14th International Conference on Applications of Statistics and Probability in Civil Engineering, ICASP14
Dublin, Ireland, July 9-13, 2023

Consistent Sensitivity Indices for p-Boxes

Clemens Hübler
Post-Doc, Leibniz Universität Hannover, ISD, ForWind, Hannover, Germany

ABSTRACT: The main objective of global sensitivity analyses is to determine the influence of uncertain input variables of a model on the outputs of the same model. Normally, for this purpose the uncertain input variables are modelled using probabilistic approaches, i.e., statistical distributions. This works fine for input variables featuring aleatory uncertainty. However, if the uncertainty is epistemic, the prerequisites for probabilistic modelling are not always given, so that possibilistic approaches, e.g., intervals or fuzzy sets, can be an alternative. If an input variable features both, aleatory and epistemic uncertainty, imprecise probabilistic uncertainty models, e.g., p-boxes, can be used. For p-boxes, standard global sensitivity analyses do not work anymore. This is why recently, research on sensitivity analyses for p-boxes has been done. However, current sensitivity indices in the context p-boxes are not defined mathematically consistently, e.g., their convergence behaviour for diminishing epistemic and/or aleatory uncertainty is misleading. Therefore, in this contribution, mathematically more consistent sensitivity indices, which can be used for p-boxes, are proposed and tested using several mathematical test functions.

Global sensitivity analyses are methods that can be used to determine the overall relevance of uncertain input variables of a (simulation) model by assessing their of influence on the outputs of the same model. However, many state-of-the-art global sensitivity analyses (Saltelli et al. (2008); Iooss and Lemaître (2015); Morris (1991); Hübler (2020)) focus on uncertainty that can be modelled using probabilistic approaches, i.e., statistical distributions. For input variables featuring aleatory uncertainty, i.e., non-reducible uncertainty, for example, due to random physical variations, this is not problematic, since probabilistic approaches are the most suitable uncertainty model for such inputs. However, input variables might also feature epistemic uncertainty. Epistemic uncertainty is reducible uncertainty, for example, due to a lack of knowledge or limited, sparse data. For variables featuring epistemic uncertainty, probabilistic approaches are not always suitable, since the prerequisites for probabilistic modelling might not be given. For example, if the uncertainty stems from limited, sparse data the requirement of a high number of data points is definitely violated. The epistemic uncertainty of such input variables is frequently modelled using possibilistic approaches, e.g., intervals or fuzzy sets. If an input variable features both, aleatory and epistemic uncertainty, imprecise probabilistic uncertainty models, e.g., p-boxes, can be used to model it (Beer et al. (2013)).

If uncertainty is either modelled using non-probabilistic approaches – i.e., possibilistic or imprecise probabilistic uncertainty models – standard sensitivity analyses can not be applied. Similarly, if effects of aleatory and epistemic uncertainty – both being modelled using probabilistic approaches – shall be differentiated, more advanced sensitivity analyses are required as well. For the latter, recently, new sensitivity methods have been developed, which focus on the computation of sensitivity indices for input variables featuring epistemic uncertainty for a given aleatory uncertainty in these or other input variables (Ehre et al. (2018); Pivovarov et al. (2019)). For the former – being the focus of this work, first approaches have already been proposed about 20 years ago (Ferson et al. (2004);

Ferson and Tucker (2006); Oberguggenberger et al. (2009)). These approaches are generally suitable to determine sensitivity indices for input variables, which are modelled using p-boxes. However, as they are feature several shortcomings, e.g., a missing or complicated differentiation of epistemic and aleatory effects or mathematically inconsistencies (see also Section 1.3), recent work focussed on new sensitivity indices for p-boxes (Schöbi and Sudret (2019); Bi et al. (2019); Wu et al. (2022)). Although some of these new approaches solved the problem of differentiating the effects of epistemic and aleatory uncertainty (Schöbi and Sudret (2019); Wu et al. (2022)), mathematical consistency is still not given. For example, the index for the epistemic uncertainty of Schöbi and Sudret (2019) is zero if the aleatory uncertainty is zero, even if the epistemic uncertainty is greater than zero. This is why in this contribution, mathematically more consistent sensitivity indices, which can be used for p-boxes, are proposed and tested using several mathematical test functions. The proposed indices improve the consistency significantly compared to state-of-the-art indices.

1. MATHEMATICAL CONSISTENCY

The objective of this work is to increase the mathematical consistency of sensitivity indices for p-boxes. To motivate this objective, in the following subsections, it is explained what is meant by mathematical consistency and why it is not given by state-of-the-art indices.

1.1. Explanation of terms

Since p-boxes are used to model epistemic and aleatory uncertainty at the same time, sensitivity analyses for p-boxes should enable a differentiation between the two. Hence, two separate indices for the epistemic and the aleatory uncertainty should be defined. Moreover, it can be valuable to define a third index covering the coupling effects between epistemic and aleatory uncertainty, as it might happen that only a combination of both types of uncertainties has an influence (see Section 2). In the following, these three indices S and the corresponding uncertainties ε are denote as $S_{\text{ep},i}$, $S_{\text{al},i}$, $S_{\text{al,ep},i}$, $\varepsilon_{\text{ep},i}$ and $\varepsilon_{\text{al},i}$, where i is the index of the input variable the sensitivity index or uncertainty corresponds to.

To achieve mathematical consistency, the following statements should be fulfilled, where j is the index of any another input variable:

I) $S_{\text{ep},i} = 0$ if $\varepsilon_{\text{ep},i} = 0$ and
$S_{\text{al},i} = 0$ if $\varepsilon_{\text{al},i} = 0$

II) $S_{\text{ep},i} > 0$ if $\varepsilon_{\text{ep},i} > 0$ even if $\varepsilon_{\text{al},i} = 0$ and
$S_{\text{al},i} > 0$ if $\varepsilon_{\text{al},i} > 0$ even if $\varepsilon_{\text{ep},i} = 0$

III) $S_{\text{al,ep},i} = 0$ if $\varepsilon_{\text{ep},i} = 0$ or $\varepsilon_{\text{al},i} = 0$

IV) $S_{\text{ep},i} \approx \text{const}$ if $\varepsilon_{\text{al},i} \neq \text{const}$ or $\varepsilon_{\text{al},j} \neq \text{const}$ and
$S_{\text{al},i} \approx \text{const}$ if $\varepsilon_{\text{ep},i} \neq \text{const}$ or $\varepsilon_{\text{ep},j} \neq \text{const}$

V) $S_{\text{al},i} \to S_i$ if $\varepsilon_{\text{ep},i} = 0$ and all $\varepsilon_{\text{ep},j} = 0$, where S_i is the standard Sobol' index for probabilistic uncertainty models

Criteria I to III are the most important ones in order to achieve reliable results. If they are not fulfilled, it can, for example, easily happen that an input variable is wrongly interpreted as non-influential, although it affects the model output. The idea of criterion IV is to focus only on one type of uncertainty. Hence, the influence of the other type of uncertainty should be as low as possible to exclude coupling effects. Otherwise, these coupling effects might lead to misinterpretations (see Section 1.3). Criterion V is less important, but helps to increase the compatibility with well-established approaches in the context of probabilistic uncertainty models.

In addition to these criteria for consistency, it is valuable, if first order and total indices for all three indices ($S_{\text{ep},i}$, $S_{\text{al},i}$ and $S_{\text{al,ep},i}$) can be defined in accordance with state-of-the-art Sobol' indices for probabilistic uncertainty models (Saltelli et al. (2008)).

1.2. State-of-the-art indices

As stated in the introduction, sensitivity methods for p-boxes allowing a differentiation of effects of aleatory and epistemic uncertainty have been proposed by Schöbi and Sudret (2019) and Wu et al. (2022). Therefore, their sensitivity indices are briefly summarised in the following. Schöbi and

Sudret (2019) propose a determination of imprecise Sobol' indices:

$$\underline{S}_i = \min S_i(\boldsymbol{\theta}) \quad \text{and} \quad \bar{S}_i = \max S_i(\boldsymbol{\theta}), \tag{1}$$

where $\boldsymbol{\theta}$ is a vector of bounded hyperparameters of length J, which encompasses all hyperparameters of all I cumulative distribution functions (CDFs): $\theta_j \in \left[\underline{\theta}_j, \bar{\theta}_j\right]$, $j = 1, \ldots, J$. Based on the imprecise Sobol' indices, they discuss interpretations of these indices. A first interpretation is the average of the bounded indices which focusses on the output variance, i.e., it can be interpreted as a measure for the aleatory uncertainty ($S_{\text{al},i}$):

$$S_{\text{al,Schöbi},i} = 0.5 \times (\bar{S}_i + \underline{S}_i). \tag{2}$$

A second one focusses on the impact of the epistemic uncertainty ($S_{\text{ep},i}$):

$$S_{\text{ep,Schöbi},i} = \bar{S}_i - \underline{S}_i. \tag{3}$$

Wu et al. (2022) explicitly define sensitivity indices $S_{\text{ep,Wu},i}$ and $S_{\text{al,Wu},i}$ for the epistemic and aleatory uncertainty, respectively. For the epistemic uncertainty, an area metric based on the work of Ferson and Tucker (2006) is proposed. For this purpose, first, the bounds of the output CDF are computed for the case that epistemic uncertainty is considered in all inputs ($\boldsymbol{\theta}$) and the case that the epistemic uncertainty of the ith input variable is eliminated ($\boldsymbol{\theta}_{\sim i}$):

$$\underline{F}_0(y) = \min F(y|\boldsymbol{\theta}),\ \bar{F}_0(y) = \max F(f|\boldsymbol{\theta}) \tag{4}$$

$$\underline{F}_{\sim i}(y) = \min F(y|\boldsymbol{\theta}_{\sim i}),\ \bar{F}_{\sim i}(y) = \max F(f|\boldsymbol{\theta}_{\sim i}). \tag{5}$$

The area between the bounding CDFs can be calculated

$$A = \int (\bar{F}(y) - \underline{F}(y))dy \tag{6}$$

and used to determine the sensitivity index $S_{\text{ep,Wu},i}$

$$S_{\text{ep,Wu},i} = (A_0 - A_{\sim i}) \times A_0^{-1}. \tag{7}$$

For the aleatory uncertainty, Wu et al. (2022) propose a metric based on the maximum variance. The maximum variance V_{max} is the variance of the (non-parametric) CDF F_{max} for which $\underline{F}(y) \leq F_{\text{max}}(y) \leq \bar{F}(y)\ \forall\ y \in \mathscr{D}_Y$ and $\text{V}_{\text{max}} = \text{Var}(F_{\text{max}}^{-1}(P)) = \max(\text{Var}(F_{\text{k}}^{-1}(P)))\ \forall\ \underline{F}(y) \leq F_{\text{k}}(y) \leq \bar{F}(y)$ and $P = [0,1]$. An illustrative example of F_{max} is shown in Figure 1. Using the maximum variance, the sensitivity index $S_{\text{al,Wu},i}$ can be determined similarly to $S_{\text{ep,Wu},i}$:

$$S_{\text{al,Wu},i} = (\text{V}_{\text{max},0} - \text{V}_{\text{max},\sim i}) \times (\text{V}_{\text{max},0})^{-1}, \tag{8}$$

where $\text{V}_{\text{max},0}$ and $\text{V}_{\text{max},\sim i}$ are the maximum variance calculated for the case that epistemic uncertainty is considered in all inputs and the case that the epistemic uncertainty of the ith input variable is eliminated, respectively.

1.3. *Inconsistencies of indices*

Although the indices presented in the last section enable a separate analysis of epistemic and aleatory effects and, therefore, are a good starting point for further developments, they are not mathematically consistent in many cases. That this why before proposing adapted sensitivity indices in the next section, the inconsistencies of state-of-the-art indices are discussed.

To demonstrate the inconsistencies of some of the indices, a very simple example following the work of Ferson and Tucker (2006) is used. A linear function $y = x_1 + x_2$ is defined. x_1 is an interval ($x_1 \in [4,5]$) and x_2 is a normally distributed random variable ($x_2 \sim \mathscr{N}(\mu = 8, \sigma = 1)$). Hence, x_1 and x_2 can be interpreted as p-boxes with $\varepsilon_{\text{al},1} = 0$ and $\varepsilon_{\text{ep},2} = 0$. The distributions are shown in Figure 1.

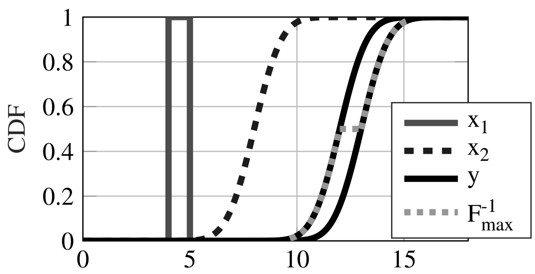

Figure 1: CDFs of x_1, x_2, y and this leading to $V_{max,0}$

Using the imprecise sensitivity indices ($\underline{S}_1 = 0$, $\bar{S}_1 = 0$, $\underline{S}_2 = 1$ and $\bar{S}_2 = 1$) the indices by Schöbi and Sudret (2019) can be computed: $S_{\text{al,Schöbi},1} = 0$, $S_{\text{al,Schöbi},2} = 1$, $S_{\text{ep,Schöbi},1} = 0$ and $S_{\text{ep,Schöbi},2} = 0$. On the one hand, it becomes obvious that the epistemic uncertainty is not correctly estimated, as $S_{\text{ep,Schöbi},1} = 0$, although $\varepsilon_{\text{ep},1} \neq 0$. Hence, the important consistency criterion I – introduced in Section 1.1 – is not fulfilled. On the other hand, $S_{\text{al,Schöbi},i}$ fulfils criteria I, II and V. Moreover, when changing the epistemic uncertainty of x_1 to $x_1 = [4, 4+z]$ and/or of x_2 to $\mu = [8, 8+z]$ with $z \in [0,2]$, $S_{\text{al,Schöbi},2}$ remains constantly 1, so that criterion IV is fulfilled as well. Therefore, it is a suitable starting point for further developments.

When computing the indices by Wu et al. (2022) ($S_{\text{al,Wu},1} = 0$, $S_{\text{al,Wu},2} = 0.88$, $S_{\text{ep,Wu},1} = 1$ and $S_{\text{ep,Wu},2} = 0$), it can be shown that the important criteria I and II are fulfilled. However, the value of $S_{\text{al,Wu},2} < 1$ already indicates that criterion IV is not met, since $S_{\text{al,Wu},2}$ has to be equal to 1 for this example according to criterion IV. When changing the epistemic uncertainty of x_1 to $x_1 = [4, 4+z]$ with $z \in [0,2]$, $S_{\text{al,Wu},2}$ varies between 1 and 0.72. Hence, $\varepsilon_{\text{ep},1}$ has a significant influence on $S_{\text{al,Wu},2}$. Therefore, criterion IV is not fulfilled and this sensitivity index is not consistent even for such a simple, linear and additive model. In contrast, the area metrics ($S_{\text{ep,Wu},1} = 1$ and $S_{\text{ep,Wu},2} = 0$) remain constant, even when changing the aleatory uncertainty of x_1 to $x_1 \sim \mathscr{U}(a = [4,5], b = [4,5] + z_1)$ and/or of x_2 to $\sigma = [1, 1+z_2]$ with $z_1 \in [0,2]$ and $z_2 \in [-0.5, 0.5]$.

To summarise: On the one hand, the index $S_{\text{ep,Schöbi},i}$ is not suitable to represent the epistemic uncertainty, as it can become zero, even if $\varepsilon_{\text{ep},i} \neq 0$ (criterion II); and the index $S_{\text{al,Wu},i}$ should not be used for aleatory uncertainty, as it is significant influenced by $\varepsilon_{\text{ep},j}$, $j \neq i$ (criterion IV) even for linear, additive models. On the other hand, the indices $S_{\text{al,Schöbi},i}$ and $S_{\text{ep,Wu},i}$ are consistent (at least for this simple linear and additive model). Therefore, they are used as a starting point for further developments.

2. FIRST ORDER INDICES

In the previous section, it was shown that $S_{\text{al,Schöbi},i}$ and $S_{\text{ep,Wu},i}$ are suitable, basically consistent sensitivity indices for the determination of the effects of pure aleatory or pure epistemic uncertainty. However, there are still two shortcomings of these indices with respect to the consistency measures stated in Section 1.1. First, in its current form $S_{\text{ep,Wu},i}$ does not differentiate between first order and total effects. And second, coupled aleatory and epistemic effects are (intensionally) not directly covered by these indices. However, coupling effects can be relevant, as demonstrated by the following example: $y = x_1^{20} + x_2$, $x_1 \sim \mathscr{N}(\mu = [-0.5, 0.5], \sigma = 0.2)$ and $x_2 \sim \mathscr{N}(\mu = [1, 10], \sigma = 4)$. For this example, $S_{\text{al,Schöbi},1} = 0.15$ and $S_{\text{ep,Wu},1} = 0.02$. Hence, the pure sensitivity indices indicate that the epistemic and perhaps even the aleatory uncertainty of x_1 can be neglected. However, x_1 only influences y significantly if $x_1 > 1$. This mainly occurs, if both, epistemic and the aleatory uncertainty, are acting. For example, the upper limit of the 99.9th percentile of y is reduced by about 15 % if the epistemic uncertainty is neglected. Hence, in addition to pure sensitivity indices, an index for coupling effects is valuable. Therefore, new sensitivity indices are defined:

$$S_{\text{ep,full},i} = S_{\text{ep,pure},i} + S_{\text{al,ep},i} \text{ and} \tag{9}$$

$$S_{\text{al,full},i} = S_{\text{al,pure},i} + S_{\text{al,ep},i}, \tag{10}$$

where

$$S_{\text{ep,pure},i} = \begin{cases} 0 & \text{for } A_0 = 0 \\ A_i \times A_0^{-1} & \text{otherwise}, \end{cases} \tag{11}$$

$$S_{\text{al,pure},i} = S_{\text{al,Schöbi},i} = 0.5 \times (\bar{S}_i + \underline{S}_i) \text{ and} \tag{12}$$

$$S_{\text{al,ep},i} = \begin{cases} 0 & \text{for } S_{\text{al,pure},i}^{-1} = 0 \\ \frac{(\bar{S}_{i,i} - \underline{S}_{i,i})}{S_{\text{al,pure},i}} & \text{otherwise}, \end{cases} \tag{13}$$

where A_i is the area metric according to Eq. 6 and $\bar{S}_{i,i}$ and $\underline{S}_{i,i}$ are the imprecise Sobol' indices according to Eq. 1 when keeping only the epistemic uncertainty of the ith input variable. Equation 11 is an adaption of the index by Wu et al. (2022) to focus on first order effects. Equation 12 is just the index for the aleatory uncertainty by Schöbi and Sudret (2019). And Eq. 13 is an adaption of the index for the epistemic uncertainty by Schöbi and Sudret (2019). In contrast to the original formulation, it only considers the effect of the ith input variable and normalises the index with $S_{\text{al,pure},i}$. This puts the relevance of coupled effects into relation with pure aleatory effects. Examples demonstrating the better consistency of the new indices are presented in Section 5.

3. TOTAL ORDER INDICES

In Section 1.1, it was mentioned that the definition of first order and total indices in accordance with classical Sobol' indices for probabilistic uncertainty models (Saltelli et al. (2008)) is valuable. Therefore, total indices corresponding to the first order indices of Section 2 are defined:

$$S_{\text{T,ep,full},i} = S_{\text{T,ep,pure},i} + S_{\text{T,al,ep},i} \text{ and} \tag{14}$$

$$S_{\text{T,al,full},i} = S_{\text{T,al,pure},i} + S_{\text{T,al,ep},i}, \tag{15}$$

where

$$S_{\text{T,ep,pure},i} = \begin{cases} 0 & \text{for } A_0 = 0 \\ (A_0 - A_{\sim i}) \times A_0^{-1} & \text{otherwise,} \end{cases} \tag{16}$$

$$S_{\text{T,al,pure},i} = 0.5 \times (\bar{S}_{\text{T},i} + \underline{S}_{\text{T},i}) \text{ and} \tag{17}$$

$$S_{\text{T,al,ep},i} = \begin{cases} 0 & \text{for } S^{-1}_{\text{T,al,pure},i} = 0 \\ \frac{(\bar{S}_{\text{T},i,\sim i} - \underline{S}_{\text{T},i,\sim i})}{S_{\text{T,al,pure},i}} & \text{otherwise,} \end{cases} \tag{18}$$

where $A_{\sim i}$ is the area metric according to Eq. 6 and $\bar{S}_{\text{T},i,\sim i}$ and $\underline{S}_{\text{T},i,\sim i}$ are the imprecise total Sobol' indices when keeping the epistemic uncertainty of all input variables $j \neq i$. Imprecise total Sobol' indices can be computed using Eq. 1 when replacing first order Sobol' indices (S_i) by total Sobol' indices ($S_{\text{T},i}$). Again examples demonstrating the performance of these indices are presented in Section 5.

4. COMPUTATION ASPECTS

In most cases, an analytical determination of the sensitivity indices is not possible. Hence, the indices have to approximated numerically. A discussion of efficient approaches to compute the indices is out of the scope of this work. For the imprecise Sobol' indices, this can be found, for example, in Schöbi and Sudret (2019). For relatively simple models – just like the test functions used in this contribution, a straight-forward approximation using numerical integration for $S_{\text{ep,pure},i}$ and $S_{\text{T,ep,pure},i}$ and Monte Carlo simulation based on quasi-random numbers, e.g., Sobol' sequences, for $S_{\text{al,pure},i}$ and $S_{\text{al,ep},i}$ and the corresponding total indices is sufficient.

5. EXAMPLE RESULTS

To assess the performance of the proposed sensitivity indices, two mathematical test functions are analysed. The first one is a simple additive, but slightly non-linear model. The second one is an adaption of the well-known Ishigami test function (Ishigami and Homma (1990)) which features non-linear and interaction effects of and between the input variables. Here, it is adapted so that the input variables feature aleatory and epistemic uncertainty.

5.1. *Additive model*

The first test function is a non-linear version of the test function by Ferson and Tucker (2006) and is defined as $y = x_1^2 + x_2$ where $x_1 \sim \mathscr{U}(a = [4,5], b = [5,6])$ and $x_2 \sim \mathscr{N}(\mu = [8,9], \sigma = 1)$. It is illustrated in Figure 2.

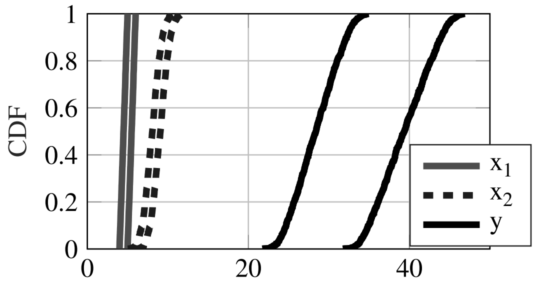

Figure 2: CDFs of x_1, x_2, $y = x_1^2 + x_2$

For this base case, all first order and total sensitivity indices are calculated using Equations 11 to 13 and 16 to 18 and are summarised in Table 1.

Analysing the results for the base case, a few observations can be made. First, the total and the first order indices are always the same. As the test function is an additive model, there are no interactions between the input variables. Therefore, the agreement of total and first order effects is correct. Second, $S_{\text{al,pure},1} > S_{\text{al,pure},2}$ and $S_{\text{ep,pure},1} > S_{\text{ep,pure},2}$. This fact might not be directly clear, since both variables feature an epistemic uncertainty of 1 on their mean values and a roughly comparable aleatory uncertainty. However, since y is a function of x_1^2 and x_2, the similar uncertainty in x_1 and x_2 contributes differently to the uncertainty uncertainty of y. Hence, it is correct that $S_{\text{al,pure},1} > S_{\text{al,pure},2}$ and that $S_{\text{ep,pure},1} > S_{\text{ep,pure},2}$. Finally, the

Table 1: Sensitivity indices for the additive but non-linear test function for different epistemic uncertainties.

Case	i	$S_{\text{al,pure}}$	$S_{\text{T,al,pure}}$	$S_{\text{ep,pure}}$	$S_{\text{T,ep,pure}}$	$S_{\text{al,ep}}$	$S_{\text{T,al,ep}}$
Base Case	1	0.89	0.89	0.91	0.91	0.04	0.04
	2	0.11	0.11	0.09	0.09	0.00	0.00
Reduced $\varepsilon_{\text{ep},1}$	1	0.88	0.88	0.83	0.83	0.02	0.02
	2	0.12	0.12	0.17	0.17	0.00	0.00
$\varepsilon_{\text{ep},1} = 0$	1	0.87	0.87	0.00	0.00	0.00	0.00
	2	0.13	0.13	1.00	1.00	0.00	0.00

coupling term $S_{\text{al,ep},i}$ is interesting. Coupling terms only become relevant if the epistemic uncertainty "shifts" the distribution to areas in the design space, where the aleatory uncertainty has a different effect. Since x_2 contributes linearly to y, $S_{\text{al,ep},2}$ has to be zero, which is the case. For x_1, there is a coupling effect, as there is a non-linear relationship between x_1 and y. However, as the absolute epistemic uncertainty of x_1 is in the same order as its aleatory uncertainty (i.e., relatively small) and the non-linearity is not very pronounced, coupling effects are small.

Having analysed this base case, now, the epistemic uncertainty of x_1 is reduced step by step, yielding $x_1 \sim \mathscr{U}(a = [4, 4.5], b = [5, 5.5])$ (case "Reduced $\varepsilon_{\text{ep},1}$") and $x_1 \sim \mathscr{U}(a = 4, b = 5)$ (case: "$\varepsilon_{\text{ep},1} = 0$"). The corresponding sensitivity indices are also summarised in Table 1. Again, a few observations can be made. First, $S_{\text{al,pure},i}$ is nearly independent of $\varepsilon_{\text{ep},1}$. This confirms the consistency criterion IV. Second, $S_{\text{ep,pure},1}$ and $S_{\text{al,ep},1}$ diminish, as expected, since $\varepsilon_{\text{ep},1}$ directly influences these indices. For $\varepsilon_{\text{ep},1} = 0$, both become zero as well (consistency criteria I and III). In return $S_{\text{ep,pure},2}$ increases up to 1, as the pure sensitivity indices are relative measures, so that a reduction of $S_{\text{ep,pure},1}$ has to yield an increase of $S_{\text{ep,pure},2}$.

To conclude, the new sensitivity indices behave as expected and feature mathematical consistency for this additive, but slightly non-linear test example.

5.2. *Ishigami function*

The second test function is an adaption of the well-known Ishigami function (Ishigami and Homma (1990)) for p-boxes. It is defined as $y = \sin(x_1) + 7 \times \sin(x_2)^2 + 0.1 \times x_3^4 \times \sin(x_1)$ where $x_i \sim \mathscr{U}(a = [-\pi, 0], b = [\pi, 2\pi])\ \forall\, i \in \{1, 2, 3\}$. For this base case, all first order and total sensitivity indices are calculated and summarised in Table 2.

For the base case, the results demonstrate that the new sensitivity indices are quite consistent even for highly non-linear models with interacting input variables. Analysing the definition of the Ishigami function, it becomes clear that x_1 and x_2 should feature first order effects and x_1 and x_3 are interacting, i.e., $S_{\text{al,pure},13} > 0$. Therefore, as expected, the following statements for the indices are true for the aleatory uncertainty: $0 < S_{\text{al,pure},1} < S_{\text{T,al,pure},1} = S_{\text{al,pure},1} + S_{\text{al,pure},13}$, $0 < S_{\text{al,pure},2} = S_{\text{T,al,pure},2}$ and $0 = S_{\text{al,pure},3} < S_{\text{T,al,pure},3} = S_{\text{al,pure},13}$. For the epistemic uncertainty, most of these statements are also true. The only exception is $S_{\text{ep,pure},3} \neq 0$. This exception is due to an interaction of $\varepsilon_{\text{ep},3}$ and $\varepsilon_{\text{al},1}$, which is not excluded in $S_{\text{ep,pure},3}$. Hence, in its current form, the area metric does not fulfil consistency criterion IV for non-additive, non-linear models. Analysing the coupling terms $S_{\text{al,ep}}$ and $S_{\text{T,al,ep}}$, it becomes apparent that they are all zero except $S_{\text{T,al,ep},3}$. This might be unexpected in the first place, since all input variables feature epistemic and aleatory uncertainty. However, it can be explained as follows. The epistemic uncertainty "shifts" the aleatory uncertainty in the design space. Since x_1 and x_2 are periodic over the entire design space, i.e., $\sin(x_1)$ and $\sin(x_2)^2$, their epistemic has no influence on the effect of their aleatory uncertainty so that $S_{\text{al,ep},1} = S_{\text{al,ep},2} = S_{\text{T,al,ep},1} = S_{\text{T,al,ep},2} = 0$. The epistemic uncertainty of x_3 definitely influences the effect of its aleatory uncertainty. Hence, there should be a coupling effect for x_3. Therefore, $S_{\text{T,al,ep},3} > 0$. Interestingly, $S_{\text{al,ep},3} = 0$, although x_3 features coupling effects. The reason for this is that these coupling effects only exist

Table 2: Sensitivity indices for the Ishigami function for different epistemic uncertainties.

Case	i	$S_{\text{al,pure}}$	$S_{\text{T,al,pure}}$	$S_{\text{ep,pure}}$	$S_{\text{T,ep,pure}}$	$S_{\text{al,ep}}$	$S_{\text{T,al,ep}}$	Analytical S_i	Analytical $S_{\text{T},i}$
Base Case	1	0.34	0.78	0.23	0.56	0.00	0.00	–	–
	2	0.22	0.22	0.11	0.11	0.00	0.00	–	–
	3	0.00	0.45	0.32	0.67	0.00	0.91	–	–
$\varepsilon_{\text{ep},1} = 0$	1	0.34	0.78	0.00	0.00	0.00	0.00	–	–
	2	0.22	0.22	0.26	0.26	0.00	0.00	–	–
	3	0.00	0.45	0.71	0.74	0.00	0.91	–	–
$\varepsilon_{\text{ep},i} = 0$ $\forall\, i \in \{1,2,3\}$	1	0.31	0.56	0.00	0.00	0.00	0.00	0.31	0.56
	2	0.44	0.44	0.00	0.00	0.00	0.00	0.44	0.44
	3	0.00	0.24	0.00	0.00	0.00	0.00	0.00	0.24

in the interactive term ($x_3^4 \times \sin(x_1)$), so that they do not influence the first order effect or, simply stated, $S_{\text{al,pure},3} = 0$ yields $S_{\text{al,ep},3} = 0$.

Having analysed this base case, now, first the epistemic uncertainty of x_1 is reduced to zero and second, all epistemic uncertainty is reduced to zero, yielding $x_1 \sim \mathscr{U}(a = -\pi, b = \pi)$ (case "$\varepsilon_{\text{ep},1} = 0$") and $x_i \sim \mathscr{U}(a = -\pi, b = \pi)\ \forall\, i \in \{1,2,3\}$ (case: "$\varepsilon_{\text{ep},i} = 0$"). The corresponding sensitivity indices are also summarised in Table 2. In addition to the observations made for the base case, the following features become apparent for the cases with reduced epistemic uncertainty. First, $S_{\text{T,al,ep},3}$ becomes zero if $\varepsilon_{\text{ep},3} = 0$. This consistent with criterion III. Second, on the one hand, the pure epistemic indices become zero, i.e., $S_{\text{ep,pure},i} = 0$, if the corresponding epistemic uncertainty is reduced to zero, i.e., $\varepsilon_{\text{ep},i} = 0$. This applies to the first order as well as the total effects and fulfils criterion I. On the other hand, the indices for which the epistemic uncertainty is not reduced, i.e., $S_{\text{ep,pure},i}$ for $i \in \{2,3\}$ in case "$\varepsilon_{\text{ep},1} = 0$", increase, as sensitivity indices are relative metrics. Third, the aleatory indices (first order and total) remain fairly constant as long as $\varepsilon_{\text{ep},3}$ is not changed (cf. base case and "$\varepsilon_{\text{ep},1} = 0$"). This is in accordance with criterion IV. However, when changing $\varepsilon_{\text{ep},3}$, criterion IV is no longer fulfilled. The significant non-linear coupling effect of $\varepsilon_{\text{ep},3}$ influences all aleatory sensitivity indices, which would not be the case if criterion IV were met. Finally, it is shown that the pure aleatory indices (first order and total), converge to the classical Sobol' indices if the epistemic uncertainty is reduced to zero. Hence criterion V is also fulfilled.

To summarise, the new sensitivity indices perform quite well for the non-additive and non-linear Ishigami function as well. Consistency criteria I, II, III and V are fulfilled. Only criterion IV is violated in case of high coupling effects of aleatory and epistemic uncertainty. An improvement of the new indices with respect to this criterion is out of the scope of this work.

6. CONCLUSIONS

Methods for global sensitivity analyses, when input variables feature epistemic and aleatory uncertainty and are modelled using p-boxes, are available in literature. However, state-of-the-art sensitivity indices are mathematically quite inconsistent. Therefore, in this work, new, more consistent sensitivity indices have been developed and tested for additive as well non-additive, non-linear models. The new indices are based on the work of Wu et al. (2022) and Schöbi and Sudret (2019).

The proposed indices feature complete consistency with respect to a convergence to zero if the corresponding uncertainty is zero (criterion I and III in Section 1.1) and are greater than zero if the corresponding uncertainty is greater than zero (criterion II in Section 1.1). Therefore, the new indices enable a more accurate determination of the effects of the epistemic and aleatory uncertainty of all input variables of a simulation compared to state-of-the-art approaches. Hence, for example, in the context of factor fixing, non-important input variables can be set to deterministic values or p-boxes of some input variables can be reduced to intervals or to pure statistical distributions based on a more scientifically sound decision.

Certainty, is has to be mentioned that the new indices are still not perfectly consistent. Especially the influence of the epistemic uncertainty on the aleatory indices and vice versa has not been excluded completely for non-additive models (criterion IV in Section 1.1). Improvements regarding a complete separation of the aleatory and epistemic uncertainties are left for future research.

Finally, the question of adequate computing times is an open research question. This question is especially important when using the new indices in an application-oriented engineering context, where computing times of simulation models are frequently quite high. The brute-force approach for the determination of the sensitivity indices used in this work will no longer be applicable. Surely, when replacing physics-based engineering models by mathematical meta-models, the computing time becomes less challenging. Nonetheless, as demonstrated by Schöbi and Sudret (2019), significantly more efficient methods to estimate imprecise sensitivity indices are possible and should be tested in the context of the proposed indices as well.

7. REFERENCES

Beer, M., Ferson, S., and Kreinovich, V. (2013). "Imprecise probabilities in engineering analyses." *Mechanical systems and signal processing*, 37(1-2), 4–29.

Bi, S., Broggi, M., Wei, P., and Beer, M. (2019). "The bhattacharyya distance: Enriching the p-box in stochastic sensitivity analysis." *Mechanical Systems and Signal Processing*, (129), 265–281.

Ehre, M., Papaiannou, I., and Straub, D. (2018). "Efficient estimation of variance-based reliability sensitivities in the presence of multi-uncertainty." *In Proceedings of the IFIP Networking 2018 Conference*.

Ferson, S., Hajagos, J., and Tucker, W. T. (2004). "Probability bounds analysis is a global sensitivity analysis." *In International Conference on Sensitivity Analysis of Model Output (SAMO)*.

Ferson, S. and Tucker, W. T. (2006). "Challenges of order reduction techniques for problems involving polymorphic uncertainty." *Reliability Engineering & System Safety*, 10-11(91), 1435–1442.

Hübler, C. (2020). "Global sensitivity analysis for medium-dimensional structural engineering problems using stochastic collocation." *Reliability Engineering & System Safety*, 195, 106749.

Iooss, B. and Lemaître, P. (2015). "A review on global sensitivity analysis methods.." *Uncertainty Management in Simulation-Optimization of Complex Systems*, 101–122.

Ishigami, T. and Homma, T. (1990). "An importance quantification technique in uncertainty analysis for computer models." *Proc. of the 1st Int. Symposium on Uncertainty Modeling and Analysis*.

Morris, M. D. (1991). "Factorial sampling plans for preliminary computational experiments." *Technometrics*, 33(2), 161–174.

Oberguggenberger, M., King, J., and Schmelzer, B. (2009). "Classical and imprecise probability methods for sensitivity analysis in engineering: A case study." *International Journal of Approximate Reasoning*, 4(50), 680–693.

Pivovarov, D. et al. (2019). "Challenges of order reduction techniques for problems involving polymorphic uncertainty." *GAMM-Mitteilungen*, 2(42), e201900011.

Saltelli, A., Ratto, M., Andres, T., Campolongo, F., Cariboni, J., Gatelli, D., Saisana, M., and Tarantola, S. (2008). *Global sensitivity analysis: the primer*. John Wiley & Sons.

Schöbi, R. and Sudret, B. (2019). "Global sensitivity analysis in the context of imprecise probabilities (p-boxes) using sparse polynomial chaos expansions." *Reliability Engineering & System Safety*, (187), 129–141.

Wu, M., Xiahou, T., Chen, J., and Liu, Y. (2022). "Differentiating effects of input aleatory and epistemic uncertainties on system output: A separating sensitivity analysis approach." *Mechanical Systems and Signal Processing*, (181), 109421.