Trinity College Dublin
Coláiste na Tríonóide, Baile Átha Cliath
The University of Dublin

Doctoral Thesis

Existence and Qualitative Properties of Solutions for Various Quasilinear Systems

---

by

Daniel Devine

Supervisor: Paschalis Karageorgis

This dissertation is submitted for the degree of
Doctor of Philosophy

January, 2025

Declaration

I declare that this thesis has not been submitted as an exercise for a degree at this or any other university and it is entirely my own work.

I agree to deposit this thesis in the University's open access institutional repository or allow the Library to do so on my behalf, subject to Irish Copyright Legislation and Trinity College Library conditions of use and acknowledgement.

I consent to the examiner retaining a copy of the thesis beyond the examining period, should they so wish (EU GDPR May 2018).

Parts of this thesis are based on published work. Chapter 2 contains results jointly obtained with Paschalis Karageorgis, and results jointly obtained with Gurpreet Singh. Chapter 3 is based on results jointly obtained with Paschalis Karageorgis. Chapter 4 is based on results jointly obtained with Marius Ghergu and Paschalis Karageorgis.

Signed: Daniel Devine

Daniel Devine
21 January, 2025

Summary

In this thesis, we study two systems of nonlinear elliptic partial differential equations, followed by a partial differential inequality.

In Chapter 2 we study a coercive quasilinear elliptic system which contains nonlinear gradient terms. This system has its origins in the study of viscous, heat-conducting fluids. We focus on positive radially symmetric solutions, and in the first part of the chapter we derive optimal conditions for the existence of these solutions. In the second part of the chapter, assuming a growth condition on the nonlinearities, we precisely describe the asymptotic profile of these solutions.

In Chapter 3 we turn our attention to the well known Hénon-Lane-Emden system. Somewhat inspired by the results of Chapter 2, we focus on the positive radial solutions, and derive a precise asymptotic expansion of these solutions. We then use these expansions to show the positive radial solutions are in fact stable solutions of the associated parabolic problem, with respect to an appropriate norm. Unlike Chapter 2, our results only apply in the so called supercritical regime.

In Chapter 4 we study a nonlocal differential inequality and derive optimal conditions for the existence of solutions, as well as the optimal decay rate of solutions. The thesis concludes with a discussion of future avenues of study.

Acknowledgements

Firstly, I would like to acknowledge and thank my supervisor, Prof. Paschalis Karageorgis, for his indefatigable support and guidance throughout my postgraduate studies. The patience and respect you have continuously shown me over these past years have been immense. It goes without saying that I have benefited immeasurably from your mathematical experience and knowledge, and I am extremely grateful for this. Working with you has been an incredibly enriching and enjoyable experience for me, and no doubt an incredibly frustrating and onerous one for you. Thank you!

I would like to thank Prof. Marius Ghergu and Dr Gurpreet Singh for collaborating with me on projects throughout my PhD studies. I thank you both for your patience and your time, and I hope to work with each of you on projects in the future.

Thank you to Ciara, Emma, Jennifer and Karen in the maths office for always being incredibly kind and helpful.
To my office mates, Christina, Koen, Nikolaos, Riccardo and Thibaut, I hope you enjoyed it as much as I did. Also to my many other friends in the School of Maths - it's been good.
To Julian, Mark and Ronan - it was excellent.

Thank you to my parents, Brendan and Eilish, for always supporting me over the years. It was probably underappreciated at times, but it was always needed. To my siblings, Ethan, Joshua, and Leah, thank you for putting up with me. Shout-out to Nelson and Scout also.

To Katie, who has had to hear more about this thesis than anyone. I don't know how you managed, but thank you.

Thank you to Prof. Jacques Giacomoni and Prof. John Stalker for examining this thesis, and for their useful suggestions for improving it. A final thank you to Prof. Andreea Nicoara for being on my thesis committee.

My postgraduate studies were funded by the Irish Research Council, under grant number GOIPG/2022/469.

Existence and Qualitative Properties of Solutions for Various Quasilinear Systems

Daniel Devine

Abstract

In this thesis we study a variety of nonlinear partial differential equations (PDE), and systems of PDE. In Chapter 2 we study a general class of coercive, quasilinear elliptic systems which have their origin in the study of viscous, heat-conducting fluids. We pose the problem in an open ball centred at the origin, or all of Euclidean space. In the bounded domain case, we derive sharp conditions for the existence of positive radial solutions with prescribed boundary behaviour. In particular, solutions which either blow up at a finite point or remain bounded are considered. We also obtain sharp conditions for the existence of solutions in the whole space, and the exact growth at infinity of such solutions is then determined.

In Chapter 3, we study a prototype reaction-diffusion system. In particular, we focus on the associated steady state problem. Assuming we lie on or above a critical curve, we derive an exact asymptotic expansion of the positive radial solutions. It was previously known that this curve is the threshold for linearly stable radial solutions to exist. We use the asymptotic expansion obtained to prove the nonlinear stability of these solutions, directly extending classical scalar results to the case of systems.

Finally, in Chapter 4 we study a nonlocal quasilinear partial differential inequality which has its origins in non-relativistic mechanics. The nonlocal interactions are modelled using the convolution operation, and a slow decay potential term is considered. Posing the problem in an exterior domain, we derive sharp conditions for the existence and nonexistence of positive solutions. In the case of existence, the optimal decay rate of solutions is also determined. Unlike the cases of fast or critical decay potential terms, exponentially decaying solutions are also shown to exist.

Publications Associated with this Project

[i] Daniel Devine. Existence of positive radial solutions of general quasilinear elliptic systems. *Adv. Nonlinear Anal.*, accepted.

[ii] Daniel Devine, Marius Ghergu, and Paschalis Karageorgis. Quasilinear elliptic inequalities with nonlinear convolution terms and potentials of slow decay. *Differential Integral Equations*, 36(1-2):1–20, 2023.

[iii] Daniel Devine and Paschalis Karageorgis. Monotonic convergence of positive radial solutions for general quasilinear elliptic systems. *Nonlinearity*, 37(3):Paper No. 035020, 14, 2024.

[iv] Daniel Devine and Paschalis Karageorgis. Stability of positive radial steady states for the parabolic Hénon-Lane-Emden system. 2024. Preprint, https://arxiv.org/abs/2406.14997.

[v] Daniel Devine and Gurpreet Singh. Existence and boundary behaviour of radial solutions for weighted elliptic systems with gradient terms. *Eur. J. Math.*, 9(4):106, 2023.

Contents

Chapter 1

Introduction

In this thesis we study the qualitative behaviour of solutions to some partial differential equations (PDE) and systems of PDE. We will study three different nonlinear elliptic problems, one in each of Chapters 2, 3 and 4. Detailed introductions and background to the problems are given at the beginning of each chapter. Broadly speaking, we will be focusing on the existence, stability and asymptotic behaviour of solutions, and in some cases we can use this information to analyse the corresponding parabolic problem. We are generally interested in problems posed on unbounded domains (either all of Euclidean space $\mathbb{R}^n$ or an exterior domain). This poses considerable technical challenges, since many commonly used tools are best suited to the bounded domain case. Chapters 2, 3 and 4 are more or less self contained. As such, there is at times an overlap in the notation used, mainly to remain consistent with the literature at large.

Chapter 2 is based on the results of [18, 20, 22]. Some of the results were jointly obtained with Paschalis Karageorgis, and some were jointly obtained with Gurpreet Singh. We study a general class of coercive, quasilinear elliptic systems which have their origin in the study of viscous, heat-conducting fluids. The first goal in Chapter 2 is to determine sharp conditions for the existence of solutions which blow up at a finite point. The study of these kinds of solutions has a rich history, going back many decades. In the 1950's, Keller [58] and Osserman [77] independently considered the following problem

$$\begin{cases} \Delta u = f(u) & \text{in } \Omega, \\ u \to \infty & \text{as } x \to \partial\Omega, \end{cases} \tag{A}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain and $f \in C^1[0, \infty)$ is non-negative and increasing. They found that problem (A) has $C^2(\Omega)$ solutions if and only if

$$\int_1^\infty \frac{ds}{\sqrt{F(s)}} < \infty, \quad \text{where} \quad F(s) := \int_0^s f(t)dt. \tag{B}$$

Since their seminal work there have been many generalisations of the above result, and conditions analogous to (B) have since become known as Keller-Osserman type conditions. In Section 2.1, we obtain optimal Keller-Osserman conditions for the system

$$\begin{cases} \Delta_p u = f_1(|x|)g_1(v)|\nabla u|^\sigma & \text{in } \Omega, \\ \Delta_p v = f_2(|x|)g_2(v)g_3(|\nabla u|) & \text{in } \Omega, \end{cases} \tag{C}$$

where $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplace operator. Note that equation (A) can be viewed as a special case of system (C) with $f_1 \equiv f_2 \equiv g_3 \equiv 1$, $p = 2$, $\sigma = 0$ and $u = v$. Our system presents many technical challenges not present in boundary value problem (A), due to the nonlinear dependence on ∇u, and the quasilinear differential operator Δ_p.

To make the problem mathematically tractable, we focus on solutions (u, v) which are radially symmetric about the origin. This leads to two natural options for the domain Ω; $\Omega = B_R$, the open ball of radius R around the origin, and $\Omega = \mathbb{R}^n$. Assuming the nonlinearities f_i, g_j are non-decreasing, we can show that any solution (u, v) is increasing and convex in both components. In the bounded domain case, this leads to two possibilities: these solutions blow up near ∂B_R, or they remain bounded. If the solution remains bounded, it can be extended to a solution in all of $\mathbb{R}^n$. In Section 2.1 we derive sharp Keller-Osserman type conditions for the existence of positive radial solutions of system (C) which blow up at a finite point, and use these to deduce sharp conditions for the existence of solutions in all of $\mathbb{R}^n$. There are several notable aspects of these results. Firstly, the hypotheses imposed on the functions f_i, g_j are quite weak. In particular, they are only assumed continuous, non-negative and non-decreasing, along with some growth conditions for g_1, g_2. Secondly, the results are valid for all values $p > 1$, which is noteworthy since qualitative differences between the cases $1 < p < 2$ and $p \geq 2$ can often occur. Finally, the Keller-Osserman conditions derived are shown to be optimal. For the $p \neq 2$ case, this required the development of a new comparison result for integrals. Namely, if f is a continuous, non-negative and non-decreasing function, and F is defined as above, we have for any $\nu > 0$ and $p > 1$ that

$$\int_1^\infty \frac{ds}{\left(\int_0^s F(t)^{\frac{1}{p-1}}dt\right)^{\nu(p-1)}} < \infty \quad \text{if and only if} \quad \int_1^\infty \frac{ds}{\left(\int_0^s f(t)^{\frac{1}{p}}dt\right)^{\nu p}} < \infty.$$

The above comparison result was previously only known in the $p = 2$ case. It is the key technical tool used in Section 2.1 to sharply distinguish between solutions of (C) which blow up, and those which exist in all of $\mathbb{R}^n$.

In the case of solutions in the whole space, the growth rate near infinity of these solutions is then studied in Section 2.2. Assuming that the functions g_j all grow polynomially, the growth rate can be calculated exactly. In fact, it is shown that the growth rate of any positive radial solution (u, v) is determined by an explicit solution $(u_0, v_0) = (C_\alpha |x|^\alpha, C_\beta |x|^\beta)$ of a simplified version of system (C). A surprising feature is that the convergence

$$\lim_{|x|\to\infty} \frac{u}{u_0} = \lim_{|x|\to\infty} \frac{v}{v_0} = 1$$

always occurs in a monotonic fashion, in the sense that both of the above quotients are decreasing at all points. This monotonic convergence result is reminiscent of some classical results concerning the Lane-Emden equation

$$-\Delta u = u^p \quad \text{in } \mathbb{R}^n \tag{D}$$

and its fourth order analogue

$$\Delta^2 u = u^p \quad \text{in } \mathbb{R}^n \tag{E}$$

where $p > 1$ in both cases. We emphasise that, in order to remain consistent with the wider literature, we have now switched to using the letter p to denote an exponent. It is by now well known that there exists a critical exponent p_S, called the Sobolev exponent, for problems (D) and (E)

$$\begin{cases} p_S = \dfrac{n+2}{n-2} & \text{for problem (D),} \\ p_S = \dfrac{n+4}{n-4} & \text{for problem (E),} \end{cases}$$

such that positive regular radial solutions exist if $p \geq p_S$, whereas no positive regular solutions (radial or not) exist if $1 < p < p_S$. For problem (D), these results were originally obtained in the seminal papers by Gidas and Spruck [45] and Ding and Ni [25], and for problem (E) by Lin [66] and Gazzola and Grunau [36].

To understand the qualitative behaviour of the solutions of problems (D) and (E), we note they both admit an explicit singular solution: $u_* = C(n, p)|x|^{-\frac{2}{p-1}}$ and $\tilde{u}_* = \tilde{C}(n, p)|x|^{-\frac{4}{p-1}}$, respectively. In his classical paper [63], Li showed that if $p > p_S$, any positive regular radial solution of (D) satisfies

$$\lim_{|x|\to\infty} \frac{u}{u_*} = 1. \tag{F}$$

Using these asymptotics, Wang [87] later showed there exists another critical exponent $p_{JL} > p_S$, named the Joseph-Lundgren exponent for the authors of [54] where the exponent first appeared, such that the following holds:

$$\begin{cases} \text{if } p \geq p_{JL} & \text{the convergence (F) occurs monotonically and} \\ & u < u_* \text{ for all } r > 0, \\ \text{if } p_{JL} > p > p_S & \text{the convergence (F) occurs in an oscillatory manner, i.e} \\ & u \text{ intersects } u_* \text{ infinitely many times.} \end{cases}$$

In [36], Gazzola and Grunau, together with their existence results, also showed that every positive regular radial solution u of (E) satisfies $u/\tilde{u}_* \to 1$ as $|x| \to \infty$ when $p > p_S$. Moreover, this convergence is eventually monotonic if $p \geq p_{JL}$ (that is, problem (E) has its own Joseph-Lundgren exponent). Three years later, in the papers [55] by Karageorgis and [30] by Ferrero, Grunau and Karageorgis, it was shown that the analogous results do indeed hold for problem (E). In other words, given a positive regular radial solution u of (E), if $p \geq p_{JL}$ we have $u < \tilde{u}_*$ for all $r > 0$, and monotonic convergence occurs. If $p_S < p < p_{JL}$, then u intersects $\tilde{u}_*$ infinitely many times, and we have oscillatory convergence.

With these now standard results in mind, we turn our attention to Chapter 3 of this thesis, which is based on the results of [21] jointly obtained with Paschalis Karageorgis. In Chapter 3, we study the positive radial solutions of the Hénon-Lane-Emden system,

$$\begin{cases} -\Delta u = |x|^k v^p & \text{in } \mathbb{R}^n, \\ -\Delta v = |x|^l u^q & \text{in } \mathbb{R}^n, \end{cases} \tag{G}$$

which arises as a natural generalisation of both problems (D) and (E). There is a vast literature on system (G), in particular the case $k = l = 0$, known as the Lane-Emden system. Beginning with the seminal papers by Mitidieri [72] and Serrin and Zou [81, 83] for the case $k = l = 0$, it is by now well known that positive regular radial solutions exist if and only if

$$\frac{k+n}{p+1} + \frac{l+n}{q+1} \leq n - 2.$$

See [8] for the case $k, l \geq 0$. It is expected that no positive solutions (including non-radial) exist when the above condition is violated. This has become known in the literature as the Hénon-Lane-Emden conjecture. Note that when $k = l = 0$, for the special case $u = v$ and $p = q$, which corresponds to problem (D), and the special case $p > q = 1$, which corresponds to problem (E), the above condition reduces to $p \geq p_S$.

It turns out the qualitative phenomena displayed by solutions of the scalar problems discussed above remain true for system (G), and this is the focus of Chapter 3. First note that system (G) has an explicit singular solution (u_*, v_*). The main result of the chapter is that, on or above the Joseph-Lundgren curve, which is an exact analogue of the Joseph-Lundgren exponents discussed above, any positive regular radial solution (u, v) of (G) satisfies

$$\lim_{|x|\to\infty} \frac{u}{u_*} = \lim_{|x|\to\infty} \frac{v}{v_*} = 1,$$

and this convergence is monotonic. The Joseph-Lundgren curve first arose in the work of Chen, Dupaigne and Ghergu [14]. The authors proved that on or above this curve, every positive regular radial solution of (G) is linearly stable and satisfies $u < u_*$ and $v < v_*$ for all $r > 0$. Using the above comparison result, we can compute a more precise asymptotic expansion of the regular radial solutions (u, v). Using this expansion, we then prove (u, v) is in fact a stable solution (with respect to an appropriate norm) of the corresponding parabolic system. This is a direct extension of the classical stability result by Gui, Ni and Wang [46] concerning the semilinear heat equation $u_t - \Delta u = u^p$ to the case of systems. To the author's knowledge this is the first such extension.

Chapter 4 of this thesis takes us in a different direction. This chapter is based on the results of [19], which was a joint work with Marius Ghergu and Paschalis Karageorgis. We study a nonlocal quasilinear partial differential inequality which has its origins in non-relativistic mechanics. In particular, we are interested in the existence, nonexistence and optimal decay rate of positive solutions of

$$-\Delta_p u + \frac{\lambda}{|x|^\gamma} u^{p-1} \geq (K * u^m) u^q \quad \text{in } \mathbb{R}^n \setminus \overline{B}_1. \tag{H}$$

We are mainly interested in convolution kernels K which behave like $(\log|x|)^\theta |x|^{\alpha-n}$ near infinity, i.e a Riesz potential with a logarithmic factor. When $\gamma < p$, which is our main focus, the potential term is said to have slow decay since, as the name suggests, in this case the potential term decays slower near infinity than the differential term. The focus of Chapter 4 is on deriving sharp conditions for the existence and nonexistence of positive solutions of (H). For the nonexistence case, we derive a priori integral estimates for solutions and use these to prove Liouville type results. Classical solutions are constructed in the case of existence, and the optimal decay rate of solutions is also determined. Unlike the cases of fast ($\gamma > p$) or critical ($\gamma = p$) decay potential terms, exponentially decaying solutions are also shown to exist; see [40, 75]. These results were previously only known in the semilinear case $p = 2$. The thesis then concludes with a brief discussion of future avenues of research.

Chapter 2

Quasilinear Systems With Gradient Terms

This chapter is based on the results of [18, 20, 22]. Some results were jointly obtained with Paschalis Karageorgis. Some results were jointly obtained with Gurpreet Singh.

2.1 Existence of Solutions

2.1.1 System of Interest

Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be either an open ball B_R centred at the origin or the whole space, and for each i, j suppose f_i, g_j are continuous, non-negative and non-decreasing functions. In this chapter we study the existence of positive, radially symmetric solutions of coercive quasilinear elliptic systems of the form

$$\begin{cases} \Delta_p u = f_1(|x|)g_1(v)|\nabla u|^\sigma & \text{in } \Omega, \\ \Delta_p v = f_2(|x|)g_2(v)g_3(|\nabla u|) & \text{in } \Omega, \end{cases} \tag{2.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplace operator, $p > 1$ and $\sigma \geq 0$. In the case $\Omega = \mathbb{R}^n$, we will also study the asymptotic profile of these solutions.

The study of elliptic systems without the presence of gradient terms has a rich history. For semilinear elliptic systems with no gradient terms present we refer to, for instance, [11, 17, 38, 59, 60, 61], and for the case of quasilinear elliptic systems without gradient terms we refer to [3, 8, 9, 10, 11, 15, 31]. More recently, systems such as (2.1) have been investigated [1, 2, 4, 7, 23, 32, 34, 39, 84], which we shall now discuss.

Their study was initiated by Díaz, Lazzo and Schmidt [23] who were interested in the following parabolic system

$$\begin{cases} u_t - \Delta u = \theta & \text{in } \Omega \times (0,T), \\ \theta_t - \Delta \theta = |\nabla u|^2 & \text{in } \Omega \times (0,T). \end{cases}$$

This system arises through a prototype model for a viscous, heat-conducting fluid. Assuming a unidirectional flow, independent of distance in the flow direction, the speed u and temperature θ satisfy the above equations. The source terms θ and $|\nabla u|^2$ represent the buoyancy force and viscous heating, respectively. Making the change of variables $v = -\theta$, steady states satisfy

$$\begin{cases} \Delta u = v & \text{in } \Omega, \\ \Delta v = |\nabla u|^2 & \text{in } \Omega. \end{cases}$$

The main result in [23] is:

Theorem ([23])**.** *For any $R > 0$, the system*

$$\begin{cases} \Delta u = v & \text{in } B_R, \\ \Delta v = |\nabla u|^2 & \text{in } B_R, \\ |(u(x), v(x))| \to \infty & \text{as } |x| \to R^-, \end{cases}$$

has a unique radially symmetric solution (u, v) with $u(0) = 0$ and $v(0) > 0$.

The map $v(0) \mapsto R$ is continuous and strictly decreasing (see [4]), and we note that the study of the above parabolic system was continued in [24]. Singh [84] later studied more general semilinear systems of the form

$$\begin{cases} \Delta u = v^m & \text{in } \Omega, \\ \Delta v = g(|\nabla u|) & \text{in } \Omega, \end{cases}$$

where $m > 0$ and $g \in C^1([0,\infty))$ is non-decreasing and positive in $(0,\infty)$. He found optimal Keller-Osserman type conditions for the existence of positive radial solutions in either $\Omega = B_R$ or $\Omega = \mathbb{R}^n$, and he also determined the asymptotic behaviour of global solutions, assuming that $g(t) = t^k$ for some $k \geq 1$ and that k, m satisfy some additional hypotheses. For $\Omega = B_R$, the following boundary behaviours are possible:

(B1) u and v are bounded in B_R;

(B2) u is bounded in B_R and $\lim_{|x|\to R^-} v(x) = \infty$;

(B3) $\lim_{|x|\to R^-} u(x) = \lim_{|x|\to R^-} v(x) = \infty$.

Singh found optimal conditions for each of the above scenarios, generalising the classical results of Keller [58] and Osserman [77] to the case of systems with gradient terms. Singh's existence results were later extended by Filippucci and Vinti [34] who studied the system

$$\begin{cases} \Delta u = v^m & \text{in } \Omega, \\ \Delta v = f(|x|)g_1(u)g_2(|\nabla u|) & \text{in } \Omega. \end{cases}$$

Here f, g_1, g_2 are non-negative, non-decreasing and continuous, g_1 is assumed to be bounded, f satisfies $f(0) > 0$, and $m > 0$. Analogous results were later obtained by Singh and the author [22], where the system

$$\begin{cases} \Delta u = |x|^a v^m & \text{in } \Omega, \\ \Delta v = |x|^b v^q g(|\nabla u|) & \text{in } \Omega, \end{cases}$$

was considered. It was assumed $a, b, m, q > 0$ and g is as above. Once again, in the case $\Omega = B_R$, optimal Keller-Osserman type conditions were found for the existence of solutions satisfying (B1)-(B3), and for $g(t) = t^k$, $k \geq 1$, the asymptotic behaviour of global solutions was determined assuming some additional hypotheses on the exponents. The main contributions of [22] were overcoming technical difficulties near the origin which arise when the weight functions are allowed to be zero there, and also the sharpness of the existence results for general functions g. We do not explicitly state the main results of [22] here, since they are particular cases of the more general results we will discuss in this chapter.

Work on the quasilinear problem began with Filippucci and Vinti in [34]. Not only did they study the semilinear system mentioned above, they also considered

$$\begin{cases} \Delta_p u = v^m & \text{in } B_R, \\ \Delta_p v = g(|\nabla u|) & \text{in } B_R, \end{cases} \tag{2.2}$$

where $m > 0$ and $g \in C^1([0, \infty))$ is non-negative and non-decreasing. In the case $p \geq 2$, the authors show that a necessary condition for system (2.2) to admit a positive radial solution

(u, v) which blows up at ∂B_R is given by

$$\int_1^\infty \frac{s^{(p-2)(p-1)/(mp+p-1)}}{\left(\int_0^s g(t)^{\frac{1}{p}} dt\right)^{mp/(mp+p-1)}} \, ds < \infty. \tag{2.3}$$

The problem of showing that condition (2.3) is sharp was left open by the authors, which motivated the study of system (2.1). The optimality of condition (2.3) was in fact shown for $g(t) = t^k$ by Ghergu, Giacomoni and Singh [39], who analysed the system

$$\begin{cases} \Delta_p u = v^{k_1} \cdot |\nabla u|^\sigma & \text{in } \Omega, \\ \Delta_p v = v^{k_2} \cdot |\nabla u|^{k_3} & \text{in } \Omega, \end{cases} \tag{2.4}$$

where $k_1, k_2, k_3, \sigma \geq 0$. Sharp results were established for the existence of positive radial solutions satisfying (B1)-(B3), and the asymptotic behaviour of global solutions was completely settled. For the case of solutions which blow up at ∂B_R, the exact rate of blow-up of solutions of (2.4) was later determined in [4].

Our aim in this chapter is to study the existence and asymptotic behaviour of the positive radial solutions of the much more general class of quasilinear elliptic systems (2.1), and as we shall see we recover each of the above mentioned results as a special case. We begin by treating the case $\Omega = B_R$ and boundary conditions (B1)-(B3), and then move on to the case $\Omega = \mathbb{R}^n$. After optimal existence conditions are established in this section, the question of asymptotic behaviour is addressed in Section 2.2. Throughout this chapter, we assume for each i, j that

(A1) f_i, g_j are continuous, non-decreasing on $[0, \infty)$ and positive on $(0, \infty)$.

When it comes to the functions g_1, g_2, we shall make one more assumption, namely that they grow polynomially. In particular, we shall also assume

(A2) there exist constants $k_1 > 0$ and $0 \leq k_2 \leq k_1$ such that for $j = 1, 2$ $g_j(t)/t^{k_j}$ is non-increasing in $(0, \infty)$ and

$$\lim_{t \to \infty} \frac{g_j(t)}{t^{k_j}} > 0.$$

Each of the systems discussed above clearly satisfy (A1)-(A2), but our assumption (A2) allows a much more general class of functions g_j to be considered. For example, each $g_j(t)$ could be any sum of non-negative powers of t with non-negative coefficients. When it comes to Theorems 2.1.1 and 2.1.2, a notable feature is that the function g_3 is only assumed to satisfy (A1), and the same is true for the weight functions f_1, f_2. To analyse the asymptotic behaviour of solutions, some additional assumptions will need to be imposed. These conditions are given in Section 2.2.

Before stating the main existence results, we again clarify that we are only interested in positive radial solutions, by which we shall always mean a pair $(u(r), v(r)), r = |x|$, such that

- $u, v \in C^2(\Omega)$ are positive and solve (2.1);
- u and v are non-constant in any neighbourhood of the origin.

In the case $\Omega = \mathbb{R}^n$, we call solutions of (2.1) global solutions. To make our main existence result easier to state, we define the function

$$\mathcal{G}(s) := \left(\int_0^s g_3 \left(t^{\frac{1}{p-1-\sigma}} \right)^{\frac{1}{p}} dt \right)^{-\frac{k_1 p}{k_1 p + p - 1 - k_2}}. \tag{2.5}$$

We note that $k_1 p + p - 1 - k_2 > 0$ by assumption, and remark that $p - 1 - \sigma > 0$ is a necessary condition for solutions to exist (see Lemma 2.1.5).

Theorem 2.1.1 (Boundedness and Blow Up). *Assume (A1),(A2), $0 \leq \sigma < p - 1$, and let $\theta = \frac{1}{p-1-\sigma}$. Suppose (u, v) is a non-constant positive radial solution of* (2.1). *Then the following holds:*

1. *(u, v) satisfies (B1) for every $R > 0$ if and only if*

$$\int_1^\infty \mathcal{G}(s)\ ds = \infty \tag{2.6}$$

2. *(u, v) satisfies (B2) for some $R > 0$ if and only if*

$$\int_1^\infty s^\theta \mathcal{G}(s)\ ds < \infty \tag{2.7}$$

3. *(u, v) satisfies (B3) for some $R > 0$ if and only if*

$$\int_1^\infty \mathcal{G}(s)\ ds < \infty \quad \text{and} \quad \int_1^\infty s^\theta \mathcal{G}(s)\ ds = \infty. \tag{2.8}$$

Theorem 2.1.2 (Global Solutions)**.** *Assume (A1),(A2), $\Omega = \mathbb{R}^n$ and $\sigma \geq 0$. Then* (2.1) *admits non-constant global positive radial solutions if and only if $\sigma < p-1$ and* (2.6) *holds.*

Remark 2.1.3. Although not explicitly treated here, the above results actually hold for several generalisations of system (2.1). For example, in the first equation we could replace $g_1(v)$ by $g_1(v) \cdot g_4(u)$ for a continuous, non-decreasing and bounded function g_4 which is positive on $(0, \infty)$. When it comes to the second equation, we could similarly replace $g_2(v)$ by $g_2(v) \cdot g_5(u)$, with g_5 subject to the same conditions as g_4. Such additional factors would not affect the arguments whatsoever. For the sake of simplicity, though, such variations are not treated explicitly here.

2.1.2 Preliminary Properties of Solutions

We now gather some properties which all non-constant positive radial solutions of (2.1) must have. We first show that if (u, v) is a positive radial solution, then both u and v are increasing and convex. This explains why solutions which exist locally, but not globally, must blow up at some point. We conclude with two comparison results. The first is for integrals, and will allow us to prove the sharpness of conditions (2.6)-(2.8). The second comparison result will allow us to compare solutions of our system (2.1) with solutions of related systems, which will be useful moving forward.

Lemma 2.1.4. *Assume (A1), $\Omega = B_R$, $\sigma \geq 0$, and suppose (u, v) is a non-constant positive radial solution of* (2.1)*. Then $u'(r), v'(r) > 0$ for all $0 < r < R$.*

Proof: Since (u, v) is a radial solution of (2.1), we can rewrite (2.1) as

$$\begin{cases} [r^{n-1}u'(r)|u'(r)|^{p-2}]' = r^{n-1} f_1(r) g_1(v(r)) |u'(r)|^\sigma & \text{for all} \quad 0 < r < R, \\ [r^{n-1}v'(r)|v'(r)|^{p-2}]' = r^{n-1} f_2(r) g_2(v(r)) g_3(|u'(r)|) & \text{for all} \quad 0 < r < R, \\ u'(0) = v'(0) = 0, u(0) > 0, v(0) > 0. \end{cases} \tag{2.9}$$

Since v is continuous, there exists a maximal subinterval $(0, R_0) \subset (0, R)$ such that $v(r) > 0$ for $0 < r < R_0$. By our assumption (A1), we thus have that $r \mapsto r^{n-1}u'(r)|u'(r)|^{p-2}$ and $r \mapsto r^{n-1}v'(r)|v'(r)|^{p-2}$ are increasing for $0 < r < R_0$, and vanish at $r = 0$. It follows that $u'(r) > 0$ and $v'(r) > 0$ for all $0 < r < R_0$. In particular, $v(r)$ cannot vanish at $r = R_0$, so we in fact have $R_0 = R$, implying $v(r) > 0$ on $[0, R)$. By the above we have that $u'(r) > 0$ and $v'(r) > 0$ on $(0, R)$. ∎

Lemma 2.1.5. *Assume (A1) and $\Omega = B_R$. If $\sigma \geq p-1$, then* (2.1) *does not admit any non-constant positive radial solutions.*

Proof: Suppose a positive radial solution (u, v) exists. From Lemma 2.1.4, we must have $u'(r) > 0$ for all $0 < r < R$, and so the first equation in (2.9) can be expressed as

$$\frac{\left[r^{n-1}u'(r)^{p-1}\right]'}{r^{n-1}u'(r)^{p-1}} = f_1(r)g_1(v(r)) \cdot u'(r)^{\sigma-p+1}$$

for all $0 < r < R$. Integrating this equation over the interval $[0, R/2]$ yields

$$\ln\left[r^{n-1}u'(r)^{p-1}\right]\Big|_0^{R/2} = \int_0^{R/2} f_1(s)g_1(v(s)) \cdot u'(s)^{\sigma-p+1}\,ds.$$

Since $\sigma \geq p-1$ and f_1, g_1 are continuous, the right hand side above is clearly finite. This is a contradiction because $u'(0) = 0$ and $p > 1$, so the left hand side is infinite. ■

Lemma 2.1.6. *Assume (A1), $\Omega = B_R$, $0 \leq \sigma < p-1$, and suppose (u, v) is a non-constant positive radial solution of* (2.1). *Then*

$$\frac{p-1-\sigma}{n(p-1-\sigma)+\sigma} f_1(r)g_1(v) \leq [u'(r)^{p-1-\sigma}]' \leq \frac{p-1-\sigma}{p-1} f_1(r)g_1(v), \tag{2.10}$$

$$\frac{1}{n} f_2(r)g_2(v)g_3(u') \leq [v'(r)^{p-1}]' \leq f_2(r)g_2(v)g_3(u'), \tag{2.11}$$

for all $0 < r < R$. In particular, both $u(r)$ and $v(r)$ are convex for all $0 < r < R$.

Proof: From Lemma 2.1.4 we see that (2.9) can be expressed as

$$\begin{cases} [u'(r)^{p-1}]' + \dfrac{n-1}{r}u'(r)^{p-1} = f_1(r)g_1(v(r))u'(r)^{\sigma} & \text{for all} \quad 0 < r < R, \\ [v'(r)^{p-1}]' + \dfrac{n-1}{r}v'(r)^{p-1} = f_2(r)g_2(v(r))g_3(u'(r)) & \text{for all} \quad 0 < r < R, \\ u'(0) = v'(0) = 0, u(0) > 0, v(0) > 0. \end{cases}$$

We can rewrite the first equation above to give us the following system

$$\begin{cases} [u'(r)^{p-1-\sigma}]' + \dfrac{\delta}{r}u'(r)^{p-1-\sigma} = \dfrac{\delta}{n-1} f_1(r)g_1(v(r)) & \text{for all} \quad 0 < r < R, \\ [v'(r)^{p-1}]' + \dfrac{n-1}{r}v'(r)^{p-1} = f_2(r)g_2(v(r))g_3(u'(r)) & \text{for all} \quad 0 < r < R, \\ u'(0) = v'(0) = 0, u(0) > 0, v(0) > 0, \end{cases} \tag{2.12}$$

where δ is defined as

$$\delta = \frac{(n-1)(p-1-\sigma)}{p-1} > 0. \tag{2.13}$$

Since $u', v' > 0$ for all $0 < r < R$ from Lemma 2.1.4, the upper bounds in (2.10) and (2.11) clearly hold. When it comes to the lower bounds, we first rewrite (2.12) as

$$\begin{cases} [r^\delta u'(r)^{p-1-\sigma}]' = \dfrac{\delta}{n-1} r^\delta f_1(r) g_1(v(r)) & \text{for all} \quad 0 < r < R, \\ [r^{n-1} v'(r)^{p-1}]' = r^{n-1} f_2(r) g_2(v(r)) g_3(u'(r)) & \text{for all} \quad 0 < r < R, \\ u'(0) = v'(0) = 0, u(0) > 0, v(0) > 0. \end{cases} \tag{2.14}$$

Now, since $v(r)$ is increasing for $0 < r < R$, recalling our assumption (A1) the first equation in (2.14) tells us

$$r^\delta u'(r)^{p-1-\sigma} = \frac{\delta}{n-1} \int_0^r t^\delta f_1(t) g_1(v(t)) dt \leq \frac{\delta}{n-1} f_1(r) g_1(v(r)) \int_0^r t^\delta dt$$

for all $0 < r < R$. So we have the estimate

$$\frac{1}{r} u'(r)^{p-1-\sigma} \leq \frac{\delta}{(\delta+1)(n-1)} f_1(r) g_1(v(r)) \tag{2.15}$$

for all $0 < r < R$, which, when combined with the first equation in (2.12) yields

$$\begin{aligned} [u'(r)^{p-1-\sigma}]' &= \frac{\delta}{n-1} f_1(r) g_1(v(r)) - \frac{\delta}{r} u'(r)^{p-1-\sigma} \\ &\geq \frac{\delta}{(\delta+1)(n-1)} f_1(r) g_1(v(r)). \end{aligned}$$

Recalling the definition of δ we see that this is precisely the lower bound in (2.10). Now, since we have just shown that $u'(r)$ is increasing, we can take the second equation in (2.14) and argue in a similar way to conclude

$$r^{n-1} v'(r)^{p-1} \leq f_2(r) g_2(v(r)) g_3(u'(r)) \int_0^r t^{n-1} dt = \frac{1}{n} r^n f_2(r) g_2(v(r)) g_3(u'(r)),$$

for all $0 < r < R$. One may now take the second equation in (2.12) and argue in an analogous manner to above to obtain the lower bound in (2.11). ■

Lemma 2.1.7. *Assume (A1), $\Omega = B_R$ and $0 \leq \sigma < p-1$. Then* (2.1) *does not admit any* *non-constant positive radial solutions (u, v) satisfying*

$$\lim_{r\to R^-} u(r) = \infty \quad \textit{and} \quad \lim_{r\to R^-} v(r) < \infty. \tag{2.16}$$

Proof: Suppose (u, v) is such a solution. For each $0 \le r < R$ we have

$$u(r) = u(0) + \int_0^r u'(t)dt,$$

so from (2.16) we see that $\lim_{r\to R^-} u'(r) = \infty$. But from (2.15), (2.16) and (A1) we also have

$$u'(r)^{p-1-\sigma} \le \frac{\delta r}{(\delta+1)(n-1)} f_1(r)g_1(v(r)) \le C$$

for all $0 < r < R$, and some constant $C > 0$. Letting $r \to R^-$ we reach a contradiction. ■

The following result is a direct extension of [84, Lemma 4.1] to all $p > 1$. It is the key technical result required to prove the sharpness of the conditions in Theorem 2.1.1.

Lemma 2.1.8. *Let $h(t)$ satisfy (A1), and $H(t) = \int_0^t h(s)ds$. Then for $s \ge 0$ we have*

$$(p-1)^{2p-1}\left(\int_0^s H(t)^{\frac{1}{p-1}}dt\right)^{p-1} \le (p-1)^{p-1}\left(\int_0^{ps} h(t)^{\frac{1}{p}}dt\right)^p \le \left(\int_0^{p^2 s} H(t)^{\frac{1}{p-1}}dt\right)^{p-1}$$

and consequently, for each $\nu > 0$

$$\int_1^\infty \frac{ds}{\left(\int_0^s H(t)^{\frac{1}{p-1}}dt\right)^{\nu(p-1)}} < \infty \quad \textit{if and only if} \quad \int_1^\infty \frac{ds}{\left(\int_0^s h(t)^{\frac{1}{p}}dt\right)^{\nu p}} < \infty.$$

Proof: We start by noting that since $p > 1$, the function $x \mapsto x^p$ is convex. Both inequalities are clearly true if $s = 0$. If we fix $s > 0$ we have

$$\begin{aligned}
\left(\int_0^s h(t)^{\frac{1}{p}}dt\right)^p = s^p\left(\frac{1}{s}\int_0^s h(t)^{\frac{1}{p}}dt\right)^p &\le s^{p-1}\int_0^s h(t)dt \quad \text{by Jensen's inequality} \\
&= (p-1)^{1-p}\left(H(s)^{\frac{1}{p-1}}\int_s^{ps} d\tau\right)^{p-1} \\
&\le (p-1)^{1-p}\left(\int_s^{ps} H(\tau)^{\frac{1}{p-1}}d\tau\right)^{p-1} \\
&\le (p-1)^{1-p}\left(\int_0^{ps} H(\tau)^{\frac{1}{p-1}}d\tau\right)^{p-1}
\end{aligned}$$

and the second inequality follows. When it comes to the first inequality, since h and H are both non-decreasing we have

$$\begin{aligned}\left(\int_0^s H(t)^{\frac{1}{p-1}}dt\right)^{p-1} &\le \left(H(s)^{\frac{1}{p-1}}\int_0^s dt\right)^{p-1}\\ &= s^{p-1}\int_0^s h(t)dt\\ &\le s^p h(s)\\ &= \left(\frac{1}{p-1}h(s)^{\frac{1}{p}}\int_s^{ps} dt\right)^p\\ &\le \left(\frac{1}{p-1}\int_0^{ps} h(t)^{\frac{1}{p}}dt\right)^p\end{aligned}$$

and the first inequality follows. ■

If we define for each $\theta > 0$

$$H_\theta(t) := \int_0^t h(s^\theta)ds, \quad t > 0, \tag{2.17}$$

then Lemma 2.1.8 tells us that for each $\nu > 0$ we have

$$\int_1^\infty \frac{ds}{\left(\int_0^s H_\theta(t)^{\frac{1}{p-1}}dt\right)^{\nu(p-1)}} < \infty \quad \text{if and only if} \quad \int_1^\infty \frac{ds}{\left(\int_0^s h(t^\theta)^{\frac{1}{p}}dt\right)^{\nu p}} < \infty.$$

It is this formulation, with $\theta = \frac{1}{p-1-\sigma}$ and $\nu = \frac{k_1}{k_1p+p-1-k_2}$, that we shall use moving forward.

For the final result of this section, we consider the following system

$$\begin{cases}\Delta_p u_0 = \tilde{f}_1(|x|)\tilde{g}_1(v_0)|\nabla u_0|^\sigma & \text{in } \Omega,\\ \Delta_p v_0 = \tilde{f}_2(|x|)\tilde{g}_2(v_0)\tilde{g}_3(|\nabla u_0|) & \text{in } \Omega,\end{cases} \tag{2.18}$$

where the functions $\tilde{f}_i, \tilde{g}_j$ are assumed to satisfy (A1) and (A2). Our goal is to compare a solution (u, v) of (2.1) with a solution (u_0, v_0) of (2.18).

Lemma 2.1.9. *Assume (A1), (A2), $n \ge 2$ and $\sigma \ge 0$. Suppose that (u, v) is a non-constant radial solution of (2.1) in $\Omega = B_R$, (u_0, v_0) is a non-constant radial solution of (2.18) in $\Omega = B_{R_0}$, and let $\overline{R} = min\{R, R_0\}$. Suppose also that (u, v) and (u_0, v_0) are positive for all $0 < r < \overline{R}$. Finally, assume that $f_i \ge \tilde{f}_i$ and $g_j \ge \tilde{g}_j$ for each i, j. If $u(0) > u_0(0)$ and*

$v(0) > v_0(0)$*, then*

$$u(r) > u_0(r), \quad v(r) > v_0(r), \quad u'(r) > u_0'(r), \quad v'(r) \geq v_0'(r) \tag{2.19}$$

throughout the whole interval $(0, \overline{R})$*.*

Proof: Let us denote by $(0, \tilde{R}) \subset (0, \overline{R})$ the maximal subinterval on which

$$u(r) > u_0(r), \qquad v(r) > v_0(r) \qquad \text{for all } 0 < r < \tilde{R}. \tag{2.20}$$

The function u satisfies the first equation of the system (2.14), so it satisfies

$$\left[r^\delta u'(r)^{p-1-\sigma} \right]' = \frac{\delta}{n-1}\, r^\delta f_1(r) g_1(v(r)),$$

where $\delta > 0$ is defined by (2.13), and $p-1-\sigma > 0$ by Lemma 2.1.5. The function u_0 satisfies the exact same equation with g_1 replaced by $\tilde{g}_1$ and f_1 replaced by $\tilde{f}_1$, hence

$$\left[r^\delta (u'(r)^{p-1-\sigma} - u_0'(r)^{p-1-\sigma}) \right]' = \frac{\delta}{n-1}\, r^\delta \left[f_1(r) g_1(v) - \tilde{f}_1(r) \tilde{g}_1(v_0) \right].$$

Along the interval $(0, \tilde{R})$, we have $v > v_0 > 0$ by (2.20), so $g_1(v) > g_1(v_0) \geq \tilde{g}_1(v_0)$ by (A2). Since f_1 and $\tilde{f}_1$ are positive by (A1), the right hand side is then positive, so it easily follows that $u'(r) > u_0'(r)$ for all $0 < r < \tilde{R}$.

Next, we turn to the functions v, v_0. Using the second equation in (2.14), we get

$$\left[r^{n-1} (v'(r)^{p-1} - v_0'(r)^{p-1}) \right]' = r^{n-1} \left[f_2(r) g_2(v) g_3(u') - \tilde{f}_2(r) \tilde{g}_2(v_0) \tilde{g}_3(u_0') \right]. \tag{2.21}$$

When it comes to the interval $(0, \tilde{R})$, we have $u' > u_0'$ by above, so $g_3(u') \geq g_3(u_0') \geq \tilde{g}_3(u_0')$ by (A1). In addition, $v > v_0$ by (2.20), and thus $g_2(v) \geq g_2(v_0) \geq \tilde{g}_2(v_0)$ by (A1), meaning the right hand side of (2.21) is non-negative. If the right hand side of (2.21) happens to be identically zero, since $v'(0) = v_0'(0) = 0$, we in fact have $v'(r) = v_0'(r)$ on $(0, \tilde{R})$. If the right hand side of (2.21) becomes positive on $(0, \tilde{R})$, we may argue as above to conclude that $v'(r) \geq v_0'(r)$ for all $0 < r < \tilde{R}$. In both cases we see that $v - v_0$ is non-decreasing on $(0, \tilde{R})$.

This shows that both $u(r) - u_0(r)$ and $v(r) - v_0(r)$ are non-decreasing throughout $(0, \tilde{R})$. As these functions are positive at $r = 0$ by assumption, they cannot possibly vanish at $r = \tilde{R}$. In other words, the maximal subinterval on which (2.20) holds is the whole interval $(0, \overline{R})$, and our argument above gives $u'(r) > u_0'(r)$ and $v'(r) \geq v_0'(r)$ for all $0 < r < \overline{R}$. ■

Using the notation from the above result, it easily follows that, if $v_0 \to \infty$ as $r \to R_0^-$, then we have $R \leq R_0$.

2.1.3 Proof of Theorem 2.1.1

Before the main proof, we first note the existence of a non-constant positive radial solution of (2.1) in a small ball B_ρ follows from a standard fixed point argument (see Appendix for details). In particular, the map

$$\mathcal{T} : C^1[0,\rho] \times C^1[0,\rho] \to C^1[0,\rho] \times C^1[0,\rho]$$

given by

$$\mathcal{T}[u,v](r) = \begin{pmatrix} \mathcal{T}_1[u,v](r) \\ \mathcal{T}_2[u,v](r) \end{pmatrix}$$

where

$$\begin{cases} \mathcal{T}_1[u,v](r) = u(0) + \displaystyle\int_0^r \left[\frac{\delta}{n-1} s^{-\delta} \int_0^s \tau^\delta f_1(\tau) g_1(v(\tau)) d\tau \right]^{\frac{1}{p-1-\sigma}} ds \\ \mathcal{T}_2[u,v](r) = v(0) + \displaystyle\int_0^r \left[s^{1-n} \int_0^s \tau^{n-1} f_2(\tau) g_2(v(\tau)) g_3(|u'(\tau)|) d\tau \right]^{\frac{1}{p-1}} ds \end{cases}$$

and δ is defined as in (2.13), has a fixed point (u,v) for sufficiently small $\rho > 0$. Note from Lemma 2.1.5 that $p - 1 - \sigma > 0$.

When it comes to part 1, we first assume that (2.1) admits a solution (u,v) satisfying

$$\lim_{r \to R^-} v(r) = \infty \tag{2.22}$$

for some $R > 0$. In what follows, C will be used to denote a positive constant which may vary on each occurrence. In general $C = C(n,p,\sigma,R,u,v)$, but its exact value will not be important. From assumption (A2), for some constant $C > 0$ we have the estimates

$$Cv(r)^{k_1} \leq g_1(v(r)) \leq \frac{g_1(v(0))}{v(0)^{k_1}} v(r)^{k_1}$$
$$Cv(r)^{k_2} \leq g_2(v(r)) \leq \frac{g_2(v(0))}{v(0)^{k_2}} v(r)^{k_2}$$

for all $r > 0$. For simplicity we shall let $w = u'$, and we note that w is positive and increasing on $(0,R)$ by Lemma 2.1.6. Multiplying the right inequality in (2.11) by v', which is positive by Lemma 2.1.4, and integrating over $[0,r]$ for $r<R$ we find

$$\int_0^r [v'(t)^{p-1}]'v'(t)dt \le \int_0^r f_2(t)g_2(v(t))g_3(w(t))v'(t)dt.$$

Now, by Lemmas 2.1.4 and 2.1.6, we know that $v(r)$ and $w(r)$ are increasing in $(0,R)$, so by our assumptions (A1)-(A2) the above inequality becomes

$$\frac{p-1}{p}\int_0^r (v'(t)^p)'dt \le Cg_3(w(r))\int_0^r v(t)^{k_2}v'(t)dt$$

from which we have, recalling $v'(0)=0$,

$$v'(r)^p \le Cg_3(w(r))\int_{v(0)}^{v(r)} t^{k_2}dt \le Cv(r)^{k_2+1}g_3(w(r))$$

for all $r\in(0,R)$. It easily follows that

$$v'(r)v(r)^{-\frac{k_2+1}{p}} \le Cg_3(w(r))^{\frac{1}{p}} \tag{2.23}$$

for all $r\in(0,R)$. Fix now some $\rho\in(0,R)$. Inequality (2.10) and our assumptions (A1)-(A2) tell us that for all $r\in[\rho,R)$ we have

$$Cv(r)^{k_1} \le \frac{1}{n}f_1(\rho)g_1(v(r)) \le \frac{1}{n}f_1(r)g_1(v(r)) \le [w(r)^{p-1-\sigma}]'.$$

Combining this with (2.23) yields

$$v'(r)v(r)^{k_1-\frac{k_2+1}{p}} \le Cg_3(w(r))^{\frac{1}{p}}[w(r)^{\frac{1}{\theta}}]',$$

where we have defined

$$\theta = \frac{1}{p-1-\sigma} > 0$$

for simplicity. Integrating over $[\rho, r]$ for any $r \in (\rho, R)$ we obtain, after changing variables,

$$
\begin{aligned}
\int_{v(\rho)}^{v(r)} t^{k_1 - \frac{k_2+1}{p}} dt &\le C \int_{w(\rho)^{\frac{1}{\theta}}}^{w(r)^{\frac{1}{\theta}}} g_3(t^\theta)^{\frac{1}{p}} dt \\
&\le C \int_0^{w(r)^{\frac{1}{\theta}}} g_3(t^\theta)^{\frac{1}{p}} dt
\end{aligned} \tag{2.24}
$$

for all $r \in (\rho, R)$. We may rewrite (2.24) as

$$
\begin{aligned}
v(r)^{\frac{k_1 p + p - 1 - k_2}{p}} &\le v(\rho)^{\frac{k_1 p + p - 1 - k_2}{p}} + C \int_0^{w(r)^{\frac{1}{\theta}}} g_3(t^\theta)^{\frac{1}{p}} dt \\
&= \left(\frac{v(\rho)^{\frac{k_1 p + p - 1 - k_2}{p}}}{\int_0^{w(r)^{\frac{1}{\theta}}} g_3(t^\theta)^{\frac{1}{p}} dt} + C \right) \int_0^{w(r)^{\frac{1}{\theta}}} g_3(t^\theta)^{\frac{1}{p}} dt \\
&\le \left(\frac{v(\rho)^{\frac{k_1 p + p - 1 - k_2}{p}}}{\int_0^{w(\rho)^{\frac{1}{\theta}}} g_3(t^\theta)^{\frac{1}{p}} dt} + C \right) \int_0^{w(r)^{\frac{1}{\theta}}} g_3(t^\theta)^{\frac{1}{p}} dt
\end{aligned}
$$

and since $\rho > 0$ is fixed, we can express the above as

$$
v(r)^{\frac{k_1 p + p - 1 - k_2}{p}} \le C \int_0^{w(r)^{\frac{1}{\theta}}} g_3(t^\theta)^{\frac{1}{p}} dt,
$$

for all $r \in [\rho, R)$. In other words

$$
v(r) \le C \left(\int_0^{w(r)^{\frac{1}{\theta}}} g_3(t^\theta)^{\frac{1}{p}} dt \right)^{\frac{p}{k_1 p + p - 1 - k_2}}
$$

for all $r \in [\rho, R)$. Combining this with (2.10) we have

$$
\begin{aligned}
\left[w(r)^{\frac{1}{\theta}} \right]' \le \frac{p - 1 - \sigma}{p - 1} f_1(r) g_1(v(r)) &\le C v(r)^{k_1} \\
&\le C \left(\int_0^{w(r)^{\frac{1}{\theta}}} g_3(t^\theta)^{\frac{1}{p}} dt \right)^{\frac{k_1 p}{k_1 p + p - 1 - k_2}} \\
&= C \mathcal{G} \left(w(r)^{\frac{1}{\theta}} \right)^{-1}
\end{aligned} \tag{2.25}
$$

for all $r \in [\rho, R)$, recalling our definition (2.5). It follows from (2.25) that for some constant $C > 0$ and all $r \in [\rho, R)$ one has

$$\mathcal{G}\left(w(r)^{\frac{1}{\theta}}\right)\left[w(r)^{\frac{1}{\theta}}\right]' \leq C.$$

We can integrate both sides of this expression over $[\rho, r]$ for $r \in (\rho, R)$ to find

$$\int_\rho^r \mathcal{G}\left(w(s)^{\frac{1}{\theta}}\right)\left[w(s)^{\frac{1}{\theta}}\right]' \, ds \leq C\int_\rho^r ds$$

or, after a change of variables

$$\int_{w(\rho)^{\frac{1}{\theta}}}^{w(r)^{\frac{1}{\theta}}} \mathcal{G}(s) \, ds \leq C(r-\rho) \leq Cr. \tag{2.26}$$

Letting $r \to R^-$, we see, recalling (2.10) and (2.22)

$$\int_{w(\rho)^{\frac{1}{\theta}}}^{\infty} \mathcal{G}(s) \, ds \leq CR < \infty.$$

Hence,

$$\int_1^\infty \mathcal{G}(s) \, ds < \infty.$$

This in fact shows that, if (2.6) holds, then any local solution (u, v) of (2.1) in a small ball B_ρ can be extended to a solution in B_R satisfying (B1) for any $R > 0$. Indeed, by Lemma 2.1.4, both u and v are increasing, and by Lemma 2.1.7 and the above argument, we know that u and v are bounded in B_R for any $R > 0$. We thus have that all positive radial solutions of (2.1) satisfy (B1) for every $R > 0$ if (2.6) holds.

Suppose now that (2.6) does not hold. We first note that system (2.1) has a positive radial solution (u, v) defined on some maximal interval $[0, R_0)$. Our goal is to show that this interval is finite. Fix $\rho \in (0, R_0)$, and recall from (2.10), (2.11) and our assumptions (A1)-(A2) that we have

$$\left[w(r)^{\frac{1}{\theta}}\right]' \leq Cv(r)^{k_1} \tag{2.27}$$

$$v(r)^{k_2} g_3(w(r)) \leq C[v'(r)^{p-1}]' \tag{2.28}$$

for all $r \in [\rho, R_0)$. Multiplying (2.27) and (2.28) and rearranging gives

$$g_3(w(r)) \left[w(r)^{\frac{1}{\theta}}\right]' \leq Cv(r)^{k_1-k_2}[v'(r)^{p-1}]'$$

for all $r \in [\rho, R_0)$. Integrating the above inequality over $[\rho, r]$ for any $r \in (\rho, R_0)$ we obtain

$$\begin{aligned}
\int_{w(\rho)^{\frac{1}{\theta}}}^{w(r)^{\frac{1}{\theta}}} g_3(t^\theta)dt &\leq C\int_\rho^r v(t)^{k_1-k_2}[v'(t)^{p-1}]'dt \\
&\leq C\int_0^r v(t)^{k_1-k_2}[v'(t)^{p-1}]'dt \\
&\leq Cv(r)^{k_1-k_2}\int_0^r [v'(t)^{p-1}]'dt \\
&\leq Cv(r)^{k_1-k_2}v'(r)^{p-1},
\end{aligned}$$

recalling that $k_1 \geq k_2$ by (A2) and $v'(0) = 0$. Taking the extreme left and right above, one has

$$\begin{aligned}
\int_0^{w(r)^{\frac{1}{\theta}}} g_3(t^\theta)dt &\leq \int_0^{w(\rho)^{\frac{1}{\theta}}} g_3(t^\theta)dt + Cv(r)^{k_1-k_2}v'(r)^{p-1} \\
&= \left(\frac{\int_0^{w(\rho)^{\frac{1}{\theta}}} g_3(t^\theta)dt}{v(r)^{k_1-k_2}v'(r)^{p-1}} + C\right) v(r)^{k_1-k_2}v'(r)^{p-1} \\
&\leq \left(\frac{\int_0^{w(\rho)^{\frac{1}{\theta}}} g_3(t^\theta)dt}{v(\rho)^{k_1-k_2}v'(\rho)^{p-1}} + C\right) v(r)^{k_1-k_2}v'(r)^{p-1} \\
&\leq Cv(r)^{k_1-k_2}v'(r)^{p-1} \qquad (2.29)
\end{aligned}$$

for all $r \in [\rho, R_0)$, recalling that $v'(r)$ is positive for $r > 0$ by Lemma 2.1.4. Using the notation from (2.17), we define

$$G_{3,\theta}(t) = \int_0^t g_3(s^\theta)\ ds,\ t > 0,$$

and remind ourselves that $\theta = \frac{1}{p-1-\sigma}$. Multiplying inequalities (2.27) and (2.29) thus gives

$$G_{3,\theta}\left(w(r)^{\frac{1}{\theta}}\right)^{\frac{1}{p-1}} \left[w(r)^{\frac{1}{\theta}}\right]' \leq Cv(r)^{\frac{k_1p-k_2}{p-1}}v'(r)$$

for all $r \in [\rho, R_0)$. A further integration over $[\rho, r]$ yields

$$\int_{w(\rho)^{\frac{1}{\theta}}}^{w(r)^{\frac{1}{\theta}}} G_{3,\theta}(t)^{\frac{1}{p-1}} dt \le Cv(r)^{\frac{k_1p+p-1-k_2}{p-1}},$$

for all $r \in (\rho, R_0)$. Similar to above, since $k_1p + p - 1 - k_2 > 0$, there exists $C > 0$ such that

$$\int_{0}^{w(r)^{\frac{1}{\theta}}} G_{3,\theta}(t)^{\frac{1}{p-1}} dt \le Cv(r)^{\frac{k_1p+p-1-k_2}{p-1}}$$

for all $r \in (\rho, R_0)$. Using (2.10) and our assumption (A2) the above inequality becomes

$$\int_{0}^{w(r)^{\frac{1}{\theta}}} G_{3,\theta}(t)^{\frac{1}{p-1}} dt \le C[v(r)^{k_1}]^{\frac{k_1p+p-1-k_2}{k_1(p-1)}} \le C([w(r)^{\frac{1}{\theta}}]')^{\frac{k_1p+p-1-k_2}{k_1(p-1)}}$$

for all $r \in (\rho, R_0)$, yielding

$$C \le \frac{[w(r)^{\frac{1}{\theta}}]'}{\left(\int_0^{w(r)^{\frac{1}{\theta}}} G_{3,\theta}(t)^{\frac{1}{p-1}} dt\right)^{\frac{k_1(p-1)}{k_1p+p-1-k_2}}}. \tag{2.30}$$

Integrating (2.30) over $[\rho, r]$ and letting $r \to R_0^-$ we find

$$\begin{aligned} C(R_0 - \rho) &\le \int_{w(\rho)^{\frac{1}{\theta}}}^{w(R_0^-)^{\frac{1}{\theta}}} \frac{ds}{\left(\int_0^s G_{3,\theta}(t)^{\frac{1}{p-1}} dt\right)^{\frac{k_1(p-1)}{k_1p+p-1-k_2}}} \\ &\le \int_{w(\rho)^{\frac{1}{\theta}}}^{\infty} \frac{ds}{\left(\int_0^s G_{3,\theta}(t)^{\frac{1}{p-1}} dt\right)^{\frac{k_1(p-1)}{k_1p+p-1-k_2}}} \end{aligned} \tag{2.31}$$

where $w(R_0^-) \in \mathbb{R}^+ \cup \{\infty\}$ is understood to mean $\lim_{r \to R_0^-} w(r)$. If (2.6) does not hold, then Lemma 2.1.8 together with (2.31) implies the maximal interval of existence $[0, R_0)$ is finite. By Lemmas 2.1.4 and 2.1.7, this implies $v(R_0^-) = \infty$, proving part 1. We now argue that, even when (2.6) does not hold, system (2.1) has a solution in B_R for any $R > 0$. We have just shown that system (2.1) has a solution (u, v) in $\Omega = B_{R_0}$. Let $R > R_0$, and consider the following system

$$\begin{cases} \Delta_p \tilde{u} = \tilde{f}_1(|x|)\tilde{g}_1(\tilde{v})|\nabla \tilde{u}|^\sigma & \text{in } \Omega, \\ \Delta_p \tilde{v} = \tilde{f}_2(|x|)\tilde{g}_2(\tilde{v})\tilde{g}_3(|\nabla \tilde{u}|) & \text{in } \Omega, \end{cases}$$

where

$$\begin{aligned} &\tilde{f}_1(r) = \lambda^{p-1-\sigma} f_1(\lambda r) && \tilde{f}_2(r) = \lambda^{p-1} f_2(\lambda r) \\ &\tilde{g}_1(t) = g_1(t) && \tilde{g}_2(t) = g_2(t) \\ &\tilde{g}_3(t) = g_3\left(\frac{t}{\lambda}\right), \end{aligned}$$

and $\lambda = \frac{R}{R_0}$. The functions $\tilde{f}_i, \tilde{g}_j$ clearly satisfy assumptions (A1) and (A2), and since g_3 doesn't satisfy condition (2.6), neither does $\tilde{g}_3$. The above analysis is thus valid for this new system. Arguing as in [34, 84], by using (2.26) and (2.31), we can show that there exists a solution $(\tilde{u}, \tilde{v})$ in B_{R_0}. We then have that

$$u(r) = \tilde{u}\left(\frac{r}{\lambda}\right) \quad v(r) = \tilde{v}\left(\frac{r}{\lambda}\right)$$

is a solution of (2.1) in B_R. Moreover, this solution satisfies $\lim\limits_{r\to\tilde{R}^-} v(r) = \infty$, for some $\tilde{R} \geq R$.

Turning now to part 2, assume that (2.22) holds and let $\Phi : (0, \infty) \to (0, \infty)$ be defined as

$$\Phi(t) := \int_t^\infty \mathcal{G}(s)\ ds.$$

We note that Φ is decreasing and by part 1 we have $\lim\limits_{t\to\infty} \Phi(t) = 0$. From (2.26) and (2.31) we see there exists $\rho \in (0, R)$ such that

$$C_1(R - r) \leq \Phi(w(r)^{\frac{1}{\theta}}) \leq C_2(R - r)$$

for all $r \in [\rho, R)$ and some constants $C_2 > C_1 > 0$. Since Φ is decreasing, this implies

$$\Phi^{-1}(C_2(R - r))^\theta \leq w(r) \leq \Phi^{-1}(C_1(R - r))^\theta.$$

Now, recalling that for all $r \in [\rho, R)$

$$u(r) = u(\rho) + \int_\rho^r w(t)dt,$$

we see that $\lim_{r \to R^-} u(r) < \infty$ if and only if

$$\int_\rho^R w(t)dt < \infty,$$

which holds if and only if

$$\int_\rho^R \Phi^{-1}(C(R-t))^\theta dt < \infty$$

for some $C > 0$. Hence if $\lim_{r \to R^-} u(r) < \infty$, after a change of variables we see

$$\int_0^{C(R-\rho)} \Phi^{-1}(s)^\theta ds < \infty,$$

which yields

$$\int_0^1 \Phi^{-1}(s)^\theta ds < \infty.$$

The change of variables $t = \Phi^{-1}(s)$ gives us that $\lim_{r \to R^-} u(r) < \infty$ if and only if

$$\int_1^\infty s^\theta \mathcal{G}(s)\ ds < \infty,$$

finishing the proof. ■

Remark 2.1.10. A possible extension of the above results would be to somehow relax assumption (A2). Without (A2), the analysis becomes somewhat technical, so we shall only briefly discuss it here. Assuming for the moment that the following integrals are well defined, one must consider the functions

$$G(t) = \int_0^t \left(\frac{g_1(s)^p}{G_2(s)}\right)^{\frac{1}{p}} ds \quad \text{and} \quad \tilde{G}(t) = \int_0^t \left(\frac{g_1(s)^p}{g_2(s)}\right)^{\frac{1}{p-1}} ds,$$
$$\text{where } G_2(t) = \int_0^t g_2(s)ds.$$

In the case $\Omega = B_R$, a necessary condition for the blow up of solutions near ∂B_R is given by

$$\int_1^\infty \frac{ds}{g_1\left(G^{-1}\left(C_1 \int_0^s g_3(t^\theta)^{\frac{1}{p}} dt\right)\right)} < \infty, \tag{2.32}$$

and a sufficient condition for blow up to occur is given by

$$\int_1^\infty \frac{ds}{g_1\left(\tilde{G}^{-1}\left(C_2\int_0^s(\int_0^t g_3(\tau^\theta)d\tau)^{\frac{1}{p-1}}dt\right)\right)} < \infty, \tag{2.33}$$

for some constants $C_1 = C_1(n,p,\sigma,R) > 0$ and $C_2 = C_2(n,p,\sigma,R) > 0$. In order to obtain sharp results analogous to the above, one must somehow show that (2.32) and (2.33) are equivalent. Intuitively speaking, if (A2) holds one has $G^{-1}(t^{\frac{1}{p}}) \sim \tilde{G}^{-1}(t^{\frac{1}{p-1}}) \sim t^{\frac{1}{k_1p+p-1-k_2}}$, so Lemma 2.1.8 allows us to compare the integrals in (2.32) and (2.33). Strictly speaking, (A2) is not necessary for $G^{-1}(t^{\frac{1}{p}}) \sim \tilde{G}^{-1}(t^{\frac{1}{p-1}})$, but determining exactly when (2.32) and (2.33) are equivalent still seems to be out of reach.

2.1.4 Proof of Theorem 2.1.2

First assume that (2.6) holds and $0 \le \sigma < p-1$. From the discussion at the beginning of Section 2.1.3, we know that when $0 \le \sigma < p-1$ we are able to construct a non-constant positive radial solution in a maximal ball. By Lemma 2.1.4, both u and v are increasing, and by Lemma 2.1.7 and Theorem 2.1.1 part 1, we know that u and v are bounded when (2.6) holds. Hence the domain of existence must be all of $\mathbb{R}^n$.

Conversely, we know from Lemma 2.1.5 that if $\sigma \ge p-1$, then no positive radial solution of (2.1) exists in B_R for any $R > 0$. So suppose now that $0 \le \sigma < p-1$, (2.6) does not hold, and (U,V) is a global positive radial solution of (2.1). We consider the related system

$$\begin{cases} \Delta_p u = \tilde{f}_1(|x|)\tilde{g}_1(v)|\nabla u|^\sigma & \text{in } \Omega, \\ \Delta_p v = \tilde{f}_2(|x|)\tilde{g}_2(v)\tilde{g}_3(|\nabla u|) & \text{in } \Omega, \end{cases} \tag{2.34}$$

where

$$\begin{aligned} &\tilde{f}_1(r) = \lambda^{p-1-\sigma}f_1(\lambda r) \quad && \tilde{f}_2(r) = \lambda^{(p-1)(1-c)}f_2(\lambda r) \\ &\tilde{g}_1(t) = g_1(\lambda^c t) && \tilde{g}_2(t) = g_2(\lambda^c t) \\ &\tilde{g}_3(t) = g_3\left(\frac{t}{\lambda}\right) \end{aligned}$$

for some $0 < c < 1$ and $\lambda > 0$. Then $(U_\lambda(r), V_\lambda(r)) = (U(\lambda r), \lambda^{-c}V(\lambda r))$ is a global positive radial solution of (2.34). The system (2.34) also satisfies the hypotheses (A1)-(A2), so by Theorem 2.1.1, since (2.6) does not hold, there exists a positive radial solution (u,v) of (2.34) satisfying, $\lim_{r\to R^-} v(r) = \infty$ for some $R > 0$. By letting $\lambda > 0$ be sufficiently small, we may assume $V_\lambda(0) > v(0) > 0$, and we note that $U_\lambda(0) = u(0)$. By Lemma 2.1.9, it follows that $V_\lambda(r) > v(r)$ for all $0 < r < R$, implying $\lim_{r\to R^-} V_\lambda(r) \geq \lim_{r\to R^-} v(r) = \infty$, contradicting the fact that (U_λ, V_λ) is a global solution of (2.34). ∎

2.2 Asymptotic Behaviour of Global Solutions

2.2.1 Background and Main Result

With optimal conditions for the existence of global positive radial solutions of (2.1) now at our disposal, we turn our attention to the behaviour of these solutions as $|x| \to \infty$. In particular, we shall focus on the asymptotic behaviour of global positive radial solutions of the following system

$$\begin{cases} \Delta_p u = c_1|x|^{m_1} \cdot g_1(v) \cdot |\nabla u|^\sigma & \text{in } \mathbb{R}^n, \\ \Delta_p v = c_2|x|^{m_2} \cdot g_2(v) \cdot g_3(|\nabla u|) & \text{in } \mathbb{R}^n, \end{cases} \tag{2.35}$$

where $c_1, c_2 > 0$ and $m_1, m_2 \geq 0$. System (2.35) is clearly a particular case of system (2.1), so Theorem 2.1.2 is valid. We are focusing on the case $f_i(|x|) = c_i|x|^{m_i}$ mainly for clarity. For this choice of weight functions, the asymptotic profile of solutions is explicit, whereas for general functions f_i we typically cannot describe the asymptotic profile of solutions in terms of elementary functions.

For the system

$$\begin{cases} \Delta u = v^m & \text{in } \mathbb{R}^n, \\ \Delta v = |\nabla u|^k & \text{in } \mathbb{R}^n, \end{cases}$$

where $m > 0$ and $k \geq 1$, Singh [84] used the theory of cooperative dynamical systems to describe the asymptotic behaviour of the positive radial solutions. In [22], a similar analysis was used, but in both cases some additional conditions on the exponents were assumed. When it come to quasilinear systems, as previously discussed, system (2.4)

$$\begin{cases} \Delta_p u = v^{k_1} \cdot |\nabla u|^\sigma & \text{in } \mathbb{R}^n, \\ \Delta_p v = v^{k_2} \cdot |\nabla u|^{k_3} & \text{in } \mathbb{R}^n, \end{cases}$$

was analysed in [39], where the asymptotic behaviour of solutions was settled assuming the existence condition

$$0 \le \sigma < p - 1, \qquad (p-1-\sigma)(p-1-k_2) > k_1 k_3. \tag{2.36}$$

Assuming that (2.36) holds, one can easily check that system (2.4) has an explicit solution of the form

$$u_0(r) = C_\alpha r^\alpha = C_\alpha |x|^\alpha, \qquad v_0(r) = C_\beta r^\beta = C_\beta |x|^\beta \tag{2.37}$$

for some $\alpha, \beta, C_\alpha, C_\beta > 0$. According to [39, Theorem 2.3], all non-constant positive radial solutions of (2.4) in $\Omega = \mathbb{R}^n$ have the same behaviour at infinity, and they satisfy

$$\lim_{r\to\infty} \frac{u(r)}{u_0(r)} = \lim_{r\to\infty} \frac{v(r)}{v_0(r)} = 1. \tag{2.38}$$

Our goal is to extend [39, Theorem 2.3] to the much more general class of quasilinear elliptic systems (2.1). The papers [22, 39, 84] all used similar approaches to determine the asymptotic behaviour of solutions. The key idea is to set $t = \log r$, and make the following nonlinear change of variables

$$X(t) = \frac{rv'(r)}{v(r)}, \quad Y(t) = \frac{rv^{k_1}(r)}{u'(r)^{p-1-\sigma}}, \quad Z(t) = \frac{rv^{k_2}(r)u'(r)^{k_3}}{v'(r)^{p-1}},$$

originally introduced in [8]. This produces an autonomous, cooperative three component dynamical system which can be analysed using classical techniques (see [51, 52] for instance). This approach, however, ceases to be useful for systems as general as (2.1). By introducing a new linear change of variables which seems more natural, we sidestep the dynamical systems theory altogether, and apply simple continuity arguments to determine the asymptotic behaviour of solutions.

When it comes to the functions g_j, we need to slightly refine the growth and regularity assumptions from Theorem 2.1.2. In particular, we will assume:

(C1) each g_j is differentiable and non-decreasing in $[0, \infty)$;

(C2) each g_j is positive in $(0, \infty)$ and g_1', g_2' are positive in $(0, \infty)$;

(C3) there exist constants $k_j \ge 0$ such that $g_j(s)/s^{k_j}$ is non-increasing in $(0, \infty)$ with

$$\lim_{s\to\infty} \frac{g_j(s)}{s^{k_j}} = 1 \qquad \text{for each } j = 1, 2, 3. \tag{2.39}$$

We will show that non-constant positive radial solutions of (2.35) in $\Omega = \mathbb{R}^n$ exhibit the same asymptotic behaviour as the corresponding solutions of the limiting system

$$\begin{cases} \Delta_p u = c_1|x|^{m_1} \cdot v^{k_1} \cdot |\nabla u|^{\sigma} & \text{in } \Omega, \\ \Delta_p v = c_2|x|^{m_2} \cdot v^{k_2} \cdot |\nabla u|^{k_3} & \text{in } \Omega, \end{cases} \tag{2.40}$$

and that they all approach $(u_0, v_0) = (C_\alpha r^\alpha, C_\beta r^\beta)$ for some parameters $\alpha, \beta, C_\alpha, C_\beta > 0$. A notable feature of this result is that the convergence actually occurs in a monotonic fashion. In fact, we shall prove that both u/u_0 and v/v_0 are decreasing for any positive radial solution (u, v) which satisfies (2.35). This observation is a key ingredient in our approach.

Theorem 2.2.1. *Assume (C1)-(C3) and* (2.36). *Let* (u, v) *be a non-constant positive radial solution of the system* (2.35), *where* $n \geq 2$, $m_1, m_2 \geq 0$ *and* $c_1, c_2 > 0$. *Then the associated system* (2.40) *has a unique solution of the form* $(u_0, v_0) = (C_\alpha r^\alpha, C_\beta r^\beta)$, *where each of the parameters* $\alpha, \beta, C_\alpha, C_\beta$ *is positive.*

(a) If $\Omega = B_R$, *then* $\frac{u(r)}{u_0(r)}, \frac{v(r)}{v_0(r)}, \frac{u'(r)}{u_0'(r)}, \frac{v'(r)}{v_0'(r)}$ *are all decreasing for each* $0 < r < R$.

(b) If $\Omega = \mathbb{R}^n$, *then* (u, v) *and* (u_0, v_0) *have the same behaviour at infinity, namely*

$$\lim_{r\to\infty} \frac{u(r)}{u_0(r)} = \lim_{r\to\infty} \frac{v(r)}{v_0(r)} = \lim_{r\to\infty} \frac{u'(r)}{u_0'(r)} = \lim_{r\to\infty} \frac{v'(r)}{v_0'(r)} = 1. \tag{2.41}$$

Remark 2.2.2. We first note that if (C1)-(C3) hold, Theorem 2.1.2 tells us that (2.36) is a sharp condition for the existence of global positive radial solutions. Next, we note that our method of proof for Theorem 2.2.1 applies verbatim for several variations of system (2.35). When it comes to the first equation, for instance, one may replace the factor $g_1(v)$ by $g_1(v) \cdot g_4(u)$, where g_4 is also subject to the assumptions (C1)-(C3). When it comes to the second equation, one may similarly replace $g_2(v)$ by $g_2(v) \cdot g_5(u)$, and also include the factor $|\nabla v|^\tau$, where $\tau \geq 0$. All these variations can be treated using the same approach by merely adjusting the existence condition (2.36). For the sake of simplicity, however, we shall not bother to treat them explicitly.

2.2.2 Monotonic comparison for general quasilinear systems

We now establish two important comparison results. To help motivate these results, let us first consider the following concrete example. Suppose we wish to study the positive radial solutions of the system

$$\begin{cases} \Delta_p u = \left(\sum_{i=1}^{k} a_i v^{b_i} \right) \cdot |\nabla u|^\sigma & \text{in } \mathbb{R}^n, \\ \Delta_p v = \left(\sum_{j=1}^{l} c_j v^{d_j} \right) \cdot \left(\sum_{q=1}^{m} e_q |\nabla u|^{f_q} \right) & \text{in } \mathbb{R}^n, \end{cases}$$

where $a_i, b_i, c_j, d_j, e_q, f_q, \sigma \geq 0$ and $p > 1$. Assuming that b_k, d_l and f_m are the largest exponents in the above polynomials, we know from Theorem 2.1.2 that positive radial solutions of this system exist if and only if $\sigma < p-1$ and $(p-1-d_l)(p-1-\sigma) > b_k f_m$. From Section 2.1, any such solution is increasing and convex in both components, so for large $|x|$ we have

$$\begin{cases} \Delta_p u \approx a_k \cdot v^{b_k} \cdot |\nabla u|^\sigma, \\ \Delta_p v \approx c_l e_m \cdot v^{d_l} \cdot |\nabla u|^{f_m}. \end{cases}$$

Heuristically speaking, our solution (u, v) seems to approximate a solution of a simpler system as $|x|$ becomes large. Assuming the existence condition holds, we can construct an explicit solution of this simplified system $(u_0, v_0) = (C_\alpha |x|^\alpha, C_\beta |x|^\beta)$ with $\alpha, \beta, C_\alpha, C_\beta > 0$ satisfying

$$\begin{cases} \Delta_p u_0 = a_k \cdot v_0^{b_k} \cdot |\nabla u_0|^\sigma & \text{in } \mathbb{R}^n, \\ \Delta_p v_0 = c_l e_m \cdot v_0^{d_l} \cdot |\nabla u_0|^{f_m} & \text{in } \mathbb{R}^n. \end{cases}$$

The results of this section aim to compare (u, v) with (u_0, v_0). The first comparison result we wish to establish is that $u > u_0$, $v > v_0$, $u' > u_0'$ and $v' > v_0'$ at all points. This easily follows from Lemma 2.1.9 applied to our systems of interest. With this in hand, we can then proceed to prove that the quotients $u/u_0, v/v_0$ are decreasing for all $|x| > 0$, and finally that both these quotients converge to 1 as $|x| \to \infty$. In this way we can explicitly describe the asymptotic profile of solutions for a system containing, for example, polynomial nonlinearities.

More generally, our overall plan is to compare solutions (u, v) of the system:

$$\begin{cases} \Delta_p u = f_1(|x|) \cdot g_1(v) \cdot |\nabla u|^\sigma & \text{in } \Omega, \\ \Delta_p v = f_2(|x|) \cdot g_2(v) \cdot g_3(|\nabla u|) & \text{in } \Omega, \end{cases} \tag{2.42}$$

with solutions (u_0, v_0) of the system

$$\begin{cases} \Delta_p u_0 = f_1(|x|) \cdot h_1(v_0) \cdot |\nabla u_0|^\sigma & \text{in } \Omega, \\ \Delta_p v_0 = f_2(|x|) \cdot h_2(v_0) \cdot h_3(|\nabla u_0|) & \text{in } \Omega, \end{cases} \tag{2.43}$$

which is obtained from (2.42) by replacing the functions g_j by the functions h_j. Intuitively speaking, as in the above example, one should think of (2.43) as a simplified version of the original system (2.42) whose solutions are expected to exhibit the same behaviour at infinity.

When it comes to the functions f_i, g_j and h_j, we impose the following assumptions:

(D1) each f_i, g_j, h_j is continuous, non-decreasing in $[0, \infty)$ and positive in $(0, \infty)$;

(D2) each g_j, h_j is differentiable in $[0, \infty)$ and g_1, g_2, h_1, h_2 are increasing in $(0, \infty)$;

(D3) the quotients $Q_j(s, t) = g_j(st)/h_j(t)$ are non-increasing in t for all $s \geq 1$, $t > 0$;

(D4) there exist constants $k_j \geq 0$ such that $\lim_{t\to\infty} Q_j(s, t) = s^{k_j}$ for each $s \geq 1$;

(D5) one has $g_j(t) \geq h_j(t)$ for all $t > 0$.

Assumption (D1) coincides with assumption (A1) from Theorem 2.1.2, but it is now also imposed on the functions h_j. For the comparison results of this section, we study local solutions in the open ball $\Omega = B_R$ without assuming (D4). In the next section, we shall turn to global solutions in the whole space $\Omega = \mathbb{R}^n$ assuming (D1)-(D4). It is easy to see that (D3) and (D4) trivially imply (D5), as they imply $g_j(st) \geq s^{k_j} h_j(t)$, more generally.

Lemma 2.2.3. *Assume (D1), (D2), (D5), $n \geq 2$ and $\sigma \geq 0$. Suppose that (u, v) and (u_0, v_0) are non-constant radial solutions of* (2.42) *and* (2.43)*, respectively, which are positive for all $0 < r < R$. If $u(0) > u_0(0)$ and $v(0) > v_0(0)$, then*

$$u(r) > u_0(r), \quad v(r) > v_0(r), \quad u'(r) > u_0'(r), \quad v'(r) > v_0'(r) \tag{2.44}$$

throughout the whole interval $(0, R)$.

Proof: The proof is almost identical to Lemma 2.1.9, with the only difference being the strict inequality $v'(r) > v_0'(r)$, which easily follows from assumption (D2). ■

Theorem 2.2.4 (Monotonic Comparison)**.** *Assume (D1)-(D3), (D5), $n \geq 2$ and $\sigma \geq 0$. Suppose that (u, v) and (u_0, v_0) are non-constant radial solutions of (2.42) and (2.43), respectively, which are positive for all $0 < r < R$. Suppose also that $u(0), v(0) > 0$ and*

$$u_0(0) = v_0(0) = 0, \qquad \lim_{r\to 0^+} \frac{u_0(r)}{u_0'(r)} < \infty, \qquad \lim_{r\to 0^+} \frac{v_0(r)}{v_0'(r)} < \infty. \tag{2.45}$$

Then each of the quotients

$$\mathcal{U}(r) = \frac{u(r)}{u_0(r)}, \qquad \mathcal{V}(r) = \frac{v(r)}{v_0(r)}, \qquad \mathcal{W}(r) = \frac{u'(r)}{u_0'(r)}, \qquad \mathcal{Y}(r) = \frac{v'(r)}{v_0'(r)} \tag{2.46}$$

has a negative derivative throughout the interval $(0, R)$.

Proof: First of all, it follows from Lemmas 2.1.4 and 2.1.5 that $\sigma < p - 1$ and that

$$u'(r),\ v'(r),\ u_0'(r),\ v_0'(r) > 0 \quad \text{for all } 0 < r < R.$$

Recalling the first equation in (2.14), we see that the functions u, u_0 satisfy the system

$$\begin{cases} \left[r^\delta u'(r)^{p-1-\sigma}\right]' = \dfrac{\delta}{n-1}\, r^\delta f_1(r) g_1(v(r)) & \text{for all } 0 < r < R, \\ \left[r^\delta u_0'(r)^{p-1-\sigma}\right]' = \dfrac{\delta}{n-1}\, r^\delta f_1(r) h_1(v_0(r)) & \text{for all } 0 < r < R, \end{cases} \tag{2.47}$$

with $\delta > 0$ defined by (2.13). We let $A(r) = r^\delta u_0'(r)^{p-1-\sigma}$ for simplicity and divide the last two equations. Using our definition (2.46) and our assumption (D3), we arrive at

$$\frac{1}{A'(r)} \left[r^\delta u'(r)^{p-1-\sigma}\right]' = \frac{g_1(v)}{h_1(v_0)} = \frac{g_1(\mathcal{V} v_0)}{h_1(v_0)} = Q_1(\mathcal{V}, v_0).$$

Since $u' = u_0' \mathcal{W}$ by our definition (2.46), the left hand side can also be expressed as

$$\frac{1}{A'(r)} \left[r^\delta u_0'(r)^{p-1-\sigma} \cdot \mathcal{W}(r)^{p-1-\sigma}\right]' = \frac{1}{A'(r)} \left[A(r) \cdot \mathcal{W}(r)^{p-1-\sigma}\right]'.$$

Once we combine the last two equations, we conclude that $\mathcal{W}(r)$ satisfies

$$\frac{A(r)}{A'(r)} \cdot \left[\mathcal{W}(r)^{p-1-\sigma}\right]' + \mathcal{W}(r)^{p-1-\sigma} = Q_1(\mathcal{V}(r), v_0(r)), \tag{2.48}$$

where $A(r) = r^\delta u_0'(r)^{p-1-\sigma}$ and Q_1 is defined in our assumption (D3).

Repeating the exact same argument, one may derive a similar equation for $\mathcal{Y}(r)$, starting with the second equation of (2.14) instead of the first. This leads to the analogue

$$\frac{B(r)}{B'(r)} \cdot \left[\mathcal{Y}(r)^{p-1}\right]' + \mathcal{Y}(r)^{p-1} = Q_2(\mathcal{V}(r), v_0(r)) \cdot Q_3(\mathcal{W}(r), u_0'(r)), \tag{2.49}$$

where $B(r) = r^{n-1} v_0'(r)^{p-1}$ and Q_2, Q_3 are defined in our assumption (D3).

We now proceed to analyse the system (2.48)-(2.49). Recalling (2.45), we note that

$$\lim_{r\to 0^+} v_0(r)\mathcal{V}'(r) = \lim_{r\to 0^+} \left[v'(r) - \frac{v(r)v_0'(r)}{v_0(r)}\right] = -v(0) \cdot \lim_{r\to 0^+} \frac{v_0'(r)}{v_0(r)} < 0.$$

Let us denote by $(0, R_1) \subset (0, R)$ the maximal subinterval on which $\mathcal{V}'(r) < 0$. Along this subinterval, the right hand side of (2.48) is easily seen to be decreasing, as $Q_1(s,t)$ is increasing in s by (D2) and non-increasing in t by (D3) for all $s \geq 1$ and $t > 0$. Using the fact that $s = \mathcal{V}(r) > 1$ by Lemma 2.2.3, we thus have

$$\frac{d}{dr} Q_1(\mathcal{V}(r), v_0(r)) = \frac{\partial Q_1}{\partial s} \cdot \mathcal{V}'(r) + \frac{\partial Q_1}{\partial t} \cdot v_0'(r) < 0$$

for all $0 < r < R_1$ because $\mathcal{V}'(r) < 0$ and $v_0'(r) > 0$ along this interval. Returning to (2.48), we conclude that the left hand side is decreasing in r, namely

$$\frac{d}{dr}\left[\frac{A(r)}{A'(r)} \cdot \left[\mathcal{W}(r)^{p-1-\sigma}\right]' + \mathcal{W}(r)^{p-1-\sigma}\right] < 0 \tag{2.50}$$

for all $0 < r < R_1$. Noting that $A(r) = r^{\delta} u_0'(r)^{p-1-\sigma}$ is positive, we now multiply the last equation by $A(r)$ and then rearrange terms to find that

$$\frac{d}{dr}\left[A(r) \cdot \frac{A(r)}{A'(r)} \cdot \left[\mathcal{W}(r)^{p-1-\sigma}\right]'\right] < 0$$

for all $0 < r < R_1$. This makes the expression within the square brackets decreasing, so

$$\frac{A(r)^2}{A'(r)} \cdot \left[\mathcal{W}(r)^{p-1-\sigma}\right]' < \lim_{r\to 0^+} \frac{A(r)^2}{A'(r)} \cdot \left[\mathcal{W}(r)^{p-1-\sigma}\right]' \tag{2.51}$$

for all $0 < r < R_1$. To compute the limit on the right hand side, we first note that

$$\begin{aligned}\lim_{r\to 0^+} \frac{A(r)^2}{A'(r)} \cdot \left[\mathcal{W}(r)^{p-1-\sigma}\right]' &= \lim_{r\to 0^+} A(r) \cdot \left[Q_1(\mathcal{V}(r), v_0(r)) - \mathcal{W}(r)^{p-1-\sigma}\right] \\ &= \lim_{r\to 0^+} \left[r^\delta u_0'(r)^{p-1-\sigma} Q_1(\mathcal{V}(r), v_0(r)) - r^\delta u'(r)^{p-1-\sigma}\right]\end{aligned}$$

by (2.48) and the definition (2.46) of $\mathcal{W}(r)$. Since $p-1-\sigma > 0$, we thus have

$$\lim_{r\to 0^+} \frac{A(r)^2}{A'(r)} \cdot \left[\mathcal{W}(r)^{p-1-\sigma}\right]' = \lim_{r\to 0^+} r^\delta u_0'(r)^{p-1-\sigma} \cdot \frac{g_1(v(r))}{h_1(v_0(r))} = 0$$

by the definition of Q_1 and since our estimate (2.15) ensures that

$$0 \le r^\delta u_0'(r)^{p-1-\sigma} \cdot \frac{g_1(v(r))}{h_1(v_0(r))} \le \frac{\delta\, r^{\delta+1}}{(n-1)(\delta+1)} \cdot f_1(r) g_1(v(r)).$$

Knowing that the limit on the right hand side of (2.51) is zero, we deduce that the left hand side is negative. Since $A'(r) > 0$ by (2.47), this implies $\mathcal{W}'(r) < 0$ for all $0 < r < R_1$.

Our next step is to show that $\mathcal{U}'(r) < 0$ for all $0 < r < R_1$. Recalling (2.46), we get

$$\frac{u_0(r)}{u_0'(r)} \cdot \mathcal{U}'(r) + \mathcal{U}(r) = \frac{u'(r)u_0(r) - u_0'(r)u(r)}{u_0(r)u_0'(r)} + \frac{u(r)}{u_0(r)} = \mathcal{W}(r). \tag{2.52}$$

Since $\mathcal{W}'(r) < 0$ throughout $(0, R_1)$ by above, one has

$$\frac{d}{dr}\left[\frac{u_0(r)}{u_0'(r)} \cdot \mathcal{U}'(r) + \mathcal{U}(r)\right] < 0$$

for all $0 < r < R_1$. Multiplying by $u_0(r)$ and rearranging terms now gives

$$\frac{d}{dr}\left[u_0(r) \cdot \frac{u_0(r)}{u_0'(r)} \cdot \mathcal{U}'(r)\right] < 0 \tag{2.53}$$

for all $0 < r < R_1$. Thus, the expression in square brackets is decreasing. Since

$$\lim_{r\to 0^+} \frac{u_0(r)^2}{u_0'(r)} \cdot \mathcal{U}'(r) = \lim_{r\to 0^+} \left[\frac{u'(r)u_0(r)}{u_0'(r)} - u(r)\right] = -u(0) < 0$$

by our assumption (2.45), it follows by the last two equations that

$$\frac{u_0(r)^2}{u_0'(r)} \cdot \mathcal{U}'(r) < 0$$

for all $0 < r < R_1$. In particular, one also has $\mathcal{U}'(r) < 0$ for all $0 < r < R_1$.

We note that the above analysis only uses the first equation (2.48) of our system. Let us now turn to the second equation (2.49), according to which

$$\frac{B(r)}{B'(r)} \cdot \left[\mathcal{Y}(r)^{p-1}\right]' + \mathcal{Y}(r)^{p-1} = Q_2(\mathcal{V}(r), v_0(r)) \cdot Q_3(\mathcal{W}(r), u_0'(r)).$$

Knowing that $\mathcal{V}'(r), \mathcal{W}'(r) < 0$ throughout $(0, R_1)$, one may easily check that the derivative of right hand side is also negative because of our assumptions (D2)-(D3) and since $u_0''(r) > 0$ by Lemma 2.1.6. In other words, one has

$$\frac{d}{dr}\left[\frac{B(r)}{B'(r)} \cdot \left[\mathcal{Y}(r)^{p-1}\right]' + \mathcal{Y}(r)^{p-1}\right] < 0$$

for all $0 < r < R_1$, an exact analogue of (2.50). Proceeding as before, one may then multiply by the positive factor $B(r)$ and integrate to deduce that $\mathcal{Y}'(r) < 0$ for all $0 < r < R_1$.

As our final step, we now relate $\mathcal{Y}(r)$ with $\mathcal{V}(r)$. According to (2.46), we have

$$\frac{v_0(r)}{v_0'(r)} \cdot \mathcal{V}'(r) + \mathcal{V}(r) = \frac{v'(r)v_0(r) - v_0'(r)v(r)}{v_0'(r)v_0(r)} + \frac{v(r)}{v_0(r)} = \mathcal{Y}(r),$$

and this provides an exact analogue of (2.52). Since $\mathcal{Y}'(r) < 0$ throughout $(0, R_1)$, this means

$$\frac{d}{dr}\left[v_0(r) \cdot \frac{v_0(r)}{v_0'(r)} \cdot \mathcal{V}'(r)\right] < 0$$

for all $0 < r < R_1$. On the other hand, $(0, R_1)$ was assumed to be the maximal subinterval on which $\mathcal{V}'(r)$ is negative. Along this interval, the expression in square brackets is decreasing and negative, so it cannot possibly vanish at $r = R_1$. This implies that $(0, R_1)$ must be the whole interval $(0, R)$. In particular, the functions $\mathcal{U}(r)$, $\mathcal{V}(r)$, $\mathcal{W}(r)$, $\mathcal{Y}(r)$ are all decreasing throughout the whole interval $(0, R)$, and the proof is complete. ∎

2.2.3 Asymptotic comparison for general quasilinear systems

Theorem 2.2.5 (Asymptotic Comparison)**.** *Assume (D1)-(D4), $n \geq 2$ and $\sigma \geq 0$. Suppose that (u,v) and (u_0,v_0) are non-constant radial solutions of* (2.42) *and* (2.43)*, respectively, which are positive for all $r>0$. If $u(0), v(0) > 0$ and (u_0,v_0) satisfies* (2.45)*, then*

$$\lim_{r\to\infty} \frac{u(r)}{u_0(r)} = \lim_{r\to\infty} \frac{v(r)}{v_0(r)} = \lim_{r\to\infty} \frac{u'(r)}{u_0'(r)} = \lim_{r\to\infty} \frac{v'(r)}{v_0'(r)} = 1.$$

Proof: Since (2.45) holds, one has $u(0) > 0 = u_0(0)$ and $v(0) > 0 = v_0(0)$, so Theorem 2.2.4 is applicable. Thus, each of $\mathcal{U}'(r), \mathcal{V}'(r), \mathcal{W}'(r), \mathcal{Y}'(r)$ must be negative for all $r > 0$. Using Lemma 2.1.4 and the fact that $\mathcal{U}'(r) < 0$, we get

$$0 > \mathcal{U}'(r) \cdot \frac{u_0(r)}{u_0'(r)} = \frac{u'(r)u_0(r) - u_0'(r)u(r)}{u_0(r)u_0'(r)} = \mathcal{W}(r) - \mathcal{U}(r), \tag{2.54}$$

and thus $\mathcal{W}(r) < \mathcal{U}(r)$ for all $r > 0$. Similarly, one finds that $\mathcal{Y}(r) < \mathcal{V}(r)$ for all $r > 0$.

Let us now recall the first equation (2.48) of our system, according to which

$$\frac{A(r)}{A'(r)} \cdot \left[\mathcal{W}(r)^{p-1-\sigma}\right]' + \mathcal{W}(r)^{p-1-\sigma} = Q_1(\mathcal{V}(r), v_0(r))$$

with $A(r) = r^\delta u_0'(r)^{p-1-\sigma}$ and Q_1 as in our assumption (D3). Note that $A'(r) > 0$ by (2.47) and that $p - 1 - \sigma > 0$ by Lemma 2.1.5. Since $\mathcal{W}'(r) < 0$, it follows by the last equation and our assumptions (D3)-(D4) that

$$\mathcal{W}(r)^{p-1-\sigma} > Q_1(\mathcal{V}(r), v_0(r)) \geq \mathcal{V}(r)^{k_1} \geq \mathcal{Y}(r)^{k_1}$$

for all $r > 0$. Applying the same argument to the second equation (2.49), we get

$$\begin{aligned} \mathcal{Y}(r)^{p-1} &> Q_2(\mathcal{V}(r), v_0(r)) \cdot Q_3(\mathcal{W}(r), u_0'(r)) \\ &\geq \mathcal{V}(r)^{k_2} \cdot \mathcal{W}(r)^{k_3} \geq \mathcal{Y}(r)^{k_2} \cdot \mathcal{W}(r)^{k_3} \end{aligned}$$

for all $r > 0$. In view of the last two estimates, one must then have

$$\mathcal{Y}(r)^{(p-1-k_2)(p-1-\sigma)} > \mathcal{W}(r)^{k_3(p-1-\sigma)} \geq \mathcal{Y}(r)^{k_1 k_3}$$

for all $r > 0$. Since $\mathcal{Y}(r) > 1$ by Lemma 2.2.3, however, this is only possible when

$$(p-1-k_2)(p-1-\sigma) > k_1 k_3. \tag{2.55}$$

Next, we analyse the behaviour of solutions as $r \to \infty$. The functions $\mathcal{U}(r)$, $\mathcal{V}(r)$, $\mathcal{W}(r)$ and $\mathcal{Y}(r)$ are all decreasing and positive, so they all attain a limit as $r \to \infty$. Let us denote their limits by $\mathcal{U}_\infty$, $\mathcal{V}_\infty$, $\mathcal{W}_\infty$ and $\mathcal{Y}_\infty$, respectively. We must then have

$$\mathcal{U}_\infty = \lim_{r\to\infty} \frac{u(r)}{u_0(r)} = \lim_{r\to\infty} \frac{u'(r)}{u_0'(r)} = \mathcal{W}_\infty \tag{2.56}$$

by L'Hôpital's rule, and similarly, $\mathcal{V}_\infty = \mathcal{Y}_\infty$. According to the system (2.47), one has

$$\begin{cases} (p-1-\sigma)\cdot u'(r)^{p-2-\sigma} u''(r) = \dfrac{\delta}{n-1}\, f_1(r) g_1(v(r)) - \dfrac{\delta}{r}\, u'(r)^{p-1-\sigma}, \\ (p-1-\sigma)\cdot u_0'(r)^{p-2-\sigma} u_0''(r) = \dfrac{\delta}{n-1}\, f_1(r) h_1(v_0(r)) - \dfrac{\delta}{r}\, u_0'(r)^{p-1-\sigma}. \end{cases}$$

Dividing these two equations and recalling our definition (2.46), one can thus write

$$\mathcal{W}(r)^{p-2-\sigma}\, \frac{u''(r)}{u_0''(r)} = \frac{\frac{\delta}{n-1}\, f_1(r) g_1(v(r)) \cdot u_0'(r)^{\sigma+1-p} - \frac{\delta}{r} \mathcal{W}(r)^{p-1-\sigma}}{\frac{\delta}{n-1}\, f_1(r) h_1(v_0(r)) \cdot u_0'(r)^{\sigma+1-p} - \frac{\delta}{r}}.$$

Taking the limit as $r \to \infty$ now leads to the identity

$$\mathcal{W}_\infty^{p-2-\sigma} \lim_{r\to\infty} \frac{u''(r)}{u_0''(r)} = \lim_{r\to\infty} \frac{g_1(v(r))}{h_1(v_0(r))} = \lim_{r\to\infty} Q_1(\mathcal{V}(r), v_0(r)) \tag{2.57}$$

with Q_1 as in our assumption (D3). The limit on the right hand side exists because

$$\lim_{r\to\infty} Q_1(\mathcal{V}(r), v_0(r)) = \lim_{r\to\infty} Q_1(\mathcal{V}_\infty, v_0(r)) = \mathcal{V}_\infty^{k_1}$$

by our assumption (D4). Thus, the limit on the left hand side of (2.57) also exists. Using this fact along with L'Hôpital's rule and (2.56), we see that (2.57) reduces to

$$\mathcal{U}_\infty^{p-1-\sigma} = \mathcal{V}_\infty^{k_1}. \tag{2.58}$$

This condition was derived by combining the equations (2.47) satisfied by $u(r)$ and $u_0(r)$.

The exact same argument applies to $v(r)$ and $v_0(r)$, so one similarly has

$$\mathcal{Y}_\infty^{p-2} \lim_{r\to\infty} \frac{v''(r)}{v_0''(r)} = \lim_{r\to\infty} Q_2(\mathcal{V}(r), v_0(r)) \cdot Q_3(\mathcal{W}(r), u_0'(r))$$

in analogy with (2.57). Since $\mathcal{Y}_\infty = \mathcal{V}_\infty$ by above, our previous approach then gives

$$\mathcal{V}_\infty^{p-1} = \mathcal{V}_\infty^{k_2} \cdot \mathcal{W}_\infty^{k_3} = \mathcal{V}_\infty^{k_2} \cdot \mathcal{U}_\infty^{k_3} \tag{2.59}$$

in analogy with (2.58). Once we combine (2.59) with (2.58), we find that

$$\mathcal{V}_\infty^{(p-1-k_2)(p-1-\sigma)} = \mathcal{U}_\infty^{k_3(p-1-\sigma)} = \mathcal{V}_\infty^{k_1k_3},$$

where $\mathcal{V}_\infty \geq 1$ by Lemma 2.2.3. In view of (2.55), the only possibility is then $\mathcal{V}_\infty = 1$, which also implies that $\mathcal{U}_\infty = 1$. This is precisely the assertion of the theorem. ■

2.2.4 Proof of Theorem 2.2.1

First of all, we seek solutions of the associated system (2.40) that have the form

$$u(r) = C_\alpha r^\alpha, \qquad v(r) = C_\beta r^\beta. \tag{2.60}$$

In view of the definition of the p-Laplace operator, it is easy to check that

$$\Delta_p(C_\alpha r^\alpha) = (\alpha C_\alpha)^{p-1} \left(\alpha(p-1) + n - p\right) \cdot r^{(p-1)\alpha - p}. \tag{2.61}$$

Inserting this identity in (2.40) and comparing powers of r, one obtains the system

$$\begin{aligned} (p-1-\sigma)\alpha - k_1\beta &= p - \sigma + m_1, \\ (p-1-k_2)\beta - k_3\alpha &= p - k_3 + m_2. \end{aligned}$$

Since (2.36) holds, this system has a unique solution (α, β) which is given by

$$\begin{aligned} \alpha &= \frac{(p-\sigma+m_1)(p-1-k_2) + k_1(p-k_3+m_2)}{(p-1-\sigma)(p-1-k_2) - k_1k_3} > 1, \\ \beta &= \frac{(p-1-\sigma)(p+m_2) + k_3(1+m_1)}{(p-1-\sigma)(p-1-k_2) - k_1k_3} > 1. \end{aligned}$$

Using (2.61) once again, we find that (2.60) is a solution of (2.40) if and only if

$$(\alpha C_\alpha)^{p-1} \cdot (\alpha(p-1)+n-p) = c_1 C_\beta^{k_1} \cdot (\alpha C_\alpha)^\sigma,$$
$$(\beta C_\beta)^{p-1} \cdot (\beta(p-1)+n-p) = c_2 C_\beta^{k_2} \cdot (\alpha C_\alpha)^{k_3}.$$

Let us set $B_\alpha = \alpha(p-1)+n-p$ for simplicity. Then we get the equivalent system

$$(\alpha C_\alpha)^{p-1-\sigma} = \frac{c_1}{\beta^{k_1} B_\alpha} (\beta C_\beta)^{k_1}, \qquad (\beta C_\beta)^{p-1-k_2} = \frac{c_2}{\beta^{k_2} B_\beta} (\alpha C_\alpha)^{k_3}.$$

Since (2.36) holds, this system has a unique solution C_α, C_β which is given by

$$(\alpha C_\alpha)^{(p-1-\sigma)(p-1-k_2)-k_1k_3} = \left(\frac{c_1}{\beta^{k_1} B_\alpha}\right)^{p-1-k_2} \left(\frac{c_2}{\beta^{k_2} B_\beta}\right)^{k_1},$$
$$(\beta C_\beta)^{(p-1-\sigma)(p-1-k_2)-k_1k_3} = \left(\frac{c_2}{\beta^{k_2} B_\beta}\right)^{p-1-\sigma} \left(\frac{c_1}{\beta^{k_1} B_\alpha}\right)^{k_3}.$$

The exact values of $\alpha, \beta, C_\alpha, C_\beta$ do not play any role in our approach, as we shall simply resort to our general Theorems 2.2.4 and 2.2.5 which apply for arbitrary solutions (u_0, v_0) that vanish at the origin. In this case, our solution (u_0, v_0) is given by (2.60), so

$$\lim_{r\to 0^+} \frac{u_0(r)}{u_0'(r)} = \lim_{r\to 0^+} \frac{r}{\alpha} = 0, \qquad \lim_{r\to 0^+} \frac{v_0(r)}{v_0'(r)} = \lim_{r\to 0^+} \frac{r}{\beta} = 0$$

and condition (2.45) holds. Let us then resort to Theorem 2.2.4 for the special case

$$f_i(r) = c_i r^{m_i}, \qquad h_j(s) = s^{k_j},$$

where $i = 1, 2$ and $j = 1, 2, 3$. Our assumptions (D1)-(D2) hold trivially, while

$$Q_j(s,t) = \frac{g_j(st)}{h_j(t)} = \frac{g_j(st)}{(st)^{k_j}} \cdot s^{k_j}$$

is non-increasing in t because of (C3). It also follows by condition (2.39) that

$$\lim_{t\to\infty} Q_j(s,t) = \lim_{t\to\infty} \frac{g_j(st)}{(st)^{k_j}} \cdot s^{k_j} = s^{k_j}$$

for each $s > 0$, so our assumptions (D1)-(D5) are all valid. Thus, part (a) of Theorem 2.2.1 follows by Theorem 2.2.4, while part (b) of Theorem 2.2.1 follows by Theorem 2.2.5.

Chapter 3

Hénon-Lane-Emden System

This chapter is based on the results of [21], which were jointly obtained with Paschalis Karageorgis.

3.1 Background and Main Results

In this chapter we study the positive radial steady states of the parabolic Hénon-Lane-Emden system

$$\begin{cases} u_t - \Delta u = |x|^k v^p & \text{in } \mathbb{R}^n \times (0, \infty), \\ v_t - \Delta v = |x|^l u^q & \text{in } \mathbb{R}^n \times (0, \infty), \end{cases} \tag{3.1}$$

where $k, l \geq 0$, $p, q \geq 1$ and $pq > 1$. In particular we are interested in the stability of these steady states. Concerning notation, note that we have now switched to using the letter p to denote an exponent. When it comes to the scalar equation $u_t - \Delta u = u^p$, in the classical paper by Gui, Ni and Wang [46] the authors show there is a critical power p_{JL} which marks a dichotomy between the positive radial steady states. If $1 < p < p_{JL}$, they are unstable, and the graphs of any two steady states intersect one another. If $p \geq p_{JL}$, they are stable, and the graphs of distinct steady states do not intersect. In this chapter we establish an analogue of the latter statement for the parabolic system (3.1).

The positive steady states of (3.1) satisfy the elliptic Hénon-Lane-Emden system

$$\begin{cases} -\Delta u = |x|^k v^p & \text{in } \mathbb{R}^n, \\ -\Delta v = |x|^l u^q & \text{in } \mathbb{R}^n, \end{cases} \tag{3.2}$$

which has been extensively studied over the last three decades. Starting with the seminal papers by Mitidieri [72] and Serrin and Zou [81, 83] for the case $k = l = 0$, it is now known that positive regular radial solutions exist if and only if

$$\frac{k+n}{p+1} + \frac{l+n}{q+1} \leq n - 2. \tag{3.3}$$

See [8] for the case $k, l \geq 0$. The same result is expected to be true for non-radial solutions, and has become known in the literature as the Hénon-Lane-Emden conjecture. The conjecture has been settled in dimensions $n \leq 4$, but only partial results are available in higher dimensions [12, 79, 85]. We remark that the case $k = l = 0$ is of most interest in the literature, and in that case is known as the Lane-Emden conjecture.

When it comes to understanding the qualitative properties of positive radial solutions of (3.2), a key role is played by an explicit radial singular solution (u_*, v_*) which has the form

$$u_*(r) = C_\alpha r^{-\alpha} = C_\alpha |x|^{-\alpha}, \qquad v_*(r) = C_\beta r^{-\beta} = C_\beta |x|^{-\beta}. \tag{3.4}$$

As in Chapter 2, we will use the symbols $\alpha, \beta, C_\alpha, C_\beta$ to denote explicit positive parameters. Note, however, the distinction between the two cases. For the coercive systems studied in Chapter 2, we had an explicit radial solution which was regular at the origin and growing, whereas here we have an explicit radial solution which is singular at the origin and decaying.

Assuming that (3.3) holds, (u_*, v_*) is easily seen to satisfy (3.2) at all points $x \neq 0$, if

$$\alpha = \frac{k+2+(l+2)p}{pq-1}, \qquad \beta = \frac{l+2+(k+2)q}{pq-1} \tag{3.5}$$

and the constants C_α, C_β are given by

$$C_\alpha^{pq-1} = Q(\alpha)Q(\beta)^p, \qquad C_\beta^{pq-1} = Q(\beta)Q(\alpha)^q, \tag{3.6}$$

where $Q(\lambda) = \lambda(n-2-\lambda)$ for all $\lambda \in \mathbb{R}$. Note that the existence condition (3.3) can also be expressed in the equivalent form $\alpha + \beta \leq n - 2$, so in particular $\alpha, \beta \in (0, n-2)$ since $pq > 1$. This also means that both $Q(\alpha)$ and $Q(\beta)$ are positive, so the coefficients C_α, C_β are well-defined as long as $pq > 1$ and (3.3) holds. The values α, β are scaling exponents for solutions of (3.1). Indeed, one can easily verify that for any $\lambda > 0$ and any solution (u, v) of (3.1), then $(u_\lambda, v_\lambda) = (\lambda^\alpha u(\lambda x, \lambda^2 t), \lambda^\beta v(\lambda x, \lambda^2 t))$ is also a solution.

The focus of this chapter is on the positive radial steady states of system (3.1), and as we shall see the singular solution (3.4) plays a key role in the analysis. Note the analogy with the results of Section 2.2, where our explicit regular solution played a central role. The singular solution (3.4) is also relevant to the study of existence of solutions of the parabolic system (3.1), which we shall now briefly discuss for completeness.

The existence and nonexistence of classical solutions of (3.1) was initially addressed in the case $k = l = 0$ in the classical paper by Escobedo and Herrero [28], while analogues of their results have since been proved for $k, l \geq 0$, as well as in more general settings [68, 73]. Assuming non-negative, bounded and continuous initial data $u(x, 0) = u_0(x), v(x, 0) = v_0(x)$, the existence of local in time classical solutions follows by standard fixed point arguments. Denote by $T = T(u_0, v_0)$ the maximal existence time of solutions for given initial data (u_0, v_0). If solutions fail to be global in time, then by standard parabolic theory one necessarily has

$$\lim_{t \to T^-} (||u(\cdot, t)||_{L^\infty(\mathbb{R}^n)} + ||v(\cdot, t)||_{L^\infty(\mathbb{R}^n)}) = \infty,$$

where $0 < T < \infty$. The main (non)existence results of [28] are as follows.

Theorem ([28])**.** *(i) Suppose max*$\{\alpha, \beta\} \geq n$. *Then* $T(u_0, v_0) < \infty$ *for any non-trivial solution of* (3.1).

(ii) Suppose max$\{\alpha, \beta\} < n$, *and either*

$$\liminf_{|x| \to \infty} |x|^a u_0 > 0 \quad \textit{or} \quad \liminf_{|x| \to \infty} |x|^b v_0 > 0$$

where $a < \alpha$ *and* $b < \beta$. *Then* $T(u_0, v_0) < \infty$ *for any non-trivial solution of* (3.1).

(iii) Suppose max$\{\alpha, \beta\} < n$, *and both*

$$\limsup_{|x| \to \infty} |x|^a u_0 < \infty \quad \textit{and} \quad \limsup_{|x| \to \infty} |x|^b v_0 < \infty$$

where $a > \alpha$ *and* $b > \beta$. *Then* $T(u_0, v_0) = \infty$ *provided* u_0, v_0 *are sufficiently small in an appropriate norm.*

Part (i) above tells us that, in order for global solutions to exist, we require the singular solution (3.4) to be integrable near 0. Moreover, from (3.3) we know that $\max\{\alpha, \beta\} \geq n$ means no regular radial solutions of (3.2) can exist either. This seems to be related to the results of [71, 86], which tell us that under certain assumptions (such as radial symmetry of solutions), finite time blow up can only occur at the origin. For global in time solutions of (3.1) to exist, we also need the initial data (u_0, v_0) to decay fast enough at infinity, and this decay rate is dictated by the singular solution (u_*, v_*). According to (ii), for global solutions to exist, we require that (u_0, v_0) decay at least as fast as (u_*, v_*) near infinity. Part (iii) tells us that the decay rate dictated by the singular solution (3.4) is optimal, which the results of this chapter will expand upon.

Returning to the problem at hand, we make the following definitions.

Definition. A solution (u, v) of (3.2) is called *linearly stable* if there exists a positive supersolution of the linearised system, i.e there exists $(\phi, \psi) \in C^2(\mathbb{R}^n)^2$ such that

$$\begin{cases} -\Delta\phi \geq p|x|^k v^{p-1}\psi & \text{in } \mathbb{R}^n, \\ -\Delta\psi \geq q|x|^l u^{q-1}\phi & \text{in } \mathbb{R}^n, \\ \quad \phi, \psi > 0 & \text{in } \mathbb{R}^n. \end{cases}$$

Definition. We shall refer to a solution (u, v) of (3.1) as *nonlinearly stable*, or simply as *stable*, with respect to a norm $||\cdot||$ if for all $\varepsilon > 0$ there is a $\delta > 0$ such that

$$||(u(x,0), v(x,0)) - (\tilde{u}(x,0), \tilde{v}(x,0))|| < \delta$$

implies

$$||(u(x,t), v(x,t)) - (\tilde{u}(x,t), \tilde{v}(x,t))|| < \varepsilon$$

at all times $t > 0$, for any other solution $(\tilde{u}, \tilde{v})$.

The above definition of nonlinear stability is obviously standard, and we remark that the definition of linear stability is equivalent to other more standard definitions of stability for elliptic problems. The above formulation seems to have been introduced by Montenegro [74], and we refer to the book [26] by Dupaigne for a comprehensive treatment of the stability of elliptic problems. The linear stability of regular positive radial solutions of (3.2) has been addressed by Chen, Dupaigne and Ghergu [14] (see also [16] for the non-radial case). The authors of [14] showed that if

$$\left[\frac{(n-2)^2 - (\alpha-\beta)^2}{4}\right]^2 \geq pqQ(\alpha)Q(\beta), \tag{3.7}$$

then all regular positive radial solutions are linearly stable, and if (3.7) does not hold, then none of them are linearly stable. Following their lead, we shall refer to (3.7) as the Joseph-Lundgren condition. It is the exact analogue of a condition that arose in the phase plane analysis of [54] for the scalar equation $-\Delta u = u^p$, and it only holds (for some p, q) in dimensions $n \geq 11$.

The left hand side of (3.7) appears as the best constant in a Hardy-Rellich type inequality that was obtained by Caldiroli and Musina [13]. In particular, in the class of radially symmetric functions $\varphi = \varphi(|x|)$ satisfying $\varphi \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$ one has

$$\left[\frac{(n-2)^2 - (\alpha-\beta)^2}{4}\right]^2 = \inf_{\varphi \neq 0} \frac{\int_\Omega |x|^{2-(\alpha-\beta)} |\Delta\varphi|^2 dx}{\int_\Omega |x|^{-2-(\alpha-\beta)} \varphi^2 dx}.$$

Its relationship with the linear stability of radial solutions is explained in [14] for systems, in [57] for the equation $-\Delta u = u^p$, and in [55] for the biharmonic equation $\Delta^2 u = u^p$ which corresponds to the case $p > q = 1$.

The Joseph-Lundgren condition is also closely related to the intersection properties of radial solutions. Assuming (3.7), in particular, the authors of [14] showed that every regular positive radial solution (u, v) satisfies

$$u_*(r) > u(r) \quad \text{and} \quad v_*(r) > v(r) \quad \text{for all } r > 0. \tag{3.8}$$

This is sometimes called the separation property for radial solutions. It was first derived by Wang [87] for the equation $-\Delta u = u^p$, and by Karageorgis [55] for the biharmonic equation $\Delta^2 u = u^p$. For the two scalar equations, it is by now well known that the separation property holds if and only if (3.7) does; see [30, 87] for more details. As a simple consequence of the results of [14], this is also true for system (3.2) in the case $k = l = 0$. The separation structure of radial solutions is an interesting question on its own, and has been extensively studied for generalisations of the above equations in $\mathbb{R}^n$, as well as on non-compact Riemannian manifolds more generally; see for instance [5, 50, 69].

The monotonic convergence of radial solutions for the scalar equation $-\Delta u = u^p$ is one of the key ingredients in the classical paper by Gui, Ni and Wang [46] towards the proof of nonlinear stability of steady states for the associated parabolic equation $u_t - \Delta u = u^p$. Here we establish the analogue for the case of system (3.2).

Theorem 3.1.1 (Monotonic convergence)**.** *Consider the elliptic Hénon-Lane-Emden system* (3.2)*, where $k, l \geq 0$, $p, q \geq 1$, $pq > 1$ and $n \geq 11$. Assume the existence condition* (3.3) *and the Joseph-Lundgren condition* (3.7)*. If (u, v) is a regular positive radial solution of* (3.2) *and* (u_*, v_*) *is the singular solution* (3.4), *then* u/u_* *and* v/v_* *are increasing with*

$$\lim_{r\to\infty} \frac{u(r)}{u_*(r)} = \lim_{r\to\infty} \frac{v(r)}{v_*(r)} = 1. \tag{3.9}$$

Remark 3.1.2. As discussed previously, when it comes to the scalar equation $-\Delta u = u^p$, $p > 1$, we have the singular solution $u_* = C_2|x|^{-\frac{2}{p-1}}$. Every positive radial solution u behaves like u_* at infinity. However, the convergence $u/u_* \to 1$ is monotonic only for supercritical powers. Although not originally expressed this way, the convergence is monotonic when

$$Q\left(\frac{n-2}{2}\right) \geq p \cdot Q\left(\frac{2}{p-1}\right). \tag{3.10}$$

A similar result holds for the scalar equation $\Delta^2 u = u^p$ which has a singular solution of the form $u_* = C_4|x|^{-\frac{4}{p-1}}$. Once again, every positive radial solution u behaves like u_* at infinity, but the convergence $u/u_* \to 1$ is monotonic only for supercritical powers p. In this case, the convergence is monotonic when

$$Q\left(\frac{n-4}{2}\right) \cdot Q\left(\frac{n}{2}\right) \geq p \cdot Q\left(\frac{4}{p-1}\right) \cdot Q\left(\frac{2(p+1)}{p-1}\right). \tag{3.11}$$

When $k = l = 0$, condition (3.7) reduces to (3.10) for $u = v$ and $p = q$, and it reduces to (3.11) when $p > q = 1$.

Following a similar approach to that in [46], we employ Theorem 3.1.1 to prove nonlinear stability of steady states, more generally, for the parabolic Hénon-Lane-Emden system (3.1). As in [46], we need to find the precise asymptotic expansion for all radial solutions at infinity. Assuming that strict inequality holds in the Joseph-Lundgren condition (3.7), we show that

$$\begin{aligned} u(r) &= C_\alpha r^{-\alpha} + D_1 r^{-\alpha-\gamma} + o(r^{-\alpha-\gamma}) \qquad \text{as } r \to \infty, \\ v(r) &= C_\beta r^{-\beta} + D_2 r^{-\beta-\gamma} + o(r^{-\beta-\gamma}) \qquad \text{as } r \to \infty, \end{aligned} \tag{3.12}$$

for some constants $D_1, D_2 < 0 < \gamma$. This asymptotic expansion was derived in [46, 63] for the equation $-\Delta u = u^p$ and in [56, 88] for the biharmonic equation $\Delta^2 u = u^p$. To prove nonlinear stability for the parabolic system (3.1), we are thus led to introduce the norm

$$||(u, v)||_\gamma = \sup_{x\in\mathbb{R}^n} \left|(1 + |x|)^{\alpha+\gamma} u(x)\right| + \left|(1 + |x|)^{\beta+\gamma} v(x)\right|. \tag{3.13}$$

If equality holds in (3.7), the asymptotic expansion (3.12) needs to be slightly modified to include logarithmic factors; see Theorem 3.3.1. In that case, a suitable norm is then

$$|||(u,v)|||_\gamma = \sup_{x\in\mathbb{R}^n} \left| \frac{(1+|x|)^{\alpha+\gamma}u(x)}{\log(2+|x|)} \right| + \left| \frac{(1+|x|)^{\beta+\gamma}v(x)}{\log(2+|x|)} \right|. \tag{3.14}$$

Theorem 3.1.3 (Nonlinear stability)**.** *Let the assumptions of the previous theorem hold, and consider the parabolic Hénon-Lane-Emden system* (3.1)*.*

(a) If strict inequality holds in (3.7)*, then* (u,v) *is a stable solution of* (3.1) *with respect to the norm* (3.13) *for some* $\gamma > 0$*.*

(b) If equality holds in (3.7)*, then* (u,v) *is a stable solution of* (3.1) *with respect to the norm* (3.14) *for some* $\gamma > 0$*.*

In each case, $\gamma > 0$ *is an explicit constant which is defined in Lemmas 3.2.2 and 3.2.4.*

To the best of our knowledge, the only scalar analogue of Theorem 3.1.3 appeared in the classical paper by Gui, Ni and Wang [46]. There are other related results, however, which establish the existence of small-amplitude solutions. In some sense, these can be regarded as nonlinear stability results for the zero solution. We refer the reader to [37] for the scalar biharmonic equation $u_t + \Delta^2 u = |u|^{p-1}u$ and to [28, 68] for the parabolic system (3.1).

3.2 Preliminary results

Lemma 3.2.1. *Suppose* (u,v) *is a regular positive radial solution of the elliptic system* (3.2)*, where* $k,l \geq 0$*,* $p,q \geq 1$ *and* $pq > 1$*. Then both* $u(r)$ *and* $v(r)$ *are decreasing with*

$$\lim_{r\to\infty} u(r) = \lim_{r\to\infty} v(r) = 0. \tag{3.15}$$

Proof: Since (u,v) is a regular positive radial solution of (3.2), $u(r)$ satisfies

$$-(r^{n-1}u'(r))' = r^{n-1+k}\, v(r)^p > 0 \quad \text{for all } r \geq 0$$

and also $u'(0) = 0$. Thus, $u(r)$ is decreasing for all $r \geq 0$, and the first limit in (3.15) exists. Let us denote this limit by $L \geq 0$. Then $u(r) \geq L$ for all $r \geq 0$, so we have

$$-(r^{n-1}v'(r))' = r^{n-1+l}\, u(r)^q \geq L^q r^{n-1+l} \quad \text{for all } r \geq 0.$$

Integrating this equation twice, we conclude that

$$-v(r)+v(0) \geq \frac{L^q r^{l+2}}{(n+l)(l+2)} \quad \text{for all } r \geq 0.$$

Since $l \geq 0$ and $v(r)$ is positive for all $r \geq 0$, this implies that $L=0$. In other words, the first limit in (3.15) is zero. The second limit must also be zero for similar reasons. ■

To study the linearised equations associated with the elliptic system (3.2), we shall need a basic fact about the associated characteristic polynomial. This has the form

$$F(\lambda) = Q(\alpha-\lambda)Q(\beta-\lambda) - pqQ(\alpha)Q(\beta), \tag{3.16}$$

where $Q(\lambda) = \lambda(n-2-\lambda)$ for each $\lambda \in \mathbb{R}$. Since the quadratic $Q(\lambda)$ is symmetric around the point $\frac{n-2}{2}$, we see that the quartic $F(\lambda)$ is symmetric around the point

$$\lambda_* = \frac{\alpha+\beta}{2} - \frac{n-2}{2}. \tag{3.17}$$

We now exploit this fact to draw some conclusions about the associated eigenvalues.

Lemma 3.2.2. *Suppose $p,q \geq 1$ and $pq > 1$, $\alpha,\beta > 0$ and $\alpha+\beta \leq n-2$. Suppose also that the Joseph-Lundgren condition* (3.7) *holds. Then $\lambda_* < 0$ and the polynomial* (3.16) *has a root in each of the intervals $(-\infty, 2\lambda_*)$, $(2\lambda_*, \lambda_*]$, $[\lambda_*, 0)$ and $(0,\infty)$. Moreover, λ_* is itself a root if and only if* (3.7) *holds with equality, in which case λ_* is actually a double root.*

Proof: Since $\alpha+\beta \leq n-2$ by assumption, we have $\lambda_* \leq 0$ by (3.17). Note that the quartic polynomial (3.16) has leading term λ^4, and it is symmetric around λ_* with

$$\lim_{\lambda\to\pm\infty} F(\lambda) = +\infty, \qquad F(2\lambda_*) = F(0) = (1-pq)Q(\alpha)Q(\beta) < 0$$

because $pq > 1$ and $0 < \alpha,\beta < n-2$. In addition, it is easy to check that

$$Q(\alpha-\lambda_*) = Q(\beta-\lambda_*) = \frac{(n-2)^2-(\alpha-\beta)^2}{4}.$$

Thus, the Joseph-Lundgren condition (3.7) can also be expressed in the form $F(\lambda_*) \geq 0$. Since $F(0) < 0$ by above, this implies that $\lambda_* \neq 0$ and thus $\lambda_* < 0$. It trivially follows by Bolzano's theorem that $F(\lambda)$ has a root in each of the intervals

$$(-\infty, 2\lambda_*), \qquad (2\lambda_*, \lambda_*], \qquad [\lambda_*, 0), \qquad (0,\infty).$$

Finally, if λ_* itself happens to be a root, then it must be a double root by symmetry. $\blacksquare$

Remark 3.2.3. If the Joseph-Lundgren condition (3.7) does not hold, then the quartic $F(\lambda)$ has a W-shaped graph which is symmetric around λ_* with $F(\lambda_*) < 0$. Thus, its unique local maximum lies below the λ-axis, and $F(\lambda)$ has two real roots only.

Lemma 3.2.4. *Let the assumptions of the previous lemma hold, and let $-\gamma < 0$ denote the largest negative root of the polynomial* (3.16). *Then the linearised system*

$$\begin{cases} -\Delta\phi = p|x|^k v_*^{p-1}\psi \\ -\Delta\psi = q|x|^l u_*^{q-1}\phi \end{cases} \qquad \text{in } \mathbb{R}^n \setminus \{0\} \tag{3.18}$$

which is associated with the singular solution (3.4) *is satisfied by the functions*

$$\phi(x) = \frac{pC_\beta^{p-1}}{Q(\alpha+\gamma)} \cdot |x|^{-\alpha-\gamma}, \qquad \psi(x) = |x|^{-\beta-\gamma}. \tag{3.19}$$

Proof: By Lemma 3.2.2, the largest negative root $-\gamma$ lies in the interval $[\lambda_*, 0)$, where λ_* is defined by (3.17). This is easily seen to imply that $Q(\alpha+\gamma) > 0$ because

$$\alpha + \gamma \leq \alpha - \lambda_* = \frac{n-2}{2} + \frac{\alpha-\beta}{2} < \frac{n-2}{2} + \frac{\alpha+\beta}{2} \leq n-2. \tag{3.20}$$

Using the fact that $-\Delta|x|^{-\lambda} = Q(\lambda)|x|^{-\lambda-2}$ for all $\lambda \in \mathbb{R}$, we now find that

$$-\Delta\phi = pC_\beta^{p-1} \cdot |x|^{-\alpha-\gamma-2} = pv_*^{p-1} \cdot |x|^{\beta(p-1)-\alpha-\gamma-2} = p|x|^k v_*^{p-1}\psi$$

by (3.4), (3.5) and (3.19). Using the exact same argument, we also have

$$-\Delta\psi = Q(\beta+\gamma) \cdot |x|^{-\beta-\gamma-2} = \frac{Q(\alpha+\gamma)Q(\beta+\gamma)}{pqC_\alpha^{q-1}C_\beta^{p-1}} \cdot q|x|^l u_*^{q-1}\phi.$$

On the other hand, $-\gamma$ is a root of the characteristic polynomial (3.16), so

$$\begin{aligned} 0 = F(-\gamma) &= Q(\alpha+\gamma)Q(\beta+\gamma) - pqQ(\alpha)Q(\beta) \\ &= Q(\alpha+\gamma)Q(\beta+\gamma) - pqC_\alpha^{q-1}C_\beta^{p-1} \end{aligned}$$

by (3.6). Combining the last two equations, we conclude that $-\Delta\psi = q|x|^l u_*^{q-1}\phi$. $\blacksquare$

The last result in this section plays a key role in our subsequent analysis. This result is due to Chen, Dupaigne and Ghergu [14]. The proof given below is only a minor modification of the original proof, and it is included here mostly for the sake of completeness.

Lemma 3.2.5. *Let* (u, v) *be a regular positive radial solution of the elliptic system* (3.2), *where* $k, l \geq 0$, $p, q \geq 1$, $pq > 1$ *and* $n \geq 11$. *Let* (u_*, v_*) *be the singular solution* (3.4). *Assume the existence condition* (3.3) *and the Joseph-Lundgren condition* (3.7). *Then*

$$u_*(r) > u(r) \quad \textit{and} \quad v_*(r) > v(r) \quad \textit{for all } r > 0. \tag{3.21}$$

Proof: We consider the difference between the two solutions and let

$$U(r) = u_*(r) - u(r), \qquad V(r) = v_*(r) - v(r). \tag{3.22}$$

Suppose first that one of $U'(r)$, $V'(r)$ does not have a zero. We may then assume, without loss of generality, that $U'(r) < 0$ for all $r > 0$. Since $\lim_{r\to\infty} U(r) = 0$ by Lemma 3.2.1, this implies that $U(r) > 0$ for all $r > 0$. In particular, $u_*(r) > u(r)$ for all $r > 0$, hence

$$(r^{n-1}V'(r))' = r^{n-1+l}\,(u(r)^q - u_*(r)^q) < 0 \quad \text{for all } r > 0.$$

On the other hand, the existence condition (3.3) ensures that $\beta < \alpha + \beta \leq n - 2$, so

$$\lim_{r\to 0^+} r^{n-1}V'(r) = \lim_{r\to 0^+} \beta C_\beta r^{n-2-\beta} = 0.$$

We conclude that $V'(r) < 0$ for all $r > 0$. Thus, $V(r)$ decreases to zero by Lemma 3.2.1, so we also have $v_*(r) > v(r)$ for all $r > 0$, and the result follows.

Suppose now that both $U'(r)$ and $V'(r)$ vanish at some point. Letting r_1 and r_2 denote the first zeros of $U'(r)$ and $V'(r)$, respectively, we must then have

$$U'(r) < 0 \quad \text{in } (0, r_1), \qquad V'(r) < 0 \quad \text{in } (0, r_2) \tag{3.23}$$

and also $U'(r_1) = V'(r_2) = 0$. Assume that $r_1 \leq r_2$ without loss of generality, and let

$$\begin{aligned} r_3 &= \inf\{r > 0 : V(r) < 0\}, \\ r_4 &= \inf\{r > 0 : U(r) < 0\}. \end{aligned}$$

If it happens that $r_1 \leq r_3$, then $V(r) > 0$ in $(0, r_1)$, and this implies that

$$(r^{n-1}U'(r))' = r^{n-1+k}\,(v(r)^p - v_*(r)^p) < 0 \quad \text{in } (0, r_1),$$

which forces $r^{n-1}U'(r)$ to be positive in $(0, r_1)$, a contradiction. We conclude that $r_3 < r_1$. Interchanging the roles of $U(r)$ and $V(r)$, one similarly finds that $r_4 < r_2$.

Next, we show that $r_4 \leq r_1$. If it happens that $r_1 < r_4$, then $r_1 < r_4 < r_2$. On the other hand, $V(r)$ is strictly decreasing in $(0, r_2)$ by (3.23), so it is negative in (r_3, r_2) and thus

$$(r^{n-1}U'(r))' = r^{n-1+k}\,(v(r)^p - v_*(r)^p) > 0 \quad \text{in } (r_3, r_2).$$

Since $r_3 < r_1 < r_2$ and $U'(r_1) = 0$ by definition, $U(r)$ is strictly increasing on (r_1, r_2). Noting that this interval contains r_4, we find that $U(r_1) < U(r_4) = 0$, which obviously contradicts the definition of r_4. We must thus have $r_4 \leq r_1 \leq r_2$ and $r_3 < r_1 \leq r_2$. Since $U(r)$ is strictly decreasing on $(0, r_1)$ and $V(r)$ is strictly decreasing on $(0, r_2)$, this also implies

$$U(r_1) \leq 0, \qquad U'(r_1) = 0, \qquad V(r_1) < 0, \qquad V'(r_1) \leq 0. \tag{3.24}$$

Finally, we use a convexity argument to finish the proof. Since $p, q \geq 1$, we have

$$\begin{cases} -\Delta U = |x|^k\,(v_*^p - v^p) \leq p|x|^k v_*^{p-1} V \\ -\Delta V = |x|^l\,(u_*^q - u^q) \leq q|x|^l u_*^{q-1} U \end{cases} \qquad \text{in } \mathbb{R}^n \setminus \{0\}.$$

Let (ϕ, ψ) be the positive radial solution of the linearised system which is defined by (3.19). Since this is a positive solution of (3.18), it follows by the last equation that

$$\begin{cases} -\psi\Delta U \leq p|x|^k v_*^{p-1} V\psi = -V\Delta\phi \\ -\phi\Delta V \leq q|x|^l u_*^{q-1} U\phi = -U\Delta\psi \end{cases} \qquad \text{in } \mathbb{R}^n \setminus \{0\}.$$

Adding the two inequalities and switching to polar coordinates, we conclude that

$$\left[r^{n-1}(U\psi' - \psi U')\right]' + \left[r^{n-1}(V\phi' - \phi V')\right]' \leq 0 \quad \text{for all } r > 0. \tag{3.25}$$

To check that the expressions in square brackets both vanish as $r \to 0^+$, we now recall our definitions (3.19) and (3.22) of ϕ, ψ, U, V. These are easily seen to imply that

$$r^{n-1}(U\psi' - \psi U') + r^{n-1}(V\phi' - \phi V') = O(r^{n-2-\alpha-\beta-\gamma}) \quad \text{as } r \to 0^+.$$

On the other hand, it follows by Lemma 3.2.2 and our computation (3.20) that

$$\alpha + \beta + \gamma \leq \alpha + \beta - \lambda_* = \lambda_* + n - 2 < n - 2.$$

In view of the last two equations, one may then integrate (3.25) to arrive at

$$U\psi' - \psi U' + V\phi' - \phi V' \leq 0 \quad \text{for all } r > 0.$$

This actually contradicts (3.24) because $\phi, \psi > 0$ and since $\phi', \psi' < 0$ for all $r > 0$. ■

3.3 Asymptotic behaviour of regular solutions

In this section, we study the asymptotic behaviour of the regular positive radial solutions of the elliptic system (3.2). Let (u, v) be such a solution, and let (u_*, v_*) be the singular solution (3.4). In order to establish Theorem 3.1.1, we need to show that

$$\lim_{r\to\infty} \frac{u(r)}{u_*(r)} = \lim_{r\to\infty} \frac{v(r)}{v_*(r)} = 1, \tag{3.26}$$

and that the convergence occurs monotonically in each case.

Proof of Theorem 3.1.1. We use the logarithmic change of variables

$$\mathcal{U}(s) = \frac{u(e^s)}{u_*(e^s)} = \frac{u(e^s)}{C_\alpha e^{-\alpha s}}, \qquad \mathcal{V}(s) = \frac{v(e^s)}{v_*(e^s)} = \frac{v(e^s)}{C_\beta e^{-\beta s}}, \tag{3.27}$$

where $s = \log r$. This is the standard Emden-Fowler transformation that has been used by several authors in the past [6, 36, 46, 55], although the constants C_α, C_β are usually omitted for simplicity. We have chosen to include them here in order to prove (3.26) directly.

First of all, we note that $0 < \mathcal{U}(s), \mathcal{V}(s) < 1$ by Lemma 3.2.5. Since (u, v) is a positive radial solution of the elliptic system (3.2), it is easy to check that $(\mathcal{U}, \mathcal{V})$ satisfies

$$\begin{cases} \mathcal{U}''(s) + (n-2-2\alpha)\,\mathcal{U}'(s) = Q(\alpha)\,(\mathcal{U}(s) - \mathcal{V}(s)^p) \\ \mathcal{V}''(s) + (n-2-2\beta)\,\mathcal{V}'(s) = Q(\beta)\,(\mathcal{V}(s) - \mathcal{U}(s)^q) \end{cases} \tag{3.28}$$

for all $s \in \mathbb{R}$. To study this autonomous system, we consider the sets

$$S_\mathcal{U} = \{s \in \mathbb{R} : \mathcal{U}(s) > \mathcal{V}(s)^p\}, \qquad S_\mathcal{V} = \{s \in \mathbb{R} : \mathcal{V}(s) > \mathcal{U}(s)^q\}. \tag{3.29}$$

If there exists a point $s_0 \notin S_\mathcal{U} \cup S_\mathcal{V}$, then $\mathcal{U}(s_0) \leq \mathcal{V}(s_0)^p \leq \mathcal{U}(s_0)^{pq}$, and this leads to the contradiction $\mathcal{U}(s_0) \geq 1$. In particular, no such point exists and $S_\mathcal{U} \cup S_\mathcal{V} = \mathbb{R}$.

Next, we claim that $\mathcal{U}'(s), \mathcal{V}'(s) \geq 0$ for all $s \in \mathbb{R}$. This is true as $s \to -\infty$ because

$$\lim_{s \to -\infty} e^{-\alpha s} \mathcal{U}'(s) = \lim_{r \to 0^+} r^{1-\alpha} \left(\frac{1}{C_\alpha} r^\alpha u(r) \right)' = \frac{\alpha u(0)}{C_\alpha} > 0,$$

and a similar statement holds for $\mathcal{V}'(s)$. Suppose that one of these functions becomes negative at some point. We may then assume, without loss of generality, that $\mathcal{U}'(s)$ is the first one to become negative. In that case, there exist $s_1 \in \mathbb{R}$ and $\delta_1 > 0$ such that

$$\mathcal{U}'(s), \mathcal{V}'(s) \geq 0 \quad \text{on } (-\infty, s_1], \qquad \mathcal{U}'(s) < 0 \quad \text{on } (s_1, s_1 + \delta_1). \tag{3.30}$$

Combining the definition of s_1 with (3.28), we now find that

$$0 \geq \mathcal{U}''(s_1) = \mathcal{U}''(s_1) + (n - 2 - 2\alpha) \mathcal{U}'(s_1) = Q(\alpha) \left(\mathcal{U}(s_1) - \mathcal{V}(s_1)^p \right). \tag{3.31}$$

This implies $s_1 \notin S_{\mathcal{U}}$ and thus $s_1 \in S_{\mathcal{V}}$. Since $S_{\mathcal{V}}$ is open, we have $(s_1 - \delta_2, s_1 + \delta_2) \subset S_{\mathcal{V}}$ for some small enough $0 < \delta_2 \leq \delta_1$. It follows by (3.28) and (3.29) that $e^{(n-2-2\beta)s} \mathcal{V}'(s)$ is strictly increasing on this interval, hence $\mathcal{U}'(s) < 0 < \mathcal{V}'(s)$ throughout $(s_1, s_1 + \delta_2)$.

Consider the largest interval $I = (s_1, s_2)$ on which $\mathcal{U}'(s) < 0 < \mathcal{V}'(s)$. Along this interval, it is clear that $\mathcal{V}(s) - \mathcal{U}(s)^q$ is strictly increasing. Since $s_1 \in S_{\mathcal{V}}$, this expression is positive when $s = s_1$, so it is positive throughout I, and thus $I \subset S_{\mathcal{V}}$. It follows by (3.28) and (3.29) that $e^{(n-2-2\beta)s} \mathcal{V}'(s)$ is strictly increasing on I. Next, we apply the same argument to the expression $\mathcal{U}(s) - \mathcal{V}(s)^p$. This is strictly decreasing on I, and it is non-positive when $s = s_1$ by (3.31), so it is negative throughout I, and $e^{(n-2-2\alpha)s} \mathcal{U}'(s)$ is strictly decreasing on I.

We now recall that $I = (s_1, s_2)$ is the largest interval on which $\mathcal{U}'(s) < 0 < \mathcal{V}'(s)$. Since $e^{(n-2-2\beta)s} \mathcal{V}'(s)$ increases on this interval and since $e^{(n-2-2\alpha)s} \mathcal{U}'(s)$ decreases, we conclude that $s_2 = \infty$ and that $\mathcal{U}(s), \mathcal{V}(s)$ are eventually monotonic. Denote their limits by

$$\mathcal{U}_\infty = \lim_{s \to \infty} \mathcal{U}(s), \qquad \mathcal{V}_\infty = \lim_{s \to \infty} \mathcal{V}(s) \tag{3.32}$$

and note that $0 \leq \mathcal{U}_\infty, \mathcal{V}_\infty \leq 1$ by Lemma 3.2.5. If it happens that $\mathcal{U}_\infty > \mathcal{V}_\infty^p$, then

$$\lim_{s \to \infty} \left[\mathcal{U}'(s) + (n - 2 - 2\alpha) \mathcal{U}(s) \right] = +\infty$$

by (3.28), and thus $\lim_{s \to \infty} \mathcal{U}(s) = +\infty$, a contradiction. If it happens that $\mathcal{U}_\infty < \mathcal{V}_\infty^p$, we get a similar contradiction. We conclude that $\mathcal{U}_\infty = \mathcal{V}_\infty^p$ and also $\mathcal{V}_\infty = \mathcal{U}_\infty^q$ for similar reasons.

Since $pq > 1$ by assumption, the only possible solutions are then

$$\mathcal{U}_\infty = \mathcal{V}_\infty = 0 \quad \text{and} \quad \mathcal{U}_\infty = \mathcal{V}_\infty = 1.$$

The former case is excluded because $0 < \mathcal{V}(s) < 1$ is eventually increasing, while the latter case is excluded because $0 < \mathcal{U}(s) < 1$ is eventually decreasing.

We may thus conclude that neither $\mathcal{U}'(s)$ nor $\mathcal{V}'(s)$ can become negative. In particular, both $\mathcal{U}(s)$ and $\mathcal{V}(s)$ are increasing for all s, and the limits in (3.32) exist. Using the exact same argument as before, one finds that both limits must be equal to 1. ■

Theorem 3.3.1. *Let (u, v) be a regular positive radial solution of the elliptic system* (3.2)*, where $k, l \geq 0$, $p, q \geq 1$, $pq > 1$ and $n \geq 11$. Let $(C_\alpha r^{-\alpha}, C_\beta r^{-\beta})$ denote the singular solution* (3.4)*. Assume the existence condition* (3.3) *and the Joseph-Lundgren condition* (3.7)*.*

(a) If strict inequality holds in (3.7)*, then there exist constants D_1, D_2 such that*

$$\begin{aligned} u(r) &= C_\alpha r^{-\alpha} + D_1 r^{-\alpha-\gamma} + o(r^{-\alpha-\gamma}) \quad \text{as } r \to \infty, \\ v(r) &= C_\beta r^{-\beta} + D_2 r^{-\beta-\gamma} + o(r^{-\beta-\gamma}) \quad \text{as } r \to \infty. \end{aligned}$$

(b) If equality holds in (3.7)*, then there exist constants D_1, D_2 such that*

$$\begin{aligned} u(r) &= C_\alpha r^{-\alpha} + D_1 r^{-\alpha-\gamma} \log r + o(r^{-\alpha-\gamma} \log r) \quad \text{as } r \to \infty, \\ v(r) &= C_\beta r^{-\beta} + D_2 r^{-\beta-\gamma} \log r + o(r^{-\beta-\gamma} \log r) \quad \text{as } r \to \infty. \end{aligned}$$

In each case, $\gamma > 0$ is an explicit constant which is defined in Lemmas 3.2.2 and 3.2.4.

Remark 3.3.2. We note that the first terms in the asymptotic expansions are given by the singular solution (3.4), while the second terms are related to the linearised solution (3.19). Although these expansions were known in the scalar case, we are not aware of any similar results in the case of systems. For the second-order equation $-\Delta u = u^p$ which corresponds to the case $p = q > 1$, we refer the reader to [46, 63]. For the biharmonic equation $\Delta^2 u = u^p$ which corresponds to the case $p > q = 1$, we refer the reader to [56, 88].

Proof: We only prove part (a), as part (b) is quite similar. Consider the variables

$$y(s) = \mathcal{U}(s) - 1 = \frac{u(e^s)}{C_\alpha e^{-\alpha s}} - 1, \qquad z(s) = \mathcal{V}(s) - 1 = \frac{v(e^s)}{C_\beta e^{-\beta s}} - 1, \tag{3.33}$$

where $s = \log r$ and $\mathcal{U}(s), \mathcal{V}(s)$ are defined by (3.27). It trivially follows by (3.28) that

$$\begin{cases} y''(s) + (n-2-2\alpha)\, y'(s) - Q(\alpha)y(s) = Q(\alpha)\,(1-(z(s)+1)^p) \\ z''(s) + (n-2-2\beta)\, z'(s) - Q(\beta)z(s) = Q(\beta)\,(1-(y(s)+1)^q) \end{cases} \tag{3.34}$$

for all $s \in \mathbb{R}$. The corresponding linearised system is then

$$\begin{cases} y''(s) + (n-2-2\alpha)\, y'(s) - Q(\alpha)y(s) = -pQ(\alpha)z(s) \\ z''(s) + (n-2-2\beta)\, z'(s) - Q(\beta)z(s) = -qQ(\beta)y(s). \end{cases}$$

Since $Q(\lambda) = \lambda(n-2-\lambda)$ for all $\lambda \in \mathbb{R}$, this can be written in the compact form

$$\begin{cases} Q(\alpha - \partial_s)\, y(s) = pQ(\alpha)z(s) \\ Q(\beta - \partial_s)\, z(s) = qQ(\beta)y(s), \end{cases}$$

where $\partial_s = \frac{\partial}{\partial s}$. In particular, the associated characteristic polynomial has the form

$$F(\lambda) = Q(\alpha-\lambda)Q(\beta-\lambda) - pqQ(\alpha)Q(\beta).$$

Focusing on part (a), we assume that the Joseph-Lundgren condition (3.7) holds with strict inequality. It then follows by Lemma 3.2.2 that $F(\lambda)$ has four distinct real roots

$$-\lambda_1 < -\lambda_2 < -\lambda_3 < 0 < \lambda_4, \tag{3.35}$$

where $\lambda_i > 0$ for each i and $\lambda_3 = \gamma$ is the positive constant introduced in Lemma 3.2.4.

To analyse the nonlinear system (3.34), we first express it in the compact form

$$\begin{cases} Q(\alpha - \partial_s)\, y(s) - pQ(\alpha)z(s) = f(z(s)) \\ Q(\beta - \partial_s)\, z(s) - qQ(\beta)y(s) = g(y(s)), \end{cases} \tag{3.36}$$

where the nonlinear terms f, g are defined by

$$f(z) = Q(\alpha)\,((z+1)^p - 1 - pz), \qquad g(y) = Q(\beta)\,((y+1)^q - 1 - qy). \tag{3.37}$$

The solution of this system can be obtained using variation of parameters. Let us mostly worry about $y(s)$, as the approach for $z(s)$ is similar. Variation of parameters gives

$$\begin{aligned} y(s) = \sum_{i=1}^{3} \left[A_i e^{-\lambda_i s} + B_i \int_{s_0}^{s} e^{-\lambda_i (s-\tau)} f(z(\tau))\, d\tau + C_i \int_{s_0}^{s} e^{-\lambda_i (s-\tau)} g(y(\tau))\, d\tau \right] \\ + A_4 e^{\lambda_4 s} + B_4 \int_{s_0}^{s} e^{\lambda_4 (s-\tau)} f(z(\tau))\, d\tau + C_4 \int_{s_0}^{s} e^{\lambda_4 (s-\tau)} g(y(\tau))\, d\tau, \end{aligned} \tag{3.38}$$

where $s_0 \in \mathbb{R}$ is arbitrary, and the constants A_i, B_i, C_i can all be determined explicitly. The constants A_i depend on both s_0 and the eigenvalues, whereas the constants B_i, C_i depend on the eigenvalues only. In view of our definition (3.33), we must have

$$\lim_{s \to \infty} y(s) = \lim_{s \to \infty} z(s) = 0 \tag{3.39}$$

by Theorem 3.1.1. Focusing on the last two integrals in (3.38), we now write

$$\begin{aligned} y(s) = \sum_{i=1}^{3} \left[A_i e^{-\lambda_i s} + B_i \int_{s_0}^{s} e^{-\lambda_i (s-\tau)} f(z(\tau))\, d\tau + C_i \int_{s_0}^{s} e^{-\lambda_i (s-\tau)} g(y(\tau))\, d\tau \right] \\ - B_4 \int_{s}^{\infty} e^{-\lambda_4 (\tau-s)} f(z(\tau))\, d\tau - C_4 \int_{s}^{\infty} e^{-\lambda_4 (\tau-s)} g(y(\tau))\, d\tau \\ + e^{\lambda_4 s} \left[A_4 + B_4 \int_{s_0}^{\infty} e^{-\lambda_4 \tau} f(z(\tau))\, d\tau + C_4 \int_{s_0}^{\infty} e^{-\lambda_4 \tau} g(y(\tau))\, d\tau \right]. \end{aligned}$$

The terms in the first two lines all approach zero as $s \to \infty$, so the same must be true for the term in the last line. Since $\lambda_4 > 0$, however, this simply means that

$$\begin{aligned} y(s) = \sum_{i=1}^{3} \left[A_i e^{-\lambda_i s} + B_i \int_{s_0}^{s} e^{-\lambda_i (s-\tau)} f(z(\tau))\, d\tau + C_i \int_{s_0}^{s} e^{-\lambda_i (s-\tau)} g(y(\tau))\, d\tau \right] \\ - B_4 \int_{s}^{\infty} e^{-\lambda_4 (\tau-s)} f(z(\tau))\, d\tau - C_4 \int_{s}^{\infty} e^{-\lambda_4 (\tau-s)} g(y(\tau))\, d\tau. \end{aligned} \tag{3.40}$$

We now use this formula to estimate $y(s)$. Recalling our definition (3.37), we have

$$f(z(\tau)) = o(\tau) \cdot |z(\tau)|, \qquad g(y(\tau)) = o(\tau) \cdot |y(\tau)| \quad \text{as } \tau \to \infty \tag{3.41}$$

by (3.39) since $p, q \geq 1$. Employing this fact in (3.40), we obtain the estimate

$$|y(s)| \leq O(e^{-\lambda_3 s}) + C \int_{s_0}^{s} e^{-\lambda_3(s-\tau)} o(\tau)|z(\tau)|\, d\tau + C \int_{s_0}^{s} e^{-\lambda_3(s-\tau)} o(\tau)|y(\tau)|\, d\tau$$
$$+ C \int_{s}^{\infty} e^{-\lambda_4(\tau-s)} o(\tau)|z(\tau)|\, d\tau + C \int_{s}^{\infty} e^{-\lambda_4(\tau-s)} o(\tau)|y(\tau)|\, d\tau$$

as $s \to \infty$ because of (3.35). A similar estimate holds for $|z(s)|$ by symmetry. If we denote their sum by $w(s) = |y(s)| + |z(s)|$, we may thus conclude that

$$w(s) \leq O(e^{-\lambda_3 s}) + C \int_{s_0}^{s} e^{-\lambda_3(s-\tau)} o(\tau) w(\tau)\, d\tau + C \int_{s}^{\infty} e^{-\lambda_4(\tau-s)} o(\tau) w(\tau)\, d\tau$$

as $s \to \infty$. This is easily seen to imply that $w(s) = O(e^{-\lambda_3 s})$ as $s \to \infty$, namely

$$|y(s)| + |z(s)| = w(s) = O(e^{-\lambda_3 s}) \quad \text{as } s \to \infty.$$

Since the functions f, g defined by (3.37) are quadratic near the origin, we also have

$$f(z(\tau)) = O(e^{-2\lambda_3 \tau}), \qquad g(y(\tau)) = O(e^{-2\lambda_3 \tau}) \quad \text{as } \tau \to \infty. \tag{3.42}$$

This provides a more precise version of our previous estimate (3.41). Combining this new version with our identity (3.40), it is now easy to verify that

$$y(s) = A_3 e^{-\lambda_3 s} + o(e^{-\lambda_3 s}) \quad \text{as } s \to \infty.$$

Since $s = \log r$ and $\gamma = \lambda_3$ by definition, it follows by (3.33) that

$$u(r) = C_\alpha r^{-\alpha} + A_3 C_\alpha r^{-\alpha-\gamma} + o(r^{-\alpha-\gamma}) \quad \text{as } r \to \infty.$$

A similar expansion holds for $v(r)$, and this completes the proof of part (a). For part (b), the exact same argument works, but $-\lambda_3$ is a double eigenvalue by Lemma 3.2.2. ■

3.4 Further properties of regular solutions

We now establish some further properties of the regular positive radial solutions of (3.2). Our starting point is a useful fact which is due to Serrin and Zou [82, 83]. If (u, v) is any such solution, then its central value $(u(0), v(0))$ lies on the graph of a function f which is both continuous and strictly increasing. This applies to general Hamiltonian systems [82]. In the case of (3.2), however, the function f can also be determined explicitly.

Lemma 3.4.1. *Consider the elliptic system* (3.2)*, where* $k, l \geq 0$*,* $p, q \geq 1$ *and* $pq > 1$*. If the existence condition* (3.3) *holds, then there exists a positive constant* $C_0 = C_0(k, l, p, q, n)$ *such that for all* $\xi > 0$*, there is a regular positive radial solution of the system satisfying*

$$u_\xi(0) = \xi^\alpha, \qquad v_\xi(0) = C_0 \xi^\beta. \tag{3.43}$$

Moreover, the one-parameter family $\{(u_\xi, v_\xi)\}_{\xi>0}$ *are the only regular positive radial solutions of system* (3.2)*.*

Proof: We refer the reader to [8] for the existence and [53] for the uniqueness of solutions. Denote by (u_1, v_1) the unique regular positive radial solution which satisfies $u_1(0) = 1$. Then every other regular positive radial solution can be obtained through the scaling

$$u_\xi(r) = \xi^\alpha u_1(\xi r), \qquad v_\xi(r) = \xi^\beta v_1(\xi r). \tag{3.44}$$

Letting $C_0 = v_1(0)$, one finds that (u_ξ, v_ξ) is the unique solution which satisfies (3.43). ■

Lemma 3.4.2. *Let the assumptions of the previous lemma hold, and assume the Joseph-Lundgren condition* (3.7) *is satisfied. Given any* $\xi_1 > \xi_2 > 0$*, we have*

$$u_{\xi_1}(r) > u_{\xi_2}(r) \quad \text{and} \quad v_{\xi_1}(r) > v_{\xi_2}(r) \quad \text{for all } r \geq 0. \tag{3.45}$$

Proof: According to Theorem 3.1.1, the functions defined by

$$\mathcal{U}_\xi(r) = r^\alpha u_\xi(r), \qquad \mathcal{V}_\xi(r) = r^\beta v_\xi(r)$$

are both increasing with respect to r. On the other hand, it follows by (3.44) that

$$\mathcal{U}_\xi(r) = (\xi r)^\alpha u_1(\xi r), \qquad \mathcal{V}_\xi(r) = (\xi r)^\beta v_1(\xi r).$$

Monotonicity with respect to r is thus equivalent to monotonicity with respect to ξ, so

$$u_{\xi_1}(r) \geq u_{\xi_2}(r) \quad \text{and} \quad v_{\xi_1}(r) \geq v_{\xi_2}(r) \quad \text{for all } r \geq 0.$$

It remains to show that strict inequality holds in each case. Since

$$(r^{n-1}(u'_{\xi_1} - u'_{\xi_2}))' = r^{n-1+k}\,(v^p_{\xi_2} - v^p_{\xi_1}) \le 0 \quad \text{for all } r \ge 0, \tag{3.46}$$

we find that $u_{\xi_1} - u_{\xi_2}$ is decreasing to zero by Lemma 3.2.1. If it vanishes at some point R, it must thus vanish throughout $[R, \infty)$. In that case, (3.46) gives $v_{\xi_1} = v_{\xi_2}$ throughout $[R, \infty)$, and this obviously contradicts the uniqueness of solutions. In particular, $u_{\xi_1} - u_{\xi_2}$ remains positive at all points, and the same is similarly true for $v_{\xi_1} - v_{\xi_2}$. ■

Remark 3.4.3. In the case $k = l = 0$, Lemma 3.4.2 is sharp. Calling $w = u_{\xi_1}(r) - u_{\xi_2}(r)$ and $z = v_{\xi_1}(r) - v_{\xi_2}(r)$, if we assume w, z are positive, since $p, q \ge 1$ we have by convexity

$$\begin{aligned} -\Delta w &= v^p_{\xi_1} - v^p_{\xi_2} \ge p v^{p-1}_{\xi_2} z \\ -\Delta z &= u^q_{\xi_1} - u^q_{\xi_2} \ge q u^{q-1}_{\xi_2} w. \end{aligned}$$

The solution (u_{ξ_2}, v_{ξ_2}) is therefore linearly stable by definition, so by [14, Theorem 1.2] condition (3.7) must be satisfied.

Proof of Theorem 3.1.3. This is a typical application of Theorem 3.3.1 and Lemma 3.4.2. In fact, one may proceed as in the classical argument by Gui, Ni and Wang [46]. We only give a sketch of the proof for part (a), as the proof of part (b) is quite similar.

To show that the solution (u_ξ, v_ξ) is stable with respect to the norm (3.13), let $\varepsilon > 0$ be given. According to Theorem 3.3.1, there exists some $R_{\xi,\varepsilon} > 0$ such that

$$u_\xi(r) = C_\alpha r^{-\alpha} + D_1(\xi) r^{-\alpha-\gamma} + E_\xi(r) \quad \text{in } [R_{\xi,\varepsilon}, \infty),$$

where $|E_\xi(r)| \le \varepsilon r^{-\alpha-\gamma}$ in $[R_{\xi,\varepsilon}, \infty)$. On the other hand, it follows by scaling that

$$u_\eta(r) = \frac{\eta}{\xi}\, u_\xi\big((\eta/\xi)^{1/\alpha}\, r\big) \quad \text{for each } \eta > 0.$$

Assume that η is sufficiently close to ξ, say $|\eta - \xi| < \frac{\xi}{2}$. Combining the last two equations, we can then find constants R_ε, C independent of η such that

$$u_\eta(r) = C_\alpha r^{-\alpha} + D_1(\eta) r^{-\alpha-\gamma} + E_\eta(r) \quad \text{in } [R_\varepsilon, \infty),$$

where $D_1(\eta) = (\frac{\xi}{\eta})^{\alpha+\gamma} D_1(\xi)$ and $|E_\eta(r)| \leq C\varepsilon r^{-\alpha-\gamma}$ in $[R_\varepsilon, \infty)$. This implies that

$$|r^{\alpha+\gamma}(u_\eta - u_\xi)| \leq |D_1(\eta) - D_1(\xi)| + 2C\varepsilon \quad \text{in } [R_\varepsilon, \infty),$$

and we also have $|(1+r)^{\alpha+\gamma}(u_\eta - u_\xi)| \to 0$ uniformly in $[0, R_\varepsilon]$ as $\eta \to \xi$. Since similar estimates hold with u_η and u_ξ replaced by v_η and v_ξ, respectively, it follows by (3.13) that

$$\lim_{\eta \to \xi} ||(u_\eta, v_\eta) - (u_\xi, v_\xi)||_\gamma = 0.$$

In particular, given any $\varepsilon > 0$, there exists some small enough $\theta \in (0, \frac{\xi}{2})$ such that

$$||(u_{\xi\pm\theta}, v_{\xi\pm\theta}) - (u_\xi, v_\xi)||_\gamma < \varepsilon. \tag{3.47}$$

We now exploit this fact in order to prove stability. According to Lemma 3.4.2, one has

$$u_{\xi-\theta} < u_\xi < u_{\xi+\theta} \quad \text{and} \quad v_{\xi-\theta} < v_\xi < v_{\xi+\theta}$$

at all points. Arguing as in [46], we can thus find some small enough $\delta > 0$ such that

$$u_{\xi-\theta} < f < u_{\xi+\theta} \quad \text{and} \quad v_{\xi-\theta} < g < v_{\xi+\theta}$$

whenever $||(f,g) - (u_\xi, v_\xi)||_\gamma < \delta$. Let (u, v) be the solution of the parabolic system (3.1) subject to $u(x, 0) = f(x)$ and $v(x, 0) = g(x)$. Since $(u_{\xi\pm\theta}, v_{\xi\pm\theta})$ satisfy the same system, one may use standard comparison theorems for parabolic systems [70] to conclude that

$$u_{\xi-\theta} < u < u_{\xi+\theta} \quad \text{and} \quad v_{\xi-\theta} < v < v_{\xi+\theta}, \tag{3.48}$$

as long as $||(f,g) - (u_\xi, v_\xi)||_\gamma < \delta$. The result now follows by (3.47) and (3.48). ■

Chapter 4

Quasilinear Choquard Type Inequality

This chapter is based on the results of [19], which were jointly obtained with Marius Ghergu and Paschalis Karageorgis.

4.1 Choquard Inequality with Slow Decay Potential

In this chapter we discuss the following quasilinear inequality:

$$-\Delta_p u + \frac{\lambda}{|x|^\gamma} u^{p-1} \geq (K * u^m) u^q \quad \text{in } \mathbb{R}^n \setminus \overline{B}_1, \tag{4.1}$$

where $n \geq 1, \lambda, m > 0$ and $\gamma, q \in \mathbb{R}$. The p-Laplace operator Δ_p, $p > 1$ is defined as in Chapter 2. There is an extensive literature on problems such as (4.1), and it is by now well understood that the value of γ in relation to p plays an important role in the qualitative study of problem (4.1). We are interested in the case $\gamma < p$. This case is referred to as a potential of slow decay, while the cases $\gamma = p$ and $\gamma > p$ are said to have a Hardy potential or a fast decay potential, respectively. These two cases are more extensively covered in the literature [40, 75], which motivated the study of a slow decay potential. The key qualitative difference, as the name suggests, is that when $\gamma < p$ the potential term decays slower than the differential term on the left hand side of (4.1) as $|x| \to \infty$. Many of the results obtained are analogous to the other two cases, but an interesting difference in this case, as we shall see below, is that we can construct solutions to (4.1) which decay exponentially as $|x| \to \infty$. This phenomenon was previously observed only for the special case $p = 2$ [75], corresponding to the Laplace operator.

Now, $K * u^m$ denotes the usual convolution operation defined by

$$(K * u^m)(x) = \int_{\mathbb{R}^n \setminus \overline{B}_1} K(x-y)u^m(y)dy \quad \text{ for all } x \in \mathbb{R}^n \setminus \overline{B}_1.$$

We assume that $K\colon \mathbb{R}^n \setminus \{0\} \to (0,\infty)$ is continuous, $K \in L^1_{loc}(\mathbb{R}^n)$ and the function

$$L(R) := \min_{|x|=R} K(x)\,, \quad R > 0, \tag{4.2}$$

is decreasing in a neighbourhood of infinity. Note that if $K \leq 0$, then (4.1) always has positive solutions. Indeed, if $\Omega \subset \mathbb{R}^n \setminus \overline{B}_1$ is a smooth and bounded domain, let $u \in C^1(\overline{\Omega})$ satisfy $-\Delta_p u = 1$ in Ω and $u = 0$ on $\partial\Omega$. Since Ω is smooth we can extend u by zero outside of Ω. Then, the left-hand side of (4.1) is nonnegative while the right-hand side of (4.1) is finite (since $u \in C^1_c(\mathbb{R}^n \setminus \overline{B}_1)$) and negative (since $K \leq 0$). Thus, u is a solution of (4.1). This justifies our assumption that $K > 0$ in $\mathbb{R}^n \setminus \overline{B}_1$. In addition, the right-hand side of (4.1) is increasing with respect to K, so if a positive solution exists for some function K, then that is trivially a solution for any other function $\widetilde{K} \leq K$.

We are interested in positive weak solutions of (4.1), that is, functions $u > 0$ belonging to the class $W^{1,p}_{loc}(\mathbb{R}^n \setminus \overline{B}_1) \cap C(\mathbb{R}^n \setminus \overline{B}_1)$ that satisfy

(i) $(K * u^m)u^q \in L^1_{loc}(\mathbb{R}^n \setminus \overline{B}_1)$;

(ii) for any $\varphi \in C^\infty_c(\mathbb{R}^n \setminus \overline{B}_1)$, $\varphi \geq 0$ we have

$$\int_{\mathbb{R}^n \setminus \overline{B}_1} \Big(|\nabla u|^{p-2}\nabla u \cdot \nabla\varphi + \frac{\lambda}{|x|^\gamma} u^{p-1}\varphi \Big) \geq \int_{\mathbb{R}^n \setminus \overline{B}_1} (K * u^m)u^q\varphi. \tag{4.3}$$

The goal of this chapter is to determine existence and nonexistence regions of solutions to (4.1) in the mq-plane. We also show that when $K(x)$ satisfies the complementary conditions

$$\limsup_{|x|\to\infty} \big|K(x)\big| \cdot \frac{|x|^{n-\alpha}}{(\log|x|)^\theta} < \infty, \tag{4.4}$$

$$\liminf_{|x|\to\infty} \big|K(x)\big| \cdot \frac{|x|^{n-\alpha}}{(\log|x|)^\theta} > 0, \tag{4.5}$$

for some $\theta \in \mathbb{R}$, $0 < \alpha < n$, the results are essentially optimal, in the sense that we account for the entire mq-plane, except for at most a line. Conditions (4.4) and (4.5) are motivated by the prototype convolution kernel used in the analysis of (4.1), namely the Riesz potential

$I_\alpha(x) = \frac{C_\alpha}{|x|^{n-\alpha}}$, where (with Γ denoting the standard Γ function)

$$(I_\alpha * u^m)(x) = C_\alpha \int_\Omega \frac{u(y)^m}{|x-y|^{n-\alpha}}\, dy, \quad \text{and} \quad C_\alpha = \frac{\Gamma(\frac{n-\alpha}{2})}{\pi^{\frac{n}{2}} 2^\alpha \Gamma(\frac{\alpha}{2})}.$$

The study of PDEs involving nonlinear convolution terms can be traced back nearly a century. Indeed, motivated by the Schrödinger equation in quantum physics, the prototype equation

$$i\psi_t - \Delta\psi = (|x|^{\alpha-n} * \psi^2)\psi \quad \text{in } \mathbb{R}^n, \quad \alpha \in (0,n), \quad n \geq 1, \tag{4.6}$$

was proposed by D.H. Hartree [47, 48, 49] in 1928 for $n = 3$ and $\alpha = 2$. The stationary case of (4.6) is more widely known as *Choquard* or *Choquard-Pekar* equation (see [78]). The first mathematical studies of the Choquard equation are due to E. Lieb [65] and P.L. Lions [67]. Since then, many mathematical methods have been devised to investigate such equations featuring nonlocal terms. With no hope of being exhaustive, we refer to, for example [41, 42, 43, 44, 62, 64, 75, 76, 80].

The study of (4.1) is inspired by the 2013 work of Moroz and Van Schaftingen [75], which deals with the semilinear inequality

$$-\Delta u + \frac{\lambda}{|x|^\gamma} u \geq (|x|^{\alpha-n} * u^m)u^q \quad \text{in } \mathbb{R}^n \setminus B_1, \tag{4.7}$$

which corresponds to (4.1) in the case $p = 2$. Using a nonlocal version of the so-called Agmon-Allegretto-Piepenbrink positivity principle (see [75, Proposition 3.2]) the authors obtain a full description of the existence and nonexistence of positive solutions to (4.7). The study of (4.7) relies heavily on the linear character of the differential operator, so many of the techniques used do not translate to the study of (4.1). As we shall see below, the analysis of (4.1) is largely dependent on various a priori estimates. More recently, the inequality

$$-\Delta_p u - \frac{\lambda}{|x|^p} u^{p-1} \geq (I_\alpha * u^m)u^q \quad \text{in } \mathbb{R}^n \setminus \overline{B}_1 \tag{4.8}$$

was discussed in [40] by Ghergu, Karageorgis and Singh, where $\lambda \in \mathbb{R}$ and I_α is the Riesz potential. In this case, there is a so-called Hardy potential term, and the authors found that the existence and nonexistence of solutions to (4.8) depends on the value of λ in relation to the best Hardy constant $C_H = \left|\frac{n-p}{p}\right|^p$. Roughly speaking, this is due to the fact that the differential and potential terms decay at the same rate as $|x| \to \infty$. Since we are dealing with a slow decay potential, our analysis is not sensitive to the exact value of $\lambda > 0$.

4.2 Main Results

We now state the main nonexistence result. The proof of this result is heavily dependent on the a priori integral estimate given in Lemma 4.3.1.

Theorem 4.2.1 (Nonexistence)**.** *Assume $n \geq 1$, $\lambda, m > 0$ and $\gamma, q \in \mathbb{R}$, and suppose one of the following conditions holds:*

(i) $m + q > p - 1$, $q < p - 1$ *and* $\limsup_{R\to\infty} R^{\min\{\gamma,p\}+n(p-1-q)/m} L(R) > 0$*;*

(ii) $q = p - 1$ *and* $\limsup_{R\to\infty} R^{\min\{\gamma,p\}} L(R) = \infty$*;*

(iii) $m + q < p - 1$ *and* $\limsup_{R\to\infty} R^{n+\min\{\gamma,p\}\frac{m}{p-1-q}} L(R) > 0$*;*

(iv) $m + q = p - 1$ *and* $\limsup_{R\to\infty} R^{n+\min\{\gamma,p\}} L(R) = \infty$*.*

Then every positive function $u \in W^{1,p}_{loc}(\mathbb{R}^n \setminus \overline{B}_1) \cap C(\mathbb{R}^n \setminus \overline{B}_1)$ satisfies

$$-\Delta_p u + \frac{\lambda}{|x|^\gamma} u^{p-1} < (K * u^m) u^q$$

at some point $x \in \mathbb{R}^n \setminus \overline{B}_1$.

As discussed in the introduction, this nonexistence result is essentially optimal when (4.5) holds. In particular we have the following:

Corollary 4.2.2. *Assume* (4.5)*, where $n \geq 1$, $\lambda, m > 0$, $0 < \alpha < n$, $\gamma < p$ and $\theta, q \in \mathbb{R}$, and suppose one of the following conditions holds:*

(i) *Case $n - \alpha \leq \gamma < p$:*

  (i1) $q < p - 1$*;*

  (i2) $q = p - 1$ *and either $\gamma > n - \alpha$ or else $\gamma = n - \alpha$ with $\theta > 0$;*

(ii) *Case $-\alpha < \gamma < \min\{n - \alpha, p\}$:*

  (ii1) $q < p - 1 + \dfrac{\alpha + \gamma - n}{n} m$*;*

  (ii2) $q = p - 1 + \dfrac{\alpha + \gamma - n}{n} m$ *and $\theta \geq 0$;*

(iii) *Case $\gamma \leq -\alpha$:*

(iii1) $q < p - 1 + \dfrac{\gamma}{\alpha}\, m$;

(iii2) $q = p - 1 + \dfrac{\gamma}{\alpha}\, m$ *and* $\theta \geq 0$ *when* $\gamma < -\alpha$;

(iii3) $q = p - 1 + \dfrac{\gamma}{\alpha}\, m$ *and* $\theta > 0$ *when* $\gamma = -\alpha$.

Then every positive function $u \in W^{1,p}_{loc}(\mathbb{R}^n \setminus \overline{B}_1) \cap C(\mathbb{R}^n \setminus \overline{B}_1)$ *satisfies*

$$-\Delta_p u + \frac{\lambda}{|x|^\gamma} u^{p-1} < (K * u^m) u^q$$

at some point $x \in \mathbb{R}^n \setminus \overline{B}_1$.

We now state the main existence result. The proof of this result is constructive, and relies on appropriately estimating the convolution term in (4.1).

Theorem 4.2.3 (Existence)**.** *Assume* (4.4), *where* $n \geq 1$, $\lambda, m > 0$, $0 < \alpha < n$, $\gamma < p$ *and* $\theta, q \in \mathbb{R}$. *If one of the following conditions holds, then* (4.1) *has a positive solution.*

(i) $q > p - 1$;

(ii) $q = p - 1$ *and either* $\gamma < n - \alpha$ *or else* $\gamma = n - \alpha$ *with* $\theta \leq 0$;

(iii) $p - 1 + \frac{\alpha + \gamma - n}{n}\, m < q < p - 1$;

(iv) $q = p - 1 + \dfrac{\alpha + \gamma - n}{n}\, m < p - 1$ *and* $\theta < -1$;

(v) $p - 1 + \frac{\gamma}{\alpha}\, m < q \leq p - 1 + \frac{\alpha + \gamma - n}{n}\, m$.

For clarity, diagrams of the nonexistence regions in the mq-plane are shown below.

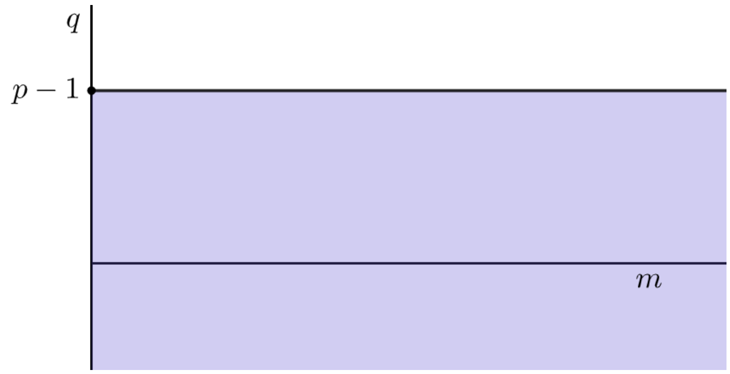

Figure 4.1: Shaded region of nonexistence when $n-\alpha \leq \gamma < p$.

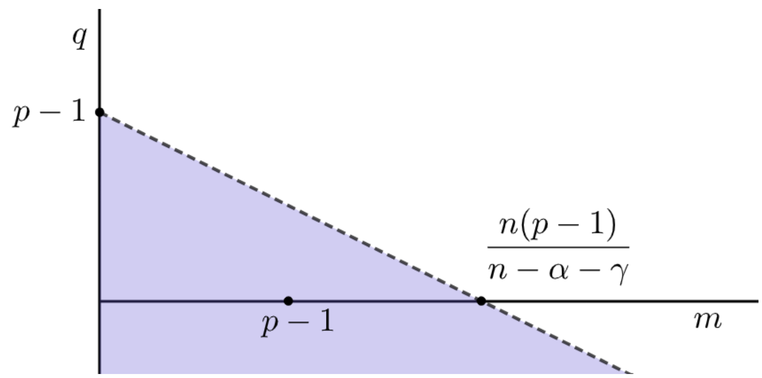

Figure 4.2: Shaded region of nonexistence when $-\alpha < \gamma < \min\{n-\alpha, p\}$.

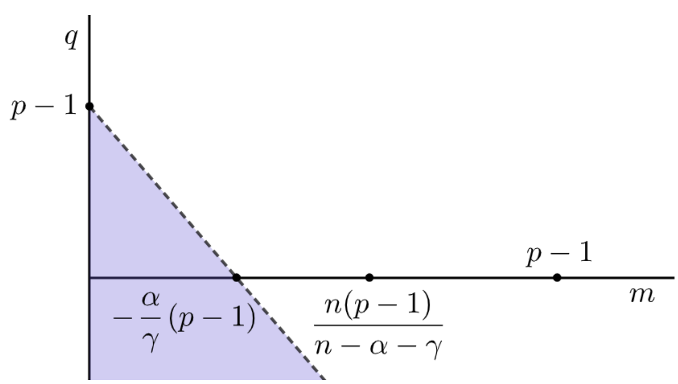

Figure 4.3: Shaded region of nonexistence when $\gamma \leq -\alpha$.

The bold line in Figure 4.1 indicates that, for the case $q = p-1$ and $\gamma = n-\alpha$, solutions may or may not exist, and this depends on whether or not $\theta > 0$. The dotted lines in Figures

4.2 and 4.3 correspond to undetermined cases of the above results. These are the cases

- $q = p - 1 + \dfrac{\alpha+\gamma-n}{n} m$ when $-\alpha < \gamma < \min\{n-\alpha, p\}$ and $-1 \le \theta < 0$;
- $q = p - 1 + \dfrac{\gamma}{\alpha} m$, when $\gamma < -\alpha$ and $\theta < 0$;
- $q = p - 1 + \dfrac{\gamma}{\alpha} m$, when $\gamma = -\alpha$ and $\theta \le 0$.

4.3 Proof of Theorem 4.2.1

To prove our main nonexistence result, we first need the following estimate. As usual, $C > 0$ will be used to denote positive constants which may vary with each appearance, and $r = |x|$.

Lemma 4.3.1. *Let u be a positive solution of (4.1) where $\gamma, q \in \mathbb{R}$ and $\lambda, m > 0$. Then, there exists a constant $C = C(n, p, m, q, \lambda, \gamma) > 0$ such that*

$$\Big(\int_{B_{2R}\setminus B_1} u^m\Big)\Big(\int_{B_{2R}\setminus B_R} u^{q-p+1}\Big) \le \frac{CR^{n-\min\{\gamma,p\}}}{L(4R)} \qquad \textit{for all } R > 2 \textit{ large.} \tag{4.9}$$

Proof: Take $\phi \in C_c^1(\mathbb{R})$ such that $0 \le \phi \le 1$, supp $\phi = [1/2, 5/2]$ and $\phi = 1$ on $[1, 2]$.

For $R > 2$ define $\phi_R(x) = \phi(|x|/R)$. Letting $\varphi = u^{1-p}\phi_R^k$, $k > p$, be a test function as in (4.3) we find

$$\begin{aligned}\int_{\mathbb{R}^n\setminus\overline{B}_1} (K * u^m) u^{q-p+1}\phi_R^k + (p-1)\int_{\mathbb{R}^n\setminus\overline{B}_1} u^{-p}|\nabla u|^p \phi_R^k \\ \le \lambda \int_{\mathbb{R}^n\setminus\overline{B}_1} \frac{\phi_R^k}{|x|^\gamma} + k\int_{\mathbb{R}^n\setminus\overline{B}_1} u^{1-p}\phi_R^{k-1}|\nabla u|^{p-2}\nabla u\cdot\nabla\phi_R.\end{aligned}$$

Applying Young's inequality $ab \le Ca^p + (p-1)b^{p/(p-1)}$, we also have

$$ku^{1-p}\phi_R^{k-1}|\nabla u|^{p-2}\nabla u\cdot\nabla\phi_R \le C(p,k)|\nabla\phi_R|^p\phi_R^{k-p} + (p-1)u^{-p}|\nabla u|^p\phi_R^k,$$

and combining the above estimates we deduce

$$\begin{aligned}\int_{\mathbb{R}^n\setminus\overline{B}_1} (K * u^m) u^{q-p+1}\phi_R^k &\le \lambda \int_{\mathbb{R}^n\setminus\overline{B}_1} \frac{\phi_R^k}{|x|^\gamma} + C\int_{\mathbb{R}^n\setminus\overline{B}_1} |\nabla\phi_R|^p\phi_R^{k-p} \\ &\le CR^{n-\min\{p,\gamma\}}.\end{aligned} \tag{4.10}$$

On the other hand, for $x, y \in B_{2R} \setminus B_1$ we have $|x-y| \le 4R$ and

$$K(x-y) \ge L(|x-y|) \ge L(4R) \qquad \text{for large } R > 2. \tag{4.11}$$

Thus, for all $x \in B_{2R} \setminus B_1$ we have

$$K * u^m(x) \ge \int_{B_{2R}\setminus B_1} K(x-y)u^m(y) \ge L(4R) \int_{B_{2R}\setminus B_1} u^m,$$

and then

$$\begin{aligned}\int_{\mathbb{R}^n \setminus \overline{B}_1} (K * u^m) u^{q-p+1} \phi_R^k &\ge \int_{B_{2R}\setminus B_R} (K * u^m) u^{q-p+1} \\ &\ge L(4R) \left(\int_{B_{2R}\setminus B_1} u^m \right) \left(\int_{B_{2R}\setminus B_R} u^{q-p+1} \right).\end{aligned}$$

Using this last estimate in (4.10) we obtain (4.9). ■

Proof of Theorem 4.2.1.

(i) Assume that $q < p-1$. By Hölder's inequality we have

$$\int_{B_{2R}\setminus B_R} 1 \le \Big(\int_{B_{2R}\setminus B_R} u^m \Big)^{\frac{p-1-q}{p+m-1-q}} \Big(\int_{B_{2R}\setminus B_R} u^{q-p+1} \Big)^{\frac{m}{p+m-1-q}}. \tag{4.12}$$

Using the estimate (4.9) we deduce

$$\begin{aligned} CR^n &\le \Big[\Big(\int_{B_{2R}\setminus B_1} u^m \Big) \Big(\int_{B_{2R}\setminus B_R} u^{q-p+1} \Big) \Big]^{\frac{p-1-q}{p+m-1-q}} \Big(\int_{B_{2R}\setminus B_R} u^{q-p+1} \Big)^{\frac{m-p+1+q}{p+m-1-q}} \\ &\le C \Big(\frac{R^{n-\min\{\gamma,p\}}}{L(4R)} \Big)^{\frac{p-1-q}{p+m-1-q}} \Big(\int_{B_{2R}\setminus B_R} u^{q-p+1} \Big)^{\frac{m-p+1+q}{p+m-1-q}} \qquad \text{for all } R > 2. \end{aligned}$$

The above estimate yields

$$\int_{B_{2R}\setminus B_R} u^{q-p+1} \ge C \Big(R^{nm+\min\{\gamma,p\}(p-1-q)} L(4R)^{p-1-q} \Big)^{\frac{1}{m-p+1+q}}, \tag{4.13}$$

for all $R > 2$. From (4.9) we have

$$\int_{B_{2R}\setminus B_R} u^{q-p+1} \le \frac{C}{\int_{B_4\setminus B_1} u^m} \cdot \frac{R^{n-\min\{\gamma,p\}}}{L(4R)} \le \frac{CR^{n-\min\{\gamma,p\}}}{L(4R)}. \tag{4.14}$$

Combining (4.14) with (4.12) we deduce

$$CR^{\min\{\gamma,p\}+n(p-1-q)/m}L(4R) \le \Big(\int_{B_{2R}\setminus B_R} u^m\Big)^{(p-1-q)/m} \qquad \text{for all } R>2.$$

Since $\limsup_{R\to\infty} R^{\min\{\gamma,p\}+n(p-1-q)/m}L(R) > 0$, one may find a positive constant $c>0$ and an increasing sequence of numbers $R_j \to \infty$ such that

$$\int_{B_{2R_j}\setminus B_{R_j}} u^m \ge c \qquad \text{for all } j \ge 1.$$

This shows that $\int_{\mathbb{R}^n\setminus B_1} u^m = \infty$ and from Lemma 4.3.1 it follows that

$$\int_{B_{2R}\setminus B_R} u^{q-p+1} = \frac{R^{n-\min\{\gamma,p\}}}{L(4R)}\, o\big(1\big) \qquad \text{as } R\to\infty.$$

We combine this estimate with (4.13) to deduce

$$C\Big(R^{nm+\min\{\gamma,p\}(p-1-q)}L(4R)^{p-1-q}\Big)^{\frac{1}{m-p+1+q}} \le \int_{B_{2R}\setminus B_R} u^{q-p+1} = \frac{R^{n-\min\{\gamma,p\}}}{L(4R)}\, o\big(1\big),$$

as $R\to\infty$. Hence,

$$\Big(R^{nm+\min\{\gamma,p\}(p-1-q)}L(4R)^{p-1-q}\Big)^{\frac{1}{m-p+1+q}} = \frac{R^{n-\min\{\gamma,p\}}}{L(4R)}\, o\big(1\big) \qquad \text{as } R\to\infty,$$

which yields

$$R^{\min\{\gamma,p\}+n(p-1-q)/m}L(4R) = o\big(1\big) \qquad \text{as } R\to\infty,$$

which cannot hold since $\limsup_{R\to\infty} R^{\min\{\gamma,p\}+n(p-1-q)/m}L(R) > 0$.

(ii) Assume $q = p-1$. Using the inequality (4.9) in Lemma 4.3.1 we find

$$\int_{B_{2R}\setminus B_1} u^m dx \le \frac{C}{R^{\min\{\gamma,p\}}L(4R)} \qquad \text{for all } R>2.$$

This yields

$$\int_{\mathbb{R}^n\setminus B_1} u^m dx \le \liminf_{R\to\infty} \frac{C}{R^{\min\{\gamma,p\}}L(4R)} = 0,$$

which is impossible since u is positive.

(iii) We start from (4.12) which we write as

$$CR^n \leq \Big(\int_{B_{2R}\setminus B_R} u^m\Big)^{\frac{p-1-m-q}{p+m-1-q}} \Big[\Big(\int_{B_{2R}\setminus B_R} u^m\Big)\Big(\int_{B_{2R}\setminus B_R} u^{q-p+1}\Big)\Big]^{\frac{m}{p+m-1-q}},$$

for all $R > 2$. We combine the above estimate with (4.9) to obtain

$$CR^n \leq \Big(\int_{B_{2R}\setminus B_R} u^m\Big)^{\frac{p-1-m-q}{p+m-1-q}} \Big(\frac{R^{N-\min\{\gamma,p\}}}{L(4R)}\Big)^{\frac{m}{p+m-1-q}},$$

that is,

$$CR^{n(p-1-q)+m\min\{\gamma,p\}} L^m(4R) \leq \Big(\int_{B_{2R}\setminus B_R} u^m\Big)^{p-1-m-q} \qquad \text{for all } R > 2. \tag{4.15}$$

On the other hand, for almost all $x \in \mathbb{R}^n \setminus B_1$ one has

$$\int_{\mathbb{R}^n\setminus\overline{B}_1} K(x-y)u^m(y)dy < \infty.$$

Thus, for almost all $x \in \mathbb{R}^n \setminus B_1$ we have

$$\int_{B_{2R}\setminus B_R} K(x-y)u^m(y)dy \to 0 \quad \text{as } R \to \infty. \tag{4.16}$$

If $x, y \in B_{2R} \setminus B_1$ we have $|x-y| \leq 4R$ and (4.11) holds. Then, from (4.16) we deduce

$$L(4R)\int_{B_{2R}\setminus B_R} u^m \to 0 \quad \text{as } R \to \infty.$$

Combining this last estimate with (4.15) we arrive at

$$CR^{n(p-1-q)+m\min\{\gamma,p\}} L^{p-1-q}(4R) \to 0 \qquad \text{as } R \to \infty,$$

that is,

$$R^{n+\min\{\gamma,p\}\frac{m}{p-1-q}} L(4R) \to 0 \quad \text{as } R \to \infty,$$

which leads to a contradiction since $\limsup_{R\to\infty} R^{n+\min\{\gamma,p\}\frac{m}{p-1-q}} L(R) > 0$.

(iv) Assume $m+q=p-1$. Using Hölder's inequality together with (4.9) we find

$$CR^{2n} = \Big(\int_{B_{2R}\setminus B_R} 1\Big)^2 \le \Big(\int_{B_{2R}\setminus B_R} u^m\Big)\Big(\int_{B_{2R}\setminus B_R} u^{-m}\Big) \le \frac{CR^{n-\min\{\gamma,p\}}}{L(4R)},$$

for $R>2$, which yields

$$R^{n+\min\{\gamma,p\}}L(4R) \le C \quad \text{ for } R>2.$$

This last estimate contradicts our hypothesis (iv) since

$$\limsup_{R\to\infty} R^{n+\min\{\gamma,p\}}L(R) = \infty.$$

■

4.4 Convolution Estimate

We now provide some estimates for Riesz potentials with logarithmic terms. These will give us upper bounds for the right hand side of (4.1) which are easier to treat. In proving Theorem 4.2.3, we will find lower bounds for the left hand side of (4.1) which we can easily compare to these estimates.

Lemma 4.4.1. *Let $0<\alpha<n$ and $\sigma>\alpha$. Assume $f,K:\mathbb{R}^n\to\mathbb{R}$ are locally integrable with*

$$\limsup_{|x|\to\infty} |K(x)|\cdot\frac{|x|^{n-\alpha}}{(\log|x|)^\theta} < \infty, \qquad \limsup_{|x|\to\infty} |f(x)|\cdot\frac{|x|^{\sigma}}{(\log|x|)^\tau} < \infty \tag{4.17}$$

for some $\theta\in\mathbb{R}$ and some $\tau>-1$. Then

$$\begin{cases} \limsup\limits_{|x|\to\infty} \dfrac{|x|^{\sigma-\alpha}}{(\log|x|)^{\theta+\tau}}\cdot|(K*f)(x)|<\infty & \text{if } \alpha<\sigma<n,\\ \limsup\limits_{|x|\to\infty} \dfrac{|x|^{n-\alpha}}{(\log|x|)^{1+\theta+\tau}}\cdot|(K*f)(x)|<\infty & \text{if } \sigma=n,\\ \limsup\limits_{|x|\to\infty} \dfrac{|x|^{n-\alpha}}{(\log|x|)^{\theta}}\cdot|(K*f)(x)|<\infty & \text{if } \sigma>n. \end{cases}$$

Proof: Suppose first that $\alpha<\sigma<n$ and fix some large enough number $R_0>2$ such that

$$\log R_0 > \frac{|\theta|}{\alpha}+\frac{|\tau|}{n-\sigma}+\frac{|\theta+\tau|}{\sigma-\alpha}. \tag{4.18}$$

In view of (4.17), we may then assume without loss of generality that

$$|K(x)| \leq |x|^{\alpha-n}(\log|x|)^{\theta} \quad \text{ and } \quad |f(x)| \leq |x|^{-\sigma}(\log|x|)^{\tau}$$

for all $|x| \geq R_0$. Our goal is to estimate the convolution term

$$|(K*f)(x)| \leq \int_{\mathbb{R}^n} |K(y)| \cdot |f(x-y)|\, dy \qquad \text{where } |x| \geq 2R_0.$$

If I_1 is the part of the integral for which $|y| \leq R_0$, then $|x-y| \geq |x|-|y| \geq \frac{1}{2}|x| \geq R_0$ within the region of integration and also $|x-y| \leq |x|+|y| \leq \frac{3}{2}|x|$, so

$$\begin{aligned} I_1 &\leq \int_{|y|\leq R_0} |K(y)| \cdot |x-y|^{-\sigma}(\log|x-y|)^{\tau}\, dy \\ &\leq C|x|^{-\sigma}(\log|x|)^{\tau} \int_{|y|\leq R_0} |K(y)|\, dy \\ &\leq C|x|^{-\sigma}(\log|x|)^{\tau}. \end{aligned}$$

If I_2 is the part for which $|x-y| \leq R_0 \leq |y|$, then $|y| \geq |x|-|x-y| \geq \frac{1}{2}|x| \geq R_0$ within the region of integration and also $|y| \leq |y-x|+|x| \leq \frac{3}{2}|x|$, so

$$\begin{aligned} I_2 &\leq \int_{|x-y|\leq R_0} |y|^{\alpha-n}(\log|y|)^{\theta} \cdot |f(x-y)|\, dy \\ &\leq C|x|^{\alpha-n}(\log|x|)^{\theta} \int_{|z|\leq R_0} |f(z)|\, dz \\ &\leq C|x|^{\alpha-n}(\log|x|)^{\theta}. \end{aligned}$$

Once we now combine the estimates for I_1 and I_2, we arrive at

$$\begin{aligned} |(K*f)(x)| &\leq C|x|^{-\sigma}(\log|x|)^{\tau} + C|x|^{\alpha-n}(\log|x|)^{\theta} \\ &\quad + C\int_{|y|\geq R_0, |x-y|\geq R_0} |y|^{\alpha-n}|x-y|^{-\sigma}\,(\log|y|)^{\theta}\,(\log|x-y|)^{\tau}\, dy. \end{aligned} \tag{4.19}$$

To treat the last integral, we split it into three parts. For the part I_3 for which $|y| \geq 2|x|$, we have $\frac{3}{2}|y| \geq |x-y| \geq |y|-|x| \geq \frac{1}{2}|y|$ and so

$$I_3 \leq C\int_{|y|\geq 2|x|} |y|^{\alpha-n-\sigma}\,(\log|y|)^{\theta+\tau}\, dy.$$

Letting I_3' denote the integral on the right, we integrate by parts to write

$$\begin{aligned} I_3' &= \int_{2|x|}^{\infty} s^{\alpha-\sigma-1} (\log s)^{\theta+\tau}\, ds \\ &= \left[\frac{s^{\alpha-\sigma}}{\alpha-\sigma} (\log s)^{\theta+\tau} \right]_{2|x|}^{\infty} + \frac{\theta+\tau}{\sigma-\alpha} \int_{2|x|}^{\infty} s^{\alpha-\sigma-1} (\log s)^{\theta+\tau-1}\, ds. \end{aligned}$$

Since $\sigma > \alpha$ by assumption, the last equation provides the estimate

$$I_3' \leq C|x|^{\alpha-\sigma} (\log |x|)^{\theta+\tau} + \frac{|\theta+\tau|}{(\sigma-\alpha)\log R_0} \cdot I_3'$$

and we may now employ our assumption (4.18) on R_0 to conclude that

$$I_3 \leq C I_3' \leq C|x|^{\alpha-\sigma} (\log |x|)^{\theta+\tau}. \tag{4.20}$$

For the part I_4 with $|y| \leq \frac{1}{2}|x|$, one has $\frac{1}{2}|x| \leq |x| - |y| \leq |x-y| \leq \frac{3}{2}|x|$ and

$$\begin{aligned} I_4 &\leq C|x|^{-\sigma} (\log |x|)^{\tau} \int_{R_0 \leq |y| \leq |x|} |y|^{\alpha-n} (\log |y|)^{\theta}\, dy \\ &= C|x|^{-\sigma} (\log |x|)^{\tau} \int_{R_0}^{|x|} s^{\alpha-1} (\log s)^{\theta}\, ds. \end{aligned}$$

Proceeding as before, we integrate by parts to write

$$\begin{aligned} \int_{R_0}^{|x|} s^{\alpha-1} (\log s)^{\theta}\, ds &= \left[\frac{1}{\alpha} s^{\alpha} (\log s)^{\theta} \right]_{R_0}^{|x|} - \frac{\theta}{\alpha} \int_{R_0}^{|x|} s^{\alpha-1} (\log s)^{\theta-1}\, ds \\ &\leq \frac{1}{\alpha} |x|^{\alpha} (\log |x|)^{\theta} + \frac{|\theta|}{\alpha \log R_0} \int_{R_0}^{|x|} s^{\alpha-1} (\log s)^{\theta}\, ds. \end{aligned}$$

In view of our assumption (4.18) on R_0, we may thus conclude that

$$I_4 \leq C|x|^{-\sigma} (\log |x|)^{\tau} \int_{R_0}^{|x|} s^{\alpha-1} (\log s)^{\theta}\, ds \leq C|x|^{\alpha-\sigma} (\log |x|)^{\theta+\tau}. \tag{4.21}$$

It remains to treat the part I_5 of the integral corresponding to $\frac{1}{2}|x| \leq |y| \leq 2|x|$. For this part of the integral, one has

$$\begin{aligned}
I_5 &\le C|x|^{\alpha-n} (\log|x|)^\theta \int_{R_0\le|x-y|\le 3|x|} |x-y|^{-\sigma} (\log|x-y|)^\tau \, dy \\
&= C|x|^{\alpha-n} (\log|x|)^\theta \int_{R_0}^{3|x|} s^{n-1-\sigma} (\log s)^\tau \, ds.
\end{aligned}$$

Letting I_5' denote the last integral, we may integrate by parts to get

$$\begin{aligned}
I_5' &= \left[\frac{s^{n-\sigma}}{N-\sigma} (\log s)^\tau\right]_{R_0}^{3|x|} - \frac{\tau}{N-\sigma} \int_{R_0}^{3|x|} s^{n-1-\sigma} (\log s)^{\tau-1} \, ds \\
&\le C|x|^{n-\sigma} (\log|x|)^\tau + \frac{|\tau|}{(n-\sigma)\log R_0} \cdot I_5'.
\end{aligned}$$

Recalling our assumption (4.18) once again, we conclude that

$$I_5 \le C|x|^{\alpha-n} (\log|x|)^\theta \cdot C|x|^{n-\sigma} (\log|x|)^\tau = C|x|^{\alpha-\sigma} (\log|x|)^{\theta+\tau}.$$

In view of our estimates (4.19), (4.20) and (4.21), we have thus found that

$$\begin{aligned}
|(K*f)(x)| &\le C|x|^{-\sigma}(\log|x|)^\tau + C|x|^{\alpha-n}(\log|x|)^\theta + C|x|^{\alpha-\sigma} (\log|x|)^{\theta+\tau} \\
&\le C|x|^{\alpha-\sigma} (\log|x|)^{\theta+\tau}.
\end{aligned}$$

This completes the proof for the case $\sigma < n$. When $\sigma > n$ or $\sigma = n$, we repeat the exact same approach as before, but we choose $R_0 > 2$ such that

$$\log R_0 > \frac{|\theta|}{\alpha} + \frac{|\theta+\tau|}{\sigma-\alpha}.$$

Our estimates for the integrals I_1, I_2, I_3, I_4 are then still applicable, so we need only worry about I_5. When $\sigma > n$, one may simply argue that

$$I_5 \le C|x|^{\alpha-n} (\log|x|)^\theta \int_{R_0}^{\infty} s^{n-1-\sigma} (\log s)^\tau \, ds \le C|x|^{\alpha-n} (\log|x|)^\theta.$$

Combining this estimate with equations (4.19), (4.20) and (4.21), we thus have

$$\begin{aligned}
|(K*f)(x)| &\le C|x|^{-\sigma}(\log|x|)^\tau + C|x|^{\alpha-n}(\log|x|)^\theta + C|x|^{\alpha-\sigma} (\log|x|)^{\theta+\tau} \\
&\le C|x|^{\alpha-n} (\log|x|)^\theta.
\end{aligned}$$

In the remaining case $\sigma = n$, finally, our assumption $\tau > -1$ gives

$$I_5 \leq C|x|^{\alpha-n} \left(\log |x|\right)^{\theta} \int_{R_0}^{3|x|} s^{-1} \left(\log s\right)^{\tau} ds \leq C|x|^{\alpha-n} \left(\log |x|\right)^{\theta+\tau+1}.$$

Once again, the desired estimate follows by combining (4.19), (4.20) and (4.21). ■

4.5 Proof of Theorem 4.2.3

We now proceed with the proof of Theorem 4.2.3. In each case, we look for a radial solution of the form

$$u(r) = \mu(A+r)^{-k} \tag{4.22}$$

for some constants $k, \mu, A > 0$ that will be specified later. Since $u = u(r)$ is radial, the definition of the p-Laplace operator leads to the identity

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \Big(|u'(r)|^{p-2}u'(r)\Big)' + \frac{n-1}{r}\,|u'(r)|^{p-2}u'(r). \tag{4.23}$$

Applying this identity to the function (4.22), it is easy to check that

$$-\frac{\Delta_p u}{u^{p-1}} = \frac{k^{p-1}}{r(A+r)^p} \cdot \Big[A(n-1) + r\big(n-p-k(p-1)\big)\Big].$$

If $k \leq \frac{n-p}{p-1}$, then the coefficient $n-p-k(p-1)$ is non-negative, so

$$-\frac{\Delta_p u}{u^{p-1}} \cdot r^{\gamma} \geq \frac{k^{p-1}\, r^{\gamma}}{r(A+r)^p} \cdot A(n-1) \geq 0 \quad \text{ for all } r > 0.$$

Otherwise, we have $k > \frac{n-p}{p-1}$ and this similarly gives

$$-\frac{\Delta_p u}{u^{p-1}} \cdot r^{\gamma} \geq \frac{k^{p-1}\, r^{\gamma}}{(A+r)^p} \cdot \big(n-p-k(p-1)\big) = -\frac{C(k,p,n)\cdot r^{\gamma}}{(A+r)^p}$$

for some constant $C(k,p,n) > 0$. Since $\gamma < p$, the right hand side converges to zero uniformly as $A \to \infty$ for all $r \geq 1$. We can thus pick $A > 0$ large enough so that

$$-\frac{\Delta_p u}{u^{p-1}} \cdot r^{\gamma} \geq -\frac{\lambda}{2} \quad \text{ for all } r \geq 1.$$

Using this fact along with our definition (4.22), we conclude that

$$-\Delta_p u + \frac{\lambda}{r^\gamma} \cdot u^{p-1} \geq \frac{\lambda}{2r^\gamma} \cdot u^{p-1} \geq C(\lambda) \cdot \mu^{p-1}(A+r)^{-k(p-1)-\gamma}. \tag{4.24}$$

When it comes to part (i), we assume $q > p-1$ and we may take any

$$k > \max\left\{\frac{n}{m}, \frac{\alpha+\gamma-n}{q-p+1}\right\}. \tag{4.25}$$

Since $km > n > \alpha$, it follows by Lemma 4.4.1 with $\sigma = km > n$ and $\tau = 0$ that

$$(K * u^m)u^q \leq C\mu^{m+q}(A+r)^{\alpha-n-kq} \cdot (\log(A+r))^\theta \quad \text{for all } r \geq R_0,$$

for some $R_0 > 1$. Thus, for all $r \geq R_0$ one has

$$\begin{aligned}(K * u^m)u^q &\leq C\mu^{m+q}(A+r)^{\alpha-n-kq} \cdot (\log(A+r))^\theta \\ &\leq C\mu^{m+q}(A+r)^{-k(p-1)-\gamma}.\end{aligned}$$

Since $(K * u^m)u^q$ and $(A+r)^{-k(p-1)-\gamma}$ are positive and continuous functions for $r \in [1, R_0]$, the above estimate holds for all $r \geq 1$. Combining the last estimate with (4.24), we conclude that u satisfies (4.1) for all small enough $\mu > 0$ because $m+q > q > p-1$.

For part (ii), we assume $q = p-1$ and either $\gamma < n-\alpha$ or else $\gamma = n-\alpha$ with $\theta \leq 0$. In particular, the above argument is still applicable if we replace (4.25) with $k > n/m$.

For part (iii), we assume $p-1+\frac{\alpha+\gamma-n}{n}\,m < q < p-1$. In view of the former condition, we can then choose k so that

$$\frac{n}{m} < k < \frac{n-\alpha-\gamma}{p-1-q}.$$

Since $km > n > \alpha$, it follows by Lemma 4.4.1 with $\sigma = km > n$ and $\tau = 0$ that

$$(K * u^m)u^q \leq C\mu^{m+q}(A+r)^{\alpha-n-kq} \cdot (\log(A+r))^\theta$$

With continuity arguments as before, the above estimate holds for all $r \geq 1$. We now compare the last equation with (4.24). Since the exponent $\alpha-n-kq$ is strictly smaller than the exponent $-k(p-1)-\gamma$, we find that u satisfies (4.1) for any fixed $\mu > 0$ and all large enough $A > 0$.

For part (iv), we assume $q = p - 1 + \dfrac{\alpha + \gamma - n}{n} m < p - 1$ and $\theta < -1$. Let

$$k = \frac{n}{m} = \frac{n - \alpha - \gamma}{p - 1 - q}.$$

Since $km = n > \alpha$, we may apply Lemma 4.4.1 with $\sigma = km = n$ and $\tau = 0$ to derive

$$\begin{aligned}(K * u^m)u^q &\leq C\mu^{m+q}(A + r)^{\alpha - n - kq} \cdot (\log(A + r))^{1+\theta} \\ &= C\mu^{m+q}(A + r)^{-k(p-1)-\gamma} \cdot (\log(A + r))^{1+\theta}.\end{aligned} \tag{4.26}$$

Let us now compare (4.24) and (4.26). Since $\theta < -1$, by taking $A > 1$ large enough we can always ensure that u defined in (4.22) is a solution of (4.1) for any fixed $\mu > 0$.

For part (v), we assume $p - 1 + \frac{\gamma}{\alpha} m < q \leq p - 1 + \frac{\alpha+\gamma-n}{n} m$, which implies that $\gamma < -\alpha$ and $m + q < p - 1$. In this case, we choose k so that

$$\frac{\alpha}{m} < k < -\frac{\alpha + \gamma}{p - 1 - m - q} \leq \frac{n}{m}.$$

Since $\alpha < km < n$, it follows by Lemma 4.4.1 with $\sigma = km < n$ and $\tau = 0$ that

$$\begin{aligned}(K * u^m)u^q &\leq C\mu^{m+q}(A + r)^{\alpha - km - kq} \cdot (\log(A + r))^{\theta} \\ &\leq C\mu^{m+q}(A + r)^{-k(p-1)-\gamma}.\end{aligned}$$

This gives the exact same estimate once again, so the result follows as before. ∎

4.6 Exponentially Decaying Solutions

In some cases it is possible to find exponentially decaying solutions to (4.1).

Theorem 4.6.1. *Assume* (4.4)*, and let* $\gamma < p$, $0 < \varepsilon < 1$, *and* $k \in \mathbb{R}$*. If one of the following conditions holds, then the function*

$$u(r) = \frac{\mu}{(A + r)^k} \exp\left(-\frac{\lambda^{\frac{2}{p}} p (1 - \varepsilon)^{\frac{2}{p}}}{(p - \gamma)(p - 1)^{\frac{1}{p}} (1 + \varepsilon)^{\frac{1}{p}}} (A + r)^{1 - \frac{\gamma}{p}} \right) \tag{4.27}$$

satisfies (4.1) *for all large enough* $A > 0$*, and small enough* $\mu > 0$*.*

(i) $q > p - 1$;

(ii) $q = p-1$ *and either* $\gamma < n-\alpha$ *or* $\gamma = n-\alpha$ *with* $\theta \le 0$.

Proof: We first consider the case $q > p-1$. For simplicity, let

$$B = 1 - \frac{\gamma}{p} \quad \text{and} \quad T = \frac{p(1-\varepsilon)^{\frac{2}{p}}}{(p-\gamma)(p-1)^{\frac{1}{p}}(1+\varepsilon)^{\frac{1}{p}}}.$$

We note that $B, T > 0$ since $\gamma < p$. Now, a quick computation gives

$$\begin{aligned} (A+r)\cdot\frac{u'(r)}{u(r)} &= -k - \lambda^{\frac{2}{p}} TB(A+r)^B \\ (A+r)^2\cdot\frac{u''(r)}{u(r)} &= k(k+1) + \lambda^{\frac{2}{p}} TB(2k-B+1)(A+r)^B + \lambda^{\frac{4}{p}} T^2B^2(A+r)^{2B}. \end{aligned} \tag{4.28}$$

Assuming that $A > 0$ is large enough it follows from (4.28) that

$$-(1+\varepsilon)\cdot\lambda^{\frac{2}{p}} TB(A+r)^B \le (A+r)\cdot\frac{u'(r)}{u(r)} \le -(1-\varepsilon)\cdot\lambda^{\frac{2}{p}} TB(A+r)^B \tag{4.29}$$

at all points. In particular, $u(r)$ is decreasing and we thus have

$$\begin{aligned} \Delta_p u =& |u'(r)|^{p-2}\cdot\left[(p-1)u''(r) + \frac{n-1}{r}u'(r)\right] \\ =& \left[(p-1)\cdot\left(\frac{k(k+1)}{(A+r)^2} + \frac{\lambda^{\frac{2}{p}} TB(2k-B+1)}{(A+r)^{2-B}} + \frac{\lambda^{\frac{4}{p}} T^2B^2}{(A+r)^{2-2B}}\right)\right. \\ &\left. + \frac{n-1}{r}\left(-\frac{k}{A+r} - \frac{\lambda^{\frac{2}{p}} TB}{(A+r)^{1-B}}\right)\right] \times \left|-\frac{k}{A+r} - \frac{\lambda^{\frac{2}{p}} TB}{(A+r)^{1-B}}\right|^{p-2} \times u^{p-1}. \end{aligned} \tag{4.30}$$

From (4.30) we see that

$$\begin{aligned}
\frac{\Delta_p u}{u^{p-1}} \cdot r^\gamma &= \left[\frac{C_0}{(A+r)^2} + \frac{C_1}{(A+r)^{2-B}} + \frac{(p-1)\lambda^{\frac{4}{p}}T^2B^2}{(A+r)^{2-2B}} + \frac{n-1}{r}\left(-\frac{k}{A+r} - \frac{\lambda^{\frac{2}{p}}TB}{(A+r)^{1-B}}\right)\right] \\
&\quad \times \left|-\frac{k}{(A+r)^B} - \lambda^{\frac{2}{p}}TB\right|^{p-2} \cdot \frac{r^\gamma}{(A+r)^{(p-2)(1-B)}} \\
&= \left[\frac{C_0}{(A+r)^{2B}} + \frac{C_1}{(A+r)^B} + (p-1)\lambda^{\frac{4}{p}}T^2B^2 + \frac{n-1}{r}\left(-\frac{k}{(A+r)^{2B-1}} - \frac{\lambda^{\frac{2}{p}}TB}{(A+r)^{B-1}}\right)\right] \\
&\quad \times \left|-\frac{k}{(A+r)^B} - \lambda^{\frac{2}{p}}TB\right|^{p-2} \cdot \frac{r^\gamma}{(A+r)^{p(1-B)}}
\end{aligned}$$

where

$$C_0 = (p-1)k(k+1) \text{ and } C_1 = (p-1)\lambda^{\frac{2}{p}}TB(2k-B+1).$$

We note here that if

$$B > 1 - \frac{\gamma}{p},$$

the above quantity grows without bound as $r \to \infty$, so $u(r)$ will not satisfy (4.1). Now, for $B = 1 - \frac{\gamma}{p}$ we have that

$$\begin{aligned}
\frac{\Delta_p u}{u^{p-1}} \cdot r^\gamma &= \left[\frac{C_0}{(A+r)^{2B}} + \frac{C_1}{(A+r)^B} + (p-1)\lambda^{\frac{4}{p}}T^2B^2 + \frac{n-1}{r}\left(-\frac{k}{(A+r)^{2B-1}} - \frac{\lambda^{\frac{2}{p}}TB}{(A+r)^{B-1}}\right)\right] \\
&\quad \times \left|-\frac{k}{(A+r)^B} - \lambda^{\frac{2}{p}}TB\right|^{p-2} \cdot \frac{r^\gamma}{(A+r)^\gamma}. \qquad (4.31)
\end{aligned}$$

For $\gamma \geq 0$, recalling (4.29), we have that for large enough $A > 0$ (4.31) implies

$$\frac{\Delta_p u}{u^{p-1}} \cdot r^\gamma \leq \left[\frac{C_0}{(A+r)^{2B}} + \frac{C_1}{(A+r)^B} + (p-1)\lambda^{\frac{4}{p}}T^2B^2\right] \cdot \left|-\frac{k}{(A+r)^B} - \lambda^{\frac{2}{p}}TB\right|^{p-2}$$

at all points, and hence

$$\lim_{A\to\infty} \frac{\Delta_p u}{u^{p-1}} \cdot r^{\gamma} \le (p-1)\lambda^{\frac{4}{p}} T^2 B^2 \left| -\lambda^{\frac{2}{p}} TB \right|^{p-2}$$
$$= (p-1)\lambda^2 T^p B^p$$
$$= \frac{(1-\varepsilon)^2 \lambda^2}{1+\varepsilon}.$$

So for all sufficiently large $A > 0$

$$\frac{\Delta_p u}{u^{p-1}} \cdot r^{\gamma} < (1-\varepsilon)\lambda^2$$

at all points. And so we see that, where $C(\lambda)$ is a positive constant,

$$-\Delta_p u + \frac{\lambda^2}{r^{\gamma}} u^{p-1} \ge \frac{\varepsilon\lambda^2}{r^{\gamma}} u^{p-1} = \frac{C(\lambda)\mu^{p-1}}{r^{\gamma}(A+r)^{k(p-1)}} \exp\left(-(p-1)\lambda^{\frac{2}{p}} T(A+r)^B\right)$$

at all points and for all large enough $A > 0$. We now estimate the right hand side of (4.1). Using Lemma 4.4.1, with $\tau = 0$ and $\sigma = n+1$ we see that

$$K * u^m \le \frac{C\mu^m \log(r)^{\theta}}{r^{n-\alpha}}$$

for some constant $C > 0$. This then implies

$$(K * u^m) u^q \le \frac{C\mu^{m+q} \log(r)^{\theta}}{r^{n-\alpha}(A+r)^{kq}} \exp\left(-q\lambda^{\frac{2}{p}} T(A+r)^B\right).$$

But we see that

$$\frac{C\mu^{m+q} \log(r)^{\theta}}{r^{n-\alpha}(A+r)^{kq}} \exp\left(-q\lambda^{\frac{2}{p}} T(A+r)^B\right) \le \frac{C(\lambda)\mu^{p-1}}{r^{\gamma}(A+r)^{k(p-1)}} \exp\left(-(p-1)\lambda^{\frac{2}{p}} T(A+r)^B\right)$$

if and only if

$$\frac{C\mu^{m+q-(p-1)} \log(r)^{\theta}}{r^{n-\alpha-\gamma}(A+r)^{k(q-p+1)}} \exp\left(-(q-p+1)\lambda^{\frac{2}{p}} T(A+r)^B\right) \le C(\lambda), \tag{4.32}$$

which is true at all points for all sufficiently large A, and sufficiently small μ. Now suppose that $\gamma < 0$. We have that

$$-\Delta_p u + \frac{\lambda^2}{r^\gamma} u^{p-1} \geq -\Delta_p u + \lambda^2 u^{p-1} \quad \text{in } \mathbb{R}^n \setminus \overline{B}_1.$$

But the right hand side above corresponds to the case $\gamma = 0$, which we have shown has a solution of form (4.27) for all sufficiently large A, and sufficiently small μ. We thus see that equation (4.1) is satisfied, and hence $u(r)$ is an exponentially decaying solution to (4.1). We now consider the case $q = p - 1$. The above discussion is still valid, and we again arrive at (4.32) which in this case implies

$$\frac{C\mu^m \log(r)^\theta}{r^{n-\alpha-\gamma}} \leq C(\lambda). \tag{4.33}$$

We thus see that if $\gamma < n - \alpha$, then (4.33) is satisfied for all sufficiently small $\mu > 0$, and again $u(r)$ is a solution to (4.1). In the case $\gamma = n - \alpha$, (4.33) is satisfied for all sufficiently small $\mu > 0$, provided $\theta \leq 0$. If $\theta > 0$, then no positive solution to (4.1) exists by Corollary 4.2.2 part (i2). ■

Remark 4.6.2. The decay rate of (4.27) is essentially optimal. Indeed, if the constant B in the proof of Theorem 4.6.1 were any larger, the proof would no longer be valid. If the constant T were any larger, then the exact value of k in relation to the parameters n, m, q, γ, α plays a crucial role in whether or not (4.27) is a solution to (4.1). The case by case analysis of values of k is quite technical and uninformative, so we have omitted it. We note, however, that in the case $p = 2$, an optimal value of k can be found (see [75, Theorem 2]).

Chapter 5

Conclusions and Further Work

To conclude, we now discuss some potential future work that could extend the findings of this thesis. In Chapter 2, we studied a general class of coercive quasilinear systems containing nonlinear gradient terms. We first determined sharp conditions for the existence of positive radial solutions which blow up as $x \to \partial B_R$, and then for the existence of solutions in all of $\mathbb{R}^n$. For these global solutions, we also described their asymptotic behaviour. We were at all times focused on solutions which are radially symmetric, so an obvious question one may ask is: *do the existence conditions apply to non-radial solutions?* This problem is by now well understood in the case of scalar coercive problems with gradient terms, see for example [29, 33] and the references therein. When it comes to systems, however, far less is known, even in the case of non-coercive systems without gradient terms, which are much more intensively studied. Indeed, this is the substance of the Lane-Emden conjecture discussed in the introduction. Seeing as this is such a long standing open problem, with such a vast literature dedicated to it, the case of coercive systems with gradient terms is likely very difficult also.

Another direction the results of Chapter 2 may take us in is to return to the corresponding parabolic problem. Recall from the introduction to Chapter 2 that the systems considered initially arose from studying steady states of the following parabolic system

$$\begin{cases} u_t - \Delta u = v & \text{in } \Omega \times (0, T), \\ v_t - \Delta v = |\nabla u|^2 & \text{in } \Omega \times (0, T). \end{cases}$$

It is known the above system has local in time strong solutions, but the existence of global in time weak solutions is still an open problem. The essential difficulty comes from the gradient term $|\nabla u|^2$, which arises when viscous heating effects are not neglected in the Boussinesq approximation. In [23], where the system was introduced, the authors conjecture that finite time blow up of solutions may occur in a suitable norm. This leads to some natural questions: *can we extend the parabolic stability results of Chapter 3 to the case of systems with gradient terms? If so, can this help shed some light on whether or not finite time blow up occurs?* These questions seem to be poorly understood at the moment.

In Chapter 3, we studied the Hénon-Lane-Emden system

$$\begin{cases} -\Delta u = |x|^k v^p & \text{in } \mathbb{R}^n, \\ -\Delta v = |x|^l u^q & \text{in } \mathbb{R}^n. \end{cases}$$

Recall that the main result of Chapter 3 states that given a positive regular radial solution (u, v), if we are on or above the Joseph-Lundgren curve one has

$$\lim_{|x|\to\infty} \frac{u}{C_\alpha |x|^{-\alpha}} = \lim_{|x|\to\infty} \frac{v}{C_\beta |x|^{-\beta}} = 1,$$

where $\alpha, \beta, C_\alpha, C_\beta$ are some explicit positive parameters. Moreover, this convergence occurs monotonically. The obvious question to ask is: *do these quantities converge to* 1 *below the Joseph-Lundgren curve?*

As discussed in the introduction, when $k = l = 0$ the Hénon-Lane-Emden system corresponds to the Lane-Emden equation

$$-\Delta u = u^p$$

in the special case $p = q, u = v$, and it corresponds to the biharmonic equation

$$\Delta^2 u = u^p$$

in the special case $p > q = 1$. For the Lane-Emden equation, Li [63] used an energy function argument to prove the quantity $|x|^\alpha u$ has a limit as $|x| \to \infty$, and this limit can then easily be shown to equal C_α. In a similar vein, for the biharmonic equation, Gazzola and Grunau [36] introduced a kind of energy function which decreases at successive critical points of $|x|^\alpha u$, and used this to prove $|x|^\alpha u$ has a limit as $|x| \to \infty$. Recall from the introduction that both of these scalar problems have a Joseph-Lundgren exponent, above which $|x|^\alpha u$ converges monotonically, and below which the convergence is oscillatory. It's worth noting that for both of these scalar problems, the fact that $|x|^\alpha u$ has a limit as $|x| \to \infty$ was derived first, and then afterwards the manner of convergence was determined. In Chapter 3, we knew a priori that the Joseph-Lundgren curve exists, thanks to the results of [14]. This motivated the use of monotonicity arguments to prove the desired convergence result above this curve, but this strategy necessarily fails below the curve. In the absence of a suitable energy function, proving that $|x|^\alpha u$ and $|x|^\beta v$ converge below the Joseph-Lundgren curve (if they even do) is quite a challenging problem.

We have seen that for the Hénon-Lane-Emden system, the separation property of solutions does not hold below the Joseph-Lundgren curve. When it comes to whether or not regular solutions oscillate around the singular solution, recent results of Dupaigne, Ghergu and Hajlaoui [27] seem to suggest that oscillations do occur in the $k = l = 0$ case at least. The authors show that regular solutions which are linearly stable outside a compact set cannot exist when $n \leq 10$, which is enough to prove they must oscillate around the singular solution. Left open was the case $n \geq 11$ and p, q lying below the Joseph-Lundgren curve.

When it comes to Chapter 4, a future line of study would be to investigate nonlocal equations containing fully nonlinear differential operators. That is, to consider equations of the form

$$F(D^2u) = (K * u^m)u^q \quad \text{in } \Omega$$

where D^2u is the Hessian of u, and K is a convolution kernel with similar assumptions to Chapter 4. For Ω, some natural choices would be a smooth, bounded domain, the exterior of a compact set, or all of $\mathbb{R}^n$. There is very little in the literature concerning fully nonlinear equations with convolution terms, so basic questions of existence, nonexistence and regularity of solutions are still not well understood.

One possibility for the differential term would be to consider a Pucci extremal type operator $\mathcal{M}^\pm$. Pucci's sup-operator $\mathcal{M}^+$ is defined as

$$\mathcal{M}^+(D^2u) = \lambda \sum_{\mu_i < 0} + \Lambda \sum_{\mu_i > 0}$$

where $0 < \lambda \leq \Lambda$ and μ_i are the eigenvalues of D^2u. Equivalently, one may define it as

$$\mathcal{M}^+(D^2u) = \sup_{A \in \mathcal{A}_{\lambda,\Lambda}} \operatorname{tr}(AD^2u)$$

where $\mathcal{A}_{\lambda,\Lambda}$ denotes the set of $n \times n$ symmetric matrices satisfying $\lambda I_n \leq A \leq \Lambda I_n$. Analogously, Pucci's inf-operator is defined as

$$\mathcal{M}^-(D^2u) = \Lambda \sum_{\mu_i<0} + \lambda \sum_{\mu_i>0} = \inf_{A\in\mathcal{A}_{\lambda,\Lambda}} \mathrm{tr}(AD^2u).$$

These operators are generalisations of the Laplacian, which corresponds to the case $\lambda = \Lambda$. They arise naturally in the context of optimal stochastic control problems, and have been used in applications to mathematical finance [35]. Studying their interaction with convolution nonlinearities is, as of now, a gap in the literature, and is a potentially fruitful research project in the future.

Appendix A

Local Existence

We now prove that system (2.1) has a positive radial solution (u, v) in a small ball B_ρ for sufficiently small $\rho > 0$. Recall that we used this fact, along with a scaling argument, to deduce that system (2.1) has a positive radial solution in B_R for any fixed $R > 0$. The following proof is standard, and uses the Schauder Fixed Point Theorem. We include it here in the Appendix for completeness, and remark that it is inspired by [34, Proposition A1]. Note from Lemma 2.1.5 that $p - 1 - \sigma > 0$, and recall our definition

$$\delta = \frac{(n-1)(p-1-\sigma)}{p-1} > 0.$$

Let $C^1[0, r_0]$, $r_0 > 0$, denote the Banach space of real valued functions which are C^1 in $[0, r_0]$, endowed with the norm $||u|| = ||u||_\infty + ||u'||_\infty$. Define $u_0(r) \equiv u_0 \in C^1[0, r_0]$ and $v_0(r) \equiv v_0 \in C^1[0, r_0]$ where $u_0, v_0 > 0$. Let $X = C^1[0, r_0] \times C^1[0, r_0]$, and consider the map

$$\mathcal{T} : X \to X$$

given by

$$\mathcal{T}[u, v](r) = \begin{pmatrix} \mathcal{T}_1[u, v](r) \\ \mathcal{T}_2[u, v](r) \end{pmatrix}$$

where

$$\begin{cases} \mathcal{T}_1[u, v](r) = u_0 + \displaystyle\int_0^r \left[\frac{\delta}{n-1} s^{-\delta} \int_0^s \tau^\delta f_1(\tau) g_1(v(\tau)) d\tau \right]^{\frac{1}{p-1-\sigma}} ds, \\ \mathcal{T}_2[u, v](r) = v_0 + \displaystyle\int_0^r \left[s^{1-n} \int_0^s \tau^{n-1} f_2(\tau) g_2(v(\tau)) g_3(|u'(\tau)|) d\tau \right]^{\frac{1}{p-1}} ds. \end{cases}$$

The operator T is clearly well defined thanks to the continuity of the functions f_i, g_j. Fix $\varepsilon > 0$ small enough so that $[u_0 - \varepsilon, u_0 + \varepsilon], [v_0 - \varepsilon, v_0 + \varepsilon] \subset \mathbb{R}^+$, and let

$$Y \coloneqq \{(u,v) \in X \mid ||(u,v) - (u_0, v_0)|| \leq \varepsilon\} \subset X,$$

where $||(u,v)|| = ||u|| + ||v||$. It is clear that Y is a closed, convex and bounded subset of X, and we see $(u,v) \in Y$ if and only if $||u - u_0||_\infty + ||v - v_0||_\infty + ||u'||_\infty + ||v'||_\infty \leq \varepsilon$. For $(u,v) \in Y$ we have

$$\begin{aligned} &u([0,r_0]) \subset [u_0 - \varepsilon, u_0 + \varepsilon],\ v([0,r_0]) \subset [v_0 - \varepsilon, v_0 + \varepsilon], \\ &u'([0,r_0]) \subset [-\varepsilon, \varepsilon],\ v'([0,r_0]) \subset [-\varepsilon, \varepsilon] \end{aligned}$$

and for $i, j = 1, 2$ we have by continuity

$$0 \leq f_i(r) \leq F_i \quad 0 < g_j(v(r)) \leq G_j$$

with $0 \leq g_3(u'(r)) \leq G_3$ for $r \in [0, r_0]$. Here we have used the result from Chapter 2 which guarantees any local solution (u,v) of system (2.1) is strictly positive with $u' > 0$ for $r > 0$.

Step 1

We first need to show that $T(Y) \subset Y$ provided r_0 is small enough. Indeed

$$\begin{aligned} ||\mathcal{T}[u,v] - (u_0, v_0)||_\infty &= \int_0^{r_0} \left[\frac{\delta}{n-1} s^{-\delta} \int_0^s \tau^\delta f_1(\tau) g_1(v(\tau)) d\tau \right]^{\frac{1}{p-1-\sigma}} ds \\ &\quad + \int_0^{r_0} \left[s^{1-n} \int_0^s \tau^{n-1} f_2(\tau) g_2(v(\tau)) g_3(|u'(\tau)|) d\tau \right]^{\frac{1}{p-1}} ds \\ &\leq r_0^{\frac{p}{p-1}} \left[(F_2 G_2 G_3)^{\frac{1}{p-1}} + c(n,p,\sigma) \, (F_1 G_1)^{\frac{1}{p-1-\sigma}} \right] \end{aligned}$$

and

$$||\mathcal{T}[u,v]'||_\infty \leq (r_0 F_1 G_1)^{\frac{1}{p-1-\sigma}} + (r_0 F_2 G_2 G_3)^{\frac{1}{p-1}}.$$

So for $r_0 > 0$ small enough we have

$$||\mathcal{T}[u,v] - (u_0, v_0)|| \leq \varepsilon,$$

or in other words $\mathcal{T}(Y) \subset Y$.

Step 2

We now show that $\mathcal{T}$ is a compact operator. Let $(u_k, v_k)_{k\geq 1}$ be a sequence in Y, which is necessarily bounded, and let $r, t \in [0, r_0]$. We have

$$\begin{aligned}
||\mathcal{T}[u_k,v_k](r) - \mathcal{T}[u_k,v_k](t)||_2 &\leq \left| \int_t^r \left[\frac{\delta}{n-1} \int_0^s f_1(\tau) g_1(v_k(\tau)) d\tau \right]^{\frac{1}{p-1-\sigma}} ds \right| \\
&\quad + \left| \int_t^r \left[\int_0^s f_2(\tau) g_2(v_k(\tau)) g_3(|u_k'(\tau)|) d\tau \right]^{\frac{1}{p-1}} ds \right| \\
&\leq |r-t| \left[\left(\frac{\delta}{n-1} r_0 F_1 g_1(v_k(r_0)) \right)^{\frac{1}{p-1-\sigma}} \right. \\
&\qquad \left. + \left(r_0 F_2 g_2(v_k(r_0)) G_3 \right)^{\frac{1}{p-1}} \right],
\end{aligned}$$

where $||\cdot||_2$ is the usual Euclidean norm on $\mathbb{R}^2$. Now, if we denote by

$$\mathcal{I}_{1,k}(r) = \int_0^r \tau^\delta f_1(\tau) g_1(v_k(\tau)) d\tau \quad \text{and} \quad \mathcal{I}_{2,k}(r) = \int_0^r \tau^{n-1} f_2(\tau) g_2(v_k(\tau)) g_3(|u_k'(\tau)|) d\tau,$$

one can similarly show that

$$\begin{aligned}
||\mathcal{T}[u_k,v_k]'(r) - \mathcal{T}[u_k,v_k]'(t)||_2 &\leq \left(\frac{\delta}{n-1} \right)^{\frac{1}{p-1-\sigma}} \left| \left(\frac{\mathcal{I}_{1,k}(r)}{r^\delta} \right)^{\frac{1}{p-1-\sigma}} - \left(\frac{\mathcal{I}_{2,k}(t)}{t^\delta} \right)^{\frac{1}{p-1-\sigma}} \right| \\
&\quad + \left| \left(\frac{\mathcal{I}_{2,k}(r)}{r^{n-1}} \right)^{\frac{1}{p-1}} - \left(\frac{\mathcal{I}_{2,k}(t)}{t^{n-1}} \right)^{\frac{1}{p-1}} \right| \\
&\leq C|r-t|
\end{aligned}$$

for some constant $C > 0$. Therefore the sequence $(\mathcal{T}[u_k, v_k])_{k\geq 1}$ is uniformly bounded and equicontinuous, which by the Arzelà-Ascoli Theorem tells us the operator $\mathcal{T}$ maps bounded sequences into relatively compact sequences with limit points in Y, since Y is closed. In other words, $\mathcal{T}$ is compact.

Step 3

We can apply the Schauder Fixed Point Theorem to conclude that $\mathcal{T}$ has a fixed point (u, v), and (u, v) is a solution of (2.1) in B_ρ for sufficiently small $\rho > 0$.

Bibliography

[1] Boumediene Abdellaoui, Ahmed Attar, and El-Haj Laamri. On the existence of positive solutions to semilinear elliptic systems involving gradient term. *Appl. Anal.*, 98(7):1289–1306, 2019.

[2] Ahmed Attar and Rachid Bentifour. Existence of positive solutions to nonlinear elliptic systems involving gradient term and reaction potential. *Electron. J. Differential Equations*, pages Paper No. 113, 10, 2017.

[3] Céline Azizieh, Philippe Clément, and Enzo Mitidieri. Existence and a priori estimates for positive solutions of p-Laplace systems. *J. Differential Equations*, 184(2):422–442, 2002.

[4] Ahmed Bachir, Jacques Giacomoni, and Guillaume Warnault. Asymptotic behavior of blowing-up radial solutions for quasilinear elliptic systems arising in the study of viscous, heat conducting fluids. *Differential Integral Equations*, 35(9-10):511–530, 2022.

[5] Soohyun Bae and Yūki Naito. Existence and separation of positive radial solutions for semilinear elliptic equations. *J. Differential Equations*, 257(7):2430–2463, 2014.

[6] Soohyun Bae and Yūki Naito. Separation structure of radial solutions for semilinear elliptic equations with exponential nonlinearity. *Discrete Contin. Dyn. Syst.*, 38(9):4537–4554, 2018.

[7] Marie-Françoise Bidaut-Véron and Marta Garcia-Huidobro. Elliptic Hamilton-Jacobi systems and Lane-Emden Hardy-Hénon equations. *Nonlinear Anal.*, 230:Paper No. 113228, 49, 2023.

[8] Marie Françoise Bidaut-Veron and Hector Giacomini. A new dynamical approach of Emden-Fowler equations and systems. *Adv. Differential Equations*, 15(11-12):1033–1082, 2010.

[9] Marie-Françoise Bidaut-Veron and Philippe Grillot. Singularities in elliptic systems with absorption terms. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 28(2):229–271, 1999.

[10] Marie-Francoise Bidaut-Veron. Local behaviour of the solutions of a class of nonlinear elliptic systems. *Adv. Differential Equations*, 5(1-3):147–192, 2000.

[11] Marie-Francoise Bidaut-Véron and Stanislav Pohozaev. Nonexistence results and estimates for some nonlinear elliptic problems. *J. Anal. Math.*, 84:1–49, 2001.

[12] Jérôme Busca and Raul Manásevich. A Liouville-type theorem for Lane-Emden systems. *Indiana Univ. Math. J.*, 51(1):37–51, 2002.

[13] Paolo Caldiroli and Roberta Musina. Rellich inequalities with weights. *Calculus of Variations and Partial Differential Equations*, 45(1):147–164, 2012.

[14] Wenjing Chen, Louis Dupaigne, and Marius Ghergu. A new critical curve for the Lane-Emden system. *Discrete Contin. Dyn. Syst.*, 34(6):2469–2479, 2014.

[15] Philippe Clément, Jacqueline Fleckinger, Enzo Mitidieri, and François de Thélin. Existence of positive solutions for a nonvariational quasilinear elliptic system. *J. Differential Equations*, 166(2):455–477, 2000.

[16] Craig Cowan. Liouville theorems for stable Lane-Emden systems and biharmonic problems. *Nonlinearity*, 26(8):2357, 2013.

[17] Florica-Corina Şt. Cîrstea and Vicenţiu D. Rădulescu. Entire solutions blowing up at infinity for semilinear elliptic systems. *J. Math. Pures Appl. (9)*, 81(9):827–846, 2002.

[18] Daniel Devine. Existence of positive radial solutions of general quasilinear elliptic systems. *Adv. Nonlinear Anal.*, accepted.

[19] Daniel Devine, Marius Ghergu, and Paschalis Karageorgis. Quasilinear elliptic inequalities with nonlinear convolution terms and potentials of slow decay. *Differential Integral Equations*, 36(1-2):1–20, 2023.

[20] Daniel Devine and Paschalis Karageorgis. Monotonic convergence of positive radial solutions for general quasilinear elliptic systems. *Nonlinearity*, 37(3):Paper No. 035020, 14, 2024.

[21] Daniel Devine and Paschalis Karageorgis. Stability of positive radial steady states for the parabolic Hénon-Lane-Emden system. 2024. Preprint, https://arxiv.org/abs/2406.14997.

[22] Daniel Devine and Gurpreet Singh. Existence and boundary behaviour of radial solutions for weighted elliptic systems with gradient terms. *Eur. J. Math.*, 9(4):106, 2023.

[23] J. I. Díaz, M. Lazzo, and P. G. Schmidt. Large solutions for a system of elliptic equations arising from fluid dynamics. *SIAM J. Math. Anal.*, 37(2):490–513, 2005.

[24] J. I. Díaz, J. M. Rakotoson, and P. G. Schmidt. Local strong solutions of a parabolic system related to the Boussinesq approximation for buoyancy-driven flow with viscous heating. *Adv. Differential Equations*, 13(9-10):977–1000, 2008.

[25] Wei Yue Ding and Wei-Ming Ni. On the elliptic equation $\Delta u + Ku^{(n+2)/(n-2)} = 0$ and related topics. *Duke Math. J.*, 52(2):485–506, 1985.

[26] Louis Dupaigne. *Stable solutions of elliptic partial differential equations*. CRC press, 2011.

[27] Louis Dupaigne, Marius Ghergu, and Hatem Hajlaoui. Asymptotically homogeneous solutions of the supercritical Lane-Emden system. *arXiv preprint arXiv:2312.07097*, 2023.

[28] M. Escobedo and M. A. Herrero. Boundedness and blow up for a semilinear reaction-diffusion system. *J. Differential Equations*, 89(1):176–202, 1991.

[29] Alberto Farina and James Serrin. Entire solutions of completely coercive quasilinear elliptic equations, II. *J. Differential Equations*, 250(12):4409–4436, 2011.

[30] Alberto Ferrero, Hans-Christoph Grunau, and Paschalis Karageorgis. Supercritical biharmonic equations with power-type nonlinearity. *Ann. Mat. Pura Appl. (4)*, 188(1):171–185, 2009.

[31] Roberta Filippucci. Nonexistence of nonnegative solutions of elliptic systems of divergence type. *J. Differential Equations*, 250(1):572–595, 2011.

[32] Roberta Filippucci. Quasilinear elliptic systems in $\mathbb{R}^N$ with multipower forcing terms depending on the gradient. *J. Differential Equations*, 255(7):1839–1866, 2013.

[33] Roberta Filippucci, Patrizia Pucci, and Marco Rigoli. Nonlinear weighted p-Laplacian elliptic inequalities with gradient terms. *Commun. Contemp. Math.*, 12(3):501–535, 2010.

[34] Roberta Filippucci and Federico Vinti. Coercive elliptic systems with gradient terms. *Adv. Nonlinear Anal.*, 6(2):165–182, 2017.

[35] Wendell H. Fleming and H. Mete Soner. *Controlled Markov processes and viscosity solutions*, volume 25 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, 1993.

[36] Filippo Gazzola and Hans-Christoph Grunau. Radial entire solutions for supercritical biharmonic equations. *Math. Ann.*, 334(4):905–936, 2006.

[37] Filippo Gazzola and Hans-Christoph Grunau. Global solutions for superlinear parabolic equations involving the biharmonic operator for initial data with optimal slow decay. *Calc. Var. Partial Differential Equations*, 30(3):389–415, 2007.

[38] Abdejabbar Ghanmi, Habib Mâagli, Vicenţiu Rădulescu, and Noureddine Zeddini. Large and bounded solutions for a class of nonlinear Schrödinger stationary systems. *Anal. Appl. (Singap.)*, 7(4):391–404, 2009.

[39] Marius Ghergu, Jacques Giacomoni, and Gurpreet Singh. Global and blow-up radial solutions for quasilinear elliptic systems arising in the study of viscous, heat conducting fluids. *Nonlinearity*, 32(4):1546–1569, 2019.

[40] Marius Ghergu, Paschalis Karageorgis, and Gurpreet Singh. Quasilinear elliptic inequalities with Hardy potential and nonlocal terms. *Proc. Roy. Soc. Edinburgh Sect. A*, 151(3):1075–1093, 2021.

[41] Marius Ghergu, Zeng Liu, Yasuhito Miyamoto, and Vitaly Moroz. Nonlinear inequalities with double Riesz potentials. *Potential Anal.*, 59(1):97–112, 2023.

[42] Marius Ghergu, Yasuhito Miyamoto, and Vitaly Moroz. Polyharmonic inequalities with nonlocal terms. *J. Differential Equations*, 296:799–821, 2021.

[43] Marius Ghergu and Steven D. Taliaferro. Pointwise bounds and blow-up for Choquard-Pekar inequalities at an isolated singularity. *J. Differential Equations*, 261(1):189–217, 2016.

[44] Marius Ghergu and Zhe Yu. Elliptic inequalities with nonlinear convolution and Hardy terms in cone-like domains. *J. Math. Anal. Appl.*, 526(1):Paper No. 127329, 21, 2023.

[45] B. Gidas and J. Spruck. Global and local behavior of positive solutions of nonlinear elliptic equations. *Comm. Pure Appl. Math.*, 34(4):525–598, 1981.

[46] Changfeng Gui, Wei-Ming Ni, and Xuefeng Wang. On the stability and instability of positive steady states of a semilinear heat equation in $\mathbf{R}^n$. *Comm. Pure Appl. Math.*, 45(9):1153–1181, 1992.

[47] D. R. Hartree. The wave mechanics of an atom with a non-Coulomb central field. part i. theory and methods. *Mathematical Proceedings of the Cambridge Philosophical Society*, 24(1):89–110, 1928.

[48] D. R. Hartree. The wave mechanics of an atom with a non-Coulomb central field. part ii. some results and discussion. *Mathematical Proceedings of the Cambridge Philosophical Society*, 24(1):111–132, 1928.

[49] D. R. Hartree. The wave mechanics of an atom with a non-Coulomb central field. part iii. term values and intensities in series in optical spectra. *Mathematical Proceedings of the Cambridge Philosophical Society*, 24(3):426–437, 1928.

[50] Shoichi Hasegawa. Separation phenomena of radial solutions to the Lane-Emden equation on non-compact Riemannian manifolds. *J. Math. Anal. Appl.*, 510(2):Paper No. 126028, 14, 2022.

[51] Morris W. Hirsch. Systems of differential equations that are competitive or cooperative. V. Convergence in 3-dimensional systems. *J. Differential Equations*, 80(1):94–106, 1989.

[52] Morris W. Hirsch. Systems of differential equations that are competitive or cooperative. IV. Structural stability in three-dimensional systems. *SIAM J. Math. Anal.*, 21(5):1225–1234, 1990.

[53] Josephus Hulshof and Robertus C. A. M. van der Vorst. Asymptotic behaviour of ground states. *Proc. Amer. Math. Soc.*, 124(8):2423–2431, 1996.

[54] Daniel D Joseph and Thomas S Lundgren. Quasilinear Dirichlet problems driven by positive sources. *Archive for Rational Mechanics and Analysis*, 49:241–269, 1973.

[55] Paschalis Karageorgis. Stability and intersection properties of solutions to the nonlinear biharmonic equation. *Nonlinearity*, 22(7):1653–1661, 2009.

[56] Paschalis Karageorgis. Asymptotic expansion of radial solutions for supercritical biharmonic equations. *NoDEA Nonlinear Differential Equations Appl.*, 19(4):401–415, 2012.

[57] Paschalis Karageorgis and Walter A. Strauss. Instability of steady states for nonlinear wave and heat equations. *J. Differential Equations*, 241(1):184–205, 2007.

[58] J. B. Keller. On solutions of $\Delta u = f(u)$. *Comm. Pure Appl. Math.*, 10:503–510, 1957.

[59] Alan V. Lair. Entire large solutions to semilinear elliptic systems. *J. Math. Anal. Appl.*, 382(1):324–333, 2011.

[60] Alan V. Lair and Aihua W. Wood. Large solutions of semilinear elliptic problems. *Nonlinear Anal.*, 37(6):805–812, 1999.

[61] Alan V. Lair and Aihua W. Wood. Existence of entire large positive solutions of semilinear elliptic systems. *J. Differential Equations*, 164(2):380–394, 2000.

[62] Xinfu Li, Shiwang Ma, and Guang Zhang. Solutions to upper critical fractional Choquard equations with potential. *Adv. Differential Equations*, 25(3-4):135–160, 2020.

[63] Yi Li. Asymptotic behavior of positive solutions of equation $\Delta u + K(x)u^p = 0$ in $\mathbf{R}^n$. *J. Differential Equations*, 95(2):304–330, 1992.

[64] Shuaishuai Liang and Yueqiang Song. Nontrivial solutions of quasilinear Choquard equation involving the p-Laplacian operator and critical nonlinearities. *Differential Integral Equations*, 35(5-6):359–370, 2022.

[65] Elliott H. Lieb. Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation. *Studies in Appl. Math.*, 57(2):93–105, 1976/77.

[66] Chang-Shou Lin. A classification of solutions of a conformally invariant fourth order equation in $\mathbf{R}^n$. *Comment. Math. Helv.*, 73(2):206–231, 1998.

[67] P.-L. Lions. The Choquard equation and related questions. *Nonlinear Anal.*, 4(6):1063–1072, 1980.

[68] Bingchen Liu and Hongyan Lin. A Cauchy problem of spatial-weighted reaction-diffusion equations. *Appl. Math. Lett.*, 92:128–133, 2019.

[69] Yi Liu, Yi Li, and Yinbin Deng. Separation property of solutions for a semilinear elliptic equation. *J. Differential Equations*, 163(2):381–406, 2000.

[70] G. Lu and B. D. Sleeman. Maximum principles and comparison theorems for semilinear parabolic systems and their applications. *Proc. Roy. Soc. Edinburgh Sect. A*, 123(5):857–885, 1993.

[71] Nejib Mahmoudi, Philippe Souplet, and Slim Tayachi. Improved conditions for single-point blow-up in reaction-diffusion systems. *J. Differential Equations*, 259(5):1898–1932, 2015.

[72] Enzo Mitidieri. Nonexistence of positive solutions of semilinear elliptic systems in $\mathbf{R}^N$. *Differential Integral Equations*, 9(3):465–479, 1996.

[73] Kiyoshi Mochizuki and Qing Huang. Existence and behavior of solutions for a weakly coupled system of reaction-diffusion equations. *Methods Appl. Anal.*, 5(2):109–124, 1998.

[74] Marcelo Montenegro. Minimal solutions for a class of elliptic systems. *Bull. London Math. Soc.*, 37(3):405–416, 2005.

[75] Vitaly Moroz and Jean Van Schaftingen. Nonexistence and optimal decay of supersolutions to Choquard equations in exterior domains. *J. Differential Equations*, 254(8):3089–3145, 2013.

[76] Vitaly Moroz and Jean Van Schaftingen. A guide to the Choquard equation. *J. Fixed Point Theory Appl.*, 19(1):773–813, 2017.

[77] Robert Osserman. On the inequality $\Delta u \geq f(u)$. *Pacific J. Math.*, 7:1641–1647, 1957.

[78] S. I. Pekar. *Untersuchungen über die Elektronentheorie der Kristalle*. De Gruyter, Berlin, Boston, 1954.

[79] Peter Poláčik, Pavol Quittner, and Philippe Souplet. Singularity and decay estimates in superlinear problems via Liouville-type theorems. I. Elliptic equations and systems. *Duke Math. J.*, 139(3):555–579, 2007.

[80] S. Rawat and K. Sreenadh. Three solutions for a fractional elliptic problem with asymmetric critical Choquard nonlinearity. *Differential Integral Equations*, 35(1-2):89–121, 2022.

[81] James Serrin and Henghui Zou. Non-existence of positive solutions of Lane-Emden systems. *Differential Integral Equations*, 9(4):635–653, 1996.

[82] James Serrin and Henghui Zou. Existence of positive entire solutions of elliptic Hamiltonian systems. *Communications in Partial Differential Equations*, 23(3-4):375–398, 1998.

[83] James Serrin and Henghui Zou. Existence of positive solutions of the Lane-Emden system. *Atti Sem. Mat. Fis. Univ. Modena*, 46:369–380, 1998.

[84] Gurpreet Singh. Classification of radial solutions for semilinear elliptic systems with nonlinear gradient terms. *Nonlinear Anal.*, 129:77–103, 2015.

[85] Philippe Souplet. The proof of the Lane-Emden conjecture in four space dimensions. *Adv. Math.*, 221(5):1409–1427, 2009.

[86] Philippe Souplet. Single-point blow-up for a semilinear parabolic system. *J. Eur. Math. Soc. (JEMS)*, 11(1):169–188, 2009.

[87] Xuefeng Wang. On the Cauchy problem for reaction-diffusion equations. *Trans. Amer. Math. Soc.*, 337(2):549–590, 1993.

[88] Michael Winkler. Refined asymptotics for entire solutions of a biharmonic equation with a supercritical nonlinearity. *Math. Ann.*, 348(3):633–666, 2010.