathbb{R}^3 \times S^1$ called calorons. At jumping points along the circle the boundary conditions from the previous section remain valid. The charge one $SU(2)$ caloron constructed by Harrington and Shepard in 1978 [29] was the first explicit solution. Calorons can be shown to be composed of constituents, each of which is like a BPS monopole [1] [13] [30] [31] [32]. The caloron solution thus shows aspects of monopole and of instanton behaviour. It has both an instanton charge and monopole charges [33]. Here we will consider only the case where the monopole charges are zero.

Caloron solutions can be constructed using Green's functions of the differential operators defined in the Nahm Transform, such as that written in Equation (1.34). In fact, this approach was used by Kraan and van Baal [1] to derive a formula

for a charge one $SU(2)$ caloron with arbitrary holonomy. Their formula is simple and can be used to write explicit solutions. Their method is to define two Green's functions in terms of ADHM operators, and to write the formula in terms of one of these Green's functions, which commutes with the quaternions. We write their result here:

$$A_\mu(x) = \frac{1}{2}\phi(x)\partial_\nu(\lambda\bar{\eta}_{\mu\nu}f_x\lambda^\dagger). \quad (1.37)$$

The Green's function is f_x , λ is a quaternionic row vector that is the ADHM data at the jumping points, and the function $\phi = (1 - \lambda f_x \lambda^\dagger)^{-1}$. In effect, the calculation of the caloron solution has been reduced to the calculation of the Green's function f_x . This is a straightforward calculation that involves solving matching conditions on the circle.

1.4 The Green's Function Method in Practice

1.4.1 The Dirac Dipole

Here we introduce the Green's function approach to the Nahm Transform by presenting some explicit examples. The idea is that if we find the Green's function of the Laplacian $\mathcal{D}_t^\dagger \mathcal{D}_t$ of the Weyl Operator, we can use it to avoid the final step of the Nahm Transform, i.e. the integration over the interval $\mathcal{I}$. We demonstrate both methods here with a simple example, the Dirac dipole. The Dirac dipole can be obtained using the Nahm Transform from noncompact Nahm data with a single jumping point, which we take to be at the origin. First of all we find an expression for the Dirac dipole by solving the Weyl Equation and calculating (Φ, A) from the formulae:

$$\Phi = \int_{\mathcal{I}} \psi^\dagger s \psi ds \quad (1.38)$$

$$A_j = \frac{i}{2} \left((\Psi, \partial_j \Psi) - (\partial_j \Psi, \Psi) \right) = \frac{i}{2} \int_{\mathcal{I}} \left(\psi^\dagger \partial_j \psi - (\partial_j \psi^\dagger) \psi \right) ds + \frac{i}{2} \left(\chi^\dagger \partial_j \chi - (\partial_j \chi^\dagger) \chi \right), \quad (1.39)$$

where the kernel of the Weyl operator is $\Psi = \begin{pmatrix} \psi \\ \chi \end{pmatrix}$, with ψ acted on by Nahm data on the interval and χ by Nahm data at the jumping point, as we shall see below. For compactness we write ∂_j for $\frac{\partial}{\partial t_j}$, where t_j is the j th component of the vector $\vec{t} \in \mathbb{R}^3$.

The second method of finding the Dirac dipole is to find $F(0,0)$, the Green's function valued at the jumping point, of the Laplacian $\mathcal{D}_t^\dagger \mathcal{D}_t$ and use the formula:

$$A_\mu = \frac{1}{4} \chi^\dagger R \partial_\nu (Q^\dagger \tilde{\eta}_{\mu\nu} F(0,0) Q) R \chi + \frac{1}{2} (\chi^\dagger R \partial_\mu \chi - \partial_\mu \chi^\dagger R \chi), \quad (1.40)$$

where $R = (1 - Q^\dagger F(0,0) Q)^{-1} = (\chi \chi^\dagger)^{-1}$. We will provide a detailed derivation of a more general version of this formula in Chapter 3, and thus do not derive it here.

Method 1

The Nahm data for a Dirac dipole is:

$$\vec{T}(s) = \begin{cases} \vec{T}_L \in \mathbb{R}^3 & \text{if } s < 0, \\ \vec{T}_R \in \mathbb{R}^3 & \text{if } s > 0. \end{cases} \quad (1.41)$$

Since we have Abelian Nahm data the commutators from Equation (1.35) vanish and we determine the jumping point Nahm data (up to a phase factor) from Equation (1.36). Letting $\vec{y} = \vec{T}_R - \vec{T}_L$, we find

$$\vec{y} = -\frac{1}{2} Q_-^\dagger \vec{\sigma} Q_- \quad (1.42)$$

$$\Rightarrow Q_- Q_-^\dagger = y - \vec{\sigma} \cdot \vec{y} = y - \mathfrak{y}. \quad (1.43)$$

The following relations will also be useful:

$$\vec{z}_L = \vec{t} - \vec{T}_L, \quad \vec{z}_R = \vec{t} - \vec{T}_R, \quad \zeta_\pm \zeta_\pm^\dagger = z \pm \mathfrak{z}, \quad (1.44)$$

where $\vec{t} \in \mathbb{R}^3$ and $\zeta_{\pm}$ are two-component spinors. The jumping point Nahm data Q_- is also a two-component spinor since we are dealing with rank 1 Nahm data. The slash notation denotes multiplication by the Pauli matrices, i.e. $\not{x} = \sigma_1 z_1 + \sigma_2 z_2 + \sigma_3 z_3$. We can now write the Weyl Equation:

$$\left(-\frac{d}{ds} + i(T_0 - t_0) - \vec{\sigma} \cdot (\vec{T} - \vec{t}) \right) \psi(s) + Q_- \chi \delta(s) = 0, \quad (1.45)$$

This reduces under a gauge transformation (that gauges $(T_0 - t_0)$ to zero) to

$$\left(-\frac{d}{ds} + \not{x} \right) \psi(s) + Q_- \chi \delta(s) = 0, \quad (1.46)$$

which has as its general solution $\psi(s) = a e^{\not{x}s} \zeta$. Now, imposing the decay conditions $\psi(s) \rightarrow 0$ as $s \rightarrow \pm\infty$, we have

$$\psi_L = a_L e^{z_L s} \zeta_{L+}, \quad (1.47)$$

$$\psi_R = a_R e^{-z_R s} \zeta_{R-}, \quad (1.48)$$

and thus the matching condition $-\psi(0+) + \psi(0-) + Q_- \chi = 0$, from integrating Equation (1.46), becomes

$$a_R \zeta_{R-} - a_L \zeta_{L+} = Q_- \chi. \quad (1.49)$$

Making use of the fact that $\zeta_{R+}^\dagger \zeta_{R-} = \zeta_{L-}^\dagger \zeta_{L+} = Q_+^\dagger Q_- = 0$, we find that

$$a_L = -\frac{\zeta_{R+}^\dagger Q_-}{\zeta_{R+}^\dagger \zeta_{L+}} \chi, \quad a_R = \frac{\zeta_{L-}^\dagger Q_-}{\zeta_{L-}^\dagger \zeta_{R-}} \chi, \quad (1.50)$$

and

$$a_R Q_+^\dagger \zeta_{R-} = a_L Q_+^\dagger \zeta_{L+}. \quad (1.51)$$

From Equations (1.50) we get

$$\begin{aligned} |a_L|^2 &= |\chi|^2 \frac{z_R y + \vec{z}_R \cdot \vec{y}}{z_L z_R + \vec{z}_L \cdot \vec{z}_R}, \\ |a_R|^2 &= |\chi|^2 \frac{z_L y + \vec{z}_L \cdot \vec{y}}{z_L z_R + \vec{z}_L \cdot \vec{z}_R}. \end{aligned} \quad (1.52)$$

The normalization condition

$$\int |\psi|^2 ds + |\chi|^2 = 1, \quad (1.53)$$

tells us that

$$|a_L|^2 + |a_R|^2 + |\chi|^2 = 1,$$

and thus

$$|\chi|^2 = \frac{z_L + z_R - y}{z_L + z_R + y}, \quad |a_L|^2 = \frac{-z_L + z_R + y}{z_L + z_R + y}, \quad |a_R|^2 = \frac{z_L - z_R + y}{z_L + z_R + y}. \quad (1.54)$$

The Higgs field can now be calculated from Equation (1.38):

$$\Phi = |a_R|^2 \zeta_{R-}^\dagger \zeta_{R-} \int_0^\infty s e^{-2z_R s} ds + |a_L|^2 \zeta_{L+}^\dagger \zeta_{L+} \int_{-\infty}^0 s e^{2z_L s} ds = \frac{1}{2z_R} - \frac{1}{2z_L}. \quad (1.55)$$

We may choose χ to be real. In this case Equation (1.39) becomes:

$$\begin{aligned} -2iA_j &= \int \left(\psi^\dagger \partial_j \psi - (\partial_j \psi^\dagger) \psi \right) ds \\ &= \frac{1}{2z_L} \left(\zeta_{L+}^\dagger \partial_j \zeta_{L+} - (\partial_j \zeta_{L+}^\dagger) \zeta_{L+} \right) + \frac{1}{2z_R} \left(\zeta_{R-}^\dagger \partial_j \zeta_{R-} - (\partial_j \zeta_{R-}^\dagger) \zeta_{R-} \right) \\ &\quad + \frac{1}{2} \partial_j \ln \frac{\zeta_{L+}^\dagger Q_- \zeta_{R-}^\dagger Q_- \zeta_{L+}^\dagger Q_+ \zeta_{R-}^\dagger Q_+}{Q_-^\dagger \zeta_{L+} Q_-^\dagger \zeta_{R-} Q_+^\dagger \zeta_{L+} Q_+^\dagger \zeta_{R-}}. \end{aligned} \quad (1.56)$$

We can gauge away the final term in Equation (1.56). Using Equations (1.44) to determine ζ_{L+} and ζ_{R-} , and noting that $dz_j = dt_j$, we find

$$A = A_j dt^j = \frac{z_L^2 dz_L^1 - z_L^1 dz_L^2}{2z_L(z_L + z_L^3)} - \frac{z_R^1 dz_R^2 - z_R^2 dz_R^1}{2z_R(z_R + z_R^3)}. \quad (1.57)$$

Method 2

Now we try the alternative method. In this case we find the expression for the Dirac Dipole using the Green's function $F(s, s_0)$ of the Laplacian $\mathcal{D}_x^\dagger \mathcal{D}_x$. The Green's function satisfies

$$\left(\left(-\frac{d}{ds} + \mathfrak{z} \right) \left(\frac{d}{ds} + \mathfrak{z} \right) + Q_- Q_-^\dagger \delta(s) \right) F(s, s_0) = \delta(s - s_0). \quad (1.58)$$

Solving for $F(s, s_0)$, we have

$$F(s, s_0 \geq 0) = \begin{cases} \frac{1}{z_L + z_R + d} e^{z_L s} e^{z_R s_0}, & \text{for } s < 0 \\ \frac{1}{2z_R} e^{-z_R |s - s_0|} - \frac{1}{2z_R} \frac{z_L - z_R + d}{z_L + z_R + d} e^{-z_R(s + s_0)}, & \text{for } s > 0 \end{cases} \quad (1.59)$$

$$F(s, s_0 \leq 0) = \begin{cases} \frac{1}{2z_L} e^{-z_L |s - s_0|} - \frac{1}{2z_L} \frac{-z_L + z_R + d}{z_L + z_R + d} e^{z_L(s + s_0)}, & \text{for } s < 0 \\ \frac{1}{z_L + z_R + d} e^{-z_R s} e^{z_L s_0}, & \text{for } s > 0 \end{cases} \quad (1.60)$$

so we have

$$F(0, 0) = \frac{1}{(z_L + z_R + d)}. \quad (1.61)$$

We use Equation (1.40) to find an expression for the Higgs field:

$$\Phi = \frac{1}{2} (\chi \chi^\dagger)^{-1} \partial_j (Q_-^\dagger \sigma^j F(0, 0) Q_-) + (\chi^{-1} \chi \Lambda - \Lambda \chi^\dagger (\chi^\dagger)^{-1}), \quad (1.62)$$

where in general $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$ for n jumping points. In this case we have a single jumping point at the origin so $\Lambda = 0$. Thus the second term vanishes and, using Equation (1.54) and noting that $Q_-^\dagger \sigma_j Q_- = -2y_j$, we find the following expression for the Higgs field:

$$\Phi = -\frac{z_L + z_R + y}{z_L + z_R - y} y_j \frac{\partial}{\partial t_j} \left(\frac{1}{z_L + z_R + y} \right) = \frac{1}{2z_R} - \frac{1}{2z_L}. \quad (1.63)$$

In the case of the gauge field Equation (1.40) becomes

$$2A_j = \frac{1}{2}\chi^{-1} \left(-\epsilon_{ijk}\partial_k(Q_-^\dagger i\sigma_i FQ_-) + Q_-^\dagger \sigma_j FQ_- \Lambda - \Lambda Q_-^\dagger \sigma_j FQ_- \right) (\chi^\dagger)^{-1} \\ + \chi^{-1}\partial_j\chi - \partial_j\chi^\dagger(\chi^\dagger)^{-1}. \quad (1.64)$$

Terms involving Λ again vanish and, since we have chosen χ to be real, the final two terms cancel. We therefore have the following expression for the gauge field:

$$-2iA_j = \frac{z_L + z_R + y}{z_L + z_R - y} \epsilon_{ijk} \partial_k \frac{y_i}{z_L + z_R + y} = \left(\frac{1}{z_L} + \frac{1}{z_R} \right) \frac{(z_L \times z_R)^j}{(z_L + z_R)^2 - y^2}. \quad (1.65)$$

Comparing Results

We can now compare our results. Expressions (1.57) and (1.65) can be compared using the formula

$$\frac{\sin(\theta_1 - \theta_2)}{1 + \cos(\theta_1 - \theta_2)} (\sin \theta_1 + \sin \theta_2) = -(\cos \theta_1 - \cos \theta_2). \quad (1.66)$$

Equation (1.65) becomes

$$-2iA_j dt_j = (\sin \theta_1 + \sin \theta_2) \frac{z_R z_L \sin(\theta_1 - \theta_2)}{2z_L z_R + 2\vec{z}_L \cdot \vec{z}_R} = \frac{1}{2} (\sin \theta_1 + \sin \theta_2) \frac{\sin(\theta_1 - \theta_2)}{1 + \cos(\theta_1 - \theta_2)}, \quad (1.67)$$

while Equation (1.57) can be written

$$A_j dt_j = \frac{z_L^2 dz_L^1 - z_L^1 dz_L^2}{2z_L(z_L + z_L^3)} - \frac{z_R^1 dz_R^2 - z_R^2 dz_R^1}{2z_R(z_R + z_R^3)} = \frac{1 - \cos \theta_1}{2} d\phi - \frac{1 - \cos \theta_2}{2} d\phi \\ = -\frac{1}{2} (\cos \theta_1 - \cos \theta_2) d\phi. \quad (1.68)$$

We see that the solutions found using each method agree.

1.4.2 The $U(2)$ Higgs Field

Here we write an expression for the Higgs field of the $U(2)$ Monopole with two singularities using the Green's function approach. This was calculated in [34] by solving the Weyl Equation and using Equations (1.33) and (1.32) to find the Higgs and gauge field. The Nahm data in this case is:

$$\vec{T}(s) = \begin{cases} \vec{T}_L \in \mathbb{R}^3 & \text{if } s < -\lambda, \\ \vec{T}_{tHP} \in \mathbb{R}^3 & \text{if } s \in (-\lambda, \lambda), \\ \vec{T}_R \in \mathbb{R}^3 & \text{if } s > \lambda. \end{cases} \quad (1.69)$$

Just as for the Dirac dipole, the following relations will be useful:

$$\vec{y}_L = \vec{T}_{tHP} - \vec{T}_L, \quad \vec{y}_R = \vec{T}_{tHP} - \vec{T}_R, \quad (1.70)$$

$$\vec{z}_L = \vec{t} - \vec{T}_L, \quad \vec{z}_R = \vec{t} - \vec{T}_R, \quad \vec{r} = \vec{t} - \vec{T}_{tHP}, \quad (1.71)$$

where $\vec{t} \in \mathbb{R}^3$ and $\vec{T}_{tHP}$ is the Nahm data for the 't Hooft-Polyakov Monopole [19].

The Higgs field Φ is obtained from the formula

$$\Phi = \frac{1}{4} \chi^{-1} \partial_j (Q^\dagger \sigma^j F(\lambda_i, \lambda_j) Q) (\chi^\dagger)^{-1} + \frac{1}{2} \left(\chi^{-1} \partial_0 \chi - (\partial_0 \chi^\dagger) (\chi^\dagger)^{-1} \right), \quad (1.72)$$

where

$$\chi = (\chi_-, \chi_+), \quad \text{and} \quad \chi_s = \frac{\zeta^\dagger}{\zeta^\dagger Q} \psi_{tHP}(s) N, \quad (1.73)$$

with ζ and Q two-component spinors as in the previous example, N a normalization constant and ψ_{tHP} the solution to the Weyl equation for the 'tHooft-Polyakov monopole:

$$\psi_{tHP}(s) = \sqrt{\frac{r}{\sinh 2\lambda r}} e^{(-ix_0 + \vec{\sigma} \cdot \vec{r})s}. \quad (1.74)$$

Using Equation (1.74) we find

$$\partial_0 \chi_s = -is\chi_s, \quad \partial_0 \chi_s^\dagger = is\chi_s^\dagger. \quad (1.75)$$

We need the Green's functions with sources at the jumping points in order to use Equation (1.72). The Green's function of $\mathcal{D}_t^\dagger \mathcal{D}_t$, where $\mathcal{D}_t^\dagger$ is the Weyl operator for the $U(2)$ Monopole, is written in [35]:

$$\begin{aligned} F(-\lambda, -\lambda) &= \frac{k_L}{\mathcal{K}}, & F(\lambda, \lambda) &= \frac{k_R}{\mathcal{K}}, \\ F(-\lambda, \lambda) &= F(\lambda, -\lambda) = \frac{r}{\mathcal{K}}, \end{aligned} \quad (1.76)$$

where, for compactness of expression, we have defined:

$$\begin{aligned} k_\alpha &= r \cosh 2\lambda r + a_\alpha \sinh 2\lambda r, \\ \mathcal{K} &= (a_L a_R + r^2) \sinh 2\lambda r + r(a_L + a_R) \cosh 2\lambda r, \end{aligned} \quad (1.77)$$

and where $a_L = z_L + y_L$ and $a_R = z_R + y_R$. We proceed to write an expression for the Higgs field in terms of the Green's function. We first consider the diagonal elements, since they are simpler than the off-diagonal terms:

$$4\Phi_{11} = (\chi_- \chi_-)^{-1} \partial_j (Q_{1-}^\dagger \sigma^j F(-\lambda, -\lambda) Q_{1-}) + 2 \left(\chi_-^{-1} \partial_0 \chi_- - (\partial_0 \chi_-^\dagger) \chi_-^{\dagger-1} \right), \quad (1.78)$$

$$4\Phi_{22} = (\chi_+ \chi_+)^{-1} \partial_j (Q_{1+}^\dagger \sigma^j F(\lambda, \lambda) Q_{1+}) + 2 \left(\chi_+^{-1} \partial_0 \chi_+ - (\partial_0 \chi_+^\dagger) (\chi_+^\dagger)^{-1} \right). \quad (1.79)$$

Using the following relations:

$$\chi_- \chi_-^\dagger = 1 - Q_{L-}^\dagger F(-\lambda, -\lambda) Q_{L-} = 1 - 2y_L F(-\lambda, -\lambda), \quad (1.80)$$

$$\chi_+ \chi_+^\dagger = 1 - Q_{R+}^\dagger F(\lambda, \lambda) Q_{R+} = 1 - 2y_R F(\lambda, \lambda), \quad (1.81)$$

$$Q_{L-}^\dagger \sigma_j Q_{L-} = -2y_L^j, \quad (1.82)$$

$$Q_{R+}^\dagger \sigma_j Q_{R+} = 2y_R^j, \quad (1.83)$$

we can write

$$\Phi_{11} = i\lambda - \frac{1}{2}(1 - 2y^L F(-\lambda, -\lambda))^{-1} y_j^L \partial_j F(-\lambda, -\lambda) = i\lambda - \frac{1}{2} \frac{\mathcal{K}}{\mathcal{K} - 2y k_L} y_L^j \partial_j \frac{k_L}{\mathcal{K}}, \quad (1.84)$$

$$\Phi_{22} = i\lambda + \frac{1}{2}(1 - 2y^R F(\lambda, \lambda))^{-1} y_j^R \partial_j F(\lambda, \lambda) = i\lambda + \frac{1}{2} \frac{\mathcal{K}}{\mathcal{K} - 2y k_R} y_R^j \partial_j \frac{k_R}{\mathcal{K}}. \quad (1.85)$$

The off-diagonal elements are

$$4\Phi_{12} = -2i\lambda \left(\chi_{-\lambda}^{-1} \chi_\lambda - \chi_{-\lambda}^\dagger (\chi_\lambda^\dagger)^{-1} \right) + (\chi_{-\lambda} \chi_\lambda^\dagger)^{-1} (Q_{1-}^\dagger \sigma^j Q_{2+}) \partial_j F(-\lambda, \lambda), \quad (1.86)$$

$$4\Phi_{21} = 2i\lambda \left(\chi_\lambda^{-1} \chi_{-\lambda} - \chi_\lambda^\dagger (\chi_{-\lambda}^\dagger)^{-1} \right) + (\chi_\lambda \chi_{-\lambda}^\dagger)^{-1} (Q_{2+}^\dagger \sigma^j Q_{1-}) \partial_j F(\lambda, -\lambda). \quad (1.87)$$

We will need expressions for $\chi_{-\lambda}$ and χ_λ ,

$$\chi_{-\lambda} = \frac{\zeta_{1-}^\dagger}{\zeta_{1-}^\dagger Q_{1-}} \sqrt{\frac{r}{\sinh 2\lambda r}} e^{-(ix_0 + \vec{\sigma} \cdot \vec{r} \lambda)} N, \quad \chi_\lambda = -\frac{\zeta_{2+}^\dagger}{\zeta_{2+}^\dagger Q_{2+}} \sqrt{\frac{r}{\sinh 2\lambda r}} e^{(-ix_0 + \vec{\sigma} \cdot \vec{r} \lambda)} N, \quad (1.88)$$

as well as the following relations:

$$\chi_{-\lambda} \chi_\lambda^\dagger = 1 - Q_{L-}^\dagger F(-\lambda, \lambda) Q_{R+}, \quad (1.89)$$

$$\chi_\lambda \chi_{-\lambda}^\dagger = 1 - Q_{R+}^\dagger F(\lambda, -\lambda) Q_{L-}. \quad (1.90)$$

We arrive at the following expressions for the two off-diagonal elements of the $U(2)$

Higgs field:

$$\begin{aligned} \Phi_{12} = i\lambda & \left(\frac{y^R (y^L z^L + y_j^L z_j^L) e^{2\vec{\sigma} \cdot \vec{r} \lambda} - y^L (y^R z^R + y_j^R z_j^R) e^{-2\vec{\sigma} \cdot \vec{r} \lambda}}{\sqrt{2(z^L z^R - z_j^L z_j^R)} (z^R y^R + z_j^R y_j^R) (z^L y^L + z_j^L y_j^L)} \right) \\ & + \frac{1}{4} (1 - \sqrt{2(y^L y^R - y_j^L y_j^R)} F_{-\lambda, \lambda})^{-1} \left(\frac{-2y^L y_j^R + 2y_j^L y^R + i(\vec{y}^L \times \vec{y}^R)^j}{\sqrt{2y^R y^L - 2y_j^R y_j^L}} \right) \partial_j F_{-\lambda, \lambda} \end{aligned} \quad (1.91)$$

$$\begin{aligned}
\Phi_{21} = i\lambda & \left(\frac{y^R(y^L z^L + y_j^L z_j^L)e^{-2\vec{\sigma} \cdot \vec{r} \lambda} - y^L(y^R z^R + y_j^R z_j^R)e^{2\vec{\sigma} \cdot \vec{r} \lambda}}{\sqrt{2(z^L z^R - z_j^L z_j^R)(z^R y^R + z_j^R y_j^R)(z^L y^L + z_j^L y_j^L)}} \right) \\
& + \frac{1}{4} \left(1 - \sqrt{2(y^R y^L - y_j^R y_j^L)F_{\lambda, -\lambda}} \right)^{-1} \left(\frac{-2y^L y_j^R + 2y^R y_j^L - i(\vec{y}^L \times \vec{y}^R)^j}{\sqrt{2y^R y^L - 2y_j^L y_j^R}} \right) \partial_j F_{\lambda, -\lambda}.
\end{aligned} \tag{1.92}$$

We conclude our introduction to instantons and the Nahm Transform with this example, and we turn to investigate the supersymmetry connection.

1.5 The Supersymmetry Connection

1.5.1 Introduction to D-branes

Let us introduce D-branes by considering a 26-dimensional bosonic string theory with the 2-dimensional string worldsheet M parametrized by the coordinates τ and σ , where $0 \leq \sigma \leq \pi$. Good references on String Theory and D-branes, which we have followed here, include [36] and [37]. The Polyakov action is

$$S_P = -\frac{1}{4\pi\alpha'} \int_M d\tau d\sigma (-\gamma)^{\frac{1}{2}} \gamma^{ab} \partial_a X^\mu \partial_b X_\mu, \tag{1.93}$$

where γ_{ab} is the Lorentzian metric on the worldsheet, $\gamma = \det \gamma_{ab}$ and α' is the Regge slope, a constant inversely related to the tension of the string. The spacetime coordinates are $X_\mu, \mu = 1, \dots, 26$.

Imposing periodic boundary conditions on the σ coordinate leads to a closed string theory, while the boundary conditions

$$\partial_\sigma X_\mu \delta X^\mu = 0, \text{ at } \sigma = 0, \pi, \tag{1.94}$$

result in an open string theory. The open string boundary conditions are Neumann for $\partial_\sigma X_\mu = 0|_{0,\pi}$ and Dirichlet if $\delta X^\mu|_{0,\pi} = 0$. The Neumann conditions correspond to open string endpoints travelling freely through spacetime at the speed of light.

Neumann conditions preserve Poincaré invariance, and result in the following mode expansion for the open string:

$$X^\mu(\tau, \sigma) = x^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'} \sum_{n \neq 0} \left(\frac{\alpha_n^\mu}{n} e^{-in\tau} \cos n\sigma \right). \quad (1.95)$$

Dirichlet boundary conditions, while they do not preserve Poincaré invariance, fix the string endpoints, thus implying the existence of objects on which the strings end. These objects are called D-branes. A Dp-brane is not only a $(p + 1)$ -dimensional hyperplane, it is a dynamical object thanks to the vibrational modes of the strings attached to it.

For simplicity, the above discussion concerned bosonic string theory, however the principle applies equally to ten-dimensional superstring theory.

1.5.2 T-duality

In the previous section we considered D-branes as objects arising from boundary conditions on open strings. Here we shall see that D-branes are required by dualities occurring in the compactification of string theory. It is of interest to examine what happens when we compactify some of the string theory dimensions since it is possible that there are, in addition to the four evident dimensions in our universe, periodic dimensions that we cannot at present detect due to their small size. Good references on this topic, which we have used here, include [36] [38].

Let us see what happens when we periodically identify a spacetime coordinate in bosonic string theory:

$$X \cong X + 2\pi R. \quad (1.96)$$

The worldsheet action and the equations of motion of the theory are unchanged by this compactification. However, it does have two effects. The first occurs since string states must be invariant under Equation (1.96). This means that the centre-of-mass

momentum is quantised:

$$k = \frac{n}{R}, n \in \mathbb{Z}.$$

The second occurs since a closed string may now wind around the compact direction, resulting in a topological winding number w .

The mass-squared formula for such a situation consists of contributions from the quantised momentum, the potential energy of the winding string, the oscillatory modes and the zero-point energy:

$$m^2 = \frac{n^2}{R^2} + \frac{w^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \tilde{N} - 2), \quad (1.97)$$

where N is the excitation level. Examining this spectrum in the limits $R \rightarrow \infty$ and $R \rightarrow 0$ we are struck by the fact that these limits are in fact the same. Equation (1.97) is invariant under

$$R \rightarrow \frac{\alpha'}{R}, \quad n \rightarrow w. \quad (1.98)$$

This symmetry is known as T-duality.

Above we considered the case of a closed string theory, where the strings can wind around the compact dimension. However in an open string theory, the winding modes are absent. This apparent discrepancy can be dealt with by requiring that open string endpoints are fixed on a $(p+1)$ -dimensional hyperplane, while the string itself can still oscillate in all directions. This hyperplane is the Dp -brane introduced in the previous section. In effect, in an open string theory T-duality transforms Neumann boundary conditions into Dirichlet conditions.

A very interesting consequence of all of this is that D-branes of given dimension can be transformed into D-branes of a different dimension by appropriately chosen compactifications and T-duality transformations. To see this, consider a Dp -brane in D -dimensional spacetime and compactify one of the $p+1$ spacetime coordinates in the Dp -brane worldvolume. There will be open strings attached to the Dp -brane. If we carry out a T-duality transformation on the compactified coordinate, the

boundary conditions on these open strings in that direction will become Neumann conditions. In other words the Dp -brane has lost a coordinate and is now a $D(p-1)$ -brane. We can even undo the compactification by choosing the large radius limit $R \rightarrow \infty$ in the T-dual theory.

In a similar fashion, we can also transform a Dp -brane into a $D(p+1)$ -brane by compactifying and T-dualising a dimension that is not in the brane worldvolume.

T-duality is a symmetry of bosonic string theory. In superstring theory, T-duality transforms one string theory into another, for example it can interchange the Type IIA and Type IIB string theories.

1.5.3 Realising Instanton Configurations in String Theory

It was demonstrated by Douglas in [10] that instantons can be realised as D-branes in string theory, in particular that the worldvolume theory of a D5-brane in type I superstring theory has, as an instanton solution, a D1-brane confined to the five-brane. This was generalised in the same work to show that $D(p-4)$ -branes turn up as instantons in the worldvolume theory of a Dp -brane. In Chapter 2 we will consider the gauge theory on a D3-brane in the Chalmers-Hanany-Witten configuration of D3-, D5-, and NS5-branes [2][3]. This particular configuration can be obtained by T-dualising a configuration of D2- and D6-branes on the product of the Taub-NUT space and six-dimensional Minkowski space - in this setup the D2-branes are instantons. This configuration was examined in [39]. The connection between D-branes and instantons can also be seen by considering D-branes as BPS states in superstring theory. They satisfy a mass-charge relation just as BPS monopoles do, and as BPS supersymmetric states they break half of the supersymmetry of the ten-dimensional theory.

1.5.4 Quantum Gauge Theory

The modes of the open strings that end on a D-brane correspond to particle states. If we take a single Dp -brane, and quantize the string states of open strings that begin and end on the brane, we will find among them a $(p + 1)$ -dimensional $U(1)$ gauge field [36]. If we consider k parallel Dp -branes, then open strings can stretch between any two of these branes. If the k branes coincide there will be massless modes from the quantized open strings stretching between them and these particles states are exactly those of a non-Abelian $U(k)$ gauge theory. As well as the gauge bosons, there will be massless scalar particles which correspond to fluctuations in the transverse directions of the brane. These lie in the adjoint representation of the gauge theory.

The configuration of interest in Chapter 2 consists of k D3-branes, intersected by n D5-branes and a single NS5-brane. There will be a 4-dimensional $SU(k)$ maximally supersymmetric gauge theory on the D3-brane worldvolume prior to the introduction of the fivebranes, as well as six massless scalar fields corresponding to fluctuations in the six dimensions transverse to the D3-branes. Three of these scalar fields will parametrize the Higgs branch of the moduli space of supersymmetric vacua of the theory, while the other three will parametrize the Coulomb branch. The Higgs branch is the part of the moduli space where hypermultiplet scalar fields have vacuum expectation values (vev's). The Coulomb branch is where vector multiplet scalar fields have vev's. When we add fivebranes at appropriate positions we will break half of the supersymmetry and introduce fundamental and bifundamental fields arising from the open strings stretching between the D3-branes and the fivebranes. The supersymmetry conditions that result from minimising the potential energy are called D- and F-flatness conditions, and as we shall see in Chapter 2, they are the Nahm Equations. The scalar fields that satisfy the D- and F-flatness conditions parametrize the Higgs branch of the moduli space of vacua of the gauge theory.

1.5.5 Moduli Spaces

The space of solutions to the self-duality equations is called a moduli space of instantons. The moduli are the coordinates on the space - they label the position, size and gauge orientation of the instanton. For given charge k and gauge group $SU(N)$, the instanton moduli space will be a $4kN$ -dimensional manifold, $\mathcal{M}_{k,N}$ [40]. The moduli space is a hyperkahler manifold, that is, it admits three complex structures I, J, K that satisfy the quaternionic relations, e.g. $IJ = -JI = K$, and three closed two-forms $\omega_{I,J,K}$, one associated to each of the complex structures.

As stated in the previous section, the scalar fields that satisfy supersymmetric vacuum conditions in the gauge theory on the D3-brane also satisfy the Nahm Equations. The Higgs branch of the space of supersymmetric vacua is a hyperkahler manifold, as required by supersymmetry [41]. In fact, the Higgs branch of the moduli space of vacua can be identified with the moduli space of solutions to the Nahm Equations for rank k Nahm data with n jumping points.

The vacuum conditions, i.e. the D- and F-flatness conditions, give the three moment maps associated to the hyperkahler quotient of this moduli space. The vacuum conditions for the field theory on the D3-brane are written in Chapter 2 while the moment maps are the Bow equations of Chapter 5.

1.6 The Taub-NUT Space

The Taub-NUT space is a curved space with self-dual Riemann curvature tensor. It is an example of an asymptotically locally flat (ALF) space. ALF spaces are hyperkahler manifolds, and are solutions of Einstein's equations. The Taub-NUT metric is

$$ds^2 = \frac{1}{4} \left(V d\vec{x}^2 + \frac{1}{V} (\omega + d\psi)^2 \right), \quad (1.99)$$

where $V = l + \frac{1}{|\vec{x}|}$. The periodic coordinate on the Taub-NUT space is $\psi \sim \psi + 4\pi$, $\vec{x}$ is a vector on $\mathbb{R}^3$, and the one-form $\omega = \omega_k dx^k$ satisfies $\partial_i \frac{1}{|\vec{x}|} = \epsilon_{ijk} \partial_j \omega_k$. The parameter l determines the asymptotic size of the periodic dimension.

The Taub-NUT space is topologically equivalent to $\mathbb{R}^4$, but it has non-zero curvature. Close to the origin it looks like $\mathbb{R}^4$ while asymptotically it is S^1 fibred over $\mathbb{R}^3$. An instanton on the Taub-NUT space shows certain similarities with calorons (instantons on $\mathbb{R}^3 \times S^1$). For example charge k calorons are composed of $2k$ monopole-like constituents, this is unlike the case for a charge k instanton on $\mathbb{R}^4$. However this constituent behaviour can also be seen in the case of a Taub-NUT instanton.

Performing a T-duality transformation on the Taub-NUT space results in an NS5-brane localised at a point on $\mathbb{R}^3 \times S^1$ [14]. Thus an instanton configuration on the Taub-NUT space can be described by an appropriate D-brane configuration that includes an NS5-brane. In Chapter 2 we describe this D-brane configuration in detail. In Chapter 4 we derive a formula for an instanton on the Taub-NUT space that corresponds to this D-brane configuration. We write an explicit expression for a charge one instanton on the Taub-NUT space in Chapter 5.

We note here that instantons in the Taub-NUT space can have monopole charges since at infinity the connection becomes independent of the periodic coordinate and thus looks like a monopole. However in this work we address only the case of charge k instantons with zero monopole charge.

1.7 Summary of Our Results

In Sections 1.2 and 1.3 we described some of the important instanton and monopole solutions that have been found, up to and including the Kraan and van Baal formula for a charge one $SU(2)$ caloron, Equation (1.37). Here we describe our results, which are derivations of new caloron and instanton formulae. We use Green's functions

from the Nahm Transform to write our solutions.

Our first result is written in Chapter 3, and it generalises the Kraan and van Baal caloron formula to arbitrary charge calorons. We give a full derivation and find the following formula for $SU(2)$ calorons of arbitrary charge:

$$(2A_\mu)^{ef} = \frac{1}{2}(\chi^{-1})_i^e \left[\partial_\nu \left(Q_{i,\gamma c}^\dagger \bar{\eta}_{\mu\nu}^{\gamma\beta} \otimes F_{cb}(\lambda_i, \lambda_j) Q_{\beta b,j} \right) \right] (\chi^{\dagger-1})_j^f \\ + (\chi^{-1})_i^e (\partial_\mu \chi)_j^f - (\partial_\mu \chi^\dagger)_i^e (\chi^{\dagger-1})_j^f. \quad (1.100)$$

To compare to the Kraan and van Baal formula we note that F is the Green's function, Q is the jumping point data, and $\phi = (\chi\chi^\dagger)^{-1}$.

In Chapter 4, we present our most important result, a formula for an arbitrary charge $SU(2)$ instanton on the Taub-NUT space. In other words we have derived a formula for finding explicit instanton solutions in a curved background. We write our formula here:

$$A = \frac{1}{4}\chi^{-1}Q^\dagger \left(\bar{\nabla} F d\bar{\tau} - d\tau \nabla F \right) Q(\chi^\dagger)^{-1} + \frac{1}{2}\chi^{-1}d\chi - \frac{1}{2}(d\chi)^\dagger (\chi^\dagger)^{-1} \\ + \frac{1}{4}\chi^{-1}Q^\dagger \begin{pmatrix} F_L & F_R \end{pmatrix} \left(\bar{\nabla}(\mathcal{L}\mathcal{L}^\dagger)d\bar{\tau} - d\tau \nabla(\mathcal{L}\mathcal{L}^\dagger) \right) \begin{pmatrix} F_L \\ F_R \end{pmatrix} Q(\chi^\dagger)^{-1} \\ - \frac{1}{4t}\chi^{-1}Q^\dagger \left(\begin{pmatrix} F_L & F_R \end{pmatrix} \mathcal{L} d\bar{\tau} \begin{pmatrix} -b_-^\dagger F_R \\ b_+^\dagger F_L \end{pmatrix} - \right. \\ \left. - \begin{pmatrix} -F_R b_- & F_L b_+ \end{pmatrix} d\tau \mathcal{L}^\dagger \begin{pmatrix} F_R \\ F_L \end{pmatrix} \right) Q\chi^{\dagger-1}. \quad (1.101)$$

Again, F is the Green's function, Q the jumping point data, and χ is related to F by $\chi\chi^\dagger = 1 - Q^\dagger F Q$. The new ingredients $\mathcal{L}$ and b are composed of the Taub-NUT bifundamental fields.

We use the formula derived in Chapter 4 to write an explicit expression for a charge one $SU(2)$ instanton on the Taub-NUT space in Chapter 5. Just as in

the case of the Kraan and van Baal formula, this problem effectively reduces to solving matching conditions on an interval to find the Green's function. This is a straightforward calculation, and thus our instanton formula is simple to use. The charge one $SU(2)$ instanton on the Taub-NUT space is written here:

$$A_\mu = \frac{1}{4}\phi\left(\bar{\eta}_{\nu\mu}^a\left(\sigma_a(\partial_\nu\phi_{00} + \Delta_{00}^\nu) + \sigma_3\sigma_a\sigma_3(\partial_\nu\phi_{33} + \Delta_{33}^\nu) - i[\sigma_a, \sigma_3](\partial_\nu\phi_{03} + \Delta_{03}^\nu) + [\sigma_a\sigma_3, \sigma_3]\rho\right)\right). \quad (1.102)$$

The functions $\phi_{\alpha\beta}$, $\Delta_{\alpha\beta}$ and ρ are simple functions of the Green's function, and are written explicitly in Chapter 5.

1.7.1 Outline of Results

1. In Chapter 2 we add a term involving bifundamental fields to the action written in [4]. This term arises due to the addition of an NS5-brane to the brane configuration considered in [4] and [12]. In the presence of the NS5-brane, the supersymmetric vacuum conditions for the gauge theory give the Bow data for an instanton on the Taub-NUT space.
2. In Chapter 3 we generalise the formula derived in [1] for charge one calorons to arbitrary charge calorons.
3. In Chapter 4 we derive a formula for an arbitrary charge $SU(2)$ instanton in the background of the Taub-NUT space.
4. In Chapter 5 we use the formula we derived in Chapter 4 to write an explicit formula for a charge one $SU(2)$ instanton on the Taub-NUT space.

Chapter 2

Supersymmetric Gauge Theory with Impurities

In this chapter we describe the configuration of intersecting D3-, D5- and NS5-branes that gives rise to the gauge theory with impurities on the threebrane worldvolume. This is the Chalmers-Hanany-Witten configuration described in [2] and [3]. It can be obtained from a Douglas and Moore-type scenario [42] in the case of an instanton on the Taub-NUT by performing T-duality along the x^6 coordinate. This was investigated in [39]. The Lagrangian for intersecting D3-D5-branes was studied by DeWolfe, Freedman and Ooguri in [12] and by Erdmenger, Gurlanik and Kirsch in [4]. Here we introduce an NS5-brane to the model. The brane configuration is chosen so that eight real supercharges are preserved. As in [12] and [4] we write the action of the theory, i.e. $N = 4, d = 4$ supersymmetric Yang-Mills theory coupled to $N = 4, d = 3$ hypermultiplets on the defects. We write these fields in terms of $N = 2, d = 3$ superfields. This involves combining both the $N = 4, d = 4$ degrees of freedom of the bulk theory and the $N = 4, d = 3$ defect hypermultiplets into $N = 2, d = 3$ supermultiplets, which are described in Section 2.3. The action of the theory, which couples half of the bulk degrees of freedom (as specified later) to the defect hypermultiplets, is given in terms of the $N = 2, d = 3$ superfields in Section

2.4, and in terms of the component fields in Appendix C. This allows us to write equations of motion for the auxiliary fields in the theory in Section 2.5. Setting these to zero we obtain the D - and F -flatness conditions for the supersymmetric gauge theory and in Section 2.6 we compare these equations to the Nahm equations.

2.1 D-brane Configuration

The brane configuration that gives rise to the impurity theory we are interested in involves intersecting D3-branes and fivebranes. We take k D3-branes with worldvolumes in the (x^0, x^1, x^2, x^6) directions. The low-energy effective field theory on the D3 worldvolumes in the absence of any fivebranes is $N = 4, d = 4$ supersymmetric Yang-Mills gauge theory with gauge group $U(k)$. Introducing fivebranes such that there are four relative transverse dimensions with the threebranes, will break half of the supersymmetry of this theory and will introduce degrees of freedom at the brane intersections, arising from open 3-5 strings stretching between the branes. We introduce n D5-branes whose worldvolume is in the $(x^0, x^1, x^2, x^3, x^4, x^5)$ directions. The presence of the D5-branes breaks the supersymmetry from sixteen to eight supercharges and gives rise to $N = 4, d = 3$ hypermultiplets transforming in the fundamental representation of the gauge group at the D3-D5 intersections. We can also introduce an NS5-brane with worldvolume in $(x^0, x^1, x^2, x^7, x^8, x^9)$ without breaking any further supersymmetry. Open string states at D3-NS5 intersections will be in the bifundamental representation of the gauge group.

Let us consider two cases prior to the introduction of the NS5-brane. If we take x^6 periodic then the supersymmetry conditions will be Nahm Equations for Nahm data on a circle, which produces a caloron under the Nahm Transform. If we have a bounded interval on x^6 then the Nahm Transform gives monopoles. With the introduction of the NS5-brane we are now considering instantons on the Taub-NUT space.

The supersymmetric gauge theory with impurities that we are interested in arises from the following brane configuration:

	0	1	2	3	4	5	6	7	8	9
D3	x	x	x				x			
D5	x	x	x	x	x	x				
NS5	x	x	x					x	x	x
Sym	SO(1,2)			SO(3) _V			-	SO(3) _H		
V.m	v_0	v_1	v_2	Re u_1	Im u_1	ρ				
H.m							Im A	Re A	Re u_2	Im u_2

The last two rows list the bosonic component fields of an $N = 1, d = 4$ vector multiplet $(v_0, v_1, v_2, \rho, u_1)$ and hypermultiplet (A, u_2) . These will be recombined into $N = 2, d = 3$ vector and chiral multiplets in the next section.

In the presence of the branes the ten-dimensional Lorentz invariance is broken from $SO(1, 9)$ to $SO(1, 2) \times SO(3)_H \times SO(3)_V$. Here $SO(3)_H$ acts on (x^7, x^8, x^9) , affecting the hypermultiplet. Scalar fields in these directions obtained from the dimensional reduction of the ten-dimensional gauge field describe fluctuations in the transverse positions of the threebranes along these directions. They parametrize the Higgs branch of the moduli space of supersymmetric vacua of the theory on the threebrane worldvolumes. We will see in Section 2.6 that these scalar fields are Nahm data for charge k instantons. The scalar fields in the (x^3, x^4, x^5) directions vanish on the Higgs branch.

2.2 Superfield Content of the Model

We can decompose $N = 2, d = 4$ multiplets into $N = 2, d = 3$ fields as described in [4], by arranging the coordinates $(\theta'_1, \bar{\theta}'^1, \theta'_2, \bar{\theta}'^2)$ into complex spinors θ_α and $\bar{\theta}_\alpha$.

This is done as follows:

$$\theta^\alpha = \theta_1^\alpha - i\theta_2^\alpha, \quad \bar{\theta}^\alpha = \theta_1^\alpha + i\theta_2^\alpha, \quad \alpha = 1, 2. \quad (2.1)$$

Here,

$$\theta_1 = \frac{1}{2}(\theta'_1 + \bar{\theta}'^1), \quad \theta_2 = \frac{1}{2}(\theta'_2 + \bar{\theta}'^2). \quad (2.2)$$

The $N = 2, d = 3$ superspace is parametrized by the spacetime coordinates x^m , $m = 0, 1, 3$, as well as the four fermionic coordinates in Equation (2.1). From these we can define new bosonic coordinates $y^m = x^m + i\theta\sigma^m\bar{\theta}$. Note that σ^m is symmetric since $m = 0, 1, 3$. The coordinate x^2 in the four-dimensional space becomes a continuous index labelling infinitely many three-dimensional fields when we move to $N = 2, d = 3$ superspace. Metric and spinor conventions as well as Fierz identities can be found in Appendix A.

2.2.1 Bulk Superfields

The standard reference on Supersymmetry, which we follow here, is [43]. Superfields in the superspace are polynomials in the fermionic coordinates. Vector superfields are defined by the constraint $V = V^\dagger$ and have a gauge boson as their component of highest spin. The vector multiplet transforms as $e^{2iV} \rightarrow e^{-2i\Lambda} e^{2iV} e^{2i\Lambda^\dagger}$, where Λ is a chiral multiplet. Here V is written in Wess-Zumino gauge, with $v^2 = \rho$:

$$V(x) = -i\theta\bar{\theta}\rho - \theta\sigma^m\bar{\theta}v_m + i(\theta\theta)\bar{\theta}\bar{\lambda} - i(\bar{\theta}\bar{\theta})\theta\lambda + \frac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta})D. \quad (2.3)$$

An $N = 2, d = 3$ linear supermultiplet

$$\Sigma = \epsilon^{\alpha\beta} \bar{D}_\alpha (e^{2iV} D_\beta e^{-2iV}), \quad (2.4)$$

is formed from V . The supersymmetry degrees of freedom contained in the linear multiplet Σ are exactly those of the vector multiplet V . Thus Σ is composed of a gauge boson, (v_m, ρ) , a fermion, λ , and a real scalar field, D , that is an auxiliary field. The linear multiplet Σ gives the kinetic term for the vector multiplet in the action. For the calculation of Σ it is useful to note that

$$e^{2iV} D_\beta e^{-2iV} = -2i D_\beta V + 2[V, D_\beta V],$$

and that $\bar{D}_\alpha = -\bar{\partial}_\alpha$ when acting on functions of y . The details of the calculation are given in Appendix B, and we find the following expression:

$$\begin{aligned} \Sigma(x) = & 4\rho - 4\theta\bar{\lambda} - 4\bar{\theta}\lambda - 4i\theta\bar{\theta}D - 4\bar{\theta}\sigma^{mn}\theta F_{mn} + 4i\theta\sigma^m\bar{\theta}[v_m, \rho] - 4i(\theta\theta)\bar{\theta}\sigma^m[v_m, \bar{\lambda}] \\ & + 2i(\bar{\theta}\bar{\theta})\theta\sigma^m\partial_m\lambda - 2i(\theta\theta)\bar{\theta}\sigma^m\partial_m\bar{\lambda} - 2(\theta\theta)(\bar{\theta}\bar{\theta})\partial_m[v^m, \rho] - (\theta\theta)(\bar{\theta}\bar{\theta})\square\rho. \end{aligned} \quad (2.5)$$

where $F_{mn} = \partial_m v_n - \partial_n v_m + [v_m, v_n]$. Although it is not Hermitian, transformations of the linear multiplet $\Sigma \rightarrow e^{-2i\Lambda}\Sigma e^{2i\Lambda}$ leave the action unchanged.

Chiral superfields are defined through the covariant constraint $\bar{D}_\alpha\Phi = 0$ (i.e. any field Φ that is a function of θ (and y) but not $\bar{\theta}$ is a chiral superfield) and have a fermion as their component of highest spin. There are three chiral superfields in the theory: Φ, U_1 and U_2 . The first of these, Φ transforms as $\Phi \rightarrow e^{-2i\Lambda}\Phi e^{2i\Lambda} + e^{-2i\Lambda}\frac{d}{ds}e^{2i\Lambda}$ under gauge transformations. The field Φ and its complex conjugate can be expanded in superspace as follows:

$$\Phi(x) = A + \sqrt{2}\theta\psi + (\theta\theta)G + i\theta\sigma^m\bar{\theta}\partial_m A + \frac{i}{\sqrt{2}}(\theta\theta)\bar{\theta}\sigma^m\partial_m\psi + \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\square A \quad (2.6)$$

$$\bar{\Phi}(x) = A^* + \sqrt{2}\bar{\theta}\bar{\psi} + (\bar{\theta}\bar{\theta})G^* - i\theta\sigma^m\bar{\theta}\partial_m A^* + \frac{i}{\sqrt{2}}(\bar{\theta}\bar{\theta})\theta\sigma^m\partial_m\bar{\psi} + \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\square A^* \quad (2.7)$$

So the components of Φ are a complex scalar field A , a fermion ψ , and a complex scalar field G . G is an auxiliary field. The other two chiral fields in the theory

transform as $U_j \rightarrow e^{-2i\Lambda} U_j e^{2i\Lambda}$, $j = 1, 2$. The chiral field U_j is expanded in its component fields below. Its complex conjugate takes the same form as that of $\bar{\Phi}$.

$$U_j = u_j + \sqrt{2}\theta\chi_j + (\theta\theta)F_j + i\theta\sigma^m\bar{\theta}\partial_m u_j + \frac{i}{\sqrt{2}}(\theta\theta)\bar{\theta}\bar{\sigma}^m\partial_m\chi_j + \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\square u_j \quad (2.8)$$

The content of each of the chiral fields U_1 and U_2 is a complex scalar field u , a fermion χ and a complex scalar field F that is again an auxiliary field.

The theory is $N = 4, d = 4$ supersymmetric Yang-Mills theory in the bulk, coupled to $N = 4, d = 3$ hypermultiplets on the defects. The action is written in $N = 2, d = 3$ superspace, so the bulk degrees of freedom are combined into the $N = 2, d = 3$ fields defined above: Σ , Φ , U_1 and U_2 . The fields Σ and U_1 together form an $N = 1, d = 4$ vector multiplet. The chiral multiplets Φ and U_2 form an $N = 1, d = 4$ hypermultiplet. We treat the coordinate $x^2 = s$ as a label.

2.2.2 Defect Superfields

The defect hypermultiplets can be in either the (anti-)fundamental or the bifundamental representation of the gauge group. The defect action was written for the fundamental hypermultiplets in [4]. Here we add a term involving bifundamental fields to the defect action. The superfields on the defects, i.e. the $N = 2, d = 3$ hypermultiplets Q_{1p} , Q_{2p} , B_1 and B_2 , are defined below.

The fields Q_{1p} transform in the fundamental representation of the gauge group, while Q_{2p} transforms in the antifundamental representation of the gauge group:

$$Q_{1p} \rightarrow e^{-2i\Lambda_p} Q_{1p},$$

$$Q_{2p} \rightarrow Q_{2p} e^{2i\Lambda_p},$$

where Λ_p is the gauge transformation at s_p and $p = 1, \dots, n$ labels the defects arising from the D3-D5 intersections. For each value of p , Q_{1p} and Q_{2p} form an $N = 4, d = 3$

hypermultiplet. Writing their expansions in $N = 2, d = 3$ superspace, we see that they are each composed of complex scalars q and J :

$$Q_{jp} = q_{jp} + i\theta\sigma^m\bar{\theta}\partial_m q_{jp} + \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\square q_{jp} + (\theta\theta)J_{jp} \quad (2.9)$$

$$\bar{Q}_{jp} = q_{jp}^* - i\theta\sigma^m\bar{\theta}\partial_m q_{jp}^* + \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\square q_{jp}^* + (\bar{\theta}\bar{\theta})J_{jp}^*, \quad (2.10)$$

where $j = 1, 2$ and $p = 1, \dots, n$.

The hypermultiplets $B_j, j = 1, 2$, transform in the bifundamental representation of the gauge group:

$$B \rightarrow e^{-2i\Lambda_L} B e^{2i\Lambda_R}.$$

Here the chiral multiplet Λ_L is the gauge transformation valued at the left end-point of s , while Λ_R is the value at the right end-point. For D3-NS5 intersections, these points are not the same. The D3-brane may split on the NS5-brane, in which case there will be a bifundamental string state from a string stretching between the left and right end-points. The bifundamental field becomes massless if we move the D3-brane ends together. The hypermultiplets are composed of complex scalars b and L :

$$B_j = b_j + i\theta\sigma^m\bar{\theta}\partial_m b_j + \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\square b_j + (\theta\theta)L_j \quad (2.11)$$

$$\bar{B}_j = b_j^* - i\theta\sigma^m\bar{\theta}\partial_m b_j^* + \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\square b_j^* + (\bar{\theta}\bar{\theta})L_j^*. \quad (2.12)$$

2.3 The Action

The action of the theory consists of the action in the 4-dimensional bulk, S_4 , plus the actions on the defects, S_{3p} and S'_3 . In this section we write the action in terms of

the superfields introduced in the previous section. The action is written in terms of the components fields in Appendix C. Since the auxiliary fields enter algebraically, by varying the auxiliary fields in the action we can find expressions for them in terms of the other component fields. We will obtain the supersymmetry conditions called the D - and F -flatness conditions by setting the auxiliary fields to zero. The special properties of integration over superspace, namely $\int d\theta\theta = 1$ and $\int d\theta 1 = 0$, mean that we have to write only $(\theta\theta)(\bar{\theta}\bar{\theta})$ terms when integrating over $d^2\theta d^2\bar{\theta}$, and only $(\theta\theta)$ $((\bar{\theta}\bar{\theta}))$ terms when integrating over $d^2\theta$ ($d^2\bar{\theta}$). This simplifies the calculation.

2.3.1 Bulk Action

The bulk action is $N = 4, d = 4$ SYM written in terms of $N = 2, d = 3$ superfields:

$$S_4 = \int dsd^3x d^2\theta d^2\bar{\theta} \text{tr} \left(-\frac{1}{16} \Sigma^2 - \frac{1}{4} (e^{2iV} (\partial_s - \bar{\Phi}) e^{-2iV} - \Phi)^2 - \frac{1}{2} e^{2iV} \bar{U}_i e^{-2iV} U_i \right) \\ + \int dsd^3x d^2\theta \text{tr} \left(-\frac{1}{4} \epsilon_{ij} U_i [-\partial_s - \Phi, U_j] \right) + \int dsd^3x d^2\bar{\theta} \text{tr} \left(-\frac{1}{4} \epsilon_{ij} [\bar{U}_j, \partial_s - \bar{\Phi}] \bar{U}_i \right). \quad (2.13)$$

Each of the terms in the bulk action is expanded and written explicitly in terms of the component fields in Appendix C. This will allow us to vary the action with respect to the auxiliary fields and thus write the flatness conditions for these fields.

2.3.2 Defect Action

On the defects, the action S_{3p} couples half the superfields in the bulk to the degrees of freedom on the defects which are contained in the superfields Q_{1p} and Q_{2p} . Only V and U_2 are coupled to the defects, resulting in the following expression for S_{3p} in terms of $N = 2, d = 3$ superfields:

$$\begin{aligned}
S_{3p} = & -\frac{1}{2} \int d^3x d^2\theta d^2\bar{\theta} (\bar{Q}_{1p} e^{-2iV} Q_{1p} + Q_{2p} e^{2iV} \bar{Q}_{2p}) + \frac{1}{2} \int d^3x d^2\theta Q_{2p} U_1 Q_{1p} \\
& + \frac{1}{2} \int d^3x d^2\bar{\theta} \bar{Q}_{1p} \bar{U}_1 \bar{Q}_{2p}.
\end{aligned}
\tag{2.14}$$

At the left and right endpoints of an interval in s , we have the D3-NS5-brane intersection that results in the defect action S'_3 . It consists of terms that couple the bulk fields V and U_2 to the bifundamental hypermultiplets on the defect:

$$\begin{aligned}
S'_3 = & -\frac{1}{2} \int d^3x d^2\theta d^2\bar{\theta} \text{tr} (e^{2iV_R} \bar{B}_1 e^{-2iV_L} B_1 + e^{2iV_L} \bar{B}_2 e^{-2iV_R} B_2) \\
& + \frac{1}{2} \int d^3x d^2\theta \text{tr} (-B_2 U_1^L B_1 + B_1 U_1^R B_2) + \frac{1}{2} \int d^3x d^2\bar{\theta} \text{tr} (-\bar{B}_1 \bar{U}_1^L \bar{B}_2 + \bar{B}_2 \bar{U}_1^R \bar{B}_1).
\end{aligned}
\tag{2.15}$$

Again, each term in S_{3p} and S'_3 is expanded in its component fields in Appendix C.

2.4 Auxiliary Fields

The D -component of the vector superfield V and the F_1 -component of the chiral superfield U_1 are auxiliary fields. Auxiliary fields are the component fields of highest dimension in their supermultiplets, and transform into total derivatives under supersymmetry transformations (since any higher powers in θ and $\bar{\theta}$ in their multiplets are spacetime derivatives of the other component fields). They can therefore be included in a supersymmetry-invariant action. We can obtain expressions for the auxiliary fields from their equations of motion and use these to eliminate them from the action. Scalar potential terms come from the F - and D -terms in the action. In Yang-Mills theory the scalar potential is the sum of the F - and D -term contributions and supersymmetric minima must satisfy $D = 0$ as well as $F = 0$, as we shall see

below.

Therefore solutions to D - and F -flatness conditions, i.e. $D = 0$ and $F = 0$, form a moduli space of supersymmetric vacuum solutions. We will see in the next section that this moduli space can be identified with the moduli space of solutions to the Nahm equations. Here we vary the action to find expressions for the auxiliary fields D and F_1 in the vector and chiral superfields.

By varying the action written in Equations (2.13), (2.14) and (2.15), (and in Appendix C in terms of its component fields) with respect to D and F_1 we find expressions for D and F_1^* :

$$D = -\frac{i}{2} \frac{d}{ds} (A + A^*) - \frac{i}{2} [A, A^*] - \frac{i}{2} [u_1, u_1^*] - \frac{i}{2} [u_2, u_2^*] + \frac{i}{2} (q_{1p} q_{1p}^* - q_{2p}^* q_{2p}) \delta(s - \lambda_p) - \frac{i}{2} (b_1^* b_1 - b_2 b_2^*) \delta(s - \lambda_R) + \frac{i}{2} (b_1 b_1^* - b_2^* b_2) \delta(s - \lambda_L), \quad (2.16)$$

$$\frac{1}{2} F_1^* = -\frac{1}{2} \frac{du_2}{ds} - \frac{1}{2} [A, u_2] + \frac{1}{2} q_{1p} q_{2p} \delta(s - \lambda_p) + \frac{1}{2} b_2 b_1 \delta(s - \lambda_R) - \frac{1}{2} b_1 b_2 \delta(s - \lambda_L). \quad (2.17)$$

Here the n fundamental fields are located at points $s = \lambda_p, p = 1, \dots, n$. The bifundamental fields are located at the left and right endpoints of the interval parametrized by s , these are λ_L and λ_R respectively. The expressions for the other auxiliary fields are written in Appendix D.

The vacuum fields are supersymmetric. Consider the supersymmetry algebra, which has the relation $\{Q, Q\} = H$. A supersymmetric vacuum state f obeys $Qf = 0$, therefore $\{Q, Q\}f = Hf = 0$, and a supersymmetric state has zero energy. Also, if a state g satisfies $Hg = 0$, then $\langle g, Hg \rangle = \langle g, \{Q, Q\}g \rangle = \langle Qg, Qg \rangle = 0$, and g is a supersymmetric state. Now consider the bulk part of the auxiliary field $D = -\frac{i}{2} \left(\frac{d}{ds} (A + A^*) + [A, A^*] + [u_1, u_1^*] + [u_2, u_2^*] \right) = -\frac{i}{2} \Omega$. Substituting back into the bulk action, we find

$$S_D = \int \left(\frac{1}{2} D^2 + \frac{i}{2} D \Omega \right) = \int \left(-\frac{1}{8} \Omega^2 + \frac{1}{4} \Omega^2 \right) = \int \frac{1}{8} \Omega^2. \quad (2.18)$$

So the potential term in the action is D^2 and thus it is minimized by setting $D = 0$, which is a supersymmetric vacuum condition. By substituting the F -terms back into the action we find similar quadratic expressions for the other potential terms, so similarly $F = 0$ is a supersymmetric vacuum condition.

2.5 Nahm Transform and Nahm Equations

As discussed in the introduction, Nahm data consist of a connection T_0 on a Hermitian bundle over an interval $\mathcal{I}$ parametrized by s and bundle endomorphisms T_j that transform as $T_j \rightarrow g^{-1}T_jg$ under the gauge group. The Nahm data satisfy the Nahm Equations:

$$\frac{dT_1}{ds} - i[T_0, T_1] + i[T_2, T_3] = 0, \quad (2.19)$$

$$\frac{dT_2}{ds} - i[T_0, T_2] + i[T_3, T_1] = 0, \quad (2.20)$$

$$\frac{dT_3}{ds} - i[T_0, T_3] + i[T_1, T_2] = 0. \quad (2.21)$$

The intervals $\mathcal{I}$ may have boundaries, which are labelled by λ_p , $p = 1, \dots, n$. The Nahm equations above hold over each interval, with the Nahm data defined on the bundle over that interval. Assuming neighbouring bundles are of equal rank there will be discontinuities at the jumping points, resulting in delta functions. Thus we may introduce source terms on the right hand side of the Nahm equations as follows:

$$\frac{dT_i}{ds} - i[T_0, T_i] + i[T_j, T_k] = -\frac{1}{2}\delta(s - \lambda_p)Q_p^\dagger\sigma_iQ_p, \quad (2.22)$$

where Q_p is a $2k$ -component row vector. The Nahm Equations on each subinterval can be rewritten in complex form as follows:

$$\frac{d}{ds}(\alpha + \alpha^\dagger) + [\alpha, \alpha^\dagger] + [\beta, \beta^\dagger] = 0 \quad (2.23)$$

$$\frac{d\beta}{ds} + [\alpha, \beta] = 0, \quad (2.24)$$

where $\alpha = -iT_0 + T_3$ and $\beta = -T_1 + iT_2$. Looking at Equations (2.16) and (2.17), we see that, setting D and F_1^* to zero, we can write them in the same form as the Nahm Equations above. Since u_1 vanishes on the Higgs branch the commutator $[u_1, u_1^*]$ in Equation (2.16) will be zero. The D- and F-flatness conditions will then have the following form:

$$\begin{aligned} \frac{d}{ds}(A^* + A) + [A, A^*] + [u_2, u_2^*] &= (q_{1p}q_{1p}^* - q_{2p}^*q_{2p})\delta(s - \lambda_p) + (b_1b_1^* - b_2^*b_2)\delta(s - \lambda_L) \\ &\quad - (b_1^*b_1 - b_2b_2^*)\delta(s - \lambda_R), \\ \frac{du_2}{ds} + [A, u_2] &= q_{1p}q_{2p}\delta(s - \lambda_p) + b_2b_1\delta(s - \lambda_R) - b_1b_2\delta(s - \lambda_L). \end{aligned} \quad (2.25)$$

We can make the following identifications:

$$\alpha = -iT_0 + T_3 = A \quad (2.26)$$

$$\beta = -T_1 + iT_2 = u_2 \quad (2.27)$$

The solutions involving scalar fields $\text{Re}(A)$ and u_2 parametrize the Higgs branch of the space of vacua of the gauge theory on the D3-branes. Here we see that these fields solve the Nahm equations, making explicit the connection between gauge theories on D-branes and instantons, as described earlier. The fundamental fields q_j that arise from intersections with D5-branes are related to the spinors Q that are Nahm data at jumping points on an interval. The bifundamental fields b appear due to intersections with NS5-branes. As we shall see in Chapter 4, these fields are Nahm data at the end-points of an interval for an instanton on the Taub-NUT space.

Chapter 3

Kraan & van Baal Formula for Calorons

Calorons are finite temperature Yang-Mills instantons on $\mathbb{R}^3 \times S^1$. Here we give a derivation of a formula for charge k calorons in terms of the Green's function of the Laplacian of the Weyl operator defined for the Nahm transform. We thus generalize the work in references [9] and [1], where similar derivations are given for $\mathbb{R}^4$ instantons and the charge one caloron, respectively. This method provides us with a simple formula for calorons in terms of the values of the Green's function at jumping points. If the Nahm data is defined over an interval $\mathcal{I}$ in $\mathbb{R}$, then the formula will give monopole solutions. If it is defined over a circle then we will have calorons. This can be compared to the situation in Chapter 2 where the D3-branes can be periodic in one coordinate and intersected by D5-branes, or can be located on an interval, suspended between two fivebranes.

The formula we derive here describes charge k calorons. This formula is related to the D3-D5-brane configuration of Chapter 2 prior to the introduction of the NS5-brane, where the D3-branes are periodic in one coordinate. The inclusion of NS5-branes has a significant effect which will be discussed in the following chapter.

Here we go through each step of the derivation of the caloron formula in detail.

We introduce the Nahm data in the next section, then in Section 3.2 we construct the Weyl operator $\mathcal{D}_t$ from the Nahm data. We write the Laplacian $\mathcal{D}^\dagger \mathcal{D}$ and consider its Green's function. Finally we derive the caloron formula in Section 3.5. In Chapter 4 we will present a derivation of an instanton formula in a curved background. Many of the details of the derivation will be similar.

3.1 Nahm Data

Nahm data T_μ are defined on a Hermitian complex vector bundle E of rank k over the circle S^1 , which we parametrize by s where $e^{\frac{2\pi i s}{l}} \in S^1$. They are $k \times k$ Hermitian matrices T that are bundle endomorphisms and transform as $T_j \rightarrow g^{-1} T_j g$ under gauge transformations, and a Hermitian connection $\nabla = \frac{d}{ds} - iT_0$ in the s direction. As discussed in the Introduction the Nahm data may also include boundary (jumping point) data. The jumping point data consist of $2k$ -component vectors Q_j for each of n jumping points λ_j , (i.e. $Q = (Q_1 \ Q_2 \ \dots \ Q_n)$ is a $2k \times n$ matrix), while T_0 and T_j are defined on each interval $(\lambda_j, \lambda_{j+1})$. In the next section we construct a Weyl operator from the Nahm data.

3.2 The Weyl Operator

The Weyl Operator $\mathcal{D}$ acts on a bundle $S \otimes E$ that is the product of a rank 2 Hermitian spinor bundle S and the Hermitian bundle E over S^1 . We use the Hermitian metric on this bundle to form the adjoint Weyl operator $\mathcal{D}^\dagger$. First we define the twisted Weyl operator:

$$\mathcal{D}_t f = \begin{pmatrix} \left(1 \otimes \left(\frac{d}{ds} - iT_0 + it_0 \right) - \vec{\sigma} \otimes (\vec{T} - \vec{t}) \right) f(s) \\ Q_1^\dagger f(\lambda_1) \\ \dots \\ Q_n^\dagger f(\lambda_n) \end{pmatrix} = \begin{pmatrix} \mathbf{D}_t f(s) \\ Q_1^\dagger f(\lambda_1) \\ \dots \\ Q_n^\dagger f(\lambda_n) \end{pmatrix},$$

where the $f(\lambda_j)$ are spinors and $f(s)$ is a continuous L^2 section. We note the form of the operator $\mathbf{D}_t$ here, which is the Weyl operator defined only on the intervals between neighbouring jumping points and not at the jumping points:

$$\mathbf{D}_t = 1 \otimes \left(\frac{d}{ds} - iT_0 + it_0 \right) - \vec{\sigma} \otimes (\vec{T} - \vec{t}). \quad (3.1)$$

Later we will use the Green's function of the operator $\mathbf{D}_t^\dagger \mathbf{D}_t$ in our derivation. We can use the Hermitian inner product on the bundle to determine $\mathcal{D}_t^\dagger$ since:

$$\langle (\psi, \chi_1, \dots, \chi_n), \mathcal{D}_t f \rangle = \langle \mathcal{D}_t^\dagger (\psi, \chi_1, \dots, \chi_n), f \rangle. \quad (3.2)$$

We write the inner product:

$$\begin{aligned} \langle (\psi, \chi_1, \dots, \chi_n), \mathcal{D}_t f \rangle = \int ds \, \psi^\dagger \left(1 \otimes \left(\frac{d}{ds} - iT_0 + it_0 \right) - \vec{\sigma} \otimes (\vec{T} - \vec{t}) \right) f + \\ + \sum_j \bar{\chi}_j Q_j^\dagger f(\lambda_j). \end{aligned} \quad (3.3)$$

Using Equation (3.2) and integrating the above expression by parts we have

$$\begin{aligned} \langle \mathcal{D}_t^\dagger (\psi, \chi_1, \dots, \chi_n), f \rangle = \int ds \left(\left(-1 \otimes \left(\frac{d}{ds} - iT_0 + it_0 \right) - \vec{\sigma} \otimes (\vec{T} - \vec{t}) \right) \psi + \right. \\ \left. + \sum_j \delta(s - \lambda_j) Q_j \chi_j^\dagger \right) f. \end{aligned} \quad (3.4)$$

Boundary terms from the integrating Equation (3.3) by parts are implicit in Equation (3.4) since we are considering a generalised derivative which includes naturally the boundary terms at jumping points, i.e. $\delta(s + \lambda)\psi(-\lambda)$ and $-\delta(s - \lambda)\psi(\lambda)$. From Equation (3.4) we see that the adjoint Weyl operator $\mathcal{D}_t^\dagger$ acts on $(\psi, \chi_1, \dots, \chi_n)$ as

follows:

$$\begin{aligned}
\mathcal{D}_t^\dagger(\psi, \chi_1, \dots, \chi_n) &= \\
&= \left(-1 \otimes \left(\frac{d}{ds} - iT_0 + it_0 \right) - \vec{\sigma} \otimes (\vec{T} - \vec{t}), \delta(s - \lambda_1)Q_1, \dots, \delta(s - \lambda_n)Q_n \right) \begin{pmatrix} \psi \\ \chi_1 \\ \dots \\ \chi_n \end{pmatrix} \\
&= \mathbf{D}_t^\dagger \psi + \sum_j \delta(s - \lambda_j) Q_j \chi_j.
\end{aligned} \tag{3.5}$$

3.3 The Kernel of the Weyl Operator

The columns of the matrix Ψ form an orthonormal basis for $\text{Ker } \mathcal{D}_t^\dagger$, i.e. $\Psi^\dagger \Psi = 1$ and $\mathcal{D}_t^\dagger \Psi = 0$. The projection onto $\text{Ker } \mathcal{D}_t^\dagger$ is

$$\Psi \Psi^\dagger = 1 - \mathcal{D}_t F \mathcal{D}_t^\dagger. \tag{3.6}$$

Writing $\Psi = \begin{pmatrix} \psi \\ \chi \end{pmatrix}$, we have

$$1 - \mathcal{D}_t F \mathcal{D}_t^\dagger = \begin{pmatrix} \psi \psi^\dagger & \psi \chi^\dagger \\ \chi \psi^\dagger & \chi \chi^\dagger \end{pmatrix}. \tag{3.7}$$

Here ψ is a $2k \times n$ matrix and $\chi = (\chi_{je})$ is an $n \times N$ matrix, where j labels the n jumping points and e labels the N solutions which are the columns of the matrix. Taking suitable components of Equation (3.7), we find the following relation, which will be useful later:

$$1 - Q^\dagger F Q = \chi \chi^\dagger. \tag{3.8}$$

This is a consequence of the orthonormalization condition.

3.4 The Green's Functions

We wish to describe the Green's function of the operator $\mathcal{D}_t^\dagger \mathcal{D}_t$ as we will write our formula in terms of this function. Importantly, we note that the Nahm Equations require that the operator $\mathcal{D}_t^\dagger \mathcal{D}_t$ commutes with the sigma matrices. To see this we write $\mathcal{D}_t^\dagger \mathcal{D}_t$:

$$\begin{aligned} \mathcal{D}_t^\dagger \mathcal{D}_t = & 1 \otimes \left(-\left(\frac{d}{ds} - iT_0 + it_0\right)^2 - \vec{z}^2 + \frac{1}{2} \text{tr}_{2 \times 2} Q Q^\dagger \delta(s - \lambda_j) \right) \\ & + \sum_i \sigma_i \otimes \left(\frac{dT_i}{ds} - i[T_0, T_i] + i[T_j, T_k] \right) + \text{vec}(Q Q^\dagger) \delta(s - \lambda_j), \end{aligned} \quad (3.9)$$

where $\vec{z} = \vec{t} - \vec{T}$ and 'vec' picks out the part of an object that is proportional to the sigma matrices, e.g.

$$\text{vec}(\sigma_0 \otimes a_0 + \sigma_1 \otimes a_1 + \sigma_2 \otimes a_2 + \sigma_3 \otimes a_3) = \sigma_1 \otimes a_1 + \sigma_2 \otimes a_2 + \sigma_3 \otimes a_3.$$

The second term in the second line of Equation (3.9) vanishes by the Nahm Equations and so for self-dual connections the operator $\mathcal{D}_t^\dagger \mathcal{D}_t$ commutes with the sigma matrices and is written

$$\mathcal{D}_t^\dagger \mathcal{D}_t = 1 \otimes \left(-\left(\frac{d}{ds} - iT_0 + it_0\right)^2 + (\vec{T} - \vec{t})^2 + \frac{1}{2} \text{tr}_{2 \times 2} Q Q^\dagger \delta(s - \lambda_j) \right). \quad (3.10)$$

It is invertible and positive definite, so there is a Green's function that satisfies:

$$\mathcal{D}_t^\dagger \mathcal{D}_t F(s, s_0) = \delta(s - s_0). \quad (3.11)$$

Since $\mathcal{D}_t^\dagger \mathcal{D}_t$ commutes with the sigma matrices its Green's function will be proportional to the identity in S and will thus be a very nice function to work with. For our derivation we will require the Green's functions of two operators. One commutes with sigma matrices, that is the Green's function of the operator introduced above. The other does not commute with sigma matrices. It will not feature in the

final instanton formula, but will be very useful in developing the formula. So we introduce the following two Green's functions:

$$1 \otimes F = (\mathcal{D}_t^\dagger \mathcal{D}_t)^{-1}, \quad G = (\mathbf{D}_t^\dagger \mathbf{D}_t)^{-1}, \quad (3.12)$$

where $\mathcal{D}^\dagger \mathcal{D}$ is as given above, and

$$\mathbf{D}_t^\dagger \mathbf{D}_t = 1 \otimes \left(-\left(\frac{d}{ds} - iT_0 + it_0 \right)^2 + \bar{z}^2 \right) + \sum_i \sigma_i \otimes \left(\frac{dT_i}{ds} - i[T_0, T_i] + i[T_j, T_k] \right). \quad (3.13)$$

The functions F and G are closely related. It is easy to see from their definitions that

$$F^{-1} = G^{-1} + \sum_j \delta(s - \lambda_j) Q_j Q_j^\dagger. \quad (3.14)$$

As mentioned above, F is easier to compute than G since it is proportional to the identity matrix in spin space and we shall need only F for the instanton formula. From Equation (3.14) we have that $G = (F^{-1} - QQ^\dagger)^{-1}$ and using $(1 - FQQ^\dagger)^{-1} = \sum_{n=0}^{\infty} (FQQ^\dagger)^n$ we obtain the following equation:

$$1 + Q^\dagger G Q = (1 - Q^\dagger F Q)^{-1}. \quad (3.15)$$

From Equation (3.14) we also find the following useful equations:

$$GQ = FQ(1 - Q^\dagger FQ)^{-1} \quad (3.16)$$

$$Q^\dagger G = (1 - Q^\dagger FQ)^{-1} Q^\dagger F. \quad (3.17)$$

We will require Equations (3.15), (3.16), and (3.17) for the derivation of the caloron formula.

3.5 Derivation of the Instanton Formula

The instanton field resulting from the Nahm Transform is given by the following expression:

$$\begin{aligned} A_\mu &= \Psi^\dagger \partial_\mu \Psi = \frac{1}{2} \Psi^\dagger \overleftrightarrow{\partial}_\mu \Psi \\ &= \frac{1}{2} (\psi^\dagger \overleftrightarrow{\partial}_\mu \psi + \chi^\dagger \overleftrightarrow{\partial}_\mu \chi). \end{aligned} \quad (3.18)$$

The notation over the derivative is understood as follows:

$$\Psi^\dagger \overleftrightarrow{\partial}_\mu \Psi = \Psi^\dagger \partial_\mu \Psi - \partial_\mu \Psi^\dagger \Psi.$$

We express ψ in terms of an operator R :

$$\begin{aligned} \mathcal{D}_t^\dagger \Psi = 0 &\Rightarrow \mathbf{D}_t^\dagger \psi + Q\chi = 0 \\ &\Rightarrow \psi = -(\mathbf{D}_t^\dagger)^{-1} Q\chi = R\chi. \end{aligned} \quad (3.19)$$

We will thus be able to deal solely with χ in our formula, eliminating any need to solve the Weyl equation for ψ . We can therefore immediately see the benefit of using this approach to constructing the caloron. From here on we will write all of the indices for each quantity in the formula explicitly. Although this may look somewhat overdecorated, it would be useful for computational purposes to have the formula written in this form. Using indices that label the rank of the Nahm data, the spinor components, the jumping points and the solutions, the expression for ψ is

$$\begin{aligned} \psi_j^f &= -(\mathbf{D}_t^\dagger)_{\alpha a, \beta b}^{-1} Q_{\beta b, j}(s, \lambda_j) \chi_j^f \\ &= R_{\alpha a, j}(s) \chi_j^f, \end{aligned} \quad (3.20)$$

where

$\alpha = 1, 2$ labels the spinor indices

$a = 1, \dots, k$ is the rank of the Nahm data

$j = 1, \dots, n$ labels the jumping points

$e = 1, \dots, N(= n)$ labels the solutions.

Our first step is to write Equation (3.18) using indices, substituting in the expression for ψ . The result will be an expression involving only R and χ . The rest of the derivation will involve using R and $R^\dagger$ to introduce the Green's functions F and G . We will then eliminate G in favour of F . We will end up with a formula comprised of χ , Q and F . Writing all of the indices and substituting for ψ , we get:

$$2A_\mu^{ef} = \int \psi_{\alpha a}^{\dagger e}(s) \overleftrightarrow{\partial}_\mu \psi_{\alpha a}^f(s) ds + \chi_i^{\dagger e} \overleftrightarrow{\partial}_\mu \chi_j^f = \chi_i^{\dagger e} R_{i,\alpha a}^\dagger \overleftrightarrow{\partial}_\mu R_{\alpha a,j} \chi_j^f + \chi_i^{\dagger e} \overleftrightarrow{\partial}_\mu \chi_j^f. \quad (3.21)$$

Expanding this we have

$$2A_\mu^{ef} = \chi_i^{\dagger e} \left(R_{i,\alpha a}^\dagger \overleftrightarrow{\partial}_\mu R_{\alpha a,j} \right) \chi_j^f + \chi_i^{\dagger e} \left(1 + R_{i,\alpha a}^\dagger R_{\alpha a,j} \right) \partial_\mu \chi_j^f - \partial_\mu \chi_i^{\dagger e} \left(1 + R_{i,\alpha a}^\dagger R_{\alpha a,j} \right) \chi_j^f. \quad (3.22)$$

To get our desired formula for A we first need to find expressions for the terms in brackets in the equation we have above. We do this for $(1 + R_{i,\alpha a}^\dagger R_{\alpha a,j})$ and come up with the following nice result:

$$\begin{aligned} 1 + R_{i,\alpha a}^\dagger R_{\alpha a,j} &= 1 + Q_{i,\gamma c}^\dagger \mathbf{D}_{t\gamma c,\alpha a}^{-1} (\mathbf{D}_t^\dagger)_{\alpha a,\beta b}^{-1} Q_{\beta b,j} \\ &= 1 + Q_{i,\gamma c}^\dagger G_{\gamma c,\beta b}(\lambda_i, \lambda_j) Q_{\beta b,j} \\ &= (1 - Q_{i,\gamma c}^\dagger F_{\gamma c,\beta b}(\lambda_i, \lambda_j) Q_{\beta b,j})^{-1} \\ &= (\chi_i^e \chi_j^{\dagger e})^{-1}. \end{aligned} \quad (3.23)$$

Here the second line follows from the definition of G , the next line from Equation (3.15), and the final line uses the projection operator in Equation (3.8) written in Section 3.4. The other expression from Equation (3.22) that we must deal with is $R_{i,\alpha a}^\dagger \overleftrightarrow{\partial}_\mu R_{\alpha a,j}$. For this we will need to use the following derivative:

$$\partial_\mu (\mathbf{D}_t^\dagger)_{\alpha a, \beta b}^{-1} = (\mathbf{D}_t^\dagger)_{\alpha a, \delta d}^{-1} \bar{\sigma}_\mu^{\delta \rho} \otimes 1_{dr} (\mathbf{D}_t^\dagger)_{\rho r, \beta b}^{-1}, \quad (3.24)$$

where, as previously, $\mathbf{D}_t^\dagger = -1 \otimes (\frac{d}{ds} - iT_0 + it_0) - \vec{\sigma} \otimes (\vec{T} - \vec{t})$. Thus we have

$$\begin{aligned} R_{i,\alpha a}^\dagger \partial_\mu R_{\alpha a,j} &= Q_{i,\gamma c}^\dagger \mathbf{D}_{t\gamma c, \alpha a}^{-1} (\mathbf{D}_t^\dagger)_{\alpha a, \delta d}^{-1} \bar{\sigma}_\mu^{\delta \rho} \otimes 1_{dr} (\mathbf{D}_t^\dagger)_{\rho r, \beta b}^{-1} Q_{\beta b,j} \\ &= Q_{i,\gamma c}^\dagger G_{\gamma c, \delta d}(\lambda_i, s) \bar{\sigma}_\mu^{\delta \rho} \otimes 1_{dr} \mathbf{D}_{t\rho r, \kappa k} G_{\kappa k, \beta b}(s, \lambda_j) Q_{\beta b,j}, \end{aligned} \quad (3.25)$$

where we have used $G = (\mathbf{D}_t^\dagger \mathbf{D}_t)^{-1}$. We can find an expression for $(\partial_\mu R_{i,\alpha a}^\dagger) R_{\alpha a,j}$ in a similar manner, so we can write:

$$R_{i,\alpha a}^\dagger \overleftrightarrow{\partial}_\mu R_{\alpha a,j} = Q_{i,\gamma c}^\dagger G_{\gamma c, \delta d}(\lambda_i, s) \bar{\sigma}_{[\mu}^{\delta \rho} \sigma_{\nu]}^{\rho \kappa} \otimes D_{dk}^\nu G_{\kappa k, \beta b}(s, \lambda_j) Q_{\beta b,j}, \quad (3.26)$$

where $D_\nu = (D_0, -D_i) = ((\frac{d}{ds} - iT_0 + it_0), -(T_i - t_i))$. Using Equations (3.16) and (3.17) we have

$$R_{i,\alpha a}^\dagger \overleftrightarrow{\partial}_\mu R_{\alpha a,j} = (1 - Q^\dagger F Q)^{-1} Q_{i,\gamma c}^\dagger F_{cd}(\lambda_i, s) \bar{\eta}_{\mu\nu}^{\delta \kappa} \otimes D_{dk}^\nu F_{kb}(s, \lambda_j) Q_{\beta b,j} (1 - Q^\dagger F Q)^{-1}. \quad (3.27)$$

Next we make use of the projection operator, Equation (3.8):

$$\begin{aligned} R_{i,\alpha a}^\dagger \overleftrightarrow{\partial}_\mu R_{\alpha a,j} &= (\chi_i^e \chi_j^{\dagger e})^{-1} Q_{i,\gamma c}^\dagger F_{cd}(\lambda_i, s) \bar{\eta}_{\mu\nu}^{\delta \kappa} \otimes \left(-\frac{1}{2} \partial_\nu F_{dk}^{-1} \right) F_{kb}(s, \lambda_j) Q_{\beta b,j} (\chi_i^f \chi_j^{\dagger f})^{-1} \\ &= \frac{1}{2} (\chi_i^e \chi_j^{\dagger e})^{-1} Q_{i,\gamma c}^\dagger \bar{\eta}_{\mu\nu}^{\gamma \beta} \otimes \partial_\nu F_{cb}(\lambda_i, \lambda_j) Q_{\beta b,j} (\chi_i^f \chi_j^{\dagger f})^{-1} \\ &= \frac{1}{2} (\chi_i^e \chi_j^{\dagger e})^{-1} \partial_\nu \left(Q_{i,\gamma c}^\dagger \bar{\eta}_{\mu\nu}^{\gamma \beta} \otimes F_{cb}(\lambda_i, \lambda_j) Q_{\beta b,j} \right) (\chi_i^f \chi_j^{\dagger f})^{-1}. \end{aligned} \quad (3.28)$$

We now have most of the work done, and all that remains is to tidy up the expression further. The full instanton formula now looks like this:

$$\begin{aligned}
(2A_\mu)^{ef} = & \chi_i^{\dagger e} \left(\frac{1}{2} (\chi_i^e \chi_j^{\dagger e})^{-1} \partial_\nu \left(Q_{i,\gamma c}^\dagger \bar{\eta}_{\mu\nu}^{\gamma\beta} \otimes F_{cb}(\lambda_i, \lambda_j) Q_{\beta b,j} \right) (\chi_i^f \chi_j^{\dagger f})^{-1} \right) \chi_j^f + \\
& + \chi_i^{\dagger e} (\chi_i^e \chi_j^{\dagger e})^{-1} \partial_\mu \chi_j^f - \partial_\mu \chi_i^{\dagger e} (\chi_i^f \chi_j^{\dagger f})^{-1} \chi_j^f.
\end{aligned} \tag{3.29}$$

The formula is given in its final form, in terms of the Green's function F , below.

$$\begin{aligned}
(2A_\mu)^{ef} = & \frac{1}{2} (\chi^{-1})_i^e \left[\partial_\nu \left(Q_{i,\gamma c}^\dagger \bar{\eta}_{\mu\nu}^{\gamma\beta} \otimes F_{cb}(\lambda_i, \lambda_j) Q_{\beta b,j} \right) \right] (\chi^{\dagger-1})_j^f \\
& + (\chi^{-1})_i^e (\partial_\mu \chi)_j^f - (\partial_\mu \chi^\dagger)_i^e (\chi^{\dagger-1})_j^f.
\end{aligned} \tag{3.30}$$

This expression generalises the formula for a charge one caloron derived in [1]. Equation (3.30) is a general formula for a caloron with no magnetic charge.

As mentioned earlier this caloron formula relates to the situation where we have k D3-branes that are periodic in one coordinate intersected by n D5-branes. In the next section we will derive a formula for the brane configuration of Section 2, in other words we will account in our formula for the effect of introducing an NS5-brane to the configuration. The derivation in the next section will be completely analogous to that above. We shall therefore not go through the same steps again but will refer to this section where necessary.

Chapter 4

Yang-Mills Instanton on the Taub-NUT Space

In this section we derive a formula for a charge k $SU(2)$ Yang-Mills instanton on the Taub-NUT space. This formula can be used to find instantons that arise from the brane configuration discussed in Chapter 2; the inclusion of NS5-branes in that model is reflected in the instanton formula by the addition of bifundamental data to the caloron formula of Chapter 3, this in turn reflects the presence of the Taub-NUT background.

Just as for the derivation in Chapter 3, we use the Green's function method to write our formula. In fact the derivation we present below is completely analogous to that in the previous chapter. In the next chapter we provide a simple example of this formula in use by writing an explicit expression for the charge one instanton. In the charge one case the Nahm data is Abelian, thus simplifying the calculation.

4.1 The Taub-NUT Space

4.1.1 The Taub-NUT Metric

The Taub-NUT space is described in [44]. It is a four-dimensional asymptotically locally flat (ALF) hyperkahler manifold with one periodic coordinate $\psi \sim \psi + 4\pi$. In the compliment of one point the Taub-NUT space can be thought of as S^1 fibred over a base $\mathbb{R}^3 \setminus \{0\}$. The Taub-NUT metric is

$$ds^2 = \frac{1}{4} \left(\left(l + \frac{1}{|\vec{x}|} \right) d\vec{x}^2 + \frac{1}{\left(l + \frac{1}{|\vec{x}|} \right)} (d\psi + \vec{\omega} \cdot d\vec{x})^2 \right), \quad (4.1)$$

where $\vec{x}$ is a vector parametrizing $\mathbb{R}^3$, and $\frac{\partial}{\partial x_i} \frac{1}{|\vec{x}|} = \epsilon_{ijk} \frac{\partial \omega_k}{\partial x_j}$. The parameter l is related to the asymptotic size of the S^1 . The Taub-NUT metric approaches the flat metric on $\mathbb{R}^4$ as $l \rightarrow 0$. To see this, we write the flat metric using quaternions, as follows:

$$ds^2 = dq d\bar{q}, \quad (4.2)$$

where $q = q_0 + iq_1 + jq_2 + kq_3$. We can write q as a product of an imaginary quaternion a and a unit complex number $e^{\frac{i\psi}{2}}$. The coordinate ψ has period 4π so that the space is smooth at the origin. Now introducing a vector $\vec{x} = -aia$ and a one-form $\omega = \frac{1}{|\vec{x}|}(daia - aida)$, we can write the flat metric as $ds^2 = \frac{1}{4} \left(\frac{1}{|\vec{x}|} d\vec{x}^2 + |\vec{x}| (d\psi + \vec{\omega} \cdot d\vec{x})^2 \right)$, the Taub-NUT metric when $l \rightarrow 0$.

An instanton on the Taub-NUT space is a connection $\nabla_\mu = \partial_\mu - iA_\mu$ on a Hermitian bundle over the Taub-NUT space, with curvature $F = dA - iA \wedge A$ that satisfies the self-duality equations $F = *F$, and with finite action $S = \int \text{tr} F \wedge *F$, where the operation $*$ is the Hodge star operator.

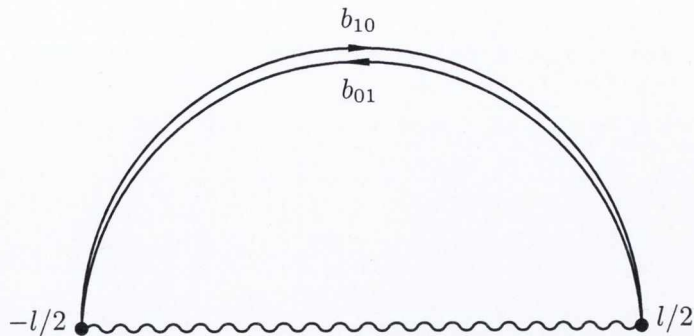


Figure 4.1: Bow Diagram for the Taub-NUT. The Nahm data t_μ is defined over the interval $[-\frac{l}{2}, \frac{l}{2}]$. It consists of a connection $\frac{d}{ds} - iT_0$ and three Hermitian endomorphisms t_i of the complex line bundle $e \rightarrow [-\frac{l}{2}, \frac{l}{2}]$. The bow data b_{01} and b_{10} are linear maps taking us between the one-dimensional vector spaces at the ends of the interval.

4.1.2 The Taub-NUT Bow Data

A description of the Taub-NUT space as a hyperkahler quotient is given in [44] [45]. We consider the interval $[-\frac{l}{2}, \frac{l}{2}]$, parametrized by the coordinate s , shown in Figure 4.1. There is a complex line bundle $e \rightarrow [-\frac{l}{2}, \frac{l}{2}]$ over this interval with a Hermitian connection $\frac{d}{ds} - it_0$ and three Hermitian bundle endomorphisms, t_1, t_2, t_3 . These t_μ are rank one Nahm data. There is a one-dimensional fibre $e_{\pm \frac{l}{2}}$ at each end of the interval, with a map b_{10} taking us from $e_{-\frac{l}{2}}$ to $e_{\frac{l}{2}}$ and a map b_{01} going from $e_{\frac{l}{2}}$ to $e_{-\frac{l}{2}}$. The data (t_μ, b_{01}, b_{10}) satisfy the Nahm equations and behave as follows under gauge transformations:

$$\begin{aligned} t_0 &\rightarrow g^{-1}t_0g + ig^{-1}\frac{dg}{ds}, \\ t_j &\rightarrow g^{-1}t_jg, \\ b_{01} &\rightarrow g^{-1}\left(-\frac{l}{2}\right)b_{01}g\left(\frac{l}{2}\right), \\ b_{10} &\rightarrow g^{-1}\left(\frac{l}{2}\right)b_{10}g\left(-\frac{l}{2}\right). \end{aligned}$$

The moment maps vanish if [44]

$$\begin{aligned} [d, \mathbf{t}] &= \delta_L b_{01} b_{10} - \delta_R b_{10} b_{01}, \\ [d^\dagger, d] + [\mathbf{t}^\dagger, \mathbf{t}] &= \delta_L (b_{01} b_{01}^\dagger - b_{10}^\dagger b_{10}) + \delta_R (b_{10} b_{10}^\dagger - b_{01}^\dagger b_{01}), \end{aligned} \quad (4.3)$$

where we have use the combinations $d = \frac{d}{ds} - it_0 - t_3$ and $\mathbf{t} = t_1 + it_2$. We can combine $(b_{01}, b_{10}, b_{01}^\dagger, b_{10}^\dagger)$ into spinors b_- and b_+ as follows:

$$b_- = \begin{pmatrix} b_{01}^\dagger \\ -b_{10} \end{pmatrix}, \quad b_+ = \begin{pmatrix} b_{10}^\dagger \\ b_{01} \end{pmatrix}.$$

Then at $s = \pm \frac{l}{2}$ we have

$$b_\pm b_\pm^\dagger = |\vec{t}| \pm \mathfrak{X}. \quad (4.4)$$

4.1.3 A Basis of Self-Dual Forms

At each point s on the Taub-NUT interval $[-\frac{l}{2}, \frac{l}{2}]$, there is a line bundle with an Abelian connection $a_s = sa$, where

$$a = \frac{d\psi + \omega}{2V}, \quad (4.5)$$

and $V = l + \frac{1}{|\vec{x}|}$. These connections are Abelian instantons since they have self-dual curvature da in the orientation (ψ, x_1, x_2, x_3) , with finite action. We note that the components of the right-hand side of the following relation provide a basis of self-dual two-forms on the Taub-NUT:

$$\left(\frac{1}{2}d\mathfrak{X} - ia\right)^\dagger \wedge \left(\frac{1}{2}d\mathfrak{X} - ia\right) = \frac{i}{2}\sigma_k \left(\frac{d\psi + \omega}{V} \wedge dx^k + \frac{1}{2}\epsilon_{ijk} dx^i dx^j\right). \quad (4.6)$$

We use coordinates (ψ, t_j) so that our basis among one-forms is (a, dt_j) . We note that $\frac{1}{2}d\mathfrak{X} - ia = -(d\mathfrak{X} + ia)$ so $(d\mathfrak{X} - ia) \wedge (d\mathfrak{X} + ia)$ is self-dual. We refer to [44] for a proof of the self-duality of the curvature of the Taub-NUT instanton. The proof

relies on the self-duality of the combinations discussed here as well as the fact that the Green's function commutes with the sigma matrices.

4.2 Connection in Curved Background

4.2.1 Setup

In this chapter we want to derive a formula for charge k connections with self-dual curvature on ALF spaces. In the previous chapter we found a formula for calorons on $\mathbb{R}^3 \times S^1$. We now plan to extend that formula to curved backgrounds. In the case of the Taub-NUT background this means including bifundamental Bow data, i.e. maps between the end-points of the Taub-NUT interval.

As before we wish to construct a Dirac-type operator from the Nahm data. In this case the data is the gauge field T_0 and scalar fields T_i on the Taub-NUT interval $\mathcal{I} \setminus \{-\lambda, \lambda\}$, the fundamental fields $Q_{\pm}$ at the jumping points $\pm\lambda$, and the bifundamental fields $B_{\pm}$ at the end-points of the interval, $\pm\frac{l}{2}$. We find the dual operator, and taking the square $\mathcal{D}^\dagger \mathcal{D}$ we end up with a Laplacian operator which has the useful property that it commutes with the sigma matrices.

The Bow data for an instanton on the Taub-NUT space was described in [44], and can be represented on a Bow diagram [44] [46] as shown in Figure 4.2. There are $k \times k$ matrices (T_0, T_j) on the Taub-NUT interval that transform as $T_0 \rightarrow g^{-1}T_0g - ig^{-1}\frac{dg}{ds}$ and $T_j \rightarrow g^{-1}T_jg$. They are defined on a Hermitian complex vector bundle $E \rightarrow \mathcal{I} \setminus \{-\lambda, \lambda\}$, of rank k . They are shown in Figure 4.3. At jumping points $s = \pm\lambda$ along the interval the Nahm data takes the form of $2k$ -component vectors $Q_{\pm} \in S \otimes E_{\pm\lambda}$ that transform in the fundamental representation of the gauge group as $Q \rightarrow g^{-1}Q$. Here $E_{\pm\lambda}$ are fibres of the bundle $E \rightarrow \mathcal{I}$, $S \approx \mathbb{C}^2$ is a rank 2 Hermitian spinor bundle and g is the identity in S . The vectors $Q_{\pm}$ are grouped

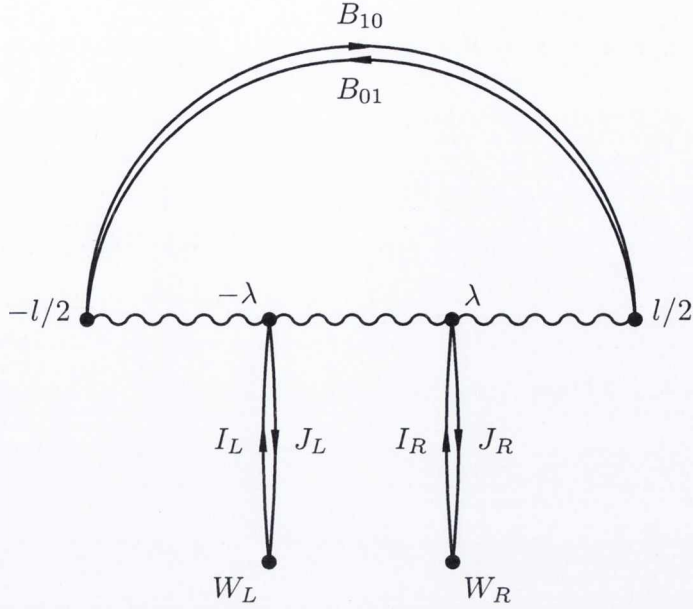


Figure 4.2: Detailed Bow Diagram for $SU(2)$ Instanton on the Taub-NUT Space. This figure shows the maps to and from the vector spaces at the end-points and jumping points of the interval.

together from the following linear maps, which are shown in Figure 4.2:

$$I_L : W_L \rightarrow E_{-\lambda}, \quad (4.7)$$

$$J_L : E_{-\lambda} \rightarrow W_L, \quad (4.8)$$

$$I_R : W_R \rightarrow E_{\lambda}, \quad (4.9)$$

$$J_R : E_{\lambda} \rightarrow W_R. \quad (4.10)$$

Here W_L and W_R are auxiliary one-dimensional complex vector spaces. We form $Q_{\pm}$ from the maps I and J as follows:

$$Q_- = \begin{pmatrix} J_L^\dagger \\ I_L \end{pmatrix}, \quad Q_+ = \begin{pmatrix} J_R^\dagger \\ I_R \end{pmatrix}, \quad (4.11)$$

We have $Q_- : W_L \rightarrow S \otimes E_{-\lambda}$ and $Q_+ : W_R \rightarrow S \otimes E_{\lambda}$. The new ingredients in the Taub-NUT formula are the linear maps $B_{01} : E_{\frac{l}{2}} \rightarrow E_{-\frac{l}{2}}$ and $B_{10} : E_{-\frac{l}{2}} \rightarrow E_{\frac{l}{2}}$. These are $k \times k$ matrices that transform as $B_{01} \rightarrow g_L^{-1} B_{01} g_R$ and $B_{10} \rightarrow g_R^{-1} B_{10} g_L$,

i.e. in the bifundamental representation of the gauge group.

Since we are considering an instanton in a Taub-NUT background, we twist the Weyl operator in the Taub-NUT space, thus the operator will provide information about the background. To twist the operator we will include the Taub-NUT Bow data $(b_{01}, b_{10}, t_0, t_j)$ described in the previous section. We will see how this is done below.

The kernel of the Weyl operator is $\Psi = (\psi, \chi, v)$, where ψ is an L^2 section of $S \otimes E \rightarrow \mathcal{I} \setminus \{-\lambda, \lambda\}$, $\chi_{\pm\lambda} \in E_{\pm\lambda}$ and $v = \begin{pmatrix} v_- \\ v_+ \end{pmatrix}$, with $v_{\pm} \in E_{\pm\frac{l}{2}}$. Therefore T acts on ψ , Q on χ and B on v . When we introduce the twisted operator we will still denote the kernel as $\Psi = (\psi, \chi, v)$ but in this case ψ is an L^2 section of $S \otimes E \otimes e \rightarrow \mathcal{I} \setminus \{-\lambda, \lambda\}$ and $v_{\pm} \in E_{\pm\frac{l}{2}} \otimes e_{\mp\frac{l}{2}}$, where e is the bundle described in Section 4.1.2. Therefore, in this case $(T - t)$ acts on ψ , Q on χ , and $(B, -b)$ on v .

4.2.2 Weyl Operator

We construct the Weyl operator $\mathcal{D}$ and its dual $\mathcal{D}^\dagger$ from the instanton and Taub-NUT Bow data. We know that this Bow data satisfies the Nahm equations, and thus we will be able to construct a Laplacian that commutes with the sigma matrices. We can derive a formula for an instanton on Taub-NUT using the Green's function of this Laplacian. A simple Bow diagram for the instanton data is shown in Figure 4.3. The Nahm data T_1 and T_2 on the middle and end intervals gives the positions of the instanton constituents. At $s = \pm\lambda$ we will have the vectors $Q_{\pm}$ while the linear maps between endpoints are given by the Bow data B_{01} and B_{10} . We define

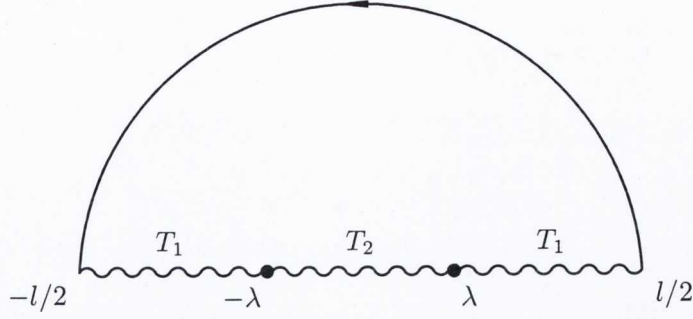


Figure 4.3: Simple Bow Diagram for $SU(2)$ Instanton on the Taub-NUT Space. This figure shows the Nahm data T_1 and T_2 . The Nahm data is the same on the left and right intervals since T_1 and T_2 are the positions of the instanton constituents.

the operator $\mathcal{D}$:

$$\mathcal{D} : f \rightarrow \begin{pmatrix} \left(\frac{d}{ds} - iT_0 - \Upsilon\right)f \\ Q_-^\dagger f(-\lambda) \\ Q_+^\dagger f(\lambda) \\ B_+^\dagger f\left(\frac{l}{2}\right) \\ B_-^\dagger f\left(-\frac{l}{2}\right) \end{pmatrix}, \quad (4.12)$$

where $B_- = \begin{pmatrix} B_{10}^\dagger \\ -B_{01} \end{pmatrix}$ and $B_+ = \begin{pmatrix} B_{01}^\dagger \\ B_{10} \end{pmatrix}$. This operator acts on the space of L^2 sections of $S \otimes E$ that are continuous on $\mathcal{I}$ and have L^2 derivatives on $\mathcal{I} \setminus \{-\lambda, \lambda\}$.

We use the hermitian inner product, which is defined as

$$\langle f, g \rangle = \int \bar{f}(s)g(s)ds,$$

to write

$$\begin{aligned} \langle \Psi, \mathcal{D}f \rangle = \int ds \psi^\dagger \left(\frac{d}{ds} - iT_0 - \Upsilon \right) f(s) &+ \left(\bar{\chi}_{-\lambda} Q_-^\dagger f(-\lambda) + \bar{\chi}_\lambda Q_+^\dagger f(\lambda) \right) \\ &+ \left(\bar{v}_+ B_+^\dagger f\left(\frac{l}{2}\right) + \bar{v}_- B_-^\dagger f\left(-\frac{l}{2}\right) \right). \end{aligned} \quad (4.13)$$

Integrating Equation (4.13) by parts and using the fact that the inner product is Hermitian, we can write an expression for the operator dual to $\mathcal{D}$:

$$\mathcal{D}^\dagger \Psi = \left(-\frac{d}{ds} + iT_0 - \Upsilon \right) \psi + \sum_{\pm\lambda} \delta(s \pm \lambda) Q_\pm \chi_{\pm\lambda}^\dagger + \delta(s - \frac{l}{2}) B_+ v_+^\dagger + \delta(s + \frac{l}{2}) B_- v_-^\dagger. \quad (4.14)$$

Here we understand $\frac{d}{ds}$ to be a generalised derivative and we extend ψ by zero outside the interval, so that $\frac{d\psi}{ds}$ has delta function terms $\delta(s + \frac{l}{2})\psi(-\frac{l}{2})$ and $-\delta(s - \frac{l}{2})\psi(\frac{l}{2})$. The operator $\mathcal{D}^\dagger$ acts on a space that is a direct sum of the space of L^2 sections of $S \otimes E$, the vector spaces $E_{\pm\lambda}$ and the vector spaces $E_{\pm\frac{l}{2}}$ [44].

We can twist the operator $\mathcal{D}$ in Taub-NUT space as follows:

$$\mathcal{D}_t : f \rightarrow \begin{pmatrix} \left(\frac{d}{ds} - i(T_0 - t_0) - (\Upsilon - \mathfrak{k})f \right) \\ Q_-^\dagger f(-\lambda) \\ Q_+^\dagger f(\lambda) \\ (-b_-, B_+)^\dagger f(\frac{l}{2}) \\ (B_-, -b_+)^\dagger f(-\frac{l}{2}) \end{pmatrix}, \quad (4.15)$$

so that now we have a family of operators over the Taub-NUT. Again, using the hermitian inner product we can write

$$\begin{aligned} \langle \Psi, \mathcal{D}_t f \rangle = \int ds \psi^\dagger \left(\frac{d}{ds} - i(T_0 - t_0) + \mathfrak{k} \right) f(s) + \sum_{\pm\lambda} \bar{\chi}_{\pm\lambda} Q_\pm^\dagger f(\pm\lambda) + \\ + \left(\bar{v}_+(-b_-, B_+) f(\frac{l}{2}) + \bar{v}_-(B_-, -b_+)^\dagger f(\frac{l}{2}) \right), \end{aligned} \quad (4.16)$$

where $\mathfrak{k} = \Upsilon - \mathfrak{k}$, and we can determine the dual twisted operator $\mathcal{D}_t^\dagger$ from $\langle \Psi, \mathcal{D}_t f \rangle = \langle \mathcal{D}_t^\dagger \Psi, f \rangle$. Integrating Equation (4.16) by parts, we get

$$\begin{aligned} \langle \Psi, \mathcal{D}_t f \rangle = \int ds \left(\left(-\frac{d}{ds} + i(T_0 - t_0) + \mathfrak{k} \right) \psi + \sum_{\pm\lambda} \delta(s \pm \lambda) Q_\pm \chi_{\pm\lambda}^\dagger + \right. \\ \left. + \delta(s - \frac{l}{2})(-b_-, B_+) v_+^\dagger + \delta(s + \frac{l}{2})(B_-, -b_+) v_-^\dagger \right) f(s), \end{aligned} \quad (4.17)$$

and so we have found an expression for $\mathcal{D}_t^\dagger$:

$$\begin{aligned} \mathcal{D}_t^\dagger \Psi = & \left(-\frac{d}{ds} + i(T_0 - t_0) + \frac{l}{2} \right) \psi + \sum_{\pm\lambda} \delta(s \pm \lambda) Q_\pm \chi_{\pm\lambda}^\dagger + \delta(s - \frac{l}{2}) (-b_-, B_+) v_+^\dagger + \\ & + \delta(s + \frac{l}{2}) (B_-, -b_+) v_-^\dagger. \end{aligned} \quad (4.18)$$

As noted above, boundary terms will not arise from the integration by parts in Equation (4.16) since we are considering a generalised derivative. We are now in a position to write the Laplacian

$$\begin{aligned} \mathcal{D}_t^\dagger \mathcal{D}_t f = & \left(-\left(\frac{d}{ds}\right)^2 - \frac{d\frac{l}{2}}{ds} + \frac{l^2}{4} \right) f(s) + \delta(s + \lambda) Q_- Q_-^\dagger f(-\lambda) + \delta(s - \lambda) Q_+ Q_+^\dagger f(\lambda) \\ & + \delta(s - \frac{l}{2}) (-b_-, B_+) \left((-b_-, B_+)^\dagger f(\frac{l}{2}) + (B_-, -b_+)^\dagger f(-\frac{l}{2}) \right) + \\ & + \delta(s + \frac{l}{2}) (B_-, -b_+) \left((-b_-, B_+)^\dagger f(\frac{l}{2}) + (B_-, -b_+)^\dagger f(-\frac{l}{2}) \right), \end{aligned} \quad (4.19)$$

where we have set T_0 to zero and gauged away t_0 . The following terms in the Laplacian will cancel due to the Nahm equations for the instanton data and for the Bow data:

$$\begin{aligned} \sum_i \sigma_i \otimes \left(\frac{dT_i}{ds} + i[T_j, T_k] \right) + \sum_{\pm\lambda} \delta(s \pm \lambda) \text{vec}(Q_\pm Q_\pm^\dagger) + \delta(s + \frac{l}{2}) \text{vec}(B_- B_-^\dagger) \\ + \delta(s - \frac{l}{2}) \text{vec}(B_+ B_+^\dagger) = 0, \end{aligned} \quad (4.20)$$

$$\sum_i \sigma_i \otimes \frac{dt_i}{ds} + i[t_j, t_k] + \delta(s + \frac{l}{2}) \text{vec}(b_+ b_+^\dagger) + \delta(s - \frac{l}{2}) \text{vec}(b_- b_-^\dagger) = 0, \quad (4.21)$$

where, as before, "vec" acts as follows:

$$\text{vec}(\sigma_0 \otimes a_0 + \sigma_1 \otimes a_1 + \sigma_2 \otimes a_2 + \sigma_3 \otimes a_3) = \sigma_1 \otimes a_1 + \sigma_2 \otimes a_2 + \sigma_3 \otimes a_3.$$

We can write the Laplacian, making evident the part that solves the Nahm equations,

as follows:

$$\begin{aligned}
\mathcal{D}_t^\dagger \mathcal{D}_t f = & 1 \otimes \left(\left(- \left(\frac{d}{ds} \right)^2 + \bar{z}^2 \right) f(s) + \frac{1}{2} \text{tr} Q_+ Q_+^\dagger f(\lambda) + \frac{1}{2} \text{tr} Q_- Q_-^\dagger f(-\lambda) \right. \\
& + \frac{1}{2} \delta(s - \frac{l}{2}) \text{tr}(b_- b_-^\dagger + B_+ B_+^\dagger) f(\frac{l}{2}) + \frac{1}{2} \delta(s + \frac{l}{2}) \text{tr}(B_- B_-^\dagger + b_+ b_+^\dagger) f(-\frac{l}{2}) \\
& \left. - \frac{1}{2} \delta(s - \frac{l}{2}) \text{tr}(b_- B_-^\dagger + B_+ b_+^\dagger) f(-\frac{l}{2}) - \frac{1}{2} \delta(s + \frac{l}{2}) \text{tr}(B_- b_-^\dagger + b_+ B_+^\dagger) f(\frac{l}{2}) \right) \\
& + \sigma_i \otimes \left(\left(\frac{dz_i}{ds} + i[T_j, T_k] - i[t_j, t_k] \right) f(s) + (Q_- Q_-^\dagger)_i f(-\lambda) + (Q_+ Q_+^\dagger)_i f(\lambda) \right. \\
& \left. + \delta(s - \frac{l}{2}) (b_- b_-^\dagger + B_+ B_+^\dagger)_i f(\frac{l}{2}) + \delta(s + \frac{l}{2}) (B_- B_-^\dagger + b_+ b_+^\dagger)_i f(-\frac{l}{2}) \right).
\end{aligned} \tag{4.22}$$

We see that the term proportional to the sigma matrices in Equation (4.22) is zero in accordance with the Nahm Equations, and we have obtained a Laplacian that is a *scalar* function in spin space. By this we mean that the operator $\mathcal{D}_t^\dagger \mathcal{D}_t$ commutes with the quaternions and the Pauli matrices. Another way of writing the Bow equations is

$$\text{vec}(\mathcal{D}_t^\dagger \mathcal{D}_t) = 0.$$

We can use the Green's function F of the Laplacian $\mathcal{D}_t^\dagger \mathcal{D}_t = 1_{2 \times 2} \otimes \Delta$ to write our formula. The Green's function solves

$$\Delta F(s, s_0) = \delta(s - s_0), \tag{4.23}$$

where s parametrizes the Taub-NUT interval. The function Δ is a scalar function in spin space, which is positive definite except at a finite number of points corresponding to zero-size instantons. Therefore it has a well-defined inverse given by the Green's function $1_{2 \times 2} \otimes F = (\mathcal{D}_t^\dagger \mathcal{D}_t)^{-1}$. In future we do not distinguish between F and $1_{2 \times 2} \otimes F$.

4.3 Deriving the Instanton Formula

Now that we have the ingredients we are in a position to derive a formula for the instanton on the Taub-NUT space in terms of the Green's function of $\mathcal{D}^\dagger \mathcal{D}$. We define the following operators, which will be useful for our derivation:

$$\mathcal{Q}_t^\dagger = -1 \otimes \left(\frac{d}{ds} - iT_0 \otimes 1 + i1 \otimes t_0 \right) - \sum_{j=1}^3 \sigma_j \otimes (T_j \otimes 1 - 1 \otimes t_j). \quad (4.24)$$

The Weyl operator is $\mathcal{D}_t = \mathbf{D} \oplus \mathcal{Q}_t^\dagger$, with $\mathbf{D}$ given by

$$\begin{aligned} \mathbf{D}^\dagger &= \mathcal{Q}_t^\dagger \oplus \left(\delta(s + \frac{l}{2})(B_- \otimes 1, -1 \otimes b_+) + \delta(s - \frac{l}{2})(-1 \otimes b_-, B_+ \otimes 1) \right) \\ &= \mathcal{Q}_t^\dagger \oplus \left(B \begin{pmatrix} \delta_L & 0 \\ 0 & \delta_R \end{pmatrix} - b \begin{pmatrix} \delta_R & 0 \\ 0 & \delta_L \end{pmatrix} \right) \\ &= \mathcal{Q}_t^\dagger \oplus Y, \end{aligned} \quad (4.25)$$

where

$$Y = \left(B \begin{pmatrix} \delta_L & 0 \\ 0 & \delta_R \end{pmatrix} - b \begin{pmatrix} \delta_R & 0 \\ 0 & \delta_L \end{pmatrix} \right).$$

We have combined $B_\pm$ and $b_\pm$ into the quaternions $B = (B_-, B_+)$ and $b = (b_-, b_+)$

We use the notation: $\delta_L = \delta(s + \frac{l}{2})$ and $\delta_R = \delta(s - \frac{l}{2})$ throughout the derivation.

It is useful to write the expression for the Laplacian $\mathbf{D}^\dagger \mathbf{D}$. Unlike $\mathcal{D}_t^\dagger \mathcal{D}_t$ this is not a scalar function since its "vector" part will not disappear by the Bow equations. This is because $\mathbf{D}$ does not include Nahm data at jumping points and thus the equations will not be exactly satisfied. Although it is not as convenient an operator to work with for calculations, it is very useful to us in our derivation, and we write it here in terms of the operators $\mathcal{Q}_t$ and Y :

$$\mathbf{D}^\dagger \mathbf{D} = \mathcal{Q}_t^\dagger \mathcal{Q}_t + YY^\dagger \quad (4.26)$$

where

$$YY^\dagger = \left(B \begin{pmatrix} \delta_L & 0 \\ 0 & \delta_R \end{pmatrix} - b \begin{pmatrix} \delta_R & 0 \\ 0 & \delta_L \end{pmatrix} \right) \left(\begin{pmatrix} \delta_L & 0 \\ 0 & \delta_R \end{pmatrix} B^\dagger - \begin{pmatrix} \delta_R & 0 \\ 0 & \delta_L \end{pmatrix} b^\dagger \right). \quad (4.27)$$

Just as in Chapter 3 we define the Green's functions

$$G = (\mathbf{D}^\dagger \mathbf{D})^{-1}, \quad F = (\mathcal{D}_t^\dagger \mathcal{D}_t)^{-1}, \quad (4.28)$$

so that Equations (3.15), (3.16), and (3.17) apply.

Each point s defines a bundle over the Taub-NUT space with a connection $d_\mu = \partial_\mu - isa_\mu$. The instanton connection is

$$A_\mu = (\Psi, d_\mu \Psi), \quad (4.29)$$

where Ψ is the kernel of $\mathcal{D}_t^\dagger$. We assemble d_μ into a single covariant differential $d = dt^\mu d_\mu$ so that the connection is $A = (\Psi, d\Psi)$. A computation completely analogous to that in the previous section leads to

$$A_\mu = \frac{1}{2} \left(\chi^{-1} Q^\dagger F \left(\mathbf{D}^\dagger [d_\mu, \mathbf{D}^\dagger]^\dagger - [d_\mu, \mathbf{D}^\dagger] \mathbf{D} \right) F Q (\chi^\dagger)^{-1} + \chi^{-1} d_\mu \chi - (d_\mu \chi)^\dagger (\chi^\dagger)^{-1} \right). \quad (4.30)$$

This formula still contains terms that are difficult to compute, notably the terms containing the operator $\mathbf{D}^\dagger \mathbf{D}$. So we would like to simplify it further, in particular we would like to replace these awkward terms with some involving $\mathcal{D}^\dagger \mathcal{D}$. By a direct computation we find an expression for $[d, \mathbf{D}^\dagger]$:

$$[d, \mathbf{D}^\dagger] = - \left((d\mathfrak{k} + ia), \left(\delta_R db_-, \delta_L db_+ \right) \right). \quad (4.31)$$

We note that

$$db_- = -\frac{1}{2t}(d\mathfrak{A} + ia)b_- \quad (4.32)$$

$$db_+ = \frac{1}{2t}(d\mathfrak{A} + ia)b_+, \quad (4.33)$$

however we will not require these expressions at present. Using Equation (4.25) we can write

$$\mathbf{D}^\dagger[d, \mathbf{D}^\dagger]^\dagger = -\mathcal{Q}_t^\dagger(d\mathfrak{A} - ia) - Y \begin{pmatrix} \delta_R & 0 \\ 0 & \delta_L \end{pmatrix} db^\dagger, \quad (4.34)$$

where $db^\dagger = \begin{pmatrix} db_-^\dagger \\ db_+^\dagger \end{pmatrix}$. We now observe that

$$\mathcal{Q}_t^\dagger = \frac{1}{2}(\mathfrak{Q} - i\partial_0)(\mathcal{Q}_t^\dagger \mathcal{Q}_t). \quad (4.35)$$

In the above and following expressions $\mathfrak{Q} \pm i\partial_0$ is understood to be the total derivative with respect to t_j and t_0 , i.e.

$$\mathfrak{Q} = \sigma^j \left(\frac{\partial}{\partial t_j} + \frac{db_\pm}{dt_j} \frac{\partial}{\partial b_\pm} + \frac{db_\pm^\dagger}{dt_j} \frac{\partial}{\partial b_\pm^\dagger} \right),$$

with a similar expression for $\frac{\partial}{\partial t_0}$. Here taking a derivative with respect to $b_\pm$ involves taking a derivative with respect to each of its components, i.e.

$$\frac{\partial}{\partial b_-} = \left(\frac{\partial}{\partial \bar{b}_{01}}, -\frac{\partial}{\partial b_{10}} \right),$$

and similarly for $\frac{\partial}{\partial b_+}$. Substituting Equation (4.26) into Equation (4.35) we can write

$$(\mathfrak{Q} - i\partial_0)(\mathcal{Q}_t^\dagger \mathcal{Q}_t) = (\mathfrak{Q} - i\partial_0)(\mathbf{D}^\dagger \mathbf{D} - YY^\dagger), \quad (4.36)$$

where $YY^\dagger$ is written in Equation (4.27). Now substituting Equation (4.36) into Equation (4.35) and Equation (4.35) into Equation (4.34) we arrive at the following

expression for $\mathbf{D}^\dagger[d, \mathbf{D}^\dagger]^\dagger$:

$$\mathbf{D}^\dagger[d, \mathbf{D}^\dagger]^\dagger = -\frac{1}{2}(\mathfrak{D} - i\partial_0)(\mathbf{D}^\dagger\mathbf{D} - YY^\dagger)(d\mathfrak{k} - ia) - Y \begin{pmatrix} \delta_R & 0 \\ 0 & \delta_L \end{pmatrix} db^\dagger. \quad (4.37)$$

We obtain an expression for $[d, \mathbf{D}^\dagger]\mathbf{D}$ in a similar manner (or note that it is the Hermitian conjugate of Equation (4.37)). Now we have

$$\begin{aligned} & \left(\mathbf{D}^\dagger[d, \mathbf{D}^\dagger]^\dagger - [d, \mathbf{D}^\dagger]\mathbf{D} \right) = \\ & = -\frac{1}{2} \left((\mathfrak{D} - i\partial_0)(\mathbf{D}^\dagger\mathbf{D} - YY^\dagger)(d\mathfrak{k} - ia) - (d\mathfrak{k} + ia)(\mathfrak{D} + i\partial_0)(\mathbf{D}^\dagger\mathbf{D} - YY^\dagger) \right) \\ & \quad - Y \begin{pmatrix} \delta_R & 0 \\ 0 & \delta_L \end{pmatrix} db^\dagger + db \begin{pmatrix} \delta_R & 0 \\ 0 & \delta_L \end{pmatrix} Y^\dagger. \end{aligned} \quad (4.38)$$

We can use the fact that

$$\mathbf{D}^\dagger\mathbf{D} = F^{-1} - QQ^\dagger, \quad (4.39)$$

to replace $\mathbf{D}^\dagger\mathbf{D}$ with the Green's function F in our formula. We note that since Q has no dependance on t_0 or t_j we will be able to drop the $QQ^\dagger$ term in $(\mathfrak{D} - i\partial_0)F$.

We obtain the following expression for the connection:

$$\begin{aligned} A &= \frac{1}{4}\chi^{-1}Q^\dagger \left((\mathfrak{D} - i\partial_0)F(d\mathfrak{k} - ia) - (d\mathfrak{k} + ia)(\mathfrak{D} + i\partial_0)F \right) Q(\chi^\dagger)^{-1} \\ &+ \frac{1}{4}\chi^{-1}Q^\dagger F \left((\mathfrak{D} - i\partial_0)YY^\dagger(d\mathfrak{k} - ia) - (d\mathfrak{k} + ia)(\mathfrak{D} + i\partial_0)YY^\dagger \right) FQ(\chi^\dagger)^{-1} \\ &- \frac{1}{2}\chi^{-1}Q^\dagger F \left(Y \begin{pmatrix} \delta_R & 0 \\ 0 & \delta_L \end{pmatrix} db^\dagger - db \begin{pmatrix} \delta_R & 0 \\ 0 & \delta_L \end{pmatrix} Y^\dagger \right) FQ(\chi^\dagger)^{-1} \\ &+ \frac{1}{2}\chi^{-1}d\chi - \frac{1}{2}(d\chi)^\dagger(\chi^\dagger)^{-1}. \end{aligned} \quad (4.40)$$

Here, the Y , $Y^\dagger$, and $YY^\dagger$ terms contain delta functions at the end-points of the Taub-NUT interval. These delta functions will hit the Green's function at either

side of the parentheses, i.e. they choose values of the Green's function at these end-points. It will therefore be useful to make this explicit in the formula.

For compactness of notation we let

$$\nabla = ie_\nu \nabla_\nu = \mathfrak{D} + i\partial_0,$$

and

$$d\tau = ie_\mu d\tau_\mu = d\mathfrak{K} + ia,$$

where the unit quaternions are $e_\mu = (1, -i\sigma_j)$. When we move the delta functions in Y out to the Green's functions, we will be left with an array of B 's and b 's only, with no delta functions remaining, and Green's functions taking values $F(\pm \frac{l}{2}, \lambda_j)$.

This we call

$$\mathcal{L} = \begin{pmatrix} B_- & -b_+ \\ -b_- & B_+ \end{pmatrix}. \quad (4.41)$$

In terms of this $\mathcal{L}$ and its conjugate we can write the instanton formula as follows:

$$\begin{aligned} A = & \frac{1}{4}\chi^{-1}Q^\dagger \left(\bar{\nabla} F d\bar{\tau} - d\tau \nabla F \right) Q(\chi^\dagger)^{-1} + \frac{1}{2}\chi^{-1}d\chi - \frac{1}{2}(d\chi)^\dagger (\chi^\dagger)^{-1} \\ & + \frac{1}{4}\chi^{-1}Q^\dagger \begin{pmatrix} F_L & F_R \end{pmatrix} \left(\bar{\nabla}(\mathcal{L}\mathcal{L}^\dagger)d\bar{\tau} - d\tau \nabla(\mathcal{L}\mathcal{L}^\dagger) \right) \begin{pmatrix} F_L \\ F_R \end{pmatrix} Q(\chi^\dagger)^{-1} \\ & - \frac{1}{2}\chi^{-1}Q^\dagger \left(\begin{pmatrix} F_L & F_R \end{pmatrix} \mathcal{L} \begin{pmatrix} db_-^\dagger F_R \\ db_+^\dagger F_L \end{pmatrix} - \begin{pmatrix} F_R db_- & F_L db_+ \end{pmatrix} \mathcal{L}^\dagger \begin{pmatrix} F_R \\ F_L \end{pmatrix} \right) Q(\chi^\dagger)^{-1}, \end{aligned} \quad (4.42)$$

or, making use of Equations (4.32) and (4.33):

$$\begin{aligned}
A = & \frac{1}{4}\chi^{-1}Q^\dagger \left(\bar{\nabla} F d\bar{\tau} - d\tau \nabla F \right) Q(\chi^\dagger)^{-1} + \frac{1}{2}\chi^{-1}d\chi - \frac{1}{2}(d\chi)^\dagger (\chi^\dagger)^{-1} \\
& + \frac{1}{4}\chi^{-1}Q^\dagger \begin{pmatrix} F_L & F_R \end{pmatrix} \left(\bar{\nabla}(\mathcal{L}\mathcal{L}^\dagger)d\bar{\tau} - d\tau \nabla(\mathcal{L}\mathcal{L}^\dagger) \right) \begin{pmatrix} F_L \\ F_R \end{pmatrix} Q(\chi^\dagger)^{-1} \\
& - \frac{1}{4t}\chi^{-1}Q^\dagger \left(\begin{pmatrix} F_L & F_R \end{pmatrix} \mathcal{L} d\bar{\tau} \begin{pmatrix} -b_-^\dagger F_R \\ b_+^\dagger F_L \end{pmatrix} - \right. \\
& \left. - \begin{pmatrix} -F_R b_- & F_L b_+ \end{pmatrix} d\tau \mathcal{L}^\dagger \begin{pmatrix} F_R \\ F_L \end{pmatrix} \right) Q\chi^{\dagger-1}. \quad (4.43)
\end{aligned}$$

This is the result that we have been looking for. Comparison with Equation (3.30) from the previous chapter shows that the first line of this instanton formula is very similar to the caloron formula we derived there, while the final line contains the distinctive $bd\bar{b}$ term, with b coming from $\mathcal{L}$ and where db can be seen explicitly in Equation (4.42), that is typical of an instanton formula.

Chapter 5

Single $SU(2)$ Taub-NUT Instanton

5.1 Bow Data

Now that we have written a formula for instantons on the Taub-NUT space, we wish to use it to find an expression for a single $SU(2)$ Taub-NUT instanton, i.e. a self-dual charge one connection in a curved background (4.1). The aim is to write this expression in terms of the Green's function of the Laplacian of the Weyl operator defined in the next section.

For a charge one $SU(2)$ instanton we have piecewise constant Abelian Nahm data. We use the Bow diagram shown in Figure 5.1 to illustrate the setup. The Nahm data $\vec{T}_1$ and $\vec{T}_2$ solve the Nahm equations on the intervals $(-\frac{l}{2}, -\lambda)$ and $(\lambda, \frac{l}{2})$,

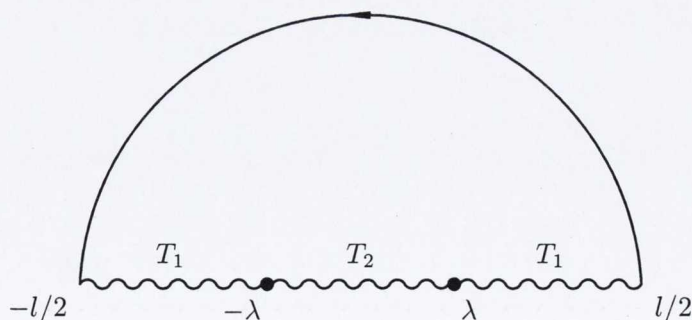


Figure 5.1: Bow Diagram for $SU(2)$ charge one instanton

and on the middle interval $(-\lambda, \lambda)$, respectively. They are the positions of the two instanton constituents. Since we have Abelian Nahm data, the Bow equations at $s = \pm \frac{l}{2}$, which are written below in Equation (5.3), match and therefore $\vec{T}_1$ on the left interval equals $\vec{T}_1$ on the right. We introduce the vectors $\vec{z}_1 = \vec{t} - \vec{T}_1$ and $\vec{z}_2 = \vec{t} - \vec{T}_2$, which will be useful when we write the twisted Weyl Operator in the next section. We also have $\vec{y} = \vec{T}_2 - \vec{T}_1$. At the jumping points $\pm\lambda$ we get additional Nahm data, $Q_\pm$ in the form of two-component spinors, which satisfy

$$Q_\pm Q_\pm^\dagger = y \pm \mathfrak{y}. \quad (5.1)$$

At the ends of the interval we have linear maps B_{10} and B_{01} between the vector spaces at $\pm \frac{l}{2}$, which are the components of the spinors B_- and B_+ introduced in Chapter 4. For a charge one instanton both these components and the components of b_- and b_+ , also introduced in the previous chapter, are complex numbers. The Bow equations at $s = \pm \frac{l}{2}$ require

$$b_\pm b_\pm^\dagger = t \pm \mathfrak{t} \quad (5.2)$$

$$B_\pm B_\pm^\dagger = T_1 \pm \mathfrak{T}_1. \quad (5.3)$$

As a consequence we find the following relations:

$$b_-^\dagger B_- = B_+^\dagger b_+ = e^{i\frac{\psi}{2}} \sqrt{(T_1 + t)^2 - z_1^2} = e^{i\frac{\psi}{2}} W, \quad (5.4)$$

$$b_+^\dagger B_+ = B_-^\dagger b_- = e^{-i\frac{\psi}{2}} \sqrt{(T_1 + t)^2 - z_1^2} = e^{-i\frac{\psi}{2}} W, \quad (5.5)$$

where we have introduced $W = \sqrt{(T_1 + t)^2 - z_1^2}$.

5.2 The Weyl Operator

The Weyl Operator for this configuration has the following form:

$$\mathcal{D}^\dagger = - \left(\frac{d}{ds} - iT_0 + \mathbb{T} \right) \oplus \sum_{\pm\lambda} \delta(s \mp \lambda) Q_\pm \oplus \left(\delta(s + \frac{l}{2}) B_-, \delta(s - \frac{l}{2}) B_+ \right). \quad (5.6)$$

We can twist this operator in Taub-NUT space as follows:

$$\mathcal{D}_t^\dagger = - \left(\frac{d}{ds} - \mathbb{X} \right) \oplus \sum_{\pm\lambda} \delta(s \mp \lambda) Q_\pm \oplus \left(\delta(s + \frac{l}{2}) (B_-, -b_+), \delta(s - \frac{l}{2}) (B_+, -b_-) \right), \quad (5.7)$$

where we have set T_0 to zero, and gauged away t_0 in favour of the phases of $B = (B_-, B_+)$ and $b = (b_-, b_+)$. Now we can write the Laplacian $\mathcal{D}_t^\dagger \mathcal{D}_t$:

$$\begin{aligned} \mathcal{D}_t^\dagger \mathcal{D}_t f = & \left(- \left(\frac{d}{ds} \right)^2 + z^2 + \sigma_i \otimes \left(\frac{dT_i}{ds} + i[T_j, T_k] - \frac{dt_i}{ds} - i[t_j, t_k] \right) \right) f(s) + \\ & + \delta(s - \lambda) Q_+ Q_+^\dagger f(\lambda) + \delta(s + \lambda) Q_- Q_-^\dagger f(-\lambda) + \\ & + \delta(s + \frac{l}{2}) \left(B_- B_-^\dagger + b_+ b_+^\dagger \right) f(-\frac{l}{2}) + \delta(s - \frac{l}{2}) \left(B_+ B_+^\dagger + b_- b_-^\dagger \right) f(\frac{l}{2}) - \\ & - \delta(s + \frac{l}{2}) \left(B_- b_-^\dagger + b_+ B_+^\dagger \right) f(\frac{l}{2}) - \delta(s - \frac{l}{2}) \left(B_+ b_+^\dagger + b_- B_-^\dagger \right) f(-\frac{l}{2}). \end{aligned} \quad (5.8)$$

A number of terms in the Laplacian will cancel or simplify due to Equations (5.1)-(5.3), and due to the Nahm equations, which arise from the vanishing of the moment map in the interior and which we write here:

$$\begin{aligned} \frac{dT_i}{ds} + i[T_j, T_k] - \frac{dt_i}{ds} - i[t_j, t_k] + \delta(s + \lambda) \text{vec}(Q_- Q_-^\dagger)_i + \delta(s - \lambda) \text{vec}(Q_+ Q_+^\dagger)_i + \\ + \delta(s + \frac{l}{2}) \text{vec} \left(B_- B_-^\dagger + b_+ b_+^\dagger \right)_i - \delta(s - \frac{l}{2}) \text{vec} \left(B_+ B_+^\dagger + b_- b_-^\dagger \right)_i = 0. \end{aligned} \quad (5.9)$$

Here, as before, "vec" picks out the part of a quantity that is proportional to the sigma matrices. After simplifying the Laplacian we can write it in the following

form:

$$\begin{aligned} \mathcal{D}_t^\dagger \mathcal{D}_t f = & \left(- \left(\frac{d}{ds} \right)^2 + z^2 \right) f(s) + \delta(s + \lambda) y f(-\lambda) + \delta(s - \lambda) y f(\lambda) \\ & + \delta_R(T_1 + t) f\left(\frac{l}{2}\right) + \delta_L(T_1 + t) f\left(-\frac{l}{2}\right) - \delta_L 2W e^{-i\frac{\psi}{2}} f\left(\frac{l}{2}\right) - \delta_R 2W e^{i\frac{\psi}{2}} f\left(-\frac{l}{2}\right). \end{aligned} \quad (5.10)$$

Now we must find the Green's function of this Laplacian. It is reasonably straightforward to find the Green's function with sources at jumping points and at end-points, which is what we need for our formula.

5.3 Matching conditions for the Green's function

We have six matching conditions on the Green's function. Solving these we can obtain an expression for the Green's function. It must satisfy Equation (4.23) at the jumping points and interval ends, and continuity conditions at the jumping points:

1.

$$- \frac{d}{ds} F(s, -\lambda + \epsilon) + \frac{d}{ds} F(s, -\lambda - \epsilon) + y F(s, -\lambda) = \int_{-\lambda - \epsilon}^{-\lambda + \epsilon} \delta(s + \lambda) ds. \quad (5.11)$$

2.

$$- \frac{d}{ds} F(s, \lambda + \epsilon) + \frac{d}{ds} F(s, \lambda - \epsilon) + y F(s, \lambda) = \int_{\lambda - \epsilon}^{\lambda + \epsilon} \delta(s - \lambda) ds. \quad (5.12)$$

3.

$$- \frac{d}{ds} F(s, -\frac{l}{2} + \epsilon) + (T_1 + t) F(s, -\frac{l}{2}) - W e^{-i\frac{\psi}{2}} F(s, \frac{l}{2}) = \int_{-\frac{l}{2} - \epsilon}^{-\frac{l}{2} + \epsilon} \delta(s + \frac{l}{2}) ds. \quad (5.13)$$

4.

$$\frac{d}{ds} F(s, \frac{l}{2} - \epsilon) + (T_1 + t) F(s, \frac{l}{2}) - W e^{+i\frac{\psi}{2}} F(-\frac{l}{2}, r) = \int_{\frac{l}{2} - \epsilon}^{\frac{l}{2} + \epsilon} \delta(s - \frac{l}{2}) ds. \quad (5.14)$$

5.

$$F(s, -\lambda - \epsilon) = F(s, -\lambda + \epsilon). \quad (5.15)$$

6.

$$F(s, \lambda - \epsilon) = F(s, \lambda + \epsilon). \quad (5.16)$$

We solve these matching conditions to find an expression for the Green's function $F(s, t)$ for the $SU(2)$ Instanton on Taub-NUT for $s, t = \pm\lambda$, i.e. for a source positioned at either of the jumping points. We then arrange these four values into a matrix:

$$F = \begin{pmatrix} F(-\lambda, -\lambda) & F(-\lambda, \lambda) \\ F(\lambda, -\lambda) & F(\lambda, \lambda) \end{pmatrix}. \quad (5.17)$$

We find the following expressions for the Green's function at the jumping points:

$$F(-\lambda, -\lambda) = F(\lambda, \lambda) = (z_1 z_2 \xi)^{-1} (z_1 C \sinh 2\lambda z_2 + z_2 S \cosh 2\lambda z_2 + y S \sinh 2\lambda z_2), \quad (5.18)$$

$$F(-\lambda, \lambda) = \overline{F(-\lambda, \lambda)} = (z_1 z_2 \xi)^{-1} (z_1 W e^{-i\frac{\psi}{2}} \sinh 2\lambda z_2 + z_2 S), \quad (5.19)$$

where

$$\begin{aligned} \xi = -W \cos \frac{\psi}{2} + C \cosh 2\lambda z_2 + \left(\frac{z_1^2 + z_2^2 + y^2}{2z_1 z_2} \right) S \sinh 2\lambda z_2 \\ + y(z_1 C \sinh 2\lambda z_2 + z_2 S \cosh 2\lambda z_2), \end{aligned} \quad (5.20)$$

and

$$C = m \cosh z_1(l - 2\lambda) + z_1 \sinh z_1(l - 2\lambda), \quad (5.21)$$

$$S = m \sinh z_1(l - 2\lambda) + z_1 \cosh z_1(l - 2\lambda). \quad (5.22)$$

In Equations (5.21) and (5.22) we have used the notation $m = T_1 + t$.

5.4 Green's Function in Caloron Limit

We compare the above expression to that computed by Kraan and van Baal for a charge one caloron in [1]. From Equations (5.21) and (5.22) we can obtain c_1 and s_1 , which are related by $C = \sqrt{m^2 - z_1^2}c_1$ and $S = \sqrt{m^2 - z_1^2}s_1$:

$$c_1 = \cosh \left(z_1(l - 2\lambda) + \sinh^{-1} \frac{z_1}{\sqrt{m^2 - z_1^2}} \right), \quad (5.23)$$

$$s_1 = \sinh \left(z_1(l - 2\lambda) + \sinh^{-1} \frac{z_1}{\sqrt{m^2 - z_1^2}} \right), \quad (5.24)$$

and we can write the Green's functions (5.18) and (5.19) in the following form:

$$F(-\lambda, -\lambda) = F(\lambda, \lambda) = (z_1 z_2 \xi)^{-1} (z_1 c_1 \sinh 2\lambda z_2 + z_2 s_1 \cosh 2\lambda z_2 + y s_1 \sinh 2\lambda z_2), \quad (5.25)$$

$$F(-\lambda, \lambda) = \overline{F(\lambda, -\lambda)} = (z_1 z_2 \xi)^{-1} \left(z_1 W e^{-i\frac{\psi}{2}} \sinh 2\lambda z_2 + z_2 s_1 \right), \quad (5.26)$$

$$\begin{aligned} \xi = -W \cos \frac{\psi}{2} + c_1 \cosh 2\lambda z_2 + \left(\frac{z_1^2 + z_2^2 + y^2}{2z_1 z_2} \right) s_1 \sinh 2\lambda z_2 \\ + y(z_1 c_1 \sinh 2\lambda z_2 + z_2 s_1 \cosh 2\lambda z_2) \end{aligned} \quad (5.27)$$

We see that as $m \rightarrow \infty$, $c_1 \rightarrow \cosh z_1(l - 2\lambda)$ and $s_1 \rightarrow \sinh z_1(l - 2\lambda)$. In this limit, where the Taub-NUT centre is at infinity while $\vec{z}_1 = \vec{t} - \vec{T}_1$ is held constant, we recover the Green's function for the charge one caloron given in [2].

5.5 Green's Function with Source at End-Points

We will also require the Green's function for a source positioned at either of the end-points of the Taub-NUT interval. Again, we solve the matching conditions

from Section 5.3. We find the following expressions:

$$F(-\frac{l}{2}, -\lambda) = F(\lambda, \frac{l}{2}) = (z_1 z_2 \xi)^{-1} \left(z_2 S' + W e^{-i\frac{\psi}{2}} \left(z_1 \cosh z_1 (\frac{l}{2} - \lambda) \sinh 2\lambda z_2 + \right. \right. \\ \left. \left. + z_2 \sinh z_1 (\frac{l}{2} - \lambda) \cosh 2\lambda z_2 + y \sinh z_1 (\frac{l}{2} - \lambda) \sinh 2\lambda z_2 \right) \right), \quad (5.28)$$

$$F(-\frac{l}{2}, \lambda) = F(-\lambda, \frac{l}{2}) = (z_1 z_2 \xi)^{-1} \left(W e^{-i\frac{\psi}{2}} z_2 \sinh z_1 (\frac{l}{2} - \lambda) + 2z_1 S' \sinh 2\lambda z_2 \right. \\ \left. - z_1 C' \sinh 2\lambda z_2 + z_2 S' \cosh 2\lambda z_2 + y S' \sinh 2\lambda z_2 \right), \quad (5.29)$$

where

$$S' = m \sinh z_1 (\frac{l}{2} - \lambda) + z_1 \cosh z_1 (\frac{l}{2} - \lambda), \quad (5.30)$$

$$C' = m \cosh z_1 (\frac{l}{2} - \lambda) + z_1 \sinh z_1 (\frac{l}{2} - \lambda). \quad (5.31)$$

Following from the symmetries of $\mathcal{D}_t^\dagger \mathcal{D}_t$, we have

$$F(-\lambda, -\frac{l}{2}) = F(\lambda, \frac{l}{2}) = \overline{F(-\frac{l}{2}, -\lambda)},$$

$$F(\lambda, -\frac{l}{2}) = F(\frac{l}{2}, -\lambda) = \overline{F(-\frac{l}{2}, \lambda)}.$$

5.6 Explicit Expression for the SU(2) Taub-NUT Instanton

5.6.1 The Calculation

The First Term

The first line of the instanton formula (4.40) contains a term

$$\frac{1}{4}(\chi^{-1})_i \partial_\nu (Q_i^\dagger \bar{\eta}_{\nu\mu} F(\lambda_i, \lambda_j) Q_j) d\tau^\mu (\chi^{\dagger-1})_j, \quad (5.32)$$

where $d\tau^\mu = (a, dt_j)$. We look at this term for the case where the Nahm data is Abelian. We choose $\vec{y} = (0, 0, y_3)$, and use the relations

$$Q_\pm Q_\pm^\dagger = y \pm \mathbb{Y}, \quad (5.33)$$

$$Q_- = i\sigma_2 \bar{Q}_+, \quad (5.34)$$

to obtain an expression for each Q_i . We get

$$Q_- = e^{i\frac{\psi}{2}} \begin{pmatrix} \sqrt{2y} \\ 0 \end{pmatrix}, \quad Q_+ = e^{-i\frac{\psi}{2}} \begin{pmatrix} 0 \\ \sqrt{2y} \end{pmatrix}, \quad (5.35)$$

and can now form a quaternion

$$\begin{aligned} (Q_-, Q_+) &= \sqrt{2y} \begin{pmatrix} \cos \frac{\psi}{2} + i \sin \frac{\psi}{2} & 0 \\ 0 & \cos \frac{\psi}{2} - i \sin \frac{\psi}{2} \end{pmatrix} \\ &= \sqrt{2y} (e_0 \cos \frac{\psi}{2} - e_3 \sin \frac{\psi}{2}) \\ &= Q_0 e_0 + Q_3 e_3, \end{aligned} \quad (5.36)$$

where $\psi = 2lt_0$ and $e_\mu = (1, -i\sigma_j)$. We use the fact that the Green's function F commutes with the quaternions to write Equation (5.32) in the following form [1]:

$$A(\text{first line}) = \frac{1}{4}(\chi^{-1})\bar{e}_\alpha\bar{\eta}_{\nu\mu}e_\beta\partial_\nu\phi_{\alpha\beta}d\tau^\mu(\chi^{\dagger-1}) = \frac{1}{4}d\tau_\mu\phi\bar{e}_\alpha\bar{\eta}_{\nu\mu}^a\sigma_a e_\beta\partial_\nu\phi_{\alpha\beta}, \quad (5.37)$$

where

$$\phi_{\alpha\beta} = Q_\alpha^\dagger F Q_\beta. \quad (5.38)$$

We can see from Equation (5.36) that $\phi_{\alpha\beta}$ will be non-zero only when both α and β are 0 or 3:

$$\begin{aligned} \phi_{00} &= 2\cos^2\frac{\psi}{2}(1 - \phi^{-1} + \text{Re}\zeta) \\ \phi_{33} &= 2\sin^2\frac{\psi}{2}(1 - \phi^{-1} - \text{Re}\zeta) \\ \phi_{03} &= \phi_{30} = -i\sin\psi\text{Im}\zeta, \end{aligned} \quad (5.39)$$

where

$$\phi = (1 - 2yF_s)^{-1}, \quad \text{and} \quad \zeta = 2yF_d. \quad (5.40)$$

Here we have used the notation

$$F_s = F(-\lambda, -\lambda) = F(\lambda, \lambda), \quad F_d = F(-\lambda, \lambda) = F(\lambda, -\lambda), \quad (5.41)$$

for the Green's function at the jumping points which we previously calculated in Section 5.3. Now we have the necessary ingredients to write a simple expression for the first line of the instanton formula. First we specify the 't Hooft tensors $\bar{\eta}_{\nu\mu}^a$:

$$\bar{\eta}_{j0}^i = -\bar{\eta}_{0j}^i = 2i\delta_{ij}, \quad (5.42)$$

$$\bar{\eta}_{jk}^i = -\bar{\eta}_{kj}^i = 2i\epsilon_{ijk}. \quad (5.43)$$

Using Equations (5.79) we can write Equation (5.37) as follows:

$$4A(\text{first line}) = d\tau_\mu \phi \bar{\eta}_{\nu\mu}^a \left(\sigma_a \partial_\nu \phi_{00} - i[\sigma_a, \sigma_3] \partial_\nu \phi_{03} + \sigma_3 \sigma_a \sigma_3 \partial_\nu \phi_{33} \right). \quad (5.44)$$

Writing A in terms of the functions ϕ and ζ we have:

$$\begin{aligned} A(\text{first line}) = & \frac{1}{2} d\tau_\mu \left(\bar{\eta}_{\nu\mu}^3 \sigma_3 \left(\partial_\nu \log \phi + \phi \partial_\nu \text{Re} \zeta \cos \psi + \right. \right. \\ & + \bar{\eta}_{\nu\mu}^1 \left(\sigma_1 (\phi \partial_\nu \text{Re} \zeta + \partial_\nu \log \phi \cos \psi) + i \sigma_2 \phi \partial_\nu \text{Im} \zeta \sin \psi \right) + \\ & \left. \left. + \bar{\eta}_{\nu\mu}^2 \left(\sigma_2 (\phi \partial_\nu \text{Re} \zeta + \partial_\nu \log \phi \cos \psi) - i \sigma_1 \phi \partial_\nu \text{Im} \zeta \sin \psi \right) \right) \right). \end{aligned} \quad (5.45)$$

This is very similar to the caloron formula derived in [1], however we have extra terms due to the presence of the phase factor ψ . The connection is

$$A = A_\mu d\tau_\mu = A_0 a + A_1 dt_1 + A_2 dt_2 + A_3 dt_3,$$

where $a = \frac{d\psi + \omega}{2V}$, and $V = l + \frac{1}{|\vec{x}|}$, as in Equation (4.5). We write the first component A_0 :

$$\begin{aligned} iA_0 = & -\sigma_3 \left(\partial_3 \log \phi + \phi \partial_3 \text{Re} \zeta \cos \psi \right) - \sigma_1 \left(\phi \partial_1 \text{Re} \zeta + \partial_1 \log \phi \cos \psi - i \phi \partial_2 \text{Im} \zeta \sin \psi \right) + \\ & - \sigma_2 \left(\phi \partial_2 \text{Re} \zeta + \partial_2 \log \phi \cos \psi + i \phi \partial_1 \text{Im} \zeta \sin \psi \right). \end{aligned} \quad (5.46)$$

The other three components of the first line of the instanton formula are written here:

$$\begin{aligned} iA_1(\text{first line}) = & \\ = & \sigma_3 \left(\partial_2 \log \phi + \phi \partial_2 \text{Re} \zeta \cos \psi \right) + \sigma_1 \left(\phi \partial_0 \text{Re} \zeta + \partial_0 \log \phi \cos \psi + i \phi \partial_3 \text{Im} \zeta \sin \psi \right) + \\ & - \sigma_2 \left(\phi \partial_3 \text{Re} \zeta + \partial_3 \log \phi \cos \psi - i \phi \partial_0 \text{Im} \zeta \sin \psi \right), \end{aligned} \quad (5.47)$$

$$\begin{aligned}
iA_2(\text{first line}) &= \\
&= -\sigma_3 \left(\partial_1 \log \phi + \phi \partial_1 \text{Re} \zeta \cos \psi \right) + \sigma_1 \left(\phi \partial_3 \text{Re} \zeta + \partial_3 \log \phi \cos \psi - i \phi \partial_0 \text{Im} \zeta \sin \psi \right) - \\
&\quad + \sigma_2 \left(\phi \partial_0 \text{Re} \zeta + \partial_0 \log \phi \cos \psi + i \phi \partial_3 \text{Im} \zeta \sin \psi \right), \quad (5.48)
\end{aligned}$$

$$\begin{aligned}
iA_3(\text{first line}) &= \\
&= \sigma_3 \left(\partial_0 \log \phi + \phi \partial_0 \text{Re} \zeta \cos \psi \right) - \sigma_1 \left(\phi \partial_2 \text{Re} \zeta + \partial_2 \log \phi \cos \psi + i \phi \partial_1 \text{Im} \zeta \sin \psi \right) + \\
&\quad + \sigma_2 \left(\phi \partial_1 \text{Re} \zeta + \partial_1 \log \phi \cos \psi - i \phi \partial_2 \text{Im} \zeta \sin \psi \right). \quad (5.49)
\end{aligned}$$

The Second Term

Now we consider the other terms in the formula. The second and third lines of the Equation (4.40) can be combined into a single term as follows:

$$A(\text{second term}) = \frac{1}{4} d\tau_\mu \phi \bar{\eta}_{\nu\mu}^a \bar{e}_\alpha \sigma_a \Delta_{\alpha\beta}^\nu e_\beta, \quad (5.50)$$

where

$$\Delta_{\alpha\beta}^\nu = \sum_{i,j} Q_\alpha^\dagger F_{iE} \left(\partial_\nu (YY^\dagger) - 2\omega_\nu \right)_{EE'} F_{E'j} Q_\beta. \quad (5.51)$$

Here the term $\partial_\nu (YY^\dagger)$ comes from the second line of Equation (4.40), while ω_ν comes from the third line. Both will be written explicitly below. The Green's function at end-points is denoted F_{iE} above, where i labels the jumping points and E labels the end-point. These functions were all written in Section 5.5. The sum in Equation (5.51) is over the jumping points. First of all we need to write $\partial_\nu (YY^\dagger)$ and ω_ν . Previously, in Equation (4.27), we had

$$YY^\dagger = (\delta_L B_- - \delta_R b_-)(B_-^\dagger \delta_L - b_-^\dagger \delta_R) + (\delta_R B_+ - \delta_L b_+)(B_+^\dagger \delta_R - b_+^\dagger \delta_L). \quad (5.52)$$

Now using

$$b_{\pm} b_{\pm}^{\dagger} = t \pm \mathfrak{k}, \quad B_{\pm} B_{\pm}^{\dagger} = T_1 \pm \mathfrak{T}_1,$$

we find

$$= \delta_L(m - \mathfrak{T}_1 + \mathfrak{k})\delta_L + \delta_R(m + \mathfrak{T}_1 - \mathfrak{k})\delta_R - \delta_L W e^{i\frac{\psi}{2}} \delta_R - \delta_R W e^{-i\frac{\psi}{2}} \delta_L. \quad (5.53)$$

We will need the derivatives of this term:

$$\partial_0(Y Y^{\dagger}) = -\delta_L i l W e^{i\frac{\psi}{2}} \delta_R + \delta_R i l W e^{-i\frac{\psi}{2}} \delta_L. \quad (5.54)$$

$$\begin{aligned} \partial_j(Y Y^{\dagger}) = & \delta_L \left(\frac{t_j}{t} + \sigma_j \right) \delta_L + \delta_R \left(\frac{t_j}{t} - \sigma_j \right) \delta_R - \delta_L \frac{1}{t W} (T_1 t_j + T_{1(j)} t) e^{i\frac{\psi}{2}} \delta_R \\ & - \delta_R \frac{1}{t W} (T_1 t_j + T_{1(j)} t) e^{-i\frac{\psi}{2}} \delta_L. \end{aligned} \quad (5.55)$$

Here j labels the components of the vectors, i.e. $j = 1, 2, 3$. Next we write ω_{ν} :

$$\omega_0 = \frac{i}{t} (\delta_R W \delta_L - \delta_L \bar{W} \delta_R), \quad (5.56)$$

$$\omega_j = \frac{1}{t} (\delta_R (t - t_j) \delta_R - \delta_L (t + t_j) \delta_L). \quad (5.57)$$

We note that in Equation (5.55) there are two terms involving sigma matrices. We shall ignore these for the moment, and exclude them from the expressions we write below for $\Delta_{\alpha\beta}^{\nu}$. By doing so, we can move e_{β} through $\Delta_{\alpha\beta}$ and write

$$A(\text{second term}) = \frac{1}{4} d\tau_{\mu} \phi \bar{\eta}_{\nu\mu}^a \bar{e}_{\alpha} \sigma_a e_{\beta} \Delta_{\alpha\beta}^{\nu}. \quad (5.58)$$

We will return to these simple terms involving sigma matrices in Equation (5.69).

Let us now write the second term in more detail. Again, excluding the sigma

matrix terms, we have the following expression for A :

$$A(\text{second term}) = \frac{1}{4} d\tau_\mu \phi \bar{\eta}_{\nu\mu}^a \left(\sigma_a \Delta_{00}^\nu + \sigma_3 \sigma_a \sigma_3 \Delta_{33}^\nu - i[\sigma_a, \sigma_3] \Delta_{03}^\nu \right). \quad (5.59)$$

We find the following values for $\Delta_{\alpha\beta}^\nu$, and write them in the form $(\Delta_{\alpha\beta}^0, \Delta_{\alpha\beta}^j)$:

$$\Delta_{00}^\nu = \frac{2}{t} \cos^2 \frac{\psi}{2} \left((2 - lt) \text{Im} Z_+, 3t_j \Phi_+^{-1} - \theta_j \text{Re} Z_+ \right), \quad (5.60)$$

$$\Delta_{33}^\nu = \frac{2}{t} \sin^2 \frac{\psi}{2} \left((2 - lt) \text{Im} Z_-, 3t_j \Phi_-^{-1} - \theta_j \text{Re} Z_- \right), \quad (5.61)$$

$$\Delta_{03}^\nu = \Delta_{30}^\nu = \frac{1}{t} \sin \psi \left(-i(2 - lt) \text{Re}(Z_+ Z_-)^{\frac{1}{2}}, \right. \quad (5.62)$$

$$\left. 2t(\Phi_+ \Phi_-)^{-\frac{1}{2}} - \frac{3t_j}{2|W|^2} ((Z_+ \bar{Z}_-)^{\frac{1}{2}} - (Z_- \bar{Z}_+)^{\frac{1}{2}}) - i\theta_j \text{Im}(Z_+ Z_-)^{\frac{1}{2}} \right). \quad (5.63)$$

where we have defined the following functions:

$$\Phi_\pm = (2y|H_\pm|^2)^{-1}, \quad Z_\pm = 2yWH_\pm^2, \quad H_\pm = F_n \pm F_f, \quad (5.64)$$

and used the following notation:

$$\theta_j = \frac{tT_{1(j)} + t_j T_1}{2(T_1 t + T_{1(j)} t_j)}, \quad (5.65)$$

$$W = \sqrt{m^2 - z_1^2} e^{-i\frac{\psi}{2}}. \quad (5.66)$$

Note that we have changed the definition of W so that it includes the phase factor from Equations (5.88) on. The Green's functions F_n and F_f are those written in Section 5.5:

$$F_n = F\left(-\frac{l}{2}, -\lambda\right) = F\left(\lambda, \frac{l}{2}\right), \quad (5.67)$$

$$F_f = F\left(-\frac{l}{2}, \lambda\right) = F\left(-\lambda, \frac{l}{2}\right). \quad (5.68)$$

The fourth and final term in Equation (4.40) will vanish if we choose χ real. We will make this gauge choice since it simplifies the result. We must now return to the terms involving sigma matrices from Equation (5.55). These terms will cancel when $\alpha = \beta$. So we just need to consider the cases where $\alpha = 0, \beta = 3$ and $\alpha = 3, \beta = 0$. The full term is

$$\frac{1}{4}\phi\bar{\eta}_{\mu\nu}^a\left(\bar{e}_0\sigma_a\Delta_{03}^\nu e_3 + \bar{e}_3\sigma_a\Delta_{03}^\nu e_0\right). \quad (5.69)$$

After some calculation we have

$$\frac{1}{4}\phi\bar{\eta}_{j\mu}^a[\sigma_a\sigma_j, \sigma_3](i\sin\psi(\Phi_+\Phi_-)^{-\frac{1}{2}}). \quad (5.70)$$

We will use the notation

$$\rho = i\sin\psi(\Phi_+\Phi_-)^{-\frac{1}{2}}. \quad (5.71)$$

In fact, this term is quite simple. It disappears when $\mu = 0$ and when $\mu = 3$, and otherwise it reduces to

$$-2\phi\sin\psi\sigma_2(\Phi_+\Phi_-)^{-\frac{1}{2}}, \quad \mu = 1,$$

$$2\phi\sin\psi\sigma_1(\Phi_+\Phi_-)^{-\frac{1}{2}}, \quad \mu = 2.$$

5.6.2 The Result

The full expression for the $SU(2)$ charge one instanton on the Taub-NUT space is

$$A = A_\mu d\tau_\mu = A_0 a + A_1 dt_1 + A_2 dt_2 + A_3 dt_3, \quad (5.72)$$

where

$$a = \frac{d\psi + \omega}{2V}, \quad (5.73)$$

and

$$A_\mu = \frac{1}{4}\phi \left(\bar{\eta}_{\nu\mu}^a \left(\sigma_a (\partial_\nu \phi_{00} + \Delta_{00}^\nu) + \sigma_3 \sigma_a \sigma_3 (\partial_\nu \phi_{33} + \Delta_{33}^\nu) - i[\sigma_a, \sigma_3] (\partial_\nu \phi_{03} + \Delta_{03}^\nu) + [\sigma_a \sigma_j, \sigma_3] \rho \right) \right). \quad (5.74)$$

5.7 Summary of the Result

We have written an expression for an $SU(2)$ charge one instanton on the Taub-NUT space in terms of number of simple functions that are related to the Green's functions of the Laplacian $\mathcal{D}_t^\dagger \mathcal{D}_t$. We write the result and a description of all of the quantities it contains in this section.

The Taub-NUT metric is

$$ds^2 = \frac{1}{4} \left(\left(l + \frac{1}{|\vec{x}|} \right) d\vec{x}^2 + \frac{1}{\left(l + \frac{1}{|\vec{x}|} \right)} (d\psi + \vec{\omega} \cdot d\vec{x})^2 \right), \quad (5.75)$$

where $\vec{x}$ is a vector parametrizing $\mathbb{R}^3$, and $\frac{\partial}{\partial x_i} \frac{1}{|\vec{x}|} = \epsilon_{ijk} \frac{\partial \omega_k}{\partial x_j}$.

The coordinates on an interval of length l are $\vec{t} = -\frac{1}{2}\vec{x}$, $2lt_0 = \psi \sim \psi + 4\pi$. The instanton constituent positions are given by $\vec{T}_1$ and $\vec{T}_2$. Relative coordinates are $\vec{z}_1 = \vec{t} - \vec{T}_1$ and $\vec{z}_2 = \vec{t} - \vec{T}_2$, while $\vec{y} = \vec{T}_2 - \vec{T}_1$ is the displacement between the constituents. We have chosen $\vec{y} = (0, 0, y_3)$ For $|\vec{y}|$, we write y , and similarly for other absolute values. The 't Hooft tensors are

$$\bar{\eta}_{j0}^i = -\bar{\eta}_{0j}^i = 2i\delta_{ij}, \quad \bar{\eta}_{jk}^i = -\bar{\eta}_{kj}^i = 2i\epsilon_{ijk}. \quad (5.76)$$

We write the result here:

$$A = A_0 a + A_1 dt_1 + A_2 dt_2 + A_3 dt_3, \quad (5.77)$$

where $a = \frac{d\psi+\omega}{V}$ and $V = l + \frac{1}{|\vec{x}|}$, and

$$A_\mu = \frac{1}{4}\phi \left(\bar{\eta}_{\nu\mu}^a \left(\sigma_a (\partial_\nu \phi_{00} + \Delta_{00}^\nu) + \sigma_3 \sigma_a \sigma_3 (\partial_\nu \phi_{33} + \Delta_{33}^\nu) - i[\sigma_a, \sigma_3] (\partial_\nu \phi_{03} + \Delta_{03}^\nu) + \right. \right. \\ \left. \left. + [\sigma_a \sigma_j, \sigma_3] \rho \right) \right). \quad (5.78)$$

The term $\phi_{\alpha\beta}$ is related to the Green's function at jumping points $\pm\lambda$ and has the following values:

$$\begin{aligned} \phi_{00} &= 2 \cos^2 \frac{\psi}{2} (1 - \phi^{-1} + \text{Re}\zeta) \\ \phi_{33} &= 2 \sin^2 \frac{\psi}{2} (1 - \phi^{-1} - \text{Re}\zeta) \\ \phi_{03} &= \phi_{30} = -i \sin \psi \text{Im}\zeta, \end{aligned} \quad (5.79)$$

where

$$\phi = (1 - 2yF(-\lambda, -\lambda))^{-1}, \quad (5.80)$$

$$\zeta = 2yF(-\lambda, \lambda), \quad (5.81)$$

and

$$F(-\lambda, -\lambda) = (z_1 z_2 \xi)^{-1} (z_1 C \sinh 2\lambda z_2 + z_2 S \cosh 2\lambda z_2 + y S \sinh 2\lambda z_2), \quad (5.82)$$

$$F(-\lambda, \lambda) = (z_1 z_2 \xi)^{-1} \left(z_1 \sqrt{(T_1 + t)^2 - z_1^2} e^{-i\frac{\psi}{2}} \sinh 2\lambda z_2 + z_2 S \right), \quad (5.83)$$

$$\xi = -\sqrt{(T_1 + t)^2 - z_1^2} \cos \frac{\psi}{2} + C \cosh 2\lambda z_2 + \left(\frac{z_1^2 + z_2^2 + y^2}{2z_1 z_2} \right) S \sinh 2\lambda z_2 \quad (5.84)$$

$$+ y(z_1 C \sinh 2\lambda z_2 + z_2 S \cosh 2\lambda z_2), \quad (5.85)$$

$$C = (T_1 + t) \cosh z_1(l - 2\lambda) + z_1 \sinh z_1(l - 2\lambda), \quad (5.86)$$

$$S = (T_1 + t) \sinh z_1(l - 2\lambda) + z_1 \cosh z_1(l - 2\lambda). \quad (5.87)$$

The term $\Delta_{\alpha\beta}^\nu$ is related to the Green's function with sources at the end-points and has the following values, written in the form $(\Delta_{\alpha\beta}^0, \Delta_{\alpha\beta}^j)$:

$$\Delta_{00}^\nu = \frac{2}{t} \cos^2 \frac{\psi}{2} \left((2 - lt) \text{Im} Z_+, 3t_j \Phi_+^{-1} - \theta_j \text{Re} Z_+ \right), \quad (5.88)$$

$$\Delta_{33}^\nu = \frac{2}{t} \sin^2 \frac{\psi}{2} \left((2 - lt) \text{Im} Z_-, 3t_j \Phi_-^{-1} - \theta_j \text{Re} Z_- \right), \quad (5.89)$$

$$\Delta_{03}^\nu = \Delta_{30}^\nu = \frac{1}{t} \sin \psi \left(-i(2 - lt) \text{Re}(Z_+ Z_-)^{\frac{1}{2}}, \right. \quad (5.90)$$

$$\left. 2t(\Phi_+ \Phi_-)^{-\frac{1}{2}} - \frac{3t_j}{2|W|^2} ((Z_+ \bar{Z}_-)^{\frac{1}{2}} - (Z_- \bar{Z}_+)^{\frac{1}{2}}) - i\theta_j \text{Im}(Z_+ Z_-)^{\frac{1}{2}} \right). \quad (5.91)$$

where

$$\Phi_\pm = \left(2y \left| F\left(-\frac{l}{2}, -\lambda\right) \pm F\left(-\frac{l}{2}, \lambda\right) \right|^2 \right)^{-1}, \quad (5.92)$$

$$Z_\pm = 2y \sqrt{(T_1 + t)^2 - z_1^2} e^{-i\frac{\psi}{2}} \left(F\left(-\frac{l}{2}, -\lambda\right) \pm F\left(-\frac{l}{2}, \lambda\right) \right)^2, \quad (5.93)$$

and

$$\theta_j = \frac{tT_{1(j)} + t_j T_1}{(T_1 + t)^2 - z_1^2}, \quad (5.94)$$

$$F\left(-\frac{l}{2}, -\lambda\right) = (z_1 z_2 \xi)^{-1} \left(z_2 S' + \sqrt{(T_1 + t)^2 - z_1^2} e^{-i\frac{\psi}{2}} \left(z_1 \cosh z_1 \left(\frac{l}{2} - \lambda\right) \sinh 2\lambda z_2 + \right. \right. \quad (5.95)$$

$$\left. + z_2 \sinh z_1 \left(\frac{l}{2} - \lambda\right) \cosh 2\lambda z_2 + y \sinh z_1 \left(\frac{l}{2} - \lambda\right) \sinh 2\lambda z_2 \right), \quad (5.96)$$

$$F\left(-\frac{l}{2}, \lambda\right) = (z_1 z_2 \xi)^{-1} \left(\sqrt{(T_1 + t)^2 - z_1^2} e^{-i\frac{\psi}{2}} z_2 \sinh z_1 \left(\frac{l}{2} - \lambda\right) + 2z_1 S' \sinh 2\lambda z_2 \right. \quad (5.97)$$

$$\left. - z_1 C' \sinh 2\lambda z_2 + z_2 S' \cosh 2\lambda z_2 + y S' \sinh 2\lambda z_2 \right), \quad (5.98)$$

$$S' = (T_1 + t) \sinh z_1 \left(\frac{l}{2} - \lambda\right) + z_1 \cosh z_1 \left(\frac{l}{2} - \lambda\right), \quad (5.99)$$

$$C' = (T_1 + t) \cosh z_1 \left(\frac{l}{2} - \lambda\right) + z_1 \sinh z_1 \left(\frac{l}{2} - \lambda\right). \quad (5.100)$$

The final term in the formula, ρ , is also written in terms of $\Phi_{\pm}$:

$$\rho = i \sin \psi (\Phi_+ \Phi_-)^{-\frac{1}{2}}. \quad (5.101)$$

These are all of the ingredients of the formula for the charge one $SU(2)$ instanton on the Taub-NUT space.

Chapter 6

Conclusion

We have derived a formula for a charge k $SU(2)$ instanton in the background of the Taub-NUT space. We used the Nahm Transform to write our formula. Our approach involved using Green's functions of operators defined in the Nahm Transform, thus we avoided an integration that normally forms part of the Transform. We also derived a formula for a charge k caloron using this method.

The instanton formula was written in terms of fields that solve the Nahm Equations. These fields also parametrize the Higgs branch of the moduli space of vacua for the supersymmetric gauge theory on the D3-brane in a Chalmers-Hanany-Witten configuration of intersecting D3-, D5- and NS5-branes. We have written the Lagrangian for this gauge theory and demonstrated that the scalar fields in the theory that parametrize the Higgs branch satisfy the Bow Equations.

We used our formula to write an explicit expression for a charge one $SU(2)$ instanton on the Taub-NUT space. We solved matching conditions on the interval to find the Green's function with sources at jumping points and end-points of a Taub-NUT interval. As a demonstration of this formula in use, we wrote the expression for the charge one instanton in terms of these functions.

Appendix A

Supersymmetry Conventions

A.1 The Metric

We use Wess and Bagger conventions. The metric is $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. The Greek indices $\mu, \nu = 0, 1, 2, 3$ label 4-dimensional spacetime, while $m, n = 0, 1, 3$ with $x^2 = s$.

A.2 Spinor Conventions

Spinor indices are raised and lowered using

$$\epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \epsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (\text{A.1})$$

So $\psi^\alpha = \epsilon^{\alpha\beta} \psi_\beta$ and $\psi_\alpha = \epsilon_{\alpha\beta} \psi^\beta$, where $\epsilon^{\alpha\beta} = i\sigma^2$. The Pauli matrices are:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (\text{A.2})$$

We have $\sigma_0 = -1_{2 \times 2}$ and $\bar{\sigma}_\mu = (\sigma_0, -\sigma_j)$. The spinor summation conventions are:

$$\psi\chi = \psi^\alpha\chi_\alpha = -\psi_\alpha\chi^\alpha = \chi^\alpha\psi_\alpha = \chi\psi \quad (\text{A.3})$$

$$\bar{\psi}\bar{\chi} = \bar{\psi}_\alpha\bar{\chi}^\alpha = -\bar{\psi}^\alpha\bar{\chi}_\alpha = \bar{\chi}_\alpha\bar{\psi}^\alpha = \bar{\chi}\bar{\psi} \quad (\text{A.4})$$

$$\psi\bar{\chi} = \psi^\alpha\bar{\chi}_\alpha = -\psi_\alpha\bar{\chi}^\alpha = \bar{\chi}^\alpha\psi_\alpha = \bar{\chi}\psi \quad (\text{A.5})$$

Some useful spinor relations are:

$$\theta^\alpha\theta^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}(\theta\theta) \quad (\text{A.6})$$

$$\bar{\theta}^\alpha\bar{\theta}^\beta = \frac{1}{2}\epsilon^{\alpha\beta}(\bar{\theta}\bar{\theta}) \quad (\text{A.7})$$

$$\theta_\alpha\theta^\beta = -\frac{1}{2}\delta_\alpha^\beta(\theta\theta) \quad (\text{A.8})$$

$$(\theta\sigma^m\bar{\theta})(\theta\sigma^n\bar{\theta}) = -\frac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta})\eta^{mn} \quad (\text{A.9})$$

A.3 Superspace Derivatives

The superspace derivatives are listed here:

$$\partial_\alpha = \frac{\partial}{\partial\theta^\alpha} \quad \partial^\alpha = \frac{\partial}{\partial\theta_\alpha} = -\epsilon^{\alpha\beta}\partial_\beta \quad (\text{A.10})$$

$$\bar{\partial}^\alpha = \frac{\partial}{\partial\bar{\theta}_\alpha} \quad \bar{\partial}_\alpha = \frac{\partial}{\partial\bar{\theta}^\alpha} = -\epsilon_{\alpha\beta}\bar{\partial}^\beta \quad (\text{A.11})$$

$$\partial_\alpha\theta^\beta = \delta_\alpha^\beta \quad \bar{\partial}^\alpha\bar{\theta}_\beta = \delta_\beta^\alpha \quad (\text{A.12})$$

$$\partial^\alpha\theta^\beta = -\epsilon^{\alpha\beta} \quad \partial_\alpha\theta_\beta = -\epsilon_{\alpha\beta} \quad (\text{A.13})$$

$$\bar{\partial}^\alpha\bar{\theta}^\beta = -\epsilon^{\alpha\beta} \quad \bar{\partial}_\alpha\bar{\theta}_\beta = -\epsilon_{\alpha\beta} \quad (\text{A.14})$$

$$\partial_\alpha(\theta\theta) = 2\theta_\alpha \quad \bar{\partial}_\alpha(\bar{\theta}\bar{\theta}) = -2\bar{\theta}_\alpha \quad (\text{A.15})$$

A.4 Superspace Integration

Integration in superspace has the following properties:

$$\int d\theta = 0 \quad (\text{A.16})$$

$$\int d\theta \theta = 1 \quad (\text{A.17})$$

$$\int d\theta f(\theta) = f_1 \quad (\text{A.18})$$

noting that the Taylor expansion of a function of a Grassman parameter θ is $f(\theta) = f_0 + \theta f_1$ (since $\{\theta, \theta\} = 0$). Integration in superspace is equivalent to differentiation:

$$\int d\theta_\alpha \theta^\beta = \partial_\alpha \theta^\beta = \delta_\alpha^\beta \quad (\text{A.19})$$

Some important notation is:

$$d^2\theta = -\frac{1}{4}d\theta^\alpha d\theta^\beta \epsilon_{\alpha\beta} \quad (\text{A.20})$$

$$d^2\bar{\theta} = -\frac{1}{4}d\bar{\theta}_\alpha d\bar{\theta}_\beta \epsilon^{\alpha\beta} \quad (\text{A.21})$$

We therefore have the following useful identities:

$$\int d^2\theta (\theta\theta) = 1 \quad (\text{A.22})$$

$$\int d^2\bar{\theta} (\bar{\theta}\bar{\theta}) = 1 \quad (\text{A.23})$$

A.5 Superspace Covariant Derivatives

Covariant derivatives acting on functions of $(x, \theta, \bar{\theta})$ are:

$$D_m = \partial_m \tag{A.24}$$

$$D_\alpha = \partial_\alpha + i\sigma_{\alpha\beta}^m \bar{\theta}^\beta \partial_m \tag{A.25}$$

$$\bar{D}_\alpha = -\bar{\partial}_\alpha - i\theta^\beta \sigma_{\beta\alpha}^m \partial_m \tag{A.26}$$

Covariant derivatives acting on functions of $(y, \theta, \bar{\theta})$ are:

$$D_m = \partial_m \tag{A.27}$$

$$D_\alpha = \partial_\alpha + 2i\sigma_{\alpha\beta}^m \bar{\theta}^\beta \partial_m \tag{A.28}$$

$$\bar{D}_\alpha = -\bar{\partial}_\alpha \tag{A.29}$$

Appendix B

The Linear Multiplet Σ

The details of the calculation of the linear multiplet Σ follow. We have

$$\Sigma = \epsilon^{\alpha\beta} \bar{D}_\alpha (e^{2iV} D_\beta e^{-2iV}). \quad (\text{B.1})$$

The vector field V , as a function of $(x, \theta, \bar{\theta})$, is written as follows in Wess-Zumino gauge:

$$V(x) = -i\theta\bar{\theta}\rho - \theta\sigma^m\bar{\theta}v_m + i(\theta\theta)\bar{\theta}\bar{\lambda} - i(\bar{\theta}\bar{\theta})\theta\lambda + \frac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta})D. \quad (\text{B.2})$$

We write it as a function of $y^m = x^m + i\theta\sigma^m\bar{\theta}$ since that will make the calculation of Σ simpler (as we shall see):

$$V(y) = -i\theta\bar{\theta}\rho - \theta\sigma^m\bar{\theta}v_m + i(\theta\theta)\bar{\theta}\bar{\lambda} - i(\bar{\theta}\bar{\theta})\theta\lambda + \frac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta})(D - i\partial_m v^m). \quad (\text{B.3})$$

We note that

$$e^{2iV} D_\beta e^{-2iV} = -2iD_\beta V + 2[V, D_\beta V], \quad (\text{B.4})$$

since powers of V higher than V^2 vanish.

Firstly we write $D_\beta V$ in components:

$$\begin{aligned}
D_\beta V &= (\partial_\beta + 2i\sigma_{\beta\gamma}^m \bar{\theta}^\gamma \partial_m) V(y) \\
&= -i\bar{\theta}_\beta \rho - \sigma_{\beta\gamma}^m \bar{\theta}^\gamma v_m + 2i\theta_\beta \bar{\theta} \bar{\lambda} - i(\bar{\theta}\bar{\theta})\lambda_\beta + \theta_\beta(\bar{\theta}\bar{\theta})D \\
&\quad + \sigma_{\beta\gamma}^m \theta^\gamma (\bar{\theta}\bar{\theta}) \partial_m \rho - i(\bar{\theta}\bar{\theta})\sigma_\beta^{mn\gamma} \theta_\gamma (\partial_m v_n - \partial_n v_m) + (\theta\theta)(\bar{\theta}\bar{\theta})\sigma_{\beta\gamma}^m \partial_m \bar{\lambda}^\gamma.
\end{aligned} \tag{B.5}$$

We can now write an expression for the commutator:

$$[V, D_\beta V] = -i(\bar{\theta}\bar{\theta})\theta^\gamma \sigma_{\beta\gamma}^m [v_m, \rho] - (\bar{\theta}\bar{\theta})\sigma_\beta^{mn\gamma} \theta_\gamma [v_m, v_n] - i(\theta\theta)(\bar{\theta}\bar{\theta})\sigma_{\beta\gamma}^m [v_m, \bar{\lambda}^\gamma]. \tag{B.6}$$

We are now in a position to write

$$\begin{aligned}
e^{2iV} D_\beta e^{-2iV} &= -2\bar{\theta}_\beta \rho + 2i\sigma_{\beta\gamma}^m \bar{\theta}^\gamma v_m - 4\theta_\beta \bar{\theta} \bar{\lambda} - 2(\bar{\theta}\bar{\theta})\lambda_\beta - 2i\theta_\beta(\bar{\theta}\bar{\theta})D \\
&\quad - 2i\sigma_{\beta\gamma}^m \theta^\gamma (\bar{\theta}\bar{\theta}) \partial_m \rho - 2i\sigma_{\beta\gamma}^m \theta^\gamma (\bar{\theta}\bar{\theta}) [v_m, \rho] - 2(\bar{\theta}\bar{\theta})\sigma_\beta^{mn\gamma} \theta_\gamma F_{mn} \\
&\quad - 2i(\theta\theta)(\bar{\theta}\bar{\theta})\sigma_{\beta\gamma}^m \partial_m \bar{\lambda}^\gamma - 2i(\theta\theta)(\bar{\theta}\bar{\theta})\sigma_{\beta\gamma}^m [v_m, \bar{\lambda}^\gamma],
\end{aligned} \tag{B.7}$$

where the curvature has the form $F_{mn} = \partial_m v_n - \partial_n v_m + [v_m, v_n]$. Since Σ is a function of y , the covariant derivative $\bar{D}_\alpha$ reduces to $-\bar{\partial}_\alpha$. So we first calculate $\Sigma(y)$ and afterwards write it as a function of x :

$$\begin{aligned}
\Sigma(y) &= 4\rho - 4\theta\bar{\lambda} - 4\bar{\theta}\lambda - 4i\theta\bar{\theta}D + 4i\theta\sigma^m \bar{\theta} \partial_m \rho + 4i\theta\sigma^m \bar{\theta} [v_m, \rho] - 4\bar{\theta}\sigma^{mn}\theta F_{mn} \\
&\quad - 4i(\theta\theta)\bar{\theta}\sigma^m \partial_m \bar{\lambda} - 4i(\theta\theta)\bar{\theta}\sigma^m [v_m, \bar{\lambda}].
\end{aligned} \tag{B.8}$$

Using the Taylor expansion $f(x) = f(y) - i\theta\sigma^m \bar{\theta} \partial_m f(y) + \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\square f(y)$, we obtain our expression for the linear multiplet Σ :

$$\begin{aligned}
\Sigma(x) &= 4\rho - 4\theta\bar{\lambda} - 4\bar{\theta}\lambda - 4i\theta\bar{\theta}D - 4\bar{\theta}\sigma^{mn}\theta F_{mn} + 4i\theta\sigma^m \bar{\theta} [v_m, \rho] - 4i(\theta\theta)\bar{\theta}\sigma^m [v_m, \bar{\lambda}] \\
&\quad + 2i(\bar{\theta}\bar{\theta})\theta\sigma^m \partial_m \lambda - 2i(\theta\theta)\bar{\theta}\sigma^m \partial_m \bar{\lambda} - 2(\theta\theta)(\bar{\theta}\bar{\theta})\partial_m [v^m, \rho] - (\theta\theta)(\bar{\theta}\bar{\theta})\square\rho.
\end{aligned} \tag{B.9}$$

Appendix C

The Action in Components

C.1 The Bulk Action in Components

We write each term in the bulk action S_4 (written in Section 2.4.1) in terms of its component fields. First we have Σ^2 :

$$-\frac{1}{16}\Sigma^2|_{(\theta\theta)(\bar{\theta}\bar{\theta})} = -\frac{1}{2}D_m\rho D^m\rho - i\lambda\sigma^m D_m\bar{\lambda} - \frac{1}{4}F_{mn}F^{mn} + \frac{1}{2}D^2, \quad (\text{C.1})$$

where $D_m\rho = \partial_m\rho + [v_m, \rho]$ and $D_m\bar{\lambda} = \partial_m\bar{\lambda} + [v_m, \bar{\lambda}]$. Now we write the second term in S_4 in components:

$$\begin{aligned} & -\frac{1}{4}(e^{2iV}(\partial_s - \bar{\Phi})e^{-2iV} - \Phi)^2|_{(\theta\theta)(\bar{\theta}\bar{\theta})} = \\ & = \frac{i}{2}D\partial_s(A + A^*) + \frac{i}{2}D[A, A^*] - \frac{1}{2}GG^* + \frac{1}{2}D_mAD^mA^* + \frac{i}{4}\bar{\psi}\bar{\sigma}^m\partial_m\psi + \frac{i}{4}\psi\sigma^m\partial_m\bar{\psi} \\ & + \sqrt{2}(\partial_s\psi\lambda - \partial_s\bar{\psi}\bar{\lambda}) + \partial_s v_m\partial^m(A - A^*) - \frac{1}{2}(\rho\partial_s\rho + v_m\partial_s v^m)(A + A^*) \\ & - \frac{1}{4}\rho A^*\partial_s\rho - \frac{1}{4}v_m A^*\partial_s v^m - \frac{1}{4}\rho\partial_s\rho A - \frac{1}{4}v_m\partial_s v^m A - \frac{1}{2}((\partial_s\rho)^2 + \partial_s v_m\partial_s v^m) \\ & - \frac{1}{2}A^{*2}(\rho^2 - v_m v^m) + \frac{1}{2}(A^*\rho A^*\rho - A^*v_m A^*v^m) + \rho A^*\rho A - \frac{1}{2}\rho^2 A^*A - \frac{1}{2}AA^*\rho^2, \end{aligned} \quad (\text{C.2})$$

where the covariant derivative $D_m A = \partial_m A + [v_m, A]$. The next term in the action is

$$\begin{aligned}
& (e^{2iV} \bar{U}_1 e^{2iV} U_1 + e^{2iV} \bar{U}_2 e^{-2iV} U_2)|_{(\theta\theta)(\bar{\theta}\bar{\theta})} = \\
& = \frac{1}{2} D_m u_1^* D^m u_1 - \frac{1}{2} F_1^* F_1 + \frac{i}{4} \bar{\chi}_1 \bar{\sigma}^m \partial_m \chi_1 + \frac{i}{4} \chi_1 \sigma^m \partial_m \bar{\chi}_1 + \frac{i}{2} D[u_1, u_1^*] + \rho u_1^* \rho u_1 \\
& - \frac{1}{2} (u_1^* \rho^2 u_1 + \rho^2 u_1^* u_1) + \frac{1}{2} D_m u_2^* D^m u_2 - \frac{1}{2} F_2^* F_2 + \frac{i}{4} \bar{\chi}_2 \bar{\sigma}^m \partial_m \chi_2 + \frac{i}{4} \chi_2 \sigma^m \partial_m \bar{\chi}_2 \\
& + \frac{i}{2} D[u_2, u_2^*] - \frac{1}{2} (u_2^* \rho^2 u_2 + \rho^2 u_2^* u_2) + \rho u_2^* \rho u_2,
\end{aligned} \tag{C.3}$$

where $D_m u_j = \partial_m u_j^* + [v_m, u_j]$. Finally,

$$\begin{aligned}
& -\frac{1}{4} \epsilon_{ij} U_i [-\partial_s - \Phi, U_j]|_{(\theta\theta)} - \frac{1}{4} \epsilon_{ij} [\bar{U}_i, \partial_s - \bar{\Phi}] \bar{U}_j|_{(\bar{\theta}\bar{\theta})} = \\
& = -\frac{1}{2} \left(F_1 \partial_s u_2 - F_2 \partial_s u_1 - [u_1, u_2] G + [u_1, A] F_2 - [u_2, A] F_1 \right) \\
& + \frac{1}{2} \left(F_1^* \partial_s u_2^* - F_2^* \partial_s u_1^* - G^* [u_2^*, u_1^*] + F_2^* [A^*, u_1^*] - F_1^* [A^*, u_2^*] \right).
\end{aligned} \tag{C.4}$$

C.2 The Defect Action in Components

We write each term in S_{3p} in terms of its component fields. First of all, we have

$$\begin{aligned}
& -\frac{1}{2} (\bar{Q}_{1p} e^{-2iV} Q_{1p} + Q_{2p} e^{2iV} \bar{Q}_{2p})|_{(\theta\theta)(\bar{\theta}\bar{\theta})} = \\
& \frac{1}{2} D_m q_1^* D^m q_1 + \frac{i}{2} D q_1 q_1^* - \frac{1}{2} J_1^* J_1 - \frac{1}{2} q_1^* \rho^2 q_1 + \frac{1}{2} D_m q_2 D^m q_2^* - \frac{i}{2} D q_2^* q_2 \\
& - \frac{1}{2} J_2 J_2^* - \frac{1}{2} q_2 \rho^2 q_2^*,
\end{aligned} \tag{C.5}$$

where the covariant derivatives are

$$D_m q_1 = \partial_m q_1 + v_m q_1,$$

$$D_m q_1^* = \partial_m q_1^* - q_1^* v_m,$$

$$D_m q_2 = \partial_m q_2 - q_2 v_m,$$

$$D_m q_2^* = \partial_m q_2^* + v_m q_2^*.$$

We also find that

$$\frac{1}{2}(Q_2 U_2 Q_1|_{(\theta\theta)} + \bar{Q}_1 \bar{U}_2 \bar{Q}_2|_{(\bar{\theta}\bar{\theta})}) = \frac{1}{2}(q_2 u_1 J_1 + q_2 F_1 q_1 + J_2 u_1 q_1 + J_1^* u_1^* q_2^* + q_1^* F_1^* q_2^* + q_1^* u_1^* J_2^*). \quad (C.6)$$

The action S'_3 is written below in terms of its component fields.

$$\begin{aligned} & (\bar{B}_1 e^{V_L} B_1 e^{-V_R} + B_2 e^{-V_L} \bar{B}_2 e^{V_R})|_{(\theta\theta)(\bar{\theta}\bar{\theta})} = \\ & = \frac{1}{2} D_m b_1^* D^m b_1 + \frac{i}{2} D_L b_1 b_1^* - \frac{i}{2} D_R b_1^* b_1 - \frac{1}{2} L_1^* L_1 + b_1^* \rho_L b_1 \rho_R - \frac{1}{2} (b_1^* b_1 \rho_R^2 + b_1 b_1^* \rho_L^2) \\ & + D_m b_2 D^m b_2^* - \frac{i}{2} D_L b_2^* b_2 + \frac{i}{2} D_R b_2 b_2^* - \frac{1}{2} L_2 L_2^* + b_2 \rho_L b_2^* \rho_R - \frac{1}{2} (b_2 b_2^* \rho_R^2 + b_2 \rho_L^2 b_2^*), \end{aligned} \quad (C.7)$$

where the covariant derivatives are

$$\begin{aligned} D_m b_1 &= \partial_m b_1 + v_{Lm} b_1 - b_1 v_{Rm}, \\ D_m b_1^* &= \partial_m b_1^* - b_1^* v_{Lm} + v_{Rm} b_1^*, \\ D_m b_2 &= \partial_m b_2 - b_2 v_{Lm} + v_{Rm} b_2, \\ D_m b_2^* &= \partial_m b_2^* + v_{Lm} b_2^* - b_2^* v_{Rm}, \end{aligned}$$

The final two terms in the action are:

$$\frac{1}{2}(B_1 U_1^R B_2 - B_2 U_1^L B_1)|_{(\theta\theta)} = \frac{1}{2} \left(L_1 (u_1^R b_2 - b_2 u_1^L) + L_2 (b_1 u_1^R - u_1^L b_1) + F_1^R b_2 b_1 - F_1^L b_1 b_2 \right), \quad (C.8)$$

and

$$\frac{1}{2}(\bar{B}_2 \bar{U}_1^R \bar{B}_1 - \bar{B}_1 \bar{U}_1^L \bar{B}_2)|_{\bar{\theta}\bar{\theta}} = \frac{1}{2} \left((b_2^* u_1^{R*} - u_1^{L*} b_2^*) L_1^* + (u_1^{R*} b_1^* - b_1^* u_1^{L*}) L_2^* + F_1^{R*} b_1^* b_2^* - F_1^{L*} b_2^* b_1^* \right). \quad (C.9)$$

Appendix D

The Auxiliary Fields

We also find expressions for other auxiliary fields in the theory in terms of the dynamical fields:

$$F_1 = \frac{du_2^*}{ds} - [A^*, u_2^*] + q_{2p}^* q_{1p} \delta(s - \lambda_p) + b_1^* b_2^* \delta(s - \lambda_R) + b_2^* b_1^* \delta(s - \lambda_L). \quad (D.1)$$

$$F_2 = -\partial_s u_1^* + [A^*, u_1^*], \quad (D.2)$$

$$F_2^* = -\partial_s u_1 - [A, u_1], \quad (D.3)$$

$$G = [u_1^*, u_2^*], \quad G^* = [u_2, u_1], \quad (D.4)$$

$$J_1 = u_1^* q_2^*, \quad J_1^* = q_2 u_1, \quad (D.5)$$

$$J_2 = q_1^* u_1^*, \quad J_2^* = u_1 q_1,$$

$$L_1 = -(b_2^* u_1^{R*} - u_1^{L*} b_2^*), \quad L_1^* = -(u_1^R b_2 - b_2 u_1^L), \quad (D.6)$$

$$L_2 = -(u_1^{R*} b_1^* - b_1^* u_1^{L*}), \quad L_2^* = (b_1 u_1^R - u_1^L b_1).$$

Bibliography

- [1] T. C. Kraan and P. van Baal, “Periodic Instantons with Non-trivial Holonomy,” Nucl. Phys. B **533**, 627 (1998) [arXiv:hep-th/9805168].
- [2] G. Chalmers and A. Hanany, “Three-Dimensional Gauge Theories and Monopoles,” Nucl. Phys. B **489**, 223 (1997) [arXiv: hep-th/9608105].
- [3] A. Hanany, E. Witten, “Type IIB Superstrings, BPS Monopoles, and Three-Dimensional Gauge Dynamics,” Nucl. Phys. B **492**, 152-190 (1997) [arXiv: hep-th/9611230].
- [4] J. Erdmenger, Z. Guralnik and I. Kirsch, “Four-Dimensional Superconformal Theories with Interacting Boundaries or Defects,” Phys. Rev. D **66**, 025020 (2002) [arXiv:hep-th/0203020].
- [5] E. B. Bogomolny, “The Stability of Classical Solutions,” Sov. J. Nucl. Phys. **24**, 49 (1976) [Yad. Fiz. **24** (1976) 861].
- [6] M. K. Prasad and C. M. Sommerfield, “An Exact Classical Solution for the ’t Hooft Monopole and the Julia-Zee Dyon,” Phys. Rev. Lett. **35**, 760 (1975).
- [7] W. Nahm, “A Simple Formalism for the BPS Monopole,” Phys. Lett. B **90**, 413 (1980)
- [8] M. Atiyah, V. G. Drinfel’d, N. J. Hitchin and Y. I. Manin, “Construction of Instantons,” Phys. Lett. A **65**, 185-187 (1978).

- [9] E. F. Corrigan, D. B. Fairlie, S. Templeton and P. Goddard, “Nahm’s Transformation for Instantons,” Nucl. Phys. B **140**, 31 (1978).
- [10] M. R. Douglas, “Branes within Branes,” [arXiv:hep-th/9512077].
- [11] D. E. Diaconescu, “D-branes, Monopoles and Nahm Equations,” Nucl. Phys. B **503**, 220 (1997) [arXiv:hep-th/9608163].
- [12] O. DeWolfe, D. Z. Freedman and H. Ooguri, “Holography and Defect Conformal Field Theories,” Phys. Rev. D **66**, 025009 (2002) [arXiv:hep-th/0111135].
- [13] T. C. Kraan and P. van Baal, “Exact T-Duality between Calorons and Taub-NUT Spaces,” Phys. Lett. B **428**, 268-276 (1998).
- [14] S. A. Cherkis, “Three-Dimensional Gauge Theories and Gravitational Instantons from String Theory,” California Institute of Technology PhD Thesis (1998).
- [15] M. F. Atiyah, “Geometry of Yang-Mills Fields,” Scuola Normal Superiore Pisa, Pisa (1978).
- [16] M. K. Prasad, “Instantons and Monopoles in Yang-Mills Gauge Field Theories,” Physica D: Nonlinear Phenomena **1**, 167 (1980).
- [17] D. Tong, “TASI Lectures on Solitons,” [arXiv:hep-th/0509216].
- [18] A. Belavin, A. Polyakov, A. Schwartz and Y. Tyupkin, “Pseudo-Particle Solutions of the Yang-Mills Equations,” Phys. Lett. B **59**, 85 (1975).
- [19] G. ’t Hooft, “Computation of the Quantum Effects due to a Four-Dimensional Pseudoparticle,” Phys. Rev. D **14**, 3432 (1976).
- [20] M. F. Atiyah and R. S. Ward, “Instantons and Algebraic Geometry,” Commun. Math. Phys. **55**, 117 (1977).
- [21] G. ’t Hooft, “Magnetic Monopoles in Unified Gauge Theories,” Nucl. Phys. B **79**, 276 (1974).

- [22] A. M. Polyakov, "Particle Spectrum in Quantum Field Theory," JETP Lett. **20**, 194 (1974).
- [23] W. Nahm, "All Self-dual Multimonopoles for Arbitrary Gauge Group," CERN-TH.3172 (1981) KEK entry
- [24] W. Nahm, "Selfdual Monopoles and Calorons," BONN-HE-83-16 SPIRES entry
Presented at 12th Colloq. on Group Theoretical Methods in Physics, Trieste, Italy, Sep 5-10, 1983
- [25] W. Nahm, "Self-dual Monopoles and Calorons," Physics **201**, Springer, New York, 1984, pp. 189–200.
- [26] P. J. Braam and P. van Baal, "Nahm's Transformation for Instantons," Commun. Math. Phys. **122**, 267 (1989).
- [27] J. Hurtubise and M. K. Murray, "On the Construction of Monopoles for the Classical Groups," Commun. Math. Phys. **122**, 35 (1989).
- [28] N. J. Hitchin, "On the Construction of Monopoles," Commun. Math. Phys. **89**, 145 (1983).
- [29] B. J. Harrington and H. K. Shepard, "Periodic Euclidean Solutions and the Finite-Temperature Yang-Mills Gas," Phys. Rev. D **17**, 2122-2125 (1978).
- [30] T. C. Kraan and P. van Baal, "Monopole Constituents inside SU(n) Calorons," Phys. Lett. B **435**, 389 (1998).
- [31] K. Lee, "Instantons and Magnetic Monopoles on $R^3 \times S^1$ with Arbitrary Simple Gauge Groups," Phys. Lett. B **426**, 323 (1998).
- [32] K. Lee and C. Lu, "SU(2) Calorons and Magnetic Monopoles," Phys. Rev. D **58**, (1998).

- [33] T. Nye, “The Geometry of Calorons,” University of Edinburgh PhD Thesis, (2001).
- [34] S. A. Cherkis and B. Durcan, “Singular Monopoles via the Nahm Transform,” JHEP **0804**, 070 (2008) [arXiv:hep-th/0712.0850].
- [35] B. Durcan, “Singular Monopoles,” Trinity College Dublin MSc Thesis (2007).
- [36] J. Polchinski, “String Theory,” Cambridge University Press (2005).
- [37] C. V. Johnson, “D-branes,” Cambridge University Press (2003).
- [38] P. DiVecchia and A. Liccardo, “D-branes in String Theory, I,” NATO Adv. Study Inst. Ser. C. Math. Phys. Sci. **556**, 1-59 (2000) [arXiv:hep-th/9912161].
- [39] S. A. Cherkis, “Moduli Spaces of Instantons on the Taub-NUT Space,” Commun. Math. Phys. **290**, 719 (2009) [arXiv:0805.1245 [hep-th]].
- [40] M. F. Atiyah and I. M. Singer, “The Index of Elliptical Operators on Compact Manifolds,” Bull. Amer. Math. Soc. **69**, 322-433 (1963)
- [41] N. J. Hitchin, A. Karlhede, U. Lindström and M. Roček, “Hyperkähler Metrics and Supersymmetry,” Commun. Math. Phys. **108**, 535 (1987).
- [42] M. R. Douglas and G. W. Moore, “D-branes, Quivers, and ALE Instantons,” [arXiv:hep-th/9603167].
- [43] J. Wess and J. Bagger, “Supersymmetry and Supergravity,” Princeton University Press (1992)
- [44] S. A. Cherkis, “Instantons on the Taub-NUT Space,” Adv. Theor. Math. Phys. **14**, no. 2 (2010) [arXiv:0902.4724 [hep-th]].
- [45] G. W. Gibbons and P. Rychenkova, “HyperKähler Quotient Construction of BPS Monopole Moduli Spaces,” Commun. Math. Phys. **186**, 585 (1997) [arXiv:hep-th/9608085].

[46] S. A. Cherkis, “Instantons on Gravitons,” [arXiv:hep-th/1007.0044].