Real Reward Testing for Probabilistic Processes
(Extended Abstract)

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Abstract. We introduce a notion of real reward testing for probabilistic processes by extending the traditional nonnegative reward testing with negative rewards. In this testing framework, the may and must preorders turn out to be the inverse relations of each other. We show that for convergent processes with finitely many states and transitions, but not in the presence of divergence, the real reward must testing preorder coincides with the nonnegative reward must testing preorder. To prove this coincidence we characterise the usual resolution based testing in terms of the weak transitions of processes, without involving policies, adversaries, schedulers, resolutions, or similar structures that are external to the process under investigation. This requires establishing the continuity of our function for calculating testing outcomes.

1 Introduction

Extending the classical testing semantics [6,8] to a setting in which probability and nondeterminism co-exist was initiated in [14]. The application of a test to a process yields a set of probabilities for reaching a success state. *Reward testing* was introduced in [9]; here the success states are labelled by nonnegative real numbers—*rewards*—to indicate degrees of success, and reaching a success state amounts to accumulating the associated reward. In [13] an infinite set of success actions is used to report success, and the testing outcomes are vectors of probabilities of performing these success actions. Compared to [9] this amounts to distinguishing different qualities of success, rather than different quantities.

In [14] and [13], both tests and testees are nondeterministic probabilistic processes, whereas [9] allows nonprobabilistic tests only, thereby obtaining a less discriminating form of testing. In [5] we strengthened reward testing by also allowing probabilistic tests. Taking rewards testing in this form we showed that for finitary processes, i.e. finite-state and finitely branching processes, all three modes of testing lead to the same testing preorders. Thus, vector based testing is no more powerful than *scalar* testing that employs only one success action, and likewise reward testing is no more powerful than the special case of reward testing in which all rewards are 1. [1]

[1] In spite of this there *is* a difference in power between the notions of testing from [14] and [13], but this is an issue that is entirely orthogonal to the distinction between scalar testing,

In certain occasions it is natural to introduce negative rewards. This is the case, for instance, in the theory of Markov decision processes [10]. Intuitively, we could understand negative rewards as costs, while positive rewards are often viewed as benefits or profits. This leads to the question: *if negative rewards are also allowed, how would the original reward testing semantics change?* We refer to the more relaxed form of testing, using positive and negative rewards, as *real reward testing* and the original one (from [9], but with probabilistic tests as in [5]) as *nonnegative reward testing*.

As remarked, [5] established that for finitary processes the nonnegative reward must testing preorder ($\sqsubseteq_{\mathbf{nr}\mathrm{must}}$) coincides with the probabilistic must testing preorder ($\sqsubseteq_{\mathrm{pmust}}$), and likewise for the may preorders. In the present paper, we show that, in contrast to the situation for nonnegative reward (or scalar) testing, for real reward testing the may and must preorders are the inverse of each other, i.e. for any processes Δ and Λ,

$$\Delta \sqsubseteq_{\mathbf{rr}\mathrm{may}} \Lambda \;\text{ iff }\; \Lambda \sqsubseteq_{\mathbf{rr}\mathrm{must}} \Delta. \tag{1}$$

Our main result is that restricted to finitary convergent processes, the real reward must preorder coincides with the nonnegative reward must preorder, i.e. for any finitary convergent processes Δ and Λ,

$$\Delta \sqsubseteq_{\mathbf{rr}\mathrm{must}} \Lambda \;\text{ iff }\; \Delta \sqsubseteq_{\mathbf{nr}\mathrm{must}} \Lambda. \tag{2}$$

Here by convergence we mean that in the pLTS generated by a processes there is no infinite sequence of internal transitions between distributions like

$$\Delta_0 \xrightarrow{\tau} \Delta_1 \xrightarrow{\tau} \cdots$$

Although it is easy to see that in (2) the former implies the latter, to prove that the latter implies the former is far from trivial. Our proof strategy employs a novel characterisation of the usual resolution based testing approach, without using any concept like *policy* [10], *adversary* [11], *scheduler* [12] or *resolution* [5] that is external to the process under investigation; instead we describe the mechanism for gathering test results in terms of the *weak τ-moves* or *derivations* [2] the investigated process can make, and hence speak of *derivation based testing*.

This allows us to exploit the failure simulation preorder $\sqsubseteq_{FS}$ that in [2] was proven to coincide with the probabilistic must testing preorder $\sqsubseteq_{\mathrm{pmust}}$ based on resolutions, at least for finitary processes. Using the derivational characterisation we can show that, for finitary convergent processes, $\sqsubseteq_{FS}$ is contained in $\sqsubseteq_{\mathbf{rr}\mathrm{must}}$. Convergence is essential here, even though it is not needed to establish that $\sqsubseteq_{FS}$ is contained in $\sqsubseteq_{\mathbf{nr}\mathrm{must}}$. Combining this with the results from [5] and [2] mentioned above leads to our required result that $\sqsubseteq_{\mathbf{nr}\mathrm{must}}$ is included in $\sqsubseteq_{\mathbf{rr}\mathrm{must}}$, as far as finitary convergent processes are concerned. Consequently, all the relations in Figure 1 collapses into one.

reward testing and vector based testing. In [13] it is the execution of a success *action* that constitutes success, whereas in [6,8,14,9] it is reaching a success *state* (even though typically success actions are used to identify those states). In [1, Ex 5.3] we showed that state-based testing is (slightly) more powerful than action-based testing. The results presented in [5] about the coincidence of scalar, reward, and vector-based testing preorders pertain to action-based version of each, but in the conclusion it is observed that the same coincidence could be obtained for their state-based versions. In the current paper we stick to state-based testing.

$$(\sqsubseteq_{\mathbf{rr}\text{may}})^{-1} \overset{\text{Thm. 2}}{=} \sqsubseteq_{\mathbf{rr}\text{must}} \overset{\text{Thm. 5}}{=} \sqsubseteq_{\mathbf{nr}\text{must}} \overset{[5]}{=} \sqsubseteq_{\text{pmust}} \overset{[2]}{=} \sqsubseteq_{FS}$$

The symbol $=$ between two relations means that they coincide for finitary convergent processes.

Fig. 1. The relationship of different testing preorders.

In order to establish the agreement of resolution based testing with derivation based testing, we need to show that in the resolution based testing approach our results-collecting function in a deterministic labelled transition system is continuous, which involves some work on the continuity of real-valued functions.

The rest of this paper is organised as follows. We start by recalling notation for probabilistic labelled transition systems. In Section 3 we review the resolution based testing approach and show that the real reward may preorder is simply the inverse of the real reward must preorder. In Section 4 we present derivation based testing. The two results gathering mechanism are compared and shown to agree in Section 5. Then in Section 6 we show that for finitary convergent processes real reward must testing coincides with nonnegative reward must testing. We also present a counterexample showing that this result does not hold in the presence of divergence. We conclude in Section 7.

Due to lack of space, we omit all proofs: they are reported in [3]. Besides the related work already mentioned above, a lot of other studies on probabilistic testing and simulation semantics have appeared in the literature. They are reviewed in [4,1].

2 Probabilistic Processes

A (discrete) probability *subdistribution* over a set S is a function $\Delta : S \rightarrow [0,1]$ with $\sum_{s \in S} \Delta(s) \leq 1$; the *support* of such a Δ is $\lceil \Delta \rceil := \{ s \in S \mid \Delta(s) > 0 \}$, and its *mass* $|\Delta|$ is $\sum_{s \in \lceil \Delta \rceil} \Delta(s)$. A subdistribution is a (total, or full) *distribution* if $|\Delta| = 1$. The point distribution $\overline{s}$ assigns probability 1 to s and 0 to all other elements of S, so that $\lceil \overline{s} \rceil = \{s\}$. With $\mathcal{D}_{sub}(S)$ we denote the set of subdistributions over S, and with $\mathcal{D}(S)$ its subset of full distributions.

Let $\{\Delta_k \mid k \in K\}$ be a set of subdistributions, possibly infinite. Then $\sum_{k \in K} \Delta_k$ is the real-valued function in $S \rightarrow \mathbb{R}$ defined by $(\sum_{k \in K} \Delta_k)(s) := \sum_{k \in K} \Delta_k(s)$. This is a partial operation on subdistributions because for some state s the sum of $\Delta_k(s)$ might exceed 1. If the index set is finite, say $\{1..n\}$, we often write $\Delta_1 + \ldots + \Delta_n$. For p a real number from $[0,1]$ we use $p \cdot \Delta$ to denote the subdistribution given by $(p \cdot \Delta)(s) := p \cdot \Delta(s)$. Finally we use ε to denote the everywhere-zero subdistribution that thus has empty support. These operations on subdistributions do not readily adapt themselves to distributions; yet if $\sum_{k \in K} p_k = 1$ for some collection of $p_k \geq 0$, and the Δ_k are distributions, then so is $\sum_{k \in K} p_k \cdot \Delta_k$. In general when $0 \leq p \leq 1$ we write $x\ {}_p\!\oplus\ y$ for $p \cdot x + (1-p) \cdot y$ where that makes sense, so that for example $\Delta_1\ {}_p\!\oplus\ \Delta_2$ is always defined, and is full if Δ_1 and Δ_2 are.

The expected value $\sum_{s \in S} \Delta(s) \cdot f(s)$ over a subdistribution Δ of a bounded nonnegative function f to the reals or tuples of them is written $\mathrm{Exp}_{\Delta}(f)$, and the image of a subdistribution Δ through a function f is written $\mathrm{Img}_f(\Delta)$ — the latter is the subdistribution over the range of f given by $\mathrm{Img}_f(\Delta)(t) := \sum_{f(s)=t} \Delta(s)$.

Definition 1. A *probabilistic labelled transition system* (pLTS) is a triple $\langle S, L, \rightarrow \rangle$, where

(i) S is a set of states,
(ii) L is a set of transition labels,
(iii) relation $\rightarrow$ is a subset of $S \times L \times \mathcal{D}(S)$.

A (non-probabilistic) labelled transition system (LTS) may be viewed as a degenerate pLTS — one in which only point distributions are used. As with LTSs, we write $s \xrightarrow{\alpha} \Delta$ for $(s, \alpha, \Delta) \in \rightarrow$, as well as $s \xrightarrow{\alpha}$ for $\exists \Delta : s \xrightarrow{\alpha} \Delta$ and $s \rightarrow$ for $\exists \alpha : s \xrightarrow{\alpha}$. A pLTS is *deterministic* if for any state s and label α there is at most one distribution Δ with $s \xrightarrow{\alpha} \Delta$. It is *finitely branching* if the set $\{\Delta \mid s \xrightarrow{\alpha} \Delta,\ \alpha \in L\}$ is finite for all states s; if moreover S is finite, then the pLTS is *finitary*. A subdistribution Δ in an arbitrary pLTS is *finitary* if restricting the state set to the states reachable from Δ yields a finitary sub-pLTS; it is *very finite* if moreover it cannot reach any loop in the pLTS.

Let Act be a set of visible actions which processes can perform, and let $\tau \notin \mathsf{Act}$ be the invisible or internal action. In this paper a *(probabilistic) process* will simply be a distribution in a pLTS with as set of transition labels $\mathsf{Act}_\tau := \mathsf{Act} \cup \{\tau\}$.

We graphically depict processes by drawing the part of the pLTS that is reachable from them as a directed graph with states represented by filled nodes $\bullet$ and distributions by open nodes $\circ$. For any state s and distribution Δ with $s \xrightarrow{\alpha} \Delta$ we draw an edge from s to Δ labelled with α; and for any distribution Δ and state s in $\lceil \Delta \rceil$, the support of Δ, we draw an edge from Δ to s labelled with $\Delta(s)$. We leave out point-distributions—diverting an incoming edge to the unique state in its support.

3 Testing probabilistic processes

A *test* is a distribution in a pLTS with as set of transition labels $\mathsf{Act}_\tau \cup \Omega$, where Ω is a set of fresh *success* actions, not already in Act_τ, for reporting test outcomes. For simplicity we may assume a fixed pLTS of processes—our results apply to any choice of such a pLTS—and a fixed pLTS of tests. Since we *need* certain tests for testing processes, we have to postulate that the pLTS of tests has some degree of universality, in the sense that any test we may need is actually present in that pLTS. In previous papers [4,1,2] we accomplished this by presenting a language, pCSP, and taking an appropriate class of pCSP expressions as the states of our pLTSs (of both tests and processes). Technically this works excellently, but it gives the impression that our work applies to one particular system description language only. Therefore we now achieve universality in a different way. Strong bisimulation between probabilistic processes is defined, for instance, in [7]. This notion can equally well be applied to relate states in different pLTSs. Now we achieve the desired degree of universality of our pLTS of tests by requiring that for any state s in any pLTS, such that $\overline{s}$ is a very finite test, there is a state in our pLTS of tests that is strongly bisimilar to it.

Although we use success *actions*, they are used merely to mark certain states, namely the sources of transitions labelled by success actions, as success states. For this reason we systematically ignore the distributions that can be reached after a success action. We impose two requirements on all states in a pLTS of tests, namely

$$\frac{t \xrightarrow{\alpha}_{\mathbf{T}} \Theta \quad \alpha \notin \mathsf{Act}}{t \| p \xrightarrow{\alpha} \Theta \| \overline{p}} \qquad \frac{p \xrightarrow{\alpha}_{\mathbf{P}} \Delta \quad \alpha \notin \mathsf{Act}}{t \| p \xrightarrow{\alpha} \overline{t} \| \Delta} \qquad \frac{t \xrightarrow{a}_{\mathbf{T}} \Theta \quad p \xrightarrow{a}_{\mathbf{P}} \Delta \quad a \in \mathsf{Act}}{t \| p \xrightarrow{\tau} \Theta \| \Delta}$$

Fig. 2. Synchronous parallel composition between tests and processes

(A) if $t \xrightarrow{\omega_1}$ and $t \xrightarrow{\omega_2}$ with $\omega_1, \omega_2 \in \Omega$ then $\omega_1 = \omega_2$.
(B) if $t \xrightarrow{\omega}$ with $\omega \in \Omega$ and $t \xrightarrow{\alpha} \Delta$ with $\alpha \in \mathsf{Act}_\tau$ then $u \xrightarrow{\omega}$ for all $u \in \lceil \Delta \rceil$.

The first condition says that a success state can have one success identity only, whereas the second condition is slight weakening of the requirement from [9] that success states must be end states; it allows further progress from an ω-success state, for some $\omega \in \Omega$, but only when staying in the realm of ω-success states. [2]

To apply test Θ to process Δ we form a parallel composition $\Theta \| \Delta$ in which *all* visible actions of Δ must synchronise with Θ. The synchronisations are immediately renamed into τ. The resulting composition is a process whose only possible actions are the elements of $\Omega_\tau := \Omega \cup \{\tau\}$. Formally, if $\langle \mathbf{P}, \mathsf{Act}_\tau, \rightarrow_{\mathbf{P}} \rangle$ and $\langle \mathbf{T}, \mathsf{Act}_\tau \cup \Omega, \rightarrow_{\mathbf{T}} \rangle$ are the pLTSs of processes and tests, then the pLTS of applications of tests to processes is $\langle \mathbf{C}, \Omega_\tau, \rightarrow \rangle$, with $\mathbf{C} = \{t \| p \mid t \in \mathbf{T} \wedge p \in \mathbf{P}\}$ and $\rightarrow$ the transition relation generated by the rules in Fig. 2. Here if $\Theta \in \mathcal{D}(\mathbf{T})$ and $\Delta \in \mathcal{D}(\mathbf{P})$, then $\Theta \| \Delta$ is the distribution given by $(\Theta \| \Delta)(t \| p) := \Theta(t) \cdot \Delta(p)$.

Note that, since the pLTS of tests satisfies conditions (A) and (B) above, so does the pLTS of applications of tests to processes; this would not be the case if we had strengthened condition (B) to require that success states must be end states.

We will define the result $\mathcal{A}(\Theta, \Delta)$ of applying the test Θ to the process Δ to be a set of testing outcomes, exactly one of which results from each resolution of the choices in $\Theta \| \Delta$. Each *testing outcome* is an Ω-tuple of real numbers in the interval [0,1], i.e. a function $o : \Omega \rightarrow [0, 1]$, and its ω-component $o(\omega)$, for $\omega \in \Omega$, gives the probability that the resolution in question will reach an *ω-success state*, one in which the success action ω is possible.

Due to the presence of nondeterminism in pLTSs, we need a mechanism to reduce a nondeterministic structure into a set of deterministic structures, each of which determines a possible outcome. Here we adapt the notion of *resolution*, defined in [5] for probabilistic automata, to pLTSs.

Definition 2 (Resolution). A *resolution* of a subdistribution $\Delta \in \mathcal{D}_{sub}(S)$ in a pLTS $\langle S, \Omega_\tau, \rightarrow \rangle$ is a triple $\langle R, \Gamma, \rightarrow_R \rangle$ where $\langle R, \Omega_\tau, \rightarrow_R \rangle$ is a deterministic pLTS and $\Gamma \in \mathcal{D}_{sub}(R)$, such that there exists a *resolving function* $f \in R \rightarrow S$ satisfying

(i) $\mathrm{Img}_f(\Gamma) = \Delta$
(ii) if $r \xrightarrow{\alpha}_R \Gamma'$ for $\alpha \in \Omega_\tau$ then $f(r) \xrightarrow{\alpha} \mathrm{Img}_f(\Gamma')$
(iii) if $f(r) \xrightarrow{\alpha}$ for $\alpha \in \Omega_\tau$ then $r \xrightarrow{\alpha}_R$.

The reader is referred to Section 2 of [5] for a detailed discussion of the concept of resolution, and the manner in which a resolution represents a run of a process; in particular

[2] Justification for imposing such restrictions can be found in Appendix A of [5].

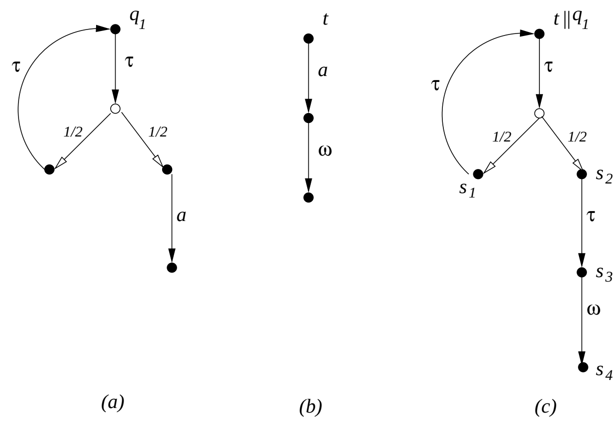

Fig. 3. Testing the process $\overline{q_1}$

in a resolution states in S are allowed to be resolved into distributions, and computation steps can be *probabilistically interpolated*. Our resolutions match the results of applying a scheduler as defined in [12].

We now explain how to associate an outcome with a particular resolution, which in turn will associate a set of outcomes with a subdistribution in a pLTS. Given a deterministic pLTS $\langle R, \Omega_\tau, \rightarrow\rangle$ consider the functional $\mathcal{R} : (R \rightarrow [0,1]^\Omega) \rightarrow (R \rightarrow [0,1]^\Omega)$ defined by

$$\mathcal{R}(f)(r)(\omega) := \begin{cases} 1 & \text{if } r \xrightarrow{\omega} \\ 0 & \text{if } r \stackrel{\omega}{\not\rightarrow} \text{ and } r \stackrel{\tau}{\not\rightarrow} \\ \mathrm{Exp}_\Delta(f)(\omega) & \text{if } r \stackrel{\omega}{\not\rightarrow} \text{ and } r \xrightarrow{\tau} \Delta. \end{cases} \tag{3}$$

We view the unit interval $[0,1]$ ordered in the standard manner as a complete lattice; this induces the structure of a complete lattice on the product $[0,1]^\Omega$ and in turn on the set of functions $R \rightarrow [0,1]^\Omega$. The functional $\mathcal{R}$ is easily seen to be monotonic and therefore has a least fixed point, which we denote by $\mathbb{V}_{\langle R,\Omega_\tau,\rightarrow\rangle}$; this is abbreviated to $\mathbb{V}$ when the resolution in question is understood.

Now we define $\mathcal{A}(\Theta, \Delta)$ to be the set of vectors

$$\mathcal{A}(\Theta, \Delta) \ := \ \{\, \mathrm{Exp}_\Gamma(\mathbb{V}_{\langle R,\Omega_\tau,\rightarrow\rangle}) \mid \langle R, \Gamma, \rightarrow\rangle \text{ is a resolution of } \Theta \| \Delta \,\} \ . \tag{4}$$

Example 1. Consider the process $\overline{q_1}$ depicted in Figure 3(a). When we apply the test $\overline{t}$ depicted in Figure 3(b) to it we get the process $\overline{t \| q_1}$ depicted in Figure 3(c). This process is already deterministic, hence has essentially only one resolution: itself. Moreover the outcome $\mathrm{Exp}_{\overline{t\|q_1}}(\mathbb{V}) = \mathbb{V}(t\|q_1)$ associated with it is the least solution of the equation

$$\mathbb{V}(t\|q_1) = \tfrac{1}{2} \cdot \mathbb{V}(t\|q_1) + \tfrac{1}{2}\overrightarrow{\omega}$$

In fact this equation has a unique solution in $[0,1]^\Omega$, namely $\overrightarrow{\omega}$, with $\overrightarrow{\omega}(\omega) = 1$ and $\overrightarrow{\omega}(\omega') = 0$ for all $\omega' \neq \omega$. Thus $\mathcal{A}(\overline{t}, \overline{q_1}) = \{\overrightarrow{\omega}\}$.

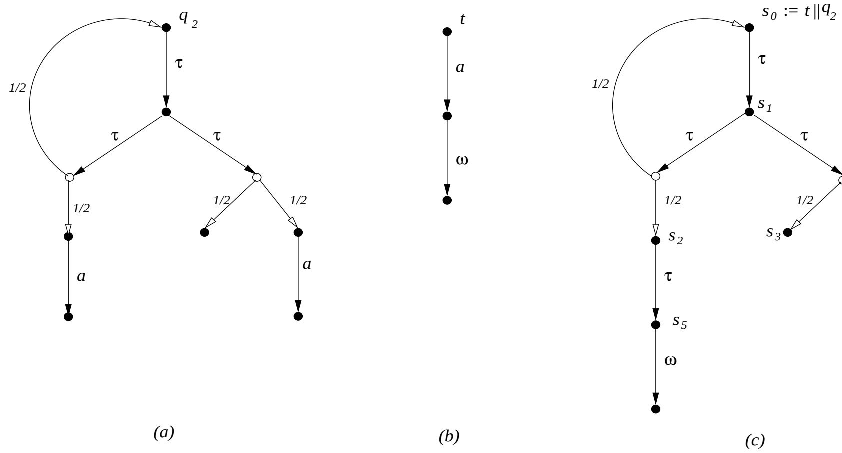

Fig. 4. Testing the process $\overline{q_2}$

Example 2. Consider the process $\overline{q_2}$ and the application of the test $\overline{t}$ to it, as outlined in Figure 4. For each $k \geq 1$ the process $\overline{t \| q_2}$ has a resolution R_k such that $\mathbb{V}(R_k) = (1 - \frac{1}{2^k})\overrightarrow{\omega}$; intuitively it goes around the loop $(k-1)$ times before at last taking the right hand τ action. Thus $\mathcal{A}(\overline{t}, \overline{q_2})$ contains $(1 - \frac{1}{2^k})\overrightarrow{\omega}$ for every $k \geq 1$. But it also contains $\overrightarrow{\omega}$, because of the resolution which takes the left hand τ-move every time. Thus $\mathcal{A}(\overline{t}, \overline{q_2})$ includes the set

$$\{(1-\tfrac{1}{2})\overrightarrow{\omega},\ (1-\tfrac{1}{2^2})\overrightarrow{\omega}, \ldots, (1-\tfrac{1}{2^k})\overrightarrow{\omega}, \ldots, \overrightarrow{\omega}\}$$

As resolutions allow any interpolation between the two τ-transitions from state s_2, $\mathcal{A}(\overline{t}, \overline{q_2})$ is actually the convex closure of the above set.

There are two standard methods for comparing two sets of outcomes:

$O_1 \leq_{\mathrm{Ho}} O_2$ if for every $o_1 \in O_1$ there exists some $o_2 \in O_2$ such that $o_1 \leq o_2$
$O_1 \leq_{\mathrm{Sm}} O_2$ if for every $o_2 \in O_2$ there exists some $o_1 \in O_1$ such that $o_1 \leq o_2$

This gives us our definition of the probabilistic may- and must-testing preorders; they are decorated with $\cdot^{\Omega}$ for the repertoire Ω of testing actions they employ.

Definition 3 (Probabilistic testing preorders).

(i) $\Delta \sqsubseteq^{\Omega}_{\text{pmay}} \Lambda$ if for every Ω-test Θ, $\mathcal{A}(\Theta, \Delta) \leq_{\mathrm{Ho}} \mathcal{A}(\Theta, \Lambda)$.
(ii) $\Delta \sqsubseteq^{\Omega}_{\text{pmust}} \Lambda$ if for every Ω-test Θ, $\mathcal{A}(\Theta, \Delta) \leq_{\mathrm{Sm}} \mathcal{A}(\Theta, \Lambda)$.

These preorders are abbreviated to $\Delta \sqsubseteq_{\text{pmay}} \Lambda$ and $\Delta \sqsubseteq_{\text{pmust}} \Lambda$ when $|\Omega| = 1$.

In [5] we established that for finitary processes $\sqsubseteq^{\Omega}_{\text{pmay}}$ coincides with $\sqsubseteq_{\text{pmay}}$ and $\sqsubseteq^{\Omega}_{\text{pmust}}$ with $\sqsubseteq_{\text{pmust}}$ for any choice of Ω. We also defined the reward testing preorders in terms of the mechanism set up so far. The idea is to associate each success action $\omega \in \Omega$ a reward, which is a nonnegative number in the unit interval $[0, 1]$, and a run of a probabilistic process in parallel with a test yields an expected reward accumulated by those states which can enable success actions. A reward tuple $h \in [0,1]^{\Omega}$ is used to assign reward $h(\omega)$ to success action ω, for each $\omega \in \Omega$. Due to the presence of nondeterminism, the application of a test Θ to a process Δ produces a set of expected rewards. Two sets of rewards can be compared by examining their supremum/infimum elements; this gives us two methods of testing called reward may/must testing. In [5] all rewards are required to be nonnegative, so we refer to that approach of testing as *nonnegative reward testing*. If we also allow negative rewards, which intuitively can be understood as costs, then we obtain an approach of testing called *real reward testing*. Technically, we simply let reward tuples h range over the set $[-1,1]^{\Omega}$. If $o \in [0,1]^{\Omega}$, we use the dot-product $h \cdot o = \sum_{\omega \in \Omega} h(\omega) \cdot o(\omega)$. It can apply to a set $O \subseteq [0,1]^{\Omega}$ so that $h \cdot O = \{h \cdot o \mid o \in O\}$. Let $A \subseteq [-1,1]$. We use the notation $\bigsqcup A$ for the supremum of set A, and $\bigsqcap A$ for the infimum.

Definition 4 (Reward testing preorders).

(i) $\Delta \sqsubseteq^{\Omega}_{\mathbf{nr}\mathrm{may}} \Lambda$ if for every Ω-test Θ and nonnegative reward tuple $h \in [0,1]^{\Omega}$, $\bigsqcup h \cdot \mathcal{A}(\Theta,\Delta) \leq \bigsqcup h \cdot \mathcal{A}(\Theta,\Lambda)$.
(ii) $\Delta \sqsubseteq^{\Omega}_{\mathbf{nr}\mathrm{must}} \Lambda$ if for every Ω-test Θ and nonnegative reward tuple $h \in [0,1]^{\Omega}$, $\bigsqcap h \cdot \mathcal{A}(\Theta,\Delta) \leq \bigsqcap h \cdot \mathcal{A}(\Theta,\Lambda)$.
(iii) $\Delta \sqsubseteq^{\Omega}_{\mathbf{rr}\mathrm{may}} \Lambda$ if for every Ω-test Θ and real reward tuple $h \in [-1,1]^{\Omega}$, $\bigsqcup h \cdot \mathcal{A}(\Theta,\Delta) \leq \bigsqcup h \cdot \mathcal{A}(\Theta,\Lambda)$.
(iv) $\Delta \sqsubseteq^{\Omega}_{\mathbf{rr}\mathrm{must}} \Lambda$ if for every Ω-test Θ and real reward tuple $h \in [-1,1]^{\Omega}$, $\bigsqcap h \cdot \mathcal{A}(\Theta,\Delta) \leq \bigsqcap h \cdot \mathcal{A}(\Theta,\Lambda)$.

This time we drop the superscript Ω if Ω is countably infinite.

It is shown in Corollary 1 of [5] that nonnegative reward testing is equally powerful as probabilistic testing.

Theorem 1 ([5]). *For any finitary processes Δ and Λ,*

(i) $\Delta \sqsubseteq_{\mathbf{nr}\mathit{may}} \Lambda$ *if and only if* $\Delta \sqsubseteq_{\mathrm{pmay}} \Lambda$.
(ii) $\Delta \sqsubseteq_{\mathbf{nr}\mathit{must}} \Lambda$ *if and only if* $\Delta \sqsubseteq_{\mathrm{pmust}} \Lambda$.

In this paper we focus on the real reward testing preorders $\sqsubseteq_{\mathbf{rr}\mathrm{may}}$ and $\sqsubseteq_{\mathbf{rr}\mathrm{must}}$, by comparing them with the nonnegative reward testing preorders $\sqsubseteq_{\mathbf{nr}\mathrm{may}}$ and $\sqsubseteq_{\mathbf{nr}\mathrm{must}}$. We first show that, although the two nonnegative reward testing preorders are in general incomparable, the two real reward testing preorders are simply the inverse relations of each other.

Theorem 2. *For any processes Δ and Λ, it holds that* $\Delta \sqsubseteq_{\mathbf{rr}\mathit{may}} \Lambda$ *if and only if* $\Lambda \sqsubseteq_{\mathbf{rr}\mathit{must}} \Delta$.

Our next task is to compare $\sqsubseteq_{\mathbf{rr}\mathrm{must}}$ with $\sqsubseteq_{\mathbf{nr}\mathrm{must}}$. The former is included in the latter, which directly follows from Definition 4. Surprisingly, it turns out that for finitary convergent processes the latter is also included in the former, thus the two preorders are in fact the same. The rest of the paper is devoted to proving this result.

4 Derivation based testing

In this section we give an alternative definition of $\mathcal{A}(\Theta, \Delta)$. Our definition has four ingredients. First of all, for technical reasons we normalise our pLTS by removing all τ-transitions that leave a success state. This way an ω-success state will only have outgoing transitions labelled ω.

Definition 5 (ω-respecting). A pLTS $\langle S, \Omega_\tau, \rightarrow\rangle$ is said to be ω-*respecting* whenever $s \xrightarrow{\omega}$, for any $\omega \in \Omega$, implies $s \overset{\tau}{\not\rightarrow}$.

It is straightforward to modify the pLTS of applications of tests to processes into one that it is ω-respecting, namely by removing all transitions $s \xrightarrow{\tau} \Delta$ for states s with $s \xrightarrow{\omega}$. With $[\Theta \| \Delta]$ we denote the distribution $\Theta \| \Delta$ in this pruned pLTS.

Secondly, we recall the definition of weak derivations from [2]. In a pLTS actions are only performed by states, in that actions are given by relations from states to distributions. But processes in general correspond to distributions over states, so in order to define what it means for a process to perform an action, we need to *lift* these relations so that they also apply to distributions. In fact we will find it convenient to lift them to subdistributions.

Definition 6. Let $(S, L, \rightarrow)$ be a pLTS and $\mathcal{R} \subseteq S \times \mathcal{D}_{sub}(S)$ be a relation from states to subdistributions. Then $\overline{\mathcal{R}} \subseteq \mathcal{D}_{sub}(S) \times \mathcal{D}_{sub}(S)$ is the smallest relation that satisfies:

(i) $s\ \mathcal{R}\ \Lambda$ implies $\overline{s}\ \overline{\mathcal{R}}\ \Lambda$, and
(ii) (Linearity) $\Delta_i\ \overline{\mathcal{R}}\ \Lambda_i$ for $i \in I$ implies $(\sum_{i\in I} p_i \cdot \Delta_i)\ \ \overline{\mathcal{R}}\ \ (\sum_{i\in I} p_i \cdot \Lambda_i)$ for any $p_i \in [0,1]$ $(i \in I)$ with $\sum_{i\in I} p_i \leq 1$.

An application of this notion is when the relation is $\xrightarrow{\alpha}$ for $\alpha \in \mathsf{Act}_\tau$; in that case we also write $\xrightarrow{\alpha}$ for $\overline{\xrightarrow{\alpha}}$. Thus, as source of a relation $\xrightarrow{\alpha}$ we now also allow distributions, and even subdistributions. A subtlety of this approach is that for any action α, we have $\varepsilon \xrightarrow{\alpha} \varepsilon$ simply by taking $I = \emptyset$ or $\sum_{i\in I} p_i = 0$ in Definition 6. That turned out to make ε especially useful for modelling the "chaotic" aspects of divergence in [2].

Definition 7 (Weak derivation). Suppose we have subdistributions $\Delta, \Delta_k^{\rightarrow}, \Delta_k^{\times}$, for $k \geq 0$, with the following properties:

$$\begin{aligned}
\Delta &= \Delta_0^{\rightarrow} + \Delta_0^{\times} \\
\Delta_0^{\rightarrow} &\xrightarrow{\tau} \Delta_1^{\rightarrow} + \Delta_1^{\times} \\
&\ \vdots \\
\Delta_k^{\rightarrow} &\xrightarrow{\tau} \Delta_{k+1}^{\rightarrow} + \Delta_{k+1}^{\times}\ . \\
&\ \vdots
\end{aligned}$$

Then we call $\Delta' := \sum_{k=0}^{\infty} \Delta_k^{\times}$ a *weak derivative* of Δ, and write $\Delta \Longrightarrow \Delta'$ to mean that Δ can make a *weak derivation* to its derivative Δ'.

There is always at least one derivative of any distribution (the distribution itself) and there can be many.

Thirdly, we identify a class of special weak derivatives called extreme derivatives.

Definition 8 (Extreme derivatives). A state s in a pLTS is called *stable* if $s \stackrel{\tau}{\not\longrightarrow}$, and a subdistribution Λ is called *stable* if every state in its support is stable. We write $\Delta \Longrightarrow\!\!\succ \Lambda$ whenever $\Delta \Longrightarrow \Lambda$ and Λ is stable, and call Λ an *extreme* derivative of Δ.

Referring to Definition 7, we see this means that in the extreme derivation of Λ from Δ at every stage a state must move on if it can, so that every stopping component can contain only states which *must* stop: for $s \in \Delta_k^{\rightarrow} + \Delta_k^{\times}$ we have $s \in \Delta_k^{\times}$ if *and now also* only if $s \stackrel{\tau}{\not\longrightarrow}$. Moreover if the pLTS is ω-respecting then whenever $s \in \Delta_k^{\rightarrow}$, it is not successful, i.e. $s \stackrel{\omega}{\not\longrightarrow}$ for every $\omega \in \Omega$.

Lemma 1 (Existence of extreme derivatives).
(i) *For every subdistribution Δ there exists some (stable) Δ' such that $\Delta \Longrightarrow\!\!\succ \Delta'$.*
(ii) *In a deterministic pLTS if $\Delta \Longrightarrow\!\!\succ \Delta'$ and $\Delta \Longrightarrow\!\!\succ \Delta''$ then $\Delta' = \Delta''$.*

It is worth pointing out that the use of subdistributions, rather than distributions, is essential here. If Δ diverges, that is if there is an infinite sequence of derivations $\Delta \xrightarrow{\tau} \Delta_1 \xrightarrow{\tau} \ldots \Delta_k \xrightarrow{\tau} \ldots$, then the only extreme derivative of Δ is the empty subdistribution ε. For example, the only transition of a state t with just a self τ-loop is $t \xrightarrow{\tau} \overline{t}$, and therefore $\overline{t}$ diverges; consequently its unique extreme derivative is ε.

The final ingredient in the definition of a set of outcomes of an application of a test to a process is the outcome of a particular extreme derivative. Note that all states $s \in \lceil \Lambda \rceil$ in the support of an extreme derivative either satisfy $s \xrightarrow{\omega}$ for a unique $\omega \in \Omega$, or have $s \not\rightarrow$.

Definition 9 (Outcomes). The outcome $\$\Lambda \in [0,1]^{\Omega}$ of a stable subdistribution Λ is given by $\$\Lambda(\omega) = \sum_{s \in \lceil \Lambda \rceil,\ s \xrightarrow{\omega}} \Lambda(s)$.

Putting all four ingredients together, we arrive at a definition of $\mathcal{A}^{\mathrm{d}}(\Theta, \Delta)$:

Definition 10. Let Δ be a pCSP process and Θ an Ω-test. Then

$$\mathcal{A}^{\mathrm{d}}(\Theta, \Delta) = \{\$\Lambda \mid [\Theta \| \Delta] \Longrightarrow\!\!\succ \Lambda\}.$$

The role of pruning in the above definition can be seen via the following example.

Example 3. Let $\overline{p}$ be a process that first does an a-action, to the point distribution $\overline{q}$, and then diverges, by engaging in a τ-loop $q \xrightarrow{\tau} \overline{q}$. Let $\overline{t}$ be the test used in Examples 1 and 2. Then $\overline{p} \| \overline{t}$ has a unique extreme derivation $\Theta \| \Delta \Longrightarrow\!\!\succ \varepsilon$, whereas $[\Theta \| \Delta]$ has a unique extreme derivation $[\Theta \| \Delta] \Longrightarrow\!\!\succ [q \| \omega]$. Here we give the name ω to the state reachable from $\overline{t}$ with the outgoing ω-transition. The outcome in $\mathcal{A}^{\mathrm{d}}(\overline{t}, \overline{p})$ shows that process $\overline{p}$ passes test $\overline{t}$ with probability 1, which is what we expect for state-based testing. Without pruning we would get an outcome saying that $\overline{p}$ passes $\overline{t}$ with probability 0.

As this example is non-probabilistic, it also illustrates how pruning enables the standard notion of non-probabilistic testing to be captured by derivation testing.

Example 4. (Revisiting Example 1.) The pLTS in Figure 3(b) is deterministic and unaffected by pruning; from part (ii) of Lemma 1 it follows that $\overline{t \| q_1}$ has a unique extreme derivative Λ. Moreover Λ can be calculated to be

$$\sum_{k \geq 1} \frac{1}{2^k} \cdot \overline{s_3},$$

which simplifies to the distribution $\overline{s_3}$. Therefore, $\mathcal{A}^{\mathrm{d}}(\overline{t}, \overline{q_1}) = \{\$\overline{s_3}\} = \{\overrightarrow{\omega}\}$.

Example 5. (Revisiting Example 2.) The application of the test $\overline{t}$ to processes $\overline{q_2}$ is outlined in Figure 4(b). Consider any extreme derivative Δ' from $s_0 = [\overline{t} \| \overline{q_2}]$; note that here again pruning actually has no effect. Using the notation of Definition 7, it is clear that $\Delta_0^{\times}$ and $\Delta_0^{\rightarrow}$ must be ε and $\overline{s_0}$ respectively. Similarly, $\Delta_1^{\times}$ and $\Delta_1^{\rightarrow}$ must be ε and $\overline{s_1}$ respectively. But s_1 is a nondeterministic state, having two possible transitions:

(i) $s_1 \xrightarrow{\tau} \Lambda_0$ where Λ_0 has support $\{s_0, s_2\}$ and assigns each of them the weight $\frac{1}{2}$
(ii) $s_1 \xrightarrow{\tau} \Lambda_1$ where Λ_1 has the support $\{s_3, s_4\}$, again dividing the mass equally among them.

So there are many possibilities for Δ_2; it is easy to see from Definition 7 that in fact Δ_2 can be of the form

$$p \cdot \Lambda_0 + (1-p) \cdot \Lambda_1 \tag{5}$$

for any choice of $p \in [0, 1]$.

Let us consider one possibility, an extreme one where p is chosen to be 0; only the transition (ii) above is used. Here $\Delta_2^{\rightarrow}$ is the subdistribution $\frac{1}{2}\overline{s_4}$, and $\Delta_k^{\rightarrow} = \varepsilon$ whenever $k > 2$. A simple calculation shows that in this case the extreme derivative generated is $\Lambda_1^e = \frac{1}{2}\overline{s_3} + \frac{1}{2}\overline{s_6}$ which implies that $\frac{1}{2}\overrightarrow{\omega} \in \mathcal{A}^{\mathrm{d}}(\overline{t}, \overline{q_2})$.

Another possibility for Δ_2 is Λ_0, corresponding to the choice of $p = 1$ in (5) above. Continuing with this derivation leads to Δ_3 being $\frac{1}{2} \cdot \overline{s_1} + \frac{1}{2} \cdot \overline{p_5}$; in other words $\Delta_3^{\times} = \frac{1}{2} \cdot \overline{p_5}$ and $\Delta_3^{\rightarrow} = \frac{1}{2} \cdot \overline{s_1}$. Now in the generation of Δ_4 from $\Delta_3^{\rightarrow}$ once more we have to resolve a transition from the nondeterministic state s_1, by choosing some arbitrary $p \in [0, 1]$ in (5). Suppose that each time this arises we systematically choose $p = 1$, that is, we ignore completely the transition (ii) above. Then it is easy to see that the extreme derivative generated is

$$\Lambda_0^e = \sum_{k \geq 1} \frac{1}{2^k} \cdot \overline{p_5}$$

which simplifies to the distribution $\overline{p_5}$. This in turn means that $\overrightarrow{\omega} \in \mathcal{A}^{\mathrm{d}}(\overline{t}, \overline{q_2})$.

We have seen two possible derivations of extreme derivatives from $\overline{s_0}$. But there are many others. In general whenever $\Delta_k^{\rightarrow}$ is of the form $q \cdot \overline{s_1}$ we have to resolve the nondeterminism by choosing a $p \in [0, 1]$ in (5) above; moreover each such choice is independent. It turns out that every extreme derivative Δ' of $\overline{s_0}$ is of the form

$$q \cdot \Lambda_0^e + (1-q)\Lambda_1^e$$

for some choice of $q \in [0, 1]$, which implies that $\mathcal{A}^{\mathrm{d}}(\overline{t}, \overline{q_2})$ is the convex closure of the set $\{\frac{1}{2}\overrightarrow{\omega}, \overrightarrow{\omega}\}$.

5 Comparison of resolution and derivation based testing

We have now seen two ways of associating sets of outcomes with the application of a test to a process. The first, in Section 3, associates with a test and a process a set of deterministic structures called resolutions, while the second, in Section 4, uses extreme derivations in which nondeterministic choices are resolved dynamically as the derivation proceeds. In this section we show that these two approaches give rise to the same outcomes.

The next proposition maintains that for each extreme derivative there is a corresponding resolution, and vice versa.

Proposition 1. *Let Δ be a subdistribution a pLTS $\langle S, \Omega_\tau, \rightarrow\rangle$.*

(i) Suppose $\Delta \Longrightarrow\!\!\succ \Delta'$. Then there is a resolution $\langle R, \Gamma, \rightarrow_R\rangle$ of Δ, with resolving function f, such that $\Gamma \Longrightarrow\!\!\succ_R \Gamma'$ for some Γ' for which $\Delta' = Img_f(\Gamma')$.
(ii) Suppose $\langle R, \Gamma, \rightarrow_R\rangle$ is a resolution of a Δ with resolving function f. Then $\Gamma \Longrightarrow\!\!\succ_R \Gamma'$ implies $\Delta \Longrightarrow\!\!\succ Img_f(\Gamma')$.

Our next step is to relate the outcomes extracted from extreme derivatives to those extracted from the corresponding resolutions. This requires some analysis of the evaluation function $\mathbb{V}(-)$. We have to show that the function $\mathcal{R}$ defined in (3) and its least fixed point $\mathbb{V}(-)$ are continuous, which is by no means trivial because it involves identifying a class of binary functions that are conditionally continuous.

Definition 11 (Continuous functions). A chain in a complete lattice L is a sequence of elements $\{\, c_n \mid n \geq 0 \,\}$ satisfying $c_i \leq c_{i+1}$. Obviously chains have least upper bounds which we denote by $\bigsqcup_{n\geq 0} c_n$. A function $f : L \rightarrow L$ is said to be continuous if it preserves the least upper bounds of chains

$$f(\bigsqcup_{n\geq 0} c_n) = \bigsqcup_{n\geq 0} f(c_n)$$

The following technical lemma states that some binary functions satisfy the property of *bounded continuity*, which allows the exchange of limit and sum operations. It plays a crucial role in proving the continuity of $\mathcal{R}$.

Lemma 2 (Bounded continuity). *Given a function $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ which satisfies the following three conditions:*

C1. *f is monotonic in its second argument, i.e. $j_1 \leq j_2$ implies $f(i, j_1) \leq f(i, j_2)$ for all $i, j_1, j_2 \in \mathbb{N}$.*
C2. *For any $i \in \mathbb{N}$, the limit $\lim_{j\rightarrow\infty} f(i, j)$ exists.*
C3. *Moreover, for any $n \in \mathbb{N}$, the partial sum $S_n = \sum_{i=0}^{n} \lim_{j\rightarrow\infty} f(i, j)$ is bounded, i.e. there exists some $c \in \mathbb{R}_{\geq 0}$ such that $S_n \leq c$ for all $n \geq 0$.*

then it holds that

$$\sum_{i=0}^{\infty} \lim_{j\rightarrow\infty} f(i, j) \;=\; \lim_{j\rightarrow\infty} \sum_{i=0}^{\infty} f(i, j).$$

In other words, the function f is continuous conditioned by the fact that its partial sums are always bounded by a constant. By exploiting the above lemma, we obtain the following nice property.

Lemma 3. *Consider a deterministic pLTS $\langle R, \Omega_\tau, \rightarrow\rangle$. The function $\mathcal{R}$ defined in (3) is continuous.*

Consequently, the least fixed point of $\mathcal{R}$, i.e. $\mathbb{V}$, is also continuous. Moreover, the function $\mathbb{V}$ can be captured by a chain of approximants. The functions $\mathbb{V}^n,\ n \geq 0$ are defined by induction on n:

$$\mathbb{V}^0(r)(\omega) = 0 \qquad \text{for all } \omega \in \Omega$$
$$\mathbb{V}^{n+1} = \mathcal{R}(\mathbb{V}^n)$$

Then $\mathbb{V} = \bigsqcup_{n\geq 0} \mathbb{V}^n$. This is used in the following central result.

Proposition 2. *Let Δ be a subdistribution in an ω-respecting deterministic pLTS. If $\Delta \Longrightarrow\!\!\!\!\succ \Delta'$ then $\mathbb{V}(\Delta) = \mathbb{V}(\Delta')$.*

We are now ready to compare the two methods for calculating the set of outcomes associated with a subdistribution:

- using resolutions and the evaluation function $\mathbb{V}$ from page 6;
- using extreme derivatives and the reward function $\$$ from Definition 9.

Corollary 1. *In an ω-respecting pLTS $\langle S, \Omega_\tau, \rightarrow\rangle$, the following statements hold.*

(i) If $\Delta \Longrightarrow\!\!\!\!\succ \Delta'$ then there is a resolution $\langle R, \Gamma, \rightarrow_R\rangle$ of Δ such that $\mathbb{V}(\Gamma) = \$(\Delta')$.
(ii) For any resolution $\langle R, \Gamma, \rightarrow_R\rangle$ of Δ, there exists an extreme derivative Δ' such that $\Delta \Longrightarrow\!\!\!\!\succ \Delta'$ and $\mathbb{V}(\Gamma) = \$(\Delta')$.

Corollary 2. *For any test Θ and process Δ we have that $\mathcal{A}^d(\Theta, \Delta) = \mathcal{A}(\Theta, \Delta)$.*

6 Agreement of nonnegative and real reward must testing

In this section we prove the agreement of $\sqsubseteq_{\mathbf{nr}\text{must}}$ with $\sqsubseteq_{\mathbf{rr}\text{must}}$ for finitary convergent processes, by using failure simulation [2] as a stepping stone. We start with defining the weak action relations $\stackrel{\alpha}{\Longrightarrow}$ for $\alpha \in \mathsf{Act}_\tau$ and the refusal relations $\stackrel{A}{\not\longrightarrow}$ for $A \subseteq \mathsf{Act}$ that are the key ingredients in the definition of the failure simulation preorder.

Definition 12. Let Δ and its variants be subdistributions in a pLTS $\langle S, \mathsf{Act}_\tau, \rightarrow\rangle$.

- For $a \in \mathsf{Act}$ write $\Delta \stackrel{a}{\Longrightarrow} \Delta'$ whenever $\Delta \Longrightarrow \Delta^{\text{pre}} \stackrel{a}{\longrightarrow} \Delta^{\text{post}} \Longrightarrow \Delta'$. Extend this to Act_τ by allowing as a special case that $\stackrel{\tau}{\Longrightarrow}$ is simply $\Longrightarrow$, i.e. including identity (rather than requiring at least one $\stackrel{\tau}{\longrightarrow}$).
- For $A \subseteq \mathsf{Act}$ and $s \in S$ write $s \stackrel{A}{\not\longrightarrow}$ if $s \stackrel{\alpha}{\not\longrightarrow}$ for every $\alpha \in A \cup \{\tau\}$; write $\Delta \stackrel{A}{\not\longrightarrow}$ if $s \stackrel{A}{\not\longrightarrow}$ for every $s \in \lceil \Delta \rceil$.
- More generally write $\Delta \Longrightarrow\stackrel{A}{\not\longrightarrow}$ if $\Delta \Longrightarrow \Delta^{\text{pre}}$ for some Δ^{pre} such that $\Delta^{\text{pre}} \stackrel{A}{\not\longrightarrow}$.

Definition 13 (Failure simulation preorder). Define $\lhd_{FS}$ to be the largest relation in $S \times \mathcal{D}_{sub}(S)$ such that if $s \lhd_{FS} \Lambda$ then

(i) whenever $\overline{s} \stackrel{\alpha}{\Longrightarrow} \Delta'$, for $\alpha \in \mathsf{Act}_\tau$, then there is a $\Lambda' \in \mathcal{D}_{sub}(S)$ with $\Lambda \stackrel{\alpha}{\Longrightarrow} \Lambda'$ and $\Delta' \;\overline{\lhd_{FS}}\; \Lambda'$, and
(ii) whenever $\overline{s} \Longrightarrow\stackrel{A}{\not\longrightarrow}$ then $\Lambda \Longrightarrow\stackrel{A}{\not\longrightarrow}$.

Any relation $\mathcal{R} \subseteq S \times \mathcal{D}_{sub}(S)$ that satisfies the two clauses above is called a *failure simulation*. The failure simulation preorder $\sqsubseteq_{FS} \subseteq \mathcal{D}_{sub}(S) \times \mathcal{D}_{sub}(S)$ is defined by letting $\Lambda \sqsubseteq_{FS} \Delta$ whenever there is a $\Lambda^\natural$ with $\Lambda \Longrightarrow \Lambda^\natural$ and $\Delta \;\overline{\lhd_{FS}}\; \Lambda^\natural$.

There are other equivalent definitions of failure simulation preorder [2]. This one is chosen because we find it convenient to use in proving Lemma 4 below.

The failure simulation preorder is a preserved under parallel composition with a test, followed by pruning, and it is sound and complete for probabilistic must testing preorder, when only finitary processes are considered.

Theorem 3 ([2]). *For finitary processes Δ and Λ,*

(i) If $\Delta \sqsubseteq_{FS} \Lambda$ then for any Ω-test Θ it holds that $[\Delta \| \Theta] \sqsubseteq_{FS} [\Lambda \| \Theta]$.
(ii) $\Delta \sqsubseteq_{FS} \Lambda$ if and only if $\Delta \sqsubseteq_{\text{pmust}} \Lambda$.

Because we prune our pLTSs before extracting values from them, we will be concerned mainly with ω-respecting structures. Moreover, we require the pLTSs to be *convergent* in the sense that there is no wholly divergent state s, i.e. with $s \Longrightarrow \varepsilon$.

Definition 14. Let Δ be a distribution in a pLTS $\langle S, \Omega_\tau, \rightarrow\rangle$. We write $\mathcal{V}(\Delta)$ for the set of testing outcomes $\{\$\Delta' \mid \Delta \Longrightarrow\!\!\!\succ \Delta'\}$.

Lemma 4. *Let Δ and Λ be two subdistributions in an ω-respecting convergent pLTS $\langle S, \Omega_\tau, \rightarrow\rangle$. If $\Lambda \sqsubseteq_{FS} \Delta$, then it holds that $\mathcal{V}(\Lambda) \supseteq \mathcal{V}(\Delta)$.*

This lemma shows that failure simulation preorder is a very strong relation in the sense that if Λ is related to Δ by the failure simulation preorder then the set of outcomes generated by Λ includes the set of outcomes given by Δ. It is mainly due to this strong requirement that we can show that the failure simulation preorder is sound for the real reward must testing preorder. Convergence is a crucial condition in this lemma.

Theorem 4. *For any finitary convergent processes Δ and Λ, if $\Delta \sqsubseteq_{FS} \Lambda$ then we have that $\Delta \sqsubseteq_{\mathbf{rr}must} \Lambda$.*

The proof of the above theorem is subtle. The failure simulation preorder is defined via weak derivations (cf. Definition 13), while the reward must testing preorder is defined in terms of resolutions (cf. Definition 4). Fortunately, we have shown in Corollary 2 that we can just as well characterise the reward must testing preorder in terms of derivations. Based on this observation, the proof can be carried out by exploiting Theorem 3(i) and Lemma 4.

Finally, by combining Theorems 1(ii) and 3(ii), together with Theorem 4, we obtain the main result of the paper which states that nonnegative reward must testing is as discriminating as real reward must testing.

Theorem 5. *For any finitary convergent processes Δ and Λ, it holds that $\Delta \sqsubseteq^{\Omega}_{\mathbf{rr}must} \Lambda$ if and only if $\Delta \sqsubseteq_{\mathbf{nr}must} \Lambda$.*

In the presence of divergence, $\sqsubseteq_{\mathbf{rr}\text{must}}$ is strictly finer than $\sqsubseteq_{\mathbf{nr}\text{must}}$. For example, let Δ be a process that diverges, by performing a τ-loop only, and Λ a process that merely performs a single action a. It holds that $\Delta \sqsubseteq_{FS} \Lambda$ because $\Delta \Longrightarrow \varepsilon$ and the empty subdistribution can failure simulate any processes. It follows from Theorems 3(ii) and 1(ii) that $\Delta \sqsubseteq_{\mathbf{nr}\text{must}} \Lambda$. However, if we apply the test $\bar{t}$ from Example 1 again, and the reward tuple h with $h(\omega) = -1$, then

$$\begin{array}{lllllll} \bigsqcup h \cdot \mathcal{A}^{\mathrm{d}}(\bar{t}, \Delta) & = & \bigsqcup h \cdot \{\varepsilon\} & = & \bigsqcup \{0\} & = & 0 \\ \bigsqcup h \cdot \mathcal{A}^{\mathrm{d}}(\bar{t}, \Lambda) & = & \bigsqcup h \cdot \{\vec{\omega}\} & = & \bigsqcup \{-1\} & = & -1 \end{array}$$

As $\bigsqcup h \cdot \mathcal{A}^{\mathrm{d}}(\bar{t}, \Delta) \not\leq \bigsqcup h \cdot \mathcal{A}^{\mathrm{d}}(\bar{t}, \Lambda)$, we see that $\Delta \not\sqsubseteq_{\mathbf{rr}\text{must}} \Lambda$.

7 Conclusion

We have studied a notion of real reward testing which extends the traditional nonnegative reward testing with negative rewards. It turned out that real reward may preorder is the inverse of real reward must preorder, and vice versa. More interestingly, for finitary convergent processes, real reward must testing preorder coincides with nonnegative reward testing preorder. In order to prove this result, we have presented two testing approaches and shown their coincidence, which involved proving some analytic properties such as the continuity of a function for calculating testing outcomes, and bounded continuity of a class of binary functions.

Although for finitary convergent processes, real reward must testing is no more powerful then nonnegative reward must testing, a similar result does not hold for may testing. This follows immediately from our result that (the inverse of) real reward may testing is as powerful as real reward must testing, that is known not to hold for nonnegative reward may and must testing. Thus, real reward may testing is strictly more discriminating than nonnegative reward may testing, even in the absence of divergence.

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