Diffusion term of the Fokker Planck equation for fractional differential equations enforced by white noise

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ABSTRACT: In many applications of engineering interest, like in the viscoelastic modelling of solids, the equation of motion can have one or more terms containing derivatives of non-integer order. It follows that finding the probability density function (PDF) of a fractional differential equation enforced by white noise is a serious problem because the Itô calculus is not still valid. Indeed, the future state of the response depends upon its entire past history, i.e. the system is not Markovian.

In this paper the loss of Markovianity problem in case of fractional differential equations enforced by a Gaussian white is overcome by using the self-similarity property and the diffusion term of the fractional Fokker Planck (FFPK) equation is calculated showing that it is ruled by a fractional derivative of order 2H, being H the Hurst index.

1. INTRODUCTION

Fractional differential equations (FDE) can be considered as an extension of the classical differential equations since the order of the derivative is a real number rather than an integer one and they are largely used to model different phenomena [1], like viscoelasticity [2-9], heat diffusion [10-16] and biomechanics [17-20]. The response of a FDE enforced by a unit step function is represented by a power law, and thus the Boltzmann superposition reverts into a fractional operator [1]. This means that in order to predict the future state of the response, the entire past history has to be considered.

The response of a FDE to a stochastic input is a stochastic process that can be fully characterized in probabilistic setting by its probability density function (PDF) calculated for each time instant. The evolution of the latter is ruled by the fractional Fokker Planck equation (FFPK).

A stochastic process is defined as self-similar if by changing the temporal scale $\left(t \to at; a > 0\right)$ the statistics in the time axis at are related to the statistics in the time axis t through a coefficient a^{nH} in which n is the order of the statistics and $H > 0$ is the Hurst index [21,22]. An important example of self-similar process is the fractional Brownian motion (fBm) that is the solution of the simplest FDE enforced by a zero-mean Gaussian white noise.

In this paper the diffusion term of the FFPK equation, calculated for the case of fBm by exploiting the self-similarity property of the stochastic process, is presented. Particularly, it is shown that the time derivative in the FFPK equation is of order $2H$. Furthermore, several numerical simulations at various values of the derivative's order have been performed to verify the self-similarity property of the fBm and to prove the validity of the proposed FFPK equation.

2. SELF-SIMILARITY OF THE FRACTIONAL BROWNIAN MOTION

The fractional Brownian motion (fBm) [23-25], labeled as $B_\beta(t)$, can be defined as the solution of the following FDE

$$\begin{cases} {}_0D_t^\beta\left(B_\beta(t)\right)=W(t) \\ B_\beta(0)=0 \quad \text{w.p.1} \end{cases}, \tag{1}$$

in which ${}_0D_t^\beta(\bullet)$ is the Riemann Liouville time fractional derivative of order $\beta>1/2$ and $W(t)$ represents a zero-mean Gaussian white noise process having constant strength equal to the product between a constant q and the unit step function $U(t)$. The fBm can also be defined as the Riemann Liouville time fractional integral of the white noise, i.e. ${}_0I_t^\beta\left(W(t)\right)$, as

$$B_\beta(t)=\frac{1}{\Gamma(\beta)}\int_0^t(t-\tau)^{\beta-1}W(\tau)\,d\tau \tag{2}$$

being $\Gamma(\beta)$ the Euler gamma function.

Since $W(t)$ is a delta-correlated process, the correlation function of the fBm can be calculated in the form [1]

$$\begin{aligned} R_{B_\beta}(t_i,t_j)&=E\left[B_\beta(t_i)B_\beta(t_i)\right]= \\ &\frac{q}{\Gamma^2(\beta)}\int_0^{t_j}\int_0^{t_i}(t_i-\tau_i)^{\beta-1}(t_j-\tau_j)^{\beta-1}\delta(\tau_j-\tau_i)\,d\tau_i d\tau_j \end{aligned} \tag{3}$$

in which $E[\bullet]$ represents the stochastic average, $\delta(\bullet)$ is the Dirac delta function and $t_j>t_i$. The closed form of Eq. (3) can be expressed as [1]

$$\begin{aligned} R_{B_\beta}(t_i,t_j)&=E\left[B_\beta(t_i)B_\beta(t_i)\right]= \\ &\frac{q(-1)^{-\beta}\left|t_j-t_i\right|^{2\beta-1}}{\Gamma^2(\beta)}B\left[-\frac{\min(t_i,t_j)}{\left|t_j-t_i\right|},\beta,\beta\right] \end{aligned} \tag{4}$$

being $B[\bullet]$ the incomplete Euler Beta function. For $t_i=t_j=t$, the correlation function in Eq. (3) coalesces with the variance of the fBm, and thus the latter can be easily obtained in the form

$$\begin{aligned} \sigma_{B_\beta}^2(t)&=\frac{q}{\Gamma^2(\beta)}\int_0^t(t-\tau)^{2\beta-2}U(\tau)\,d\tau \\ &=\frac{q}{\Gamma^2(\beta)}\frac{t^{2\beta-1}}{2\beta-1} \end{aligned}. \tag{5}$$

A stochastic process is defined self-similar if

$$s_n(at_1,\dots,at_n)=a^{nH}s_n(t_1,\dots,t_n) \tag{6}$$

where s_n represents the statistic of order n, $a>0$ is a coefficient and H is the so called Hurst index. Since $B_\beta(t)$ is Gaussian, the fBm is self-similar if Eq. (6) is verified for the second order statistics.

By changing the temporal scale in Eq. (4) and Eq. (5) the correlation function and the variance of the fBm become, respectively,

$$R_{B_\beta}(at_i,at_j)=a^{2\beta-1}R_{B_\beta}(t_i,t_j) \tag{7}$$

and

$$\sigma_{B_\beta}^2(at)=a^{2\beta-1}\sigma_{B_\beta}^2(t). \tag{8}$$

From Eq.(7) and Eq.(8) it is clear that the fBm is self-similar with Hurst index equal to $\beta-1/2$. Obviously, for $\beta=1$ the fBm coalesces with the classical Brownian motion that is self-similar with Hurst index equal to $1/2$.

3. FRACTIONAL FOKKER PLANCK EQUATION

In this section the FFPK equation related to the FDE in Eq. (1) is presented. In order to do this, the evolution of the variance has to be calculated starting from Eq. (5). Particularly, if Eq. (5) is multiplied and divided for $\Gamma(2H)$, it becomes

$$\sigma_{B_\beta}^2(t)=\frac{q\Gamma(2H)}{\Gamma^2(\beta)}\frac{1}{\Gamma(2H)}\int_0^t(t-\tau)^{2\beta-2}U(\tau)\,d\tau. \tag{9}$$

Since the process $B_\beta(t)$ is self-similar with Hurst index $\beta-1/2$, then the integral in Eq. (9) represents the Riemann Liouville fractional integral of order $2H$ and thus Eq. (9) can be rewritten in the form

$$\sigma_{B_\beta}^2(t)=\frac{q\Gamma(2H)}{\Gamma^2(\beta)}\,{}_0I_t^{2H}\left(U(t)\right). \tag{10}$$

By performing the Riemann Liouville fractional derivative of order $2H$ of both members of Eq. (10), the evolution of the variance can be obtained in the form

$$ {}_0D_t^{2H}\left(\sigma_{B_\beta}^2(t)\right)=\frac{q\Gamma(2H)}{\Gamma^2(\beta)}U(t). \tag{11}$$

By exploiting the self-similarity properties of the fBm, the FFPK equation related to Eq. (1) can be expressed as

$$\begin{cases} {}_0D_t^{2H}\left(p_{B_\beta}(b_\beta,t)\right)=\dfrac{q\Gamma(2H)}{2\Gamma^2(\beta)}\dfrac{\partial^2 p_{B_\beta}(b_\beta,t)}{\partial b_\beta^2} \\ p_{B_\beta}^{(r)}\left(b_\beta^{(r)},0\right)=\delta\left(b_\beta^{(r)}\right) \end{cases} \tag{12}$$

in which b_β is the domain of $B_\beta(t)$ and the apex (r) represents the derivative of integer order r.

The validity of Eq. (12) can be simply verified by multiplying both members of Eq. (12) for $b_\beta^2 db_\beta$ and integrating from $-\infty$ to ∞. In this way, Eq. (12) becomes

$$ {}_0D_t^{2H}\left(\sigma_{B_\beta}^2(t)\right)=\frac{q\Gamma(2H)}{2\Gamma^2(\beta)}\int_{-\infty}^{\infty} b_\beta^2\frac{\partial^2 p_{B_\beta}(b_\beta,t)}{\partial b_\beta^2}db_\beta. \tag{13}$$

Considering that $B_\beta(t)$ is a Gaussian process, its PDF can be expressed as

$$ p_{B_\beta}(b_\beta,t)=\frac{1}{\sqrt{2\pi\sigma_{B_\beta}^2(t)}}\exp\left(-\frac{b_\beta^2}{2\sigma_{B_\beta}^2(t)}\right) \tag{14}$$

and thus

$$\int_{-\infty}^{\infty} b_\beta^2\frac{\partial^2 p_{B_\beta}(b_\beta,t)}{\partial b_\beta^2}db_\beta=2U(t). \tag{15}$$

By putting Eq. (15) in Eq. (13), it may be observed that Eq. (13) coalesces with Eq. (11). Moreover, by performing the Riemann Liouville fractional integral of Eq. (13), the closed form of the variance in Eq. (10) can be obtained and thus Eq. (12) can be considered valid. For $\beta=1$ Eq. (12) reverts into the classical Fokker Planck equation, i.e.

$$\begin{cases} \dfrac{\partial p_{B_\beta}(b_\beta,t)}{\partial t}=\dfrac{q}{2}\dfrac{\partial^2 p_{B_\beta}(b_\beta,t)}{\partial b_\beta^2} \\ p_{B_\beta}(b_\beta,0)=\delta(b_\beta) \end{cases}. \tag{16}$$

4. NUMERICAL SIMULATIONS

To verify the self-similarity property of the fBm and prove the validity of the proposed FFPK equation, several numerical simulations at various values of the derivative's order have been performed. Particularly, three different cases have been considered: $\beta=0.75$, $\beta=1.00$ and $\beta=1.25$. For each value of β, 10000 samples of fBm having a duration of 60 s discretized with a time sampling step $\Delta t=0.01\,\mathrm{s}$ have been generated. The variance of the fBm, numerically calculated, has been used to verify the self-similarity condition and, in Figures 1-3, the results obtained have been compared with the closed form of the variance.

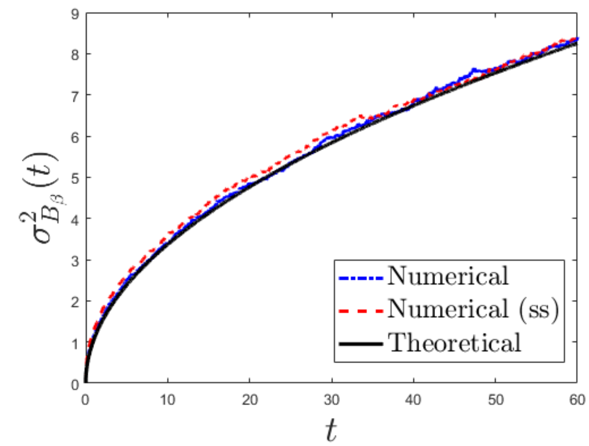

Figure 1 – Theoretical variance (continuous line), numerical variance (dash-dotted line) and numerical variance calculated by exploiting the self-similarity property (dashed line): q=0.8; β=0.75.

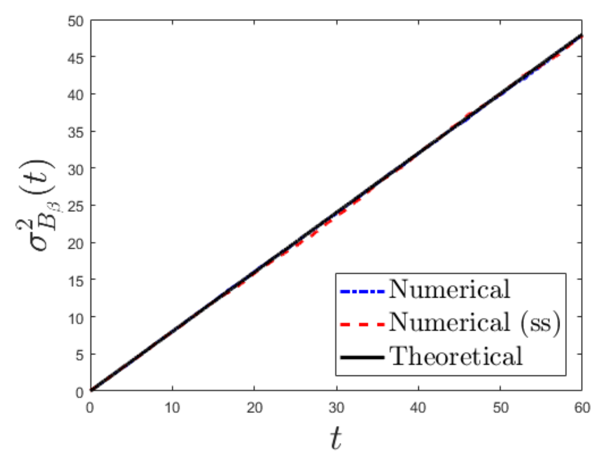

Figure 2 – Theoretical variance (continuous line), numerical variance (dash-dotted line) and numerical

variance calculated by exploiting the self-similarity property (dashed line): q=0.8; β=1.00.

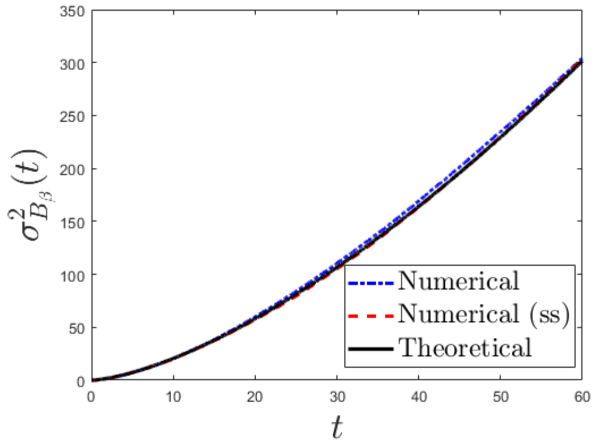

Figure 3 – Theoretical variance (continuous line), numerical variance (dash-dotted line) and numerical variance calculated by exploiting the self-similarity property (dashed line): q=0.8; β=1.25.

From Figures 1-3, it can be observed that the variance obtained by exploiting the self-similarity property provides a good approximation of the closed form.
In Figures 4-6 the PDF of the fBm, numerically calculated, has been compared with the exact one. From Figures 4-6, it can be observed that, once again, the numerical results provide a good approximation of the closed form.

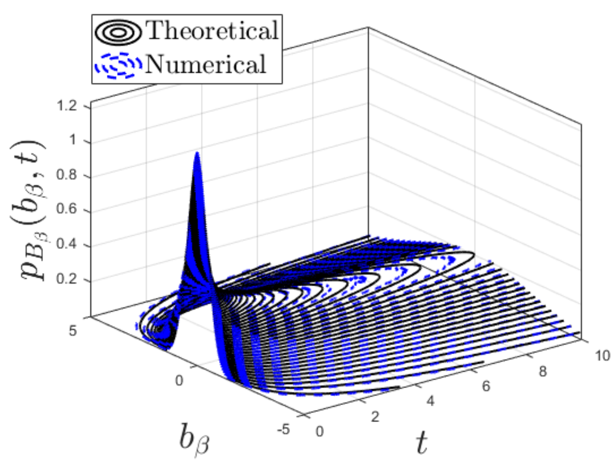

Figure 4 – Theoretical probability density function (continuous line) and numerical probability density function (dashed line): q=0.8; β=0.75.

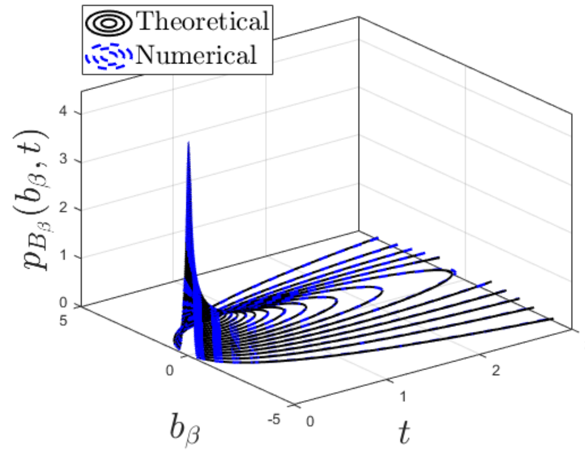

Figure 5 – Theoretical probability density function (continuous line) and numerical probability density function (dashed line): q=0.8; β=1.00.

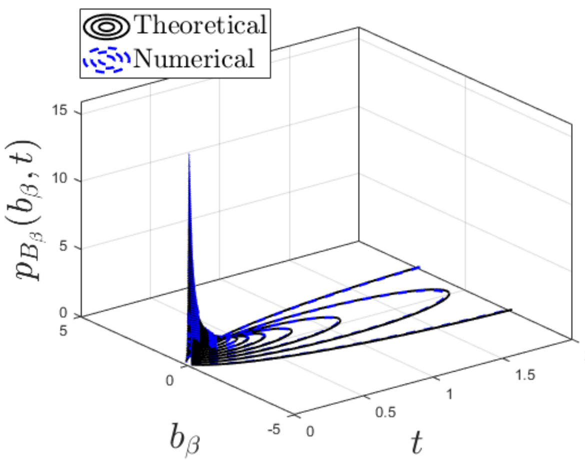

Figure 6 – Theoretical probability density function (continuous line) and numerical probability density function (dashed line): q=0.8; β=1.25.

5. CONCLUSIONS

In this paper the fractional Fokker Planck equation (FFPK) for the case of fractional Brownian motion (fBm) is found by exploiting the self-similarity property. It is shown that the fractional derivative in the diffusive term is of order $2H$ being H the Hurst index of the process that, in case of fBm, is equal to $\beta-1/2$. Several numerical simulations have been performed at various values of the fractional derivative and the results obtained in terms of variance and PDF have been compared with the closed forms, showing that there is a perfect correspondence between the theoretical and the numerical results.

6. APPENDIX – FRACTIONAL OPERATORS

Fractional derivatives and fractional integrals can be considered as an extension to the real order of derivatives and integrals of integer order [26,27]. They are convolutional integrals having a kernel described by a power law. The Caputo's fractional derivative and the Riemann Liouville fractional integral are defined, respectively, as

$$ {}_0^C D_t^{\beta}\left(f(t)\right)=\frac{1}{\Gamma(m-\beta)}\int_0^t (t-\tau)^{m-\beta-1}\frac{d^m f(\tau)}{d\tau^m}d\tau \quad \text{(A.1)}$$

$$ {}_0 I_t^{\beta}\left(f(t)\right)=\frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1} f(\tau)d\tau \quad \text{(A.2)}$$

in which $\Gamma(\bullet)$ represents the Euler Gamma function and m is the integer part of β. If the function $f(t)$ is equal to zero up to $t=0$, then the Riemann Liouville fractional derivative coalesces with the Caputo's fractional derivative. Fractional operators have different properties, like linearity, integration by parts and semi-groups property. Particularly, the following relations are valid:

$$ {}_0 D_t^{\beta_1}\left({}_0 D_t^{\beta_2}\left(f(t)\right)\right)={}_0 D_t^{\beta_2}\left({}_0 D_t^{\beta_1}\left(f(t)\right)\right); \quad \text{(A.3)}$$

$$ {}_0 D_t^{\beta_1}\left({}_0 D_t^{\beta_2}\left(f(t)\right)\right)={}_0 D_t^{\beta_1+\beta_2}\left(f(t)\right); \quad \text{(A.4)}$$

$$ {}_0 I_t^{\beta_1}\left({}_0 I_t^{\beta_2}\left(f(t)\right)\right)={}_0 I_t^{\beta_2}\left({}_0 I_t^{\beta_1}\left(f(t)\right)\right); \quad \text{(A.5)}$$

$$ {}_0 I_t^{\beta_1}\left({}_0 I_t^{\beta_2}\left(f(t)\right)\right)={}_0 I_t^{\beta_1+\beta_2}\left(f(t)\right); \quad \text{(A.6)}$$

$$ {}_0 D_t^{\beta_2}\left({}_0 I_t^{\beta_1}\left(f(t)\right)\right)={}_0 D_t^{\beta_2-\beta_1}\left(f(t)\right);\ \beta_2>\beta_1. \quad \text{(A.7)}$$

Moreover, if the function $f(t)$ is equal to zero up to $t=0$, then

$$ {}_0 I_t^{\beta_1}\left({}_0 D_t^{\beta_2}\left(f(t)\right)\right)={}_0 D_t^{\beta_2}\left({}_0 I_t^{\beta_1}\left(f(t)\right)\right). \quad \text{(A.8)}$$

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