



Trinity College Dublin
Coláiste na Tríonóide, Baile Átha Cliath
The University of Dublin

Fluctuations and Geometry in Quantum Thermodynamics

CANDIDATE:

Laetitia Paula Bettmann

SUPERVISOR:

Prof. John Goold

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Signed: Lupita Sittmann

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Fluctuations and Geometry in Quantum Thermodynamics

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ABSTRACT

Understanding thermodynamics at the quantum scale is a central challenge in modern physics. Thermodynamic concepts were originally formulated for macroscopic systems, where fluctuations are negligible and quantum features are effectively washed out. For quantum systems operating far from equilibrium, these assumptions break down, motivating the development of nonequilibrium quantum thermodynamics. At the same time, engineered quantum platforms, ranging from superconducting circuits and cavity QED to mesoscopic conductors, now provide experimentally accessible settings in which this theoretical framework can be systematically tested and, increasingly, applied to practical tasks.

This thesis addresses a broad range of problems in the nonequilibrium thermodynamics of open quantum systems, organised around two central themes: fluctuations and geometry. Our work spans measurement-induced effects, stochastic dynamics beyond the Markovian weak-coupling regime, information-geometric aspects, and dissipative critical phenomena, and accordingly employs a wide array of theoretical tools.

Specifically, we propose a model of measurement-boosted quantum transport within the framework of repeated interactions, and demonstrate that measurement backaction can enhance both the performance and stability of quantum heat engines. We then introduce a method for directly sampling the time- and energy-resolved statistics of thermodynamic quantities in strongly coupled, driven open quantum systems, by combining Markovian embeddings with quantum-jump trajectory unravellings.

Using tools from information geometry, we relate the temporal quantum Fisher information to entropic fluxes in open quantum systems and develop a geometric description of the thermodynamic cost associated with slow transitions between nonequilibrium steady states. This approach yields general bounds on excess entropy flux and nonadiabatic entropy production and, owing to their geometric origin, naturally suggests dissipation-minimising protocols based on geodesic paths. Finally, we apply thermodynamic geometry to dissipative quantum phase transitions, revealing dynamical universality in the nonadiabatic entropy production under finite-speed driving. These results constitute a far-from-equilibrium analogue of the celebrated Kibble-Zurek mechanism.

LIST OF PUBLICATIONS

The results exposed in this thesis are based, primarily, on the following publications and preprints:

1. **L. P. Bettmann**, M. J. Kewming, J. Goold,
Thermodynamics of a continuously monitored double-quantum-dot heat engine in the repeated interactions framework,
[Phys. Rev. E **107**, 044102 \(2023\)](#)
2. **L. P. Bettmann**, M. J. Kewming, G. T. Landi, J. Goold, M. T. Mitchison,
Quantum stochastic thermodynamics in the mesoscopic-leads formulation,
[Phys. Rev. E **112**, 014105 \(2025\)](#)
3. **L. P. Bettmann**, J. Goold,
Information geometry approach to quantum stochastic thermodynamics,
[Phys. Rev. E **111**, 014133 \(2025\)](#)
4. A. M. Lacerda, **L. P. Bettmann**, J. Goold,
Information geometry of transitions between quantum nonequilibrium steady states,
[Phys. Rev. E **112**, L022101 \(2025\)](#)
5. **L. P. Bettmann**, A. M. Lacerda, M. T. Mitchison, J. Goold,
Thermodynamic universality across dissipative quantum phase transitions,
[arXiv:2512.05074 \[quant-ph\] \(2025\)](#)

The following are other publications and preprints of the author which, although relevant to a certain extent to this thesis and completed during the doctoral training, are not covered in detail in the following document:

1. S. L. Jacob, **L. P. Bettmann**, A. M. Lacerda, K. Zawadzki, S. R. Clark, J. Goold, J. J. Mendoza-Arenas,
Dephasing-assisted transport in a tight-binding chain with a linear potential,
[Front. Phys. **12**, 1474018 \(2024\)](#)
2. A. Summer, M. Moroder, **L. P. Bettmann**, X. Turkeshi, I. Marvian, J. Goold,
Resource-theoretical unification of Mpemba effects: classical and quantum,
[Phys. Rev. X **16**, 011065 \(2026\)](#)
3. W. Alford, **L. P. Bettmann**, G. T. Landi,
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Introduction

The contributions of this thesis to the thermodynamics in open quantum systems can be organised around two main themes: fluctuations and geometry. Focusing on fluctuations, we demonstrate that measurement backaction can stabilise the output power of a quantum heat engine and develop a method to access stochastic thermodynamics beyond the Markovian weak-coupling regime (Parts [II-III](#)). With respect to geometry, we reveal a fundamental relation between quantum information geometry and entropy dynamics, construct a geometric description of the entropic cost incurred during slow-driving protocols connecting different nonequilibrium steady states, and apply this framework to uncover universal thermodynamic scaling in dissipative quantum phase transitions (Parts [IV-V](#)). In the following, we motivate these two perspectives by emphasizing their significance for both fundamental understanding and practical applications.

Thermodynamics provides a universal framework for describing how energy flows and transforms in physical systems, originating in the study of thermal machines such as heat engines or refrigerators. Central to this framework are two foundational principles, known as the first and second laws of thermodynamics [\[1\]](#). The first law expresses the conservation of energy and identifies heat and work as its modes of transfer, while the second law reveals a fundamental arrow of time in physical processes and, in its Clausius form [\[2, 3\]](#), implies that entropy is produced during irreversible transformations. The breadth of this insight into “how the world works” was famously captured by Boltzmann [\[4\]](#): *The general struggle for existence of living beings is therefore not a fight for the elements — the elements of all organisms are available in abundance in air, water, and soil — nor for energy, which is plentiful in the form of heat, unfortunately untransformable, in every body. Rather it is a struggle for entropy that becomes available through the flow of energy from the hot Sun to the cold Earth.*

Viewed from this broader perspective, the remarkable success of thermodynamics in diverse physical contexts, reflecting its largely model-independent structure, motivates a careful examination of how its principles manifest at small scales. As attention turned to such systems, it became increasingly clear that classical thermodynamics, originally formulated in terms of averages over macroscopic ensembles, had to be extended to incorporate fluctuations and intrinsically quantum effects.

Accordingly, this stimulated the development of stochastic thermodynamics [5], a framework in which thermodynamic quantities, such as work, heat, and entropy production, are defined along individual stochastic trajectories, allowing one to access their fluctuations. A key achievement of this theory is the set of fluctuation theorems [6], which quantify, for example, the exponentially suppressed probability of entropy-decreasing events and thereby provide a trajectory-level refinement of the second law.

In parallel, the field of quantum thermodynamics developed, seeking to understand how classical thermodynamic concepts and laws emerge in the quantum domain [7–10]. Open quantum systems play an important role in this context, offering a description of how quantum systems interact with their surroundings through energy and information exchange. Such interactions induce decoherence and dissipation and, under suitable conditions, give rise to nonequilibrium steady states (NESSs): stationary states that sustain persistent currents and continuously produce entropy [11]. Within this setting, a natural question concerns the performance and practical usefulness of artificial quantum thermodynamic devices [12–14], for instance in supporting the operation of existing quantum hardware [15–17] or in thermodynamic computing [18]. An important class of these devices are autonomous machines [19, 20]. These require no time-dependent, external control: heat exchanged with thermal reservoirs is converted into work through steady currents. A prominent example is provided by quantum absorption refrigerators, where cooling is achieved solely by exploiting thermal gradients [21]. Such devices have recently been shown to be practically useful for the autonomous reset of a superconducting qubit [17], a task of central importance for quantum information processing. Remarkably, the coefficient of performance was found to be comparable to that of macroscopic devices, while the reset operation can be completed in under $1\ \mu\text{s}$. While this scheme does not yet match state-of-the-art active reset times $\lesssim 100\ \text{ns}$ [22], it nonetheless represents a promising direction.

The experimental characterization of such quantum devices often relies on continuous measurement. Even weak measurements, however, perturb the system's dynamics and can influence energy transport [23], motivating a careful investigation of how it affects the thermodynamic performance of quantum devices. In this context, solid-state platforms provide a particularly versatile setting in which

quantum point contacts (QPCs) act as sensitive charge sensors capable of monitoring electron dynamics in real time [24–27]. A recent experiment employed such a QPC to quantify the entropic cost of extracting classical ticks from a quantum clock implemented as a double quantum dot (DQD) coupled to two thermal baths [28].

Interestingly, the same DQD setup can also function as a quantum heat engine. In Chapter II.4 of this thesis, we show that the measurement backaction due to the QPC can enhance the engine’s net output power while simultaneously reducing both the current’s relative fluctuations and the entropic cost required to achieve a given signal-to-noise ratio [29]. Thus, contrary to the common perception of measurement as a purely detrimental disturbance, continuous monitoring can be harnessed to improve the stability and thermodynamic performance of quantum devices, extending previous reports [23, 30]. Our results are derived within the repeated-interactions framework [31, 32], which provides a thermodynamically consistent method for describing systems locally coupled to an environment and evolving under the corresponding Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) master equations [33, 34].

GKSL master equations have proven highly successful in describing a wide range of experimental observations, particularly in quantum optics [11]. Their validity, however, relies on several restrictive assumptions. In particular, the Markovian approximation neglects memory effects, and the system-environment coupling is assumed to be weak, suppressing the buildup of correlations between the two. Such correlations can play a crucial role in the statistics of thermodynamic quantities when the coupling strength increases or when the reservoirs exhibit structured spectral densities [35]. This motivates the development of approaches capable of capturing fluctuations beyond the Markovian and weak-coupling limits. Achieving this in practice is difficult: traditional methods, such as nonequilibrium Green’s functions and scattering theory [36–39] are typically restricted to weakly interacting or slowly driven regimes [40–42], while path-integral techniques [43, 44] are limited by system size [45, 46].

In Chapter III.6 of this thesis, we address this challenge. Our approach combines the mesoscopic-lead framework, in which an infinite-dimensional macroscopic bath is replaced by a finite subsystem (the lead) that is itself weakly coupled to an idealised thermal reservoir [47–50], with quantum-jump trajectory unravelings [51]. This hybrid framework enables the time- and energy-resolved construction of distributions of thermodynamic quantities, while providing trajectory-resolved insight into the physical mechanisms underlying rare fluctuations [52].

A parallel line of research seeks to extend quantum geometric tools, originally developed for closed quantum systems, to open-system settings. These tools are powerful: their well-developed mathematical structure allows us to leverage es-

established results while often providing clear geometric intuition. The effort is further motivated by recent experimental advances in condensed matter platforms, which now enable direct probing of quantities such as the quantum geometric tensor [53–55]. The imaginary part of this tensor corresponds to the Berry curvature, which encodes local geometrical properties of energy bands and underlies a variety of fascinating phenomena, including the quantum Hall effect, Chern insulators, and topological superconductors, through its connection to global topological invariants such as the Chern number, see, e.g., the review [56], and references therein. Its real part is proportional to the quantum Fisher information (QFI), which captures a state’s sensitivity to infinitesimal changes in external parameters. More generally, the Fisher information and its quantum extensions appear widely across physics, from metrology [57] to thermodynamics [58], with the latter being of particular interest in this thesis. In this context, we focus on the Fisher information defined with respect to time, which admits a dynamical interpretation: the temporal Fisher information quantifies the system’s intrinsic rate of change. This interpretation forms the basis of fundamental quantum and classical speed limits on state evolution [59–66]. Importantly, although seemingly abstract, such geometric quantum speed limits have recently been explored experimentally [67, 68], underscoring that they are not merely mathematical constructs but experimentally accessible constraints.

In classical stochastic systems, the temporal Fisher information can be related to entropic currents [69, 70], revealing a fundamental link between temporal information geometry and thermodynamics. In Chapter IV.8 of this thesis, we provide a fully quantum mechanical derivation of this connection. This allows us to generalise bounds on the rate of change of a system’s von Neumann entropy through a non-negative quantum contribution that can be identified as the coherent part of the so-called information geometric action with respect to the temporal QFI metric [71]. This contribution can be interpreted as a “kinetic energy” associated with the quantum state’s coherent evolution through state space.

Thermodynamic geometry, closely related to information geometry, provides a framework for quantifying the thermodynamic cost of driving a system along a path in its parameter space. This is achieved by endowing the space of control parameters of the system’s dynamical generator with a thermodynamic metric, so that the cost of a protocol can be expressed as an action functional over the path [72–78]. Operationally, this geometric structure is significant because geodesic paths in parameter space correspond to optimal control protocols that minimise excess dissipation, a feature that is particularly relevant for applications such as the control of quantum heat engines, see Ref. [77], and references therein. In this thesis, we focus on slow, finite-time protocols within Markovian, weak-coupling regimes. Thermodynamic geometry in such settings has been extensively

explored for classical thermal-state transitions [72, 74], for classical NESSs [75], and more recently for transitions between quantum thermal states [76].

A comprehensive treatment of the most general case, transitions between quantum NESSs, has, however, remained elusive. We present our analysis of this problem in Section IV.7.5. Within the so-called Hatano-Sasa framework [79, 80, 5], the total entropy production and the corresponding entropic fluxes can be decomposed into two distinct contributions: one quantifies the housekeeping dissipation required to sustain the system's NESS currents, while the other captures the excess, nonadiabatic dissipation generated by external driving. We show that the thermodynamic metric governing the dissipation incurred during the slow driving of a parameter connecting different NESSs, as quantified by the nonadiabatic entropy production, is closely related to the so-called Kubo-Mori-Bogoliubov (KMB) quantum Fisher information metric, which naturally arises in quantum thermodynamics due to its connection with the quantum relative entropy [78].

Among the manifold of NESSs supported by open quantum systems, a particularly compelling class arises in systems that exhibit dissipative quantum phase transitions (DQPTs): phase transitions that occur in the steady state of an open quantum system, rather than in the ground state of a closed system [81]. DQPTs arise from the nontrivial interplay between coherent unitary dynamics, generated by the system Hamiltonian, external driving, and irreversible dissipative processes induced by coupling to an environment, and exhibit rich and distinctly nonequilibrium phenomenology [81]. Moreover, they provide the intriguing opportunity to probe critical behaviour nondestructively, thanks to the system's continuous, intrinsic interaction with its environment [82]. They have been explored theoretically in photonic platforms [83–95], lossy polariton condensates [96–98], and spin systems [99–107]. Experimentally, they have been observed in platforms ranging from ultracold atomic gases [108, 109] to solid-state optical [110–112] and electronic [113] devices. A canonical example of a DQPT is provided by the open Dicke model (see, e.g., Ref. [114]), which describes a large ensemble of atoms collectively interacting with a single mode of a lossy optical cavity. As the atom-light coupling strength is increased, the system undergoes a transition from a normal phase, characterised by a negligible cavity photon occupation, to a superradiant phase in which the cavity mode becomes macroscopically populated. Importantly, this transition is experimentally accessible: it was first realised by Baumann *et al.*, who coupled a Bose-Einstein condensate to a lossy optical cavity [108]. More recently, Brunelli *et al.* [115] showed experimentally that the irreversible entropy production in this system diverges as the critical point is approached, providing compelling evidence that DQPTs are accompanied by a distinctive thermodynamic signature.

While these results elucidate critical dissipative behaviour in stationary states,

they raise the question of how criticality manifests thermodynamically under driving. Addressing this question falls within the domain of dynamical universality, a theoretical framework that explains how universal nonequilibrium behaviour emerges from the universal equilibrium properties of continuous phase transitions [116]. Near the critical point, both the equilibrium correlation length and the intrinsic relaxation time diverge, giving rise to critical slowing down. As a consequence, a system driven at a finite rate cannot adiabatically follow its instantaneous equilibrium state in the vicinity of the transition. The Kibble-Zurek mechanism (see, e.g., the review [116]) explains how this breakdown of adiabaticity causes nonequilibrium dynamical response to be governed by equilibrium universality classes. When a control parameter is slowly ramped across the transition, the competition between the externally imposed driving timescale and the intrinsic relaxation time gives rise to a characteristic freeze-out scale, reflecting the breakdown of adiabatic following. This scale depends only on equilibrium critical exponents and the driving rate, and leads to universal power-law scaling of observables such as the excitation density and the excess work (the additional work performed beyond the adiabatic, quasi-static limit), independent of microscopic details [117–123]. In open quantum systems, the quantity playing an analogous role to the excess work in closed systems is the nonadiabatic entropy production. As discussed above, in the slow-driving regime this irreversibility cost admits a natural geometric interpretation: it is governed by a thermodynamic metric defined on the manifold of NESSs, which is closely related to the KMB QFI. Since the KMB QFI acts as a generalised susceptibility and is expected to diverge at critical points [124], the associated thermodynamic metric is likewise expected to be highly sensitive to criticality.

In Chapter V.10 of this thesis, we show that this sensitivity gives rise to an emergent universal thermodynamical scaling in open quantum systems undergoing continuous DQPTs. Specifically, departures from the NESS manifold induced by finite-time driving give rise to nonadiabatic entropy production that obeys Kibble-Zurek-type scaling laws, governed by the same universal dynamical principles that control critical response in driven closed quantum systems [125]. Importantly, our results, which we verify numerically for driven-dissipative Dicke and Kerr models, are directly amenable to experimental verification. As mentioned above, experiments on cavity-QED platforms have measured the cavity photon flux to monitor entropy production [115], making it an ideal probe for the universal dynamical scaling we predict.

Throughout this thesis, we work in natural units and set $\hbar = k_B = 1$.

PART I:

Theory and methods

Chapter I.1

Dynamics of open quantum systems

This chapter is based on Refs. [51, 11].

In realistic settings, quantum systems are never perfectly isolated but exchange energy and information with their surroundings. While such couplings can hinder quantum technologies, for example in quantum computation [126], they also underpin a wide range of physical phenomena and enable applications in quantum thermodynamics [127, 19, 128], transport [129], and metrology [57].

This chapter introduces the basic tools used throughout the thesis to describe open quantum systems. We start from completely positive and trace-preserving (CPTP) quantum maps [11] and their Kraus representation. We then specialise to the Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) master equation [33, 34], which captures the Markovian dynamics of systems weakly coupled to their environments (see Section I.1.1). This framework serves as the workhorse of open-system theory throughout this thesis. Next, we discuss the adjoint (Heisenberg-picture) evolution of operators (see Section I.1.2), quantum-jump unravellings and their interpretation in terms of continuous monitoring (see Section I.1.3), and the Liouvillian spectrum in the Liouville representation (see Section I.3.3). We conclude with nonequilibrium steady states and the Drazin inverse (see Sections I.1.5-I.1.6), and briefly contrast global and local GKSL constructions (see Section I.1.7).

I.1.1 CPTP maps and GKSL master equations

We consider a quantum system S interacting with an environment E . The total Hilbert space is the tensor product $\mathcal{H}_{\text{tot}} = \mathcal{H}_S \otimes \mathcal{H}_E$. The Hamiltonian of the

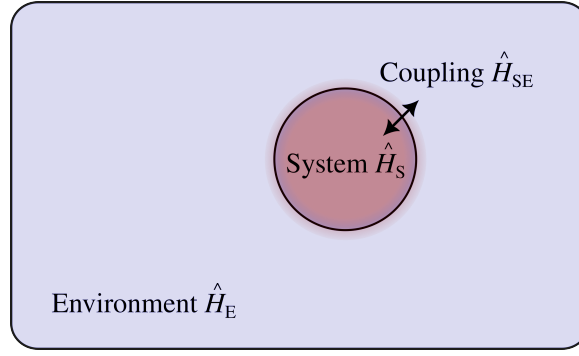


Figure I.1.1: Sketch of an open quantum system. A quantum system, described by the Hamiltonian $\hat{H}_S$, interacts with its environment, described by $\hat{H}_E$, via the coupling Hamiltonian $\hat{H}_{SE}$.

combined system takes the generic form

$$\hat{H}_{\text{tot}} = \hat{H}_S \otimes \hat{\mathbb{I}}_E + \hat{\mathbb{I}}_S \otimes \hat{H}_E + \hat{H}_{SE}, \quad (\text{I.1.1})$$

where $\hat{H}_S$ acts on the system, $\hat{H}_E$ on the environment, and $\hat{H}_{SE}$ represents their interaction. The total density operator $\hat{\rho}_{\text{tot}}(t)$ evolves unitarily according to the Liouville-von Neumann equation,

$$\frac{d}{dt}\hat{\rho}_{\text{tot}}(t) = -i[\hat{H}_{\text{tot}}, \hat{\rho}_{\text{tot}}(t)], \quad (\text{I.1.2})$$

and the system and environment are assumed to be uncorrelated initially, so that $\hat{\rho}_{\text{tot}}(0) = \hat{\rho}_S(0) \otimes \hat{\rho}_E(0)$. The system of interest is described by the reduced density matrix $\hat{\rho}_S(t) = \text{Tr}_E[\hat{\rho}_{\text{tot}}(t)]$, which contains all information accessible through system observables. Formally, the reduced dynamics define a family of linear, completely positive, and trace-preserving maps (CPTP maps¹), also referred to as quantum channels, $\mathcal{E}_t$ acting on $\hat{\rho}_S$:

$$\hat{\rho}_S(t) = \mathcal{E}_t[\hat{\rho}_S(0)]. \quad (\text{I.1.3})$$

In finite dimensions, any such map can be written in Kraus form [130, 11],

$$\mathcal{E}_t[\hat{\rho}_S] = \sum_k \hat{K}_k(t) \hat{\rho}_S \hat{K}_k^\dagger(t), \quad \sum_k \hat{K}_k^\dagger \hat{K}_k = \hat{\mathbb{I}}. \quad (\text{I.1.4})$$

From a physical perspective, the Kraus operators $\hat{K}_k(t)$ encode how interactions with the environment modify the system's state. They can be obtained explicitly as [11]

$$\hat{K}_{ij}(t) = \sqrt{p_j} \langle i | \hat{U}_t | j \rangle, \quad (\text{I.1.5})$$

¹A map is CPTP if it is linear ($\mathcal{E}(c_1\hat{\rho}_1 + c_2\hat{\rho}_2) = c_1\mathcal{E}(\hat{\rho}_1) + c_2\mathcal{E}(\hat{\rho}_2)$), trace-preserving ($\text{Tr}[\mathcal{E}(\hat{\rho})] = \text{Tr}[\hat{\rho}]$), as well as completely-positive ($\mathcal{E} \otimes \mathbb{I}_N$ maps positive operators to positive operators).

where $\{p_j\}$ and $\{|j\rangle\}$ are the eigenvalues and eigenstates of the initial environmental state $\hat{\rho}_E(0)$, and

$$\hat{U}_t = \mathcal{T} \exp \left[-i \int_0^t d\tau \hat{H}_{\text{tot}}(\tau) \right] \quad (\text{I.1.6})$$

is the unitary time-evolution operator of the full system-environment composite, with $\mathcal{T}$ denoting the time-ordering operator. The pair of indices (i, j) should be understood as a single composite index labeling the Kraus operators.

This expression provides physical intuition: each Kraus operator represents a conditional evolution of the system associated with a particular transition of the environment from $|j\rangle$ (weighted by its probability p_j) to $|i\rangle$ under the joint unitary $\hat{U}_t$. In other words, the environment performs a (generally unobserved) “measurement-like” interaction on the system, and the reduced dynamics arise from averaging over all such conditional outcomes. We expand on this perspective in Section I.1.3.

When the reduced dynamics is Markovian², the maps $\{\mathcal{E}_t\}_{t \geq 0}$ are CP-divisible, meaning that for any $t \geq s \geq 0$ there exists a completely positive and trace-preserving propagator $\mathcal{E}_{t:s}$ such that $\mathcal{E}_t = \mathcal{E}_{t:s} \circ \mathcal{E}_s$. In this case, the evolution can be written in time-ordered exponential form,

$$\mathcal{E}_t = \mathcal{E}_{t:0} = \mathcal{T} \exp \left[\int_0^t d\tau \mathcal{L}(\tau) \right], \quad (\text{I.1.7})$$

where $\mathcal{L}(t)$ is the (generally time-dependent) generator of the dynamics, known as the Liouvillian.

Its structure is strongly constrained by physical requirements: weak coupling to the environment (Born approximation), factorised initial state, environment correlation functions decaying on timescales much shorter than the system dynamics (Markov approximation), and the environment’s influence on the systems being much slower than the system’s internal dynamics (secular approximation). Under such conditions, the evolution of the system state is governed by a GKSL master equation [33, 34, 11, 51], whose general structure is guaranteed by the GKSL theorem by which the generator $\mathcal{L}$ of a quantum dynamical semigroup must take the form

$$\frac{d}{dt} \hat{\rho}(t) = \mathcal{L}[\hat{\rho}] = -i[\hat{H}, \hat{\rho}(t)] + \sum_j \mathcal{D}[\hat{L}_j] \hat{\rho}(t), \quad \mathcal{D}[\hat{L}] \hat{\rho} = \hat{L} \hat{\rho} \hat{L}^\dagger - \frac{1}{2} \{ \hat{L}^\dagger \hat{L}, \hat{\rho} \}, \quad (\text{I.1.8})$$

²Quantum Markovianity admits several inequivalent definitions, such as complete positive divisibility, absence of information backflow, or validity of regression relations, reflecting different operational notions of memory, see e.g. Ref. [131]. Here, we adopt the CP-divisible notion, which is equivalent to the existence of a time-local GKSL generator.

with $\hat{L}_j$ known as Lindblad or jump operators. For notational simplicity, we here and in the following omit the subscript S on system operators. This mathematical form ensures the desired trace preservation and complete positivity of the reduced dynamics.

In summary, the GKSL master equation emerges both as a fundamental structural result and as the effective description of a system's dynamics when weakly coupled to memoryless environments: it encodes the interplay of coherent Hamiltonian evolution and irreversible dissipative processes due to the interaction with an environment, and provides a standard framework for the study of open quantum systems.

I.1.2 Adjoint GKSL evolution of operators

In the preceding section, we discussed the evolution of quantum states under the GKSL master equation (see Section I.1.1). The corresponding evolution of operators in the Heisenberg picture is governed by the adjoint Liouvillian $\mathcal{L}^\dagger$, which acts on system operators $\hat{A}$ as

$$\frac{d\hat{A}}{dt} = \mathcal{L}^\dagger[\hat{A}] = i[\hat{H}, \hat{A}] + \sum_j \left(\hat{L}_j^\dagger \hat{A} \hat{L}_j - \frac{1}{2} \{ \hat{L}_j^\dagger \hat{L}_j, \hat{A} \} \right). \quad (\text{I.1.9})$$

This equation defines the dynamics of observables in the Heisenberg picture, ensuring consistency with the Schrödinger-picture evolution through the trace relation

$$\text{Tr}[\hat{A} \mathcal{L}[\hat{\rho}]] = \text{Tr}[\mathcal{L}^\dagger[\hat{A}] \hat{\rho}], \quad (\text{I.1.10})$$

which establishes $\mathcal{L}^\dagger$ as the adjoint of $\mathcal{L}$ with respect to the Hilbert-Schmidt inner product $\langle \hat{A}, \hat{B} \rangle = \text{Tr}[\hat{A}^\dagger \hat{B}]$. The adjoint Liouvillian preserves the identity operator, $\mathcal{L}^\dagger[\hat{\mathbb{I}}] = 0$, reflecting trace preservation of the density operator under $\mathcal{L}$, i.e., $\text{Tr}[\mathcal{L}[\hat{\rho}]] = 0$. This property ensures that expectation values evolve consistently in time:

$$\frac{d}{dt} \langle \hat{A} \rangle_t = \text{Tr} \left[\hat{A} \frac{d\hat{\rho}}{dt} \right] = \text{Tr} \left[\mathcal{L}^\dagger[\hat{A}] \hat{\rho} \right] = \langle \mathcal{L}^\dagger[\hat{A}] \rangle_t. \quad (\text{I.1.11})$$

Here, we have focused on the Liouvillian $\mathcal{L}$, but similar considerations apply to any linear superoperator, since the definition of an adjoint with respect to the Hilbert-Schmidt inner product and the consistency between Schrödinger- and Heisenberg-picture evolution hold generally.

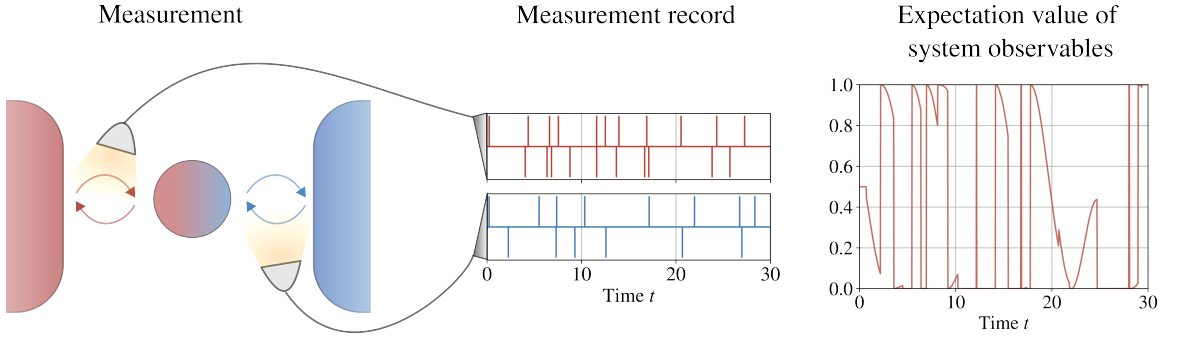


Figure I.1.2: Sketch of continuous monitoring of particle transfer between a quantum system and its environment. The measurement outcomes are stored in a measurement record. The system state evolves conditioned on this record, and expectation values of system observables are computed with respect to the conditional state. As a result, observables evolve stochastically: abruptly when jumps are recorded, and smoothly when no jumps occur.

I.1.3 Quantum jump trajectories

In this section, we review the formalism of quantum jump trajectories. From a mathematical perspective, quantum jump trajectories arise by unravelling the GKSL master equation into stochastic evolutions of pure or mixed states [132]. From a measurement perspective, they describe the stochastic evolution of the state conditioned on a sequence of detection events recorded in continuous measurements, e.g., photon or charge counting [133]. This provides direct access to trajectory-level fluctuations: expectation values become stochastic variables along individual realisations (see Fig. I.1.2). We will make use of this in Chapter III.6.

To uncover the stochastic structure underlying the GKSL master equation, it is useful to decompose the Liouvillian into contributions associated with continuous no-jump evolution and discrete jump events, $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$, where

$$\mathcal{L}_0[\hat{\rho}] = -i \left(\hat{H}_{\text{eff}} \hat{\rho} - \hat{\rho} \hat{H}_{\text{eff}}^\dagger \right), \quad \mathcal{L}_1[\hat{\rho}] = \sum_j \mathcal{L}_{1,j}[\hat{\rho}] = \sum_j \hat{L}_j \hat{\rho} \hat{L}_j^\dagger. \quad (\text{I.1.12})$$

Here, we have introduced the effective non-Hermitian Hamiltonian³

$$\hat{H}_{\text{eff}} = \hat{H} - \frac{i}{2} \sum_j \hat{L}_j^\dagger \hat{L}_j. \quad (\text{I.1.13})$$

Here, $\mathcal{L}_0$ generates a continuous, non-unitary evolution corresponding to the absence of detected quantum jumps, while $\mathcal{L}_1$ accounts for discontinuous state updates associated with detection events in the environment. This decomposition suggests an interpretation in terms of continuous measurement. To make this ex-

³Indeed, the conditional no-jump evolution is a very natural way of implementing non-Hermitian physics [134].

PLICIT, we discretise time into small intervals of duration dt . Over a single time step, the system evolution may be represented in terms of measurement operators acting on the system state,

$$\hat{M}_0 = \hat{\mathbb{I}} - i\hat{H}_{\text{eff}} dt, \quad \hat{M}_j = \sqrt{dt} \hat{L}_j, \quad (\text{I.1.14})$$

corresponding respectively to a no-jump outcome and a jump of type j occurring within the interval dt .

These operators satisfy the completeness relation up to order dt ,

$$\hat{M}_0^\dagger \hat{M}_0 + \sum_j \hat{M}_j^\dagger \hat{M}_j = \hat{\mathbb{I}} + \mathcal{O}(dt^2), \quad (\text{I.1.15})$$

ensuring conservation of total probability. The measurement outcomes define a measurement record $r = \{(t_1, j_1), (t_2, j_2), \dots\}$, consisting of a sequence of jump events labeled by their occurrence times t_k and channels j_k , or equivalently by the sequence of outcomes observed in each discrete time step. Conditioned on a given measurement record r , the system state is described by an unnormalised density matrix $\tilde{\rho}_r(t)$, obtained by applying the corresponding sequence of measurement operators to the initial state. For example, if no jump occurs during a time interval $[0, t]$, the conditional state evolves as

$$\tilde{\rho}_{\text{nj}}(t) = e^{t\mathcal{L}_0}[\hat{\rho}(0)] = e^{-i\hat{H}_{\text{eff}}t} \hat{\rho}(0) e^{i\hat{H}_{\text{eff}}^\dagger t}. \quad (\text{I.1.16})$$

The trace of the unnormalised state,

$$p_r(t) = \text{Tr} [\tilde{\rho}_r(t)], \quad (\text{I.1.17})$$

has a direct probabilistic meaning: it is the probability for observing the measurement record r up to time t . In particular,

$$P_{\text{surv}}(t) = \text{Tr} [\tilde{\rho}_{\text{nj}}(t)] \quad (\text{I.1.18})$$

is the survival probability, i.e. the probability that no jump has occurred up to time t . The normalised conditional state is obtained as

$$\hat{\rho}_r(t) = \frac{\tilde{\rho}_r(t)}{\text{Tr} [\tilde{\rho}_r(t)]}. \quad (\text{I.1.19})$$

In this sense, the evolution of $\hat{\rho}_r(t)$ can be viewed as a quantum analogue of a Bayesian update, with each measurement outcome updating the conditional state based on newly acquired information.

Since the formal solution of the GKSL master equation is $\hat{\rho}(t) = e^{t\mathcal{L}}[\hat{\rho}(0)]$, we

may use the decomposition $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$ to expand the propagator using a Dyson series,

$$e^{t\mathcal{L}} = \sum_{n=0}^{\infty} \int_0^t dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_2} dt_1 e^{(t-t_n)\mathcal{L}_0} \mathcal{L}_1 e^{(t_n-t_{n-1})\mathcal{L}_0} \cdots \mathcal{L}_1 e^{t_1\mathcal{L}_0}. \quad (\text{I.1.20})$$

Each term in this expansion corresponds to a quantum trajectory with exactly n jumps occurring at times $0 < t_1 < \cdots < t_n < t$, with non-unitary evolution generated by $\mathcal{L}_0$ in between. The unnormalised density matrix associated with a given measurement record r is therefore

$$\tilde{\rho}_r(t) = e^{(t-t_n)\mathcal{L}_0} \mathcal{L}_{1,j_n} \cdots \mathcal{L}_{1,j_1} e^{t_1\mathcal{L}_0} [\hat{\rho}(0)], \quad (\text{I.1.21})$$

and $\mathcal{L}_1 = \sum_j \mathcal{L}_{1,j}$. Summing over all possible records yields

$$\hat{\rho}(t) = \sum_r \tilde{\rho}_r(t), \quad (\text{I.1.22})$$

establishing that the solution of the GKSL master equation is the ensemble average over all quantum jump trajectories.

In the continuous-time limit, the evolution of the unnormalised conditional state may be written as a stochastic differential equation,

$$d\tilde{\rho}_r(t) = \mathcal{L}_0[\tilde{\rho}_r(t)] dt + \sum_j \mathcal{L}_{1,j}[\tilde{\rho}_r(t)] dN_j, \quad (\text{I.1.23})$$

while the normalised conditional state evolves under

$$d\hat{\rho}_r = -i[\hat{H}, \hat{\rho}_r] dt - \frac{1}{2} \sum_j \mathcal{H}[\hat{L}_j^\dagger \hat{L}_j] \hat{\rho}_r dt + \sum_j \mathcal{G}[\hat{L}_j] \hat{\rho}_r dN_j, \quad (\text{I.1.24})$$

with

$$\mathcal{G}[\hat{O}] \hat{\rho}_r = \frac{\hat{O} \hat{\rho}_r \hat{O}^\dagger}{\text{Tr}[\hat{O} \hat{\rho}_r \hat{O}^\dagger]} - \hat{\rho}_r, \quad \mathcal{H}[\hat{O}] \hat{\rho}_r = \hat{O} \hat{\rho}_r + \hat{\rho}_r \hat{O}^\dagger - \text{Tr}[\hat{O} \hat{\rho}_r + \hat{\rho}_r \hat{O}^\dagger] \hat{\rho}_r, \quad (\text{I.1.25})$$

and the stochastic increments satisfy [135]

$$\mathbb{E}[dN_j] = \langle \hat{L}_j^\dagger \hat{L}_j \rangle_{\hat{\rho}_r} dt, \quad dN_j dN_{j'} = \delta_{jj'} dN_j. \quad (\text{I.1.26})$$

Here, $\mathcal{G}$ describes instantaneous updates due to detected jumps ($dN_j = 1$), while $\mathcal{H}$ describes continuous evolution between jumps ($dN_j = 0$).

I.1.4 Liouville representation and Liouville spectrum

For both analytical and numerical purposes, it is convenient to represent density operators as vectors in a larger Hilbert space. This approach is known as the Liouville representation or vectorization of the density matrix [11]. Given a density operator $\hat{\rho}$ acting on a Hilbert space $\mathcal{H}$ of dimension d , we define its vectorised form $|\hat{\rho}\rangle\rangle \in \mathbb{C}^{d^2}$ by stacking the columns of $\hat{\rho}$ into a single column vector

$$\hat{\rho} = \sum_{i,j} \rho_{ij} |i\rangle\langle j| \quad \longrightarrow \quad |\hat{\rho}\rangle\rangle = \sum_{i,j} \rho_{ij} |i\rangle \otimes |j\rangle. \quad (\text{I.1.27})$$

Here, $\{|i\rangle\}$ is an orthonormal basis of $\mathcal{H}$. In this representation, the action of a linear map $\mathcal{E}$ on $\hat{\rho}$ can be written as a matrix multiplication

$$|\mathcal{E}(\hat{\rho})\rangle\rangle = \underline{\mathcal{E}} |\hat{\rho}\rangle\rangle, \quad (\text{I.1.28})$$

where $\underline{\mathcal{E}} \in \mathbb{C}^{d^2 \times d^2}$ is the matrix corresponding to $\mathcal{E}$. For example, if $\hat{\mathcal{E}}[\hat{\rho}] = \hat{A}\hat{\rho}\hat{B}$ with operators $\hat{A}$ and $\hat{B}$ on $\mathcal{H}$, then in vectorised form

$$|\mathcal{E}[\hat{\rho}]\rangle\rangle = (\hat{B}^T \otimes \hat{A}) |\hat{\rho}\rangle\rangle. \quad (\text{I.1.29})$$

The Liouville representation is particularly useful when dealing with the GKSL master equation, as it allows one to treat the evolution of $\hat{\rho}$ as a linear differential equation in a higher-dimensional vector space

$$\frac{d}{dt} |\hat{\rho}(t)\rangle\rangle = \underline{\mathcal{L}} |\hat{\rho}(t)\rangle\rangle. \quad (\text{I.1.30})$$

While this form is particularly convenient for numerical simulations, it also allows us to analyse the spectral properties of the Liouvillian in matrix form. Since $\underline{\mathcal{L}}$ is not Hermitian, its spectrum may be complex with complex conjugate eigenpairs, and distinct left and right eigenvectors

$$\underline{\mathcal{L}} |\hat{r}_i\rangle\rangle = \lambda_i |\hat{r}_i\rangle\rangle, \quad \langle\langle \hat{l}_i | \underline{\mathcal{L}} = \lambda_i \langle\langle \hat{l}_i |, \quad (\text{I.1.31})$$

where the left and right eigenvectors fulfil the biorthonormality condition $\langle\langle \hat{l}_i | \hat{r}_j \rangle\rangle = \delta_{ij}$, and, importantly, $\langle\langle \hat{l}_i | \neq (|\hat{r}_i\rangle\rangle)^\dagger$. Further, $\text{Re } \lambda_i \leq 0$ ensures the physical requirement that density matrices evolve toward a steady state without unphysical exponential growth. In fact, the steady state is the zero-eigenvalue right eigenvector $\underline{\mathcal{L}} |\hat{\pi}_{ss}\rangle\rangle = 0$, with the vectorised identity matrix corresponding to the left eigenvector, ensuring unit trace. Another important spectral feature is the so-called Liouvillian gap

$$\Delta = \min_{\lambda_i \neq 0} \{-\text{Re } \lambda_i\}. \quad (\text{I.1.32})$$

Its dynamical importance becomes apparent when we decompose the evolution of the system from some initial state into that of Liouvillian eigenmodes. For convenience, we order the Liouvillian eigenvalues by descending real parts, so that $\dots, \text{Re } \lambda_2 \leq \text{Re } \lambda_1 < \lambda_0 = 0$ and we, for simplicity, assume that λ_1 is real. The state evolves as

$$|\hat{\rho}(t)\rangle\rangle = e^{t\mathcal{L}}|\hat{\rho}(0)\rangle\rangle = |\hat{\pi}_{\text{ss}}\rangle\rangle + \sum_{i>0} c_i e^{\lambda_i t} |\hat{r}_i\rangle\rangle, \quad (\text{I.1.33})$$

where the spectral weights c_i are set by the initial state, and trace preservation is ensured since all Liouvillian eigenmodes, apart from the steady state, are traceless. Since $\text{Re } \lambda_i \leq 0$, at long times

$$|\hat{\rho}(t)\rangle\rangle \approx |\hat{\pi}_{\text{ss}}\rangle\rangle + c_1 e^{\lambda_1 t} |\hat{r}_1\rangle\rangle, \quad (\text{I.1.34})$$

and $\lambda_1 = -\Delta$ sets the relaxation time scale. In Part V, this quantity will play a central role: the closing of the Liouvillian gap is precisely what gives rise to critical slowing down in DQPTs.

I.1.5 Nonequilibrium steady states

When a quantum system is coupled to one or several environments, its state typically relaxes toward a time-independent configuration known as a NESS. Formally, the NESS is defined as a stationary solution of the GKSL master equation,

$$\frac{d}{dt}\hat{\rho}(t) = \mathcal{L}[\hat{\rho}(t)], \quad (\text{I.1.35})$$

satisfying $\mathcal{L}[\hat{\pi}_{\text{ss}}] = 0$. In thermal equilibrium with an environment at inverse temperature $\beta = 1/T$, the so-called detailed balance condition⁴ ensures that the steady state is a Gibbs state $\hat{\rho}_{\text{eq}} \propto e^{-\beta\hat{H}}$ [11], provided the dynamics is ergodic. More generally, open quantum systems can be driven out of equilibrium by competing dissipative processes, such as coupling to reservoirs at different temperatures or chemical potentials, or by coherent external driving. In such situations, the stationary state $\hat{\pi}_{\text{ss}}$ supports nonvanishing steady-state fluxes of energy, particles, or other observables, signaling a breakdown of detailed balance and giving rise to a finite entropy production rate. For a discussion of entropy production and related thermodynamic quantities, we refer to Chapter I.2.

⁴For open quantum systems, detailed balance expresses a form of time-reversal symmetry of the dissipative dynamics with respect to its stationary state. A GKSL generator satisfies quantum detailed balance if it is self-adjoint with respect to a steady-state-weighted inner product, such as the Kubo-Martin-Schwinger (KMS) inner product. Detailed balance implies the absence of steady-state probability currents and vanishing entropy production, while its violation characterises NESSs; see, e.g., Ref. [136].

I.1.6 Drazin inverse of the Liouvillian

Because the steady state corresponds to the zero eigenvalue of $\mathcal{L}$, the Liouvillian is generally non-invertible. Nevertheless, in many contexts we are interested in a notion of an “inverse” for $\mathcal{L}$. To address this, it is convenient to introduce the Drazin inverse $\mathcal{L}^+$, a generalised inverse that acts on the complement of the steady-state subspace. Effectively, we “invert what we can”. It proves particularly useful for formally solving GKSL evolution equations, as we will demonstrate with several examples throughout this thesis. The Drazin inverse $\mathcal{L}^+$ satisfies

$$\mathcal{L}\mathcal{L}^+\mathcal{L} = \mathcal{L}, \quad \mathcal{L}^+\mathcal{L}\mathcal{L}^+ = \mathcal{L}^+, \quad \text{and} \quad \mathcal{L}\mathcal{L}^+ = \mathcal{L}^+\mathcal{L} = \mathbb{I} - \mathcal{P}_0, \quad (\text{I.1.36})$$

where $\mathcal{P}_0$ is the projector onto the kernel of $\mathcal{L}$ (i.e., the steady-state subspace, and we assume a unique steady state).

If the Liouvillian $\mathcal{L}$ is diagonalizable (see Section I.1.4), then the Drazin inverse $\mathcal{L}^+$ can then be expressed in matrix form as

$$\underline{\mathcal{L}}^+ = \sum_{i \neq 0} \frac{1}{\lambda_i} |\hat{r}_i\rangle\rangle \langle\langle \hat{l}_i|, \quad (\text{I.1.37})$$

where the sum runs over all nonzero eigenvalues. In this notation, the projector onto the steady-state subspace is $\underline{\mathcal{P}}_0 = |\hat{r}_0\rangle\rangle \langle\langle \hat{l}_0| = |\hat{\pi}_{\text{ss}}\rangle\rangle \langle\langle \hat{I}|$, with $\langle\langle \hat{l}_0| = \langle\langle \hat{I}|$ and $|\hat{r}_0\rangle\rangle = |\hat{\pi}_{\text{ss}}\rangle\rangle$. Alternatively, the Drazin inverse admits an integral representation in terms of the propagator,

$$\mathcal{L}^+ = \int_0^\infty dt (e^{t\mathcal{L}} - \mathcal{P}_0), \quad (\text{I.1.38})$$

which converges if all nonzero eigenvalues of $\underline{\mathcal{L}}$ have negative real parts [137].

I.1.7 Local versus global approaches

This section is based on Refs. [138–140], to which we refer for more detailed discussions.

Depending on how the system-bath coupling is treated in the derivation of the GKSL master equation, one distinguishes between global and local GKSL master equations. While both generate CPTP dynamics, they differ substantially in their physical assumptions and thermodynamic properties. The global approach is justified when internal interactions are comparable to system-bath coupling and level splittings are well resolved, whereas local master equations are appropriate in the limit of weak inter-system interactions or when baths couple on timescales short compared to internal equilibration.

In the global approach, the GKSL master equation is derived after diagonalizing the full interacting system Hamiltonian,

$$\hat{H}_S = \sum_{\alpha} \hat{H}_{\alpha} + \hat{H}_{\text{int}}, \quad (\text{I.1.39})$$

where $\hat{H}_{\text{int}}$ describes interactions between subsystems. The jump operators are constructed from eigenoperators $\hat{A}_{\alpha}(\omega)$ of $\hat{H}_S$, satisfying $[\hat{H}_S, \hat{A}_{\alpha}(\omega)] = -\omega \hat{A}_{\alpha}(\omega)$. For a single thermal reservoir, the steady state is the Gibbs state $\hat{\omega}_{\beta} \propto e^{-\beta \hat{H}_S}$, ensuring detailed balance with respect to the full interacting Hamiltonian. From a thermodynamic perspective, global GKSL master equations are consistent: heat currents, work, and entropy production can be defined unambiguously, and the second law of thermodynamics is satisfied. However, the approach becomes computationally prohibitive for large many-body systems, as the jump operators are highly nonlocal and depend on the full spectrum of $\hat{H}_S$. Further, while thermodynamically consistent, the global approach may fail to capture internal transport properties, e.g. vanishing internal particle or energy currents in the NESS, even when such currents are expected on physical grounds [141]. This occurs because the baths induce transitions between global energy eigenstates, effectively bypassing the local gradients that drive transport.

In contrast, local GKSL master equations are obtained by coupling each bath to a local subsystem operator, for example to a site at the edges of a spin chain, neglecting inter-system interactions during the derivation of the dissipator. In this approach, one exploits the decomposition (I.1.39) and constructs the dissipator from eigenoperators of the local Hamiltonians $\hat{H}_{\alpha}$, defined through $[\hat{H}_{\alpha}, \hat{L}_{\alpha}(\omega)] = -\omega \hat{L}_{\alpha}(\omega)$, thereby neglecting the effect of inter-system interactions on the bath-induced transitions. Consequently, the dissipator is often not diagonal in the energy eigenbasis of $\hat{H}_S$. While local master equations are particularly natural for boundary-driven transport setups, for example, with spin chains or fermionic lattices, they may lack thermodynamic consistency. Since dissipation does not respect the full eigenstructure of $\hat{H}_S$, the steady state of a locally thermalised system is not, in general, a Gibbs state of the interacting Hamiltonian. As a result, local master equations may predict entropy production rates that are inconsistent with the second law [142], as discussed in Section I.2.4.2. Although this violation was later shown to be only apparent, arising beyond the order of the employed approximation [143], it can nevertheless be advantageous to adopt alternative descriptions of thermodynamic currents. One such approach is provided by repeated-interaction (collisional) models, which restore thermodynamic consistency [31] and will be described in Section I.2.4.3.

Chapter I.2

Thermodynamics of open quantum systems

Quantum thermodynamics seeks to understand how the laws of thermodynamics emerge in the quantum regime, while also leveraging these universal principles to both elucidate quantum phenomena and guide the design of useful quantum devices [7–10]. Guided by these objectives, we explore a range of complementary scenarios in open quantum systems.

This chapter establishes the thermodynamic framework used throughout the thesis. We begin from a microscopic description of a central system coupled to thermal baths and formulate quantum thermodynamics for this standard class of open quantum systems (see Section I.2.1). We then specialise to the particularly tractable GKSL regime (see Section I.2.2) and summarise key notions for assessing the performance of heat engines (see Section I.2.3). We subsequently discuss thermodynamics in the local GKSL approach (see Section I.2.4), highlighting apparent thermodynamic inconsistencies and their resolution within the repeated-interactions framework, illustrated using a boundary-driven double quantum dot (DQD) heat engine. Both the repeated-interactions framework and the DQD heat engine play a central role in Chapter II.4. We then introduce the entropic cost of nonstationarity (see Section I.2.5), which is relevant for Chapters IV.8 and V.10. Finally, we review stochastic entropy production along quantum jump trajectories and the associated fluctuation theorems (see Section I.2.6), which are relevant for Chapter III.6.

At a later stage, in Chapter III.5, we extend this thermodynamic framework beyond the Markovian weak-coupling regime using the mesoscopic-leads approach. Combined with quantum jump trajectories, this provides a methodology for studying stochastic thermodynamics under strong coupling and non-Markovian dynamics, as explored in Chapter III.6.

The connections between information geometry and quantum thermodynamics

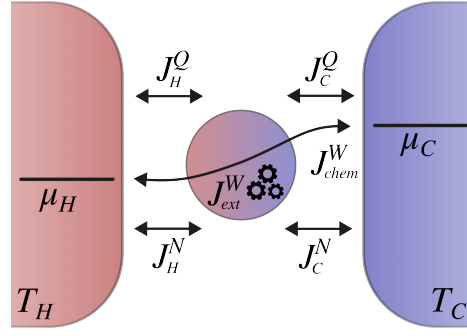


Figure I.2.1: Illustration of a microscopic system described by a time-dependent Hamiltonian, which acts as a source or sink of external work. The system is coupled to two thermal reservoirs characterised by temperatures T_α and chemical potentials μ_α . Chemical work is performed when particles are driven against a chemical bias $\Delta\mu$. The system exchanges particles and heat with the reservoirs via the particle and heat currents J_α^N and J_α^Q , respectively.

that are relevant for this thesis are introduced in Chapter IV.7 and further developed in Chapters IV.8 and V.10.

I.2.1 Microscopic description

This section is based on Ref. [144]. The presentation closely follows this reference.

In the following, we consider a microscopic system S described by a time-dependent Hamiltonian $\hat{H}_S(t)$, coupled to at least one fermionic or bosonic bath, as schematically shown in Fig. I.2.1. The total Hamiltonian is given by

$$\hat{H}_{\text{tot}}(t) = \hat{H}_S(t) + \sum_{\alpha} (\hat{H}_{\alpha} + \hat{H}_{S\alpha}), \quad (\text{I.2.1})$$

where $\hat{H}_{\alpha}$ is the free Hamiltonian of bath α , and $\hat{H}_{S\alpha}$ describes the coupling between the bath α and the system. Further, we assume that the bath Hamiltonians preserve their respective particle numbers, so that $[\hat{H}_{\alpha}, \hat{N}_{\alpha}] = 0$, where $\hat{N}_{\alpha}$ is the number operator of the α^{th} bath. Further, we define the particle current from reservoir α into the system as $J_{\alpha}^N(t) = -\frac{d}{dt}\langle \hat{N}_{\alpha} \rangle$. Finally, we assume that the system and the baths, prepared in thermal states at inverse temperature $\beta_{\alpha} = 1/T_{\alpha}$ and chemical potential μ_{α} , are initially uncorrelated, so that the global state is

$$\hat{\rho}_{\text{tot}}(0) = \hat{\rho}_S(0) \bigotimes_{\alpha} \hat{\omega}_{\alpha}, \quad \text{where} \quad \hat{\omega}_{\alpha} = \frac{e^{-\beta_{\alpha}(\hat{H}_{\alpha} - \mu_{\alpha}\hat{N}_{\alpha})}}{Z_{\alpha}}. \quad (\text{I.2.2})$$

Here, $Z_{\alpha} = \text{Tr} \left[e^{-\beta_{\alpha}(\hat{H}_{\alpha} - \mu_{\alpha}\hat{N}_{\alpha})} \right]$ denotes the partition function. The global state evolves unitarily under $\hat{H}_{\text{tot}}(t)$, and the reduced system state is obtained by tracing

out the baths, see Chapter I.1. Having this set-up in mind, we now focus on the average thermodynamic currents in it. Since the global system is closed, any change in its total energy must stem from work performed by an external protocol:

$$W_{\text{ext}}(t) = \langle \hat{H}_{\text{tot}}(t) \rangle - \langle \hat{H}_{\text{tot}}(0) \rangle = \int_0^t dt' \frac{d\langle \hat{H}_{\text{tot}}(t') \rangle}{dt'}. \quad (\text{I.2.3})$$

Since we assumed that only the system Hamiltonian is time-dependent, this reduces to the time integral of the expectation value of its instantaneous rate of change

$$W_{\text{ext}}(t) = \int_0^t dt' \left\langle \frac{\partial \hat{H}_S(t')}{\partial t'} \right\rangle. \quad (\text{I.2.4})$$

Here, we have also used that since the global state evolves unitarily, $\text{Tr}[\dot{\hat{\rho}}_{\text{tot}} \hat{H}_{\text{tot}}] \propto \text{Tr}[[\hat{\rho}_{\text{tot}}, \hat{H}_{\text{tot}}] \hat{H}_{\text{tot}}] = 0$, due to the cyclicity of the trace. Due to the nonzero chemical potential of the baths, the system may also perform chemical work resulting in an increase in the particle number within the baths. This is reflected in

$$W_{\text{chem}}(t) = - \sum_{\alpha} \mu_{\alpha} (\langle \hat{N}_{\alpha}(t) \rangle - \langle \hat{N}_{\alpha}(0) \rangle). \quad (\text{I.2.5})$$

Combining these two contributions yields the total work: $W(t) = W_{\text{chem}}(t) + W_{\text{ext}}(t)$. Next, following the general framework introduced by Spohn [145–149], we define the entropy production as

$$\Sigma(t) = D \left(\hat{\rho}_{\text{tot}}(t) \parallel \hat{\rho}_S(t) \bigotimes_{\alpha} \hat{\omega}_{\alpha} \right). \quad (\text{I.2.6})$$

Here,

$$D(\hat{\rho} \parallel \hat{\sigma}) = \text{Tr}[\hat{\rho}(\log \hat{\rho} - \log \hat{\sigma})] \quad (\text{I.2.7})$$

denotes the quantum relative entropy between the two density matrices $\hat{\rho}$ and $\hat{\sigma}$. By construction, the second law of thermodynamics, which can be stated mathematically as $\Sigma(t) \geq 0$, is then automatically satisfied, since $D(\hat{\rho} \parallel \hat{\sigma}) \geq 0$. Further, this allows us to express the second law in the standard Clausius form¹ [151, 2, 3, 146, 148]

$$\Sigma(t) = \Delta S_S(t) + \sum_{\alpha} \beta_{\alpha} Q_{\alpha}(t) \geq 0. \quad (\text{I.2.8})$$

Here, $\Delta S_S(t)$ denotes the change in the system's von Neumann entropy, $\Delta S_S(t) =$

¹Here, we implicitly used our assumption that the baths are macroscopic, such that their temperatures and chemical potentials can be taken to be constant. Otherwise, for finite baths, this assumption pertaining to the temperatures and chemical potentials does not hold beyond a finite time [150, 149].

$S_{\text{vN}}(\hat{\rho}(t)) - S_{\text{vN}}(\hat{\rho}(0))$ with

$$S_{\text{vN}}(\hat{\rho}) = -\text{Tr} [\hat{\rho} \log \hat{\rho}], \quad (\text{I.2.9})$$

and the heat transferred into the α -th bath,

$$Q_\alpha(t) = \langle \hat{H}_\alpha(t) \rangle - \langle \hat{H}_\alpha(0) \rangle - \mu_\alpha (\langle \hat{N}_\alpha(t) \rangle - \langle \hat{N}_\alpha(0) \rangle), \quad (\text{I.2.10})$$

weighted by the inverse bath temperatures β_α . The second term,

$$\Pi = \sum_\alpha \beta_\alpha Q_\alpha(t), \quad (\text{I.2.11})$$

is typically referred to as the entropy flux from the system into its environment. The balance (I.2.8) clarifies why Σ is called the entropy production: unlike conserved quantities, entropy does not satisfy a continuity equation and can be irreversibly generated [151, 2, 3]. For a review on the topic of entropy production we refer to Ref. [148].

Expressing the first law of thermodynamics, the change in internal energy of the system is defined by the difference between work done on the system and heat transferred into the environment, so that $\Delta U(t) = W(t) - \sum_\alpha Q_\alpha(t)$. Further intuition can be gained by using the definitions of both the total work and heat, as well as $\hat{H}_{\text{tot}}$ in Eq. (I.2.1), to find that the change in internal energy,

$$\Delta U(t) = \left\langle \hat{H}_S(t) + \sum_\alpha \hat{H}_{S\alpha}(t) \right\rangle - \left\langle \hat{H}_S(0) + \sum_\alpha \hat{H}_{S\alpha}(0) \right\rangle, \quad (\text{I.2.12})$$

depends only on the system-related terms in the Hamiltonian. Note that here we included the average interaction energy between the system and the baths in the definition of the internal system energy. This choice, albeit self-consistent, is not unique and is of particular relevance in the strong-coupling regime which we will return to shortly [35].

At first glance, the above statements of work and heat seem to demand access to the full microscopic details of the baths. Fortunately, this is not so, as the energy current from the α -th bath can be expressed as

$$J_\alpha^E(t) = -\frac{d \langle \hat{H}_\alpha \rangle}{dt} = -i \langle [\hat{H}_{\text{tot}}(t), \hat{H}_\alpha] \rangle = i \langle [\hat{H}_\alpha, \hat{H}_{S\alpha}] \rangle. \quad (\text{I.2.13})$$

Together with the particle current,

$$J_\alpha^N(t) = -\frac{d \langle \hat{N}_\alpha \rangle}{dt} = -i \langle [\hat{H}_{\text{tot}}(t), \hat{N}_\alpha] \rangle = i \langle [\hat{N}_\alpha, \hat{H}_{S\alpha}] \rangle, \quad (\text{I.2.14})$$

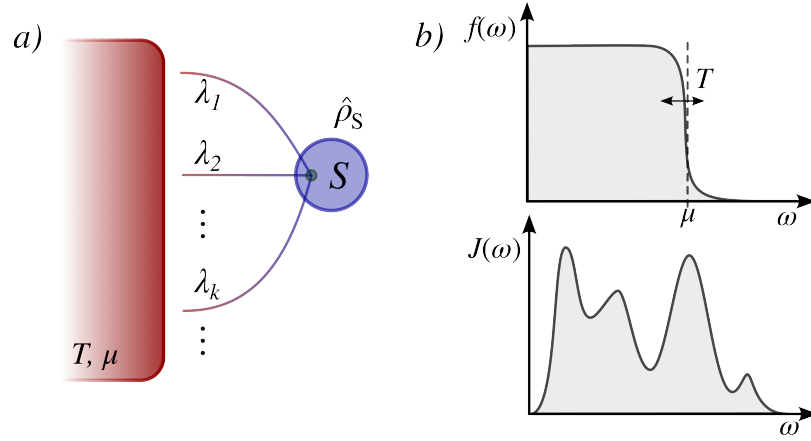


Figure I.2.2: The dynamics of a system coupled to a single thermal fermionic bath, shown in a), is determined by b) the bath's spectral density $J(\omega)$, together with the bath's Fermi-Dirac distribution $f(\omega)$, set by its chemical potential μ and temperature T .

the heat current from the α -th bath can be rewritten as

$$J_\alpha^Q(t) = J_\alpha^E(t) - \mu_\alpha J_\alpha^N(t). \quad (\text{I.2.15})$$

This means that both the heat flowing into the α -th bath as well as contributions to the total work take a significantly simpler form:

$$\begin{aligned} Q_\alpha(t) &= - \int_0^t dt' J_\alpha^Q(t'), & W_{\text{chem}}(t) &= \sum_\alpha \mu_\alpha \int_0^t dt' J_\alpha^N(t'), \\ W_{\text{ext}}(t) &= \Delta U(t) - \sum_\alpha \int_0^t dt' J_\alpha^E(t'). \end{aligned} \quad (\text{I.2.16})$$

We thus have arrived at a description of average thermodynamics quantities relying only on energy and the particle currents from the baths, the state of the system and the system-bath coupling energy. To proceed, we assume that the baths consist of infinitely many fermionic or bosonic modes and are initially prepared in thermal states (see Eq. (I.2.2)). That is, we take them to be idealised reservoirs. We set the α -th reservoir Hamiltonian to

$$\hat{H}_\alpha = \sum_{j=1}^{\infty} \omega_{j\alpha} \hat{b}_{j\alpha}^\dagger \hat{b}_{j\alpha}, \quad (\text{I.2.17})$$

with $\hat{b}_{j\alpha}$ being fermionic or bosonic annihilation operators, and the subscript indicates the mode j of the respective reservoir α . In addition, we let the system-reservoir coupling take the form

$$\hat{H}_{S\alpha} = \sum_{j=1}^{\infty} \left(\lambda_{j\alpha} \hat{c}_\alpha^\dagger \hat{b}_{j\alpha} + \lambda_{j\alpha}^* \hat{b}_{j\alpha}^\dagger \hat{c}_\alpha \right), \quad (\text{I.2.18})$$

Here, the operator $\hat{c}_\alpha$ denotes the system operator to which the α -th reservoir couples. Under these assumptions (including (I.2.2)), which yield the canonical model of thermal reservoirs, it can be shown that the dynamics of the system, the currents from the reservoirs and the system-reservoir coupling energy are all fully determined by the reservoir spectral functions [11],

$$\mathcal{J}_\alpha(\omega) = 2\pi \sum_{m=1}^{\infty} |\lambda_{m\alpha}|^2 \delta(\omega - \omega_{m\alpha}), \quad (\text{I.2.19})$$

as well as the particle statistics in the reservoirs, i.e. the Fermi-Dirac or the Bose-Einstein distributions, $f_\alpha(\omega) = (e^{\beta_\alpha(\omega - \mu_\alpha)} \pm 1)^{-1}$, as schematically shown for the fermionic case in Fig. I.2.2. This is quite remarkable: these quantities depend only on the macroscopic characteristics of the reservoirs. Consequently, any collection of reservoirs with identical spectral functions, temperatures, and chemical potentials give rise to the same system dynamics and the same currents exchanged between the system and the reservoirs. This observation is central to understanding the working principle of the mesoscopic-leads approach introduced in Chapter III.5. However, there is a caveat: we have assumed that the reservoirs contain infinitely many modes, which makes the calculation of these thermodynamic quantities intractable. Moreover, without further approximations, the system's reduced dynamics is generally non-Markovian. In fact, only in the absence of external driving, and when the system is quadratic in bosonic or fermionic creation and annihilation operators, can the long-time (steady-state) results be obtained exactly using standard analytical techniques such as nonequilibrium Green's functions [152] or the Landauer-Büttiker formalism [153–155]. In the presence of external driving, even for quadratic systems, these approaches typically require perturbative methods [156–158].

We may consider two scenarios. In the first, the system couples only weakly to its environment, and additionally, this interaction induces Markovian dynamics. In such cases, the system's evolution is governed by a GKSL master equation (see Section I.1.1). This framework renders the thermodynamic description more tractable, since all thermodynamic quantities depend only on the instantaneous state of the system, and it removes ambiguities associated with assigning interaction energy between the system and the reservoirs, which can then be consistently neglected. We examine the thermodynamics under these conditions in detail in Section I.2.2. Although certain caveats remain, such as potential thermodynamic inconsistencies under specific circumstances (see Section I.2.4.2), these can be addressed relatively easily, for instance, by employing the framework of repeated interactions (see Section I.2.4.3).

However, when the system couples strongly to its environment or exhibits non-Markovian dynamics, the most straightforward numerical approach is to model

finite, but sufficiently large baths and evolve the total system under full unitary dynamics. A systematic implementation of this strategy is provided by chain-mapping techniques [159–167], which allow one to choose a microscopic representation of the baths that exactly preserves their spectral functions, leaving both the dynamics and thermodynamics unchanged. This method, however, has its practical limitations since the number of bath modes required grows with the time the joint system is evolved for.

A versatile (and also physically motivated) alternative to treat strong-coupling dynamics and thermodynamics is the mesoscopic-leads formalism. We introduce it in Chapter III.5, as it underpins results presented in the subsequent Chapter III.6.

I.2.2 GKSL regime

In the following, we restrict our attention to the particularly well-controlled GKSL regime, in which the coupling between the system and the thermal reservoirs is weak and can be assumed to be additive², and the resulting system dynamics is Markovian. Under these conditions, the system's evolution is accurately captured by a GKSL master equation, as discussed in Section I.1.1.

We consider a system with Hamiltonian $\hat{H}$ that conserves its particle number $\hat{N}$ (i.e. $[\hat{H}, \hat{N}] = 0$) coupled to at least one thermal reservoir labeled by α , each characterised by an inverse temperature β_α and chemical potential μ_α . The system dynamics are governed by the GKSL master equation (in the form of Eq. (I.1.8)),

$$\frac{d\hat{\rho}}{dt} = \mathcal{L}[\hat{\rho}] = -i[\hat{H}, \hat{\rho}] + \sum_{\alpha} \mathcal{D}_{\alpha}[\hat{\rho}], \quad (\text{I.2.20})$$

where, for convenience, we collect into the dissipator $\mathcal{D}_{\alpha}$ all contributions arising from the system's coupling to reservoir α . To formalise this, we define

$$\mathcal{L}_{\alpha}[\hat{\rho}] = -i[\hat{H}, \hat{\rho}] + \mathcal{D}_{\alpha}[\hat{\rho}], \quad (\text{I.2.21})$$

as the generator corresponding to reservoir α alone, and we assume that under $\mathcal{L}_{\alpha}$, the system would relax to the Gibbs state

$$\mathcal{L}_{\alpha}[\hat{\omega}_{\alpha}] = 0, \quad \hat{\omega}_{\alpha} = \frac{e^{-\beta_{\alpha}(\hat{H} - \mu_{\alpha}\hat{N})}}{Z_{\alpha}}, \quad Z_{\alpha} = \text{Tr}[e^{-\beta_{\alpha}(\hat{H} - \mu_{\alpha}\hat{N})}]. \quad (\text{I.2.22})$$

Note that this constitutes a strong assumption. We adopt it here because, as will be demonstrated below, it ensures a thermodynamically consistent description. When this assumption fails, however, a more refined treatment becomes necessary.

²This assumption must be treated with care, as discussed in Ref. [168].

This is the case, for instance, for local master equations (see Section I.1.7).

We begin by examining the energetic exchanges between the system and its environment, and define the system's internal energy as $U(t) = \text{Tr}[\hat{\rho}(t) \hat{H}]$, while the energy and particle currents, $J^E = \frac{d}{dt} \langle \hat{H} \rangle$ and $J^N = \frac{d}{dt} \langle \hat{N} \rangle$, can be decomposed into additive contributions from each reservoir,

$$J_\alpha^E = \text{Tr}[\hat{\rho} \mathcal{D}_\alpha^\dagger[\hat{H}]], \quad J_\alpha^N = \text{Tr}[\hat{\rho} \mathcal{D}_\alpha^\dagger[\hat{N}]], \quad (\text{I.2.23})$$

where $\mathcal{D}_\alpha^\dagger$ denotes the adjoint of the dissipator $\mathcal{D}_\alpha$ (see Section I.1.2). The rate at which mechanical work is performed on the system by an external drive, i.e. the power, is [169]

$$J_{\text{mech}}^W(t) = \text{Tr}[\hat{\rho}(t) \dot{\hat{H}}], \quad (\text{I.2.24})$$

while particle exchange with the reservoirs gives rise to the chemical work rate

$$J_{\text{chem}}^W(t) = \sum_\alpha \mu_\alpha J_\alpha^N. \quad (\text{I.2.25})$$

The total power is thus $J^W(t) = J_{\text{mech}}^W(t) + J_{\text{chem}}^W(t)$. Differentiating the internal energy yields the first law of thermodynamics as a rate balance

$$\frac{dU}{dt} = J^W(t) + J^Q(t), \quad (\text{I.2.26})$$

with $J^Q(t) = \sum_\alpha J_\alpha^Q$ and the heat currents flowing from the reservoirs into the system are here defined by

$$J_\alpha^Q = J_\alpha^E - \mu_\alpha J_\alpha^N. \quad (\text{I.2.27})$$

We now turn to entropic quantities. For a single reservoir with stationary Gibbs state $\hat{\omega}$ (see Eq. (I.2.22)) at inverse temperature β and with chemical potential μ , the entropy production rate in state $\hat{\rho}(t)$ is [145]

$$\dot{\Sigma}(t) = -\frac{d}{dt} D(\hat{\rho}(t) \| \hat{\omega}) \geq 0. \quad (\text{I.2.28})$$

Using $\log \hat{\omega} = -\beta(\hat{H} - \mu \hat{N}) - \log Z$, one obtains

$$\dot{\Sigma}(t) = \frac{dS_{\text{vN}}}{dt} - \beta J^Q(t), \quad (\text{I.2.29})$$

with the system's von Neumann entropy S_{vN} (see Eq. (I.2.9)) and heat current $J^Q = J^E - \mu J^N$. Equivalently, it may be written as

$$\dot{\Sigma}(t) = -\text{Tr}[\mathcal{D}[\hat{\rho}(t)](\log \hat{\rho}(t) - \log \hat{\omega})] \geq 0, \quad (\text{I.2.30})$$

which expresses the second law of thermodynamics. In the presence of more than

one reservoir, the system generally relaxes to a NESS that is non-Gibbsian, and the entropy production is expressed in terms of the heat currents as

$$\dot{\Sigma}(t) = \frac{dS_{\text{vN}}}{dt} - \sum_{\alpha} \beta_{\alpha} J_{\alpha}^Q(t). \quad (\text{I.2.31})$$

This definition is widely used due to its intuitive interpretation and practical relevance. Its usefulness becomes particularly apparent in the steady state, where the von Neumann entropy of the system is constant and $\dot{\Sigma}(t)$ can be inferred solely from current measurements. To elucidate its mathematical structure and connect it to the single-reservoir expression in terms of the relative entropy, we first note that the time derivative of the von Neumann entropy can be written as

$$\frac{dS_{\text{vN}}(t)}{dt} = - \sum_{\alpha} \text{Tr}[\mathcal{L}_{\alpha}[\hat{\rho}(t)] \log \hat{\rho}(t)], \quad (\text{I.2.32})$$

where we have used the decomposition of the generator introduced above and the fact that each $\mathcal{L}_{\alpha}$ admits a Gibbs fixed point (see Eq. (I.2.22)). Furthermore, the heat current associated with reservoir α can be expressed as

$$\beta_{\alpha} J_{\alpha}^Q(t) = -\text{Tr}[\mathcal{L}_{\alpha}[\hat{\rho}(t)] \log \hat{\omega}_{\alpha}], \quad (\text{I.2.33})$$

where we have used the explicit form of $\hat{\omega}_{\alpha}$ and the assumption that the system Hamiltonian conserves particle number. Combining these expressions, the entropy production rate takes the form

$$\dot{\Sigma}(t) = \sum_{\alpha} \text{Tr}[\mathcal{L}_{\alpha}[\hat{\rho}(t)] (\log \hat{\omega}_{\alpha} - \log \hat{\rho}(t))] \geq 0. \quad (\text{I.2.34})$$

This constitutes a generalised version of Spohn's inequality (see Eq. (I.2.28)), which guarantees the validity of the second law. It is important to emphasise, however, that this non-negativity relies crucially on the assumption that each generator $\mathcal{L}_{\alpha}$ admits a Gibbs state as its fixed point with respect to the full system Hamiltonian. When this assumption fails, as is generally the case for local master equations in which reservoirs couple to subsystems rather than to the system as a whole (see Section I.1.7), the entropy production defined via heat currents may become negative. We will make this explicit in Sections I.2.4.2.

I.2.3 A primer on heat engines

Thermal devices convert energy between heat and work and can operate as engines, refrigerators, or heat pumps, enabled by their interaction with thermal reservoirs.

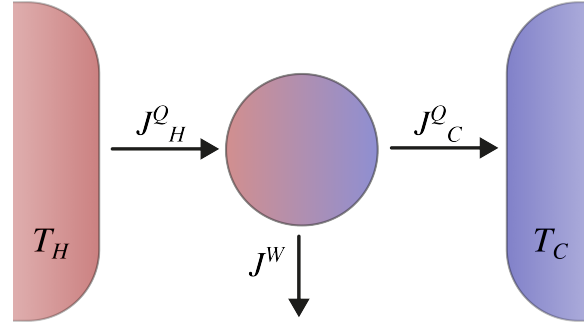


Figure I.2.3: Illustration of a heat engine. The central working medium is coupled to hot and cold reservoirs at temperatures T_H and T_C . Heat J_H^Q absorbed from the hot reservoir is partially converted into power J^W , while the remainder is released to the cold reservoir via the heat current J_C^Q .

The efficiency of a heat engine, as shown schematically in Fig. I.2.3, is defined as

$$\eta = \frac{J^W}{J_H^Q}, \quad (\text{I.2.35})$$

and the second law bounds it by the Carnot efficiency, which is reached only in the reversible limit ($\dot{\Sigma} = 0$), and finite entropy production reduces the steady-state efficiency according to

$$\eta = \eta_C - \frac{T_C}{J_H^Q} \dot{\Sigma} \leq \eta_C = 1 - \frac{T_C}{T_H}. \quad (\text{I.2.36})$$

Fluctuations of thermodynamic currents are significant at small scales. In stochastic thermodynamics, the thermodynamic uncertainty relation (TUR) links the diffusion coefficient of a NESS current J to its mean $\langle J \rangle$ and total entropy production rate $\dot{\Sigma}$ [170, 171]:

$$\frac{D\dot{\Sigma}}{\langle J \rangle^2} \geq 2, \quad (\text{I.2.37})$$

with diffusion coefficient defined as

$$D = \lim_{t \rightarrow \infty} \frac{d}{dt} \text{Var} \int_0^t dt' J(t'). \quad (\text{I.2.38})$$

This implies that achieving high precision (small relative fluctuations) requires greater dissipation, constraining the performance of nonequilibrium engines.

These relations have recently been extended to the quantum regime (see, e.g., Refs. [172, 173]).

The concepts presented here provide the foundation for analysing the performance of a continuously monitored DQD heat engine in Chapter II. To further establish the basis for this discussion, we now detail the dynamics and NESS thermodynamics of the DQD heat engine in the local GKSL approach.

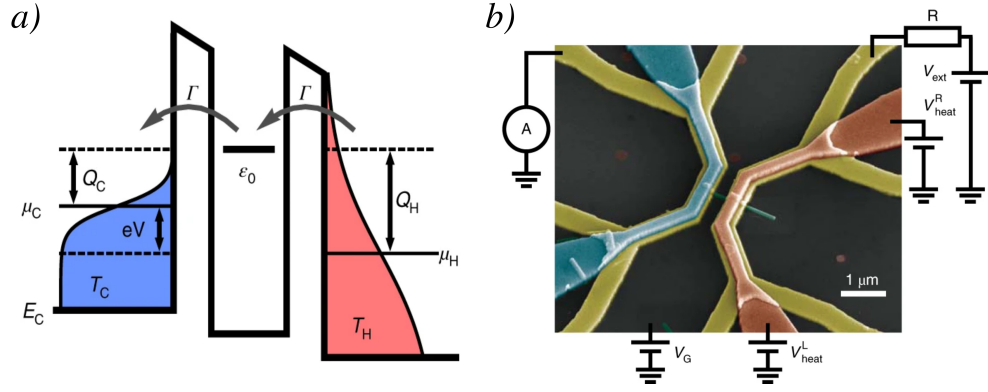


Figure I.2.4: a) Illustration of a quantum-dot-based heat engine with resonance energy ε_0 . The quantum dot is tunnel-coupled at rate Γ to hot and cold electron reservoirs with Fermi distributions characterised by T_H , μ_H and T_C , μ_C , respectively. An electron that traverses the quantum dot at energy ε_0 removes heat Q_H from the hot reservoir, converts part of it into useful work, eV , and deposits the remaining part as heat, Q_C , in the cold reservoir. Adapted from Ref. [177].

I.2.4 Thermodynamics in the local GKSL approach

This section is based on Ref. [29]. The presentation closely follows this reference.

I.2.4.1 The DQD heat engine

The boundary-driven DQD³, schematically shown in Fig. I.2.5, can operate as an autonomous heat engine (see, e.g., Ref. [175]). Experiments on quantum-dot-based autonomous thermal machines, such as the device displayed in Fig. I.2.4, have shown that their efficiencies can approach the Curzon–Ahlborn limit⁴ at maximum power, demonstrating the potential for nanoscale thermoelectric energy conversion near fundamental limits. This makes this DQD system a well-controlled and experimentally relevant platform for exploring the impact of measurement backaction on thermodynamic currents and device performance, as discussed in Chapter II.4 and originally reported in Ref. [29]. The following discussion provides the necessary foundation for this analysis.

We start from a microscopic model: the fermionic DQD in contact with two thermal fermionic reservoirs is described by the Hamiltonian

$$\hat{H} = \hat{H}_{\text{DQD}} + \sum_{\alpha \in \{H,C\}} \hat{H}_{\alpha} + \hat{H}_{\text{DQD},R}, \quad (\text{I.2.39})$$

³A quantum dot is a nanoscale semiconductor structure in which electrons are confined in all three spatial dimensions, resulting in discrete, atom-like energy levels. Quantum dots are often referred to as “artificial atoms” and serve as platforms for studying quantum transport and implementing quantum information processing [174].

⁴The Curzon–Ahlborn efficiency is the theoretical efficiency at maximum power for an endoreversible heat engine operating between two thermal reservoirs [176].

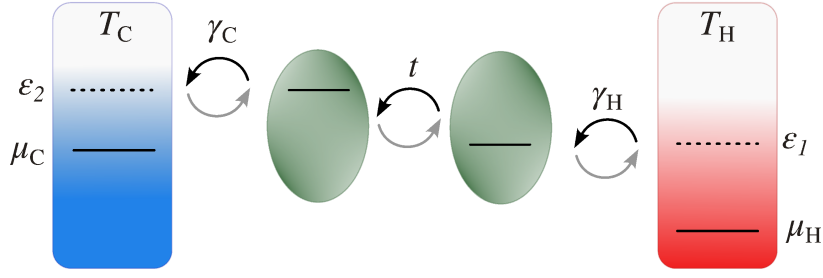


Figure I.2.5: Schematic of the DQD heat engine. Two quantum dots are coupled via a tunnelling amplitude t . Each dot exchanges particles and energy with an independent thermal reservoir, characterised by the rates $\gamma_{H(C)}$, chemical potentials $\mu_{H(C)}$, and temperatures $T_{H(C)}$. Adapted from Ref. [29].

where the DQD Hamiltonian reads

$$\hat{H}_{\text{DQD}} = \epsilon_1 \hat{c}_1^\dagger \hat{c}_1 + \epsilon_2 \hat{c}_2^\dagger \hat{c}_2 + t \left(\hat{c}_1^\dagger \hat{c}_2 + \hat{c}_2^\dagger \hat{c}_1 \right), \quad (\text{I.2.40})$$

with $\hat{c}_{1,2}^{(\dagger)}$ the fermionic annihilation (creation) operators of each quantum dot, with onsite energies $\epsilon_{1,2}$, and interdot tunnelling amplitude t . The reservoirs are modelled as non-interacting fermionic reservoirs

$$\hat{H}_\alpha = \sum_k \epsilon_{\alpha,k} \hat{c}_{\alpha,k}^\dagger \hat{c}_{\alpha,k}, \quad (\text{I.2.41})$$

and their coupling to the DQD is described by

$$\hat{H}_{\text{DQD,R}} = \sum_k \left(g_{H,k} \hat{c}_1^\dagger \hat{c}_{H,k} + g_{C,k} \hat{c}_2^\dagger \hat{c}_{C,k} + \text{h.c.} \right), \quad (\text{I.2.42})$$

with tunnelling amplitudes $g_{\alpha,k}$. Under standard assumptions (weak system-reservoir coupling, Born-Markov approximation, wide-band limit, and reservoirs maintained in local thermal equilibrium, see Section I.1.1) tracing out the reservoirs yields the additive local GKSL master equation

$$\dot{\hat{\rho}}_{\text{DQD}} = \mathcal{L}[\hat{\rho}_{\text{DQD}}] = -i[\hat{H}_{\text{DQD}}, \hat{\rho}_{\text{DQD}}] + \mathcal{L}_H[\hat{\rho}_{\text{DQD}}] + \mathcal{L}_C[\hat{\rho}_{\text{DQD}}], \quad (\text{I.2.43})$$

$$\mathcal{L}_\alpha[\hat{\rho}_{\text{DQD}}] = \gamma_\alpha f_\alpha(\epsilon_\alpha) \mathcal{D}[\hat{c}_\alpha^\dagger] \hat{\rho}_{\text{DQD}} + \gamma_\alpha (1 - f_\alpha(\epsilon_\alpha)) \mathcal{D}[\hat{c}_\alpha] \hat{\rho}_{\text{DQD}}, \quad (\text{I.2.44})$$

where $f_\alpha(\epsilon_\alpha) = [1 + \exp((\epsilon_\alpha - \mu_\alpha)/T_\alpha)]^{-1}$ denotes the reservoir Fermi functions.

I.2.4.2 Thermodynamic inconsistency

Having set up the model, we are now interested in its NESS thermodynamics. However, apparent thermodynamic inconsistencies may arise in local master

equations such as Eq. (I.2.43), as first discussed by Levy and Kosloff in Ref. [142]⁵. Here, we briefly summarise their analysis of an apparent violation of the second law of thermodynamics in this setting. For simplicity, we set $\mu_H = \mu_C = 0$. The heat current flowing from the hot reservoir into the DQD is defined as (see Section I.2.2)

$$J_H^Q = \text{Tr} [\hat{H}_{\text{DQD}} \mathcal{L}_H [\hat{\rho}_{\text{DQD}}]] . \quad (\text{I.2.45})$$

Evaluating the trace, in the NESS one finds that this expression can be written as

$$J_H^Q = (f_H(\epsilon_1) - f_C(\epsilon_2)) \mathcal{F}, \quad (\text{I.2.46})$$

where $\mathcal{F}$ is a positive function of the DQD parameters. It is then evident that when $\beta_C \epsilon_2 < \beta_H \epsilon_1$, the term $(f_H(\epsilon_1) - f_C(\epsilon_2))$ becomes negative (since $\mu_H = \mu_C = 0$), implying that the heat current from the hot reservoir is negative. In the NESS, $J_H^Q = -J_C^Q$, so that the corresponding entropy production rate (see Eq. (I.2.31)),

$$\dot{\Sigma} = (\beta_C - \beta_H) J_H^Q, \quad (\text{I.2.47})$$

is negative, suggesting an unphysical violation of the second law. This apparent inconsistency arises from the definition of entropy production in terms of the heat currents, which is guaranteed to be thermodynamically consistent only when derived from a global master equation satisfying detailed balance (see Section I.2.2). The example thus illustrates that the local GKSL master equation may fail to capture the correct thermodynamics in systems such as the boundary-driven DQD [139, 178, 140]. While alternative approaches exist [143, 179–182, 139], a widely used solution is to employ a framework that is thermodynamically consistent by construction and reproduces the same reduced dynamics in the appropriate limit: the repeated-interaction framework [31], which we introduce in the following section.

I.2.4.3 Framework of repeated interactions

To address the issue of thermodynamic inconsistencies in local GKSL master equations, Barra [31], building on earlier work [146, 147], formulated a description within the repeated-interactions (or collisional) framework (see review [32] and references therein). This approach derives effective GKSL generators from a fully microscopic model in which the system interacts sequentially with independent, identically prepared units (or ancillas) drawn from thermal reservoirs. Each unit

⁵Also global GKSL equations have shortcomings: for example, in some nonequilibrium setups they may predict vanishing currents in the bulk even in a true NESS, failing to capture local transport phenomena correctly [178].

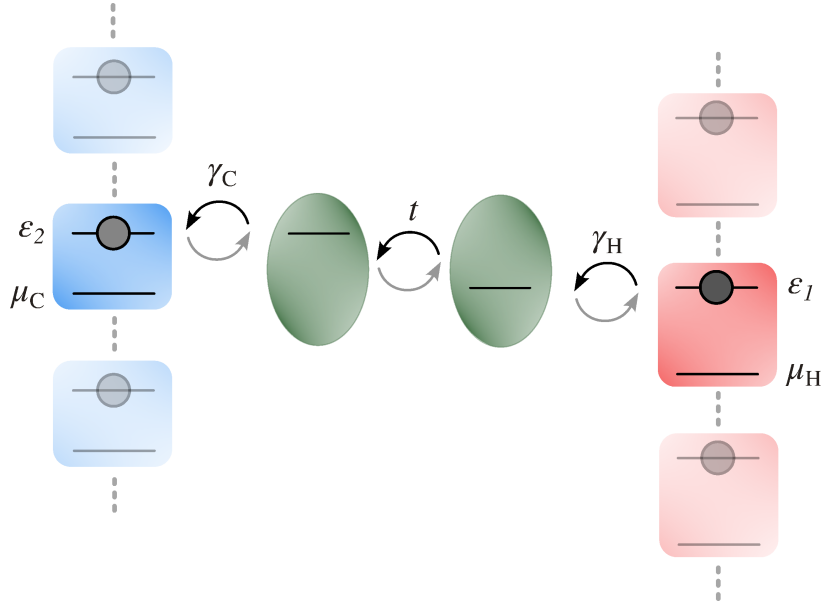


Figure I.2.6: In the repeated-interaction framework, the hot and cold reservoirs are represented by sequences of thermal qubits with energy splitting $\epsilon_1 - \mu_H$ and $\epsilon_2 - \mu_C$, respectively. During each interaction of duration τ , one qubit per reservoir couples to the DQD. Afterward, both qubits are replaced by new ones prepared in the same thermal state. The populations of the qubit levels are determined by the temperature and chemical potential of the respective reservoir. Adapted from Ref. [29].

interacts with the system for a finite duration before being replaced by the next. Note that repeated-interaction schemes are also experimentally accessible, for instance in trapped-ion platforms [183].

We now apply this framework to the example of a DQD heat engine (see Section I.2.4.1). In Fig. I.2.6, we schematically illustrate how the two thermal reservoirs are modelled by periodically refreshed units, each consisting of a single thermal qubit with energy splitting $\epsilon_1 - \mu_H$ and $\epsilon_2 - \mu_C$, respectively, so that the qubit populations are set by the temperatures and chemical potentials of the corresponding reservoirs, $T_\alpha = \beta_\alpha^{-1}$ and μ_α .

The microscopic setup is described by the bare DQD Hamiltonian $\hat{H}_{\text{DQD}}$ from Eq. (I.2.40) and the Hamiltonians of the reservoir qubits,

$$\hat{H}_\alpha = (\epsilon_\alpha - \mu_\alpha) \hat{c}_\alpha^\dagger \hat{c}_\alpha, \quad (\text{I.2.48})$$

with $\alpha \in \{\text{H}, \text{C}\}$. The DQD interacts with the qubit units via

$$\hat{v}_\text{H} = \sqrt{\gamma_\text{H}} (\hat{c}_\text{H}^\dagger \hat{c}_1 + \hat{c}_1^\dagger \hat{c}_\text{H}), \quad \hat{v}_\text{C} = \sqrt{\gamma_\text{C}} (\hat{c}_\text{C}^\dagger \hat{c}_2 + \hat{c}_2^\dagger \hat{c}_\text{C}). \quad (\text{I.2.49})$$

The total Hamiltonian is thus

$$\hat{H} = \hat{H}_{\text{DQD}} + \sum_\alpha \hat{H}_\alpha + \sum_\alpha \hat{V}_\alpha, \quad (\text{I.2.50})$$

where $\hat{V}_\alpha = \hat{v}_\alpha / \sqrt{\tau}$ ⁶. The periodic resetting of the qubit units introduces an explicit time dependence into the total Hamiltonian, implying that external work is required to sustain the dissipative dynamics generated by the local GKSL equation [31]. Each reservoir unit is initialised in a thermal state,

$$\hat{\omega}_{\beta_H} = \begin{pmatrix} f_H(\epsilon_1) & 0 \\ 0 & 1 - f_H(\epsilon_1) \end{pmatrix}, \quad \hat{\omega}_{\beta_C} = \begin{pmatrix} f_C(\epsilon_2) & 0 \\ 0 & 1 - f_C(\epsilon_2) \end{pmatrix}. \quad (\text{I.2.51})$$

At the beginning of each interaction, the total density matrix is

$$\hat{\rho} = \hat{\rho}_{\text{DQD}} \otimes \underbrace{\hat{\omega}_{\beta_H} \otimes \hat{\omega}_{\beta_C}}_{\hat{\omega}_R} \quad (\text{I.2.52})$$

and evolves unitarily for a duration τ under $\hat{U} = \exp(-i\tau\hat{H})$. After the n^{th} interaction, the change in the system state is

$$\hat{\rho}_{\text{DQD}}(n\tau + \tau) - \hat{\rho}_{\text{DQD}}(n\tau) = \text{Tr}_R \left[\hat{U} \hat{\rho}_{\text{DQD}}(n\tau) \otimes \hat{\omega}_R \hat{U}^\dagger - \hat{\rho}_{\text{DQD}}(n\tau) \otimes \hat{\omega}_R \right]. \quad (\text{I.2.53})$$

Expanding $\hat{U}$ to second order in τ and taking the limit of infinitely many infinitesimal interactions ($n \rightarrow \infty, \tau \rightarrow 0$) yields the GKSL master equation

$$\dot{\hat{\rho}}_{\text{DQD}} = -i[\hat{H}_{\text{DQD}}, \hat{\rho}_{\text{DQD}}] + \sum_{\alpha \in \{H, C\}} \mathcal{D}_\alpha[\hat{\rho}_{\text{DQD}}], \quad (\text{I.2.54})$$

where the dissipators arising from the two reservoirs read

$$\mathcal{D}_\alpha[\hat{\rho}_{\text{DQD}}] = \text{Tr}_\alpha [\hat{v}_\alpha (\hat{\rho}_{\text{DQD}} \otimes \hat{\omega}_{\beta_\alpha}) \hat{v}_\alpha] - \frac{1}{2} \text{Tr}_\alpha \left\{ \hat{v}_\alpha^2, (\hat{\rho}_{\text{DQD}} \otimes \hat{\omega}_{\beta_\alpha}) \right\}. \quad (\text{I.2.55})$$

Substituting Eq. (I.2.49) into Eq. (I.2.55) yields the GKSL master equation (I.2.43). The heat and work currents associated with reservoir α can be defined unambiguously as [31]

$$J_\alpha^Q = -D_\alpha(\hat{H}_\alpha), \quad J_\alpha^W = D_\alpha(\hat{H}_{\text{DQD}} + \hat{H}_\alpha), \quad (\text{I.2.56})$$

where

$$D_\alpha(\hat{A}) = \text{Tr} \left[(\hat{v}_\alpha \hat{A} \hat{v}_\alpha - \frac{1}{2} \{ \hat{v}_\alpha^2, \hat{A} \}) (\hat{\rho}_{\text{DQD}} \otimes \hat{\omega}_\alpha) \right] \quad (\text{I.2.57})$$

is the dissipator contribution of reservoir α in the limit $\tau \rightarrow 0$.

⁶In repeated-interaction (collision) models, the interaction Hamiltonian is often scaled as $\hat{V}_\alpha = \hat{v}_\alpha / \sqrt{\tau}$ so that the cumulative effect of many short collisions remains finite in the continuous-time limit $\tau \rightarrow 0$. Mathematically, this ensures that the limit produces a well-defined GKSL generator: $\mathcal{L}[\rho] = \lim_{\tau \rightarrow 0} \frac{\mathcal{E}[\rho] - \rho}{\tau}$, where $\mathcal{E}$ is the reduced map after a single collision. The $\sqrt{\tau}$ scaling guarantees that second-order contributions in $\hat{V}_\alpha$ yield a finite dissipation rate per unit time.

In the NESS, one finds that the average particle current from the hot reservoir into the DQD is given by

$$J_H^N = \gamma_H \left(f_H(\epsilon_1) - \langle \hat{c}_1^\dagger \hat{c}_1 \rangle \right) = \frac{4|t|^2 \Delta f \gamma_H \gamma_C \tilde{\gamma}}{\gamma_H \gamma_C (4\Delta\epsilon^2 + \tilde{\gamma}^2) + 4|t|^2 \tilde{\gamma}^2}. \quad (\text{I.2.58})$$

Here, $\Delta\epsilon = \epsilon_2 - \epsilon_1$ denotes the detuning between the two quantum dots in the DQD, $\Delta f = f_H(\epsilon_1) - f_C(\epsilon_2)$ is the difference in the Fermi distributions of the reservoirs evaluated at the quantum dot energies, and $\tilde{\gamma} = (\gamma_H + \gamma_C)$. Further, the work currents into the DQD in the NESS are given by

$$\begin{aligned} \dot{W}_H &= \left(\frac{\gamma_H \Delta\epsilon}{\tilde{\gamma}} + \mu_H \right) J_H^N, & \dot{W}_C &= \left(\frac{\gamma_C \Delta\epsilon}{\tilde{\gamma}} - \mu_C \right) J_H^N, \\ \dot{W}_{\text{tot}} &= -[(\epsilon_1 - \mu_H) - (\epsilon_2 - \mu_C)] J_H^N. \end{aligned} \quad (\text{I.2.59})$$

Similarly, for the heat currents into the DQD in the NESS, we find

$$\begin{aligned} \dot{Q}_H &= (\epsilon_1 - \mu_H) J_H^N, & \dot{Q}_C &= -(\epsilon_2 - \mu_C) J_H^N, \\ \dot{Q}_{\text{tot}} &= [(\epsilon_1 - \mu_H) - (\epsilon_2 - \mu_C)] J_H^N. \end{aligned} \quad (\text{I.2.60})$$

Note here that the energy currents depend linearly on the particle current. This is the so-called tight-coupling limit. It is a consequence of the weak-coupling approximation made in deriving the GKSL equation and does not hold in general.

By the first law, $\dot{W}_{\text{tot}} + \dot{Q}_{\text{tot}} = \langle \dot{\hat{H}}_{\text{DQD}} \rangle$. Here, $\langle \dot{\hat{H}}_{\text{DQD}} \rangle = \sum_\alpha D_\alpha (\hat{H}_{\text{DQD}})$, and in the NESS, this quantity vanishes. Clearly, $\dot{W}_{\text{tot}} = -\dot{Q}_{\text{tot}}$.

Moreover, if the boundary-driven DQD is operated as a heat engine, then its efficiency η is defined as

$$\eta = \frac{-\dot{W}_{\text{tot}}}{\dot{Q}_H} = 1 - \frac{\epsilon_2 - \mu_C}{\epsilon_1 - \mu_H}. \quad (\text{I.2.61})$$

Note now that the DQD operates as a heat engine, if the work current, i.e. the power, from the DQD $-\dot{W}_{\text{tot}} > 0$. This is so, if $(f_H(\epsilon_1) - f_C(\epsilon_2)) > 0$, so that $J_H^N > 0$, and $(\epsilon_1 - \mu_H) - (\epsilon_2 - \mu_C) > 0$. By the first condition,

$$\frac{(\epsilon_2 - \mu_C) / T_C}{(\epsilon_1 - \mu_H) / T_H} > 1, \quad (\text{I.2.62})$$

and we find that η is bounded from above by the Carnot efficiency η_C , since

$$\eta < 1 - \frac{T_C}{T_H} = \eta_C. \quad (\text{I.2.63})$$

Consistency with the second law can be shown by recognising that in the NESS,

the entropy production rate can be expressed in standard thermodynamic form

$$\dot{\Sigma} = - \sum_{\alpha \in \{H,C\}} \frac{\dot{Q}_\alpha}{T_\alpha} = \left(\frac{\epsilon_2 - \mu_C}{T_C} - \frac{\epsilon_1 - \mu_H}{T_H} \right) J_H^N \geq 0. \quad (\text{I.2.64})$$

I.2.5 Entropic cost of nonstationarity

This section is based on Ref. [78]. The presentation closely follows this reference.

This section distinguishes the entropy production and entropy fluxes that are intrinsic to NESSs from additional contributions that arise only in transient or driven settings. We assume that the system evolves under GKSL dynamics (see Sections I.1.1 and I.2.2).

As is standard in nonequilibrium thermodynamics [184], we start by expressing the entropy balance in terms of the system's von Neumann entropy change $\dot{S}_{\text{vN}}$, the entropy flux to the environment $\dot{\Pi}$, and the total entropy production rate $\dot{\Sigma}$. Although we have encountered it previously, we restate it here:

$$\frac{dS_{\text{vN}}}{dt} = \dot{\Sigma} - \dot{\Pi}. \quad (\text{I.2.65})$$

We decompose the total entropy production into adiabatic and nonadiabatic contributions [185, 186]

$$\dot{\Sigma} = \dot{\Sigma}_{\text{ad}} + \dot{\Sigma}_{\text{na}}, \quad \dot{\Sigma}_{\text{na}} = - \text{Tr} \left[\dot{\hat{\rho}}_t (\log \hat{\rho}_t - \log \hat{\pi}_{\vec{\theta}_t}) \right], \quad (\text{I.2.66})$$

where $\hat{\pi}_{\vec{\theta}_t}$ denotes the instantaneous steady state of the Liouvillian $\mathcal{L}_{\vec{\theta}_t}$, parameterised by a set of time-dependent parameters $\vec{\theta}_t$. The term $\dot{\Sigma}_{\text{na}}$ represents the instantaneous version of Spohn's entropy production [145] relative to $\hat{\pi}_{\vec{\theta}_t}$ and is non-negative, $\dot{\Sigma}_{\text{na}} \geq 0$, vanishing only when $\hat{\rho}_t = \hat{\pi}_{\vec{\theta}_t}$. The adiabatic entropy production rate $\dot{\Sigma}_{\text{ad}} = \dot{\Sigma} - \dot{\Sigma}_{\text{na}}$ quantifies the continuous dissipation required to maintain the NESS. In the steady state, $\dot{\Sigma}_{\text{na}} = 0$, so that $\dot{\Sigma}_{\text{NESS}} = \dot{\Sigma}_{\text{ad}}$. In the special case where the steady state is an equilibrium Gibbs state, the adiabatic contribution vanishes, and no entropy is produced. The splitting of the entropy production into adiabatic and nonadiabatic contributions has a clear physical interpretation: $\dot{\Sigma}_{\text{na}}$ quantifies the dissipation associated with driving or relaxation toward the instantaneous steady state, while $\dot{\Sigma}_{\text{ad}}$ captures the steady-state dissipation required to sustain the NESS. Historically, the first decomposition of entropy flux was formulated at the level of heat currents [187, 79, 188, 189]. Here, we consider the

complementary decomposition of the entropy flux into two contributions

$$\dot{\Pi} = \dot{\Pi}_{\text{hk}} + \dot{\Pi}_{\text{ex}}, \quad (\text{I.2.67})$$

where $\dot{\Pi}_{\text{hk}}$ is the housekeeping entropy flux and $\dot{\Pi}_{\text{ex}}$ is the excess entropy flux [5]. By construction, the housekeeping flux coincides with the adiabatic entropy production,

$$\dot{\Pi}_{\text{hk}} = \dot{\Sigma}_{\text{ad}}, \quad (\text{I.2.68})$$

since it is the only nonvanishing contribution in the steady state and quantifies the entropy supplied by the environment to maintain the NESS. The excess flux, which accounts for the entropy exchange due to changes in the control parameters, is then defined as

$$\dot{\Pi}_{\text{ex}} = \dot{\Pi} - \dot{\Pi}_{\text{hk}}, \quad (\text{I.2.69})$$

and can be rewritten in the compact form

$$\dot{\Pi}_{\text{ex}} = \text{Tr} \left[\dot{\rho}_t \log \hat{\pi}_{\vec{\theta}_t} \right]. \quad (\text{I.2.70})$$

Combining Eqs. (I.2.65), (I.2.66), and (I.2.70), one obtains the generalised Clausius inequality (see Refs. [79, 188] for a discussion in classical settings)

$$\frac{dS}{dt} + \dot{\Pi}_{\text{ex}} = \dot{\Sigma}_{\text{na}} \geq 0, \quad (\text{I.2.71})$$

where the first equality constitutes a quantum generalisation of the Hatano-Sasa relation [79]. The relation (I.2.71) provides a refined formulation of the second law for driven open quantum systems. It expresses the instantaneous nonadiabatic entropy production as the sum of the system's entropy change and the excess entropy flux, thereby extending the Hatano-Sasa theorem to the quantum regime [78].

In both classical and quantum systems, this decomposition admits an information-geometric interpretation in the slow-driving regime: the nonadiabatic entropy production can be viewed as a path action defined with respect to a thermodynamic metric related to the (quantum) Fisher information [78, 75]. This connection is discussed in detail in Section IV.7.5.

I.2.6 Stochastic entropy production and fluctuation theorems

This section is based on Refs. [190, 52]. The presentation closely follows these references.

Fluctuation theorems are central to modern nonequilibrium statistical physics, as they provide exact relations for the probability distributions of work, heat, and entropy production, even far from equilibrium [5]. They deal with symmetries of the entropy production under time-reversal, and quantify the likelihood of transient violations of the second law in small systems or over short timescales. In doing so, they link microscopic reversibility to macroscopic irreversibility. The theoretical development of fluctuation theorems in the 1990s was motivated by the study of entropy production statistics in NESSs [191–193]. Within the framework of stochastic thermodynamics, this line of research led to the Jarzynski equality [194] and the Crooks fluctuation theorem [195], which characterise the statistics of work performed during finite-time, time-dependent driving protocols and relate nonequilibrium work distributions to equilibrium free energy differences. Fluctuation theorems have since been generalised to the quantum regime [36, 196] and were first experimentally verified using nuclear magnetic resonance (NMR) systems [197]. Subsequent demonstrations have been carried out on other platforms, including quantum computers [198] and, more recently, photonic systems [199]. While the second law of thermodynamics asserts that entropy tends to increase in irreversible processes, this holds strictly only on average. At mesoscopic scales, fluctuations can be significant, and individual trajectories may temporarily decrease entropy, though such events are exponentially suppressed. In its most general form, the stochastic entropy production generated along a single trajectory of duration τ is given by

$$\Sigma_{\text{tot}}(r_{[0,\tau]}) = \log \left(\frac{P_F(r)}{P_B(\tilde{r})} \right). \quad (\text{I.2.72})$$

Here, $P_F(r)$ and $P_B(\tilde{r})$ denote the probabilities of observing, respectively, the forward trajectory $r_{[0,\tau]}$ and its time-reversed counterpart $\tilde{r}_{[0,\tau]}$. The ratio $P_F(r)/P_B(\tilde{r})$ thus quantifies how much more likely a given sequence of events is in the natural forward-time evolution of the system compared to the corresponding reversed process. In other words, it provides a measure of the irreversibility associated with that particular trajectory: the larger the ratio, the more strongly the trajectory contributes to the overall increase of entropy⁷.

In this section, we establish a trajectory-level formulation of entropy production suitable for quantum jump dynamics (see Section I.1.3). This framework provides the basis for accessing fluctuation relations, entropy production statistics, and refined decompositions of entropy production that will be used in Chapter III.6.

⁷In terms of surprisal, the negative logarithm of the probability of an event, Σ_{tot} measures the difference between the surprisal of the trajectory in the forward process and that in the backward process. A larger value of Σ_{tot} indicates that the trajectory is much more expected in the forward-time evolution than in the reversed dynamics.

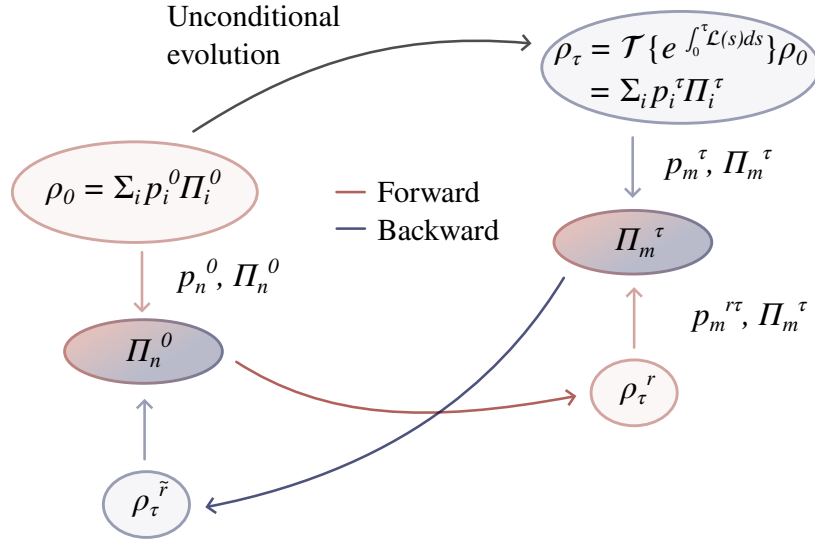


Figure I.2.7: Schematic of the TPM scheme. In the first measurement, the initial state $\hat{\rho}_0 = \hat{\rho}(0)$ is projected into its n^{th} eigenstate $\hat{\Pi}_n^0$ with probability p_n^0 . Subsequently, this state evolves according to the conditional evolution into $\hat{\rho}_\tau^r = \hat{\rho}_r(\tau)$. Finally, $\hat{\rho}_r(\tau)$ is projected into the m^{th} eigenstate $\hat{\Pi}_m^\tau$ of the unconditionally evolved initial state $\hat{\rho}_\tau = \hat{\rho}(\tau)$ with probability $p_m^{r\tau}$. The probability for the reverse trajectory to start in the eigenstate $\hat{\Pi}_m^\tau$ is given by p_m^τ . Adapted from Ref. [52].

I.2.6.1 Two-point measurement scheme

To assess the entropy produced along individual quantum jump trajectories, we adopt the two-point measurement (TPM) scheme, illustrated schematically in Fig. I.2.7 (we refer to Ref. [190] for a detailed account). In this framework, the stochastic time evolution of the system over the open interval $(0, \tau)$, which produces a measurement record denoted by $r_{(0,\tau)}$, is framed by two projective measurements: one performed at the initial time $t = 0$ and another at the final time $t = \tau$. The full trajectory, including these two projective measurements, thus spans the closed interval $[0, \tau]$ and is referred to as the forward trajectory. We denote its full measurement record by $r_{[0,\tau]}$.

The initial state of the forward trajectory can be written in its spectral decomposition as $\rho(0) = \sum_i p_i^0 \Pi_i^0$. Under unconditional evolution, the system state, having evolved for a duration τ , is given by $\hat{\rho}(\tau) = \mathcal{T} \left\{ e^{\int_0^\tau \mathcal{L}(s) ds} \right\} \hat{\rho}(0) = \sum_i p_i^\tau \hat{\Pi}_i^\tau$. Accordingly, $\rho(0)$ and $\rho(\tau)$ represent the ensemble-averaged initial and final states, respectively. In the TPM scheme, the first projective measurement prepares the initial state of each trajectory. Specifically, this measurement projects the ensemble-averaged initial state into its eigenstate $\hat{\Pi}_n^0$, with probability p_n^0 . Thus, the initial state of the trajectory is $\hat{\rho}_r(0) = \Pi_n^0$. This state then evolves conditionally up to time τ . During this time, it follows the stochastic dynamics generating the corresponding measurement record r . The resulting final state of the trajectory is denoted by $\hat{\rho}_r(\tau)$. At $t = \tau$, a second projective measurement is performed in

the eigenbasis of the ensemble-averaged final state, yielding the outcome $\hat{\Pi}_m^\tau$ with conditional probability $p_m^{\tau} = \text{Tr} [\hat{\Pi}_m^\tau \hat{\rho}_r(\tau)]$. For the corresponding time-reversed trajectory, the probability to start in $\hat{\Pi}_m^\tau$ is given by its weight in the decomposition of $\hat{\rho}(\tau)$, i.e. p_m^τ .

In the following, we assume that the jump operators come in pairs, corresponding to particle transfer into ($\sigma = +$) or out of ($\sigma = -$) the system due to interactions with thermal reservoirs. Each jump in channel k is associated with energy and particle exchanges ΔE_k^σ and ΔN_k^σ , respectively. These paired jump operators are assumed to satisfy a local detailed balance condition [200, 186],

$$\hat{L}_k^+ = \hat{L}_k^{-\dagger} e^{\Delta s_k^+/2}, \quad (\text{I.2.73})$$

with associated entropy flux

$$\Delta s_k^+ = - \left(\frac{\Delta E_k^+}{T_\alpha} - \mu_\alpha \frac{\Delta N_k^+}{T_\alpha} \right), \quad (\text{I.2.74})$$

where T_α and μ_α are the temperature and chemical potential of the reservoir α that facilitates the jump. The measurement record between the two projective measurements for the forward trajectory can then be written as

$$r_{(0,\tau)} = \{(t_{J_1}, k_1, \sigma_1), \dots, (t_{J_{N_j}}, k_{N_j}, \sigma_{N_j})\}, \quad (\text{I.2.75})$$

where N_j denotes the number of recorded jumps.

For the corresponding time-reversed experiment, we consider the case in which each projective measurement yields the same outcome as in the forward process. Consequently, the measurement record of the backward trajectory between the two measurements, $\tilde{r}_{(0,\tau)}$, is simply the time-reversed version of the forward measurement record $r_{(0,\tau)}$. Explicitly,

$$\tilde{r}_{(0,\tau)} = \{(\tau - t_{J_{N_j}}, k_{N_j}, -\sigma_{N_j}), (\tau - t_{J_{N_j-1}}, k_{N_j-1}, -\sigma_{N_j-1}), \dots, (\tau - t_{J_1}, k_1, -\sigma_1)\}, \quad (\text{I.2.76})$$

as discussed in Ref. [190].

I.2.6.2 Trajectory-level entropy production

The projection probabilities p_n^0 and p_m^τ are of particular interest to us here, as they enter directly into the expression for the total entropy production along a trajectory $r_{[0,\tau]}$ (see Eq. (I.2.72)), which within the TPM framework takes the form [190]

$$\Sigma_{\text{tot}}(r_{[0,\tau]}) = \log \left(\frac{p_n^0}{p_m^\tau} \right) + \Pi_{r_{(0,\tau)}}. \quad (\text{I.2.77})$$

Here, we have separated $\Sigma_{\text{tot}}(r_{[0,\tau]})$ into two contributions: a stochastic entropy associated with the system along a trajectory, and the entropy flux exchanged with the environment. The first term captures changes in the Shannon self-information, or surprisal, of the system between the two projective measurements, and takes the same form as in the classical counterpart [195, 201]. The entropy flux is given by

$$\Pi_{r_{(0,\tau)}} = - \sum_k \sum_{\sigma \in \{+,-\}} \int_0^\tau \frac{\Delta E_k^\sigma - \mu_\alpha \Delta N_k^\sigma}{T_\alpha} dN_k(t). \quad (\text{I.2.78})$$

I.2.6.3 Integral fluctuation theorem

Taking the ensemble average over the forward trajectories r of the exponentiated negative of the total stochastic entropy production, we obtain an integral fluctuation relation, a central result of stochastic thermodynamics [192, 6, 202, 36, 203]

$$\langle e^{-\Sigma_{\text{tot}}(r_{[0,\tau]})} \rangle_r = 1. \quad (\text{I.2.79})$$

Applying Jensen's inequality [204], $\langle e^x \rangle \geq e^{\langle x \rangle}$, directly leads to the second law

$$\langle \Sigma_{\text{tot}}(r_{[0,\tau]}) \rangle_r \geq 0. \quad (\text{I.2.80})$$

I.2.6.4 Uncertainty and martingale entropy production

It is well established that the total entropy production in nonequilibrium processes can be decomposed in several physically meaningful ways [185, 205]. In earlier sections, we have discussed decompositions into changes of the system entropy and entropy fluxes into the environment (I.2.77), as well as into adiabatic and nonadiabatic contributions (I.2.66)⁸.

Here, we instead adopt a different decomposition of the trajectory-dependent total entropy production Σ_{tot} , separating it into a classical-like exponential martingale contribution and a genuinely quantum term. Remarkably, both obey integral fluctuation theorems [205]. This perspective can be motivated by an operational question: given access to trajectory-level information about entropy production, how useful is this information? In particular, can it be exploited to implement feedback or stopping strategies that reduce entropy production and extract useful resources, in the spirit of Maxwell-demon-like protocols [206, 207]? These questions have been addressed both theoretically and experimentally in Ref. [208].

The mathematical framework underlying such questions is provided by martingale theory, which plays a central role in probability theory [209] and also quan-

⁸Under suitable conditions, related to commutation relations between the jump operators and the NESS, these quantities can also be evaluated along individual quantum trajectories [186].

tative finance [210]. A martingale is a stochastic process whose conditional expectation at any later time equals its present value, reflecting the absence of systematic trends once all available information has been taken into account. More precisely, a stochastic process $M(t)$ is a martingale if it is bounded, $\langle M(t) \rangle < \infty$, and satisfies

$$\langle M(t) | M_{\{0,\tau\}} \rangle = M(\tau), \quad (\text{I.2.81})$$

for all $t \geq \tau$, where the conditional expectation is taken over all realisations consistent with the history $M_{\{0,\tau\}}$ up to time τ [209].

In Ref. [205], Manzano *et al.* extended martingale theory to the quantum domain within the framework of quantum jump trajectories, and demonstrated that the total entropy production accumulated along a measurement record $r_{[0,\tau]}$ can be decomposed as [190]

$$\Sigma_{\text{tot}}(r_{[0,\tau]}) = \Sigma_{\text{mart}}(r_{[0,\tau]}) + \Sigma_{\text{unc}}(r_{[0,\tau]}). \quad (\text{I.2.82})$$

The martingale entropy production is defined as

$$\Sigma_{\text{mart}}(r_{[0,\tau]}) = \log p_n^0 + S_\rho(\tau) + \Pi_{r_{(0,\tau)}}, \quad (\text{I.2.83})$$

and depends on the probability p_n^0 of the initial projective measurement outcome and on the trajectory-resolved entropy flux $\Pi_{r_{(0,\tau)}}$. Its defining property is that the negative exponential $e^{-\Sigma_{\text{mart}}}$ constitutes a martingale along quantum trajectories, i.e.,

$$\langle e^{-\Sigma_{\text{mart}}(t)} | r_{[0,\tau]} \rangle = e^{-\Sigma_{\text{mart}}(\tau)}, \quad (\text{I.2.84})$$

for all $t \geq \tau$, where $\langle \cdot | r_{[0,\tau]} \rangle$ denotes conditional expectation given the past measurement record [209]. Notably, unlike in classical stochastic thermodynamics, the quantity $e^{-\Sigma_{\text{tot}}}$ does not in general form a martingale in quantum systems. This is due to the existence of the additional term, the uncertainty entropy production, defined as

$$\Sigma_{\text{unc}}(r_{[0,\tau]}) = -\log p_m^\tau - S_\rho(\tau), \quad (\text{I.2.85})$$

where p_m^τ denotes the probability of the outcome of the final projective measurement and

$$S_\rho(\tau) = -\log[\text{Tr}[\hat{\rho}_r(\tau)\hat{\rho}(\tau)]] . \quad (\text{I.2.86})$$

Here, $\hat{\rho}_r(\tau)$ is the conditional state of the system given the measurement record prior to the final projective measurement, while $\hat{\rho}(\tau)$ is the corresponding unconditional state. Equation (I.2.85) therefore quantifies the extra information gained by knowing the specific measurement outcome, and hence the conditional state, beyond what is contained in the unconditional density operator.

Both contributions individually satisfy integral fluctuation theorems,

$$\left\langle e^{-\Sigma_{\text{unc}}} \right\rangle_r = 1, \quad \left\langle e^{-\Sigma_{\text{mart}}} \right\rangle_r = 1, \quad (\text{I.2.87})$$

and are therefore non-negative on average, $\langle \Sigma_{\text{unc}} \rangle_r \geq 0$ and $\langle \Sigma_{\text{mart}} \rangle_r \geq 0$ [190].

This decomposition separates trajectory-level entropy production into a classical-like component and an intrinsically quantum contribution. The uncertainty entropy production Σ_{unc} originates from the final projective measurement and vanishes in classical systems. It is therefore non-extensive in time. By contrast, the martingale contribution Σ_{mart} can grow extensively, for instance in NESSs, where it is dominated by the accumulated entropy flux into the environment.

We revisit this decomposition in Chapter III.6.

Chapter I.3

Methods for Gaussian models

When dealing with systems of bosons or fermions, many physically relevant quantities admit compact closed-form expressions when the states and processes under consideration are Gaussian [211–213]. If so, all correlations are fully characterised by first and second moments, and thus Gaussian states constitute a natural extension of Gaussian distributions in classical probability. Importantly, all relevant objects reduce to matrices of dimension at most $2N$, where N is the number of modes in the system. This leads to a drastic simplification: even for multimode or otherwise complex configurations, Gaussian descriptions remain computationally efficient and analytically tractable. Beyond their mathematical convenience, Gaussian states are important in quantum physics because they are straightforward to prepare, control, and measure in a wide variety of experimental architectures, see Refs. [213–216] and references therein. These properties make the Gaussian framework an indispensable tool throughout this thesis. In particular, it forms the basis of the analysis of fermionic dynamics in Chapter III.6 and of bosonic dynamics in Chapter V.10.

To provide a unified treatment of bosonic and fermionic systems, we consider a set of N annihilation operators $\hat{b}_i$, which satisfy either the bosonic commutation relations $[\hat{b}_i, \hat{b}_j^\dagger] = \delta_{ij}$ or the fermionic anticommutation relations $\{\hat{b}_i, \hat{b}_j^\dagger\} = \delta_{ij}$. Throughout, when the notation $\pm$ appears, the upper sign refers to bosons and the lower sign to fermions. We define quadratures (for bosons) or Majorana operators (for fermions) as

$$\hat{q}_i = \frac{1}{\sqrt{2}}(\hat{b}_i + \hat{b}_i^\dagger), \quad \hat{p}_i = \frac{i}{\sqrt{2}}(\hat{b}_i^\dagger - \hat{b}_i), \quad (\text{I.3.1})$$

and construct the column vector $\vec{R} = (\hat{q}_1, \dots, \hat{q}_N, \hat{p}_1, \dots, \hat{p}_N)^T$.

I.3.1 Gaussian states

This section is based on Refs. [212, 217, 218].

General Gaussian states can be written in the form

$$\hat{\rho} = \frac{e^{-\frac{1}{2}\vec{R}^\dagger \cdot M \cdot \vec{R}}}{Z}, \quad Z = \begin{cases} \frac{1}{\sqrt{\det(1 - e^{-i\Omega M})}} & (\text{bosons}) \\ \sqrt{\det(1 - e^{-M})} & (\text{fermions}). \end{cases} \quad (\text{I.3.2})$$

with $M^\dagger = M$. For bosons, $M^\dagger = M = M^T$ real and symmetric, while for fermions $M^T = -M$ purely imaginary and skew-symmetric. The matrix M and the covariance matrix Θ are related via

$$\Theta = \begin{cases} \frac{1}{2} \coth\left(\frac{i\Omega M}{2}\right) i\Omega & (\text{bosons}) \\ \frac{1}{2} \tanh\left(\frac{M}{2}\right) & (\text{fermions}). \end{cases} \quad (\text{I.3.3})$$

The matrices M and Θ can be diagonalised in a suitable canonical (symplectic) basis, which allows for a simple characterization of Gaussian states in terms of their symplectic eigenvalues.

For bosonic systems, the covariance matrix Θ satisfies the symplectic condition $\Theta + i\Omega/2 \geq 0$, known as the Robertson-Schrödinger uncertainty principle [219]. It guarantees that the quantum state described by Θ is physically valid. Intuitively, it enforces the fundamental quantum constraint that the fluctuations of conjugate variables cannot be arbitrarily small, and, for a single mode, it reduces to the familiar Heisenberg uncertainty relation $\text{Var}(\hat{q}) \text{Var}(\hat{p}) \geq \frac{1}{4}$. More generally, it ensures that the symplectic eigenvalues of Θ , which can be interpreted as effective mode occupations or “quantum uncertainties” in each canonical degree of freedom, are bounded below by $1/2$. In this way, the inequality encodes the minimal quantum noise present in any physical bosonic Gaussian state and guarantees the positivity of the corresponding density matrix. The symplectic eigenvalues $\{\nu_j\}$ of Θ are defined as the positive eigenvalues of $i\Omega\Theta$, i.e.,

$$\text{Spec}(i\Omega\Theta) = \{\pm\nu_1, \dots, \pm\nu_N\}. \quad (\text{I.3.4})$$

Equivalently, one can define symplectic eigenvalues $\{\epsilon_j\}$ of M via the diagonalization

$$M = S^T \bigoplus_j \epsilon_j \mathbb{I}_2 S, \quad (\text{I.3.5})$$

where S is a symplectic transformation ($S\Omega S^T = \Omega$). The two sets of symplectic

eigenvalues are related by $v_j = \frac{1}{2} \coth \frac{\epsilon_j}{2}$.

For fermionic systems, the covariance matrix satisfies $\Theta^T = -\Theta$. Further, an additional constraint due to the anticommuting nature of fermionic operators arises. Specifically, Θ must obey $0 \leq \Theta \leq 1$, which ensures that the occupation numbers of the fermionic modes remain physically valid, i.e., between 0 and 1, enforcing the Pauli exclusion principle, which prevents any single-particle state from being doubly occupied. Equivalently, the eigenvalues of Θ correspond to the mode occupations, and the inequality guarantees that the density matrix $\hat{\rho}$ remains positive semidefinite and properly normalised. The symplectic (or more accurately, skew-symmetric) eigenvalues $\{v_j\}$ of Θ are defined via the spectrum of $i\Theta$

$$\text{Spec}(i\Theta) = \{\pm v_1, \dots, \pm v_N\}, \quad 0 \leq v_j \leq 1. \quad (\text{I.3.6})$$

Similarly, M can be block-diagonalised by an orthogonal transformation O into

$$M = O^T \bigoplus_j \epsilon_j \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} O, \quad (\text{I.3.7})$$

and the eigenvalues are related to the covariance matrix via $v_j = \frac{1}{2} \tanh(\epsilon_j/2)$. In both the bosonic and fermionic cases, the symplectic eigenvalues completely characterise the Gaussian state. We will use this property extensively in Chapters III.6 and V.10.

I.3.2 GKSL dynamics

This section is based on Ref. [217]. The presentation closely follows this reference.

We consider general Gaussian dynamics governed by a GKSL master equation with jump operators that are linear in the creation and annihilation operators,

$$\mathcal{L}[\hat{\rho}] = -i[\hat{H}, \hat{\rho}] + \sum_i \gamma_i^- \mathcal{D}[\hat{b}_i] \hat{\rho} + \gamma_i^+ \mathcal{D}[\hat{b}_i^\dagger] \hat{\rho}, \quad (\text{I.3.8})$$

where $\gamma_i^\pm$ denote the corresponding jump rates, and a system Hamiltonian that is quadratic

$$\hat{H} = \sum_{ij} A_{ij} \hat{b}_i^\dagger \hat{b}_j + \frac{1}{2} (B_{ij} \hat{b}_i^\dagger \hat{b}_j^\dagger + B_{ij}^* \hat{b}_j \hat{b}_i), \quad (\text{I.3.9})$$

with $A = A^\dagger$, $B = \pm B^T$. The Hamiltonian in Eq. (I.3.9) can be written as

$$\hat{H} = \frac{1}{2} \vec{R}^T \cdot H_R \cdot \vec{R}. \quad (\text{I.3.10})$$

Here,

$$H_R = \frac{1}{2}(\mathbb{I}_2 \otimes (A \pm A^T) + \sigma_z \otimes (B \pm B^*) + \sigma_y \otimes (A^T \mp A) - i\sigma_x \otimes (B \mp B^*)). \quad (\text{I.3.11})$$

The covariance matrix is defined as

$$\Theta_{ij} = \frac{i}{2}\Omega_{ij} \pm \langle \hat{R}_j \hat{R}_i \rangle - \langle \hat{R}_i \rangle \langle \hat{R}_j \rangle, \quad (\text{I.3.12})$$

with symplectic form $\Omega = (i\sigma_y) \otimes \mathbb{I}_N$ for bosons and $\Omega = -i\mathbb{I}_{2N}$ for fermions. To express the dynamics in terms of the covariance matrix, we use Wick's theorem [220], which states that for a Gaussian state $\hat{\rho}$, any fourth-order correlator can be written in terms of products of second-order correlators. Let $\hat{O}_i$ denote operators that are arbitrary linear combinations of bosonic or fermionic creation and annihilation operators. Wick's theorem then yields

$$\begin{aligned} \langle \hat{O}_1 \hat{O}_2 \hat{O}_3 \hat{O}_4 \rangle_{\hat{\rho}} &= \langle \hat{O}_1 \hat{O}_2 \rangle_{\hat{\rho}} \langle \hat{O}_3 \hat{O}_4 \rangle_{\hat{\rho}} \pm \langle \hat{O}_1 \hat{O}_3 \rangle_{\hat{\rho}} \langle \hat{O}_2 \hat{O}_4 \rangle_{\hat{\rho}} \\ &+ \langle \hat{O}_1 \hat{O}_4 \rangle_{\hat{\rho}} \langle \hat{O}_2 \hat{O}_3 \rangle_{\hat{\rho}} - 2 \langle \hat{O}_1 \rangle_{\hat{\rho}} \langle \hat{O}_2 \rangle_{\hat{\rho}} \langle \hat{O}_3 \rangle_{\hat{\rho}} \langle \hat{O}_4 \rangle_{\hat{\rho}}, \end{aligned} \quad (\text{I.3.13})$$

where the upper (lower) sign applies to bosons (fermions). analysing the evolution of the covariance-matrix elements under GKSL dynamics in the Heisenberg picture (see Section I.1.2), one finds that the covariance matrix satisfies a Lyapunov differential equation,

$$\frac{d\Theta}{dt} = -(W\Theta + \Theta W^\dagger) + F, \quad (\text{I.3.14})$$

where W , known as the drift matrix, encodes the drift generated by the Hamiltonian and dissipation, and F , commonly referred to as the diffusion matrix, collects the noise contributions:

$$W = -\Omega H_R + \frac{1}{2}\mathbb{I}_2 \otimes \Gamma, \quad F = \begin{cases} \frac{1}{2}\mathbb{I}_2 \otimes (\gamma_+ + \gamma_-) & (\text{bosons}), \\ -\frac{1}{2}\sigma_y \otimes (\gamma_- + \gamma_+) & (\text{fermions}), \end{cases} \quad (\text{I.3.15})$$

and $\Gamma = \gamma_- - \gamma_+$. Consequently, the steady-state covariance matrix solves the Lyapunov equation

$$W\Theta + \Theta W^\dagger = F. \quad (\text{I.3.16})$$

I.3.3 Liouville spectrum

This section is based on Refs. [221, 222].

The spectrum of the Liouvillian can be constructed directly from that of the drift

matrix W . Assuming that $\mathcal{L}$ is diagonalisable, its spectrum consists of all finite linear combinations of the eigenvalues of W with coefficients determined by the allowed occupation numbers [221, 222]. For bosonic systems, these coefficients take non-negative integer values,

$$\text{Spec}(\mathcal{L}) = \left\{ -\sum_i n_i \lambda_i \mid \lambda_i \in \text{Spec}(W), n_i \in \mathbb{N}_0 \right\}, \quad (\text{I.3.17})$$

while for fermionic systems they are restricted to $n_i \in \{0, 1\}$,

$$\text{Spec}(\mathcal{L}) = \left\{ -\sum_i n_i \lambda_i \mid \lambda_i \in \text{Spec}(W), n_i \in \{0, 1\} \right\}. \quad (\text{I.3.18})$$

Therefore, the Liouvillian gap is given by

$$\Delta = \min_{w \in \text{Spec}(W)} |\text{Re}(w)|, \quad (\text{I.3.19})$$

for both bosonic and fermionic quadratic systems.

PART II:

Stability of monitored quantum devices

Chapter II.4

DQD heat engine under continuous measurement

The results presented in this chapter are based on Ref. [29]. I performed the analytical and numerical calculations, carried out the data analysis, and wrote the manuscript. The co-authors contributed to the conceptualisation of the work, discussed the results, and provided feedback and revisions on the manuscript. The presentation closely follows this reference.

In quantum mechanics, measurements differ fundamentally from their classical counterparts in that they cannot, in general, be performed without disturbing the system being observed. A quantum system is described by a state that encodes the probabilities of different measurement outcomes. When a measurement is performed, the outcome is not merely revealed but actively produced through the interaction between the system and the measuring apparatus. This interaction gives rise to measurement backaction, whereby the system's state is updated (or "collapsed") conditioned on the measurement result. More concretely, measuring an observable typically projects the system onto an eigenstate, or more generally onto a post-measurement state described by a measurement operator associated with the observed outcome (see Section I.1.3). Unless the system was already in such a state, this projection alters its subsequent evolution. Even weak or continuous measurements, which extract information only gradually, introduce backaction in the form of additional noise or dephasing. As a result, any attempt to monitor a quantum system inevitably perturbs it [51].

Since measurements require coupling the system to an external detector, they are naturally described within the framework of open quantum systems (see Chapter I.1). Motivated by the broader aim of this thesis to explore nonequilibrium quantum thermodynamics, the central question addressed here is: How does con-

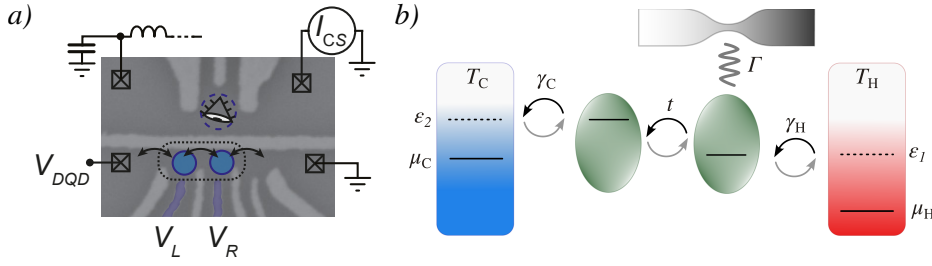


Figure II.4.1: a) Scanning electron microscope image of a device similar to the one modelled in this section. Blue circles mark the two dots forming the DQD and the eye represents a charge sensor dot used for classical readout. V_L and V_R are the voltages applied to gate electrodes that control the number of charges within the left and right dot respectively. Charge transport can occur from either source to drain, drain to source, or any intermediary process (black arrows), influenced by an applied source-drain voltage V_{DQD} across the DQD. Adapted from Ref. [28]. b) Schematic representation of a DQD in the two-terminal set-up, continuously monitored via an interaction between one quantum dot in the DQD and a QPC with measurement strength Γ . The particle current through the QPC depends on the occupation of the monitored quantum dot. Adapted from Ref. [29].

tinuous measurement affect the performance and stability of autonomous quantum thermal machines?

In solid-state quantum devices, the state of a charge qubit is typically monitored via weak, continuous measurement using a quantum point contact (QPC) positioned near the qubit [223]. The QPC conductance depends on the local electrostatic environment, allowing partial information about the qubit's charge state to be extracted from the measured current [51]. This information gain is unavoidably accompanied by backaction, manifesting as measurement-induced dephasing. To investigate the thermodynamic consequences of such continuous monitoring on quantum thermal devices, three ingredients are required: (i) a quantum-coherent system, (ii) thermodynamic gradients driving the system into a NESS capable of delivering power, and (iii) a measurement mechanism that introduces back-action. We adopt a minimal model satisfying these criteria: a boundary-driven autonomous DQD quantum heat engine (see Section I.2.4.1), with one dot continuously monitored by a QPC acting as a weak charge sensor [224, 225]. This setup is of high experimental relevance. It has enabled bidirectional single-electron counting [226, 227], provided the first experimental verification of the current fluctuation theorem in mesoscopic systems [228], and, more recently, facilitated studies of the entropic cost of extracting classical ticks from a quantum clock [28]. Figure II.4.1 shows a scanning electron microscope image of a device similar to the one modelled here, along with a schematic representation of the setup.

II.4.1 Set-up

We now proceed to derive the GKSL dynamics of the continuously monitored, boundary-driven DQD. The total Hamiltonian in the microscopic description of this configuration is

$$\hat{H} = \hat{H}_{\text{DQD}} + \sum_{\alpha \in \{H, C\}} \hat{H}_{\alpha} + \hat{H}_{\text{QPC}} + \hat{H}_{\text{DQD}, R} + \hat{H}_{\text{DQD}, \text{QPC}}, \quad (\text{II.4.1})$$

where $\hat{H}_{\text{DQD}}$, $\hat{H}_{\alpha}$, and $\hat{H}_{\text{DQD}, R}$ are defined in Eqs. (I.2.40), (I.2.41) and (I.2.42), respectively. The QPC is modelled as a tunnel junction

$$\hat{H}_{\text{QPC}} = \sum_{\alpha \in \{L, R\}, k} \omega_{\alpha, k} \hat{a}_{\alpha, k}^{\dagger} \hat{a}_{\alpha, k} + \sum_{k, k'} \mathcal{T}_{k, k'} \left(\hat{a}_{R, k}^{\dagger} \hat{a}_{L, k'} + \hat{a}_{L, k'}^{\dagger} \hat{a}_{R, k} \right), \quad (\text{II.4.2})$$

with $\hat{a}_{\alpha, k}^{(\dagger)}$ fermionic operators, and $\mathcal{T}_{k, k'}$ the tunnelling amplitude. The interaction between the first quantum dot and the QPC reads

$$\hat{H}_{\text{DQD}, \text{QPC}} = \sum_{k, k'} \chi_{k, k'} \hat{c}_1^{\dagger} \hat{c}_1 \left(\hat{a}_{R, k}^{\dagger} \hat{a}_{L, k'} + \hat{a}_{L, k'}^{\dagger} \hat{a}_{R, k} \right), \quad (\text{II.4.3})$$

where $\chi_{k, k'}$ quantifies the change in tunnelling amplitude when the quantum dot is occupied. Under the standard assumptions of low temperature, a narrow transport window and unidirectional QPC current (see, e.g., Refs. [229, 230]), and upon neglecting the baseline QPC tunnelling contributions, this yields

$$\dot{\hat{\rho}}_{\text{DQD}, \Gamma} = \mathcal{L}[\hat{\rho}_{\text{DQD}, \Gamma}] + \Gamma \mathcal{D} \left[\hat{c}_1^{\dagger} \hat{c}_1 \right] \hat{\rho}_{\text{DQD}, \Gamma} = \mathcal{L}_{\Gamma} [\hat{\rho}_{\text{DQD}, \Gamma}], \quad (\text{II.4.4})$$

where Γ is the effective measurement-induced dephasing rate, and is proportional to the sensitivity of the QPC current to the dot occupation $n_1 = \langle \hat{c}_1^{\dagger} \hat{c}_1 \rangle$, and $\mathcal{L}[\hat{\rho}]$ is defined in Eq. (I.2.43).

II.4.2 Composite QPC unit

We now incorporate the QPC within the repeated interaction framework and model it as a composite unit consisting of two qubits, each representing one of the QPC leads. The qubits have energy splitting $\Omega - \mu_R$ and $\Omega - \mu_L$, respectively, and are coupled via a tunnelling term with amplitude $\mathcal{T}$. The bare Hamiltonians of the DQD and the reservoir units, Eqs. (I.2.40) and (I.2.48), remain unchanged, while the QPC unit Hamiltonian, Eq. (II.4.2), becomes

$$\hat{H}_{\text{QPC}} = (\Omega - \mu_R) \hat{a}_R^{\dagger} \hat{a}_R + (\Omega - \mu_L) \hat{a}_L^{\dagger} \hat{a}_L + \mathcal{T} \left(\hat{a}_R^{\dagger} \hat{a}_L + \hat{a}_L^{\dagger} \hat{a}_R \right), \quad (\text{II.4.5})$$

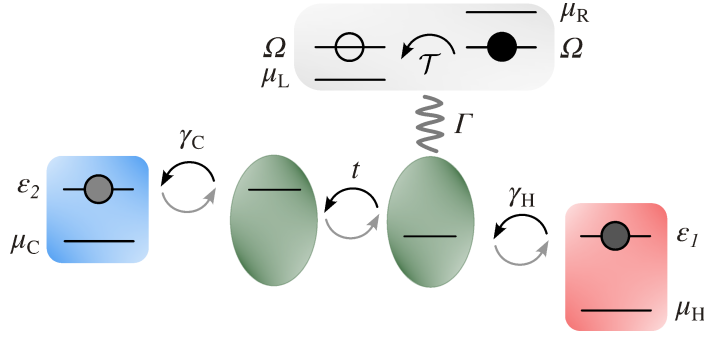


Figure II.4.2: In the framework of repeated interactions, we model the interaction of the DQD with the QPC with periodically refreshed units comprised of two qubits, i.e. one qubit per QPC lead. The energy splitting of the qubits is $(\Omega - \mu_{R(L)})$, respectively. They are coupled via a tunnelling interaction with amplitude $\mathcal{T}$. The qubits are initialised so that the qubit in the right lead is occupied while the qubit in the left lead is empty, resulting in a unidirectional current from right to left. Adapted from Ref. [29].

with the convention $\mu_R \geq \mu_L$.

At the beginning of each interaction, we assume that the global density matrix factorises into independent states of the DQD, the two reservoir units, and the QPC unit. Furthermore, the QPC unit is assumed to be initially separable into $\hat{\omega}_L$ and $\hat{\omega}_R$ (see Fig. II.4.2). Accordingly, the initial state of the total system reads

$$\hat{\rho} = \hat{\rho}_{\text{DQD},\Gamma} \otimes \underbrace{\hat{\omega}_{\beta_H} \otimes \hat{\omega}_{\beta_C} \otimes \hat{\omega}_R \otimes \hat{\omega}_L}_{\hat{\rho}_E}, \quad (\text{II.4.6})$$

where $\hat{\omega}_{\beta_H}$ and $\hat{\omega}_{\beta_C}$ are given in Eq. (I.2.51), respectively.

In the regime in which Eq. (II.4.4), including the dissipator accounting for the QPC, is valid, namely $|eV|, T_{\text{QPC}} \ll \mu_L, \mu_R$, so that the QPC operates in a narrow transport window and the current is effectively unidirectional, and assuming low temperatures, we set

$$\hat{\omega}_R = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{\omega}_L = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \quad (\text{II.4.7})$$

Since the QPC couples only to the first quantum dot with strength $\sqrt{\Gamma}$, we write the interaction Hamiltonian as

$$\hat{v}_{\text{QPC}} = \sqrt{\Gamma} \hat{c}_1^\dagger \hat{c}_1 \left(\hat{a}_R^\dagger \hat{a}_L + \hat{a}_L^\dagger \hat{a}_R \right), \quad (\text{II.4.8})$$

while the DQD-reservoir interactions $\hat{v}_H$ and $\hat{v}_C$ are given by Eq. (I.2.49).

Applying the repeated interaction framework and following the derivation outlined in Section I.2.4.3, we find that the GKSL master equation for the DQD is

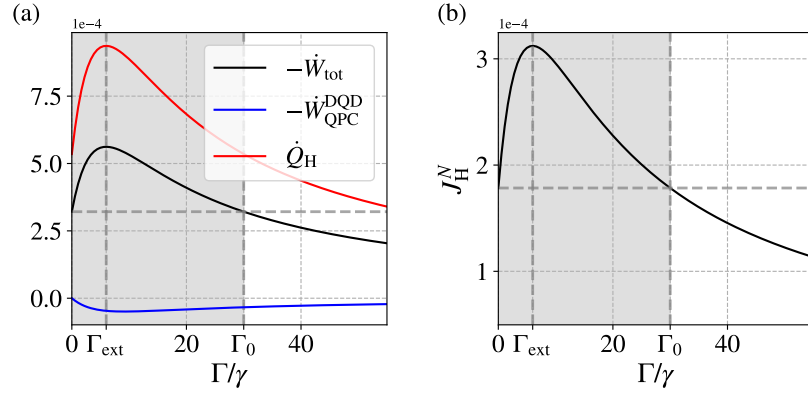


Figure II.4.3: Work, heat and particle currents as a function of the measurement strength. (a) In the NESS, the total work current out of the DQD $-\dot{W}_{\text{tot}}$ (black solid line) exhibits an extremum at the measurement strength Γ_{ext} and is larger than its value in the absence of measurement up a measurement strength of Γ_0 , as indicated by the dashed grey lines and the grey box. Here, also the work current between DQD and QPC $-\dot{W}_{\text{QPC}}^{\text{DQD}}$ (blue solid line) and the heat current between the hot reservoir and the DQD $\dot{Q}_H$ (red solid line) are shown. (b) The average particle current attains a maximum at the measurement strength Γ_{ext} and exceeds its magnitude in the absence of measurement up a measurement strength of Γ_0 , as indicated by the dashed grey lines and the grey box. Parameters: $T_H = 3$, $T_C = 1$, $\mu_H = 0.5$, $\mu_C = 1.5$, $\epsilon_1 = 2$, $\epsilon_2 = 2.1$, $t = 0.025$, $\gamma_H = \gamma_C = 0.025$. Adapted from Ref. [29].

given by

$$\dot{\rho}_{\text{DQD},\Gamma} = -i[\hat{H}_{\text{DQD}}, \hat{\rho}_{\text{DQD},\Gamma}] + \sum_{\alpha \in \{H, C, \text{QPC}\}} \mathcal{D}_\alpha[\hat{\rho}_{\text{DQD},\Gamma}], \quad (\text{II.4.9})$$

with the dissipators $\mathcal{D}_\alpha$ defined in Eq. (I.2.55). This procedure reproduces the DQD-QPC master equation presented in Eq. (II.4.4).

II.4.3 Thermodynamic description

Equipped with this microscopic model, we now study NESS currents, which are either induced by, labelled with subscript “QPC”, or altered in the presence of the QPC. As a first step, we validate that our description is thermodynamically consistent. Employing the formalism described in Section I.2.4.2, we find that the heat and work currents are given by

$$J_{\text{QPC}}^Q = -eV\Gamma\langle\hat{c}_1^\dagger\hat{c}_1\rangle, \quad J_H^Q = (\epsilon_1 - \mu_H)J_{H,\Gamma}^N, \quad J_C^Q = -(\epsilon_2 - \mu_C)J_{H,\Gamma}^N \quad (\text{II.4.10})$$

and

$$\begin{aligned} J_{\text{QPC}}^W &= \frac{\Gamma\Delta\epsilon}{\Gamma + \tilde{\gamma}}J_{H,\Gamma}^N + eV\Gamma\langle\hat{c}_1^\dagger\hat{c}_1\rangle, & J_H^W &= \left(\frac{\gamma_H\Delta\epsilon}{\Gamma + \tilde{\gamma}} + \mu_H\right)J_{H,\Gamma}^N, \\ J_C^W &= \left(\frac{\gamma_C\Delta\epsilon}{\Gamma + \gamma_C + \gamma_H} - \mu_C\right)J_{H,\Gamma}^N, \end{aligned} \quad (\text{II.4.11})$$

where $eV = \mu_R - \mu_L$, $\Delta\epsilon = \epsilon_2 - \epsilon_1$ denotes the detuning between the two quantum dots in the DQD and $\tilde{\gamma} = (\gamma_H + \gamma_C)$. The particle current across the DQD in the presence of the QPC, indicated by the subscript Γ , is given by

$$J_{H,\Gamma}^N = \gamma_H \left(f_H(\epsilon_1) - \langle \hat{c}_1^\dagger \hat{c}_1 \rangle \right) = \frac{4\Delta f \gamma_C \gamma_H (\Gamma + \tilde{\gamma}) |t|^2}{\tilde{\gamma} \left(4\Delta\epsilon^2 + (\Gamma + \tilde{\gamma})^2 \right) + 4\tilde{\gamma} (\Gamma + \tilde{\gamma}) |t|^2}, \quad (\text{II.4.12})$$

where $\Delta f = f_H(\epsilon_1) - f_C(\epsilon_2)$ is the difference in the Fermi distributions of the reservoirs evaluated at the quantum dot energies. Consistency with the first law of thermodynamics for the NESS immediately follows since

$$\langle \dot{\hat{H}}_{\text{DQD}} \rangle_{\text{tot}} = \sum_{\alpha=\text{QPC,H,C}} J_\alpha^W + J_\alpha^Q = 0. \quad (\text{II.4.13})$$

Note that the QPC's work and heat current can be divided into two contributions: one associated with the particle current through the DQD, $J_{H,\Gamma}^N$, and the other associated with the change in particle current through the QPC as a function of the monitored dot occupation, $\Delta J_{\text{QPC}}^N = \Gamma \langle \hat{c}_1^\dagger \hat{c}_1 \rangle$. We express this in the form

$$J_{\text{QPC}}^Q = -eV \Delta J_{\text{QPC}}^N, \quad J_{\text{QPC}}^W = \frac{\Gamma \Delta\epsilon}{\Gamma + \tilde{\gamma}} J_{H,\Gamma}^N + eV \Delta J_{\text{QPC}}^N = J_{\text{QPC}}^{W\text{DQD}} + J_{\text{QPC}}^{W\text{QPC}}. \quad (\text{II.4.14})$$

As expected, the energy current associated with changes in the QPC particle current, ΔJ_{QPC}^N , satisfies Watt's law¹.

We now examine how continuous measurement influences the thermodynamics of the DQD when it operates as a heat engine. In this setting, the particle current through the DQD, $J_{H,\Gamma}^N$, is the relevant energy carrier. Consequently, the heat and work currents arising from ΔJ_{QPC}^N , which are internal to the QPC unit, should not contribute to the engine's efficiency or to the entropic cost associated with driving the DQD particle current. We therefore define

$$J_{\text{tot}}^W = J_H^W + J_C^W + J_{\text{QPC}}^{W\text{DQD}}. \quad (\text{II.4.15})$$

As shown in Fig. II.4.3, the magnitudes of all NESS thermodynamic currents, Eqs. (II.4.10), (II.4.11) and (II.4.12), are initially enhanced as the measurement strength Γ increases. Moreover, the total output power $-J_{\text{tot}}^W$ exhibits an extremum at Γ_{ext} , a point to which we will return below. Since the DQD heat engine operates in the tight-coupling regime, the nontrivial dependence on Γ originates entirely from the particle current in Eq. (II.4.12). This naturally raises the question of whether the measurement provides any advantage in terms of efficiency. We argue that the

¹Watt's law, in the context of electronic transport, states that the energy current (the power) is given by the product of the voltage drop and the particle current ($P = VI$) [231].

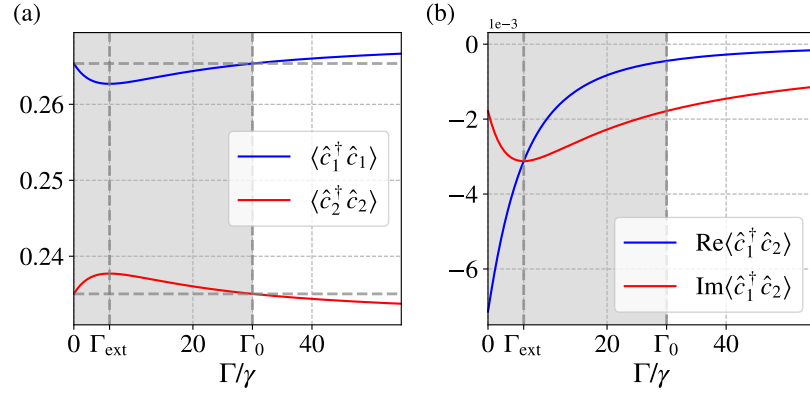


Figure II.4.4: Quantum dot occupations and coherences as a function of the measurement strength. (a) The single quantum dot occupations $n_{1(2)} = \langle \hat{c}_{1(2)}^\dagger \hat{c}_{1(2)} \rangle$ exhibit an extremum at the measurement strength Γ_{ext} and return to their original value in the absence of measurement at measurement strength Γ_0 , as indicated by the dashed grey lines and the grey box. (b) At Γ_{ext} the real and imaginary part of the coherence $\langle \hat{c}_1^\dagger \hat{c}_2 \rangle$ coincide and $\text{Im}\langle \hat{c}_1^\dagger \hat{c}_2 \rangle$, which in the NESS is proportional to the average particle current, exhibits an extremum. Parameters: $T_H = 3$, $T_C = 1$, $\mu_H = 0.5$, $\mu_C = 1.5$, $\epsilon_1 = 2$, $\epsilon_2 = 2.1$, $t = 0.025$, $\gamma_H = \gamma_C = 0.025$. Adapted from Ref. [29].

relevant notion of efficiency is given by

$$\eta = \frac{-J_{\text{tot}}^W}{J_H^Q}, \quad (\text{II.4.16})$$

which counts only the energy currents associated with particle transport through the DQD. In the tight-coupling limit, this efficiency is independent of the dephasing strength. However, this conclusion may change once the weak-coupling approximation is relaxed. Lastly, the entropy production rate associated with the thermodynamic cost of sustaining the NESS particle current through the DQD is

$$\dot{\Sigma}_{\text{DQD}} = -(\beta_H J_H^Q + \beta_C J_C^Q) = \left(\frac{\epsilon_1 - \mu_H}{T_H} - \frac{\epsilon_2 - \mu_C}{T_C} \right) J_{H,\Gamma}^N \geq 0. \quad (\text{II.4.17})$$

The total entropy production rate, $\dot{\Sigma}_{\text{tot}} = \dot{\Sigma}_{\text{DQD}} + \dot{\Sigma}_{\text{QPC}} \geq 0$, satisfies the second law, since also $\dot{\Sigma}_{\text{QPC}} = -\beta_{\text{QPC}} J_{\text{QPC}}^Q = \beta_{\text{QPC}} eV J_{\text{QPC}}^N \geq 0$, with $\beta_{\text{QPC}} = 1/T_{\text{QPC}}$.

II.4.4 Measurement-boosted transport

In the previous section, we found that the particle current $J_{H,\Gamma}^N$ exhibits a marked dependence on the measurement strength Γ . Continuous monitoring induces dephasing within the DQD [24], and as Γ becomes large, the loss of coherence due to the dephasing ultimately suppresses transport altogether via the quantum Zeno effect [232]. In this strong-measurement regime, the DQD can no longer sustain

coherent tunnelling, and its steady state reduces to a tensor product of the states of the two dots, each individually thermalised with its corresponding reservoir,

$$\lim_{\Gamma \rightarrow \infty} \hat{\rho}_{\text{DQD}} = \begin{pmatrix} f_H(\epsilon_1) & 0 \\ 0 & 1 - f_H(\epsilon_1) \end{pmatrix} \otimes \begin{pmatrix} f_C(\epsilon_2) & 0 \\ 0 & 1 - f_C(\epsilon_2) \end{pmatrix}. \quad (\text{II.4.18})$$

However, for an intermediate range of measurement strengths, which depends both on the system parameters and the couplings to the reservoirs, we observe dephasing-enhanced particle transport, in agreement with previously reported results [233–239]. To understand this effect, we analyse the NESS particle current in the steady state through the rate equations for the single quantum dot occupations, $n_{1(2)} = \langle \hat{c}_{1(2)}^\dagger \hat{c}_{1(2)} \rangle$, and their coherence $\langle \hat{c}_1^\dagger \hat{c}_2 \rangle$

$$\frac{d\langle \hat{c}_1^\dagger \hat{c}_1 \rangle}{dt} = \gamma_H \left(f_H - \langle \hat{c}_1^\dagger \hat{c}_1 \rangle \right) + i \left(t \langle \hat{c}_1^\dagger \hat{c}_2 \rangle + t^* \langle \hat{c}_2^\dagger \hat{c}_1 \rangle \right) = J_{H,\Gamma}^N + J_{12}^N, \quad (\text{II.4.19})$$

$$\frac{d\langle \hat{c}_1^\dagger \hat{c}_2 \rangle}{dt} = \left[i\Delta\epsilon - \frac{\tilde{\gamma} + \Gamma}{2} \right] \langle \hat{c}_1^\dagger \hat{c}_2 \rangle + it^* \left(\langle \hat{c}_2^\dagger \hat{c}_2 \rangle + \langle \hat{c}_1^\dagger \hat{c}_1 \rangle \right), \quad (\text{II.4.20})$$

$$\frac{d\langle \hat{c}_2^\dagger \hat{c}_1 \rangle}{dt} = \left[-i\Delta\epsilon - \frac{\tilde{\gamma} + \Gamma}{2} \right] \langle \hat{c}_2^\dagger \hat{c}_1 \rangle - it \left(\langle \hat{c}_2^\dagger \hat{c}_2 \rangle + \langle \hat{c}_1^\dagger \hat{c}_1 \rangle \right), \quad (\text{II.4.21})$$

$$\frac{d\langle \hat{c}_2^\dagger \hat{c}_2 \rangle}{dt} = \gamma_C \left(f_C - \langle \hat{c}_2^\dagger \hat{c}_2 \rangle \right) - i \left(t \langle \hat{c}_1^\dagger \hat{c}_2 \rangle + t^* \langle \hat{c}_2^\dagger \hat{c}_1 \rangle \right) = J_{C,\Gamma}^N - J_{12}^N. \quad (\text{II.4.22})$$

Here, $J_{H/C,\Gamma}^N$ denotes the NESS average particle current from the hot/cold reservoir into the DQD, while J_{12}^N represents the NESS average current between the two quantum dots within the DQD. In the NESS, $J_{H,\Gamma}^N = -J_{C,\Gamma}^N = -J_{12}^N$ and therefore it suffices to focus on $J_{H,\Gamma}^N$. Figure II.4.4 shows the impact of the measurement strength Γ on the NESS occupations of the individual quantum dots and on their coherence. While the occupations of the single dots, n_1 and n_2 , vary with Γ , the total DQD occupation $n = n_1 + n_2$ remains constant. While the QPC performs work on the DQD, as shown in Fig. II.4.3, this work only redistributes the occupancy of the eigenstates of the DQD Hamiltonian H_{DQD} associated with total occupation $n = 1$. And since the QPC does not exchange particles with the DQD, the total occupation n is conserved despite the redistribution. Solving the system of equations above, we find that the occupations of the individual quantum dots, n_1 and n_2 , reach extremal values at

$$\Gamma_{\text{ext}} = 2|\Delta\epsilon| - \tilde{\gamma}. \quad (\text{II.4.23})$$

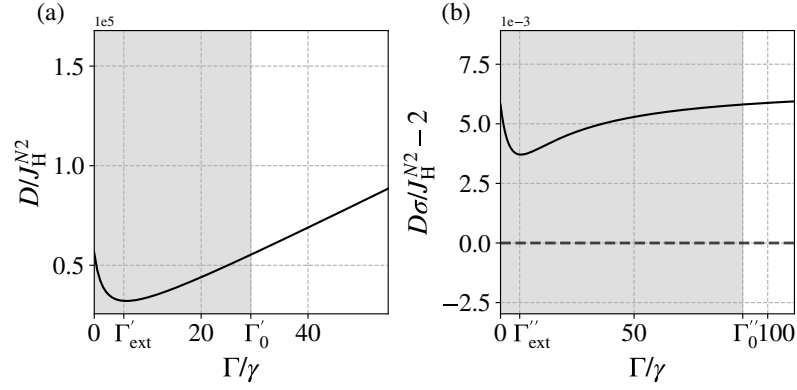


Figure II.4.5: Relative fluctuations and the thermodynamic uncertainty relation ratio (TURR) of the particle current as a function of the measurement strength. (a) Dephasing due to measurement-induced backaction can reduce the relative fluctuations of the particle current up to a measurement strength of Γ'_0 . The relative fluctuations are minimal at measurement strength Γ'_{ext} . (b) Similarly, the TURR, expressing the trade-off between the entropic cost and the relative fluctuations of the particle current, is minimal at Γ''_{ext} and is reduced up to Γ''_0 , which is significantly larger than Γ_0 and Γ'_0 . Parameters: reservoir temperatures $T_H, T_C = 3, 1$; we set $T = (T_H + T_C)/2$; reservoir chemical potentials $\mu_H, \mu_C = 0.5T, 1.5T$; quantum dot energies $\epsilon_1, \epsilon_2 = 2T, 2.1T$; interdot hopping amplitude $t = 0.025T$; coupling to reservoirs $\gamma_C, \gamma_H = \gamma = 0.025T$. Adapted from Ref. [29].

Additionally, the measurement strength at which the dot occupations coincide with their values in the absence of measurement, $n_{1/2}(\Gamma_0) = n_{1/2}(0)$, is

$$\Gamma_0 = \frac{4\Delta\epsilon^2}{\tilde{\gamma}} - \tilde{\gamma}. \quad (\text{II.4.24})$$

indicating that measurement-assisted particle transport occurs for $\Gamma \leq \Gamma_0$, as illustrated in Fig. (II.4.3).

II.4.5 Fluctuations and entropic cost

TURs capture a fundamental trade-off between the relative fluctuations of a stochastic thermodynamic current in classical Markovian NESS heat engines and the entropic cost of their driving [170, 240, 241, 171, 242], as discussed in Section I.2.3.

Here, we numerically investigate the relative fluctuations of the particle current $J_{H,\Gamma}^N$, as well as the TUR ratio (TURR), defined as $D\dot{\Sigma}_{\text{DQD}}/J_{H,\Gamma}^N{}^2$. The corresponding diffusion coefficient D can be straightforwardly computed using established methods [243, 36, 244, 51] based on the power spectrum $S(\omega)$, which is often more accessible experimentally. In particular, D is obtained from the zero-frequency component, $D = S(\omega = 0)$. The diffusion coefficient can be expressed in vec-

torised form (see Section I.1.4) as [243]

$$D = S(0) = M - 2\langle\langle\hat{\mathbb{I}}|\mathcal{L}_1\mathcal{L}^+|\hat{\rho}\rangle\rangle, \quad (\text{II.4.25})$$

where $|\hat{\mathbb{I}}\rangle\rangle$ denotes the vectorised identity, $|\hat{\rho}\rangle\rangle$ the vectorised steady-state density matrix, and

$$\mathcal{L}_1[\hat{\rho}] = \sum_k \mu_k \hat{L}_k \hat{\rho} \hat{L}_k^\dagger. \quad (\text{II.4.26})$$

Here, $\hat{L}_k$ is the jump operator corresponding to a detector click with outcome μ_k , and $\mathcal{L}^+$ is the Drazin inverse of the Lindbladian $\mathcal{L}$ associated with the GKSL master equation (see Section I.1.6). The dynamical activity M is defined as

$$M = \sum_k \mu_k^2 \text{Tr} [\hat{L}_k \hat{\rho} \hat{L}_k^\dagger]. \quad (\text{II.4.27})$$

As we evaluate the diffusion coefficient of the fluctuating particle current between the DQD and the hot reservoir, we select the measurement operators

$$\hat{L}_k \in \left\{ \sqrt{\gamma_H f_H(\epsilon_1)} \hat{c}_1^\dagger, \sqrt{\gamma_H (1 - f_H(\epsilon_1))} \hat{c}_1 \right\}, \quad (\text{II.4.28})$$

with associated outcomes $\mu_k \in \{+1, -1\}$. We observe that the relative fluctuations of the particle current, quantified by $D / \left(J_{H,\Gamma}^N \right)^2$, can be suppressed maximally at Γ'_{ext} and remain suppressed up to Γ'_0 , effectively stabilizing the current, as illustrated in Fig. (II.4.5). This behaviour occurs provided that the system parameters and reservoir couplings are suitably chosen. Next, we observe that the TURR can approach its lower bound even for very weak measurement strengths Γ , as shown in Fig. (II.4.5) for certain parameter regimes, indicating that dephasing induced by the measurement backaction can reduce the TURR even beyond the regime of dephasing-assisted transport. It is important to note that in this scenario, the TURR varies while the engine efficiency remains constant. This behaviour, however, is a consequence of the tight coupling between energy and particle currents and should be regarded as a pathological effect.

II.4.6 Conclusion

Fluctuations in thermodynamic currents play a central role when assessing the practical usefulness and stability of quantum heat engines. In this Part, we have shown that continuous measurement, often regarded as detrimental to coherent quantum devices, can instead play a constructive role, simultaneously enhancing the average particle current while suppressing its relative fluctuations. To demonstrate this effect, we considered a minimal yet genuinely quantum and ex-

perimentally accessible setup: a boundary-driven DQD continuously monitored via a QPC coupled to one of the dots. A crucial step in our analysis was to show that the commonly used local GKSL master equation for the DQD, interacting with two thermal reservoirs and monitored by a QPC, admits a rigorous microscopic derivation within the thermodynamically consistent repeated-interactions framework. Within this setting, we identified the relevant entropy production rate associated with sustaining the particle current through the DQD as determined solely by the heat exchanged with the thermal reservoirs and their temperatures. Importantly, this contribution is positive and depends on the QPC temperature only through the induced measurement strength Γ . In the formal limit of a zero-temperature QPC, Γ remains finite but the entropy production associated with the QPC diverges, pointing to an open conceptual issue: how a realistic measurement apparatus should be incorporated into a consistent thermodynamic description of continuously monitored quantum systems. Our results further show that continuous monitoring can lower the entropic cost required to maintain a fixed signal-to-noise ratio in the particle current. Consequently, the entropy associated with the precision of the particle flux, which is often more directly accessible experimentally than energy flow, can be reduced without violating the thermodynamic uncertainty relation.

This study offers a solid starting point for more refined future investigations. Natural extensions include models incorporating realistic reservoir spectral densities using more sophisticated collision-based approaches [245–247], or within the mesoscopic-leads approach [144, 50, 248, 52], which we discuss next.

PART III:

Fluctuations in the mesoscopic-leads approach

Chapter III.5

Ensemble description

This chapter is based on Refs. [52, 249, 50, 144].

Recent advances in nanotechnology have enabled unprecedented control over quantum devices, including implementations of quantum thermal machines [12, 13, 19]. These devices have begun to demonstrate practical functionality, such as the autonomous reset of a superconducting qubit [17]. Among the most promising platforms for realizing autonomous quantum thermal devices are electronic setups of this kind, as well as semiconductor quantum dots [177]. Such progress has broadened experimental investigations in mesoscopic physics beyond conventional charge transport, opening the way to controlled manipulation of heat currents in nanoscale systems [177, 250]. Providing a faithful theoretical description of such devices is, however, highly nontrivial. One must simultaneously account for coherent quantum dynamics, finite-temperature effects, strong system-reservoir coupling, and, in some cases, significant many-body interactions within the working medium. Widely used approaches based on local quantum master equations [11] (see Section I.2.2) rely on perturbative and Markovian assumptions that can severely restrict their range of validity. In particular, they may fail to capture far-from-equilibrium quantum coherences [141, 251, 168] and, at finite temperatures, can even lead to predictions that are incompatible with fundamental thermodynamic principles [142, 140] (see Section I.2.4.2). Moreover, external driving can induce pronounced non-Markovian effects [252, 253], further challenging the applicability of these standard descriptions.

What is therefore needed is a framework that remains thermodynamically consistent while accurately describing strong system-reservoir coupling and far-from-equilibrium dynamics, yet remains computationally tractable for driven many-body systems. Several alternatives have been developed in this direction, in-

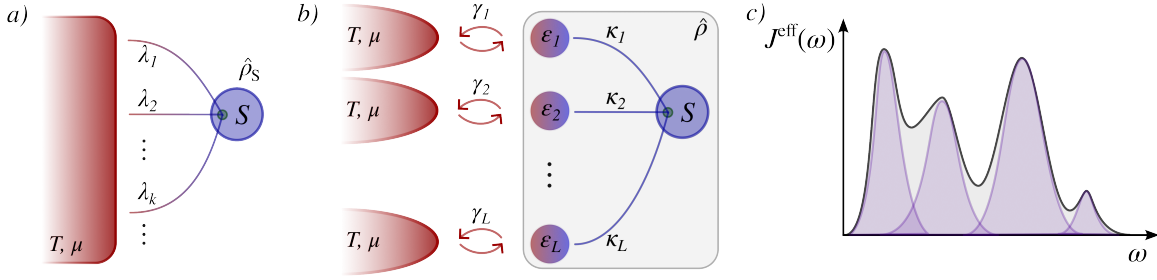


Figure III.5.1: A central system S coupled to a single reservoir at temperature T and chemical potential μ (a) approximated in the mesoscopic-leads formalism (b). All L lead modes with on-site energies ϵ_k couple to the same site in the central system, with hopping interaction of strength κ_k . The lead modes are damped through their coupling to a residual reservoir at the same temperature T and chemical potential μ , with damping rate γ_k . The extended system, comprised of the central system and the lead modes, in state $\hat{\rho}$ is shaded in grey. c) The mesoscopic lead comprised of damped modes gives rise to an effective spectral density $\mathcal{J}^{\text{eff}}(\omega)$ that is a sum of Lorentzian spectral densities. By tuning the damping of each mode and its coupling to the system, one can approximate the spectral density of the infinite bath depicted in Fig. 1.2.2. Subfigures a)-b) are adapted from Ref. [52].

cluding nonequilibrium Green's function techniques [254] and path-integral methods [255]. While powerful, these approaches are perturbative in the system-bath coupling and can become difficult to apply in the presence of strong interactions or time-dependent driving, motivating the search for complementary methods.

The mesoscopic-leads approach (see Refs. [52, 249, 50, 144] and references therein) provides such a framework for fermionic, potentially interacting systems coupled to macroscopic thermal reservoirs. In this approach, each infinite reservoir is replaced by a finite set of modes that are continuously damped toward thermal equilibrium via Lindblad-type jump operators, yielding an accurate approximation of continuum baths while preserving a Markovian GKSL description at the level of an extended system. A key advantage of this construction is that it enables the use of standard analytical and numerical techniques for GKSL dynamics, such as quantum-jump trajectory methods, which we employ in Chapter III.6.

The basic setup is illustrated in Fig. III.5.1 for the case of a single reservoir. Specifically, in the mesoscopic-leads approach, each infinite reservoir, labelled by α , at temperature T_α and chemical potential μ_α and with spectral density $\mathcal{J}_\alpha(\omega)$, is replaced by a finite mesoscopic lead composed of l_α modes, each coupled to the same system site p_α ,

$$\hat{H}_\alpha = \sum_{k=1}^{l_\alpha} \epsilon_{\alpha,k} \hat{a}_{\alpha,k}^\dagger \hat{a}_{\alpha,k}, \quad \hat{H}_{S\alpha} = \sum_{k=1}^{l_\alpha} \left(\kappa_{\alpha k} \hat{d}_{p_\alpha}^\dagger \hat{a}_{\alpha,k} + \kappa_{\alpha k}^* \hat{a}_{\alpha,k}^\dagger \hat{d}_{p_\alpha} \right). \quad (\text{III.5.1})$$

The system comprises N_S modes with annihilation operators $\{\hat{d}_j\}_{j=1}^{N_S}$ and Hamiltonian $\hat{H}_S(t)$. The residual reservoirs of each lead mode are chosen to have a

flat spectral density [48, 49], and effective damping rate $\gamma_{\alpha,k}$ (typically one sets $\gamma_{\alpha,k} = \epsilon_{\alpha,k+1} - \epsilon_{\alpha,k}$, corresponding to a uniform discretization of the reservoir spectrum) [256–261]. Consequently, the coupling rate between the system site and modes in lead α is given by $\kappa_{\alpha,k} = \sqrt{\mathcal{J}_\alpha(\omega)\gamma_{\alpha,k}/2\pi}$, and the effective spectral density perceived by the system is a sum of Lorentzian spectral densities, each centred at a lead mode's energy $\epsilon_{\alpha,k}$,

$$\mathcal{J}_\alpha^{\text{eff}}(\omega) = \sum_{k=1}^{l_\alpha} \frac{|\kappa_{\alpha,k}|^2 \gamma_k}{(\omega - \epsilon_{\alpha,k})^2 + (\gamma_{\alpha,k}/2)^2}, \quad (\text{III.5.2})$$

which can be tuned to approximate $\mathcal{J}_\alpha(\omega)$ with high accuracy. This is shown schematically in Fig. III.5.1 c). In the limit $l_\alpha \rightarrow \infty$, and for suitably chosen parameters, the dynamics in the presence of the original infinite reservoirs are recovered. As discussed in Section 1.2.1, this is crucial: within this limit, the system's (thermo)dynamics are fully determined by the reservoirs' spectral function together with their macroscopic properties, such as temperatures and chemical potentials. For finite l_α , finite-size effects or deviations from Markovianity may arise, but these vanish as $l_\alpha \rightarrow \infty$, ensuring a faithful representation of the infinite-reservoir dynamics. For a detailed discussion of subtleties and convergence limitations in parameter selection, we refer to Ref. [262].

III.5.1 Dynamics

Crucially, the dynamics of the extended system, composed of the system and the lead, are Markovian and evolve under a GKSL master equation,

$$\frac{d\hat{\rho}}{dt} = -i[\hat{H}, \hat{\rho}] + \sum_{\alpha=1}^{N_R} \sum_{k=1}^{l_\alpha} \sum_{\sigma=\pm} \mathcal{D} [\hat{L}_{\alpha,k}^\sigma] \hat{\rho}, \quad (\text{III.5.3})$$

with $\hat{L}_{\alpha,k}^+ = \sqrt{\gamma_{\alpha,k} f_{\alpha,k}} \hat{a}_{\alpha,k}^\dagger$ and $\hat{L}_{\alpha,k}^- = \sqrt{\gamma_{\alpha,k} (1 - f_{\alpha,k})} \hat{a}_{\alpha,k}$, where $\hat{H} = \hat{H}_S + \sum_\alpha \hat{H}_\alpha + \hat{H}_{S\alpha}$, and $f_{\alpha,k} = f(\epsilon_{\alpha,k})$ denotes the Fermi occupation of the residual reservoir of mode k in lead α evaluated at the lead mode's onsite energy $\epsilon_{\alpha,k}$. For convenience, we define

$$\mathcal{L}_\alpha [\hat{\rho}] = \sum_{k=1}^{l_\alpha} \sum_{\sigma=\pm} \mathcal{D} [\hat{L}_{\alpha,k}^\sigma] \hat{\rho}. \quad (\text{III.5.4})$$

Unlike conventional weak-coupling master equations, the mesoscopic-leads approach treats the coherent system-lead coupling explicitly. Strong-coupling and memory effects are naturally encoded in the dynamics of the lead modes, while the overall evolution of the extended system remains Markovian. This provides a

bridge between fully microscopic and phenomenological reservoir models, while retaining computational efficiency and the toolbox that comes with GKSL master equations. In Chapter III.6, we will make use of this advantage in the following way: we apply a quantum jump trajectory unravelling, discussed in Section I.1.3, to the GKSL dynamics of the extended system.

In this thesis, we restrict our analysis to the case where the extended system is governed by a non-interacting and non-pairing fermionic Hamiltonian and is initially prepared in a Gaussian fermionic state. This restriction allows the full dynamics to be reduced to closed equations for two-point correlations, leading to an efficient covariance-matrix description. Here, a fully general description in terms of Majorana operators, as presented in Section I.3.2, is not required and working with standard (non-Hermitian) fermionic operators is sufficient. Then, the Hamiltonian of the extended system may be written in a quadratic form $\hat{H} = \sum_{i,j=1}^{N_S+L} H_{ij} \hat{c}_i^\dagger \hat{c}_j$, with annihilation operators of the extended system $\{\hat{c}_i\}_{i=1}^{N_S+L}$. Here, for notational convenience, we have made the transformation

$$\{\hat{d}_1, \dots, \hat{d}_{N_S}, \hat{a}_{1,1}, \dots, \hat{a}_{1,l_1}, \dots, \hat{a}_{N_R,1}, \dots, \hat{a}_{N_R,l_{N_R}}\} \rightarrow \{\hat{c}_1, \dots, \hat{c}_{N_S+L}\}. \quad (\text{III.5.5})$$

Since the GKSL dynamics (III.5.3) preserves Gaussianity (see Section I.3.2), at any time t , the density matrix of the extended system can be written in the Gaussian form

$$\hat{\rho}(t) = \frac{e^{-\vec{c}^\dagger \cdot M(t) \cdot \vec{c}}}{Z(t)}, \quad (\text{III.5.6})$$

where $\vec{c}$ denotes the vector of system and lead fermionic operators, $\vec{c}_i = \hat{c}_i$. The matrix $M(t)$ and the partition function $Z(t)$, ensuring normalization of the state, are given by

$$M(t) = \log \left(\frac{1 - C(t)}{C(t)} \right), \quad Z(t) = \frac{1}{\det[1 - C(t)]}. \quad (\text{III.5.7})$$

Since the state $\hat{\rho}(t)$ is now fully determined by its covariance matrix $C(t)$, the dynamics can be formulated in terms of matrices of size $(N_S + L) \times (N_S + L)$, where $L = \sum_\alpha l_\alpha$, allowing for efficient computations even for systems with many modes. Within the Heisenberg picture, the ensemble (unconditional) dynamics of the covariance matrix elements $C_{ij} = \text{Tr} [\hat{c}_j^\dagger \hat{c}_i \hat{\rho}]$ is determined by

$$\frac{dC_{ij}}{dt} = \left\langle \frac{d\hat{c}_j^\dagger \hat{c}_i}{dt} \right\rangle = \left\langle i [\hat{H}, \hat{c}_j^\dagger \hat{c}_i] + \sum_{\alpha=1}^{N_R} \sum_{k=1}^{l_\alpha} \mathcal{L}_{\alpha,k}^+ [\hat{c}_j^\dagger \hat{c}_i] \right\rangle, \quad (\text{III.5.8})$$

where $\mathcal{L}_{\alpha,k}^+$ is the adjoint dissipator to $\mathcal{L}_{\alpha,k}$ (see Section I.1.2) and $\mathcal{L}_\alpha^+ [\hat{A}] =$

$\sum_{k=1}^{l_\alpha} \mathcal{L}_{\alpha,k}^+ [\hat{A}]$. In particular,

$$\mathcal{L}_{\alpha,k}^+ [\hat{A}] = \sum_{\sigma \in \{+, -\}} \mathcal{D}^+ [\hat{L}_{\alpha,k}^\sigma] \hat{A}, \quad (\text{III.5.9})$$

where $\mathcal{D}^+ [\hat{L}] \hat{A} = \hat{L}^\dagger \hat{A} \hat{L} - \frac{1}{2} \{ \hat{L}^\dagger \hat{L}, \hat{A} \}$. The covariance matrix C of the extended system then evolves under the Lyapunov differential equation [263], Eq. (I.3.14), with

$$W = iH + \frac{1}{2}\Gamma, \quad \Gamma_{kk} = \gamma_k \quad \text{and} \quad F_{kk} = \gamma_k f_k. \quad (\text{III.5.10})$$

Here, both Γ and F are diagonal matrices. In the steady state, $dC/dt = 0$, and the corresponding covariance matrix therefore solves the Lyapunov equation (I.3.16).

III.5.2 Thermodynamics

Within the mesoscopic-leads framework, we identify the relevant currents as those exchanged between the lead modes and the residual reservoir, termed external currents, rather than the internal currents exchanged between the system and the leads. This distinction is crucial, since in mesoscopic experiments (e.g., measurements performed via an ammeter), the observed signal typically corresponds to reservoir currents, especially in the strong-coupling regime where coherences and hybridization blur the boundary between system and environment. For a detailed treatment of internal versus external currents in the mesoscopic-leads formalism, see Refs. [144, 249].

Particle current The average particle current $I_N(t)$ flowing into the extended system is defined through the continuity equation for the total particle number operator, $\hat{N} = \sum_{j=1}^{N_S+L} \hat{c}_j^\dagger \hat{c}_j$,

$$I_N(t) = \frac{d\langle \hat{N} \rangle}{dt} = \sum_{\alpha=1}^{N_R} I_{N_\alpha}(t), \quad (\text{III.5.11})$$

where $I_{N_\alpha}(t) = \langle \mathcal{L}_\alpha^+ [\hat{N}] \rangle$ denotes the particle current flowing into lead α , and $I_N(t) = \sum_{\alpha=1}^{N_R} I_{N_\alpha}(t)$ is the net current entering the extended system. Note that the total particle current arises entirely from the contributions of the residual reservoirs. In terms of the matrices F_α and Γ_α , the average current into lead α can be written as

$$I_{N_\alpha}(t) = \text{Tr} [F_\alpha - \Gamma_\alpha C(t)]. \quad (\text{III.5.12})$$

Inserting the explicit forms of F_α and Γ_α then yields

$$I_{N_\alpha}(t) = \sum_{k=1}^{l_\alpha} \gamma_{k,\alpha} (f_{k,\alpha} - \langle \hat{c}_{k,\alpha}^\dagger \hat{c}_{k,\alpha} \rangle). \quad (\text{III.5.13})$$

In this form, the expression highlights that the unconditional particle current is determined entirely by the occupations of the lead modes relative to their corresponding reservoir distributions.

Energy current The average total energy current $I_E(t)$ flowing into the extended system is analogously defined via the continuity equation for the Hamiltonian $\hat{H}$ of the extended system,

$$\frac{d\langle\hat{H}\rangle}{dt} = \sum_{\alpha=1}^{N_R} I_{E_\alpha}(t), \quad (\text{III.5.14})$$

with $I_{E_\alpha}(t) = \langle\mathcal{L}_\alpha^+[\hat{H}]\rangle$, and denotes the energy current flowing into lead α . The net energy current entering the extended system is then $I_E(t) = \sum_{\alpha=1}^{N_R} I_{E_\alpha}(t)$.

In terms of the matrices F and Γ , the energy current can be expressed as

$$I_E(t) = \text{Tr} \left[FH(t) - \frac{1}{2}\Gamma(C(t)H(t) + H(t)C(t)) \right]. \quad (\text{III.5.15})$$

Substituting the explicit form of the Hamiltonian, we recover

$$I_E(t) = \sum_{k=1}^L \gamma_k \epsilon_k (f_k - \langle\hat{c}_k^\dagger \hat{c}_k\rangle) - \frac{1}{2} \sum_{k=1}^L \gamma_k \langle \kappa_{kp} \hat{d}_p^\dagger \hat{a}_k + \kappa_{kp}^* \hat{a}_k^\dagger \hat{d}_p \rangle, \quad (\text{III.5.16})$$

which agrees with results derived in Ref. [50]. This expression explicitly shows that the energy current is determined both by the occupations of the lead modes relative to their reservoir distributions and by the hybridization between system and lead modes.

Heat current and entropy production The total average external heat current is given by

$$I_Q(t) = \sum_{\alpha} I_{Q_\alpha}, \quad \text{with} \quad I_{Q_\alpha} = I_{E_\alpha}(t) - \mu_\alpha I_{N_\alpha}(t). \quad (\text{III.5.17})$$

Further, we express the entropy production rate, see Refs. [146, 148, 147], as the difference between the temperature-weighted dissipated heat current, the entropy flux, and the change in the instantaneous von Neumann entropy of the extended system, so that the entropy production rate is given by

$$\dot{\Sigma}(t) = \dot{S}_{\text{SL}}(t) - \sum_{\alpha} \beta_{\alpha} I_{Q_\alpha}(t), \quad (\text{III.5.18})$$

with von Neumann entropy of the extended system $S_{\text{SL}}(t) = S_{\text{vN}}(\hat{\rho}_{\text{SL}}(t))$ (see Eq. (I.2.9)). For a detailed analysis of entropy production in the mesoscopic-leads framework, we refer to Ref. [249].

This ensemble-level description provides the foundation for accessing not only

average thermodynamic currents, but also their fluctuations via quantum trajectory unravellings of the extended GKSL dynamics, which we develop in the next chapter.

Chapter III.6

Stochastic description

The results presented in this chapter are based on Ref. [52]. I performed the analytical and numerical calculations, carried out the data analysis, and wrote the manuscript. The co-authors contributed to the conceptualisation of the work, discussed the results, and provided feedback and revisions on the manuscript. The presentation closely follows this reference.

The fluctuating flows of particles and heat exchanged between an open system and its environment are governed not only by thermodynamic biases, but also by the reservoir spectra, which can be highly structured [35]. Capturing the resulting full counting statistics, rather than only average currents, is particularly challenging in a regime highly relevant to mesoscopic thermal devices: strong system-reservoir coupling, finite temperature, and arbitrary time-dependent driving. In this regime, standard weak-coupling Markovian master equations break down, while established alternatives face complementary limitations. Nonequilibrium Green's functions and scattering theory [37, 36, 38, 39] are typically restricted to weakly interacting or slowly driven settings [40–42], and path-integral approaches [43, 44, 264] are so far confined to small, nonintegrable systems [265, 46] and become prohibitively costly in larger many-body problems [266]. Moreover, many methods provide access only to characteristic functions of integrated currents, so extracting distributions or higher moments requires numerical differentiation or Fourier transforms, which can be ill-conditioned and introduce substantial errors.

In this chapter, we present our approach [52], which addresses these obstacles within the mesoscopic-leads framework introduced in Chapter III.5. The key idea is to embed the non-Markovian, strongly coupled problem into an extended sys-

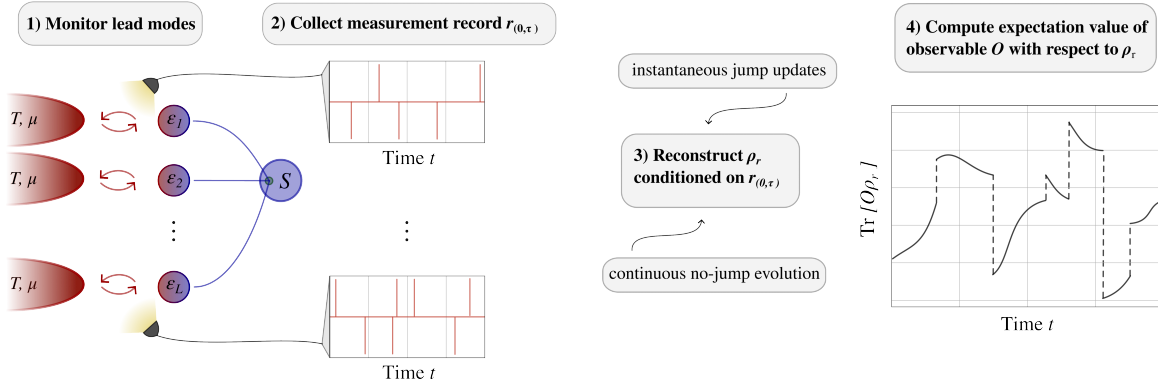


Figure III.6.1: Particle transfer between each lead mode and its associated residual reservoir is monitored and recorded in the measurement record r . The expectation value of an observable $\hat{O}$ is computed with respect to the state conditioned on r . When a quantum jump is recorded, the state undergoes an abrupt update, leading to an instantaneous change in the expectation value of the observable. During the intervals between jumps, the state evolves smoothly following the no-jump conditional evolution. Adapted from Ref. [52].

tem, the central system plus a finite set of damped lead modes, whose dynamics obey a GKSL master equation. Non-Markovianity here refers to the reduced dynamics of the central system obtained after tracing out the mesoscopic lead, while the embedded system-lead evolution itself remains Markovian. This Markovian embedding enables the use of quantum-trajectory unravellings (see Section I.1.3) to evaluate thermodynamic quantities along single stochastic realizations. By direct sampling over trajectories, we obtain the full probability distributions of charge, heat, and entropy production in driven, strongly coupled open quantum systems, without relying on numerically delicate transform procedures. Thermodynamic consistency at the stochastic level is established by verifying integral fluctuation theorems for the total, uncertainty, and martingale entropy production [190, 205]. As an application, we investigate heat fluctuations during finite-time information erasure in a driven quantum dot strongly coupled to a fermionic reservoir. Altogether, the results extend trajectory-level quantum thermodynamics to regimes that combine strong coupling, reduced non-Markovian dynamics, and arbitrary time-dependent driving.

III.6.1 Dynamics

In the mesoscopic-leads formalism, the stochastic dynamics of the extended system are unravelled by monitoring particle exchange between each lead mode and its residual reservoir (see Section I.1.3). This monitoring is described by the fol-

lowing measurement operators,

$$\hat{M}_0 = \hat{\mathbb{I}} - \frac{dt}{2} \sum_{k=1}^L \sum_{\sigma=\pm} \hat{L}_k^{\sigma\dagger} \hat{L}_k^{\sigma} - i dt \hat{H}, \quad \hat{M}_k^{\sigma} = \sqrt{dt} \hat{L}_k^{\sigma}, \quad 1 \leq k \leq L, \quad (\text{III.6.1})$$

with $\hat{L}_k^+ = \sqrt{\gamma_k f_k} \hat{c}_k^{\dagger}$ and $\hat{L}_k^- = \sqrt{\gamma_k (1 - f_k)} \hat{c}_k$ describing absorption and emission events, and $\hat{\mathbb{I}}$ the identity operator. Here, the lead index α and the mode index within each lead are combined into a single index k running over all lead modes. The measurement record is $r_{(0,\tau)} = \{(t_j, k, \sigma) \mid 0 < t_j < \tau\}$, registering a jump at time t_j in mode k , with $\sigma = + (-)$ indicating particle transfer onto (off) the lead mode. We neglect the possibility of multiple simultaneous jumps. At each time $t \in (0, \tau)$, the system state is updated conditioned either on no jump or on the occurrence of a jump. Under these measurement operators, the evolution of $\hat{\rho}$ unravels into a continuous ‘no-jump’ evolution, punctuated by instantaneous jump-induced updates of the conditioned density matrix $\hat{\rho}_r$, described by the stochastic jump equation (see Section I.1.3)

$$d\hat{\rho}_r = -i [\hat{H}, \hat{\rho}_r] dt - \frac{1}{2} \sum_{k=1}^L \sum_{\sigma \in \{+, -\}} \mathcal{H} [\hat{L}_k^{\sigma\dagger} \hat{L}_k^{\sigma}] \hat{\rho}_r dt + \sum_{k=1}^L \sum_{\sigma \in \{+, -\}} \mathcal{G} [\hat{L}_k^{\sigma}] \hat{\rho}_r dN_k^{\sigma}. \quad (\text{III.6.2})$$

The superoperators, $\mathcal{G}$ and $\mathcal{H}$, acting on a state $\hat{\rho}_r$, are defined in Eq. (I.1.25). While here we assume ideal detection efficiency in all measurement channels, a more general treatment that accounts for imperfect detection or alternative measurement setups is detailed in Ref. [52]. The expectation values conditioned on the measurement record, are defined as $\langle \cdot \rangle_r = \text{Tr} [\hat{\rho}_r \cdot]$. For a Gaussian fermionic system, both ensemble averages and individual trajectories can be fully described using the covariance matrix [52, 267, 217, 268]. In particular, for single trajectories, the conditioned covariance matrix is defined as $C_{ij}^r = \text{Tr} [\hat{c}_j^{\dagger} \hat{c}_i \hat{\rho}_r]$. In the following, we discuss the evolution of C^r during no-jump periods and its update upon detection of jumps. We first focus on the interval between two consecutive jumps, $(t_{J_{m-1}}, k_{m-1}, \sigma_{m-1})$ and (t_{J_m}, k_m, σ_m) . For times $t \in (t_{J_{m-1}}, t_{J_m})$, no jumps occur, $dN_k^{\sigma}(t) = 0$ for all k and σ , and the covariance matrix evolves smoothly according to

$$dC_{ij}^r(t) = i \left\langle [\hat{H}, \hat{c}_j^{\dagger} \hat{c}_i] \right\rangle_r dt - \frac{1}{2} \sum_{k=1}^L \sum_{\sigma \in \{+, -\}} \left\langle \mathcal{H}^+ [\hat{L}_k^{\sigma\dagger} \hat{L}_k^{\sigma}] \hat{c}_j^{\dagger} \hat{c}_i \right\rangle_r dt. \quad (\text{III.6.3})$$

Here, $\mathcal{H}^+$ is the adjoint superoperator to $\mathcal{H}$ (see Eq. (I.1.25) and Section I.1.2). Expanding Eq. (III.6.3) generates terms that are quartic in the creation and annihilation operators of the extended system. Crucially, we can exploit the fact that the

initial state is Gaussian and that the conditional evolution along any single trajectory preserves Gaussianity [267]. Using Wick's theorem [220] (see Section I.3.2 and Eq. (I.3.13)), we obtain a Riccati-type differential equation governing the evolution of the no-jump conditioned covariance matrix of the extended system

$$\frac{dC^r}{dt} = -(VC^r + C^rV^\dagger) + C^rBC^r, \quad (\text{III.6.4})$$

where $B = \Gamma - 2F$, $V = iH + \frac{1}{2}B = W - F$. W and F are defined in Eq. (III.5.10). Between jumps, the survival probability P_{surv} (see Eq. (I.1.18)) evolves according to the differential equation

$$\frac{dP_{\text{surv}}(t)}{dt} = -P_{\text{surv}}(t) \sum_{k=1}^L \sum_{\sigma \in \{+, -\}} \left\langle \hat{L}_k^{\sigma\dagger} \hat{L}_k^\sigma \right\rangle_r, \quad (\text{III.6.5})$$

and decays as

$$P_{\text{surv}}(t) = P_{\text{surv}}(t_{J_{m-1}}) \exp \left(- \int_{t_{J_{m-1}}}^t K(s) ds \right), \quad (\text{III.6.6})$$

where $K(s) = \text{Tr}[F\bar{C}^r(s)] + \text{Tr}[(\Gamma - F)C^r(s)]$, and $\bar{C}^r = 1 - C^r$.

We now consider the jump updates. When a jump of type σ_m occurs in lead mode k_m at time t_{J_m} , recorded in the measurement record as (t_{J_m}, k_m, σ_m) , the conditioned covariance matrix C^r is updated instantaneously. At the moment of the jump, $t = t_{J_m}$, the stochastic increment satisfies $dN_k^\sigma(t_{J_m}) = \delta_{kk_m} \delta_{\sigma\sigma_m} \in \{0, 1\}$, so that

$$dC_{ij}^r(t_{J_m}) = \left\langle \mathcal{G}^+ \left[\hat{\rho}_r, \hat{L}_{k_m}^{\sigma_m} \right] \hat{c}_j^\dagger \hat{c}_i \right\rangle_r (t_{J_m}), \quad (\text{III.6.7})$$

where $\mathcal{G}^+$ denotes the adjoint superoperator to $\mathcal{G}$ (see Section I.1.2). The jump update resets the survival probability to unity, $P_{\text{surv}}(t_{J_m}^+) = 1$, while the conditioned covariance matrix is updated according to $C^r(t_{J_m}^+) = C^r(t_{J_m}^-) + dC_{ij}^r(t_{J_m})$. Here, for any time-dependent quantity $A(t)$, we adopt the shorthand $A(t^\pm) = \lim_{s \rightarrow t^\pm} A(s)$. For a jump onto the lead mode k_m ($\sigma_m = +$), then

$$dC^r(t_{J_m}) = \bar{C}^r \left(\frac{dN^+(t_{J_m})}{\bar{C}^r} \right) \bar{C}^r. \quad (\text{III.6.8})$$

Otherwise, for a jump off the lead mode k_m ($\sigma_m = -$)

$$dC^r(t_{J_m}) = -C^r \left(\frac{dN^-(t_{J_m})}{C^r} \right) C^r. \quad (\text{III.6.9})$$

For notational ease, we use the shorthand

$$\left(\frac{dN^+(t)}{\bar{C}^r}\right)_{kk} = \frac{dN_k^+(t)}{1 - C_{kk}^r}, \quad \left(\frac{dN^-(t)}{C^r}\right)_{kk} = \frac{dN_k^-(t)}{C_{kk}^r}. \quad (\text{III.6.10})$$

Combining the smooth no-jump evolution with the instantaneous jump-induced updates, we obtain a stochastic master equation that generates the conditional trajectories C^r

$$dC^r(t) = \left[-(VC^r + C^rV^\dagger) + C^rBC^r \right] dt + \bar{C}^r \left(\frac{dN^+(t)}{\bar{C}^r} \right) \bar{C}^r - C^r \left(\frac{dN^-(t)}{C^r} \right) C^r. \quad (\text{III.6.11})$$

Applying Itô's lemma [135], the stochastic increments satisfy

$$\mathbb{E}[dN_k^\sigma(t) f(\hat{\rho}_r(t))] = dt \mathbb{E} \left[\left\langle \hat{L}_k^{\sigma\dagger} \hat{L}_k^\sigma \right\rangle_r(t) f(\hat{\rho}_r(t)) \right]. \quad (\text{III.6.12})$$

Averaging over trajectories therefore eliminates the stochastic terms and yields the unconditional evolution of the covariance matrix, recovering Eq. (I.3.14).

III.6.2 Monte Carlo trajectory sampling

We now provide a brief overview of the numerical implementation of trajectory sampling based on the covariance matrix, which relies on the following Monte Carlo algorithm:

- Draw a random number R_1 between 0 and 1 (uniformly distributed).
- Time evolve the survival probability P_{surv} and the covariance matrix C^r , conditioned on the measurement record r , for a duration of length t_J such that $P_{\text{surv}}(t_J) = R_1$.
- The probability for a jump prescribed by the collapse operator L_k^σ after time t_J to occur is given by

$$P_{k,\sigma}(t_J) = \frac{\langle \hat{L}_k^{\sigma\dagger} \hat{L}_k^\sigma \rangle_r(t_J)}{1 - R_1}. \quad (\text{III.6.13})$$

- Draw a second random number R_2 between 0 and 1 (uniformly distributed) to determine which jump to perform: the corresponding collapse operator $L_{k_m}^{\sigma_m}$ is the first one for which the ordered sum

$$\sum_{k=1}^L \sum_{\sigma \in \{+, -\}} P_{k,\sigma}(t_J) \geq R_2. \quad (\text{III.6.14})$$

Note that by definition, $\sum_{k=1}^L \sum_{\sigma \in \{+, -\}} P_{k,\sigma}(t_J) = 1$. Finally, the covariance matrix is updated according to the corresponding jump as detailed in Section III.6.1.

Crucially, $P_{k,\sigma}(t_J)$ can be easily expressed in terms of the diagonal elements of the covariance matrix, since $\langle \hat{L}_k^{++} \hat{L}_k^+ \rangle_r = \gamma_k f_k (1 - C_{kk}^r)$ and $\langle \hat{L}_k^{-+} \hat{L}_k^- \rangle_r = \gamma_k (1 - f_k) C_{kk}^r$.

III.6.3 Particle and energy currents

Within our framework, we discuss current fluctuations through the statistics of the integrated current along individual quantum trajectories.

By interpreting the conditional state $\hat{\rho}_r$ as our best guess of the system's state given the measurement record, we compute the corresponding currents with respect to $\hat{\rho}_r$. This state evolves as new measurement data is acquired, reflecting a Bayesian update of the observer's knowledge, consistent with the standard theory of quantum trajectories [51]. We now show that the resulting stochastic currents recover standard average values while producing thermodynamically consistent fluctuations (see Section III.6.4).

As for the average currents, the corresponding stochastic currents can also be expressed in terms of the conditioned covariance matrix C^r . The results presented here assume perfect detection in all measurement channels. A more general treatment, including imperfect detection and alternative measurement configurations, is provided in Ref. [52].

Particle current We define the net particle current conditioned on the measurement record through a continuity equation, and thus, along a single trajectory, $I_N^r(t)$, it is given by

$$\frac{d\langle \hat{N} \rangle_r}{dt} = \sum_{\alpha=1}^{N_R} I_{N_\alpha}^r(t), \quad (\text{III.6.15})$$

with $I_N^r(t) = \sum_{\alpha=1}^{N_R} I_{N_\alpha}^r(t)$. Here, the inferred particle current into lead α reads

$$I_{N_\alpha}^r(t) dt = -\frac{1}{2} \sum_{k=1}^{l_\alpha} \sum_{\sigma \in \{+, -\}} \text{Tr} \left[\hat{N} \mathcal{H} \left[\hat{L}_k^{\sigma\dagger} \hat{L}_k^\sigma \right] \hat{\rho}_r \right] dt + \sum_{k=1}^{l_\alpha} \sum_{\sigma \in \{+, -\}} \text{Tr} \left[\hat{N} \mathcal{G} \left[\hat{L}_k^\sigma \right] \hat{\rho}_r \right] dN_k^\sigma. \quad (\text{III.6.16})$$

We express it in terms of C^r and find

$$I_{N_\alpha}^r(t) dt = \text{Tr} [C^r B_\alpha C^r - B_\alpha C^r] dt + \text{Tr} \left[\bar{C}^r \left(\frac{dN_\alpha^+(t)}{\bar{C}^r} \right) \bar{C}^r \right] - \text{Tr} \left[C^r \left(\frac{dN_\alpha^-(t)}{C^r} \right) C^r \right], \quad (\text{III.6.17})$$

where $B_\alpha = \Gamma_\alpha - 2F_\alpha$ with

$$(\Gamma_\alpha)_{kk}, (F_\alpha)_{kk} = \begin{cases} \Gamma_{kk}, F_{kk} & \text{if } k\text{-mode in lead } \alpha, \\ 0 & \text{otherwise.} \end{cases} \quad (\text{III.6.18})$$

We use the shorthand

$$\begin{aligned} \left(\frac{dN_\alpha^+(t)}{\bar{C}^r} \right)_{kk} &= \begin{cases} \frac{dN_{k,\alpha}^+(t)}{\bar{C}_{kk}^r} & \text{if } k\text{-mode in lead } \alpha, \\ 0 & \text{otherwise,} \end{cases} \\ \left(\frac{dN_\alpha^-(t)}{C^r} \right)_{kk} &= \begin{cases} \frac{dN_{k,\alpha}^-(t)}{C_{kk}^r} & \text{if } k\text{-mode in lead } \alpha, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (\text{III.6.19})$$

The Bayesian character of the current is explicit in Eq. (III.6.17). In particular, even a detected extraction event (corresponding to the last term) does not necessarily yield a current of -1 . The reason is that the covariance matrix encodes also our prior uncertainty about the particle number in the extended system. Consequently, while a quantum jump unambiguously signals the detection of an electron, it does not imply a deterministic change of -1 in the system's occupation. Instead, the inferred change reflects an update of our probabilistic knowledge conditioned on the measurement outcome. Performing an ensemble average we recover Eq. (III.5.12).

Energy current Similarly, along a single trajectory, the energy current into the extended system via lead α , inferred on the basis of the measurement record as discussed in Section III.6.3, is given by

$$\begin{aligned} I_{E_\alpha}^r(t)dt &= \text{Tr} \left[B_\alpha \left(C^r H C^r - \frac{1}{2}(H C^r + C^r H) \right) \right] dt \\ &+ \text{Tr} \left[H \bar{C}^r \left(\frac{dN_\alpha^+(t)}{\bar{C}^r} \right) \bar{C}^r \right] - \text{Tr} \left[H C^r \left(\frac{dN_\alpha^-(t)}{C^r} \right) C^r \right], \end{aligned} \quad (\text{III.6.20})$$

summing to the total inferred energy current

$$I_E^r(t) = \sum_{\alpha=1}^{N_R} I_{E_\alpha}^r(t). \quad (\text{III.6.21})$$

Accordingly, the change in energy in the extended system along a single trajectory in time interval $[0, \tau]$ conditioned on the measurement record is given by

$$\Delta E^r(\tau, 0) = \int_0^\tau I_E^r(t) dt. \quad (\text{III.6.22})$$

Performing an ensemble average, we recover Eq. (III.5.15). Further, we define the total inferred heat current as

$$I_Q^r(t) = \sum_{\alpha=1}^{N_R} I_{Q_\alpha}^r(t), \quad (\text{III.6.23})$$

where $I_{Q_\alpha}^r(t) = I_{E_\alpha}^r(t) - \mu_\alpha I_{N_\alpha}^r(t)$. Analogously, the total heat exchange is given by

$$\Delta Q^r(\tau, 0) = \int_0^\tau I_Q^r(t) dt. \quad (\text{III.6.24})$$

Measurement energy In trajectory-level energy balance relations derived from stochastic master equations with jump operators that are not eigenoperators of the system Hamiltonian, a particular contribution can be identified [269, 270, 142]. The physical interpretation of this term is, however, ambiguous: depending on the perspective adopted, it has been associated either with a “quantum heat” [271] or with a “measurement work” [269, 270, 190].

This measurement energy current along an individual trajectory is defined as [190]

$$\begin{aligned} I_{E_M} dt = & -\frac{dt}{2} \sum_{k=1}^L \sum_{\sigma \in \{+, -\}} \text{Tr} \left[\hat{H} \{ \hat{L}_k^{\sigma\dagger} \hat{L}_k^\sigma, \hat{\rho}_r \} \right] + dt \sum_{k=1}^L \sum_{\sigma \in \{+, -\}} \langle \hat{H} \rangle_r \langle \hat{L}_k^{\sigma\dagger} \hat{L}_k^\sigma \rangle_r \\ & + \sum_{k=1}^L \sum_{\sigma \in \{+, -\}} dN_k^\sigma \frac{\text{Tr} \left[\hat{L}_k^{\sigma\dagger} \hat{H}_{\text{int}} \hat{L}_k^\sigma \hat{\rho}_r \right]}{\langle \hat{L}_k^{\sigma\dagger} \hat{L}_k^\sigma \rangle_r} \\ & + \frac{1}{2} \sum_{k=1}^L \sum_{\sigma \in \{+, -\}} dN_k^\sigma \frac{\text{Tr} \left[\{ \hat{L}_k^{\sigma\dagger} \hat{L}_k^\sigma, \hat{H}_0 \} \hat{\rho}_r \right]}{\langle \hat{L}_k^{\sigma\dagger} \hat{L}_k^\sigma \rangle_r} - \sum_{k=1}^L \sum_{\sigma \in \{+, -\}} dN_k^\sigma \langle \hat{H} \rangle_r. \end{aligned} \quad (\text{III.6.25})$$

For the contribution through lead α we find

$$\begin{aligned} I_{E_{M,\alpha}} dt = & \text{Tr} \left[B_\alpha \left(C^r H C^r - \frac{1}{2} (H C^r + C^r H) \right) \right] dt \\ & + \text{Tr} \left[\left(\frac{dN_\alpha^+}{\bar{C}^r} \right) \bar{C}^r H_{\text{int}} \bar{C}^r \right] - \text{Tr} \left[\left(\frac{dN_\alpha^-}{C^r} \right) C^r H_{\text{int}} C^r \right] \\ & - \frac{1}{2} \text{Tr} \left[\left(\frac{dN_\alpha^+}{\bar{C}^r} \right) (\bar{C}^r H_0 C^r + C^r H_0 \bar{C}^r) \right] \\ & - \frac{1}{2} \text{Tr} \left[\left(\frac{dN_\alpha^-}{C^r} \right) (\bar{C}^r H_0 C^r + C^r H_0 \bar{C}^r) \right]. \end{aligned} \quad (\text{III.6.26})$$

The total measurement energy current is obtained simply as the sum over the contributions from the different leads

$$I_{E_M} = \sum_{\alpha=1}^{N_R} I_{E_{M,\alpha}}. \quad (\text{III.6.27})$$

For the ensemble average, we find

$$E[I_{E_M}] = -\frac{1}{2} \text{Tr} [\Gamma(H_{\text{int}}C + CH_{\text{int}})]. \quad (\text{III.6.28})$$

Interestingly, upon inserting the explicit form of the system-lead interaction (see Chapter III.5),

$$\hat{H}_{\text{int}} = \hat{H}_{\text{SL}}(t) = \sum_{\alpha=1}^{N_R} \sum_{k=1}^{l_\alpha} \kappa_{\alpha,k} \hat{d}_{p_\alpha}^\dagger \hat{a}_k + \text{h.c.}, \quad (\text{III.6.29})$$

one obtains

$$E[I_{E_M}] = -\frac{1}{2} \sum_{\alpha=1}^{N_R} \sum_{k=1}^{l_\alpha} \gamma_{\alpha,k} \left\langle \kappa_{\alpha,k} \hat{d}_{p_\alpha}^\dagger \hat{a}_k + \kappa_{\alpha,k}^* \hat{a}_k^\dagger \hat{d}_{p_\alpha} \right\rangle. \quad (\text{III.6.30})$$

The quantity $E[I_{E_M}]$ therefore coincides precisely with the contribution to the energy current arising from coherent hopping between the lead modes and their respective coupling sites in the central system, see Eq. (III.5.16). Terms of this form have appeared in a variety of related settings, including the repeated-interactions framework [138], which we introduced in Section I.2.4.3.

III.6.4 Entropy production

In Section I.2.6, we discussed the entropy produced in the TPM scheme,

$$\Sigma_{\text{tot}}(r_{[0,\tau]}) = \log \left(\frac{p_n^0}{p_m^\tau} \right) + \Pi_{r_{(0,\tau)}}, \quad (\text{III.6.31})$$

as well as the corresponding integral fluctuation theorem. In the following, we show that the statistics of entropy production along single trajectories in the mesoscopic-leads formalism also satisfy the integral fluctuation theorem, thereby demonstrating thermodynamic consistency at the level of fluctuations.

To this end, we now show how to efficiently obtain the projection probabilities p_n^0 and p_m^τ , as well as the covariance matrix of the post-measurement state $C^r(0)$ following the first projective measurement.

III.6.4.1 Efficient computation along trajectories

Probability for the forward trajectory to start in $|s_n\rangle$

We assume the initial state of the extended system, $\hat{\rho}(0)$, to be Gaussian. A convenient way to express such a state is in the form

$$\begin{aligned}\hat{\rho}(0) &= \frac{e^{-\vec{c}^\dagger \cdot M(0) \cdot \vec{c}}}{Z(0)}, \quad \vec{c} = (\hat{c}_1, \hat{c}_2, \dots, \hat{c}_{N_S+L}), \\ &= \prod_{i=1}^{N_S+L} \frac{e^{-\mu_i^0 \hat{d}_i^\dagger \hat{d}_i}}{Z_i(0)},\end{aligned}\tag{III.6.32}$$

where μ_i^0 are the eigenvalues of $M(0)$, and $Z_i(0) = 1 + e^{-\mu_i^0}$. The 2^{N_S+L} eigenvalues of $\hat{\rho}(0)$ are

$$p_n^0 = \prod_{i=1}^{N_S+L} \frac{e^{-\mu_i^0 s_n^i}}{Z_i(0)}.\tag{III.6.33}$$

Here, $s_n = s_n^1 s_n^2 \dots s_n^{N_S+L}$ is a binary string, with $s_n^i \in \{0, 1\}$. Further, $\mu_i^0 = (U^\dagger M(0) U)_{ii'}$, and U diagonalises the covariance matrix of the initial state $C(0)$. The 2^{N_S+L} eigenvectors of $\hat{\rho}(0)$ are given by

$$|s_n\rangle = \prod_{i=1}^{N_S+L} (\hat{d}_i^\dagger)^{s_n^i} |0\rangle.\tag{III.6.34}$$

It is important to note that fermionic Fock states are Gaussian as they are connected to the vacuum by a Gaussian unitary

$$|s_n\rangle = U_s |0\rangle, \quad U_s = \prod_{i=1}^{N_S+L} (\hat{d}_i^\dagger \hat{d}_i)^{s_n^i},\tag{III.6.35}$$

and

$$\hat{d}_j' = U_s \hat{d}_j U_s^\dagger = \begin{cases} \pm \hat{d}_j & (s_m^j = 0), \\ \pm \hat{d}_j^\dagger & (s_m^j = 1), \end{cases}\tag{III.6.36}$$

where $\pm$ stands for a phase factor which is not relevant for whether state is Gaussian or not. We then find $\{\hat{d}_j', \hat{d}_k'^\dagger\} = \delta_{jk}$, and so U_s is Gaussian. The probability of projecting $\hat{\rho}(0)$ onto its eigenstate $|s_n\rangle$ in the first projective measurement is given p_n^0 .

Covariance matrix after the first projective measurement

Given the covariance matrix $C(0)$, computed with respect to the ensemble-averaged initial state $\hat{\rho}(0)$, we can diagonalise it to obtain the single-particle eigenmodes of the extended system. This is done by finding the unitary matrix U_0 that di-

agonalises $C(0)$, i.e. $C(0) = U_0 \Lambda U_0^\dagger$, where $\Lambda = \text{diag}(\text{eigenvals}(C(0)))$. Each eigenvalue thus corresponds to the occupation probability of one of the eigenmodes defined by U_0 . The covariance matrix $C^r(0)$, evaluated with respect to the eigenstate $|s_n\rangle$ of the initial density matrix $\hat{\rho}(0)$, can be constructed from the diagonal occupation matrix in the eigenmode basis as $C^r(0) = U_0 n_{s_n} U_0^\dagger$, where the elements of n_{s_n} are given by $(n_{s_n})_{ij} = \langle s_n | \hat{d}_j^\dagger \hat{d}_i | s_n \rangle$.

The set of all such matrices $\{n_{s_i}\}$ corresponds to the 2^{N_S+L} possible diagonal matrices whose entries are binary strings of length $N_S + L$, with each entry being either 0 (unoccupied) or 1 (occupied). For example, for $N_S + L = 2$, the set is $\{n_{s_i}\} = \{\text{diag}(0,0), \text{diag}(0,1), \text{diag}(1,0), \text{diag}(1,1)\}$. This construction provides an explicit way to obtain the covariance matrix of each possible initial eigenstate.

Probability to project into m^{th} - eigenstate of $\hat{\rho}(\tau)$

Since the ensemble average final state of the extended system $\hat{\rho}(\tau)$ is Gaussian, it takes the form

$$\begin{aligned} \hat{\rho}(\tau) &= \frac{e^{-\vec{c}^\dagger \cdot M(\tau) \cdot \vec{c}}}{Z(\tau)}, \quad \vec{c} = (\hat{c}_1, \hat{c}_2, \dots, \hat{c}_{N_S+L}), \\ &= \prod_{i=1}^{N_S+L} \frac{e^{-\mu_i^\tau \hat{d}_i^\dagger \hat{d}_i}}{Z_i(\tau)}, \end{aligned} \quad (\text{III.6.37})$$

where μ_i^τ are the eigenvalues of $M(\tau)$, and $Z_i(\tau) = 1 + e^{-\mu_i^\tau}$. The eigenvalues of $\hat{\rho}(\tau)$ are

$$p_m^\tau = \prod_{i=1}^{N_S+L} \frac{e^{-\mu_i^\tau s_m^i}}{Z_i(\tau)}, \quad (\text{III.6.38})$$

and the eigenvectors of $\hat{\rho}(\tau)$ are given by

$$|s_m\rangle = \prod_{i=1}^{N_S+L} (\hat{d}_i^\dagger)^{s_m^i} |0\rangle. \quad (\text{III.6.39})$$

Our aim is now to compute the probability of the final state of the trajectory $\hat{\rho}_r(\tau)$ to be projected into the eigenstate $|s_m^o\rangle$ (here in the original, operational basis) of the ensemble average density matrix of the final state $\hat{\rho}(\tau)$, with $|s_m^o\rangle \langle s_m^o| = \Pi_m^\tau$,

$$p_m^{r\tau} = \text{Tr} [\hat{\rho}_r(\tau) |s_m^o\rangle \langle s_m^o|]. \quad (\text{III.6.40})$$

Importantly, we are looking for a way to compute it given only the covariance matrix $C(\tau)$ of the ensemble average of the final state $\hat{\rho}(\tau)$ and the covariance matrix $C^r(\tau)$ of the final state of the trajectory $\hat{\rho}_r(\tau)$.

To this end, we first write $\hat{\rho}_r(\tau)$ in the eigenmode basis of $\hat{\rho}(\tau)$

$$\hat{\rho}'_r(\tau) = \frac{e^{-\vec{d}^\dagger \cdot \mathbf{Q} \cdot \vec{d}}}{Y}, \quad (\text{III.6.41})$$

so that $p_m^{r\tau} = \text{Tr} [\hat{\rho}'_r(\tau) |s_m\rangle \langle s_m|]$.

Using the functional determinant formula [272, 273], which maps a many-body expectation value onto a determinant in single-particle space, we find

$$p_m^{r\tau} = \frac{1}{Y} \det [1 - n_{s_m} + n_{s_m} e^{-Q}], \quad (\text{III.6.42})$$

where $|s_m\rangle = U |s_m^0\rangle$ and $n_{s_m} = \text{diag}(\{s_m^1, \dots, s_m^{N_S+L}\})$. The above expression can be further simplified by expressing both Q and Y in terms of the covariance matrix $C^{r'}(\tau)$, defined in the eigenbasis of $C(\tau)$. This representation helps to avoid numerical instabilities that may arise from eigenvalues in the spectrum that are close to 0 or 1. That is,

$$Q = \log \left[\frac{1 - C^{r'}(\tau)}{C^{r'}(\tau)} \right], \quad Y = \det \left[\frac{1}{1 - C^{r'}(\tau)} \right], \quad (\text{III.6.43})$$

so that

$$p_m^{r\tau} = \det [(1 - n_{s_m})(1 - C^{r'}(\tau)) + n_{s_m} C^{r'}(\tau)]. \quad (\text{III.6.44})$$

Probability for the backward trajectory to start in $|s_m\rangle$

What remains to be determined is the probability for the reverse trajectory to begin in the eigenstate $|s_m\rangle$ of the ensemble-averaged final state $\hat{\rho}(\tau)$. This probability corresponds to the eigenvalue p_m^τ of $\hat{\rho}(\tau)$ associated with $|s_m\rangle$, as defined in Eq. (III.6.38). It can therefore be computed using the same procedure outlined in Section III.6.4.1.

III.6.4.2 Uncertainty and martingale entropy production

In Section I.2.6.4, we introduced the uncertainty and martingale entropy production (see Eq. (I.2.85) and Eq. (I.2.83), respectively), which each fulfil an integral fluctuation theorem [205]. To compute these quantities efficiently, we first note that the entropy flux $\Pi_{r(0,\tau)}$ (see Eq. (I.2.78)) can be evaluated directly from the measurement record. Moreover, in Section III.6.4.1, we established that the projection probabilities p_n^0 and p_m^τ can be expressed in terms of covariance matrices. In the following, we show that the remaining contribution,

$$S_\rho(\tau) = -\log(\text{Tr}[\hat{\rho}_r(\tau) \hat{\rho}(\tau)]), \quad (\text{III.6.45})$$

where $\hat{\rho}_r(\tau)$ denotes the state conditioned on the measurement record prior to the second projective measurement and $\hat{\rho}(\tau)$ the corresponding unconditional final state, can likewise be computed solely in terms of covariance matrices. To this end, note that for two Gaussian fermionic states $\hat{\rho}_1 = e^{-\vec{c}^\dagger \cdot M_1 \cdot \vec{c}} / Z_1$ and $\hat{\rho}_2 = e^{-\vec{c}^\dagger \cdot M_2 \cdot \vec{c}} / Z_2$, using the Baker-Campbell-Hausdorff formula,

$$e^{-\vec{c}^\dagger \cdot M_1 \cdot \vec{c}} e^{-\vec{c}^\dagger \cdot M_2 \cdot \vec{c}} = -\vec{c}^\dagger \cdot M_1 \cdot \vec{c} - \vec{c}^\dagger \cdot M_2 \cdot \vec{c} - \vec{c}^\dagger \cdot [M_1, M_2] \cdot \vec{c} + \dots$$

One can then show that $e^{-\vec{c}^\dagger \cdot M_1 \cdot \vec{c}} e^{-\vec{c}^\dagger \cdot M_2 \cdot \vec{c}} = e^{-\vec{c}^\dagger \cdot \chi \cdot \vec{c}}$, with $\chi = \log(e^{M_1} e^{M_2})$. Using the relation

$$\text{Tr} \left[e^{-\vec{c}^\dagger \cdot \chi \cdot \vec{c}} \right] = \det [1 + e^{-\chi}], \quad (\text{III.6.46})$$

we find

$$\text{Tr} [\hat{\rho}_1 \hat{\rho}_2] = \frac{\det (1 + e^{-M_1} e^{-M_2})}{Z_1 Z_2}, \quad (\text{III.6.47})$$

so that

$$S_\rho(\tau) = -\log \left(\frac{\det (1 + e^{-M_{\hat{\rho}_r(\tau)}} e^{-M_{\hat{\rho}(\tau)}})}{Z_{\hat{\rho}_r(\tau)} Z_{\hat{\rho}(\tau)}} \right). \quad (\text{III.6.48})$$

III.6.4.3 Integral fluctuation theorems for entropy production

To provide a proof of principle that the integral fluctuation theorems for the total, martingale, and uncertainty entropy productions are correctly verified within our mesoscopic-leads description, we consider the simple setup of a single quantum dot, described by the Hamiltonian $\hat{H}_S = \epsilon \hat{d}^\dagger \hat{d}$, coupled to a fermionic reservoir within the mesoscopic-leads formalism, as illustrated in Fig. III.6.2. For simplicity, we assume the reservoir has a flat spectral density,

$$J(\omega) = \begin{cases} \Gamma, & \omega \in [-\omega_{\max}, \omega_{\max}], \\ 0, & \text{otherwise,} \end{cases} \quad (\text{III.6.49})$$

where Γ denotes the coupling strength between the quantum dot and the reservoir, and $\omega_{\max}$ is a hard cutoff. The lead-mode energies $\{\epsilon_k\}$, the couplings $\{\kappa_k\}$ between the dot and the lead modes, and the damping rates $\{\gamma_k\}$ of the residual reservoirs are chosen following the procedure in Chapter III.5, with $\{\epsilon_k\}$ obtained via linear discretization. Trajectories are generated using the TPM scheme, performing projective measurements in the eigenbasis of the extended system's steady state at both the start and end of each trajectory, as described in Section III.6.4.

We find that the integral fluctuation theorems for $\Sigma_{\text{tot}}(r_{[0,\tau]}), \Sigma_{\text{unc}}(r_{[0,\tau]}),$ and

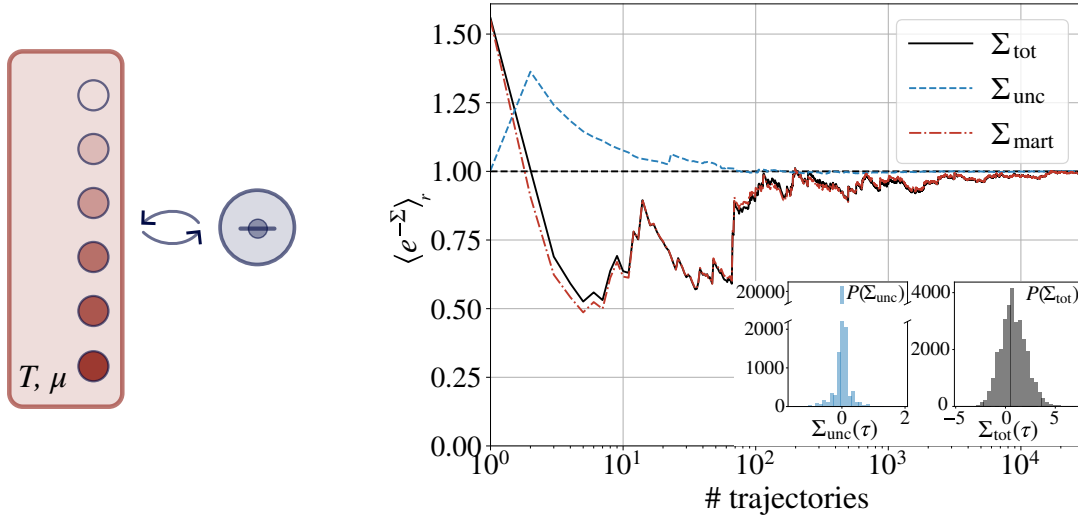


Figure III.6.2: Left: Schematic of a single dot coupled to a fermionic reservoir in the mesoscopic-leads formalism. Right: Verification of the integral fluctuation theorem in the TPM scheme for the total entropy production (black), the uncertainty entropy production (blue) and the martingale entropy production (red) in the steady state. The insets show the distribution of Σ_{tot} (left) and Σ_{unc} (right). The estimated probability densities $P(\Sigma_{\text{tot}})$ and $P(\Sigma_{\text{unc}})$, respectively, are obtained by dividing the number of counts over the total number of trajectories. Parameters: $L = 10$, up to 30000 trajectories, $\omega_{\text{max}} = 1$, $\epsilon = \omega_{\text{max}}/4$, $\Gamma = \omega_{\text{max}}/8$, $T = \omega_{\text{max}}$, $\mu = \omega_{\text{max}}/16$, $\Gamma\tau = 50$. Both figures are adapted from Ref. [52].

$\Sigma_{\text{mart}}(r_{[0,\tau]})$ are each individually satisfied for their respective distributions after the second projective measurement. Specifically, the functionals $\langle e^{-\Sigma_{\text{tot}}(r_{[0,\tau]})} \rangle_r$, $\langle e^{-\Sigma_{\text{unc}}(r_{[0,\tau]})} \rangle_r$, and $\langle e^{-\Sigma_{\text{mart}}(r_{[0,\tau]})} \rangle_r$ all converge to 1 as the number of trajectories in the simulations increases, as shown in Fig. III.6.2. Notably, convergence for the uncertainty entropy production Σ_{unc} is achieved most rapidly, consistent with the findings in [190]. For all cases, approximate convergence occurs after about 100 trajectories, which aligns with the typical scale required for the unconditional density operator in standard quantum trajectory simulations. However, full convergence of Σ_{tot} and Σ_{mart} demands roughly two orders of magnitude more trajectories, reflecting the need to adequately sample the tails of the distributions when probing such detailed statistical properties of the reservoir. By verifying the fluctuation theorems for entropy production, we have confirmed the thermodynamic consistency of the mesoscopic-leads approach at the stochastic level, representing a central result of this Part. Importantly, our framework inherently accounts for measurement conditioning along individual trajectories, enabling an analysis of how entropy production decomposes into contributions from uncertainty and martingale terms. The inset of Fig. III.6.2 displays the estimated distributions of the total and uncertainty entropy production, $P(\Sigma_{\text{tot}})$ and $P(\Sigma_{\text{unc}})$, respectively. The distribution of the total entropy production closely resembles a shifted Gaussian with a positive mean, consistent with the second law. In contrast, the un-

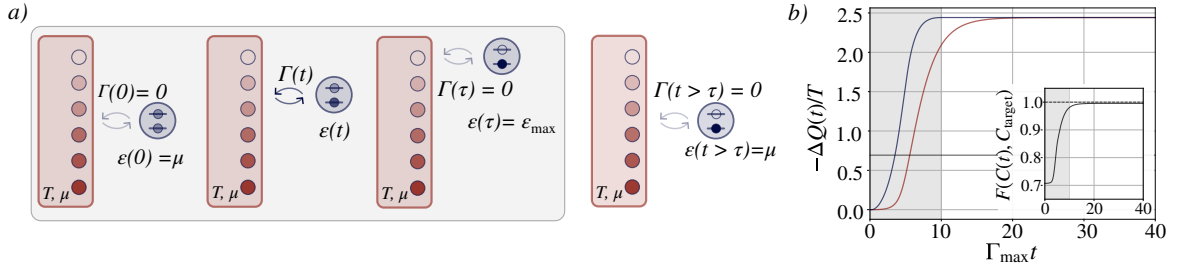


Figure III.6.3: a) Driving protocol of both site energy $\epsilon(t)$ and coupling strength $\Gamma(t)$: the equilibration period (outside the grey shaded region) allows equilibration of the lead modes with respect to their residual reservoirs. b) Time evolution of the dissipated heat from the system (blue) and the extended system (red) during the execution of the driving protocol and equilibration time. As we allow for equilibration of the lead modes to their residual reservoirs after uncoupling from the system site, they agree. The grey line indicates Landauer's bound. The inset shows the time evolution of the fidelity to the target state. Adapted from Ref. [52].

certainty entropy production exhibits a prominent peak near zero, accompanied by smaller secondary peaks on either side. We stress that the capacity to directly sample and visualise such detailed features of the distributions constitutes a major advantage of our approach.

We briefly comment on why entropy production along individual trajectories remains nonzero, and why its average does not vanish, even in this example of an equilibrium process without external driving. This behaviour stems from the definition of the total entropy production (see Eq. (I.2.77)) under a local GKSL master equation, where transitions induced by the baths do not necessarily connect eigenstates of the system. As a result, the relative surprisal of measurement outcomes, $\log(p_n^0/p_m^\tau)$, cannot be unambiguously associated with a specific heat exchange, unlike in a global GKSL treatment. Consequently, even in equilibrium, the total entropy flux exhibits a nontrivial distribution. The nonzero mean arises from the exclusion of the measurement energy (see Eq. (III.6.25)) in Eq. (I.2.77). Including this term would, however, violate the integral fluctuation theorem (I.2.79) [190]. That the average measurement energy (III.6.30) is nonzero is a known subtlety of the mesoscopic-leads approach, discussed previously in Refs. [50, 144, 249]. Importantly, this discrepancy diminishes as the number of lead modes increases. A more detailed discussion of this aspect is provided in Ref. [52].

III.6.5 Heat statistics in fast information erasure

Having established the thermodynamic consistency of our framework, we now turn to a concrete application: investigating heat fluctuations during finite-time information erasure. According to Landauer's principle, there is a fundamental

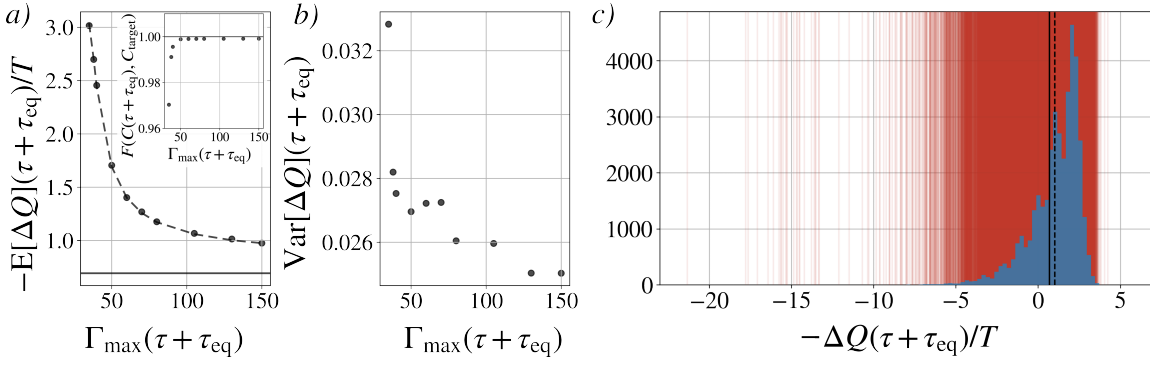


Figure III.6.4: The mean a) and variance b) of the distribution of dissipated heat during the information erasure protocol both decrease as a function of the protocol duration. The full grey line in a) indicates the Landauer bound. The black dots represent data derived from the distribution of dissipated heat obtained through trajectory sampling. In contrast, the dashed line plot illustrates the results obtained when considering the unconditional evolution of the system subject to the erasure protocol. The inset shows the quantum fidelity to the target state as a function of the driving duration. c) Trajectory-sampling of the distribution of dissipated heat (blue) reveals non-Gaussian statistics, even at slow-driving conditions with $\Gamma_{\max}\tau = 100$. The red vertical lines show the dissipated heat along the individual trajectories. The distribution's mean is represented by the black dashed line, while the Landauer bound is indicated by the solid line. We observe large deviations from the mean, highlighted by the markedly different scales in a) and c). Parameters: 40000 trajectories, $L = 20$, $\omega_{\max} = 1$, $T = \omega_{\max}/10$, $\mu = -0.8\omega_{\max}$, $\epsilon_{\max} = 0.8\omega_{\max}$, $\Gamma_{\max} = \epsilon_{\max}/\pi$, $\Gamma_{\max}\tau_{eq} = 30$. Adapted from Ref. [52].

lower bound on the heat cost of erasing a single bit of information:

$$-\Delta Q \geq T \log(2), \quad (\text{III.6.50})$$

where $-\Delta Q$ denotes the heat dissipated into the environment over the course of the erasure. This bound is only reached in the idealised limit of a reversible, isothermal process, which requires an infinite duration. In realistic finite-time protocols, additional thermodynamic contributions arise. Recent works have quantified these corrections [274–278], but predominantly in regimes of weak system-bath coupling or slow driving. Our mesoscopic-leads approach allows us to explore these finite-time effects beyond such limitations, providing access to the full statistics of heat exchange. Specifically, we employ our method to investigate how the speed of driving affects heat fluctuations during information erasure under strong system-bath coupling. We encode a single bit of information in the occupation of a fermionic mode (the “bit mode”) and erase it by varying its time-dependent Hamiltonian, $H_S(t) = \epsilon(t)\hat{d}^\dagger\hat{d}$, while the system remains coupled to a heat bath at temperature T and chemical potential μ . We model this thermal reservoir as a mesoscopic lead coupled to residual reservoirs (see Chapter III.5). Initially, the bit mode and the lead modes are uncorrelated. The bit mode starts with an occupation of $\frac{1}{2}$, while the lead modes are populated according to the

Fermi-Dirac distribution $f(\epsilon, T, \mu)$. The goal of the protocol is to drive the bit mode into the target state with zero occupation, while restoring the lead modes to their original Fermi-Dirac populations, again uncorrelated with the bit mode. Achieving this corresponds to the successful erasure of a single bit of information. Accordingly, an external drive is applied, so that the energy of the mode changes as

$$\epsilon(t) = \mu + (\epsilon_\tau - \mu) \left(\frac{t}{\tau} - \frac{\sin(2\pi t/\tau)}{2\pi} \right), \quad (\text{III.6.51})$$

where ϵ_τ denotes the energy splitting of the bit mode at the final time. The overall coupling Γ also varies with time as

$$\Gamma(t) = \frac{\epsilon_\tau}{\pi} \sin^2 \left(\pi \frac{t}{\tau} \right). \quad (\text{III.6.52})$$

Here, we again assume a flat spectral density

$$\mathcal{J}(\omega_k) = \begin{cases} \Gamma(t) & \omega_k \in [-\omega_{\max}, \omega_{\max}], \\ 0 & \text{otherwise,} \end{cases} \quad (\text{III.6.53})$$

where $\omega_{\max}$ is a hard cut-off. As before, we linearly discretise the reservoir into L energy modes so that $\Delta\omega = \omega_{k+1} - \omega_k = 2\omega_{\max}/L$ and choose the damping rate $\gamma_k = \Delta\omega$. The coupling strength between the bit mode and the k -th lead modes follows $\kappa_k(t) = \sqrt{\frac{\Gamma(t)\Delta\omega}{2\pi}}$. Once the driving protocol concludes, the lead modes are allowed to relax for an additional equilibration time τ_{eq} . This step ensures that the erasure process constitutes a full cycle, returning the lead modes to their initial state, where they remain uncorrelated and populated according to the Fermi-Dirac distribution $f(\epsilon, T, \mu)$. Moreover, it guarantees that the total average heat transferred from the extended system to the residual reservoirs,

$$\Delta Q^r(t, 0) = \Delta E^r(t, 0) - \mu \Delta N^r(t, 0), \quad (\text{III.6.54})$$

coincides with the total average heat exchanged between the charge bit and the lead modes,

$$\Delta Q_S^r(t, 0) = \int_0^t dt' \left[J^E(t') - \mu J^N(t') \right], \quad (\text{III.6.55})$$

as illustrated in Fig. III.6.3 b). Here, the mesoscopic-leads formalism defines the internal average particle and energy currents exchanged between the system and lead α as [144]

$$J_\alpha^N(t) = i \langle [N_{L_\alpha}, H_{SL_\alpha}] \rangle, \quad J_\alpha^E(t) = i \langle [H_{L_\alpha}, H_{SL_\alpha}] \rangle + \text{Tr} \left[H_{SL_\alpha} \mathcal{L}_\alpha[\rho] \right], \quad (\text{III.6.56})$$

where $N_{L_\alpha} = \sum_{k=1}^{l_\alpha} a_{\alpha,k}^\dagger a_{\alpha,k}$, and the total currents are given by $J^E(t) = \sum_{\alpha=1}^{N_R} J_\alpha^E(t)$ and $J^N(t) = \sum_{\alpha=1}^{N_R} J_\alpha^N(t)$. We find that the average heat dissipated during the protocol approaches Landauer's limit as the driving speed is decreased, in agreement with theoretical expectations (see Fig. III.6.4 a)). Further, the quantum fidelity to the target state, which for fermionic Gaussian states is given by [249, 279]

$$F(\hat{\rho}_1 || \hat{\rho}_2) = \text{Tr} \left[\sqrt{\hat{\rho}_1 \hat{\rho}_2} \right] = \frac{\det \left(\mathbb{I} + \sqrt{e^{-M_1} e^{-M_2}} \right)}{\sqrt{Z_1 Z_2}}, \quad (\text{III.6.57})$$

approaches unity, as shown in the inset of Fig. III.6.4 a)). Although fluctuations of the internal particle current in mesoscopic leads can be computed using full counting statistics [248], an analogous treatment for the external energy and heat currents is not yet established. In our approach, we access the distribution of dissipated heat through direct sampling and observe that, as the protocol duration τ increases, the variance of the dissipated heat decreases, as illustrated in Fig. III.6.4b)). Fig. III.6.4c) presents a histogram of the heat distribution obtained from trajectory sampling, revealing pronounced non-Gaussian features even for slow-driving protocols. Previous studies in the weak-coupling regime have shown that non-Gaussian heat statistics emerge during slow information erasure whenever quantum coherence is generated along the protocol [274, 276]. In our setup, strong system-bath coupling naturally induces coherence. However, unlike the weak-coupling case, we observe a negatively skewed heat distribution. Moreover, rare trajectories exhibit extremely large heat flow into the system, while the majority comply with the expectation of heat dissipation into the bath, consistent with the bound (III.6.50). But because our bath model fundamentally differs from that in Ref. [274], exact agreement is not anticipated. More importantly, Fig. III.6.4c) highlights the novel and nontrivial features of heat fluctuations that may arise under strong system-bath coupling. We further note that the heat distribution shown in Fig. III.6.4c) displays fine structure arising from the discrete spectrum of the lead. Such features can be smoothed out either by increasing the number of lead modes or by employing coarser histogram binning. Consequently, the number of lead modes effectively sets the fundamental energy resolution achievable within our approach.

III.6.6 Conclusion

In this chapter, we have presented a trajectory-level framework for the quantum stochastic thermodynamics of fermionic systems coupled to thermal reservoirs. Our approach accounts for non-Markovian effects, accommodates strong system-environment coupling, and remains valid under arbitrary driving [52].

Within the mesoscopic-leads approach, we derived thermodynamically consistent analytic expressions for the particle, energy, and entropic currents exchanged between the extended system and its residual reservoirs, formulated directly at the trajectory level. Importantly, these external currents are the natural theoretical counterparts of reservoir currents measured in mesoscopic transport experiments. In particular in the strong-coupling regime, where the distinction between internal and external exchanges becomes blurred, our framework ensures that the experimentally measurable integrated currents, e.g., those detected by a standard ammeter, can be directly compared with theoretically computed stochastic currents, without requiring the identification or resolution of individual mesoscopic leads or residual reservoirs. In addition, we verified the thermodynamic consistency of the formalism by showing that several key fluctuation relations hold: integral fluctuation theorems for total, uncertainty, and martingale entropy production are satisfied. To illustrate the approach, we applied it to finite-time information erasure in a charge qubit, and sampled the heat statistics as a function of the driving speed. This captured non-Gaussian statistics of dissipated heat, even in the slow-driving regime. Because our method accesses individual quantum trajectories, it naturally reveals features beyond conventional cumulant-based analyses, including the microscopic mechanisms underlying rare fluctuations [274]. A further strength of our approach is computational efficiency. Because the mesoscopic-leads method requires only the covariance matrix of the extended system (for quadratic Hamiltonians), stochastic trajectories for fermionic currents can be generated at low numerical cost. The general expressions for stochastic currents derived here, however, are not restricted to non-interacting systems: they can in principle be combined with tensor-network methods to address interacting problems [50]. Although sampling eigenvalues of many-body density operators is challenging in tensor-network representations (and was only needed here to evaluate entropy production statistics), trajectory sampling of charge and energy currents involves only local operations and is therefore compatible with tensor-network algorithms. Although we focused on reservoirs with flat spectral densities for clarity, the method naturally extends to structured environments with non-trivial spectra.

Finally, the framework we developed opens a path toward applying the full toolbox of continuous quantum measurement and feedback control in thermodynamically consistent settings at the mesoscopic scale.

PART IV:

Information geometry in thermodynamics

Chapter IV.7

Temporal distinguishability and thermodynamic cost

This chapter is based on Refs. [10, 69, 71, 78, 70, 124, 280].

In this thesis, we explore the fundamental relationship between the information geometry of quantum state spaces and thermodynamics from two complementary perspectives. First, we consider far-from-equilibrium dynamics and nonstationary states. In this regime, the system's entropy dynamics are shown to be intimately linked to information-geometric notions of temporal distinguishability (see Chapter IV.8). Second, we study slowly driven transformations between NESSs. Here, the thermodynamic cost of externally controlled driving can be expressed as the path action associated with the so-called thermodynamic metric (see Section IV.7.5). In Chapter V.10, we use this framework to reveal dynamical universality in DQPTs.

These geometric connections naturally emerge when one focuses on two special manifolds in state space¹. The first manifold comprises states uniquely parameterised by time, obtained by evolving a fixed initial state under a given Liouvillian generator. The second consists of NESSs associated with a family of Liouvillian generators that depend on external control parameters, with each NESS uniquely determined by the chosen parameters. Both manifolds are equipped with a metric given by, or directly related to, the (quantum) Fisher information metric [281], which quantifies how sensitively a (quantum) state responds to infinitesimal parameter variations.

To build intuition, we first examine the relation between temporal Fisher information and dynamics before turning to its connection with thermodynamics: we show that the temporal Fisher information of a time-evolving state provides an

¹These may consist of probability distributions in classical settings or density matrices in the quantum case.

intrinsic measure of the system's speed of evolution. We develop this intuition from two complementary viewpoints: one based on the celebrated Cramér-Rao bound² [282, 283], and the other rooted in a geometric interpretation [64].

We consider a system that can occupy a finite set of M discrete configurations $\{x_i\}_{i=1}^M$. Its dynamics are described by the time-dependent probabilities $p_{x_i}(t)$ of being in configuration x_i , collected in the probability vector, the state,

$$\vec{p}(t) = (p_{x_1}(t), p_{x_2}(t), \dots, p_{x_M}(t))^T, \quad (\text{IV.7.1})$$

which evolves, for example, either due to external driving or as a result of relaxation from an initial nonstationary state. In many situations, our interest lies in configuration-dependent observables $O(x_i)$ and their ensemble averages,

$$\langle O \rangle_t = \sum_{i=1}^M O(x_i) p_{x_i}(t), \quad (\text{IV.7.2})$$

as well as in their fluctuations, $\langle \Delta O^2 \rangle_t = \langle O^2 \rangle_t - \langle O \rangle_t^2$. The Cramér-Rao bound [282, 283] connects these two: if time t is treated as the parameter and $\langle O \rangle_t$ is used as a biased estimator of t , it takes the form

$$|\partial_t \langle O \rangle_t| \leq \sqrt{\langle \Delta O^2 \rangle_t} v_F(t), \quad (\text{IV.7.3})$$

where $v_F(t) = \sqrt{F(t)}$ and

$$F(t) = \sum_{i=1}^M \frac{(\partial_t p_{x_i}(t))^2}{p_{x_i}(t)} \quad (\text{IV.7.4})$$

is the Fisher information with respect to the time parameter. We refer to this quantity as the temporal Fisher information. It measures how sensitively a state responds to infinitesimal changes in time, thereby quantifying the rate at which the state evolves under a given dynamical process. Importantly, while the inequality in Eq. (IV.7.3) holds for any observable O , the quantity $v_F(t)$ depends only on the underlying dynamics. The Fisher information therefore defines an intrinsic speed of evolution: a fundamental upper bound on how rapidly any observable can change, measured relative to its typical fluctuations. This interpretation was first brought forward by Ito and Dechant in 2020 [64] and forms part of a broader

²The Cramér-Rao bound commonly arises in discussions of parameter estimation based on measurement outcomes. When measurements are performed, different outcomes occur according to a probability distribution that depends on an unknown parameter. Metrology concerns both the sampling of this distribution and the construction of an estimator for the parameter from the observed data. The Cramér-Rao bound states that the variance of any unbiased estimator is bounded from below by the inverse of the Fisher information associated with the measurement statistics.

line of research exploring the interplay between dynamical evolution and information geometry, see, e.g., Refs. [284–286].

We now turn to the geometric viewpoint. By Čencov’s theorem, the Fisher information endows the space of probability distributions with a unique (up to a constant factor) monotone Riemannian metric³ [287, 288], from which one can define an infinitesimal line element

$$ds = \frac{1}{2} \sqrt{F(t)} dt. \quad (\text{IV.7.5})$$

In this form, the analogy with the familiar relation, distance = speed $\times$ time, becomes explicit: the square root of the Fisher information with respect to time plays the role of an instantaneous speed in probability space. Having now identified a line element, the statistical distance between two states follows as the integral of ds along the path connecting them.

We now move beyond treating time itself as the relevant parameter and instead consider an externally controlled, time-dependent parameter⁴. Specifically, we consider a state $\vec{p}(t) = \vec{p}(\theta(t))$, where the parameter θ varies smoothly in time. Then, $\partial_t \vec{p}(t) = \dot{\theta} \partial_\theta \vec{p}(\theta)$, and correspondingly, $F(t) = \dot{\theta}^2 F(\theta)$, where $F(\theta)$ denotes the Fisher information of the state $\vec{p}(\theta)$ with respect to the parameter θ . In this formulation, all explicit time dependence enters solely through the parameter velocity $\dot{\theta}$, and the Fisher information $F(\theta)$ defines a monotone Riemannian metric on the space of states parameterised by θ . States exhibiting this type of parametric dependence arise as unique stationary states of parameterised dynamical generators, such as Liouvillian superoperators or classical Pauli rate matrices.

The remainder of this chapter introduces the Fisher information in a systematic manner. We begin by reviewing the Fisher information in classical systems (see Section IV.7.1), before extending the discussion to the quantum setting (see Section IV.7.2) and highlighting the key conceptual differences. We then examine important key connections between thermodynamics and information geometry. First, we discuss the link between temporal Fisher information and classical stochastic thermodynamics (see Section IV.7.3), which underpins the results presented in Chapter IV.8. We then highlight, in Section IV.7.4, the connections between the (Q)FI, fluctuations, and response functions, linking information geometry to the first theme of this thesis. Finally, we explore the relation between information geometry and thermodynamic geometry (see Section IV.7.5), provid-

³A metric g_p is called monotone if it contracts under any stochastic map Φ , i.e., $g_{\Phi(p)} \leq g_p$ for all states p . A Riemannian metric is a smoothly varying inner product $g_p : T_p M \times T_p M \rightarrow \mathbb{R}$ on the tangent space at each point p , satisfying bilinearity, symmetry, and positive definiteness $g_p(X, X) > 0$ for all $X \neq 0$. This contractivity ensures that the resulting notion of distance is compatible with the data-processing inequality.

⁴For simplicity, we restrict to the case in which only a single parameter is varied. The generalization to the multiparameter case, however, is straightforward.

ing the conceptual foundation for some of the results developed in Chapter V.10.

IV.7.1 Fisher information metric

In classical information geometry, we regard probability distributions as points on a statistical manifold. We access the geometry of this manifold through the Fisher information metric, the unique monotone Riemannian metric on the space of probability distributions [287, 288]. To describe the evolution of a classical system in probabilistic terms, consider a set of N discrete states X . The state of the system at time t is given by the probability vector $\vec{p}(t) \in \mathbb{R}^N$, with components $(\vec{p}(t))_x = p_x(t)$, with $x \in X$, where $p_x(t)$ is the probability to be in configuration x at time t . As the system evolves over a time interval of duration τ , it traces a path through the space of all possible states. Suppose now that the evolution of the system is governed by a finite set of externally controlled parameters, denoted by $\vec{\theta}(t) = (\theta_1(t), \dots, \theta_M(t))^T$. The family of probability distributions parameterised by these controls defines a statistical manifold, $\Theta = \{ \vec{p}(\vec{\theta}) : \vec{\theta} \in \mathbb{R}^M \}$, whose coordinates are given by the control parameters $\vec{\theta}$. We quantify distances on this manifold through the Fisher information metric [281, 289, 282, 290]

$$m_{ij}(\vec{\theta}) = \sum_x p_x(\vec{\theta}) \frac{\partial \log p_x(\vec{\theta})}{\partial \theta_i} \frac{\partial \log p_x(\vec{\theta})}{\partial \theta_j}, \quad (\text{IV.7.6})$$

which measures the infinitesimal statistical distinguishability between neighbouring distributions, and gives rise to the line element via $ds^2 = \frac{1}{4} \sum_{i,j} m_{ij}(\vec{\theta}) d\theta_i d\theta_j$. To further build intuition before we extend this framework to quantum settings, it is useful to introduce the score function, associated with the parameter θ_i ,

$$S_{i,x}(\vec{\theta}) = \frac{\partial}{\partial \theta_i} \log p_x(\vec{\theta}). \quad (\text{IV.7.7})$$

The score quantifies how sensitively the log-likelihood changes under an infinitesimal variation of the parameter θ_i , and thus captures the direction in which the probability distribution deforms when θ_i is perturbed. Then, the Fisher information matrix takes the form $m_{ij}(\vec{\theta}) = \sum_x p_x(\vec{\theta}) S_{i,x}(\vec{\theta}) S_{j,x}(\vec{\theta})$, revealing it as the covariance matrix of the scores. Alternatively, we may identify the Fisher information as a probability-weighted inner product on the tangent space, defined by $\langle f, g \rangle_{\vec{p}} = \sum_x p_x(\vec{\theta}) f(x) g(x)$, so that the metric tensor elements take the compact form

$$m_{ij}(\vec{\theta}) = \langle \vec{S}_i, \vec{S}_j \rangle_{\vec{p}}. \quad (\text{IV.7.8})$$

The line element ds^2 is thus determined by the inner-product structure induced by the distribution itself. The statistical distance accumulated along a path γ ,

parameterised by t , is given by

$$\mathcal{L}[\gamma] = \int_{\gamma} ds = \frac{1}{2} \int_0^{\tau} \sqrt{\sum_{i,j} m_{ij}(\vec{\theta}(t)) \frac{d\theta_i}{dt} \frac{d\theta_j}{dt}} dt. \quad (\text{IV.7.9})$$

Further, the geometric path action is given by

$$\mathcal{S}[\gamma] = \int_{\gamma} ds^2 = \frac{1}{4} \int_0^{\tau} \sum_{i,j} m_{ij}(\vec{\theta}(t)) \frac{d\theta_i}{dt} \frac{d\theta_j}{dt} dt. \quad (\text{IV.7.10})$$

Note that here, the metric m_{ij} plays the role of a mass tensor, while the derivatives $\dot{\theta}_i = d\theta_i/dt$ act as velocities in parameter space, so that $\mathcal{S}$ formally resembles the kinetic energy integrated along the path. Further, we introduce the statistical divergence along the path:

$$\mathcal{J}[\gamma] = \tau \mathcal{S}[\gamma]. \quad (\text{IV.7.11})$$

To relate these quantities, we apply the standard Cauchy-Schwarz inequality for integrals⁵, and find the bound

$$\mathcal{J}[\gamma] = \tau \mathcal{S}[\gamma] \geq \mathcal{L}^2[\gamma]. \quad (\text{IV.7.12})$$

Equality is achieved if and only if the “velocity”, i.e. $\sum_{i,j} m_{ij}(\vec{\theta}(t)) \dot{\theta}_i \dot{\theta}_j$, is constant along the path, corresponding to a geodesic in the metric m_{ij} . This geodesic path, the path of shortest length, satisfies the standard geodesic equation

$$\frac{d^2\theta_i}{dt^2} + \sum_{j,k} \Gamma_{jk}^i \frac{d\theta_j}{dt} \frac{d\theta_k}{dt} = 0, \quad (\text{IV.7.13})$$

where Γ_{jk}^i are the Christoffel symbols associated with the metric m_{ij} .

For two probability distributions, represented by the vectors $\vec{p} = \{p_i\}$ and $\vec{q} = \{q_i\}$, the geodesic distance between them can be written in closed form as

$$\mathcal{L}_{\text{geo}}(\vec{p}, \vec{q}) = \arccos\left(\sum_i \sqrt{p_i q_i}\right), \quad (\text{IV.7.14})$$

where the sum inside the arccos is known as the Bhattacharyya coefficient or classical fidelity.

⁵The standard Cauchy-Schwarz inequality for integrals implies that for any square-integrable function $f(t) \geq 0$, $(\int_0^{\tau} f(t) dt)^2 \leq \tau \int_0^{\tau} f(t)^2 dt$.

	Classical	Quantum
State	$\vec{p}(\vec{\theta})$	$\hat{\rho}(\vec{\theta})$
Tangent vector	$\partial_i \vec{p}$	$\partial_i \hat{\rho}$
Logarithmic representation	$\vec{S}_i = \partial_i \ln \vec{p}$	$\hat{L}_i$ defined by $\partial_i \hat{\rho} = J_{\hat{\rho}}(\hat{L}_i)$
Relation between representations	$\partial_i \vec{p} = \vec{p} \circ \vec{S}_i$	$\partial_i \hat{\rho} = J_{\hat{\rho}}(\hat{L}_i)$
Metric	$m_{ij} = \sum_x \frac{\partial_i p_x \partial_j p_x}{p_x}$	$g_{ij} = \text{Tr}[\partial_i \hat{\rho} J_{\hat{\rho}}^{-1}(\partial_j \hat{\rho})]$
Metric (logarithmic form)	$m_{ij} = \langle \vec{S}_i, \vec{S}_j \rangle_{\vec{p}}$	$g_{ij} = \langle \hat{L}_i, \hat{L}_j \rangle_{\hat{\rho}}$

Table IV.7.1: Overview of the classical-quantum analogy in information geometry.

IV.7.2 Quantum Fisher information metric

In classical statistics, the Fisher information is the unique Riemannian metric on the space of probability distributions that is contractive under stochastic maps. In the quantum setting, however, the state space we consider consists of density matrices, and the notion of stochastic maps is replaced by CPTP maps. Crucially, the classical uniqueness of the Fisher information no longer holds: instead, there exists an infinite family of inequivalent Riemannian metrics that remain monotone under CPTP maps.

To lift the Fisher information metric to the quantum realm, a natural starting point is to endow the tangent space at a quantum state $\hat{\rho} = \hat{\rho}(\vec{\theta})$ with a state-dependent inner product, in analogy with the probability-weighted inner product in the classical setting (IV.7.8). Tangent vectors at $\hat{\rho}$ are represented by traceless, Hermitian operators, corresponding to infinitesimal variations of the density matrix, i.e. $\{\partial_i \hat{\rho}\}$. We introduce a family of bilinear forms parameterised by a linear, positive superoperator $J_{\hat{\rho}}$, defined as

$$\langle \hat{A}, \hat{B} \rangle_{\hat{\rho}} = \text{Tr}[\hat{A} J_{\hat{\rho}}(\hat{B})]. \quad (\text{IV.7.15})$$

Here, $\hat{A}$ and $\hat{B}$ (as well as $\hat{X}$ and $\hat{Y}$ below) denote arbitrary elements of the tangent space at $\hat{\rho}$, i.e. traceless, Hermitian operators representing infinitesimal variations of the density matrix. Provided that $J_{\hat{\rho}}$ is invertible on the tangent space, this bilinear form induces a Riemannian metric on the manifold of density matrices. In terms of the inverse superoperator $J_{\hat{\rho}}^{-1}$, the metric reads

$$g_{\hat{\rho}}(\hat{X}, \hat{Y}) = \text{Tr}[\hat{X} J_{\hat{\rho}}^{-1}(\hat{Y})]. \quad (\text{IV.7.16})$$

Not every choice of $J_{\hat{\rho}}$ leads to a physically meaningful metric. In particular, we demand that the metric be monotone under CPTP maps, meaning that statistical

distinguishability between nearby quantum states cannot increase under physical processes. Remarkably, this requirement completely characterises the admissible metrics. Petz's theorem states that a Riemannian metric is monotone if and only if the associated superoperator $J_{\hat{\rho}}$ can be expressed in terms of left and right multiplication by the density matrix,

$$J_{\hat{\rho}} = m_f(L_{\hat{\rho}}, R_{\hat{\rho}}), \quad (\text{IV.7.17})$$

where the superoperators $L_{\hat{\rho}}(\hat{A}) = \hat{\rho} \hat{A}$ and $R_{\hat{\rho}}(\hat{A}) = \hat{A} \hat{\rho}$ act by left and right multiplication, respectively [291]. This family of metrics has no classical analogue and arises precisely because, in the quantum setting, the state $\hat{\rho}$ and its infinitesimal variations $\partial_i \hat{\rho}$ need not commute. The generalised mean $m_f(x, y)$ is defined as

$$m_f(x, y) = x f(x^{-1}y). \quad (\text{IV.7.18})$$

The so-called Morozova-Čencov (MC) function f must satisfy three conditions: it must be operator monotone, self-inversive, $f(x) = xf(1/x)$, and normalised, $f(1) = 1$. These properties ensure natural properties of the mean $m_f(x, y)$, e.g. its symmetry $m_f(x, y) = m_f(y, x)$, as well as $m_f(x, x) = x$. Each admissible choice of f therefore defines a distinct monotone Riemannian metric on the quantum state space [291, 292].

With these ingredients in place, for a family of states $\hat{\rho}(\vec{\theta})$ depending smoothly on parameters $\vec{\theta} = (\theta_1, \dots, \theta_n)$, the quantum Fisher information (QFI) metric tensor associated with a given MC function f takes the form

$$g_{ij}^{(f)} = g^{(f)}(\partial_i \hat{\rho}, \partial_j \hat{\rho}) = \text{Tr}[\partial_i \hat{\rho} m_f^{-1}(L_{\hat{\rho}}, R_{\hat{\rho}})(\partial_j \hat{\rho})]. \quad (\text{IV.7.19})$$

We now introduce the quantum analogues of the score function, the so-called logarithmic derivatives $\hat{L}_i^{(f)}$, which are implicitly defined by

$$\partial_i \hat{\rho} = J_{\hat{\rho}}(\hat{L}_i^{(f)}), \quad (\text{IV.7.20})$$

and allow us to express the metric elements as

$$g_{ij}^{(f)} = \text{Tr}[J_{\hat{\rho}}(\hat{L}_i^{(f)}) \hat{L}_j^{(f)}] = \text{Tr}[\partial_i \hat{\rho} \hat{L}_j^{(f)}] = \langle \hat{L}_i^{(f)}, \hat{L}_j^{(f)} \rangle_{\hat{\rho}}, \quad (\text{IV.7.21})$$

which mirrors the classical expression in (IV.7.8). From the line element,

$$ds_f^2 = \frac{1}{4} \sum_{i,j} g_{ij}^{(f)}(\vec{\theta}) d\theta_i d\theta_j, \quad (\text{IV.7.22})$$

one can define the statistical length of a smooth path γ connecting two states $\hat{\rho}_{\vec{\theta}(0)}$

and $\hat{\rho}_{\vec{\theta}(\tau)}$,

$$\mathcal{L}_f[\gamma] = \int_{\gamma} ds_f = \frac{1}{2} \int_0^{\tau} \sqrt{\sum_{i,j} g_{ij}^{(f)}(\vec{\theta}(t)) \frac{d\theta_i}{dt} \frac{d\theta_j}{dt}} dt. \quad (\text{IV.7.23})$$

The corresponding geometric action can be written as

$$\mathcal{S}_f[\gamma] = \int_{\gamma} ds_f^2 = \frac{1}{4} \int_0^{\tau} \sum_{i,j} g_{ij}^{(f)}(\vec{\theta}(t)) \frac{d\theta_i}{dt} \frac{d\theta_j}{dt} dt, \quad (\text{IV.7.24})$$

and the statistical divergence is given by

$$\mathcal{J}_f[\gamma] = \tau \mathcal{S}_f[\gamma]. \quad (\text{IV.7.25})$$

Analogously to the classical case,

$$\mathcal{J}_f[\gamma] = \tau \mathcal{S}_f[\gamma] \geq \mathcal{L}_f^2[\gamma], \quad (\text{IV.7.26})$$

and geodesic paths can be obtained as the solution to the geodesic equation (IV.7.13), where we replace the Christoffel symbols with those associated with the metric $g_{ij}^{(f)}$. Although determining the explicit protocol for traversing the quantum state space along a geodesic path is generally challenging, in some cases closed-form expressions for the geodesic length exist. Specifically, when using either the symmetric-logarithmic derivative (SLD, $f_{\text{SLD}}(x) = (1+x)/2$) [293], or the Wigner-Yanase (WY, $f_{\text{WY}}(x) = ((1+\sqrt{x})/2)^2$) [294], QFI metrics, the geodesic distance between two density matrices $\hat{\rho}_1$ and $\hat{\rho}_2$ can be expressed as

$$\mathcal{L}_{\text{geo}}^{(\text{SLD})} = \arccos \sqrt{F(\hat{\rho}_1, \hat{\rho}_2)}, \quad \mathcal{L}_{\text{geo}}^{(\text{WY})} = \arccos A(\hat{\rho}_1, \hat{\rho}_2), \quad (\text{IV.7.27})$$

where

$$F(\hat{\rho}_1, \hat{\rho}_2) = \text{Tr} \left[\sqrt{\sqrt{\hat{\rho}_1} \hat{\rho}_2 \sqrt{\hat{\rho}_1}} \right]^2 \quad (\text{IV.7.28})$$

is the Uhlmann fidelity, and

$$A(\hat{\rho}_1, \hat{\rho}_2) = \text{Tr} \left[\sqrt{\hat{\rho}_1} \sqrt{\hat{\rho}_2} \right] \quad (\text{IV.7.29})$$

is known as the quantum affinity. Among the infinite family of QFI, the SLD QFI takes the smallest value, whereas the harmonic mean (HM) QFI, $f_{\text{HM}} = 2x/(x+1)$, takes the largest. Another important member of the QFI family is the Kubo-Mori-Bogoliubov (KMB) QFI metric. It corresponds to choosing $f_{\text{KMB}}(x) = (x-1)/\log x$, which translates to the superoperator

$$J_{\hat{\rho}}^{(\text{KMB})}(\hat{A}) = \int_0^1 \hat{\rho}^s \hat{A} \hat{\rho}^{1-s} ds, \quad (\text{IV.7.30})$$

which in turn defines the KMB inner product. The corresponding metric takes the form⁶

$$g_{ij}^{(\text{KMB})} = \int_0^1 \text{Tr}[\hat{\rho}^s \hat{L}_i^{(\text{KMB})} \hat{\rho}^{1-s} \hat{L}_j^{(\text{KMB})}] ds, \quad (\text{IV.7.31})$$

and the KMB-logarithmic derivative is given by $\hat{L}_i^{(\text{KMB})} = \partial_i \log \hat{\rho}$. This metric is distinguished by its thermodynamic significance, which emerges in the study of entropy production due to its relation to the relative entropy (see Eq. (I.2.28)).

We take a closer look at this connection now. For simplicity we assume that the state depends only on a single parameter. The quantum relative entropy between the two density matrices $\hat{\rho}(\theta)$ and $\hat{\rho}(\theta + d\theta)$ is given by

$$D(\hat{\rho}(\theta) \parallel \hat{\rho}(\theta + d\theta)) = \text{Tr}[\hat{\rho}(\theta) (\log \hat{\rho}(\theta) - \log \hat{\rho}(\theta + d\theta))]. \quad (\text{IV.7.32})$$

We expand this expression to second order in $d\theta$. For notational convenience, we denote $\hat{\rho} = \hat{\rho}(\theta)$. Using a Taylor expansion of $\log \hat{\rho}(\theta + d\theta)$ about $\hat{\rho}$,

$$\log(\hat{\rho}(\theta + d\theta)) = \log \hat{\rho} + \partial_\theta \log \hat{\rho} d\theta + \frac{1}{2} \partial_\theta^2 \log \hat{\rho} d\theta^2 + O(d\theta^3), \quad (\text{IV.7.33})$$

we obtain

$$D(\hat{\rho}(\theta) \parallel \hat{\rho}(\theta + d\theta)) = -\text{Tr}[\hat{\rho} \partial_\theta \log \hat{\rho}] d\theta - \frac{1}{2} \text{Tr}[\hat{\rho} \partial_\theta^2 \log \hat{\rho}] d\theta^2 + O(d\theta^3). \quad (\text{IV.7.34})$$

The linear term vanishes because $\text{Tr}[\hat{\rho} \partial_\theta \log \hat{\rho}] = \partial_\theta \text{Tr}[\hat{\rho}] = 0$, so the leading non-trivial contribution is quadratic, given by

$$D(\hat{\rho}(\theta) \parallel \hat{\rho}(\theta + d\theta)) = -\frac{1}{2} \text{Tr}[\hat{\rho} \partial_\theta^2 \log \hat{\rho}] d\theta^2 + O(d\theta^3). \quad (\text{IV.7.35})$$

We note that

$$\text{Tr}[\hat{\rho} \partial_\theta^2 \log \hat{\rho}] = \partial_\theta \text{Tr}[\hat{\rho} \partial_\theta \log \hat{\rho}] - \text{Tr}[\partial_\theta \hat{\rho} \partial_\theta \log \hat{\rho}] = -\text{Tr}[\partial_\theta \hat{\rho} \hat{L}_\theta^{(\text{KMB})}]. \quad (\text{IV.7.36})$$

We identify $g_\theta^{(\text{KMB})} = \text{Tr}[\partial_\theta \hat{\rho} \hat{L}_\theta^{(\text{KMB})}]$, and $D(\hat{\rho} \parallel \hat{\rho} + d\hat{\rho}) = \frac{1}{2} g_\theta^{(\text{KMB})} d\theta^2 + O(d\theta^3)$, thereby establishing the KMB QFI's natural connection to entropy production.

Coming back to the general form of the metric (IV.7.19), we now show that it can always be split into an incoherent (classical-like) and a coherent (quantum) contribution. To this end, we express the density matrix in its instantaneous eigenbasis. Let $\hat{\rho} = \sum_x p_x |x\rangle \langle x|$ be the spectral decomposition of $\hat{\rho}$, with $p_x > 0$ (full-rank). In this basis, an arbitrary tangent vector $\partial_i \rho$ has matrix elements $(\partial_i \hat{\rho})_{xy} = \langle x | \partial_i \hat{\rho} | y \rangle$.

⁶Here, we use the Duhamel (or Fréchet) derivative formula for the logarithm of an operator: $d\hat{\rho} = \int_0^1 ds \hat{\rho}^s (d \log \hat{\rho}) \hat{\rho}^{1-s}$. It is valid for full-rank density matrices $\hat{\rho}$ and expresses the differential $d\hat{\rho}$ in terms of the differential of $\log \hat{\rho}$ [295].

In the eigenbasis of $\hat{\rho}$, the action of $m_f^{-1}(L_{\hat{\rho}}, R_{\hat{\rho}})$ becomes

$$\langle x | m_f^{-1}(L_{\hat{\rho}}, R_{\hat{\rho}}) (\partial_i \hat{\rho}) | y \rangle = m_f^{-1}(p_x, p_y) (\partial_i \hat{\rho})_{xy}, \quad (\text{IV.7.37})$$

so that the metric takes the form

$$g_{ij}^{(f)} = \sum_{xy} m_f^{-1}(p_x, p_y) (\partial_i \hat{\rho})_{xy} (\partial_j \hat{\rho})_{yx}. \quad (\text{IV.7.38})$$

This expression splits into two contributions⁷. The first is a classical-like (incoherent) part obtained from the diagonal terms. Since $m_f^{-1}(p_x, p_x) = 1/p_x$ for all operator monotone functions f , this reproduces the classical Fisher information of the eigenvalues $\{p_x\}$:

$$g_{ij}^{(f), \text{ic}} = \sum_x \frac{(\partial_i p_x)(\partial_j p_x)}{p_x}. \quad (\text{IV.7.39})$$

The second is a quantum-like (coherent) part obtained from the off-diagonal terms

$$g_{ij}^{(f), \text{c}} = \sum_{x \neq y} m_f^{-1}(p_x, p_y) (\partial_i \hat{\rho})_{xy} (\partial_j \hat{\rho})_{yx}. \quad (\text{IV.7.40})$$

This part encodes the contribution to the QFI arising from the variation of the eigenvectors of $\hat{\rho}$. Thus, any QFI metric, may be split accordingly

$$g_{ij}^{(f)} = g_{ij}^{(f), \text{ic}} + g_{ij}^{(f), \text{c}}. \quad (\text{IV.7.41})$$

This decomposition highlights the two distinct sources of distinguishability in quantum systems: changes in populations (diagonal) and changes in coherences (off-diagonal), which are absent in classical probability distributions.

If the parameter is chosen to be time, the decomposition of the QFI metric into incoherent and coherent contributions acquires an interesting physical interpretation. In such a scenario, any smooth evolution of the density matrix can be thought of as arising from a combination of two distinct processes: (i) changes in the eigenvalues of the state, corresponding to population dynamics typically associated with dissipative processes, and (ii) rotations of the eigenvectors, corresponding to coherent, unitary dynamics.

Mathematically, any smoothly varying density matrix $\hat{\rho}_t$ can be decomposed as

$$\hat{\rho}_t = \hat{U}_t \hat{\chi}_t \hat{U}_t^\dagger, \quad (\text{IV.7.42})$$

⁷Another useful way to arrive at what follows is to switch to the matrix representation of the superoperators $\underline{L}_{\hat{\rho}} = \hat{\rho} \otimes \hat{\mathbb{I}}$ and $\underline{R}_{\hat{\rho}} = \hat{\mathbb{I}} \otimes \hat{\rho}^T$, which act on vectorised versions of the regular operators. That is, for an operator O , we denote its vectorised form in an arbitrary basis as (see also Section I.1.4) $|\hat{O}\rangle\rangle = \sum_{ij} O_{ij} |ij\rangle\rangle$. The, in the eigenbasis of $\hat{\rho} = \sum_j p_j |j\rangle\langle j|$, we can write $\underline{L}_{\hat{\rho}} = \sum_{ij} p_i |ij\rangle\rangle\langle\langle ij|$ and $\underline{R}_{\hat{\rho}} = \sum_{ij} p_j |ij\rangle\rangle\langle\langle ij|$. In this basis, $m^f(L_{\hat{\rho}}, R_{\hat{\rho}}) = \sum_{ij} m^f(p_i, p_j) |ij\rangle\rangle\langle\langle ij|$, where we use that $m^f(p_i, p_j) = p_i f(p_i^{-1} p_j)$ [68].

where $\hat{\chi}_t$ is diagonal in some fixed reference basis and $\hat{U}_t$ is unitary, transforming the reference eigenvectors into the instantaneous ones. In this decomposition, the dynamics of $\hat{\chi}_t$ describe the evolution of the populations, i.e., the incoherent part of the evolution, while the dynamics of $\hat{U}_t$ encode changes in the eigenbasis, i.e., the coherent part. To describe the coherent evolution, we introduce the Hermitian operator $\hat{H}_t = i \left(\frac{d\hat{U}_t}{dt} \right) \hat{U}_t^\dagger$, which can be interpreted as an effective Hamiltonian generating the coherent part of the evolution. The time derivative of $\hat{\rho}_t$ then reads [296–298]

$$\partial_t \hat{\rho}_t = -i[\hat{H}_t, \hat{\rho}_t] + \hat{U}_t \partial_t \hat{\chi}_t \hat{U}_t^\dagger. \quad (\text{IV.7.43})$$

The temporal QFI, which we denote by $I_Q^{(f)}(t)$, is then

$$\begin{aligned} I_Q^{(f)}(t) &= g_{tt}^{(f)} = I_Q^{(f),\text{ic}}(t) + I_Q^{(f),\text{c}}(t), \\ I_Q^{(f),\text{ic}}(t) &= \sum_x \frac{(\partial_t p_x(t))^2}{p_x(t)}, \\ I_Q^{(f),\text{c}}(t) &= \sum_{x \neq y} m_f^{-1}(p_x, p_y) (p_y(t) - p_x(t))^2 |\langle x(t) | \hat{H}_t | y(t) \rangle|^2. \end{aligned} \quad (\text{IV.7.44})$$

In the introduction to this chapter, we showed that the classical temporal Fisher information quantifies a system's intrinsic speed of evolution. The same interpretation carries over to the quantum setting: any QFI $I_Q^{(f)}(t)$ provides a natural measure of the instantaneous speed of quantum evolution and thus a powerful tool for deriving bounds on dynamical processes, known as geometric quantum speed limits⁸ [63, 65], as we will see now. As discussed above, there exists an infinite family of QFI, each associated with a contractive Riemannian metric on the space of quantum states. Correspondingly, one obtains a family of quantum speed limits, rather than a single one as in classical case. Importantly, Refs. [68, 66] show that, generalising the classical speed limit used to build intuition in the introduction, for any valid operator monotone function f ,

$$|\dot{a}| \leq (\Delta^{(f)} \hat{A}) \sqrt{I_Q^{(f)}}. \quad (\text{IV.7.45})$$

Here, $\dot{a} = \text{Tr}[\hat{A} \dot{\hat{\rho}}]$ denotes the rate of change of the expectation value of an arbitrary observable $\hat{A}$ in the state $\hat{\rho}$, while $(\Delta^{(f)} \hat{A})^2 = \text{Tr}[\hat{A}_0 m_f(L_{\hat{\rho}}, R_{\hat{\rho}})(\hat{A}_0)]$ is the generalised f -variance of $\hat{A}$ [301], with $\hat{A}_0 = \hat{A} - \text{Tr}[\hat{\rho} \hat{A}]$ ⁹.

In direct analogy with the Mandelstam-Tamm formulation of time-energy uncer-

⁸Comprehensive reviews of quantum speed limits are provided in Refs. [59, 299, 300].

⁹That is, the variance compatible with the inner product specified by f . For $f = f_{\text{SLD}}$, one recovers the usual variance with respect to the Hilbert-Schmidt inner product.

tainty [61], one may introduce a characteristic timescale

$$\tau_a = \frac{\Delta^{(f)} \hat{A}}{|\dot{\hat{A}}|}, \quad (\text{IV.7.46})$$

which quantifies the time required for the expectation value of $\hat{A}$ to change by one f -standard deviation¹⁰. This immediately yields a time-information uncertainty relation [66],

$$\tau_a \sqrt{I_Q^{(f)}} \geq 1. \quad (\text{IV.7.47})$$

Moreover, exploiting the interpretation of any QFI as a Riemannian metric, integration along the dynamical path yields a geometric bound on the time-averaged characteristic velocity,

$$\int_0^t ds \frac{|\dot{\hat{A}}(s)|}{(\Delta^{(f)} \hat{A})(s)} \leq \int_0^t ds \sqrt{I_Q^{(f)}(s)} = 2\mathcal{L}_f \leq 2\mathcal{L}_f^{\text{geo}}. \quad (\text{IV.7.48})$$

Here, $\mathcal{L}_f$ denotes the length of the actual dynamical path, while $\mathcal{L}_f^{\text{geo}}$ is the corresponding geodesic distance induced by the metric specified by f . The tightness of these bounds depends on the particular choice of QFI and also on the physical setting under consideration. One may optimise over f to obtain the tightest possible bound for a given dynamical process and observable.

These generalised quantum speed limits will be used in Chapter IV.8 to analyse anomalous thermalisation and excess entropic flux in arbitrary finite-speed protocols in open quantum systems. Finally, we note that, although seemingly abstract, geometric quantum speed limits have recently been explored experimentally [67, 68], underscoring that they are physically meaningful and experimentally accessible constraints.

IV.7.3 Temporal FI in classical thermodynamics

We now review recent developments connecting temporal information geometry with stochastic thermodynamics [70, 69]. These classical results motivate our quantum analysis [71], discussed in Chapter IV.8.

We consider a discrete-state system with $N > 1$ accessible states. The system is weakly coupled to one or more thermal reservoirs, which induce stochastic transitions between these states. Its dynamics are fully characterised by the time-dependent probability distribution $\vec{p}_t = (p_1(t), \dots, p_N(t))^T$, where $p_x(t)$ denotes the probability of finding the system in state $x \in X$ at time t , and X is the set of

¹⁰Mandelstam and Tamm established the time-energy uncertainty relation $\tau \Delta \hat{H} \geq 1/2$, valid for isolated systems evolving under a Hamiltonian $\hat{H}$ [61].

all accessible states. The temporal evolution of the probabilities is governed by a continuous-time Markovian Pauli master equation,

$$\dot{p}_x(t) = \sum_y w^{xy}(t) p_y(t), \quad (\text{IV.7.49})$$

where $w^{xy}(t) > 0$ (for $x \neq y$) denotes the transition rate from state y to x . The diagonal elements are defined as $w^{yy}(t) = -\sum_{x \neq y} w^{xy}(t)$, which guarantees probability conservation, $\sum_x w^{xy}(t) = 0$, and thus the normalization of $\vec{p}_t$ at all times. It is convenient to express the master equation in terms of the probability currents between states, defined as $j^{xy}(t) = w^{xy}(t) p_y(t) - w^{yx}(t) p_x(t)$, which represent the net flow of probability from state y to state x at time t . Then, the master equation becomes

$$\dot{p}_x(t) = \sum_y j^{xy}(t). \quad (\text{IV.7.50})$$

To connect the dynamics with thermodynamics, we impose the local detailed balance condition. This condition relates the transition rates to the stochastic entropy change in the environment, $\phi^{xy}(t)$, associated with a single transition $y \rightarrow x$ at time t ,

$$\phi^{xy}(t) = \log \frac{w^{xy}(t)}{w^{yx}(t)}. \quad (\text{IV.7.51})$$

Following Shannon's foundational work [302], the information content associated with occupying a given state y , also known as the surprisal or self-information, is defined as $I_y(t) = -\log p_y(t)$. This quantity measures how unexpected it is to find the system in state y at time t : the smaller the probability $p_y(t)$, the larger the surprisal. The local difference in surprisal between two states y and x defines the relative surprisal, $I_{xy}(t) = -\log(p_y(t)/p_x(t))$. This quantity captures the change in information content associated with transitions between the two states. Using these definitions, the time derivative of the Shannon entropy, $S(t) = -\sum_x p_x(t) \log p_x(t)$, can be expressed as a current-weighted average of the relative surprisal [70, 80]. Introducing the notation $\langle\langle A \rangle\rangle = \frac{1}{2} \sum_{x,y} j^{xy}(t) A_{xy}$, for the current-weighted average of any transition-dependent asymmetric function A_{xy} , we obtain

$$\dot{S}(t) = -\sum_x \dot{p}_x(t) \log p_x(t) = -\frac{1}{2} \sum_{x,y} j^{xy}(t) I_{xy}(t) = -\langle\langle I(t) \rangle\rangle. \quad (\text{IV.7.52})$$

Hence, the entropy production rate is directly linked to the probability currents and the relative surprisal between connected states.

From a physical perspective, rather than a purely information-theoretic one, the time derivative of the Shannon entropy can be decomposed into two distinct contributions: the entropy production rate within the system, $\dot{S}(t)$, and the entropy

flux to the environment, $\dot{\Pi}(t)$, $\dot{S}(t) = \dot{\Sigma}(t) - \dot{\Pi}(t)$, where [303]

$$\dot{\Sigma}(t) = \left\langle \left\langle \log \frac{w^{xy}(t)p_y(t)}{w^{yx}(t)p_x(t)} \right\rangle \right\rangle, \quad \dot{\Pi}(t) = \left\langle \left\langle \log \frac{w^{xy}(t)}{w^{yx}(t)} \right\rangle \right\rangle. \quad (\text{IV.7.53})$$

To gain additional insight, we introduce the generalised thermodynamic force [304] associated with transitions from state y to x , defined as

$$f^{xy}(t) = \log \left[\frac{w^{xy}(t)p_y(t)}{w^{yx}(t)p_x(t)} \right]. \quad (\text{IV.7.54})$$

This force quantifies the local “driving” responsible for sustaining probability currents in NESSs, and it vanishes when detailed balance¹¹ is satisfied. Using this structure, both the entropy production and entropy flux can be expressed as current-weighted averages,

$$\dot{\Sigma}(t) = \langle \langle f(t) \rangle \rangle, \quad \dot{\Pi}(t) = \langle \langle \phi(t) \rangle \rangle. \quad (\text{IV.7.55})$$

The Fisher information provides a complementary perspective by quantifying fluctuations in the rate of change of the system’s surprisal,

$$F(t) = \sum_x p_x(t) \left(\frac{d}{dt} \log p_x(t) \right)^2. \quad (\text{IV.7.56})$$

Alternatively, it can be written in terms of the probability currents as $F(t) = \langle \langle \dot{I}(t) \rangle \rangle$. Importantly, the surprisal rate $\dot{I}(t)$ vanishes only when the system is in a stationary state. Hence, the Fisher information captures how strongly the state changes with time: that is, how effectively it encodes the time parameter through its nonequilibrium evolution. The second derivative of the Shannon entropy, the entropic acceleration, can be written as [69]

$$\ddot{S}(t) = -\frac{d}{dt} \langle \langle I(t) \rangle \rangle = -\sum_x \dot{p}_x(t) \log p_x(t) - F(t). \quad (\text{IV.7.57})$$

To gain intuition, consider the term $\mathcal{B} = -\sum_x \dot{p}_x(t) \log p_x(t)$. The first derivative of the Shannon entropy, $\dot{S}(t) = -\langle \langle I(t) \rangle \rangle$, represents the rate at which the system’s average information content changes in time. The second derivative, $\ddot{S}(t) = -\frac{d}{dt} \langle \langle I(t) \rangle \rangle$, therefore quantifies how this rate itself evolves: it measures the acceleration of the mean information change, or equivalently, the rate of change of the current-averaged relative surprisal. In contrast, the Fisher information $F(t) = \langle \langle \dot{I}(t) \rangle \rangle$ reflects the average rate of change of the surprisal between

¹¹A system satisfies detailed balance if, in the steady state, the probability flux between any pair of states is balanced, i.e., $w^{xy}p_y^{\text{ss}} = w^{yx}p_x^{\text{ss}}$ for all states x, y . In this case, there are no net probability currents, and the system is in equilibrium.

individual microscopic transitions $y \rightarrow x$. Hence, while $F(t)$ captures the instantaneous local variation of surprisal along all transition paths, the derivative of $\langle\langle I(t) \rangle\rangle$ captures the evolution of the overall information flow at the ensemble level. The difference between these two quantities,

$$\mathcal{B} = \left\langle \left\langle \frac{d}{dt} I(t) \right\rangle \right\rangle - \frac{d}{dt} \langle\langle I(t) \rangle\rangle, \quad (\text{IV.7.58})$$

thus represents the discrepancy between the local and global rates of information change. Further, using the definition of the generalised thermodynamic force in Eq. (IV.7.54), one can show [70] that the Fisher information may be rewritten as

$$\begin{aligned} F(t) &= -\frac{1}{2} \sum_{x,y} j^{xy}(t) \frac{df^{xy}(t)}{dt} + \frac{1}{2} \sum_{x,y} j^{xy}(t) \frac{d}{dt} \log \frac{w^{xy}(t)}{w^{yx}(t)} \\ &= -\left\langle \left\langle \frac{df(t)}{dt} \right\rangle \right\rangle + \left\langle \left\langle \frac{d\phi(t)}{dt} \right\rangle \right\rangle. \end{aligned} \quad (\text{IV.7.59})$$

This compact expression gives the Fisher information a direct thermodynamic interpretation. It shows that the Fisher information quantifies how rapidly the generalised thermodynamic forces $f^{xy}(t)$ and the corresponding entropic fluxes $\phi^{xy}(t)$ evolve in time, weighted by the microscopic probability currents $j^{xy}(t)$. This relationship provides a bridge between information geometry and stochastic thermodynamics for classical Markovian systems that obey local detailed balance. Because the Fisher information is non-negative, we have

$$\left\langle \left\langle \frac{d\phi(t)}{dt} \right\rangle \right\rangle \geq \left\langle \left\langle \frac{df(t)}{dt} \right\rangle \right\rangle, \quad (\text{IV.7.60})$$

with equality occurring only in a stationary state, where the temporal Fisher information vanishes [70]. Physically, this relation implies that the time variation of the generalised thermodynamic forces is fully “transferred” into the rate of entropy change in the environment when the system is stationary. Any discrepancy between these two quantities indicates nonstationarity, representing a partial loss in the conversion of the system’s internal force fluctuations into environmental entropy changes [70].

IV.7.4 Fisher information, fluctuations and response functions

In this section, we establish the connection between the (Q)FI, fluctuations and thermodynamic response functions, thereby linking information geometry to the first central theme of this thesis.

In classical equilibrium statistical mechanics, the probability distribution of a system in the canonical ensemble, whose energy levels $E_x(\theta)$ depend smoothly on a control parameter θ , is given by [1]

$$p_x(\theta) = e^{\beta(F_\beta(\theta) - E_x(\theta))}, \quad (\text{IV.7.61})$$

where $\beta = 1/T$ and $F_\beta(\theta) = -\beta^{-1} \ln \sum_x \exp(-\beta E_x(\theta))$ is the free energy. Substituting this into the definition of Fisher information shows that [305, 72]

$$F(\theta) = \beta^2 \left\langle (\partial_\theta E - \langle \partial_\theta E \rangle)^2 \right\rangle, \quad (\text{IV.7.62})$$

i.e., the variance of the generalized force conjugate to θ . If we choose $\theta = \beta$, then the Fisher information is given by the energy fluctuations: $F(\beta) = \langle (E - \langle E \rangle)^2 \rangle$. In the thermodynamic limit, the Fisher information further relates to derivatives of the free energy, yielding conventional equilibrium response functions: $F(\theta) = -\partial_\theta^2(\beta F_\beta(\theta)) = \beta^2 \text{Var}[\partial_\theta E]$ [305, 72]. Now, consider a thermal Gibbs state

$$\hat{\omega}_\beta(\theta) = \frac{e^{-\beta \hat{H}(\theta)}}{Z(\theta)}, \quad Z(\theta) = \text{Tr}[e^{-\beta \hat{H}(\theta)}]. \quad (\text{IV.7.63})$$

For the KMB-score operator we find $\hat{L}_\theta^{(\text{KMB})} = -\beta(\partial_\theta \hat{H}(\theta) - \langle \partial_\theta \hat{H}(\theta) \rangle_{\hat{\omega}_\beta})$, and the KMB QFI becomes

$$I_Q^{(\text{KMB})}(\theta) = \beta^2 \langle \partial_\theta \hat{H}(\theta) - \langle \partial_\theta \hat{H}(\theta) \rangle_{\hat{\omega}_\beta}, \partial_\theta \hat{H}(\theta) - \langle \partial_\theta \hat{H}(\theta) \rangle_{\hat{\omega}_\beta} \rangle_{\hat{\omega}_\beta}. \quad (\text{IV.7.64})$$

This shows explicitly that the KMB QFI can also be understood as a thermally weighted variance of the generalised force $\partial_\theta \hat{H}(\theta)$, with the weighting defined by the KMB inner product. Since the KMB inner product governs correlation functions in linear response theory, the KMB QFI directly generalizes the classical relation between equilibrium fluctuations and response functions [306, 307]. From the perspective of this thesis, this relation makes explicit how the QFI serves as a unifying quantity: it simultaneously quantifies equilibrium fluctuations and governs the corresponding thermodynamic response.

If we consider nonequilibrium situations, however, the interpretation becomes less straightforward. While we may express the NESS $\hat{\pi}_\theta$ in exponential form, using the so-called nonequilibrium potential [270], $\hat{\Phi}_\theta = -\log \hat{\pi}_\theta$, so that $\hat{\pi}_\theta = e^{-\hat{\Phi}_\theta}$, and therefore,

$$I_Q^{(\text{KMB})}(\theta) = \langle \partial_\theta \hat{\Phi}_\theta - \langle \partial_\theta \hat{\Phi}_\theta \rangle_{\hat{\pi}_\theta}, \partial_\theta \hat{\Phi}_\theta - \langle \partial_\theta \hat{\Phi}_\theta \rangle_{\hat{\pi}_\theta} \rangle_{\hat{\pi}_\theta}, \quad (\text{IV.7.65})$$

the direct interpretation, e.g., in terms of thermally weighted fluctuations of the energy or equilibrium linear response is generally lost, because NESSs break de-

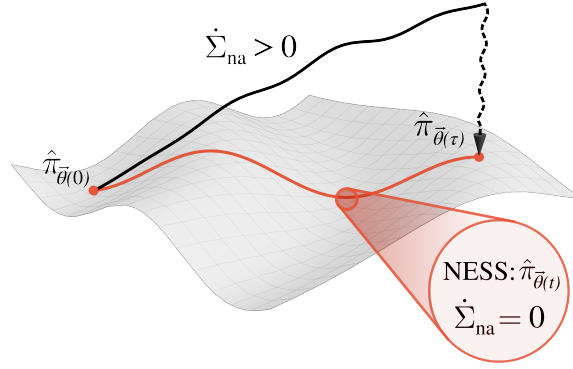


Figure IV.7.1: Schematic of a protocol driving the system between two NESSs. The system is initialised at a NESS $\hat{\pi}_{\vec{\theta}(0)}$ specified by $\vec{\theta}(0)$. In a quasistatic protocol connecting parameter(s) $\vec{\theta}(0)$ and $\vec{\theta}(\tau)$ (orange line), the system remains in the manifold of NESS (grey surface), with $\hat{\pi}_{\vec{\theta}(t)}$, at all times, and the nonadiabatic entropy production rate vanishes $\dot{\Sigma}_{na} = 0$. Along the finite-time transition (black line), $\dot{\Sigma}_{na} > 0$.

tailed balance. Nevertheless, as we will demonstrate in the next section, the KMB QFI is central to understanding the thermodynamic response of such systems to slow driving.

IV.7.5 Thermodynamic geometry

The results presented in this section are based on Refs. [10, 78]. For Ref. [10], I contributed to the writing of Chapter “Quantum thermodynamic geometry”. For Ref. [78], I contributed to selected analytical derivations, carried out the numerical example, and provided input during the writing and revision of the manuscript. The main theoretical derivations and the primary manuscript preparation were carried out by the lead author.

Traditional thermodynamic analysis often assumes quasi-static transformations, in which systems remain arbitrarily close to equilibrium and entropy production is negligible. Under these idealised conditions, quantities such as work and efficiency can be described using simple, static relations. Realistic processes, however, occur over finite times and inevitably generate dissipation, making the quasi-static assumption inadequate. To address this, the framework of thermodynamic geometry was introduced in the 1980s as a means to quantify, and if possible minimise, the unavoidable entropy production in finite-time processes [73, 308, 309]. At the heart of this approach is the concept of a thermodynamic length, a geometric measure in the space of control parameters $\vec{\theta}(t)$ (e.g., temperature, chemical potential, or Hamiltonian couplings) that provides a lower bound on the nonadiabatic entropy (see Section I.2.5) generated during a protocol of duration τ ,

$$\Sigma_{na} \geq \frac{\ell^2(\vec{\theta}(0), \vec{\theta}(\tau))}{\tau}. \quad (\text{IV.7.66})$$

Here, the thermodynamic length, $\ell(\vec{\theta}(0), \vec{\theta}(\tau))$, depends on a metric capturing both the (non)equilibrium (static) state space and relaxation dynamics, and minimizing dissipation corresponds to steering the system along the geodesic of this metric. By framing thermodynamic control as a geometric problem, one can identify optimal paths that reduce irreversibility and energetic costs.

Originally developed for macroscopic systems, thermodynamic geometry has since been extended to classical stochastic thermodynamics [74, 72, 75], where it has proven valuable for optimising finite-time processes, including the energetically efficient erasure of information (see Ref. [310] for a review). More recently, these ideas have been generalised to the quantum regime [77, 78, 76], enabling the design of low-dissipation quantum heat engines [311, 312] as well as the simultaneous suppression of their stochastic work fluctuations [313]. The strength of thermodynamic geometry in many instances relies on its ability to connect fundamental principles of thermodynamics with information geometry, providing both practical guidance for minimal-dissipation protocols and deep insight into the nature of irreversibility in complex, out-of-equilibrium systems.

In the following, we explore this connection in the context of the slow-driving thermodynamics of quantum NESS transitions, as shown in Fig. IV.7.1, which provide a particularly general setting: within this framework, slow classical NESS transitions, classical thermal-state transformations, and quantum thermal-state transformations all arise as special cases. Specifically, we show how the thermodynamic metric in these slow protocols is directly connected to information geometry through the KMB QFI, which we discussed in detail in Section IV.8.1. The results reviewed here were published in Ref. [78]. For the purposes of this thesis, we restrict attention to the simpler scenario in which a single control parameter is varied. The generalisation to the multiparameter case is straightforward and discussed in Ref. [78].

We consider an open quantum system governed by a GKSL master equation (see Section I.1.1) that admits a unique NESS (see Section I.1.5). A Lindbladian parameter θ is slowly varied in time, and we denote by $\hat{\pi}_{\theta(t)}$ the corresponding instantaneous NESS, defined by $\mathcal{L}_{\theta(t)}[\hat{\pi}_{\theta(t)}] = 0$. Initially, the system is prepared in the NESS $\hat{\pi}_{\theta(0)}$. For sufficiently slow driving, the system will remain close to $\hat{\pi}_{\theta(t)}$, up to a small correction. We consider the expansion

$$\hat{\rho}_t = \hat{\pi}_{\theta(t)} + \delta\hat{\rho}_t, \quad (\text{IV.7.67})$$

where $\text{Tr}[\delta\hat{\rho}_t] = 0$. Inserting this expansion into $d\hat{\rho}/dt = \mathcal{L}_{\theta(t)}[\hat{\rho}]$ and using that $\mathcal{L}_{\theta(t)}[\hat{\pi}_{\theta(t)}] = 0$, we obtain a differential equation for $\delta\hat{\rho}_t$ [76, 75, 314]

$$\left(\mathcal{L}_{\theta(t)} - \frac{d}{dt} \right) [\delta\hat{\rho}_t] = \frac{d}{dt} \hat{\pi}_{\theta(t)}, \quad (\text{IV.7.68})$$

which can be formally solved using the Drazin inverse [315] of the Liouvillian superoperator, $\mathcal{L}_{\theta(t)}^+$ (see Section I.1.6). By applying $\mathcal{L}_{\theta(t)}^+$ on both sides, we obtain the expansion

$$\hat{\rho}_t = \hat{\pi}_{\theta(t)} + \sum_{n=1}^{\infty} \left(\mathcal{L}_{\theta(t)}^+ \frac{d}{dt} \right)^n [\hat{\pi}_{\theta(t)}]. \quad (\text{IV.7.69})$$

Notice that the n^{th} term in this series contains an n^{th} -order derivative of $\hat{\pi}_{\theta(t)}$, and is thus proportional to $\dot{\theta}^n$, which means that Eq. (IV.7.69) can be seen as an expansion in powers of the driving speed $\dot{\theta}$. Using the above expansion of the state, the nonadiabatic entropy production, introduced in Section I.2.5, along a slow driving protocol $\theta(t)$ is, to leading order, [78]

$$\Sigma_{\text{na}}(\tau) \approx \int_0^\tau dt \dot{\theta}^2(t) \zeta_{\theta(t)}, \quad (\text{IV.7.70})$$

where ζ_θ denotes the thermodynamic metric in the space of NESS,

$$\zeta_\theta = -\text{Tr} \left[\frac{d \log \hat{\pi}_\theta}{d\theta} \mathcal{L}_\theta^+ \left[\frac{d \hat{\pi}_\theta}{d\theta} \right] \right]. \quad (\text{IV.7.71})$$

Equivalently, this metric can be expressed as a correlation integral,

$$\zeta_\theta = \int_0^\infty d\tau \langle \hat{L}_\theta^{(\text{KMB})}, \hat{L}_\theta^{(\text{KMB})}(\tau) \rangle_{\hat{\pi}_\theta}, \quad (\text{IV.7.72})$$

where the KMB inner product is

$$\langle \hat{A}, \hat{B} \rangle_{\hat{\pi}_\theta} = \int_0^1 ds \text{Tr} \left[\hat{\pi}_\theta^s \hat{A}^\dagger \hat{\pi}_\theta^{1-s} \hat{B} \right], \quad (\text{IV.7.73})$$

and $\hat{L}_\theta^{(\text{KMB})}$ is the KMB logarithmic derivative operator, see Eq. (IV.7.20), evolved in the Heisenberg picture as $\hat{L}_\theta^{(\text{KMB})}(\tau) = e^{\mathcal{L}_\theta \tau} [\hat{L}_\theta^{(\text{KMB})}]$.

The metric is closely related to the KMB QFI,

$$I_Q^{(\text{KMB})}(\theta) = \langle \hat{L}_\theta^{(\text{KMB})}, \hat{L}_\theta^{(\text{KMB})} \rangle_{\hat{\pi}_\theta}, \quad (\text{IV.7.74})$$

via the so-called integral relaxation time [316]

$$\tau_\theta = \int_0^\infty d\tau \frac{\langle \hat{L}_\theta^{(\text{KMB})}, \hat{L}_\theta^{(\text{KMB})}(\tau) \rangle_{\hat{\pi}_\theta}}{\langle \hat{L}_\theta^{(\text{KMB})}, \hat{L}_\theta^{(\text{KMB})} \rangle_{\hat{\pi}_\theta}}. \quad (\text{IV.7.75})$$

The thermodynamic metric can then be expressed as their product

$$\zeta_\theta = \tau_\theta I_Q^{(\text{KMB})}(\theta). \quad (\text{IV.7.76})$$

Chapter IV.8

Temporal QFI in quantum thermodynamics

The results presented in this chapter are based on Refs. [71, 78]. For Ref. [71], I performed the analytical and numerical calculations, carried out the data analysis, and wrote the manuscript. The co-authors contributed to the conceptualisation of the work, discussed the results, and provided feedback and revisions on the manuscript. For Ref. [78], I contributed to selected analytical derivations, carried out the numerical example, and provided input during the writing and revision of the manuscript. The main theoretical derivations and the primary manuscript preparation were carried out by the lead author. The presentation closely follows these references.

In this chapter, we generalise established connections between classical stochastic thermodynamics and information geometry, presented in Refs. [70, 69] and reviewed in Section IV.7.3, to Markovian, weakly coupled open quantum systems. For such settings, we further derive an information-geometric bound on the excess entropy flux in nonstationary processes.

IV.8.1 Temporal QFI and entropy dynamics

We consider open quantum systems evolving under GKSL dynamics (I.1.8), and require local detailed balance [200, 186], so that the jump operators appear in pairs (k, k') satisfying

$$\hat{L}_k = e^{\phi^k/2} \hat{L}_{k'}^\dagger, \quad \phi^k = -\phi^{k'}. \quad (\text{IV.8.1})$$

This requirement lets us identify ϕ^k as the entropy change in the environment associated with the action of the jump operator $\hat{L}_k$. We express the density matrix in its instantaneous eigenbasis as $\hat{\rho}_t = \sum_x p_x(t) |x(t)\rangle \langle x(t)|$. Crucially, in this basis, one can define effective transition rates associated with each jump operator

$\hat{L}_k$ as $w_k^{xy}(t) = |\langle x(t)|\hat{L}_k|y(t)\rangle|^2$ [317, 172], which describe the probability per unit time for the system to transition from state $|y(t)\rangle$ to $|x(t)\rangle$ due to the action of $\hat{L}_k$. Taking the time derivative of $p_x(t) = \langle x(t)|\hat{\rho}_t|x(t)\rangle$ leads to a classical Pauli master equation for the instantaneous spectrum of the density matrix [317, 172]

$$\frac{dp_x(t)}{dt} = \sum_{k,y (y \neq x)} [w_k^{xy}(t)p_y(t) - w_{k'}^{yx}(t)p_x(t)]. \quad (\text{IV.8.2})$$

We define the probability current between the instantaneous eigenstates $|y(t)\rangle$ and $|x(t)\rangle$, $j_k^{xy}(t) = w_k^{xy}(t)p_y(t) - w_{k'}^{yx}(t)p_x(t)$, and

$$f_k^{xy}(t) = \log \frac{w_k^{xy}(t)p_y(t)}{w_{k'}^{yx}(t)p_x(t)}, \quad \phi_k^{xy}(t) = \log \frac{w_k^{xy}(t)}{w_{k'}^{yx}(t)}, \quad (\text{IV.8.3})$$

where $f_k^{xy}(t)$, analogous to the classical case, can be interpreted as a generalised thermodynamic force driving the current, while $\phi_k^{xy}(t)$ corresponds to the entropy flow associated with the action of the jump operator $\hat{L}_k$, resulting in the current $j_k^{xy}(t)$ from $|y(t)\rangle$ to $|x(t)\rangle$ at time t . The rate equation for the instantaneous eigenvalues $p_x(t)$ can be compactly rewritten in terms of the probability currents between eigenstates

$$\frac{dp_x(t)}{dt} = \sum_{k,y (y \neq x)} j_k^{xy}(t). \quad (\text{IV.8.4})$$

As discussed in Section IV.7.2, the quantum Fisher information (QFI) can always be decomposed into incoherent and coherent contributions. Importantly, the incoherent part depends only on the instantaneous spectrum of the state. This allows us to draw a direct analogy to the classical case and relate the incoherent QFI to thermodynamic quantities. In particular, the rate of change of the von Neumann entropy (see Eq. (I.2.9)), i.e. the Shannon entropy of the instantaneous spectrum, $\dot{S}_{\text{vN}}(t)$, can be expressed in terms of the probability currents and further decomposed into the entropy production rate $\dot{\Sigma}(t)$ and the entropy flux to the environment $\dot{\Pi}(t)$ [172]

$$\begin{aligned} \dot{S}_{\text{vN}}(t) &= - \sum_{x,y,k (x \neq y)} j_k^{xy}(t) \log p_x(t) \\ &= \frac{1}{2} \sum_{x,y,k} j_k^{xy}(t) \log \frac{w_k^{xy}(t)p_y(t)}{w_{k'}^{yx}(t)p_x(t)} - \frac{1}{2} \sum_{x,y,k} j_k^{xy}(t) \log \frac{w_k^{xy}(t)}{w_{k'}^{yx}(t)} \\ &= \dot{\Sigma}(t) - \dot{\Pi}(t). \end{aligned} \quad (\text{IV.8.5})$$

The entropy production rate can be expressed as a current-weighted average of the generalised thermodynamic force, $\dot{\Sigma}(t) = \langle\langle f(t) \rangle\rangle$, while the entropy flux is given by $\dot{\Pi}(t) = \langle\langle \phi(t) \rangle\rangle$. The incoherent contribution to the QFI can be expressed

as

$$I_Q^{(f),ic}(t) = \mathcal{B} - (\ddot{\Sigma}(t) - \ddot{\Pi}(t)), \quad (\text{IV.8.6})$$

where $\mathcal{B} = -\sum_x \dot{p}_x \log p_x$, mirroring the classical result in [69]. Equivalently, it may be written as [70]

$$I_Q^{(f),ic}(t) = -\left\langle\left\langle\frac{df(t)}{dt}\right\rangle\right\rangle + \left\langle\left\langle\frac{d\phi(t)}{dt}\right\rangle\right\rangle. \quad (\text{IV.8.7})$$

This constitutes the first central result. As in the classical case [70], discussed in Section IV.7.3, the thermodynamic inequality

$$\left\langle\left\langle\frac{d\phi(t)}{dt}\right\rangle\right\rangle \geq \left\langle\left\langle\frac{df(t)}{dt}\right\rangle\right\rangle \quad (\text{IV.8.8})$$

holds, linking the rate of change of the generalised thermodynamic force to the rate of entropy change in the environment. This relation carries over naturally to the quantum regime. Although the coherent contribution to the QFI is generally finite and contributes to the total statistical length, it does not enter this thermodynamic balance.

IV.8.2 Geometric bound on the von Neumann entropy rate

In Ref. [69], a trade-off relation was established between the geometric uncertainty associated with the path a system takes through classical state space and the time-averaged rate of change of information. In this Section, we demonstrate that this uncertainty relation remains valid for quantum dynamics.

The geometric uncertainty, given by the time-averaged variance of the QFI, provides a quantitative measure of how “unevenly” the system evolves along its path in state space. While originally defined in the classical setting [69], it straightforwardly generalises to

$$\delta_f = 4\frac{\mathcal{J}_f - \mathcal{L}_f^2}{T^2} = \mathbb{E}\left[I_Q^{(f)}\right] - \mathbb{E}\left[\sqrt{I_Q^{(f)}}\right]^2 \geq 0, \quad (\text{IV.8.9})$$

for any evolution time T , where $0 \leq T \leq \tau$. Here, $\mathcal{J}_f$ and $\mathcal{L}_f$ denote, respectively, the statistical divergence (IV.7.25) and the statistical length (IV.7.23) accumulated up to time T . This expression captures the cumulative deviation of the system’s path from the geodesic path in the statistical manifold. The geodesic corresponds to the path of constant speed, i.e., constant QFI, connecting the initial and final states. Consequently, the difference $\mathcal{J}_f - \mathcal{L}_f^2$ vanishes for motion along it, while

any nonzero value reflects fluctuations in the instantaneous rate of information change, indicating a departure from uniform evolution along the optimal path. Since the quantities $\mathcal{J}_f$, $\mathcal{L}_f^2$, and δ_f are all non-negative by definition, it immediately follows that

$$\delta_f \leq \frac{4\mathcal{J}_f}{T^2}. \quad (\text{IV.8.10})$$

It is useful to express this relation in terms of the time-averaged Fisher information,

$$\mathcal{I}_f = \mathbb{E}[I_Q^{(f)}] = \frac{1}{T} \int_0^T dt I_Q^{(f)}(t), \quad (\text{IV.8.11})$$

which quantifies the mean rate of information change along the path. The inequality can then be rewritten as

$$\frac{\mathcal{I}_f}{\delta_f} \geq 1. \quad (\text{IV.8.12})$$

That is, the geometric uncertainty in the path lower-bounds the time-averaged rate of information change. To explore the implications of the uncertainty relation in more detail, we decompose the QFI into its incoherent and coherent contributions, $I_Q^{(f)} = I_Q^{(f),\text{ic}} + I_Q^{(f),\text{c}}$, with $I_Q^{(f),\text{ic}} = \mathcal{B} - \ddot{S}_{\text{vN}}$. Integrating over a finite time interval $[0, T]$, we obtain

$$\int_0^T I_Q^{(f)}(t) dt = \int_0^T \left(\mathcal{B}(t) + I_Q^{(f),\text{c}}(t) - \ddot{S}_{\text{vN}}(t) \right) dt. \quad (\text{IV.8.13})$$

From this relation, it follows that the change in the entropy rate between the initial and final times is bounded from above

$$\dot{S}_{\text{vN}}(T) - \dot{S}_{\text{vN}}(0) \leq \mathcal{C} - T\delta_f + \int_0^T I_Q^{(f),\text{c}}(t) dt, \quad (\text{IV.8.14})$$

where $\mathcal{C} = \int_0^T \mathcal{B}(t) dt$ quantifies the integrated difference between the local information rate and the bulk information rate over the interval, δ_f is the geometric uncertainty associated with the path, and $\int_0^T I_Q^{(f),\text{c}}(t) dt$ captures the contribution from coherent quantum dynamics. This bound provides a clear, intuitive picture: the maximum change in the entropy rate is limited by three factors. First, $\mathcal{C}$ reflects the intrinsic fluctuations of information at the local level. Second, $T\delta_f$ encodes the penalty due to deviations from a geodesic, i.e., from a constant-speed path in state space. Finally, the coherent contribution $\int_0^T I_Q^{(f),\text{c}}(t) dt$ represents purely quantum effects arising from changes in the eigenbasis of the density matrix: it quantifies the geometric action associated with the coherent dynamics. Importantly, this bound is meaningful for nonstationary dynamics, as all three terms vanish trivially in stationary states. It is also worth noting that while our bound holds for any

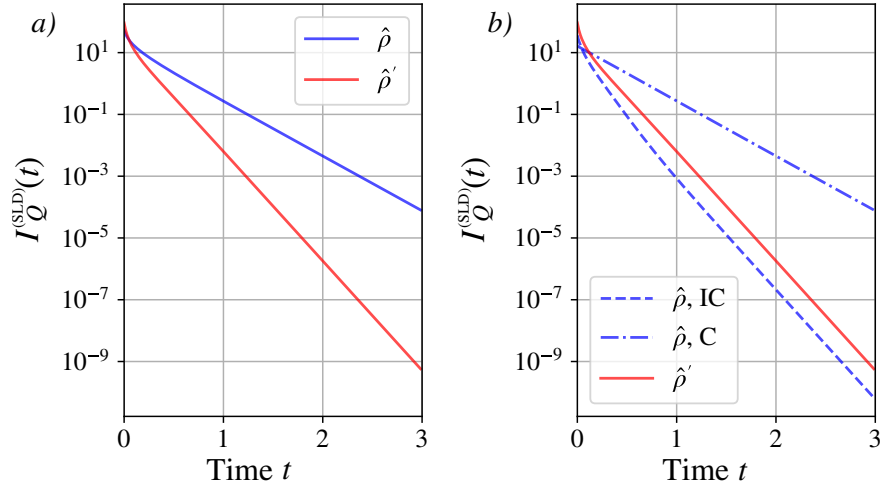


Figure IV.8.1: a) Time-evolution of the total SLD QFI of the reference state $\hat{\rho}$ (blue) and the rotated state $\hat{\rho}'$ (red). b) The rotated state evolves classically (red), therefore the QFI reduces to the incoherent contribution. The reference state, however, is coherent in the energy eigenbasis and thus has both incoherent (dashed blue) and coherent (dash-dotted) contributions. We find that the incoherent contribution decays exponentially faster than the coherent contribution. Adapted from Ref. [71].

choice of QFI metric, the tightest bound is obtained by choosing the SLD QFI [71].

IV.8.3 Example: quantum Mpemba effect

We now employ the information-geometric framework introduced above to analyse the (thermo)dynamics underlying a quantum Mpemba effect, that occurs during the thermal relaxation of a qubit weakly coupled to a bosonic bath, as recently investigated by Moroder *et al.* [318].

The classical Mpemba effect [319, 320] describes the counterintuitive scenario in which a hotter system cools faster than a colder one. For example, consider two identical containers of liquid initially at equilibrium but at different temperatures T_h and T_c with $T_h > T_c$. When placed in a cooler environment at temperature $T^* < T_c$, the Mpemba effect occurs if, after some time t_M , the initially hotter liquid becomes cooler than the initially colder one and remains so thereafter. This phenomenon has spurred extensive theoretical and experimental investigations in classical systems [321–327]. More recently, analogous Mpemba-like effects have been predicted and observed in quantum systems [328–334].

Following Moroder *et al.* [318], we consider a qubit weakly coupled to a thermal bosonic environment at inverse temperature β . The qubit has ground and excited states, $|g\rangle$ and $|e\rangle$, separated by an energy ϵ , and is governed by the Hamiltonian $\hat{H} = \frac{\epsilon}{2}\hat{\sigma}_z$. Transitions between the two levels are induced by the environment, occurring at rates $\gamma_+ = \gamma f_B(\epsilon, \beta)$, and $\gamma_- = \gamma[1 + f_B(\epsilon, \beta)]$, where $f_B(\epsilon, \beta) = (e^{\beta\epsilon} - 1)^{-1}$ is the Bose-Einstein occupation number and γ characterises

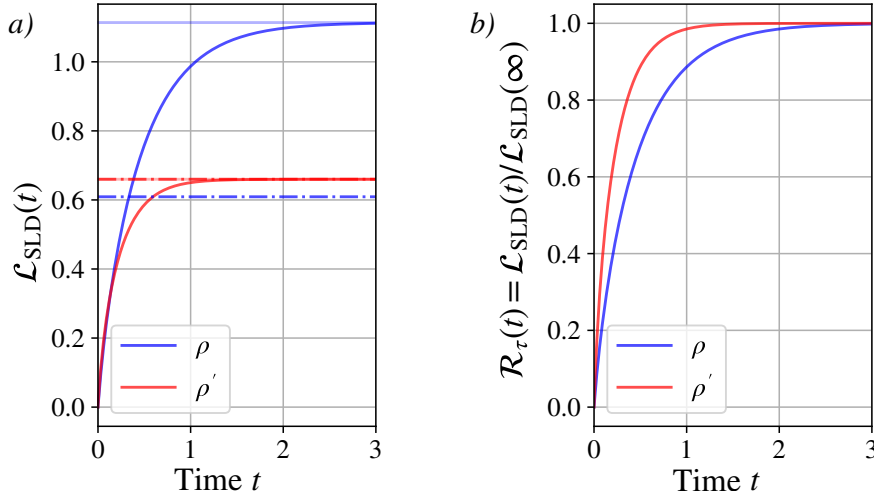


Figure IV.8.2: a) Statistical distance (SLD) traced by the path of the reference state $\hat{\rho}$ (blue solid) and the rotated state $\hat{\rho}'$ (red solid) in state space. The semi-transparent lines indicate the distance traced by the respective paths in the infinite time limit, indicating that sufficient thermalisation is reached within the chosen time period. The statistical distance for the rotated state is shorter than for the reference state. However, the SLD geodesic distance between the reference state and the thermal state (blue dash-dotted) is shorter than that between the rotated state and the thermal state (red dash-dotted), which coincides with the statistical distance. b) Furthermore, we find that the rotated state (red) completes the path at a faster rate as shown by the higher ratio of completion at all times. Adapted from Ref. [71].

the system-environment coupling strength. The reduced dynamics of the qubit is governed by the GKSL master equation

$$\frac{d}{dt}\hat{\rho}(t) = -i[\hat{H}, \hat{\rho}(t)] + \mathcal{D}[\hat{L}_+] \hat{\rho}(t) + \mathcal{D}[\hat{L}_-] \hat{\rho}(t), \quad (\text{IV.8.15})$$

where the jump operators $\hat{L}_+ = \sqrt{\gamma_+}|e\rangle\langle g|$ and $\hat{L}_- = \sqrt{\gamma_-}|g\rangle\langle e|$ describe absorption and emission events. The qubit is initialised in an arbitrary mixed state $\hat{\rho}_0 = \frac{1}{2}(1 + \vec{r} \cdot \vec{\sigma})$, with $\vec{\sigma} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)^T$ and Bloch vector $\vec{r} = (r_x, r_y, r_z)^T$. We choose initial states with non-zero coherences in the energy eigenbasis, i.e., r_x or $r_y \neq 0$. A unitarily rotated state diagonal in the energy basis can be expressed as $\hat{\rho}'_0 = \text{diag}(\lambda_1, \lambda_2)$, where $\lambda_1 \geq \lambda_2$ are the eigenvalues of $\hat{\rho}_0$. Importantly, this rotation ensures that the rotated state starts out with higher nonequilibrium free energy¹ [318]. The numerical results, discussed in the following, are obtained for the reference state and its unitarily rotated counterpart, characterised by the Bloch vectors $\vec{r} = (-0.41760, -0.60647, 0.47879)^T$ and $\vec{r}' = (0, 0, 0.87836)^T$, respectively. The remaining parameters defining the setup are the system-bath coupling

¹The nonequilibrium free energy is defined as $F_{\text{neq}}(\hat{\rho}) = \beta^{-1}D(\hat{\rho}||\hat{\omega}_\beta) + F_{\text{eq}}$, with the equilibrium free energy $F_{\text{eq}} = -\beta^{-1}\log Z_\beta$, $Z_\beta = \text{Tr}[e^{-\beta\hat{H}}]$, the quantum relative entropy (I.2.7), and thermal fixed point $\hat{\omega}_\beta = \frac{e^{-\beta\hat{H}}}{Z_\beta}$ associated with the system Hamiltonian $\hat{H}$.

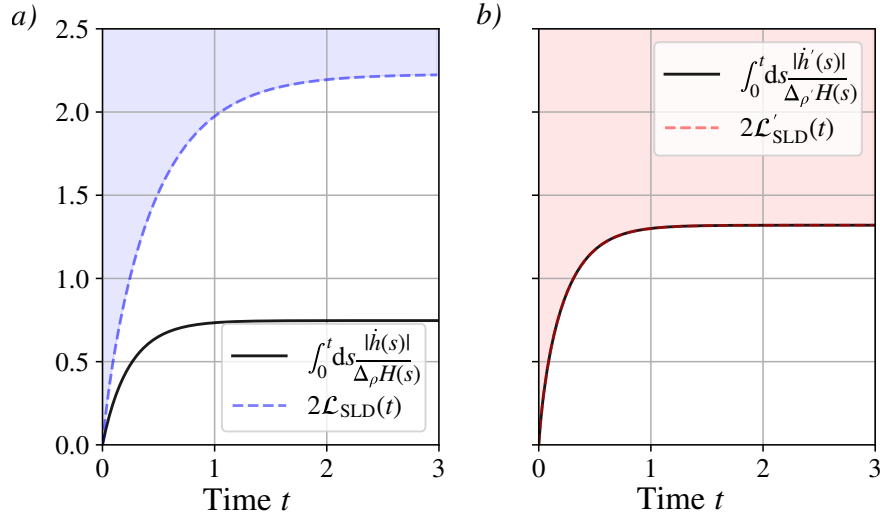


Figure IV.8.3: The path-dependent time-integrated ratio between the heat current and the standard deviation of the Hamiltonian with respect to the time-evolving state is upper bounded by twice the statistical distance associated with the path of the time-evolving density matrix in state space, shown for a) the reference state $\hat{\rho}$ and b) the rotated state $\hat{\rho}'$. The path traced by the rotated state lies on the geodesic curve (even though it is not traced at constant speed). Therefore, the bound set by the statistical distance coincides with the generally tightest one provided by the length of the geodesic. The bound is saturated because the state evolves purely incoherently. Adapted from Ref. [71].

strength $\gamma = 1$, the temperature $T = 10$, and the qubit energy splitting $\epsilon = 5$. Note that in this scenario, the system-environment interaction cannot generate coherence in the energy eigenbasis. Consequently, under the GKSL dynamics of Eq. (IV.8.15), the rotated state remains diagonal in the Hamiltonian eigenbasis at all times. Recalling the general decomposition of time evolution into evolving eigenvalues $\hat{\chi}(t)$ and a time-dependent eigenbasis $\hat{U}_t$ (see Section IV.7.2), we observe that here the eigenbasis is static, so $\hat{U}_t = \mathbb{I}$, and the dynamics are fully captured by the evolution of the eigenvalues. Accordingly, the QFI with respect to time reduces to its incoherent contribution, corresponding to the classical Fisher information of the evolving spectrum. In contrast, the reference state possesses initial coherence in the energy basis, so its QFI contains both incoherent and coherent contributions, as illustrated in Fig. IV.8.1b). Since $\sqrt{I_Q^{(f)}(t)}$ measures the instantaneous speed of a state along its path in state space it vanishes at equilibrium. Our results indicate that the rotated state, initially with higher nonequilibrium free energy, relaxes faster than the coherent reference state, as seen from the more rapid decay of its total QFI (see Fig. IV.8.1a). Decomposing the QFI clarifies this difference: for the reference state, the coherent contribution from Hamiltonian dynamics decays more slowly than the incoherent, dissipative contribution, thus setting the overall relaxation timescale. Since populations and coherences evolve independently in this Davies-type model [335, 336], the incoherent part of the QFI decays identically for both states, as shown in Fig. IV.8.1. Fig. IV.8.2 shows that

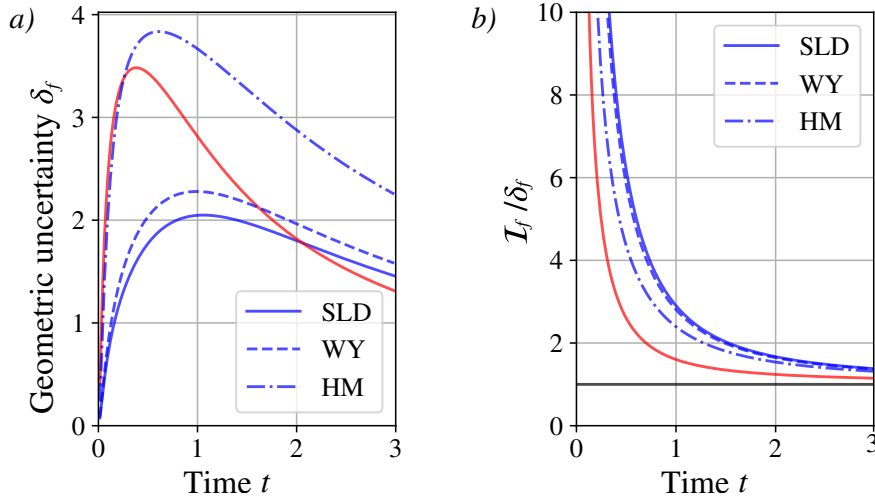


Figure IV.8.4: a) Geometric uncertainty δ_f about the path of the rotated state $\hat{\rho}'$ (red). We compute the geometric uncertainty of the time-evolving reference state $\hat{\rho}$ using the SLD, WY and HM QFI metric (blue). The SLD yields the smallest geometric uncertainty. Deviations from the geodesic connecting the reference state and rotated state to the thermal state, respectively, are accumulated at short times. b) Trade-off between the time-averaged QFI and the geometric uncertainty. Adapted from Ref. [71].

the SLD statistical length of the path for the incoherent, rotated initial state (light red) is markedly shorter than that of the coherent reference state (light blue), i.e., $\mathcal{L}_{\text{SLD}}^\infty(\hat{\rho}'_0) < \mathcal{L}_{\text{SLD}}^\infty(\hat{\rho}_0)$, with

$$\mathcal{L}_\infty(\hat{\rho}_0) = \lim_{t \rightarrow \infty} \int_0^t \sqrt{I_Q^{(f)}(\hat{\rho}(t'))} dt'. \quad (\text{IV.8.16})$$

By contrast, the SLD geodesic lengths (dashed lines, Eq. (IV.7.27)) exhibit the opposite ordering, as seen in Fig. IV.8.2a). Although the rotated state does not follow a geodesic, so the QFI varies along its path, its instantaneous path length coincides with that of a geodesic representing monotonic motion along the z -axis of the Bloch sphere.

To quantify how far a state has progressed along its path, we introduce the ratio of completion, defined as the fraction of the total SLD statistical length traversed at time t : $\mathcal{R}_\tau(t) = \mathcal{L}_{\text{SLD}}(\hat{\rho}(t)) / \mathcal{L}_{\text{SLD}}^\infty(\hat{\rho}_0)$. As shown in Fig. IV.8.2b), the rotated state consistently exhibits a higher ratio of completion than the coherent reference state, $\mathcal{R}'_\tau(t) > \mathcal{R}_\tau(t)$, highlighting its faster approach to equilibrium. We now turn to a geometric bound on the time evolution of observables and their fluctuations [66, 68], discussed in Section IV.7.2,

$$\int_0^\tau \frac{|\dot{o}(t)|}{\Delta_{\hat{\rho}_t} \hat{O}} dt \leq \int_0^\tau \sqrt{I_Q^{(\text{SLD})}} dt = 2\mathcal{L}_{\text{SLD}}, \quad (\text{IV.8.17})$$

where $|\dot{o}(t)| = \left| \text{Tr} \left[\hat{O} \frac{d\hat{\rho}_t}{dt} \right] \right|$ denotes the instantaneous rate of change of the ob-

servable due to the state evolution, and $\Delta_{\hat{\rho}_t} \hat{O} = \sqrt{\text{Tr}[\hat{\rho}_t \hat{O}^2] - \text{Tr}[\hat{\rho}_t \hat{O}]^2}$ is its standard deviation in the instantaneous state. This inequality shows that the time-integrated ratio of observable change to fluctuation is bounded by twice the SLD statistical distance. Since $\mathcal{L}_{\text{SLD}} \geq \mathcal{L}_{\text{SLD}}^{\text{geo}}$ (see Eq. (IV.7.27)), the same ratio evaluated along a geodesic path yields the tightest possible bound.

Building on the general geometric bound for arbitrary observables discussed above, we now focus on the Hamiltonian $\hat{H}$, as its instantaneous change due to the state evolution corresponds to the heat current, denoted $|\dot{h}(t)| = |\dot{Q}(t)|$. In Fig. IV.8.3, we compare the integrated ratio $\int_0^t \frac{|\dot{h}(t')|}{\Delta_{\hat{\rho}_{t'}} \hat{H}} dt'$ (black line) with its bounds determined by the statistical distance (blue and red dashed lines), where the coloured regions indicate areas excluded by the bounds. Notably, for the incoherent, rotated state $\hat{\rho}'$, the statistical distance between $\hat{\rho}'$ and $\hat{\rho}'(t)$ coincides with the geodesic length at all times, yielding identical bounds (see Fig. IV.8.3b)). Moreover, the bound is saturated throughout the evolution, a consequence of the factorization of the covariance between the SLD operator $\hat{L}_{\text{SLD}}$ and $\hat{H}$ along the incoherent path

$$|\dot{h}(t)| = \text{cov}_{\hat{\rho}'_t}(\hat{H}, \hat{L}_{\text{SLD}}) = \Delta_{\hat{\rho}'_t} \hat{H} \Delta_{\hat{\rho}'_t} \hat{L}_{\text{SLD}} = \Delta_{\hat{\rho}'_t} \hat{H} \sqrt{I_Q^{(f)}(t)}, \quad (\text{IV.8.18})$$

as detailed in [66]. In contrast, this saturation is not observed for the initially coherent reference state, as illustrated in Fig. IV.8.3a). We now turn to the geometric uncertainty, which, for the evolution of the coherent reference state, exhibits metric dependence. As illustrated in Fig. IV.8.4a), the cumulative deviation from the respective geodesic paths for both initial states is primarily accumulated during the early stages of relaxation. Notably, the geometric uncertainty is initially larger for the incoherent, rotated state, independent of the chosen QFI metric. However, the time at which the geometric uncertainty of the rotated state surpasses that of the reference state varies depending on the specific metric employed. Finally, we analyse the trade-off between the time-averaged QFI $\mathcal{I}_f$ and the geometric uncertainty δ_f , as discussed in Section IV.8.2. Initially, all paths deviate strongly from their respective geodesics, and the bound implies that the high geometric uncertainty, reflecting a large time-variance in the speed of evolution, is accompanied by a correspondingly large time-averaged speed. Our results show that this ratio remains consistently lower, and therefore closer to the bound, throughout the relaxation of the rotated state, as illustrated in Fig. IV.8.4b). At long times, the bound is nearly saturated, as the state approaches the thermal steady state where $I_Q^{(f)}$ vanishes.

IV.8.4 Geometric bound on the excess entropy flux

NESs generally sustain nonvanishing currents, including entropic fluxes generated, for example, by heat exchange with thermal reservoirs. Additional contributions arise in nonstationary settings, such as during finite-speed driving protocols. To isolate these nonstationary effects, the total entropy flux is commonly decomposed into a housekeeping component, which sustains the NESS, and an excess component, which vanishes in the steady state [79], as discussed in Section I.2.5. The aim of this section is to derive an information-geometric bound on the excess entropy flux.

We assume GKSL dynamics, with a unique steady state $\hat{\pi}$ (see Section I.1.1). Then, using the nonequilibrium potential $\hat{\Phi} = -\log \hat{\pi}$ [270], the excess entropy flux can be written as $\dot{\Pi}_{\text{ex}} = -\text{Tr}[\dot{\hat{\rho}}\hat{\Phi}]$. This form allows us to apply the general observable speed limit (IV.7.45), yielding

$$|\dot{\Pi}_{\text{ex}}| \leq \sigma_{\hat{\rho}}^f[\hat{\Phi}] \sqrt{I_Q^{(f)}}. \quad (\text{IV.8.19})$$

Integrating this inequality over the driving time T gives

$$\int_0^T dt |\dot{\Pi}_{\text{ex}}| \leq \int_0^T dt \sigma_{\hat{\rho}}^f[\hat{\Phi}] \sqrt{I_Q^{(f)}}, \quad (\text{IV.8.20})$$

which provides a general upper bound on the total excess entropy flux, more precisely, on its integrated absolute value, i.e., the accumulated flux irrespective of its direction. Alternatively, using the geometric interpretation of $\sqrt{I_Q^{(f)}}$ as the instantaneous speed in state space, integration yields the statistical length,

$$\int_0^T dt \frac{|\dot{\Pi}_{\text{ex}}|}{\sigma_{\hat{\rho}}^f[\hat{\Phi}]} \leq \int_0^T dt \sqrt{I_Q^{(f)}} = 2 \mathcal{L}_f. \quad (\text{IV.8.21})$$

These bounds show that, for any GKSL evolution with a unique steady state, the geometry of the path traced in state space constrains the possible dissipation dynamics. Notably, the inequality (IV.8.21) is saturated when the system evolves along a geodesic of the chosen QFI metric. For certain metrics, such as those associated with the symmetric logarithmic derivative or the Wigner-Yanase skew information [337], the corresponding geodesic lengths can be obtained analytically, see Eq. (IV.7.27).

IV.8.5 Conclusion

For nonstationary irreversible quantum processes, establishing a connection between thermodynamics and information geometry is challenging due to the absence of strong constraints on the admissible quantum states far from equilibrium. Unlike equilibrium states or NESS, where stationarity, detailed balance, or the spectral properties of the Liouvillian impose significant restrictions, nonstationary quantum dynamics allow a much broader range of possible states, complicating both their thermodynamic characterization and their geometric description.

Recent works in classical stochastic thermodynamics have begun to bridge this gap, demonstrating that for processes arbitrarily far from equilibrium, entropy production and entropy flow rates can be bounded in terms of the geometric uncertainty associated with the path connecting any two states [70, 69].

In this chapter, we have established a corresponding connection in open quantum systems evolving through nonstationary states [71]. Specifically, for GKSL dynamics satisfying local detailed balance, we demonstrated that the QFI is directly related to the accelerations of entropy production and entropy flow. By decomposing the QFI into coherent and incoherent contributions, we found that the entropic acceleration is entirely determined by the incoherent part. Further, we strengthened the classical uncertainty relation linking the geometric uncertainty along a path in state space to the time-integrated rate of information change, and extended the classical bound on the rate of change of the von Neumann entropy by explicitly including the non-negative contribution from coherent dynamics. This bound is tightest when evaluated using the SLD QFI. Applying this framework, we demonstrated that the thermodynamic quantum Mpemba effect [318] can be analysed through geometric quantities such as the ratio of completion and deviations from the geodesic path. Furthermore, we established geometric bounds that constrain the excess entropy flux exchanged with the environment under arbitrary nonstationary GKSL dynamics with unique fixed points [78].

More generally, this approach provides a systematic framework for analysing information-geometric constraints on quantum dynamics and thermodynamic behaviour, and may help clarify the mechanisms underlying anomalous phenomena such as Mpemba-like effects in both closed and open quantum systems.

PART V:

Thermodynamic geometry of DQPTs

Chapter V.9

Dissipative quantum phase transitions

This chapter is based on Refs. [114, 125, 338, 339].

Phase transitions describe qualitative changes in a system's macroscopic behaviour induced by variations in external parameters such as temperature, external fields, or interaction strengths. Their study reveals how collective phenomena emerge from microscopic interactions, how fluctuations and correlations become long-ranged near critical points, and why small parameter changes can produce dramatic effects across a wide range of physical systems [340, 341].

Continuous phase transitions, the class of transitions that will be the focus of this chapter, are distinguished not only by diverging fluctuations and correlation lengths, but also by the emergence of universality. As the critical point is approached, microscopic details become increasingly irrelevant: long-wavelength collective modes dominate, and the macroscopic behaviour is governed by only a few general features, like dimensionality or symmetries. Consequently, systems with vastly different microscopic constituents can display identical critical behaviour. Despite the extensive number of degrees of freedom involved, the physics near criticality is fully characterised by a handful of universal critical exponents, which define a universality class. Remarkably, only a limited number of such classes are observed in nature, despite the enormous diversity of physical materials and engineered platforms [340, 341].

We now formalise these ideas, starting with thermal phase transitions (TPT). Here, continuous phase transitions are conveniently described in terms of an order parameter ϕ that distinguishes the disordered and ordered phases: its thermal average $\langle\phi\rangle$ vanishes on one side of the transition and becomes finite on the other. Although $\langle\phi\rangle = 0$ in the disordered phase, fluctuations do not vanish. Instead,

	TPT	QPT	DQPT
System	Hamiltonian	Hamiltonian	Liouvillian
operator	$\hat{H} = \hat{H}^\dagger$	$\hat{H} = \hat{H}^\dagger$	$\mathcal{L}$
Relevant quantity	Free energy $F_\beta(\hat{\rho}) = \langle \hat{H} \rangle_{\hat{\rho}} - \beta^{-1} S_{\text{vN}}(\hat{\rho})$	Energy eigenvalues $E_\psi : \hat{H} \psi\rangle = E_\psi \psi\rangle$	Liouvillian eigenvalues $\lambda_i : \mathcal{L}\hat{\rho}_i = \lambda_i \hat{\rho}_i$
State	Gibbs state $\hat{\omega}_\beta = \arg \min_{\hat{\rho}} [F_\beta(\hat{\rho})]$ $\hat{\omega}_\beta \propto e^{-\beta \hat{H}}$	Ground state $ \psi_0\rangle = \arg \min_{\psi} \langle \psi H \psi \rangle$ $[\hat{H} - E_{\psi_0}] \psi_0\rangle = 0$	NESS $\hat{\pi}_{\text{ss}} = \arg \min_{\hat{\rho}_i \ \hat{\rho}_i \ _{\text{tr}}=1} \ \mathcal{L} \hat{\rho}_i \ _{\text{tr}}$ $\mathcal{L} \hat{\pi}_{\text{ss}} = 0$
Phase transition	Nonanalyticity in $F_\beta(\hat{\omega}_\beta)$	$\Delta_H = E_{\psi_1} - E_{\psi_0}$ vanishes	$\Delta = -\min_{\lambda_i \neq 0} [\text{Re}(\lambda_i)]$ vanishes

Table V.9.1: Comparison of thermal phase transitions (TPT), quantum phase transitions (QPT), and dissipative quantum phase transitions (DQPT). Adapted from Ref. [101].

temporal and spatial correlations grow and ultimately diverge at the critical point, defining characteristic scales: the correlation length ξ and the relaxation time τ . These quantities obey power laws in terms of the distance from criticality $|g - g_c|$:

$$\xi(g) \sim |g - g_c|^{-\nu}, \quad \tau(g) \sim \xi^z \sim |g - g_c|^{-z\nu}, \quad (\text{V.9.1})$$

where ν and z are the correlation-length and dynamical critical exponents. The divergence of τ is often referred to as critical slowing down, reflecting the increasingly long time required for the system to recover from small disturbances near criticality. The simultaneous divergence of ξ and τ implies that there is no intrinsic length or time scale left in the system. As a result, the system becomes scale invariant at criticality. In this regime, the singular parts of observables such as the order parameter mean $\langle \phi \rangle$, susceptibility χ , and heat capacity C are governed by the diverging correlation length ξ , and thus also display power-law scaling:

$$\langle \phi \rangle \sim |g - g_c|^\beta, \quad \chi \sim |g - g_c|^{-\gamma}, \quad C \sim |g - g_c|^{-\alpha}, \quad (\text{V.9.2})$$

where β, γ, α are standard critical exponents, determined by the scaling dimensions of the corresponding observables. The universality of these exponents means they depend only on general features of the system [341, 342], so that systems sharing these features are said to belong to the same universality class and exhibit identical critical behaviour.

Closely related critical phenomena arise in equilibrium many-body quantum systems at zero temperature [340]. A quantum phase transition (QPT) occurs when, in the thermodynamic limit, the ground-state expectation value of an observable $\hat{O}$ becomes nonanalytic at the critical point $g = g_c$:

$$\lim_{g \rightarrow g_c} \left| \frac{\partial^M}{\partial g^M} \lim_{N \rightarrow \infty} \langle \psi_0(g, N) | \hat{O} | \psi_0(g, N) \rangle \right| = +\infty, \quad (\text{V.9.3})$$

where $|\psi_0(g, N)\rangle$ denotes the ground state of the Hamiltonian $\hat{H}(g)$ [340]. The hallmark of such a transition is the closing of the energy gap $\Delta_H(g)$ between the ground state and the excited states, typically following $\Delta_H(g) \sim |g - g_c|^{z\nu}$. This gap closing gives rise to critical slowing down: the Hamiltonian gap sets the intrinsic timescale for the coherent evolution of low-energy excitations, which diverges as

$$\tau_R \sim \Delta_H^{-1} \sim |g - g_c|^{-z\nu}. \quad (\text{V.9.4})$$

Here, τ_R is not a relaxation timescale associated with dissipation, but an intrinsic timescale for the system's coherent dynamics, in contrast to thermal transitions.

Several well-studied quantum many-body systems provide illustrative examples of the phenomenology of continuous QPTs. For instance, in the transverse-field Ising model, a paramagnet becomes ferromagnetic as the transverse field is tuned [340]. The Bose-Hubbard model undergoes a superfluid-to-Mott-insulator transition [343], the Lipkin-Meshkov-Glick model displays collective spin ordering [344], and the Dicke model exhibits a superradiant phase with macroscopic cavity occupation [114].

Even richer phenomenology emerges in open quantum systems, where phase transitions occur in NESSs as a control parameter is varied [81]. Unlike equilibrium QPTs, these dissipative quantum phase transitions (DQPTs) combine critical behaviour with genuinely nonequilibrium features, such as irreversible entropy production associated to NESS currents, and the interplay of coherent and driven-dissipative dynamics. DQPTs have been explored theoretically in photonic platforms [83–95], lossy polariton condensates [96–98], and spin systems [99–107]. Experimentally, they have been observed in platforms ranging from ultra-cold atomic gases [108, 109] to solid-state optical [110–112] and electronic [113] devices, enabling time-resolved studies of critical fluctuations [345] and entropy production across a DQPT [115]. Table V.9.1 summarises the key differences and analogies between TPTs, QPTs, and DQPTs. Unlike TPTs, which are organised by the properties of a free energy and are therefore comparatively well understood [341], and QPTs, which have been extensively studied using many-body and renormalization-group techniques [340], DQPTs remain far less explored. In contrast to equilibrium quantum phase transitions, the NESS of an open system is neither the ground state of a Hamiltonian nor, in general, a thermal state. Consequently, standard frameworks that treat dissipation as a weak perturbation to an underlying equilibrium structure are inadequate [346]. In equilibrium, critical exponents govern the scaling of observables near criticality. Remarkably, they also predict universal nonequilibrium responses when a system is driven towards or across a critical point, as captured by the celebrated Kibble-Zurek mechanism (KZM), originally formulated for classical phase transitions and later extended to

quantum critical points [347–349, 116]. While signatures of KZM-like dynamical scaling near DQPTs have been observed experimentally [350], in numerical studies of continuous DQPTs [351], and in theoretical analyses of pure dephasing models [352], a comprehensive account for general Markovian weakly coupled open quantum systems has remained lacking. We fill this gap in Ref. [125] and present the results in Chapter V.10.

The goal of this chapter is to provide the conceptual groundwork needed to follow that discussion. We begin with a concise review of dynamical universality underpinning the Kibble-Zurek mechanism and its central predictions (see Section V.9.1). We then introduce DQPTs in open quantum systems obeying GKSL dynamics (see Section I.1.1) and identify the quantities relevant to dynamical scaling in this setting (see Section V.9.2). Next, we discuss two paradigmatic and experimentally accessible models exhibiting continuous DQPTs: the driven-dissipative Dicke model (see Section V.9.3.1) and the driven-dissipative Kerr oscillator (see Section V.9.3.2). Finally, we outline the finite-size scaling approach appropriate for such fully connected models (see Section V.9.4).

V.9.1 Kibble-Zurek mechanism

This section is based on Ref. [116].

When a classical or closed quantum system is driven across a second-order (continuous) phase transition at a finite rate, diverging intrinsic timescales, arising from critical slowing down or gap closing, inevitably lead to a breakdown of adiabatic evolution. The Kibble-Zurek mechanism (KZM) provides a unifying framework for this nonequilibrium response, predicting universal dynamical scaling laws based solely on equilibrium critical exponents. Its central insight is that near the critical point the system can no longer follow the instantaneous equilibrium state within a finite window around g_c , defining the so-called impulse regime. Outside this region, the system can evolve adiabatically, but within it, the dynamics effectively “freeze”, leading to the formation of defects or excitations whose density is determined by universal scaling. The onset of the impulse regime is therefore typically referred to as the freeze-out point $\hat{t}$, as the time at which the system’s intrinsic timescale τ exceeds the time scale set by the driving. For a linear ramp protocol, $\varepsilon(t) = (g(t) - g_c)/g_c = \frac{t}{\tau_Q}$, $\hat{t}$ is defined by the condition

$$\tau(\varepsilon(\hat{t})) \sim \left| \frac{\varepsilon(\hat{t})}{\dot{\varepsilon}(\hat{t})} \right|, \quad (\text{V.9.5})$$

As a result, the system “falls out of equilibrium”. This yields the characteristic Kibble-Zurek scaling relations

$$\hat{t} \sim \tau_Q^{\frac{z\nu}{1+z\nu}}, \quad \hat{\xi} \sim \tau_Q^{\frac{\nu}{1+z\nu}}. \quad (\text{V.9.6})$$

The length scale $\hat{\xi}$ determines the typical size of correlated domains that form during the passage through criticality. Correspondingly, the density of defects is set by the inverse of this scale, $n_{\text{KZ}} \sim \hat{\xi}^{-d}$ in d spatial dimensions. These “defects” may be topological objects such as vortices in superfluid helium or magnetic domain walls in ferromagnets (see the review [116] and references therein), while in quantum many-body systems, they correspond to excitations above the ground state (see the reviews [349, 353] and references therein). Crucially, the KZM predicts that n_{KZ} depends only on τ_Q and the critical exponents z and ν , demonstrating that the nonequilibrium response to finite-rate driving is itself universal. The same universal scaling extends to thermodynamic quantities like the dissipated work¹, defined as the excess work beyond the reversible free-energy change, as well as its cumulants [119–121, 123, 354], reflecting the intrinsic irreversibility of finite-rate driving to or through critical points.

V.9.2 DQPTs within GKSL dynamics

This section is based on Refs. [338, 125].

We now focus on systems displaying a DQPT whose dynamics are governed by a GKSL master equation with parameterised Liouvillian generator $\mathcal{L}_g$ (see Section I.1.1)². A DQPT occurs when the NESS $\hat{\pi}_g$ exhibits nonanalytic behaviour in the thermodynamic limit at the critical point g_c [338]. More specifically, in the thermodynamic limit $N \rightarrow \infty$, a transition between two different phases is characterised by the nonanalytical behaviour of some g -independent observable $\hat{O}$ when g tends to the critical value g_c . A phase transition of order M occurs if

$$\lim_{g \rightarrow g_c} \left| \frac{\partial^M}{\partial g^M} \lim_{N \rightarrow \infty} \text{Tr} [\hat{\pi}_{g,N} \hat{O}] \right| = +\infty. \quad (\text{V.9.7})$$

Since $\hat{O}$ is independent of g , the discontinuity is due to a discontinuous behaviour in $\hat{\pi}_{g,N \rightarrow \infty}$. Under the standard assumptions of a unique NESS away from crit-

¹The dissipated work quantifies irreversibility and is defined as $W_{\text{diss}} = W - \Delta F$, where W is the total work performed and $\Delta F = F_{\text{final}} - F_{\text{initial}}$ is the free-energy difference of the system, with $F = -k_B T \ln Z$ and Z the equilibrium partition function.

²A spectral theory for DQPTs in non-Markovian settings was recently proposed in Ref. [355].

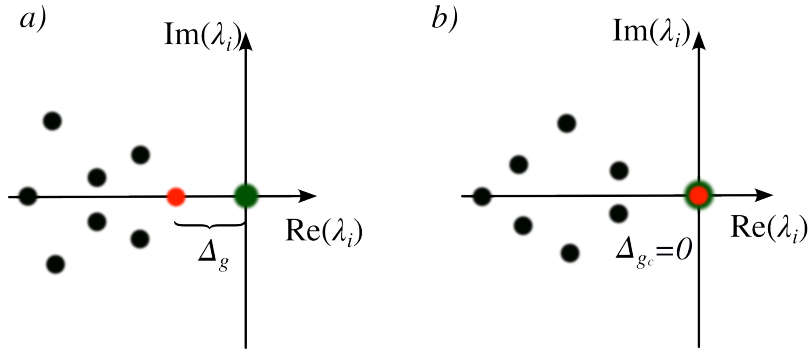


Figure V.9.1: a) The Liouvillian spectrum $\{\lambda_i\}$, shown in the complex plane, has a finite Liouvillian gap Δ_g away from the critical point. The eigenvalue associated with the slowest decaying mode is shown in red, and the zero eigenvalue corresponding to the unique steady state in green. b) At the critical coupling g_c , the Liouvillian gap closes. Adapted from Ref. [125].

icality and a regular (diagonalizable) Liouvillian spectrum, a discontinuity of an eigenmatrix is associated with a level crossing in the spectrum of the Liouvillian [356], and since $\hat{\pi}_{g,N}$ is associated with the zero-eigenvalue of the Liouvillian, the phase transition must coincide with the closure of the Liouvillian gap [101, 357], as shown schematically in Fig. V.9.1. The closing of this gap signals critical slowing down, since the relaxation time scales as $\tau_R \sim 1/\Delta_g$ (see Section I.3.3). This phenomenon is the dynamical analogue of the vanishing Hamiltonian gap in QPTs, see Table V.9.1. Near criticality, for a continuous DQPT³, when the Liouvillian control parameter g is tuned towards its critical value g_c , the Liouvillian gap is expected to close algebraically,

$$\Delta_g \sim |g - g_c|^\gamma. \quad (\text{V.9.8})$$

In this setting, the Liouvillian gap Δ_g and its critical exponent γ play roles analogous to the Hamiltonian gap Δ_H and $z\nu$ in equilibrium quantum phase transitions, and they set the relevant timescale for universal dynamical behaviour: the relaxation time $\tau_R \sim 1/\Delta_g$. The relaxation time determines how quickly the system can adjust to changes in the control parameter and thus sets the boundary between adiabatic and impulse regimes during a finite-time driving protocol. In addition, the structure of the NESS as well as the slowest Liouvillian eigenmode that governs deviations from it become central objects. Identifying how these quantities scale near criticality provides the foundation for revealing dynamical universality in open quantum systems and for establishing the appropriate analogue of the Kibble-Zurek framework in the dissipative setting, as we will see in Chapter V.10.

³This means that the order of the phase transition is $M > 1$.

V.9.3 Examples

We now introduce two paradigmatic driven-dissipative models: the Dicke model (see Section V.9.3.1) and the Kerr oscillator (see Section V.9.3.2). Both systems exhibit continuous DQPTs in the thermodynamic limit, characterised by a spontaneous $\mathbb{Z}_2$ symmetry⁴-breaking bifurcation of the NESS as a control parameter is tuned [114, 358]. At finite system size, the steady states are generally non-Gaussian, but in the thermodynamic limit (large atom number for the Dicke model, or large photon number for the Kerr oscillator) the dynamics are accurately described by Gaussian bosonic approximations. These models will serve as concrete examples of dissipative criticality and will be revisited in Chapter V.10.

V.9.3.1 Driven-dissipative Dicke model

This section is based on Refs. [114, 125].

The Dicke model describes the collective interaction of N two-level atoms with a single-mode electromagnetic field in a cavity [359]. In its closed-system form, the model exhibits the phenomenon of superradiance, in which the atoms emit light collectively, leading to a macroscopic occupation of the cavity mode once a critical atom-field coupling is reached. This behaviour can be understood as a QPT from a normal phase, with the cavity nearly empty, to a superradiant phase, where the field acquires a nonzero expectation value and the atomic ensemble becomes collectively polarised [360–362]. In the open Dicke model, cavity photon loss (and possibly other dissipative processes) are explicitly included. Remarkably, the open system still displays a continuous phase transition that mirrors the equilibrium superradiant transition: as the atom-field coupling crosses a critical value, the NESS undergoes a $\mathbb{Z}_2$ symmetry-breaking bifurcation, the cavity becomes macroscopically occupied, and the atoms collectively polarise [114, 363, 350, 364–366, 82]. At the same time, the steady-state photon flux emitted from the cavity provides a direct experimental signature of the transition [350]. This model has been realised in several experimental platforms, such as Bose-Einstein condensates coupled to high-finesse optical cavities [108, 350], superconducting quantum circuits [367] or trapped-ion arrays [368].

The Dicke Hamiltonian describes the collective light-matter interaction between

⁴A $\mathbb{Z}_2$ symmetry is a simple discrete symmetry corresponding to a two-element group, typically represented by a sign flip. In quantum systems, it can be implemented as a unitary operator $\hat{U}$ with $\hat{U}^2 = \mathbb{I}$, such as $\hat{U} = \prod_i \hat{\sigma}_i^x$ in spin models. In bosonic systems, a $\mathbb{Z}_2$ symmetry can be realised as a parity transformation that flips the sign of the field operators: $\hat{a} \rightarrow -\hat{a}$ and $\hat{a}^\dagger \rightarrow -\hat{a}^\dagger$, corresponding to the unitary operator $\hat{U} = e^{i\pi\hat{a}^\dagger\hat{a}}$.

N two-level atoms and a single-mode electromagnetic field and reads

$$\hat{H} = \omega_c \hat{a}^\dagger \hat{a} + \omega_z \hat{J}_z + \frac{g}{\sqrt{N}} (\hat{a} + \hat{a}^\dagger) (\hat{J}_+ + \hat{J}_-), \quad (\text{V.9.9})$$

where $\hat{a}, \hat{a}^\dagger$ are the annihilation and creation operators of the cavity mode, $\hat{J}_z, \hat{J}_\pm$ are collective spin operators for N two-level atoms, ω_c is the cavity frequency, ω_z is the atomic transition frequency, and g is the atom-light coupling strength. Note that the Hamiltonian possesses a $\mathbb{Z}_2$ parity symmetry associated with simultaneous sign changes of the field and collective spin. To treat the normal phase of the Dicke model in the thermodynamic limit, it is useful to employ the Holstein-Primakoff transformation [369], which maps collective spin operators to bosonic modes [369, 363]

$$\hat{J}_+ = \hat{b}^\dagger \sqrt{2J - \hat{b}^\dagger \hat{b}} \approx \sqrt{2J} \hat{b}^\dagger, \quad \hat{J}_- = \sqrt{2J - \hat{b}^\dagger \hat{b}} \hat{b} \approx \sqrt{2J} \hat{b}, \quad \hat{J}_z = \hat{b}^\dagger \hat{b} - J, \quad (\text{V.9.10})$$

where $J = N/2$ and $\hat{b}$ is a bosonic operator with $[\hat{b}, \hat{b}^\dagger] = 1$. Here, we have used that in the thermodynamic limit $N \rightarrow \infty$, the system, in the normal phase, is assumed to remain close to the collective spin ground state, which justifies the expansion $\sqrt{2J - \hat{b}^\dagger \hat{b}} \approx \sqrt{2J}$ and allows one to treat the spin fluctuations as a harmonic bosonic mode. In other words, the Holstein-Primakoff boson describes small quantum fluctuations around a fully polarised collective spin, with $\langle \hat{b}^\dagger \hat{b} \rangle / N \ll 1$. In this limit, substituting into Eq. (V.9.9), the Hamiltonian becomes for the normal phase ($g < g_c$)

$$\hat{H}_N \approx \omega_c \hat{a}^\dagger \hat{a} + \omega_z \hat{b}^\dagger \hat{b} + g(\hat{a} + \hat{a}^\dagger)(\hat{b} + \hat{b}^\dagger), \quad (\text{V.9.11})$$

which is a quadratic Hamiltonian in the bosonic modes $\hat{a}$ and $\hat{b}$. To include dissipation, we add a Lindblad term modelling single photon loss out of the cavity, so that

$$\dot{\hat{\rho}} = -i[\hat{H}, \hat{\rho}] + \kappa \mathcal{D}[\hat{a}]\hat{\rho}, \quad (\text{V.9.12})$$

where κ is the cavity decay rate. In Section I.3.2, we introduced GKSL dynamics for Gaussian bosonic and fermionic systems. Adopting the notation established there, the driven-dissipative Dicke model can be expressed in terms of the following matrices

$$A_N = \begin{pmatrix} \omega_c & g \\ g & \omega_z \end{pmatrix}, \quad B_N = \begin{pmatrix} 0 & g \\ g & 0 \end{pmatrix} \quad (\text{V.9.13})$$

for the normal phase Hamiltonian. Further,

$$\Gamma = \begin{pmatrix} 2\kappa & 0 \\ 0 & 0 \end{pmatrix}. \quad (\text{V.9.14})$$

The critical point in the atom-light coupling strength can be obtained by solving $\det(W) = 0$ [114]:

$$g_c = \frac{1}{2} \sqrt{\frac{\omega_c^2 + \kappa^2}{\omega_c}} \omega_z. \quad (\text{V.9.15})$$

V.9.3.2 Driven-dissipative Kerr oscillator

This section is based on Refs. [358, 125].

Another widely studied platform exhibiting DQPTs is the driven-dissipative Kerr oscillator, a single bosonic cavity mode with a Kerr nonlinearity⁵ subject to coherent two-photon (parametric) driving and single-photon loss [370, 358]. In an appropriate thermodynamic limit, semiclassical and mean-field analyses reveal a $\mathbb{Z}_2$ symmetry-breaking bifurcation: as the drive strength crosses a critical value the cavity acquires a macroscopic occupation, leading to a continuous DQPT [370, 90, 358, 338]. Although not the subject of our analysis, we note that the model also features a discontinuous DQPT [372]. Interestingly, this symmetry breaking has found applications in quantum error correction [373]. Further, for several driving and dissipation configurations exact steady-state solutions are known [90], making the Kerr resonator a paradigmatic and analytically tractable model for universal phenomena in open quantum many-body physics. Further, this model has been realised in several experimental platforms such as semiconductor microcavities (see, for example, Refs. [111, 112]) or superconducting quantum circuits (see, for example, Refs. [374, 375]).

In a reference frame rotating at the parametric pump frequency, the Hamiltonian reads [358]

$$\hat{H} = -\Delta \hat{a}^\dagger \hat{a} + \frac{U}{2} \hat{a}^{\dagger 2} \hat{a}^2 + \frac{1}{2} (G \hat{a}^{\dagger 2} + G^* \hat{a}^2), \quad (\text{V.9.16})$$

where Δ is the detuning between the cavity and pump, U the Kerr interaction strength, and G the amplitude of the parametric drive, which creates or annihilates photons in pairs. The open system dynamics is induced by single-photon loss at rate κ out of the cavity. The density operator $\hat{\rho}$ evolves according to the GKSL

⁵The Kerr nonlinearity describes a shift of the oscillator frequency that depends on its photon number, which makes the oscillator anharmonic. It represents a photon-photon interaction and is responsible for effects such as optical bistability [370] and the generation of nonclassical states in cavity and circuit-QED systems [371].

master equation

$$\frac{d}{dt}\hat{\rho} = -i[\hat{H}, \hat{\rho}] + \kappa\mathcal{D}[\hat{a}]\hat{\rho}, \quad (\text{V.9.17})$$

We now define the mean field $\alpha(t) = \langle \hat{a}(t) \rangle$. Taking the expectation value of the Heisenberg equation (and including the damping contribution from the Lindblad term) yields the semiclassical equation of motion

$$\frac{d}{dt}\alpha = i\Delta\alpha - iU|\alpha|^2\alpha - iG\alpha^* - \frac{\kappa}{2}\alpha. \quad (\text{V.9.18})$$

Steady states α_{ss} satisfy $d\alpha/dt = 0$. The semiclassical theory predicts a critical point at $\Delta_c = -\sqrt{G^2 - \frac{\kappa^2}{4}}$, below which $\alpha_{ss} = 0$ is the unique stable solution, and above which finite-amplitude solutions appear marking spontaneous phase symmetry breaking [372]. To capture small quantum fluctuations, we decompose the field operator as $\hat{a} = \alpha_{ss} + \hat{d}$, with $\langle \hat{d} \rangle = 0$. Keeping terms up to quadratic order in $(\hat{d}, \hat{d}^\dagger)$ yields a Gaussian theory for the fluctuations. We find the linearised equation of motion

$$\frac{d}{dt}\vec{R}_c = M\vec{R}_c + \vec{\xi}(t), \quad \vec{R}_c = \begin{pmatrix} \hat{d} \\ \hat{d}^\dagger \end{pmatrix}, \quad (\text{V.9.19})$$

where $\vec{\xi}(t)$ denotes the input noise from the bath. The drift matrix M is

$$M = \begin{pmatrix} i(\Delta - 2U|\alpha_{ss}|^2) - \frac{\kappa}{2} & -i(U\alpha_{ss}^2 + G) \\ i(U\alpha_{ss}^{*2} + G^*) & -i(\Delta - 2U|\alpha_{ss}|^2) - \frac{\kappa}{2} \end{pmatrix}. \quad (\text{V.9.20})$$

Below threshold ($\alpha_{ss} = 0$) this matrix simplifies and describes a stable linear damped parametric oscillator. We now introduce the canonical quadratures

$$\hat{X} = \frac{\hat{d} + \hat{d}^\dagger}{\sqrt{2}}, \quad \hat{P} = \frac{\hat{d} - \hat{d}^\dagger}{i\sqrt{2}}, \quad \vec{R}_q = \begin{pmatrix} \hat{X} \\ \hat{P} \end{pmatrix}. \quad (\text{V.9.21})$$

The transformation to quadrature variables is given by

$$\vec{R}_q = V\vec{R}_c, \quad V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}. \quad (\text{V.9.22})$$

The linearised Langevin equation in quadrature form is

$$\frac{d}{dt}\vec{R}_q = W\vec{R}_q + \vec{\eta}(t), \quad W = VMV^{-1}, \quad \vec{\eta}(t) = V\vec{\xi}(t), \quad (\text{V.9.23})$$

with W the drift matrix in quadrature coordinates and $\vec{\eta}$ the corresponding noise vector. The input noise is Gaussian and white with zero mean, $\langle \vec{\eta}(t) \rangle = \vec{0}$, and

symmetrised correlation

$$\frac{1}{2}\langle\vec{\eta}_i(t)\vec{\eta}_j(t') + \vec{\eta}_j(t')\vec{\eta}_i(t)\rangle = Y_{ij}\delta(t-t'). \quad (\text{V.9.24})$$

For single-photon loss into a vacuum bath the diffusion (noise) matrix in the quadrature basis is $Y = \frac{\kappa}{2} \mathbb{I}_2$, which corresponds to vacuum input noise with strength set by the loss rate. The linearised dynamics we discuss here, in agreement with those derived from Keldysh action [358] by which it can be shown that non-Gaussian fluctuations can be neglected in the thermodynamic limit, preserve Gaussianity, so the state is fully specified by the first moments and the covariance matrix (see Section I.3.2).

V.9.4 Finite-size scaling without spatial degrees of freedom

This section is based on Refs. [125, 339].

Scaling theory describes how physical observables behave near a critical point, where fluctuations span all scales and correlation lengths diverge [376, 342, 340]. Initially developed for classical equilibrium phase transitions, it links macroscopic behaviour to universal critical exponents and identifies the relevant scaling variables [377]. In conventional systems, spatial dimensionality governs the universality class and scaling relations. In models without spatial structure, such as zero-dimensional systems or fully connected mean-field models, as in the driven-dissipative Dicke and Kerr models we treat in this thesis, length scales and spatial correlations lose their standard meaning. Here, finite-size (or rather generalised) scaling is controlled by an extensive quantity, like a particle or mode number. In infinitely coordinated systems, the so-called coherence number replaces the correlation length, diverging in the thermodynamic limit [339, 378], $N_c \sim |g - g_c|^{-\nu^*}$. Assume we are interested in the finite-size scaling behaviour of a quantity that diverges in the thermodynamic limit as the system approaches the critical point

$$A_\infty \sim |g - g_c|^{-\alpha_A}. \quad (\text{V.9.25})$$

The scaling hypothesis postulates the existence of a regular scaling function F_A such that, near g_c and for large N ,

$$A_N \sim |g - g_c|^{-\alpha_A} F_A\left(\frac{N}{N_c}\right). \quad (\text{V.9.26})$$

Consequently, one expects a collapse of the curves when plotting

$$A_N |g - g_c|^{\alpha_A} \quad \text{vs.} \quad N |g - g_c|^{\nu^*}, \quad (\text{V.9.27})$$

and, at the critical point,

$$A_N(g_c) \sim N^{\alpha_A/\nu^*}. \quad (\text{V.9.28})$$

Interestingly, the effective exponent ν^* is related to the mean-field correlation length exponent ν_{MF} and the upper critical dimension d_c of a corresponding short-range model via

$$\nu^* = \nu_{\text{MF}} d_c. \quad (\text{V.9.29})$$

The approach detailed here will be adopted to investigate finite-size scaling at the continuous DQPT in the open Kerr parametric oscillator in Section [V.10.5](#).

Chapter V.10

Thermodynamic universality in DQPTs

The results presented in this chapter are based on Ref. [125]. I performed the analytical and numerical calculations, carried out the data analysis, and wrote the manuscript. The co-authors contributed to the conceptualisation of the work, discussed the results, and provided feedback and revisions on the manuscript. The presentation closely follows this reference.

A comprehensive understanding of the KZM and its universal thermodynamic signatures in DQPTs (see Chapter V.9 for an overview) has remained elusive. A central difficulty lies in identifying a meaningful measure of dissipation that exhibits universal scaling, analogous to the excess work in the near-equilibrium KZM, given that NESSs dissipate energy continuously, even without external driving. Moreover, in contrast to closed quantum systems, there is no natural notion of “excitations” above a ground state in open dynamics: only the NESS is a normalised density operator, while all other Liouvillian eigenmodes are traceless decay modes rather than physical states (see Section I.3.3). Their “occupation” (i.e., spectral weight) therefore carries no interpretation as populations of excited states. In this chapter, we resolve this conceptual challenge by leveraging recent advances in the information geometry of quantum NESSs [78]. We show that in DQPTs, the nonadiabatic entropy production naturally quantifies deviations from the NESS manifold caused by finite-time driving. For continuous DQPTs, both the nonadiabatic entropy production rate and its time-integrated counterpart display Kibble-Zurek-like universal power-law scaling with the driving speed, determined by the Liouvillian gap exponent γ and the KMB QFI critical exponent α . Remarkably, for Gaussian bosonic systems we find that, to leading order, the total nonadiabatic entropy production becomes independent of the driving speed. This highlights a

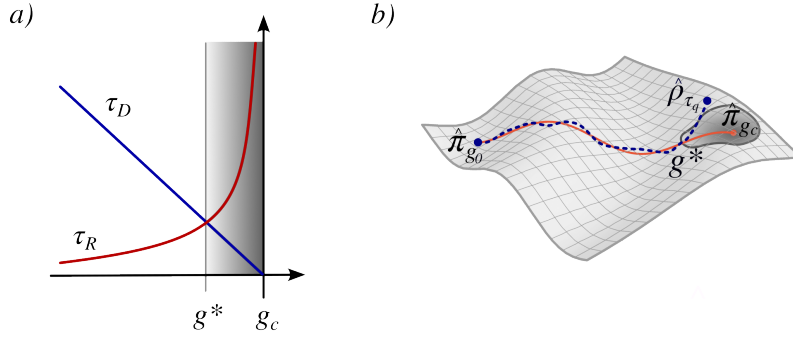


Figure V.10.1: a), b) During a finite-time slow protocol approaching criticality, the evolving state (blue line in b) eventually fails to adiabatically follow the instantaneous steady state (red line in b) and departs from the steady-state manifold. This defines the freeze-out point, $g^* = g(t^*)$, occurring when the intrinsic relaxation time τ_R exceeds the driving time scale τ_D (V.10.3). The protocol thus separates into a quasiadiabatic and an impulse regime, indicated in grey. Adapted from Ref. [125].

distinctive dynamical universality specific to Gaussian bosonic criticality. We validate our analytical predictions numerically in the driven-dissipative Dicke and Kerr models.

V.10.1 Set-up

We consider a continuous DQPT in an open quantum system governed by a GKSL master equation $d\hat{\rho}/dt = \mathcal{L}_g[\hat{\rho}]$ (see Eq. (I.1.8)), where the Liouvillian superoperator $\mathcal{L}_g$ depends on a control parameter g . The corresponding NESS $\hat{\pi}_g$, defined by $\mathcal{L}_g[\hat{\pi}_g] = 0$, undergoes a DQPT as g is tuned through its critical value g_c (see Section V.9.2). Relaxation toward the NESS is governed by the Liouvillian gap Δ_g (see Section I.1.4). Near a continuous DQPT, this gap closes algebraically [358, 379, 380, 101],

$$\Delta_g \sim |g - g_c|^\gamma, \quad (\text{V.10.1})$$

leading to a diverging relaxation time $\tau_R = \Delta_g^{-1}$. The quantities Δ_g and γ thus play roles analogous to the Hamiltonian gap Δ_H and critical exponent combination $z\nu$ in equilibrium quantum phase transitions (see Section V.9.2). Dynamical critical behaviour in open quantum systems characterises the universal nonequilibrium response of the system to slow parameter variations in the vicinity of a DQPT. To probe this behaviour, we consider a linear ramp of a control parameter g towards its critical value g_c ,

$$g(t) = g_0 + (g_c - g_0) \frac{t}{\tau_q}, \quad \dot{g}(t) \sim \frac{1}{\tau_q}, \quad t \in [0, \tau_q], \quad (\text{V.10.2})$$

where τ_q denotes the total quench duration. Far away from criticality, τ_R is short compared to the external time scale of the drive, and the system has sufficient

time to relax (close) to the instantaneous NESS. However, close to g_c , the intrinsic relaxation becomes too slow for the system to remain adiabatic. As discussed in Section V.9.1, to formalise the boundary between these two regimes, it is useful to invoke the standard reasoning behind the KZM. The key idea is to compare two characteristic time scales: the driving time scale, which quantifies the rate at which changes are imposed on the system, $\tau_D = |\Delta_g / \dot{\Delta}_g|$, and the intrinsic relaxation time due to the dissipative dynamics $\tau_R(g)$. For a DQPT, the freeze-out point is then defined by the moment at which these time scales become equal. If we denote this time by $t^* = \tau_q - \bar{t}$, with corresponding control-parameter value $g^* = g(t^*)$, the condition reads

$$\tau_D(g^*) = \left| \frac{\Delta_{g^*}}{\dot{\Delta}_{g^*}} \right| = \tau_R(g^*), \quad g^* = g(t^*). \quad (\text{V.10.3})$$

Before this time, the system evolves quasiadiabatically. After this time, it departs from the instantaneous NESS and enters the so-called impulse regime. This behaviour is illustrated schematically in Fig. V.10.1. Importantly, the solution of Eq. (V.10.3) yields the characteristic scaling relations that define the dissipative KZM. Using the fact that the Liouvillian gap closes near criticality as $\Delta_g \sim |g - g_c|^\gamma$, one finds the standard KZM power laws for the impulse duration $\bar{t}$ and for the distance from criticality at freeze-out

$$\bar{t} \sim \tau_q^{\frac{\gamma}{1+\gamma}}, \quad |g^* - g_c| \sim \tau_q^{-\frac{1}{1+\gamma}}, \quad (\text{V.10.4})$$

as we discussed in Section V.9.1. These expressions thus generalise the familiar results for isolated systems [116–118, 381, 382] to open quantum dynamics, with the crucial modification that the Liouvillian gap replaces the Hamiltonian energy gap as the quantity governing adiabaticity and critical slowing down. Thus, despite the fundamentally nonequilibrium nature of DQPTs, the scaling behaviour of slow quenches follows a structure directly analogous to the known KZM.

V.10.2 Impulse-regime identification through fidelity dynamics

To test the adiabatic-impulse approximation when approaching a DQPT, we track the Uhlmann fidelity to the instantaneous NESS along slow driving protocols (see, e.g., Ref. [383]). The breakdown of adiabatic following then defines the impulse regime and allows extraction of freeze-out scaling near the critical point. We apply this approach to the open Dicke model, introduced in Section V.9.3.1. In the thermodynamic limit, this model admits a Gaussian bosonic description, and

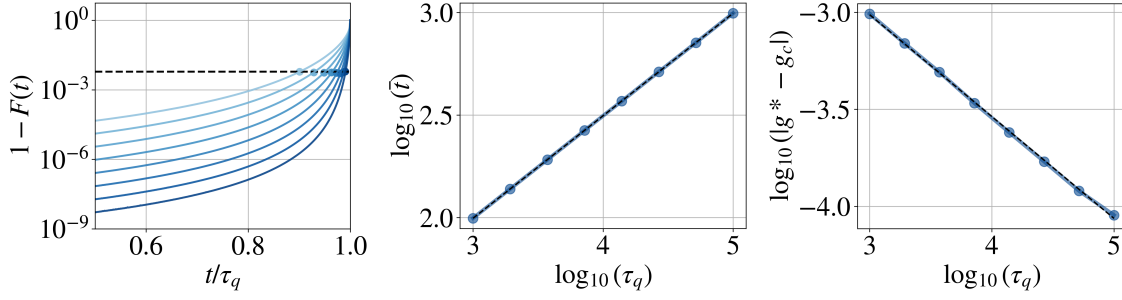


Figure V.10.2: To identify the onset of the impulse regime, we compute the infidelity to the instantaneous NESS along linear quenches of duration $\tau_q = 10^3, \dots, 10^5$ from the normal phase to the critical point in the open Dicke model. Left: Infidelity as a function of time for different quench durations. The freeze-out time t^* coincides with the point along the protocol at which the infidelity first exceeds 6×10^{-3} , signaling the breakdown of adiabatic following. Center: Scaling of the freeze-out time $\tilde{t} = |t^* - \tau_q|$ with the quench duration τ_q . A power-law fit to the four largest values of τ_q yields an exponent $x = 0.4996 \pm 0.0002$, in excellent agreement with the theoretical freeze-out condition (V.10.4) for $\gamma = 1$. Right: Scaling of the distance of the control parameter from criticality at the freeze-out point. A power-law fit to the four largest values of τ_q gives an exponent $x = -0.525 \pm 0.005$, again consistent with the theoretical prediction (V.10.4) for $\gamma = 1$. Parameters: $g_0 = g_c - 10^{-2}$, $g_f = g_c - 10^{-15}$, $\omega_c = 1$, $\omega_z = 1$, and $\kappa = 0.2$.

the fidelity between two states can be computed using their respective covariance matrices, as we detail in Appendix A.6. We find clear signatures of the adiabatic-impulse crossover when the system is driven to the critical point. The freeze-out time t^* is identified as the first time at which the infidelity exceeds a small threshold (here 6×10^{-3}). The extracted scaling exponents are robust under moderate variations of this choice. Scaling of both the freeze-out time and the distance of the control parameter from criticality with the protocol duration is consistent with the predicted Kibble-Zurek scaling (V.10.4), confirming that the fidelity to the instantaneous steady state provides a robust probe of the breakdown of adiabatic following, as we show in Fig. V.10.2.

V.10.3 Universality of dissipation

In driven open quantum systems, any finite rate of change of external parameters inevitably drives the system away from its instantaneous NESS. Quantifying this departure is essential for characterizing the additional irreversibility and dissipation that underlie KZM-like scaling. This role is played by the nonadiabatic entropy production Σ_{na} (see Section I.2.5), which provides a rigorous measure of how strongly the actual evolution deviates from the adiabatic trajectory that would perfectly follow the family of NESSs $\hat{\pi}_g$ [79, 384, 186]. Operationally, Σ_{na} monitors how distinguishable the evolving state is from the instantaneous NESS. This is most transparent when expressed in terms of the quantum relative entropy. The instantaneous rate of nonadiabatic entropy production is given by (see

Section I.2.5)

$$\dot{\Sigma}_{\text{na}}(t) = - \left. \frac{d}{ds} D(\hat{\rho}_s \parallel \hat{\pi}_{g(t)}) \right|_{s=t}, \quad (\text{V.10.5})$$

where $\hat{\rho}_s$ is the system's state at time s , and $\hat{\pi}_{g(t)}$ is the instantaneous NESS of $\mathcal{L}_{g(t)}$. This expression makes the physical meaning explicit: $\dot{\Sigma}_{\text{na}}(t)$ measures the instantaneous rate at which the system's state diverges from the steady state associated with the current value of the control parameter. Whenever the driving prevents the system from perfectly tracking this manifold of NESSs, additional entropy is produced, reflecting the added irreversibility inherent to finite-time protocols. When the system is driven slowly enough that $\hat{\rho}_t$ remains close to $\hat{\pi}_{g(t)}$, the deviation can be treated perturbatively. In particular, one expands $\hat{\rho}_t$ around $\hat{\pi}_{g(t)}$ in powers of the driving speed $\dot{g} \sim 1/\tau_q$, which plays the role of a small parameter in the slow-driving expansion [314, 78, 75, 74, 195, 76]. Under this expansion, and as shown in Ref. [78] and reviewed in Section IV.7.5, the leading nonvanishing contribution to the entropy-production rate is quadratic in the driving speed, yielding

$$\dot{\Sigma}_{\text{na}}(t) \approx \dot{g}(t) \zeta_{g(t)} \dot{g}(t). \quad (\text{V.10.6})$$

Here, ζ_g is the thermodynamic metric on the space of control parameters (see Section IV.7.5). Remarkably, this metric factorises into two conceptually distinct quantities:

$$\zeta_g = \tau_g I_Q^{(\text{KMB})}(g), \quad (\text{V.10.7})$$

where τ_g is a generalised integral relaxation known from linear-response theory [385], and $I_Q^{(\text{KMB})}(g)$ is the KMB QFI associated with the instantaneous steady state $\hat{\pi}_g$ (see Section IV.7.5). As discussed in Section IV.7.2, the latter quantifies the sensitivity of $\hat{\pi}_g$ to changes in the parameter g , thereby linking nonequilibrium thermodynamics with information geometry. Importantly, the assumptions underlying Eq. (V.10.6) remain valid up to the freeze-out point in a slow finite-time ramp. This behaviour is explicitly demonstrated in Fig. V.10.3 a), which shows that the metric approximation continues to hold until the system enters the impulse regime.

To derive scaling relations for the nonadiabatic entropy production evaluated at the freeze-out point, it is essential to understand how the two components of the thermodynamic metric behave near the dissipative critical point. Specifically, we must determine the scaling of the generalised integral relaxation time τ_g and of the KMB QFI. We begin by analysing the generalised integral relaxation time, introduced in Eq. (IV.7.75). To make its critical behaviour transparent, it is useful to express the relevant correlation function in the spectral decomposition of the Liouvillian. For this purpose, we expand operators in the biorthonormal eigenbasis

$\{\lambda_n, |\hat{r}_n\rangle\rangle, \langle\langle \hat{l}_n| \}$ of the vectorised Liouvillian (see Section I.1.4), defined through

$$\underline{\mathcal{L}}_g |\hat{r}_n\rangle\rangle = \lambda_n |\hat{r}_n\rangle\rangle, \quad \langle\langle \hat{l}_n| \underline{\mathcal{L}}_g = \lambda_n \langle\langle \hat{l}_n|, \quad (\text{V.10.8})$$

and satisfying the biorthonormality condition $\langle\langle \hat{l}_n| \hat{r}_m\rangle\rangle = \delta_{nm}$. In this basis, the time-evolved operator $|\hat{F}_g(\tau)\rangle\rangle = e^{\underline{\mathcal{L}}_g \tau} |\hat{F}_g\rangle\rangle$ admits the expansion

$$|\hat{F}_g(\tau)\rangle\rangle = \sum_n e^{\lambda_n \tau} |\hat{r}_n\rangle\rangle \langle\langle \hat{l}_n| \hat{F}_g\rangle\rangle. \quad (\text{V.10.9})$$

Substituting this expression into the correlation function appearing in the definition of the integral relaxation time yields

$$\langle\langle \hat{F}_g, \hat{F}_g(\tau) \rangle\rangle_{\hat{\pi}_g} = \sum_n e^{\lambda_n \tau} \langle\langle \hat{l}_n| \hat{F}_g\rangle\rangle \langle\langle \hat{F}_g, \hat{r}_n \rangle\rangle_{\hat{\pi}_g}. \quad (\text{V.10.10})$$

The integral relaxation time is then obtained by integrating this correlation function over τ ,

$$\tau_g = - \frac{\sum_{n>0} \lambda_n^{-1} \langle\langle \hat{l}_n| \hat{F}_g\rangle\rangle \langle\langle \hat{F}_g, \hat{r}_n \rangle\rangle_{\hat{\pi}_g}}{\sum_{n>0} \langle\langle \hat{l}_n| \hat{F}_g\rangle\rangle \langle\langle \hat{F}_g, \hat{r}_n \rangle\rangle_{\hat{\pi}_g}}. \quad (\text{V.10.11})$$

Here, the sums run over all nonstationary modes $n > 0$. Since the Liouvillian of a stable open quantum system satisfies $\text{Re}(\lambda_n) < 0$ for all $n > 0$, the integral is well-defined and convergent. Near a dissipative phase transition, the spectral gap of the Liouvillian closes, and the dynamics become dominated by the slowest-decaying mode, corresponding to $n = 1$. Noting that the Liouvillian gap is defined as $\Delta_g = -\text{Re}(\lambda_1)$, the above expression simplifies asymptotically, yielding the scaling

$$\tau_g \sim \frac{1}{\Delta_g}. \quad (\text{V.10.12})$$

Thus, the generalised integral relaxation time diverges with the inverse Liouvillian gap, mirroring the familiar critical slowing down observed in equilibrium phase transitions. Indeed, a compatible result was shown in Ref. [121] for closed quantum systems, where the integral relaxation time diverges with the inverse of the closing Hamiltonian gap. We now turn to the second constituent of the thermodynamic metric, the KMB QFI. Since the KMB QFI quantifies the sensitivity of the instantaneous steady state $\hat{\pi}_g$ to changes in the control parameter g , it plays the role of a generalised susceptibility. Consequently, near a second-order DQPT, it is expected to diverge algebraically,

$$I_Q^{(\text{KMB})}(g) \sim |g - g_c|^{-\alpha}, \quad (\text{V.10.13})$$

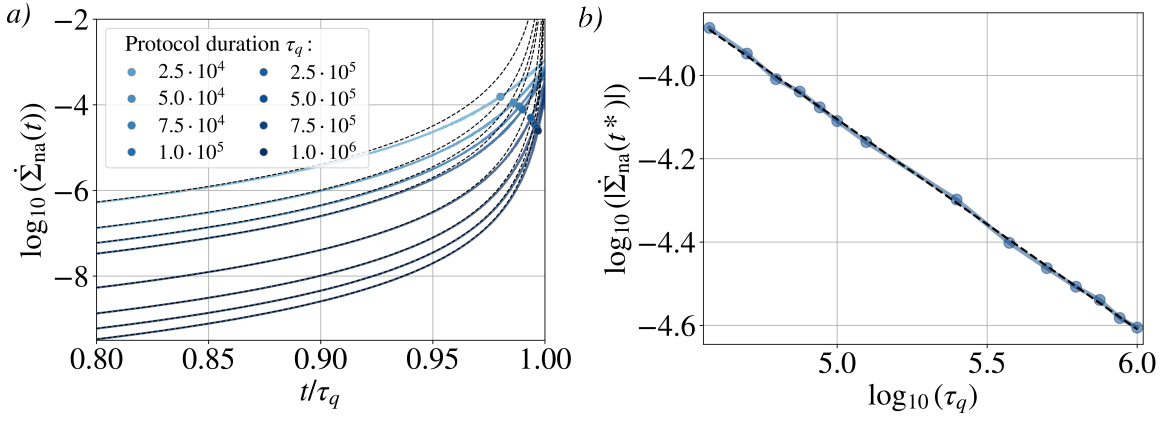


Figure V.10.3: Example of a DQPT in the open Dicke model. a) The nonadiabatic entropy production rate is shown as a function of time for linear ramps of the coupling strength g from the normal phase to the critical point g_c , with protocol durations τ_q specified in the legend. Scatter points mark their respective freeze-out points t^* , set by (V.10.3), solid lines show the exact result (V.10.5), and black dashed lines correspond to the slow-driving approximation (V.10.6). b) Scaling of the nonadiabatic entropy production rate at the onset of the impulse regime t^* as a function of the protocol duration τ_q for a linear ramp of the coupling strength g from the normal phase to the critical point g_c . The scaling exponent is found to be $\dot{\Sigma}_{\text{na}}(t^*) \sim \tau_q^y$ with $y = -0.508 \pm 0.001$, in excellent agreement with the theoretical prediction $y = (\alpha - 2 - \gamma)/(\gamma + 1)$ with $\alpha = 2$ and $\gamma = 1$. Parameters: cavity frequency $\omega_c = 1$, atomic frequency $\omega_z = 1$, photon loss rate $\kappa = 0.2$, initial coupling strength $g_0 = g_c - 10^{-2}$, final coupling strength $g_f = g_c - 10^{-15}$. Adapted from Ref. [125].

with a critical exponent α characterizing the information-geometric response of the NESS. Combining this divergence with the previously derived scaling of the generalised relaxation time, $\tau_g \sim \Delta_g^{-1} \sim |g - g_c|^{-\gamma}$, we obtain the scaling of the thermodynamic metric

$$\zeta_g \sim |g - g_c|^{-(\alpha+\gamma)}. \quad (\text{V.10.14})$$

This result directly connects the information-geometric properties of the NESS manifold and spectral properties of the Liouvillian to the dissipative response to the external driving. We are now equipped to evaluate the nonadiabatic entropy-production rate at the freeze-out time t^* . Using the Kibble-Zurek scaling (V.10.4) for the distance to criticality at t^* , together with the geometric expression for the entropy-production rate (V.10.5), we find that for a linear ramp,

$$\dot{\Sigma}_{\text{na}}(t^*) \sim \tau_q^{\frac{\alpha-2-\gamma}{\gamma+1}}. \quad (\text{V.10.15})$$

This relation, which constitutes our first principal result, shows that the nonadiabatic entropy-production rate at freeze-out obeys a universal power-law scaling with the quench duration, determined solely by the critical exponents α and γ . While the previous result concerns the rate evaluated at the onset of the impulse regime t^* , we now shift our focus to the total nonadiabatic entropy generated over the entire protocol of duration τ_q . Related analyses of total entropy production

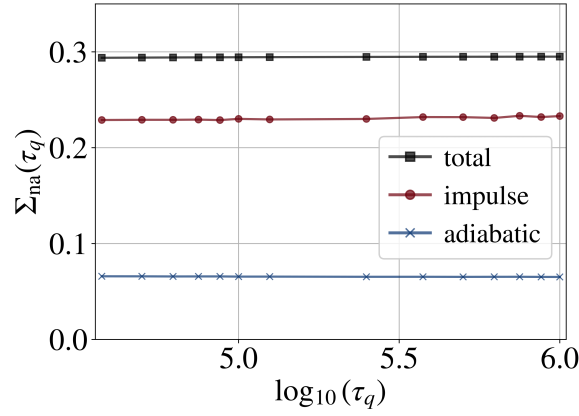


Figure V.10.4: Example of a DQPT in the open Dicke model. The nonadiabatic entropy production as a function of time for different protocol durations τ_q of a linear ramp of the coupling strength g from the normal phase to the critical point g_c . We split the contribution to the total (black squared markers) nonadiabatic entropy production into two contributions, stemming from the adiabatic (blue cross markers) and the impulse regime (red dot markers). Parameters: cavity frequency $\omega_c = 1$, atomic frequency $\omega_z = 1$, photon loss rate $\kappa = 0.2$, initial coupling strength $g_0 = g_c - 10^{-2}$, final coupling strength $g_f = g_c - 10^{-15}$. Adapted from Ref. [125].

within scaling contexts have been explored in both classical phase transitions [119] and closed-system quantum critical dynamics [121].

The total nonadiabatic entropy production follows from integrating the rate over the adiabatic and impulse regimes

$$\Sigma_{\text{na}}(0, \tau_q) = \int_0^{t^*} dt \dot{\Sigma}_{\text{na}}(t) + \int_{t^*}^{\tau_q} dt \dot{\Sigma}_{\text{na}}(t) = \Sigma_{\text{na}}(0, t^*) + \Sigma_{\text{na}}(t^*, \tau_q). \quad (\text{V.10.16})$$

Importantly, Eq. (V.10.6) is valid only so long as the system's state remains close to the corresponding instantaneous steady state. This condition is satisfied throughout the adiabatic regime but ceases to hold once the evolution enters the impulse regime (see Fig. V.10.3a). Accordingly, the two regimes must be analysed on different conceptual footings.

Up to the freeze-out point, the nonadiabatic entropy production can be faithfully approximated as the geometric path action associated with the thermodynamic metric [78]

$$\Sigma_{\text{na}}(0, t^*) = \int_0^{t^*} dt \dot{g}(t) \zeta_g \dot{g}(t). \quad (\text{V.10.17})$$

Using the freeze-out scaling relations (V.10.4), this becomes

$$\Sigma_{\text{na}}(0, t^*) \sim \frac{1}{\tau_q} \int_{g_0}^{g(t^*)} dg |g - g_c|^{-(\gamma+\alpha)} \sim \begin{cases} \frac{1}{\tau_q} \ln \tau_q, & \gamma + \alpha = 1, \\ \tau_q^{\frac{\alpha-2}{\gamma+1}}, & \text{otherwise.} \end{cases} \quad (\text{V.10.18})$$

In the impulse regime ($t \in (t^*, \tau_q]$), critical slowing down prevents the system from adjusting to the changing parameter, so the state remains essentially frozen while g approaches g_c . We can therefore approximate $\mathcal{L}_{g(t)} \approx \mathcal{L}_{g_c}$ and model the dynamics as relaxation under the final generator with steady state $\hat{\pi}_{g_c}$. To be explicit, we evaluate Eq. (V.10.5) under the approximation that $\hat{\rho}_t \approx e^{\mathcal{L}_{g_c}(t-t^*)}\hat{\rho}_{t^*}$ and $\hat{\pi}_{g(t)} \approx \hat{\pi}_{g_c}$. This corresponds to the standard formulation of entropy production for evolution under a fixed Lindblad generator $\mathcal{L}_{g_c}$ [145]. Since the steady state is strictly ill-defined in the thermodynamic limit at $g = g_c$, $\hat{\pi}_{g_c}$ should be understood as the limiting NESS approached from below. This approximation is valid in the slow-driving regime of large τ_q , where $\delta g = |g^* - g_c|$ can be treated as a small parameter. The integrated nonadiabatic entropy production is then

$$\Sigma_{\text{na}}(t^*, \tau_q) = D(\hat{\rho}_{t^*} \parallel \hat{\pi}_{g_c}) - D(\hat{\rho}_{\tau_q} \parallel \hat{\pi}_{g_c}). \quad (\text{V.10.19})$$

This quantity is positive since, for Markovian dynamics, the quantum relative entropy is contractive under CPTP maps, implying the monotonic decrease of $D(\hat{\rho}_t \parallel \hat{\pi}_{g_c})$. Consequently, the second term is either subleading compared to the first or scales in the same way in the large- τ_q limit. Therefore, the scaling of $\Sigma_{\text{na}}(t^*, \tau_q)$ is determined by the first term: it either dominates the second term, or the leading contributions cancel, leaving only subleading corrections. Focusing on the first term in Eq. (V.10.19) and using $\hat{\rho}_{t^*} \approx \hat{\pi}_{g^*}$ in the slow-driving limit, we find (see Section IV.7.2)

$$\Sigma_{\text{na}}(t^*, \tau_q) \sim D(\hat{\pi}_{g^*} \parallel \hat{\pi}_{g_c}) = \frac{1}{2}(\delta g)^2 \mathcal{I}_{g^*}^{(\text{KMB})} + O(\delta g^3). \quad (\text{V.10.20})$$

It then follows that

$$\Sigma_{\text{na}}(t^*, \tau_q) \sim |g^* - g_c|^{2-\alpha}. \quad (\text{V.10.21})$$

Using the standard Kibble-Zurek scaling of δg with the quench time τ_q , we finally obtain

$$\Sigma_{\text{na}}(t^*, \tau_q) \sim \tau_q^{\frac{\alpha-2}{\gamma+1}}. \quad (\text{V.10.22})$$

Remarkably, this coincides with the scaling found for the quasiadiabatic contribution in the quasiadiabatic regime, so long as $\gamma + \alpha \neq 1$. From the preceding analysis, we arrive at our second central result: the total nonadiabatic entropy produced during a linear ramp toward the critical point, over a quench of duration τ_q , exhibits a universal scaling behaviour governed by the critical exponents of the DQPT. Explicitly, we have

$$\Sigma_{\text{na}}(0, \tau_q) \sim \tau_q^{\frac{\alpha-2}{\gamma+1}}. \quad (\text{V.10.23})$$

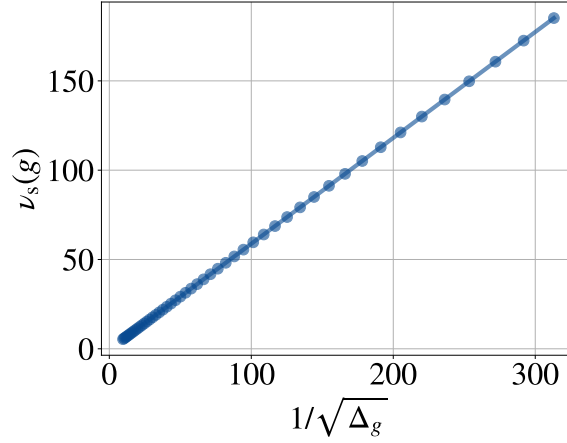


Figure V.10.5: Scaling of the symplectic eigenvalue of the soft mode with the square-root of the inverse Liouvillian gap in the dissipative Dicke model. Parameters: cavity frequency $\omega_c = 1$, atomic frequency $\omega_z = 1$, photon loss rate $\kappa = 0.2$. Adapted from Ref. [125].

This relation highlights that the total entropy production is not merely a system-specific quantity, but rather displays dynamical KZM-like universality, determined solely by τ_q and the exponents α and γ associated with the KMB QFI and the Liouvillian gap, respectively.

V.10.4 Gaussian bosonic systems

We now emphasise a striking property of DQPTs within the mean-field, Ising-like universality class [386]. This class encompasses prototypical fully connected open quantum systems that exhibit spontaneous $\mathbb{Z}_2$ symmetry-breaking, including the open Dicke [366], Rabi [378], Kerr [358], and Lipkin-Meshkov-Glick [387] models. Although these systems are generally non-Gaussian at finite size, their critical behaviour in the thermodynamic limit can be captured by an effectively Gaussian bosonic description with critical behaviour dominated by a single soft mode. As discussed in Section I.3.3, Gaussian states are fully characterised by their symplectic eigenvalues. Their scaling will be crucial for the scaling analysis of observable expectation values as we show in the following.

V.10.4.1 Critical scaling of the symplectic eigenvalues of Θ

We consider a Gaussian bosonic system whose covariance matrix evolves under a Lyapunov differential equation (I.3.14), so that the NESS covariance matrix solves the Lyapunov equation (I.3.16). In this section, we aim to understand the behaviour the soft-mode component of the steady-state covariance matrix in the vicinity of a critical point g_c in a continuous DQPT. Let λ_s denote the eigenvalue of W , assumed to be diagonalisable, with smallest absolute real part (Liouvillian

gap), and

$$Wv_s = \lambda_s v_s, \quad w_s^\dagger W = \lambda_s w_s^\dagger, \quad (\text{V.10.24})$$

where v_s is the right eigenvector and w_s is the left eigenvector. We define the projection of Θ onto the subspace of the slowest-decaying (soft) mode as

$$\theta_s = w_s^\dagger \Theta w_s. \quad (\text{V.10.25})$$

Multiplying Eq. (I.3.16) from the left by $w_s^\dagger$ and from the right by w_s gives

$$(w_s^\dagger W) \Theta w_s + w_s^\dagger \Theta (W^\dagger w_s) = w_s^\dagger Y w_s. \quad (\text{V.10.26})$$

Using the eigenvector relations, we obtain the equation

$$\lambda_s \theta_s + \lambda_s^* \theta_s = y_s, \quad (\text{V.10.27})$$

where $y_s = w_s^\dagger Y w_s$, which we assume remains finite and non-zero across the transition. Thus,

$$\theta_s = \frac{y_s}{\lambda_s + \lambda_s^*} = \frac{y_s}{2 \operatorname{Re} \lambda_s}. \quad (\text{V.10.28})$$

To understand the structure of the divergences, recall that the Lyapunov equation admits the integral solution

$$\Theta = \int_0^\infty e^{tW} Y e^{tW^\dagger} dt = \sum_{i,j} \frac{v_i (w_i^\dagger Y w_j) v_j^\dagger}{\lambda_i + \lambda_j^*}, \quad (\text{V.10.29})$$

using $e^{tW} = \sum_i e^{t\lambda_i} v_i w_i^\dagger$ and $e^{tW^\dagger} = \sum_j e^{t\lambda_j^*} w_j v_j^\dagger$. As $\Delta_g = |\operatorname{Re} \lambda_s| \rightarrow 0^+$, the dominant term is the contribution

$$\Theta = \frac{y_s}{\lambda_s + \lambda_s^*} v_s v_s^\dagger + \Theta_{\text{reg}}, \quad (\text{V.10.30})$$

where Θ_{reg} represents the regular component and remains finite at the transition. The divergent part is rank-one, hence in the two-dimensional canonical subspace of the soft mode, Θ_s has one eigenvalue $\propto \Delta_g^{-1}$ and the other $\propto \mathcal{O}(1)$. The covariance matrix of the slowest-decaying mode, in canonical coordinates, can be written as

$$\Theta_s = \begin{pmatrix} \langle \hat{X}_s^2 \rangle & \frac{1}{2} \langle \hat{X}_s \hat{P}_s + \hat{P}_s \hat{X}_s \rangle \\ \frac{1}{2} \langle \hat{P}_s \hat{X}_s + \hat{X}_s \hat{P}_s \rangle & \langle \hat{P}_s^2 \rangle \end{pmatrix}, \quad (\text{V.10.31})$$

where $(\hat{X}_s, \hat{P}_s)$ are the associated quadratures, chosen such that they satisfy canonical commutation relations. The symplectic eigenvalue of this matrix is

$$\nu_s = \sqrt{\det \Theta_s}. \quad (\text{V.10.32})$$

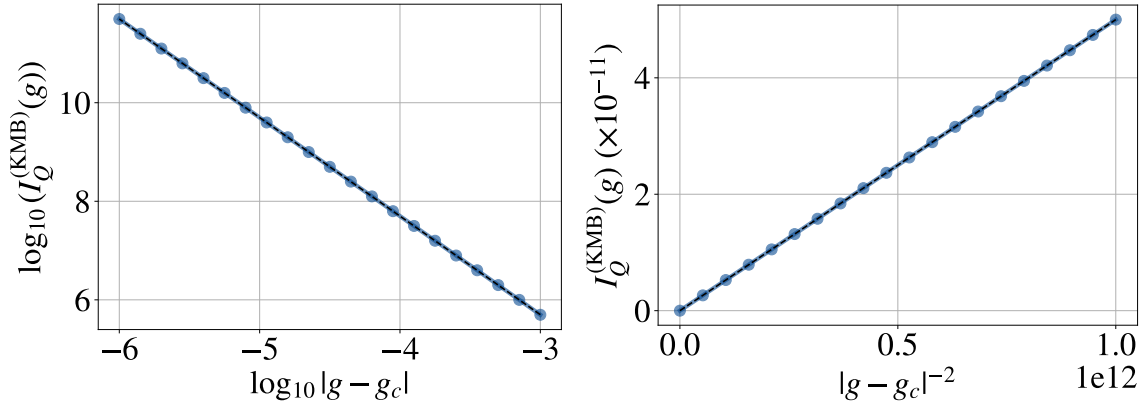


Figure V.10.6: Left: The KMB QFI scales with a power law in the distance to the critical point $I_Q^{(\text{KMB})}(g) \sim \frac{\gamma^2}{2}|g - g_c|^{-\alpha}$, with $\alpha = 2 \pm 1 \cdot 10^{-4}$, in the Gaussian open Dicke model. Right: Here, we show that the slope obtained from a linear fit is $0.500001 \pm 1 \cdot 10^{-6}$, which is in excellent agreement with the predicted $\gamma^2/2$, with $\gamma = 1$. Parameters: cavity frequency $\omega_c = 1$, atomic frequency $\omega_z = 1$, photon loss rate $\kappa = 0.2$. Adapted from Ref. [125].

From the rank-one structure above, we have

$$\det \Theta_s \sim \Delta_g^{-1}, \quad \nu_s \sim \Delta_g^{-1/2}. \quad (\text{V.10.33})$$

The scaling behaviour of the soft mode symplectic eigenvalue is confirmed in the Gaussian driven-dissipative Dicke model, see Fig. V.10.5.

V.10.4.2 Critical scaling of the KMB QFI

We now use the scaling behaviour of the soft-mode symplectic eigenvalue of the NESS covariance matrix to understand why the KMB QFI diverges as $\frac{\gamma^2}{2}|g - g_c|^{-2}$ near a critical point for Gaussian bosonic states with a single soft mode, implying $\alpha = 2$. Notably, this exponent is universal within Gaussian bosonic models in the sense that it is independent of the Liouvillian gap exponent or any other critical exponent that typically appears in scaling predictions, making it a distinctive feature of these systems.

In Appendix A.2, we show that for zero mean Gaussian bosonic states, the KMB QFI can be expressed as

$$I_Q^{(\text{KMB})}(g) = \frac{1}{2} \int_0^1 ds \text{Tr}[A \check{\Theta} S^T(s) A S(s) \check{\Theta}^T], \quad (\text{V.10.34})$$

with $S(s) = e^{si\Omega M}$ and $\check{\Theta} = \Theta + i\Omega/2$. Let us first express $X = i\Omega M$ in its biorthonormal basis

$$X = \sum_i \lambda_i v_i w_i^T, \quad w_j^T v_i = \delta_{ij}, \quad (\text{V.10.35})$$

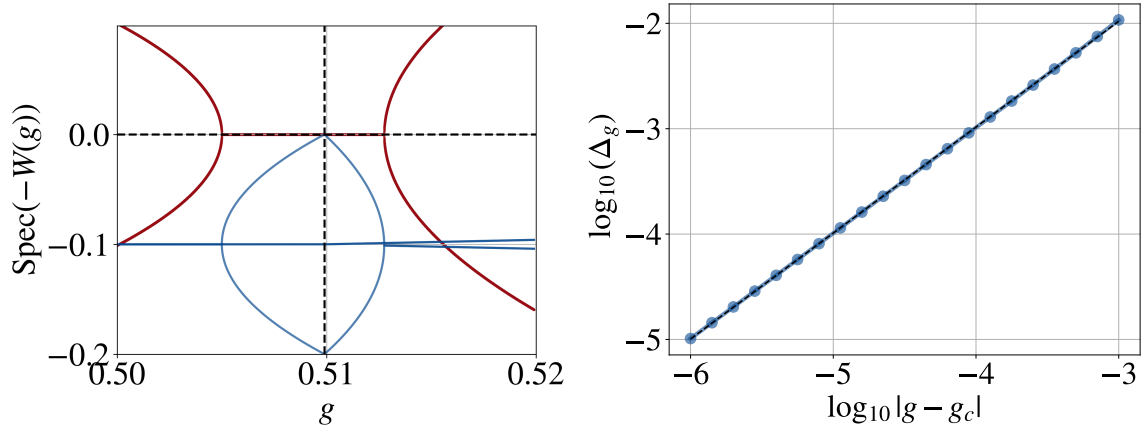


Figure V.10.7: Left: Complex spectrum of the drift matrix $-W$ as a function of the coupling strength g in the dissipative Dicke model (blue: real part, red: imaginary part). The vertical dashed black line indicates the critical point g_c at which the Liouvillian gap closes. Right: Scaling of the Liouvillian gap with proximity of the coupling strength g to the critical point g_c : $\Delta_g \sim |g - g_c|^\gamma$, in the dissipative Dicke model. From a numerical fit, we obtain $\gamma = 1.009 \pm 0.001$. Parameters: cavity frequency $\omega_c = 1$, atomic frequency $\omega_z = 1$, photon loss rate $\kappa = 0.2$. Adapted from Ref. [125].

so that

$$S(s) = \sum_j e^{\lambda_j s} v_j w_j^T, \quad S^T(s) = \sum_j e^{\lambda_j s} w_j v_j^T. \quad (\text{V.10.36})$$

The KMB QFI reads

$$\begin{aligned} I_Q^{(\text{KMB})}(g) &= \frac{1}{2} \sum_{ij} \text{Tr} \left[A \check{\Theta} \int_0^1 e^{(\lambda_i + \lambda_j)s} ds w_i v_i^T A v_j w_j^T \check{\Theta}^T \right] \\ &= \frac{1}{2} \sum_{ij} \frac{e^{\lambda_i + \lambda_j} - 1}{\lambda_i + \lambda_j} \text{Tr} \left[A \check{\Theta} w_i v_i^T A v_j w_j^T \check{\Theta}^T \right]. \end{aligned} \quad (\text{V.10.37})$$

Equation (V.10.37) contains the factors $\check{\Theta} = \Theta + i\Omega/2$ and $\check{\Theta}^T = \Theta - i\Omega/2$. We expect the $\Theta\Theta$ -contribution to dominate near the critical point, since, as we have shown in Section V.10.4.1, $\nu_s \sim 1/\sqrt{\Delta_g}$ and $\Delta_g \rightarrow 0$. We further use that

$$\begin{aligned} A &= \partial_g M = i\Omega \partial_g X \\ &= \sum_j \underbrace{i\Omega \partial_g \lambda_j v_j w_j^T}_{A_j^\lambda} + \underbrace{i\Omega \lambda_j \partial_g (v_j w_j^T)}_{A_j^P} \\ &= \sum_j A_j^\lambda + A_j^P, \end{aligned} \quad (\text{V.10.38})$$

so that A can be split into components involving the derivatives of the symplectic eigenvalues (A_j^λ) the projectors (A_j^P), respectively. We now perform the explicit computation of the term involving the g -derivatives of λ_i , labelled with the super-

script λ . We express $\Theta = (\Theta i\Omega)i\Omega = \sum_j v_j v_j w_j^T i\Omega$, and find

$$\begin{aligned}
 I_Q^{(\text{KMB}),\lambda}(g) &= \frac{1}{2} \sum_{ijklmn} \frac{e^{\lambda_i + \lambda_j} - 1}{\lambda_i + \lambda_j} \times \\
 &\quad \times \text{Tr} \left[\underbrace{i\Omega \partial_g \lambda_k v_k w_k^T}_{A_k^\lambda} \underbrace{v_l v_l w_l^T i\Omega}_{\Theta_l} w_i v_i^T \underbrace{i\Omega \partial_g \lambda_m v_m w_m^T}_{A_m^\lambda} v_j w_j^T \underbrace{v_n v_n w_n^T i\Omega}_{\Theta_n} \right] \\
 &= \frac{1}{2} \sum_{ij} \frac{e^{\lambda_i + \lambda_j} - 1}{\lambda_i + \lambda_j} (\partial_g \lambda_j v_j)^2 w_j^T i\Omega w_i v_i^T i\Omega v_j,
 \end{aligned}$$

where we have used that $(i\Omega)^2 = 1$, the cyclic property of the trace as well as the biorthonormality condition. Since $X = 2 \coth^{-1}(2\Theta i\Omega)$, $\Theta i\Omega$ shares an eigenbasis with X and their respective eigenvalues are related via $\lambda_j = 2 \coth^{-1}(2v_j)$, and

$$\partial_g \lambda_j = -\frac{4\partial_g v_j}{4v_j^2 - 1} \xrightarrow{g \rightarrow g_c} -\frac{\partial_g v_j}{v_j^2}. \quad (\text{V.10.39})$$

From our analysis of the soft mode in Section V.10.4.1, we know that in the vicinity of the critical point, the soft mode symplectic eigenvalue of the covariance matrix diverges as $v_s \sim 1/\sqrt{\Delta_g}$. Therefore, the soft mode symplectic eigenvalue of M $\lambda_s \sim 1/v_s$ vanishes. Further, the factors $w_j^T i\Omega w_i$ and $v_i^T i\Omega v_j$ are non-zero only for the labels i and j referring to symplectic eigenvalues of opposite sign. Therefore, there are two non-zero identical contributions to the sum, relevant to the scaling $(s, -s)$ and $(-s, s)$. Further, we note that prefactor $\frac{e^{\lambda_i + \lambda_j} - 1}{\lambda_i + \lambda_j}$ reduces to 1, for the dominant soft mode contribution, since $\lambda_s \rightarrow 0$, and find

$$I_Q^{(\text{KMB}),\lambda}(g) \sim (\partial_g \lambda_j)^2 v_j^2 \approx \left(\frac{\partial_g v_s}{v_s} \right)^2 \sim \left(\frac{\gamma}{2} \right)^2 |g - g_c|^{-2}, \quad (\text{V.10.40})$$

Similarly, we find for the contribution involving the projector derivatives

$$\begin{aligned}
 I_Q^{(\text{KMB}),P}(g) &= \frac{1}{2} \sum_{ijklmn} \frac{e^{\lambda_i + \lambda_j} - 1}{\lambda_i + \lambda_j} \times \\
 &\quad \times \text{Tr} \left[\underbrace{i\Omega \lambda_k \partial_g (v_k w_k^T)}_{A_k^P} \underbrace{v_l v_l w_l^T i\Omega}_{\Theta_l} w_i v_i^T \underbrace{i\Omega \lambda_m \partial_g (v_m w_m^T)}_{A_m^P} v_j w_j^T \underbrace{v_n v_n w_n^T i\Omega}_{\Theta_n} \right] \\
 &\sim (\partial_g \lambda_j)^2 v_j^2 \sim \left(\frac{\gamma}{2} \right)^2 |g - g_c|^{-2}
 \end{aligned}$$

All other terms can be computed in an analogous fashion and can be shown to be subleading or vanishing by the biorthonormality condition. Combining these

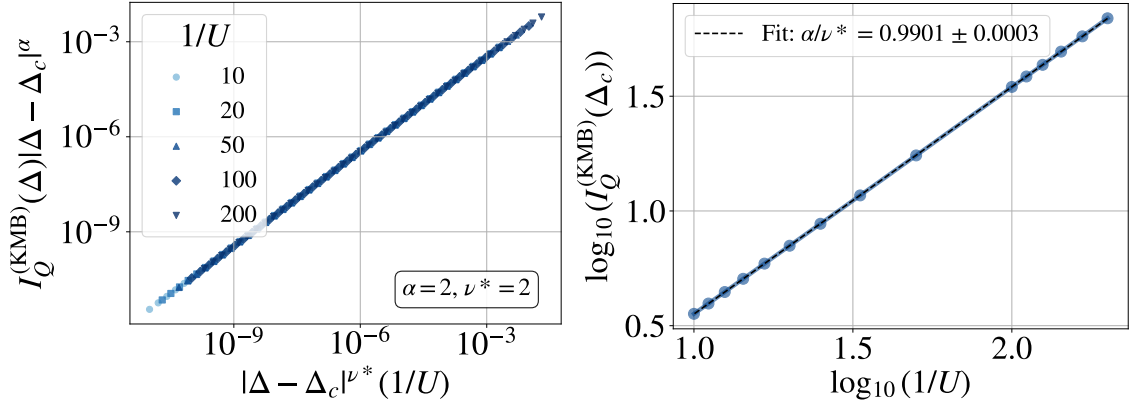


Figure V.10.8: Left: Finite-size scaling analysis of the KMB QFI in the dissipative Kerr model, revealing excellent agreement with the predicted $\alpha = 2$. Parameters: parametric drive amplitude $G = 1.5$, photon loss rate $\kappa = 1$. Adapted from Ref. [125].

findings, we obtain that the dominant scaling of the KMB QFI is

$$I_Q^{(\text{KMB})}(g) \sim \frac{\gamma^2}{2} |g - g_c|^{-2}, \quad (\text{V.10.41})$$

so that the KMB QFI critical exponent is given by $\alpha = 2$. Note that the prefactor $\frac{\gamma^2}{2}$ depends on the gap exponent, indicating that the KMB QFI indeed picks up on the underlying equilibrium critical exponents.

V.10.5 Examples

We now illustrate our results in the context of the driven-dissipative Dicke (see Section V.9.3.1) and Kerr model (see Section V.9.3.2), which both serve as paradigmatic examples of open quantum systems exhibiting a DQPT and, importantly, are experimentally realisable [108, 350, 367, 368, 111, 112, 115]. We begin by analysing the closure of the Liouvillian gap in the open Dicke model in the thermodynamic limit, with the goal of determining the critical exponent that characterises its algebraic vanishing near the transition. As discussed in Section I.3.2, the Liouvillian gap can be extracted from the spectrum of the drift matrix W . Figure V.10.7 displays the full spectrum of W as a function of the coupling strength g and highlights the behaviour of the Liouvillian gap Δ_g near the critical point. A numerical fit in the normal phase yields $\gamma = 1.009 \pm 0.001$, confirming that $\gamma \approx 1$. This result is in excellent agreement with previous studies [380]. Furthermore, we find that the KMB QFI exhibits a clear algebraic divergence near the critical point, scaling as

$$I_g^{\text{KMB}} \approx \frac{\gamma^2}{2} |g - g_c|^{-2}, \quad (\text{V.10.42})$$

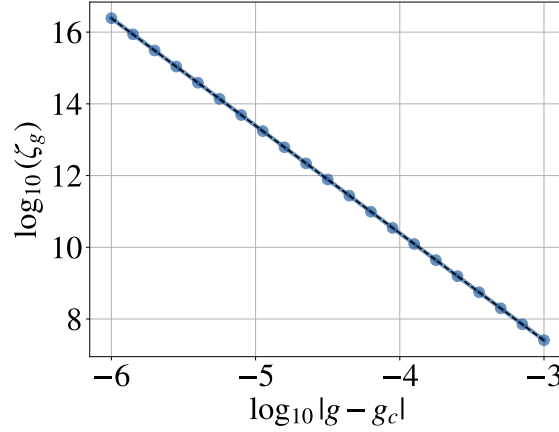


Figure V.10.9: Scaling of the thermodynamic metric ζ_g in the proximity to the critical point: $\zeta_g \sim |g - g_c|^x$, in the dissipative Dicke model. We obtain $x = -2.9953 \pm 0.0009$ via a numerical fit, which is in excellent agreement with our theoretical prediction $x = -(\alpha + \gamma)$, with $\alpha = 2$ and $\gamma = 1$. Parameters: cavity frequency $\omega_c = 1$, atomic frequency $\omega_z = 1$, photon loss rate $\kappa = 0.2$. Adapted from Ref. [125].

as shown in Fig. V.10.6, where both the exponent and the prefactor are confirmed numerically. This behaviour is in excellent agreement with the analytical scaling predictions presented in Section V.10.4.2, demonstrating that the KMB QFI faithfully captures the increasing sensitivity of the steady state as the system approaches criticality. We further validate our scaling prediction for the KMB QFI through finite-size scaling analysis (see Section V.9.4) in the driven-dissipative Kerr model, which admits an effective Gaussian bosonic description near the critical point in the thermodynamic limit [358]. Our numerical analysis shows excellent agreement with our theoretical predictions: Figure V.10.8 confirms a critical exponent $\alpha = 2$, consistent with the mean-field Ising universality class with $\mathbb{Z}_2$ symmetry, characterised by the critical dimension $d_c = 4$ and mean-field correlation-length exponent $\nu_{\text{MF}} = 1/2$. Furthermore, these results are in agreement with similar analyses performed for other models within the same universality class [378, 388]. Building on these results, we observe that the thermodynamic metric (V.10.7) exhibits the expected scaling $\zeta(g) \sim |g - g_c|^{-3}$, as illustrated, for the open Dicke model, in Fig. V.10.9. Consequently, the nonadiabatic entropy-production rate at the freeze-out point scales as

$$\dot{\Sigma}_{\text{na}}(t^*) \sim \tau_q^{-\frac{\gamma}{\gamma+1}}, \quad (\text{V.10.43})$$

with the exponent determined solely by the Liouvillian gap exponent γ . Remarkably, this scaling has a striking implication for the total nonadiabatic entropy generated over the entire protocol: to leading order, the total nonadiabatic entropy production becomes independent of the quench duration,

$$\Sigma_{\text{na}}(0, \tau_q) \sim \tau_q^0. \quad (\text{V.10.44})$$

In DQPTs, nonadiabatic effects can generate dissipation even for arbitrarily slow driving to the critical point, unlike in isolated, fully connected quantum systems, where finite-speed driving to a QPT produces excess work that vanishes universally as $\tau_q \rightarrow \infty$ [389, 390].

Finally, we confirm the predicted scaling of both the nonadiabatic entropy production rate and its integrated value in the open Dicke model, as shown in Fig. V.10.3b) and Fig. V.10.4, respectively.

V.10.6 Conclusion

In this chapter, we develop a comprehensive framework for characterizing critical quantum dynamics in open quantum systems, extending the Kibble-Zurek paradigm to continuous dissipative quantum phase transitions. A central result is the emergence of universal scaling laws for the nonadiabatic entropy production, which quantifies excess dissipation and plays a role analogous to that of dissipated work in closed systems [119–121, 123, 354]. We show that these scaling laws are governed by the simultaneous closing of the Liouvillian gap and the divergence of the KMB QFI, which together control the thermodynamic signatures of irreversibility near criticality.

Information-geometric methods, and in particular the QFI metric, are well established as sensitive probes of criticality, capable of detecting phase transitions and delineating phase diagrams even when no local order parameter is available [279, 391–397, 305] (see Appendix A.1). Building on this insight, we demonstrate that the information geometry of quantum NESS is directly tied to thermodynamic irreversibility induced by critical slowing down. This connection establishes a comprehensive framework for understanding dissipative universality classes that extends beyond both near-equilibrium thermodynamics and purely unitary quantum dynamics.

For Gaussian bosonic systems governed by a single soft mode, we find that the quadratic divergence of the KMB QFI near criticality implies a leading-order nonadiabatic entropy production that becomes independent of the driving speed. We corroborate this prediction explicitly in the driven-dissipative Dicke model and further substantiate it via finite-size scaling in the driven-dissipative Kerr oscillator. More broadly, the open Kerr, Rabi, and Lipkin-Meshkov-Glick models fall into the same universality class as the open Dicke model, namely the classical mean-field Ising class [388]. As a result, they are expected to display identical thermodynamic signatures in the vicinity of criticality, reflecting the emergent classical dynamics of a $\mathbb{Z}_2$ -symmetric order parameter.

Our results also underscore the potential for experimental exploration of entropic currents in quantum devices. While stochastic entropy production has been mea-

sured in classical systems to verify fluctuation relations such as the Hatano-Sasa relation [79, 80, 398–402], only a few quantum experiments have directly accessed such entropic currents [17, 28, 403–407, 115].

Looking forward, these results motivate time-resolved measurements also of other experimentally accessible currents, such as the cavity photon flux in Dicke or Kerr setups, which also follow Kibble-Zurek-type scaling, as we show in Appendix C.3. Since the cavity photon flux is directly related to the entropic currents [408] and has been experimentally measured [115], combined measurements of the Wigner function and photon emission provide a promising route to reconstructing the nonadiabatic entropy production in dissipative quantum critical dynamics. This establishes a concrete bridge between theoretical predictions and feasible experimental probes of nonequilibrium universality in open quantum systems.

Final remarks and future work

This thesis investigated nonequilibrium thermodynamics in open quantum systems from multiple complementary perspectives, spanning regimes from weak to strong system-environment coupling, incorporating both stochastic and ensemble-averaged descriptions, and addressing dynamics both far from stationarity and in nonequilibrium steady states (NESSs).

First, in Chapter II.4, we investigated the thermodynamic role of measurement backaction on NESS currents in an autonomous quantum heat engine, based on Ref. [29]. We studied a minimal yet experimentally relevant setup: a double quantum dot coupled to two fermionic reservoirs, with one dot continuously monitored via a quantum point contact. By formulating the problem within a repeated-interactions framework, we obtained a thermodynamically consistent description of continuous measurement. Within this setting, we showed that measurement backaction, in contrast to the conventional view, is not inherently detrimental to thermodynamic performance. On the contrary, it can enhance steady-state currents, improve their signal-to-noise ratio, and reduce the entropic cost required to maintain currents at a fixed precision. This insight is particularly relevant for realistic quantum thermal devices, where continuous monitoring is often unavoidable, for instance, to enable feedback or stabilisation protocols. Our results therefore suggest that measurement-induced effects can be deliberately harnessed as a functional resource in the design and optimisation of nonequilibrium quantum machines.

A second, methodological contribution of this thesis addresses current fluctuations beyond the Markovian weak-coupling regime using the mesoscopic-leads framework [47, 48, 50], which enables a thermodynamically consistent treatment of strong coupling, non-Markovian dynamics, and time-dependent driving. In Chapter III.6, we presented the results of Ref. [52], demonstrating direct access to the full counting statistics of thermodynamic observables along individual quantum trajectories with both temporal and energetic resolution. This level of access connects to approaches based on large deviation theory and the thermodynamic

analysis of quantum jump trajectory ensembles, which have been used, for instance, to reveal dynamical phase transitions in continuously monitored open systems [409–412]. While our explicit numerical implementations focused on noninteracting fermionic systems, the general expressions for stochastic currents derived here are valid for arbitrary systems. This opens the door to future applications in interacting settings, for example via tensor-network representations of the extended system. More broadly, it lays the foundation for integrating quantum measurement-based control into a consistent thermodynamic framework for mesoscopic systems.

Finally, this thesis explored the role of information geometry in nonequilibrium quantum thermodynamics, highlighting both the connection between temporal quantum Fisher information (QFI) and entropy dynamics far from stationarity, based on Ref. [71] (see Chapter IV.8), and its role in quantifying the entropic cost of slow driving between NESSs via the nonadiabatic entropy production (see Section IV.7.5), based on Refs. [78, 125]. In the first context, we extended classical results of nonequilibrium thermodynamics [70, 69] to the quantum regime [71], showing that temporal QFI is directly linked to the entropy dynamics in open systems and establishing quantum generalisations of classical entropy bounds. More broadly, the geometric framework developed here provides a promising foundation for future studies of anomalous thermalisation phenomena, including Mpemba-like effects [413], in both open and closed quantum systems. Studying the temporal QFI at the single-trajectory level would be particularly interesting, as it could reveal the stochastic fluctuations underlying the entropic dynamics of these phenomena. First steps in this direction have recently been reported in Refs. [414, 415]. In the second context, we showed that the Kubo-Mori-Bogoliubov (KMB) QFI emerges as a central component of the thermodynamic metric quantifying the entropic cost of slowly driving a system from one NESS to another, based on Ref. [78].

Most notably, in Chapter V.10, we demonstrated further that this information-geometric framework reveals genuine dynamical universality in continuous dissipative quantum phase transitions (DQPTs) through thermodynamic quantities, in particular the nonadiabatic entropy production, as originally demonstrated in Ref. [125]. This observable captures the irreversibility of finite-time driving and displays universal Kibble-Zurek-type scaling near critical points. Historically, the Kibble-Zurek mechanism has linked the scaling of equilibrium correlations to the system's response under finite-time driving across continuous phase transitions. Our results indicate that the same principles govern the response of driven open quantum systems: the divergence of the KMB QFI metric near dissipative critical points reflects this mechanism, likely imposing fundamental limits on statistical and thermodynamic length minimisation and on the feasibility of optimal or

counterdiabatic control. A similar connection has recently been established for classical second-order phase transitions [416] and has been explored in quantum systems through quantum speed limits [417]. Extending these insights to DQPTs would be a natural and exciting direction for future research. Importantly, we also predict Kibble-Zurek-type scaling for experimentally accessible quantities, such as the cavity photon flux in Kerr or Dicke-type models, making verification possible in commonly used platforms [115]. Another interesting direction is to extend our framework to settings beyond the Markovian weak-coupling regime. In this context, a spectral theory of dissipative quantum phase transitions under non-Markovian dynamics has been developed recently [355], providing a natural starting point for exploring universal nonequilibrium scaling and thermodynamic geometry in such open system settings.

Although not yet exploited in this thesis, our investigation of information geometry led us to identify a compact extension of the adiabatic gauge potential (AGP) to density matrices evolving non-unitarily under a Liouvillian. We refer to this object as the quasistatic gauge potential (QGP) (see Appendix D). In closed, unitarily evolving systems, the AGP generates adiabatic evolution and counteracts nonadiabatic transitions. The QGP plays an analogous role in open systems, see also Ref. [418]. Moreover, the norm of the AGP is directly related to the QFI, and establishing a similar connection for the QGP would be of significant interest. This generalisation also naturally raises questions about counterdiabatic control and its approximations in open systems, inspired by closed-system studies [419], pointing to another promising avenue for future research.

More generally, quantum geometry in open quantum systems remains largely unexplored. An intriguing route toward addressing this gap is to extend quantum geometric tensors, originally defined for pure states, to the setting of mixed states. In this context, the real part of the quantum geometric tensor naturally involves the QFI. However, since mixed states admit an entire family of inequivalent QFI metrics, one should correspondingly expect a family of quantum geometric tensors for density matrices. In particular, exploring analogues of the Berry curvature for mixed states could provide valuable insights. To date, progress has been limited, with only a few works addressing the so-called Uhlmann curvature, which in some sense complements the QFI obtained using the symmetric logarithmic derivative. Potential applications of this framework include the study of DQPTs [420], topological states in open systems or in finite-temperature settings [421], and a systematic investigation of the differences between strong and weak symmetry breaking in open quantum systems [422].

Appendices

A Supplementary results on Gaussian models

A.1 Thermodynamic metric (Fermionic)

In this appendix, we derive an explicit expression for the thermodynamic metric for Gaussian fermionic NESSs in terms of their covariance matrix.

For generality, we choose to work in the Majorana basis ($\hat{y}_i$) instead of the complex fermionic basis ($\hat{c}_i$), using the convention

$$\begin{aligned} \hat{y}_i &= \hat{c}_i^\dagger + \hat{c}_i, & \hat{y}_{i+N} &= i(\hat{c}_i^\dagger - \hat{c}_i), \\ \hat{\pi}_{\vec{\lambda}} &= \frac{e^{\frac{1}{4}\vec{Y}^\dagger \cdot \mathbf{M} \cdot \vec{Y}}}{Z}, & \vec{Y} &= (\hat{y}_1, \hat{y}_2, \dots, \hat{y}_{2N}), \\ \hat{\Sigma}_{ij} &= \frac{1}{2} [\hat{y}_i, \hat{y}_j], & \Theta_{\vec{\lambda}, ij} &= \text{Tr} [\hat{\Sigma}_{ij} \hat{\pi}_{\vec{\lambda}}], \end{aligned} \quad (\text{A.1})$$

where N denotes the number of complex fermionic modes and $\{\hat{y}_i, \hat{y}_j\} = 2\delta_{ij}$. We assume a Gaussian fermionic NESS, $\hat{\pi}_{\vec{\lambda}}$, as the fixed point of Gaussianity-preserving GKSL dynamics with Liouvillian $\mathcal{L}_{\vec{\lambda}}$ which depends on parameters $\vec{\lambda}$ with quadratic Hamiltonian and jump operators that are a linear combination of the Majorana operators

$$\hat{H} = \frac{1}{4} \vec{Y}^\dagger \cdot \mathbf{h} \cdot \vec{Y}, \quad \hat{L}_r = \sum_i c_i^r \hat{y}_i. \quad (\text{A.2})$$

For a more detailed discussion, we refer to Chapter [I.3](#). The metric is given by $\zeta_{\mu\nu}$, where $\zeta_{\mu\nu} = \frac{1}{2}(\tilde{\zeta}_{\mu\nu} + \tilde{\zeta}_{\nu\mu})$ is the symmetric part of

$$\tilde{\zeta}_{\mu\nu} = -\text{Tr} \left[\frac{\partial \log \hat{\pi}_{\vec{\lambda}}}{\partial \lambda_\nu} \mathcal{L}_{\vec{\lambda}}^\dagger \frac{\partial \hat{\pi}_{\vec{\lambda}}}{\partial \lambda_\mu} \right]. \quad (\text{A.3})$$

First, we use the integral representation of the Drazin inverse

$$\mathcal{L}^+[\hat{A}] = \int_0^\infty d\nu e^{\nu\mathcal{L}}(\hat{\pi} \text{Tr}[\hat{A}] - \hat{A}) = - \int_0^\infty d\nu e^{\nu\mathcal{L}}(\hat{A} - \hat{\pi} \text{Tr}[\hat{A}]), \quad (\text{A.4})$$

so that we find

$$\mathcal{L}^+ \left[\frac{\partial \hat{\pi}}{\partial \lambda_\mu} \right] = - \int_0^\infty d\nu e^{\nu\mathcal{L}} \left(\frac{\partial \hat{\pi}}{\partial \lambda_\mu} - \text{Tr} \left[\frac{\partial \hat{\pi}}{\partial \lambda_\mu} \right] \hat{\pi} \right) = - \int_0^\infty d\nu e^{\nu\mathcal{L}} \left(\frac{\partial \hat{\pi}}{\partial \lambda_\mu} \right), \quad (\text{A.5})$$

since $\text{Tr} \left[\frac{\partial \hat{\pi}}{\partial \lambda_\mu} \right] = 0$. We now compute $\frac{\partial \hat{\pi}_{\bar{\lambda}}}{\partial \lambda_\mu}$ using the identity

$$\frac{\partial e^{\hat{X}}}{\partial \lambda} = \int_0^1 ds e^{s\hat{X}} \frac{\partial \hat{X}}{\partial \lambda} e^{(1-s)\hat{X}}, \quad (\text{A.6})$$

with $\hat{X} = \log \hat{\pi}$

$$\frac{\partial \hat{\pi}_{\bar{\lambda}}}{\partial \lambda_\mu} = \int_0^1 ds \hat{\pi}^s \frac{\partial \log \hat{\pi}}{\partial \lambda_\mu} \hat{\pi}^{1-s} = \int_0^1 ds \hat{\pi}^s \frac{\partial \log \hat{\pi}}{\partial \lambda_\mu} \hat{\pi}^{-s} \hat{\pi}. \quad (\text{A.7})$$

We find

$$\begin{aligned} \frac{\partial \log \hat{\pi}}{\partial \lambda_\mu} &= \frac{1}{4} \vec{Y}^\dagger \cdot \frac{\partial M}{\partial \lambda_\mu} \cdot \vec{Y} - 1/Z \text{Tr} \left[\frac{\partial e^{\frac{1}{4} \vec{Y}^\dagger \cdot M \cdot \vec{Y}}}{\partial \lambda_\mu} \right] \\ &= \frac{1}{4} \vec{Y}^\dagger \cdot \frac{\partial M}{\partial \lambda_\mu} \cdot \vec{Y} - \frac{1}{4} \text{Tr} \left[\int_0^1 ds \hat{\pi}^s \vec{Y}^\dagger \cdot \frac{\partial M}{\partial \lambda_\mu} \cdot \vec{Y} \hat{\pi}^{1-s} \right] \\ &= \frac{1}{4} \vec{Y}^\dagger \cdot \frac{\partial M}{\partial \lambda_\mu} \cdot \vec{Y} - \frac{1}{4} \text{Tr} \left[\vec{Y}^\dagger \cdot \frac{\partial M}{\partial \lambda_\mu} \cdot \vec{Y} \hat{\pi} \right] \\ &= \tilde{X}_\mu, \end{aligned} \quad (\text{A.8})$$

where we have defined the conjugate forces $\tilde{X}_\mu = \hat{X}_\mu - \text{Tr}[\hat{X}_\mu \hat{\pi}]$ and $\hat{X}_\mu = \frac{1}{4} \vec{Y}^\dagger \cdot \frac{\partial M}{\partial \lambda_\mu} \cdot \vec{Y}$. Using this, we find

$$\frac{\partial \hat{\pi}_{\bar{\lambda}}}{\partial \lambda_\mu} = \int_0^1 ds \hat{\pi}^s \tilde{X}_\mu \hat{\pi}^{-s} \hat{\pi}. \quad (\text{A.9})$$

Combining these results, we find

$$\begin{aligned} \xi_{\mu\nu} &= - \text{Tr} \left[\frac{\partial \log \hat{\pi}_{\bar{\lambda}}}{\partial \lambda_\nu} \mathcal{L}_{\bar{\lambda}}^+ \frac{\partial \hat{\pi}_{\bar{\lambda}}}{\partial \lambda_\mu} \right] \\ &= \int_0^\infty d\nu \int_0^1 ds \text{Tr} \left[\tilde{X}_\mu e^{\nu\mathcal{L}} \hat{\pi}^s \tilde{X}_\nu \hat{\pi}^{-s} \hat{\pi} \right] \\ &= \int_0^\infty d\nu \int_0^1 ds \text{Tr} \left[\mathcal{M}(\nu) [\tilde{X}_\mu] \mathcal{U}(is) [\tilde{X}_\nu] \hat{\pi} \right], \end{aligned} \quad (\text{A.10})$$

where we made use of the identity $\text{Tr}[\hat{A}e^{\nu\mathcal{L}}[\hat{B}]] = \text{Tr}[e^{\nu\mathcal{L}^\dagger}[\hat{A}]\hat{B}]$, and $\mathcal{L}^\dagger$ denotes the Liouvillian's adjoint, and we defined the dissipative-like Heisenberg propagator and unitary-like propagator

$$\mathcal{M}(\nu)[\tilde{X}_i] = e^{\nu\mathcal{L}^\dagger}[\tilde{X}_i], \quad \mathcal{U}(\nu)[\tilde{X}_i] = e^{-\frac{i}{4}\nu\vec{Y}^\dagger \cdot M \cdot \vec{Y}} \tilde{X}_i e^{\frac{i}{4}\nu\vec{Y}^\dagger \cdot M \cdot \vec{Y}}, \quad (\text{A.11})$$

respectively. Previously, we have introduced the conjugate forces

$$\tilde{X}_i = \hat{X}_i - \text{Tr}[\hat{X}_i \hat{\pi}] \hat{\mathbb{I}}, \quad (\text{A.12})$$

where $\hat{X}_i = \frac{1}{4}\vec{Y}^\dagger \cdot \chi_i \cdot \vec{Y}$, with $\chi_i = \frac{\partial M}{\partial \lambda_i}$. We next analyse $\hat{X}_i$:

$$\hat{X}_i = \frac{1}{4}\vec{Y}^\dagger \cdot \chi_i \cdot \vec{Y} = \frac{1}{4} \sum_{jk} \chi_i^{jk} \hat{y}_j \hat{y}_k = \frac{1}{4} (\text{Tr}[\chi_i] \hat{\mathbb{I}} - \text{Tr}[\chi_i \Sigma]), \quad (\text{A.13})$$

where we have defined $\hat{\Sigma}_{jk} = \frac{1}{2}[\hat{y}_j, \hat{y}_k]$ and used its antisymmetry $\Sigma = -\Sigma^T$. Similarly, for $\text{Tr}[\hat{X}_i \hat{\pi}]$ we find

$$\text{Tr}[\hat{X}_i \hat{\pi}] = \frac{1}{4} \sum_{jk} \chi_i^{jk} \text{Tr}[\hat{y}_j \hat{y}_k \hat{\pi}] = \frac{1}{4} (\text{Tr}[\chi_i] - \text{Tr}[\chi_i \Theta]), \quad (\text{A.14})$$

where we have defined the covariance matrix $\Theta_{jk} = \text{Tr}[\hat{\Sigma}_{jk} \hat{\pi}]$. Therefore, combining the two terms yields

$$\tilde{X}_i = \frac{1}{4} (\text{Tr}[\chi_i] \hat{\mathbb{I}} - \text{Tr}[\chi_i \hat{\Sigma}]) - \frac{1}{4} (\text{Tr}[\chi_i] - \text{Tr}[\chi_i \Theta]) \hat{\mathbb{I}} = -\frac{1}{4} \text{Tr}[\chi_i (\hat{\Sigma} - \Theta)]. \quad (\text{A.15})$$

Dissipative-like Heisenberg propagation

We now focus on the dissipative-like Heisenberg evolution under the superoperator $\mathcal{M}$ of the $\hat{\Sigma}$ matrix of operators. To this end, we remember that in the Heisenberg picture the evolution of operators is governed by

$$\dot{\hat{O}} = \mathcal{L}^\dagger \hat{O} = i[\hat{H}, \hat{O}] + \sum_r \hat{L}_r^\dagger \hat{O} \hat{L}_r - \frac{1}{2} \{\hat{L}_r^\dagger \hat{L}_r, \hat{O}\}, \quad (\text{A.16})$$

where $\mathcal{L}^\dagger$ is the adjoint Liouvillian and we assume that the Hamiltonian is quadratic in the Majorana operators, $\hat{H} = \frac{1}{4}\vec{Y}^\dagger \cdot h \cdot \vec{Y}$, and the jump operators are linear in the Majorana operators, so that $\hat{L}_r = \sum_j c_j^r \hat{y}_j$.

We next analyse the unitary contribution. To this end, we compute

$$\begin{aligned}
 i[\hat{H}, \hat{y}_j \hat{y}_k] &= \frac{i}{4} \sum_{lm} h_{lm} [\hat{y}_l \hat{y}_m, \hat{y}_j \hat{y}_k] \\
 &= \frac{i}{2} \sum_{lm} h_{lm} (\delta_{mj} \hat{y}_l \hat{y}_k - \delta_{lj} \hat{y}_m \hat{y}_k + \delta_{mk} \hat{y}_j \hat{y}_l - \delta_{lk} \hat{y}_j \hat{y}_m) \\
 &= \frac{i}{2} \sum_m (h_{mj} \hat{y}_m \hat{y}_k - h_{jm} \hat{y}_m \hat{y}_k + h_{mk} \hat{y}_j \hat{y}_m - h_{km} \hat{y}_j \hat{y}_m),
 \end{aligned} \tag{A.17}$$

where we have used the canonical anticommutation relations (CARs). Similarly,

$$i[\hat{H}, \hat{y}_k \hat{y}_j] = \frac{i}{2} \sum_m (h_{mk} \hat{y}_m \hat{y}_j - h_{km} \hat{y}_m \hat{y}_j + h_{mj} \hat{y}_k \hat{y}_m - h_{jm} \hat{y}_k \hat{y}_m), \tag{A.18}$$

so that

$$i[\hat{H}, \hat{\Sigma}] = i(h^T \hat{\Sigma} + \hat{\Sigma} h) = -i(h \hat{\Sigma} + \hat{\Sigma} h^T), \tag{A.19}$$

where we have used the antisymmetry of $h = -h^T$.

Next, we examine the contribution from the dissipator. To this end, we compute using the CARs

$$\begin{aligned}
 \hat{L}_r^\dagger \hat{y}_m \hat{y}_n \hat{L}_r - \frac{1}{2} \{ \hat{L}_r^\dagger \hat{L}_r, \hat{y}_m \hat{y}_n \} &= \sum_{k,l} c_k^{r*} c_l^r \left(\hat{y}_k \hat{y}_m \hat{y}_n \hat{y}_l - \frac{1}{2} (\hat{y}_k \hat{y}_l \hat{y}_m \hat{y}_n + \hat{y}_m \hat{y}_n \hat{y}_k \hat{y}_l) \right) \\
 &= \sum_k (c_k^{r*} c_n^r \hat{y}_k \hat{y}_m - c_k^{r*} c_m^r \hat{y}_k \hat{y}_n) + (c_m^{r*} c_k^r \hat{y}_n \hat{y}_k - c_n^{r*} c_k^r \hat{y}_m \hat{y}_k),
 \end{aligned} \tag{A.20}$$

so that

$$\begin{aligned}
 \sum_r \hat{L}_r^\dagger \hat{\Sigma}_{mn} \hat{L}_r - \frac{1}{2} \{ \hat{L}_r^\dagger \hat{L}_r, \hat{\Sigma}_{mn} \} &= \sum_k B_{nk} \hat{y}_k \hat{y}_m - B_{kn} \hat{y}_m \hat{y}_k + B_{km} \hat{y}_n \hat{y}_k - B_{mk} \hat{y}_k \hat{y}_n \\
 &= -2(\hat{\Sigma} \text{Re}[B]^T + \text{Re}[B] \hat{\Sigma})_{mn} + 4\text{Im}[B]_{mn},
 \end{aligned} \tag{A.21}$$

where we have defined $B_{ij} = \sum_r c_i^r c_j^{r*}$, and used that $B_{ij} = B_{ji}^*$, and $\text{Re}[B] = \frac{B+B^*}{2}$ and $i\text{Im}[B] = \frac{B-B^*}{2}$.

Combining the above results, we finally find the Lyapunov differential equation for $\hat{\Sigma}$

$$\dot{\hat{\Sigma}} = -(W_D \hat{\Sigma} + \hat{\Sigma} W_D^T) + Y, \tag{A.22}$$

with

$$W_D = ih + 2\text{Re}[B], \quad Y = 4\text{Im}(B). \tag{A.23}$$

The solution for the matrix $\hat{\Sigma}$ is formally obtained as

$$\hat{\Sigma}(t) = F(t)\hat{\Sigma}F^T(t) + G(t), \quad (\text{A.24})$$

with

$$F(t) = e^{-W_D t}, \quad G(t) = \int_0^t ds F(s) Y F(s)^T. \quad (\text{A.25})$$

We now compute the corresponding evolution of the covariance matrix (multiplied by the identity operator). We first note that since the identity operator is the fixed point of $\mathcal{L}^\dagger$, $\mathcal{M}(\nu)[\Theta\hat{\mathbb{I}}] = \Theta\hat{\mathbb{I}}$. On the other hand, by multiplying with the fixed point and taking the trace

$$\begin{aligned} \Theta\hat{\mathbb{I}} &= \text{Tr}[\hat{\Sigma}(t)\hat{\pi}]\hat{\mathbb{I}} = \text{Tr}\left[F(t)\hat{\Sigma}F^T(t)\hat{\pi}\right]\hat{\mathbb{I}} + \text{Tr}[G(t)\hat{\pi}]\hat{\mathbb{I}} \\ &= F(t) \text{Tr}[\hat{\Sigma}\hat{\pi}]F^T(t) + G(t) \text{Tr}[\hat{\pi}]\hat{\mathbb{I}} \\ &= F(t)\Theta F^T(t) + G(t)\hat{\mathbb{I}}, \end{aligned} \quad (\text{A.26})$$

so that

$$\mathcal{M}(\nu)[\hat{\Sigma} - \Theta\hat{\mathbb{I}}] = F(\nu)(\hat{\Sigma} - \Theta\hat{\mathbb{I}})F^T(\nu). \quad (\text{A.27})$$

Finally, we arrive at

$$\mathcal{M}(\nu)[\tilde{X}_j] = -\frac{1}{4} \text{Tr}\left[\chi_j F(\nu)(\hat{\Sigma} - \Theta\hat{\mathbb{I}})F^T(\nu)\right]. \quad (\text{A.28})$$

Unitary-like Heisenberg propagation

We now treat the unitary-like Heisenberg evolution with

$$\mathcal{U}(\nu)[\tilde{X}_i] = e^{-\frac{i}{4}\nu\vec{Y}^\dagger \cdot M \cdot \vec{Y}} \tilde{X}_i e^{\frac{i}{4}\nu\vec{Y}^\dagger \cdot M \cdot \vec{Y}}. \quad (\text{A.29})$$

To this end, we note that, by analogy with the results derived for the unitary contribution in the previous subsection, the equation of motion of the Σ matrix of operators is

$$\dot{\hat{\Sigma}} = -\frac{i}{4}[\vec{Y}^\dagger \cdot M \cdot \vec{Y}, \hat{\Sigma}] = -(W_U \hat{\Sigma} + \hat{\Sigma} W_U^T), \quad (\text{A.30})$$

with $W_U = -iM$. Further, we define $S(t) = e^{-W_U t} = e^{iMt}$. Therefore, we obtain

$$\hat{\Sigma}(t) = S(t)\hat{\Sigma}(0)S(t)^T. \quad (\text{A.31})$$

Therefore, the unitarily propagated conjugate forces are (using that Θ should be understood as $\Theta\hat{\mathbb{I}}$)

$$\begin{aligned}\mathcal{U}(\nu)[\tilde{X}_j] &= -\frac{1}{4}e^{-\frac{i}{4}\nu(\vec{Y}^\dagger \cdot M \cdot \vec{Y})} \text{Tr}[\chi_i(\hat{\Sigma} - \Theta)]e^{\frac{i}{4}\nu(\vec{Y}^\dagger \cdot M \cdot \vec{Y})} \\ &= -\frac{1}{4} \text{Tr}\left[\chi_i e^{-\frac{i}{4}\nu(\vec{Y}^\dagger \cdot M \cdot \vec{Y})}(\hat{\Sigma} - \Theta)e^{\frac{i}{4}\nu(\vec{Y}^\dagger \cdot M \cdot \vec{Y})}\right] \\ &= -\frac{1}{4} \text{Tr}\left[\chi_i S(\nu)(\hat{\Sigma} - \Theta)S(\nu)^T\right].\end{aligned}\tag{A.32}$$

Final Expression for the Metric

In the previous sections, we obtained

$$\begin{aligned}\mathcal{U}(\nu)[\tilde{X}_j] &= -\frac{1}{4} \text{Tr}\left[\chi_j S(\nu)(\hat{\Sigma} - \Theta)S(\nu)^T\right], \\ \mathcal{M}(\nu)[\tilde{X}_i] &= -\frac{1}{4} \text{Tr}\left[\chi_i (F(\nu)(\hat{\Sigma} - \Theta)F(\nu)^T)\right].\end{aligned}\tag{A.33}$$

We now insert these into the previously obtained expression for the metric

$$\begin{aligned}\xi_{ij} &= \int_0^\infty d\nu \int_0^1 ds \text{Tr}\left[\mathcal{M}(\nu)[\tilde{X}_\mu]\mathcal{U}(is)[\tilde{X}_\mu]\hat{\pi}\right] \\ &= \frac{1}{16} \int_0^\infty d\nu \int_0^1 ds \text{Tr}\left[\hat{\pi} \text{Tr}\left[\chi_i (F(\nu)(\hat{\Sigma} - \Theta)F(\nu)^T)\right] \text{Tr}\left[\chi_j S(is)(\Sigma - \Theta)S(is)^T\right]\right].\end{aligned}\tag{A.34}$$

The terms in the trace are

$$\begin{aligned}\text{Tr}\left[\chi_i (F(\nu)(\hat{\Sigma} - \Theta)F(\nu)^T)\right] &= -\text{Tr}\left[\chi_i F(\nu)\Theta F(\nu)^T\right] + \sum_{jkmn} \chi_i^{jk} F_{km} \hat{\Sigma}_{mn} F_{nj}^T, \\ \text{Tr}\left[\chi_j S(\nu)(\Sigma - \Theta)S(\nu)^T\right] &= -\text{Tr}\left[\chi_j S(\nu)\Theta S(\nu)^T\right] + \sum_{xykl} \chi_j^{xy} S_{yk} \hat{\Sigma}_{kl} S_{lx}^T.\end{aligned}\tag{A.35}$$

To arrive at the final expression, we now multiply these and take the trace

$$\begin{aligned}
 & \text{Tr} \left[\hat{\pi} \text{Tr} \left[\chi_i (F(\nu) (\hat{\Sigma} - \Theta) F(\nu)^T) \right] \text{Tr} \left[\chi_j (S(is) \Sigma S(is)^T - \Theta) \right] \right] \\
 &= \text{Tr} \left[\chi_i F(\nu) \Theta F(\nu)^T \right] \text{Tr} \left[\chi_j S(\nu) \Theta S(\nu)^T \right] \\
 &\quad - \text{Tr} \left[\chi_j S(\nu) \Theta S(\nu)^T \right] \underbrace{\sum_{jkmn} \chi_i^{jk} F_{km} \text{Tr} [\hat{\Sigma}_{mn} \hat{\pi}] F_{nj}^T}_{\text{Tr} [\chi_i F \Theta F^T]} \\
 &\quad - \text{Tr} \left[\chi_i F(\nu) \Theta F(\nu)^T \right] \underbrace{\sum_{xykl} \chi_j^{xy} S_{yk} \text{Tr} [\hat{\Sigma}_{kl} \hat{\pi}] S_{lx}^T}_{\text{Tr} [\chi_j S \Theta S^T]} \\
 &\quad + \sum_{xykl} \sum_{ijmn} \chi_j^{xy} S_{yk} S_{lx}^T \chi_i^{ij} F_{jm} \text{Tr} [\hat{\Sigma}_{kl} \hat{\Sigma}_{mn} \hat{\pi}] F_{ni}^T.
 \end{aligned} \tag{A.36}$$

To compute the last term above, we make use of Wick's theorem

$$\text{Tr} [y_{\alpha_1} y_{\alpha_2} \dots y_{\alpha_{2n}} \hat{\pi}] = (-i)^n \text{Pf} \left[i \Theta_{|\alpha_1 \alpha_2 \dots \alpha_{2n}} \right], \tag{A.37}$$

where Θ_{ijkl} is the corresponding sub-matrix of Θ . We find that

$$\text{Tr} [y_i \hat{y}_j \hat{y}_k y_l \hat{\pi}] = \Theta_{ij} \Theta_{kl} - \Theta_{ik} \Theta_{jl} + \Theta_{il} \Theta_{jk}. \tag{A.38}$$

Further,

$$\begin{aligned}
 \text{Tr} [\hat{\Sigma}_{kl} \hat{\Sigma}_{mn} \hat{\pi}] &= \frac{1}{4} \text{Tr} [[\hat{y}_k, \hat{y}_l] [\hat{y}_m, \hat{y}_n] \hat{\pi}] \\
 &= \frac{1}{4} \text{Tr} [(4 \hat{y}_k \hat{y}_l \hat{y}_m \hat{y}_n - 2 \delta_{mn} \hat{y}_k \hat{y}_l - 2 \delta_{kl} \hat{y}_m \hat{y}_n + \delta_{mn} \delta_{kl}) \hat{\pi}].
 \end{aligned} \tag{A.39}$$

Therefore,

$$\begin{aligned}
 & \sum_{xykl} \sum_{ijmn} \chi_j^{xy} S_{yk} S_{lx}^T \chi_i^{ij} F_{jm} \text{Tr} [\hat{\Sigma}_{kl} \hat{\Sigma}_{mn} \hat{\pi}] F_{ni}^T \\
 &= \text{Tr} [\chi_i F \Theta F^T] \text{Tr} [\chi_j S \Theta S^T] - 2 \text{Tr} [F^T \chi_i F \Theta S^T \chi_j S \Theta].
 \end{aligned} \tag{A.40}$$

With this, we finally arrive at

$$\begin{aligned}
 \xi_{ij} &= \int_0^\infty d\nu \int_0^1 ds \text{Tr} [\mathcal{M}(\nu) [\tilde{X}_\mu] \mathcal{U}(is) [\tilde{X}_\mu] \hat{\pi}] \\
 &= -\frac{1}{8} \int_0^\infty d\nu \int_0^1 ds \text{Tr} [F(\nu)^\dagger \chi_i F(\nu) (\Theta - \mathbb{I}) S(is)^\dagger \chi_j S(is) (\Theta + \mathbb{I})].
 \end{aligned} \tag{A.41}$$

Note that the term in the trace is a product of matrices of dimension $(2N \times 2N)$, as desired.

The most efficient (and stable) way of computing the integral $\int_0^\infty d\nu F(\nu)^T \chi_i F(\nu)$

is to notice that $\int_0^\infty dv F(v)^T \chi_i F(v)$ is the formal solution to the continuous Lyapunov equation $(W_D^T X + X W_D) = \chi_i$, for which numerical solvers are readily available.

We now compute the S -evolution integral making use of the eigendecomposition of M :

$$\begin{aligned} \int_0^1 ds S(s)^T \chi_j S(s) &= \int_0^1 ds e^{Ms} \chi_j e^{-Ms} \\ &= \sum_{kl} \int_0^1 ds U e^{\mu_k s} |k\rangle \langle k| U^\dagger \chi_j U e^{-\mu_l s} U^\dagger \\ &= U \sum_{kl} \tilde{\chi}_{kl} |k\rangle \langle l| \frac{e^{\mu_k - \mu_l} - 1}{\mu_k - \mu_l} U^\dagger. \end{aligned} \quad (\text{A.42})$$

First, we diagonalised the Hermitian matrix M via a unitary transformation U , such that $M = U \Lambda U^\dagger$, where $\Lambda = \text{diag}(\mu_i)$ contains the eigenvalues. The operator χ_i is then transformed into the eigenbasis as $\chi_i^{\text{diag}} = U^\dagger \chi_i U$. A new matrix A is constructed in the eigenbasis, with

$$A_{ij} = \begin{cases} (\chi_i^{\text{diag}})_{ij} \frac{e^{\mu_i - \mu_j} - 1}{\mu_i - \mu_j}, & \text{for } i \neq j, \\ (\chi_i^{\text{diag}})_{ii}, & \text{for } i = j. \end{cases} \quad (\text{A.43})$$

For obtaining the diagonal elements we used that $\int_0^1 ds e^{\mu_k s} e^{-\mu_l s} |_{m=n} = \int_0^1 1 ds = 1$. Finally, the result is transformed back to the original basis via $U A U^\dagger$. This approach is numerically efficient and stable.

Example: Boundary-driven XY chain

We investigate a dissipative quantum phase transition (DQPT) in the boundary-driven XY spin chain with Markovian noise [423], using the thermodynamic metric. Our analysis is inspired by earlier work employing the quantum Fisher information (QFI) metric [279].

The XY Hamiltonian is given by

$$H = \sum_{i=1}^{n-1} \left(\frac{1+\kappa}{2} \sigma_i^x \sigma_{i+1}^x + \frac{1-\kappa}{2} \sigma_i^y \sigma_{i+1}^y \right) + h_0 \sum_{i=1}^n \sigma_i^z, \quad (\text{A.44})$$

where σ_j^α denote the Pauli operators acting on the j th spin. The boundary spins are coupled to two reservoirs, which is described by the Lindblad jump operators

$$L_L^\pm = \sqrt{\Gamma_L^\pm} \sigma_1^\pm, \quad L_R^\pm = \sqrt{\Gamma_R^\pm} \sigma_n^\pm, \quad (\text{A.45})$$

with $\sigma_j^\pm = (\sigma_j^x \pm i\sigma_j^y)/2$. The rates $\Gamma_{L/R}^\pm$ depend on the reservoir parameters

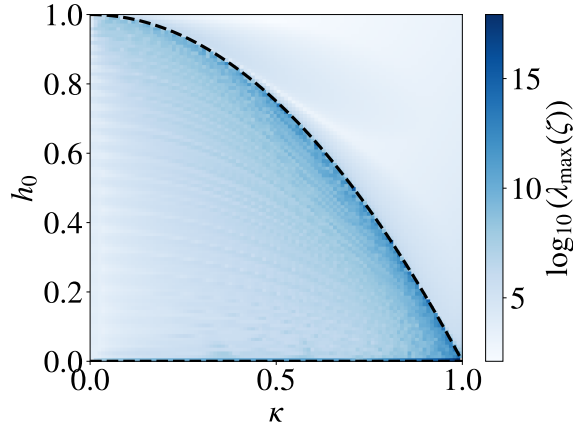


Figure A.1: Phase diagram of the dissipative XY model in terms of the largest eigenvalue, $\lambda_{\max}$, of the thermodynamic metric ζ . The black dashed lines show the theoretically predicted critical lines. Parameters: $N = 50$, $J = 1$, $\gamma_L = 0.8$, $\gamma_R = 0.6$, $f_L = 0.275$, $f_R = 0.167$.

according to

$$\Gamma_{L/R}^+ = \sqrt{\gamma_{L/R} f_{L/R}}, \quad \Gamma_{L/R}^- = \sqrt{\gamma_{L/R} (1 - f_{L/R})}. \quad (\text{A.46})$$

Using the Jordan-Wigner transformation [424], the model can be mapped exactly onto a quadratic Majorana master equation [279]. As a consequence, the NESS of the resulting Markovian dynamics is Gaussian, and distinct phases can be identified as functions of the Hamiltonian parameters (h_0, κ) . In particular, the model exhibits a DQPT at the critical field

$$h_c = 1 - \kappa^2. \quad (\text{A.47})$$

Although additional transition lines are present, we do not explore them here, as the model serves only as a representative example. A detailed analysis can be found in Ref. [423].

We find that, similarly to the QFI [423], the thermodynamic metric provides a clear diagnostic of the critical lines, as illustrated in Fig. A.1.

A.2 KMB QFI (Bosonic)

The Kubo-Mori-Bogoliubov (KMB) quantum Fisher information about parameter g is defined as

$$I_Q^{(\text{KMB})}(g) = \int_0^1 ds \text{Tr} \left[\hat{\rho}^s (\partial_g \log \hat{\rho}) \hat{\rho}^{1-s} (\partial_g \log \hat{\rho}) \right]. \quad (\text{A.48})$$

Performing an analogous calculation to that presented in Appendix A.1, we find that for Gaussian bosonic states with zero mean, the KMB QFI (see Chapter IV.7)

is given by

$$I_Q^{(\text{KMB})}(g) = \frac{1}{2} \int_0^1 ds \text{Tr}[\chi_g \check{\Theta} S^T(s) \chi_g S(s) \check{\Theta}^T], \quad (\text{A.49})$$

with $S(s) = e^{si\Omega M}$ and $\check{\Theta} = \Theta + i\Omega/2$.

A.3 Thermodynamic metric (Bosonic)

Generalising the results of Ref. [425] from Gaussian bosonic thermal states to Gaussian bosonic NESS and assuming zero mean ($\langle \vec{R} \rangle = 0$), we find that the single-parameter thermodynamic metric (see Section IV.7.5) is given by

$$\zeta_g = \frac{1}{2} \text{Tr} \left[\mathcal{I}(\chi_g) \left(\Theta - \frac{i}{2} \Omega \right) \mathcal{F}(\chi_g) \left(\Theta + \frac{i}{2} \Omega \right) \right]. \quad (\text{A.50})$$

Here, we define $\chi_g = \partial_g M$, with M being the matrix defining the Gaussian NESS (see Eq. (I.3.2)). Furthermore,

$$\mathcal{I}(\chi_g) = \int_0^1 ds (e^{is\Omega M})^T \chi_g e^{is\Omega M}, \quad \mathcal{F}(\chi_g) = \int_0^\infty d\nu (e^{\nu A})^T \chi_g e^{\nu A}, \quad (\text{A.51})$$

with $A = \Omega(H - \text{Im}(CC^\dagger))$, where the matrix $C = (\vec{c}_1^T, \dots, \vec{c}_{N_L}^T)^T$ encodes the jump rates of the associated linear Lindblad jump operators $\hat{L}_i = \vec{c}_i^T \cdot \vec{R}$. In Appendix A.3, we discuss how this metric can be evaluated efficiently numerically. A derivation of the thermodynamic metric for Gaussian fermionic states is presented in Appendix A.1.

Efficient evaluation

In Section A.3, we presented Eq. (A.50), which allows the thermodynamic metric of Gaussian bosonic states to be computed directly from their covariance matrix, rather than from the full state. In this appendix, we show how the individual terms composing the metric can be evaluated, simplifying both analytical calculations and the avoidance of certain numerical errors.

Unitary contribution We first evaluate the integral in $\mathcal{I}(\chi_g)$. To this end, we define

$$A = (i\Omega M)^T, \quad B = i\Omega M. \quad (\text{A.52})$$

Diagonalising yields

$$A = U_A D_A U_A^{-1}, \quad B = U_B D_B U_B^{-1}, \quad (\text{A.53})$$

with D_A, D_B diagonal, and we can write

$$e^{sA} = U_A e^{sD_A} U_A^{-1}, \quad e^{sB} = U_B e^{sD_B} U_B^{-1}. \quad (\text{A.54})$$

Substituting into $\mathcal{I}(\chi_g)$, we find

$$\begin{aligned} \mathcal{I}(\chi_g) &= \int_0^1 ds e^{sA} \chi_g e^{sB} = U_A \left[\int_0^1 ds e^{sD_A} \tilde{\chi}_g e^{sD_B} \right] U_B^{-1}, \\ \tilde{\chi}_g &= U_A^{-1} \chi_g U_B. \end{aligned}$$

Let $D_A = \text{diag}(\lambda_1, \dots, \lambda_{2n})$ and $D_B = \text{diag}(\mu_1, \dots, \mu_{2n})$. Then

$$[e^{sD_A} \tilde{\chi}_g e^{sD_B}]_{ij} = \tilde{\chi}_{gij} e^{s(\lambda_i + \mu_j)}. \quad (\text{A.55})$$

Hence,

$$\int_0^1 ds e^{s(\lambda_i + \mu_j)} = \begin{cases} \frac{e^{\lambda_i + \mu_j} - 1}{\lambda_i + \mu_j}, & \lambda_i + \mu_j \neq 0, \\ 1, & \lambda_i + \mu_j = 0. \end{cases} \quad (\text{A.56})$$

We define

$$J_{ij} = \tilde{\chi}_{gij} f(\lambda_i + \mu_j), \quad f(z) = \begin{cases} \frac{e^z - 1}{z}, & z \neq 0, \\ 1, & z = 0, \end{cases} \quad (\text{A.57})$$

and find

$$\int_0^1 ds e^{s(i\Omega M)^T} \chi_g e^{si\Omega M} = U_A J U_B^{-1}. \quad (\text{A.58})$$

Dissipative-like contribution For the dissipative-like term we evaluate

$$\mathcal{F}(\chi_g) = \int_0^\infty d\nu e^{\nu A^T} \chi_g e^{\nu A}. \quad (\text{A.59})$$

This integral is the solution of the continuous Lyapunov equation [426]

$$A^T Y + Y A = -\chi_g, \quad (\text{A.60})$$

which can be solved efficiently using standard numerical routines.

A.4 Relative entropy (Bosonic)

Given two bosonic Gaussian states (see Section I.3.1), and assuming zero mean,

$$\hat{\rho} = \frac{e^{-\frac{1}{2} R^T \cdot M \cdot R}}{Z} \quad \text{and} \quad \hat{\sigma} = \frac{e^{-\frac{1}{2} R^T \cdot N \cdot R}}{Y}, \quad (\text{A.61})$$

we may use the symmetry of M and N and the definition of the covariance matrix, to show that the relative entropy is given by [211]

$$D(\hat{\rho}||\hat{\sigma}) = -\frac{1}{2}\text{Tr}[(M - N)\Theta] - \log \frac{Z}{Y}, \quad (\text{A.62})$$

where Θ denotes the covariance matrix corresponding to the state $\hat{\rho}$.

A.5 Von Neumann Entropy (Bosonic)

The symplectic eigenvalues $\{\nu_j\}$ of the covariance matrix are given by the positive eigenvalues of the matrix $|i\Omega\Theta|$ (see Section I.3.1). In particular, the von Neumann entropy can be expressed solely in terms of the symplectic eigenvalues as [212]

$$S_{\text{vN}} = \sum_{j=1}^N f(\nu_j), \quad f(\nu) = \left(\nu + \frac{1}{2}\right) \log\left(\nu + \frac{1}{2}\right) - \left(\nu - \frac{1}{2}\right) \log\left(\nu - \frac{1}{2}\right). \quad (\text{A.63})$$

A.6 Uhlmann fidelity (Bosonic)

The Uhlmann fidelity is defined as

$$F(\rho_1, \rho_2) = \text{Tr} \sqrt{\sqrt{\rho_1} \rho_2 \sqrt{\rho_1}}. \quad (\text{A.64})$$

Given two bosonic Gaussian states, it can be computed as [212]

$$F(\rho_1, \rho_2) = F_0(\rho_1, \rho_2) e^{-\frac{1}{4}(\bar{r}_1 - \bar{r}_2)^T (\Theta_1 + \Theta_2)^{-1} (\bar{r}_1 - \bar{r}_2)}, \quad (\text{A.65})$$

with

$$F_0(\rho_1, \rho_2) = \left[\frac{\det \left(2 \left(\sqrt{1 + \frac{1}{4}(V\Omega)^{-2}} + 1 \right) V \right)}{\det(\Theta_1 + \Theta_2)} \right]^{1/4}, \quad (\text{A.66})$$

and

$$V = \Omega^T (\Theta_1 + \Theta_2)^{-1} \left(\frac{1}{4} \Omega + \Theta_2 \Omega \Theta_1 \right). \quad (\text{A.67})$$

B Dissipative Lipkin-Meshkov-Glick model

This appendix is based on Ref. [387].

The Lipkin-Meshkov-Glick (LMG) model is a paradigmatic model of a fully connected spin system, originally introduced to study collective interactions and quantum phase transitions in nuclei and many-body systems [427]. In the con-

text of open quantum systems, it also serves as a minimal platform to explore DQPTs [387, 428, 429]. Physically, the model describes an ensemble of N effective spin- $\frac{1}{2}$ degrees of freedom with all-to-all anisotropic interactions, subject to dissipation induced by coupling to a memoryless environment. In solid-state realisations, examples include engineered magnetic moments on metallic surfaces interacting with spin-polarised conduction electrons; in quantum optics or cold atoms, similar collective spin dynamics can be realised by coupling atoms to optical cavities or reservoirs [387].

The coherent part of the dynamics is given by a collective spin Hamiltonian acting in the maximally symmetric sector of total spin $s = N/2$:

$$\hat{H} = -h \hat{S}_z - \frac{\gamma_x}{2s} \hat{S}_x^2 - \frac{\gamma_y}{2s} \hat{S}_y^2, \quad (\text{B.1})$$

where $\hat{\vec{S}} = (\hat{S}_x, \hat{S}_y, \hat{S}_z)^T$ are collective spin operators with $[\hat{S}_\alpha, \hat{S}_\beta] = i\varepsilon_{\alpha\beta\gamma}\hat{S}_\gamma$. Here, h is an external magnetic field, and γ_x, γ_y parameterise the strength of anisotropic collective spin-spin interactions. The explicit $1/s$ scaling ensures a well-defined thermodynamic limit $s \rightarrow \infty$ with extensive energy [387].

To model dissipation, the system is coupled to memoryless reservoirs that induce quantum jumps. Under conditions such as large bias voltage and weak coupling to electronic leads, the environmental influence on the spin degrees of freedom can be treated in the Born-Markov approximation, yielding a GKSL master equation [387]:

$$\frac{d}{dt}\hat{\rho} = \mathcal{L}[\hat{\rho}] = -i[\hat{H}, \hat{\rho}] + \sum_i \mathcal{D}[\hat{L}_i]\hat{\rho}, \quad (\text{B.2})$$

with the dissipator

$$\mathcal{D}[\hat{L}_i]\hat{\rho} = \hat{L}_i\hat{\rho}\hat{L}_i^\dagger - \frac{1}{2}\{\hat{L}_i^\dagger\hat{L}_i, \hat{\rho}\}. \quad (\text{B.3})$$

The set of jump operators is chosen in accordance with the polarisation of the reservoir and reads [387]

$$\hat{L}_z = \sqrt{\frac{\Gamma}{2s}} \tilde{S}_z, \quad \hat{L}_+ = \sqrt{\frac{\Gamma}{2s} \frac{1-p}{2}} \tilde{S}_+, \quad \hat{L}_- = \sqrt{\frac{\Gamma}{2s} \frac{1+p}{2}} \tilde{S}_-. \quad (\text{B.4})$$

Here, Γ denotes the overall quantum jump rate, proportional to the absolute value of the applied bias voltage, while the dimensionless parameter $p \in [-1, 1]$ characterises the polarisation of the reservoir and controls the imbalance between raising and lowering spin-flip processes. The tilde indicates that the collective spin operators are quantised along the polarisation axis of the reservoir. We restrict to a parallel set-up, so that one has $\tilde{S}_\alpha = S_\alpha$.

In the thermodynamic limit $N \rightarrow \infty$, collective spin models admit a systematic semiclassical description. One defines rescaled spin components $\hat{m} = \langle \hat{\vec{S}} \rangle / s$,

which evolve according to nonlinear mean-field equations derived from $\mathcal{L}$. Fluctuations around the mean-field solutions can be treated via a Holstein-Primakoff mapping (see Section V.9.3.1), which maps small deviations of the collective spin away from a classical orientation onto a bosonic mode. In the large- s limit (thermodynamic limit), the Liouvillian becomes quadratic in the bosonic operators, and the steady state is Gaussian, allowing analytical access to spectral properties of the Liouvillian (see Section I.3.3), including the closing of the Liouvillian gap at the DQPT [387].

The model used in Ref. [387] exhibits a variety of NESS phases and the authors identify a second order phase transition between the phases they label by $I_{\parallel}$ (unique stable NESS) and $II_{\parallel}$ (two steady states related by symmetry). For our purposes it suffices to restrict to these two phases, and we thus refer to the $I_{\parallel}$ phase as the normal phase and the $II_{\parallel}$ phase as the symmetry-broken phase. Further, it is found that at this second order DQPT, the Liouvillian spectral gap vanishes as $\Delta \sim s^{-1}$, and the von Neumann entropy exhibits logarithmic divergence $S_{\text{vN}} \sim \ln s$ [387]. In the thermodynamic limit, treated within the Holstein-Primakoff approximation (see Section V.9.3.1), when introducing the coefficients

$$\begin{aligned}\kappa &= \eta \left[(2h + \gamma_x + \gamma_y)^2 - 4p(h + \gamma_x)(h + \gamma_y) + (1 - p)p^2\Gamma^2 \right], \\ \delta &= \eta(\gamma_y - \gamma_x)(2h + \gamma_x + \gamma_y + ip\Gamma), \\ \eta^{-1} &= 2p \left[4(h + \gamma_x)(h + \gamma_y) + \Gamma^2 p^2 \right],\end{aligned}\tag{B.5}$$

the steady-state von Neumann entropy can be computed using

$$S_{\text{vN}} = p_+ \ln p_+ + p_- \ln p_-, \tag{B.6}$$

where

$$p_{\pm} = \frac{1}{2} \left(1 \pm \sqrt{(1 + 2\kappa)^2 - 4\bar{\delta}\delta} \right) \tag{B.7}$$

are related to the eigenvalues of the fluctuation mode covariance matrix of the Holstein-Primakoff boson.

In Appendix C.1 we show that, in the thermodynamic limit, this entropy exhibits a logarithmic divergence with the distance to the critical point of the $I_{\parallel} \leftrightarrow I_{\parallel}$ continuous phase transition as the collective spin-spin coupling γ_x is varied, in agreement with the analytical prediction presented in the same section.

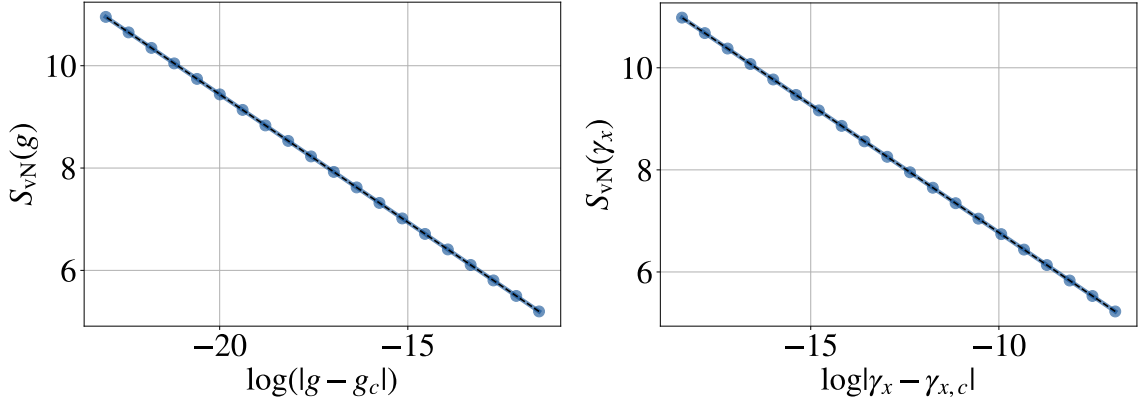


Figure C.1: Logarithmic scaling of the von Neumann entropy in the proximity to the critical point: $S_{\text{vN}} \sim -\frac{\gamma}{2} \log |g - g_c|$. Left: driven-dissipative Dicke model. We obtain $\gamma/2 = 0.49999977 \pm 4 \cdot 10^{-8}$ via a numerical fit. Parameters: cavity frequency $\omega_c = 1$, atomic frequency $\omega_z = 1$, photon loss rate $\kappa = 0.2$. Right: open LMG model. We obtain $\gamma/2 = 0.500015 \pm 4 \cdot 10^{-6}$ via a numerical fit. Parameters: magnetic field strength $h = 1$, collective spin-spin coupling along the y -axis $\gamma_y = 2$, jump rate $\Gamma = 0.2$, reservoir polarisation $p = 0.2$.

C Supplementary results on critical scaling in Gaussian bosonic models

C.1 Critical scaling of the von Neumann entropy

We consider a Gaussian bosonic system whose covariance matrix evolves under a Lyapunov differential equation (I.3.14), so that the NESS covariance matrix solves the Lyapunov equation (I.3.16). Here, we aim to understand the behaviour of the von Neumann entropy of the NESS in the vicinity of a critical point g_c in a continuous DQPT. To this end we use the scaling of the symplectic eigenvalues with the distance to the critical point, presented in Section V.10.4.1. For large ν_s , the von Neumann entropy contribution of this mode (see Appendix A.5),

$$s(\nu_s) \simeq \log \nu_s + \mathcal{O}(1), \quad (\text{C.1})$$

diverges logarithmically as $\log(\Delta_g^{-1/2})$. Therefore, the total von Neumann entropy is dominated by the soft mode contribution,

$$S_{\text{vN}} \sim -\frac{1}{2} \log(\Delta_g). \quad (\text{C.2})$$

Assuming an algebraic vanishing of the Liouvillian gap ($\Delta \sim |g - g_c|^\gamma$), it follows that

$$S_{\text{vN}} \sim -\frac{\gamma}{2} \log(|g - g_c|). \quad (\text{C.3})$$

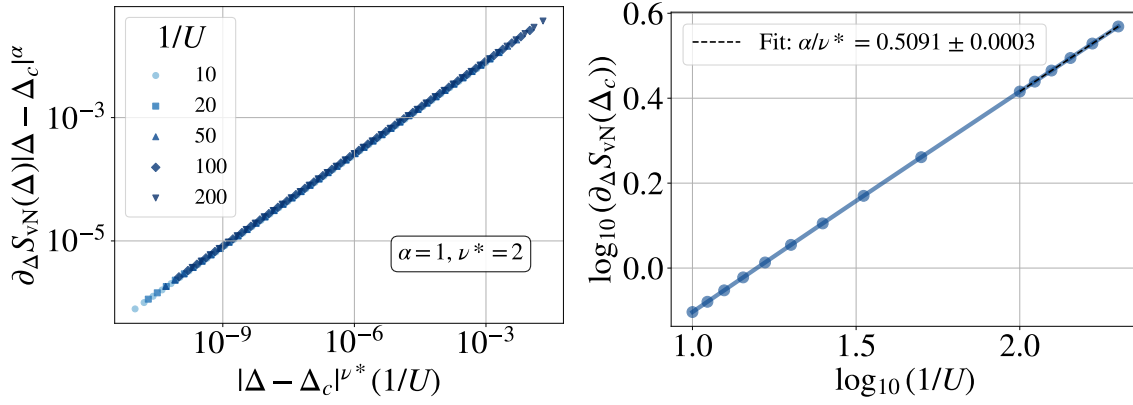


Figure C.2: Finite-size scaling analysis of the von Neumann entropy in the dissipative Kerr model as a function of the distance to the critical point (left panel) and at the critical point (right panel). Here, $1/U \rightarrow \infty$ corresponds to the thermodynamic limit [430]. Parameters: parametric drive amplitude $G = 1.5$, photon loss rate $\kappa = 1$.

We validate the predicted logarithmic divergence of the NESS von Neumann entropy at the critical point in Gaussian bosonic systems both in the thermodynamic limit in the open Dicke model, as shown in Fig. C.1, the open LMG model, as shown in Fig. C.1 (see Appendix B and Ref. [387] for the model), and via finite-size scaling (see Section V.9.4) of the driven-dissipative Kerr model (see Section V.9.3.2). Following standard practice, we analyse the derivative of the entropy with respect to the control parameter, which converts the logarithmic divergence into a power-law scaling. The results, in excellent agreement with the theoretical prediction, are shown in Fig. C.2.

C.2 Critical scaling of the excess entropy flux

The excess entropy flux, to first order in the driving speed, is given by

$$\dot{\Pi}_{\text{ex}}(t) \approx -\dot{g} \partial_g S_{\text{vN}}(\hat{\pi}_g). \quad (\text{C.4})$$

In Appendix C.1, we have shown that for bosonic Gaussian models, the von Neumann entropy diverges logarithmically with the closing Liouvillian gap at the transition (see Eq. (C.2)). Since we assume an algebraic vanishing of the Liouvillian gap, we obtain

$$\partial_g S_{\text{vN}}(\hat{\pi}_g) \sim \partial_g \log |g - g_c| \sim \frac{1}{|g - g_c|}. \quad (\text{C.5})$$

Therefore,

$$\dot{\Pi}_{\text{ex}}(t) \sim \frac{\dot{g}}{|g - g_c|}. \quad (\text{C.6})$$

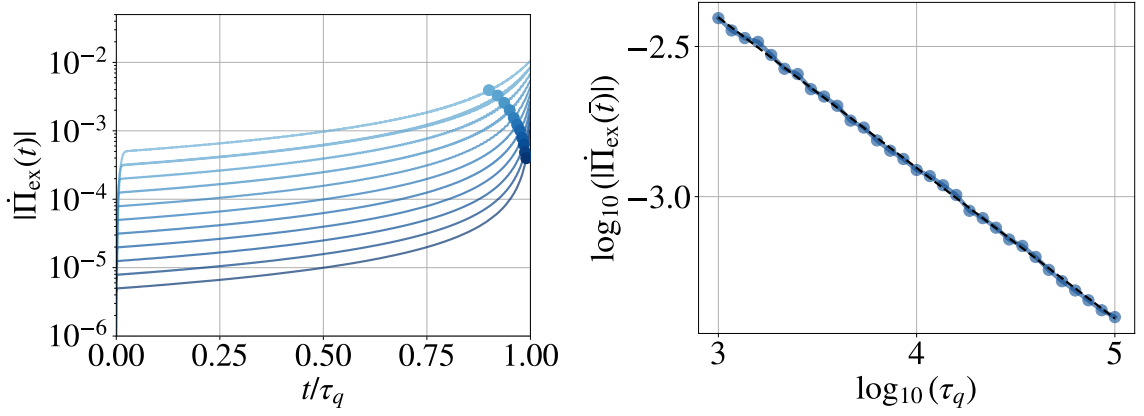


Figure C.3: Example: DQPT in the open Dicke model. Left: The excess entropy flux as a function of time for different protocol durations $\tau_q = 10^2 \dots 10^5$ of a linear ramp of the coupling strength g from the normal phase to the critical point g_c . The scatter points indicate the onset of the impulse regime. Right: Scaling of the excess entropy flux upon entering the impulse regime with the protocol duration τ_q of a linear ramp of the coupling strength g from the normal phase to the critical point g_c . We find $\dot{\Pi}_{\text{ex}}(\bar{t}) \sim \tau_q^x$ with $x = -0.501 \pm 0.004$. This numerical finding is in excellent agreement with our theoretical prediction $x = -\frac{\gamma}{\gamma+1}$ with $\gamma = 1$. Parameters: cavity frequency $\omega_c = 1$, atomic frequency $\omega_z = 1$, photon loss rate $\kappa = 0.2$, initial coupling strength $g_0 = g_c - 5 \times 10^{-2}$, final coupling strength $g_f = g_c - 10^{-15}$.

Note that given a linear protocol of duration τ_q , the impulse regime sets in at $|\bar{g} - g_c| \sim \tau_q^{-\frac{1}{\gamma+1}}$. Therefore,

$$\dot{\Pi}_{\text{ex}}(t^*) \sim \frac{\dot{g}}{|\bar{g} - g_c|} \sim \tau_q^{-1} \tau_q^{\frac{1}{\gamma+1}} = \tau_q^{-\frac{\gamma}{\gamma+1}}. \quad (\text{C.7})$$

We find excellent agreement with this theoretical prediction for the dissipative quantum Dicke model in the thermodynamic limit (see Section V.9.3.1), for which $\gamma = 1$ (see Section V.10.5), as shown in Fig. C.3.

C.3 Critical scaling of the cavity photon flux

To obtain the cavity photon flux in terms of the covariance matrix, it is useful to first write the general jump superoperator as [217]

$$\mathcal{J}[\hat{\rho}] = \sum_{ij} V_{ji} \hat{R}_i \hat{\rho} \hat{R}_j, \quad (\text{C.8})$$

with $V = \frac{1}{2} \mathbb{I}_2 \otimes (\nu_+ \gamma_+ + \nu_- \gamma_-) + \frac{1}{2} \sigma_y \otimes (\nu_+ \gamma_+ - \nu_- \gamma_-)$, where $\gamma_{\pm}$ are the photon absorption and emission rate, respectively (see Eq. (I.3.8)). The average current then is given by [217]

$$J = \text{Tr}[\mathcal{J}[\hat{\rho}]] = \text{Tr} \left[\sum_{ij} V_{ji} \hat{R}_i \hat{\rho} \hat{R}_j \right] = \sum_{ij} V_{ji} \langle \hat{R}_j \hat{R}_i \rangle = \text{Tr}[V \tilde{\Theta}] + r^T V r, \quad (\text{C.9})$$

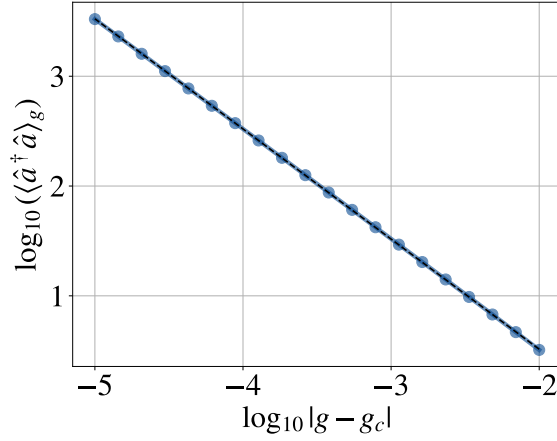


Figure C.4: Scaling of the NESS cavity photon number in the proximity to the critical point: $\langle \hat{a}^\dagger \hat{a} \rangle_g \sim |g - g_c|^{-\gamma}$, in the driven-dissipative Dicke model. We obtain $\gamma = 1.0032 \pm 0.0006$ via a numerical fit, which is in excellent agreement with our theoretical prediction $x = 1$. Parameters: cavity frequency $\omega_c = 1$, atomic frequency $\omega_z = 1$, photon loss rate $\kappa = 0.2$.

with weights ν_i^- associated with jump operators describing the extraction of particles ($\hat{b}_i \hat{\rho} \hat{b}_i^\dagger$), and ν_i^+ associated with injection ($\hat{b}_i^\dagger \hat{\rho} \hat{b}_i$), and

$$\tilde{\Theta} = \Theta - i \frac{\Omega}{2}, \quad (\text{C.10})$$

using the convention $\Omega = (i\sigma_y) \otimes \mathbb{I}_N$. For the settings considered in the driven-dissipative Dicke and Kerr models, dissipation arises solely from a single jump operator associated with cavity photon loss. We therefore set $\nu^- = 1$ and $\nu^+ = 0$. In this case, the NESS photon flux is directly proportional to the cavity photon occupation,

$$J = \gamma_- \langle \hat{a}^\dagger \hat{a} \rangle. \quad (\text{C.11})$$

Consequently, any divergence of the cavity photon number at a dissipative critical point is directly reflected in a corresponding divergence of the photon flux.

We now turn to the scaling of the cavity photon number. This follows from the divergence of the soft-mode occupation, which can be inferred directly from the structure of the steady-state covariance matrix. As shown in Section V.10.4.1, in the vicinity of the dissipative critical point the dominant contribution to the covariance matrix is rank one and scales as $\Theta_s \sim \Delta_g^{-1} v_s v_s^\dagger$, while all remaining contributions remain finite. When restricted to the canonical phase-space subspace of the soft mode, this implies that only a single quadrature variance diverges as Δ_g^{-1} , whereas the conjugate quadrature remains $\mathcal{O}(1)$. The occupation of the soft mode,

$$n_s = \frac{1}{2} (\text{Tr } \Theta_s - 1), \quad (\text{C.12})$$

therefore diverges as $n_s \sim \Delta_g^{-1} \sim |g - g_c|^{-\gamma}$.

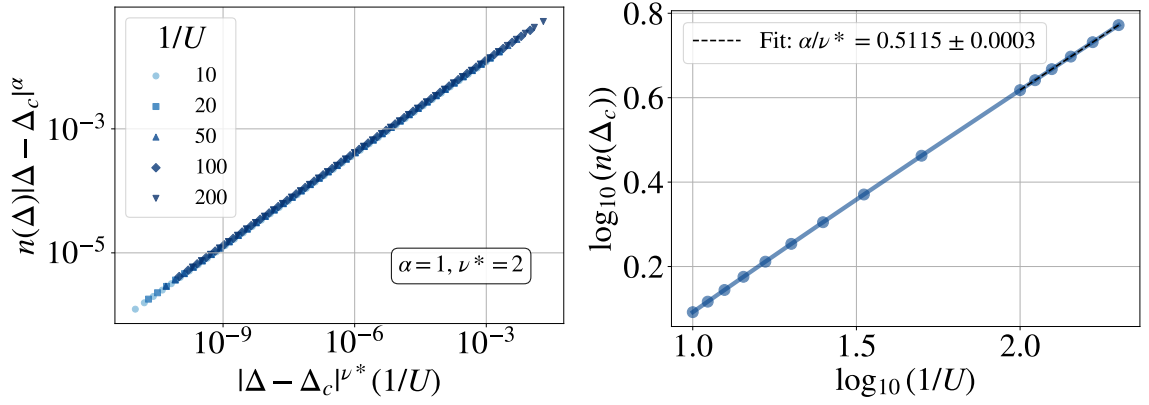


Figure C.5: Finite-size scaling analysis of the cavity photon flux in the open Kerr model as a function of the distance to the critical point (left panel) and at the critical point (right panel). Here, $1/U \rightarrow \infty$ corresponds to the thermodynamic limit [430]. Parameters: parametric drive amplitude $G = 1.5$, photon loss rate $\kappa = 1$.

As a result, the photon flux exhibits the same critical divergence,

$$J \sim |g - g_c|^{-\gamma}. \quad (\text{C.13})$$

We confirm this scaling prediction in the open Dicke model in the thermodynamic limit, as shown in Fig. C.4, and via finite-size scaling in the driven-dissipative Kerr oscillator, as shown in Fig. C.5. Further, evaluating this expression at the freeze-out point (V.10.4), we obtain

$$J(g^*) \sim |g^* - g_c|^{-\gamma} \sim \tau_q^{\gamma/(\gamma+1)}. \quad (\text{C.14})$$

Hence, the experimentally accessible photon flux evaluated at the freeze-out point obeys Kibble-Zurek-type scaling.

D Quasistatic gauge potential

In this appendix, we discuss the quasistatic gauge potential, an open-system analogue of the adiabatic gauge potential. It serves as the generator of quasistatic transformations for systems evolving under GKSL dynamics. The adiabatic theorem states that a quantum system initially prepared in an eigenstate of a Hamiltonian will remain in that eigenstate if the Hamiltonian varies sufficiently slowly. This principle underpins many applications in quantum state preparation, quantum computation, and thermodynamics. For open systems, an analogous statement can be formulated: a quantum system that starts in a NESS of a Liouvillian will remain in that state (or rather “close” to it), provided the Liouvillian changes sufficiently slowly.

D.1 Derivation

To derive the form of the quasistatic gauge potential, we begin from the instantaneous steady-state condition

$$\mathcal{L}_{g(t)}[\hat{\pi}_{g(t)}] = 0. \quad (\text{D.1})$$

Applying the time derivative, we find that

$$\frac{d}{dt}\mathcal{L}_{g(t)}[\hat{\pi}_{g(t)}] = \dot{\mathcal{L}}_{g(t)}[\hat{\pi}_{g(t)}] + \mathcal{L}_{g(t)}\left[\frac{d}{dt}\hat{\pi}_{g(t)}\right] = 0. \quad (\text{D.2})$$

Solving for $\frac{d}{dt}\hat{\pi}_{g(t)}$, we obtain

$$\frac{d}{dt}\hat{\pi}_{g(t)} = -\mathcal{L}_{g(t)}^+\dot{\mathcal{L}}_{g(t)}[\hat{\pi}_{g(t)}]. \quad (\text{D.3})$$

Alternatively, in terms of the derivative with respect to the parameter g (applying the chain rule), we find

$$\partial_g \hat{\pi}_g = -\mathcal{L}_g^+ \partial_g \mathcal{L}_g[\hat{\pi}_g]. \quad (\text{D.4})$$

These expressions motivate the definition of a quasistatic response superoperator $\mathcal{A}_g$, which serves as an open-system extension of the adiabatic gauge potential, defined as (compare to unitary case: $i[\hat{H}_g, [\hat{H}_g, A_g]] = [\partial_g \hat{H}_g, \hat{H}_g]$):

$$\mathcal{A}_t[\hat{\pi}_{g(t)}] = \frac{d}{dt}\hat{\pi}_{g(t)} = -\mathcal{L}_{g(t)}^+ \frac{d}{dt}\mathcal{L}_{g(t)}[\hat{\pi}_{g(t)}]. \quad (\text{D.5})$$

and

$$\mathcal{A}_g[\hat{\pi}_g] = \partial_g \hat{\pi}_g = -\mathcal{L}_g^+ \partial_g \mathcal{L}_g[\hat{\pi}_g]. \quad (\text{D.6})$$

This can be derived starting from $\mathcal{L}_g[\hat{\pi}_g] = 0$ with $\rho_{g+\delta g} \approx \hat{\pi}_g + \delta g \frac{d\hat{\pi}}{dg}$. In the case of Hamiltonian driving, so that $\hat{H}(t) = \sum_k \lambda_k(t) \hat{X}_k$, one finds that $\mathcal{A}_k[\cdot] = i\mathcal{L}^+[\hat{X}_k, \cdot]$.

The above equation may be expanded in the vectorised Liouvillian eigenmodes, simply using $\mathcal{L}_{g(t)}^+ = \sum_{n>0} \frac{1}{\lambda_n(g(t))} |n(g(t))\rangle\rangle \langle\langle n(g(t))|$ and $\text{vec}(\hat{\pi}_{g(t)}) = |\hat{\pi}_{g(t)}\rangle\rangle$:

$$\frac{d}{dt}|\hat{\pi}_{g(t)}\rangle\rangle = - \sum_{n>0} \frac{1}{\lambda_n(g(t))} |n(g(t))\rangle\rangle \langle\langle n(g(t))| \dot{\mathcal{L}}_{g(t)} |\hat{\pi}_{g(t)}\rangle\rangle. \quad (\text{D.7})$$

D.2 Connection between QGP and AGP

Next, we establish a connection between the quasistatic gauge potential for GKSL dynamics and the adiabatic gauge potential for unitary evolution. To this end, let $\hat{H} = \sum_n E_n |n\rangle\langle n|$ be the spectral decomposition of the Hamiltonian. Consider the

superoperator $\mathcal{L}_U[\cdot] = -i[\hat{H}, \cdot]$. In the eigenbasis of $\hat{H}$, the operator $|n\rangle\langle m|$ is an eigenoperator of $\mathcal{L}_U$ with eigenvalue $-i(E_n - E_m)$,

$$\mathcal{L}_U[|n\rangle\langle m|] = -i[\hat{H}, |n\rangle\langle m|] = -i(E_n - E_m)|n\rangle\langle m|. \quad (\text{D.8})$$

In vectorised form, using the identity $\text{vec}(\hat{A}\hat{X}\hat{B}) = (\hat{B}^T \otimes \hat{A}) \text{vec}(\hat{X})$, we write

$$\text{vec}(\mathcal{L}_U[\hat{X}]) = -i(\hat{\mathbb{I}} \otimes \hat{H} - \hat{H}^T \otimes \hat{\mathbb{I}}) \text{vec}(\hat{X}). \quad (\text{D.9})$$

Thus, the vectorised representation of $\mathcal{L}_U$ is

$$\mathcal{L}_U = -i(\hat{\mathbb{I}} \otimes \hat{H} - \hat{H}^T \otimes \hat{\mathbb{I}}), \quad (\text{D.10})$$

and its eigenvalues are $-i(E_n - E_m)$, with eigenvectors $\text{vec}(|n\rangle\langle m|)$. Now, we let $\mathcal{L}_U[\cdot] = -i[\hat{H}_g, \cdot]$ be the coherent Liouvillian generated by a family of Hamiltonians $\hat{H}_g$, and let $\hat{\rho} = |0\rangle\langle 0|$ be the instantaneous ground state of $\hat{H}_g$ with eigenvalue E_0 . We aim to compute how the quasistatic gauge potential acts on the instantaneous Hamiltonian ground state density matrix

$$\mathcal{A}_g^U[\hat{\rho}] = -\mathcal{L}_U^+(\partial_g \mathcal{L}_U[\hat{\rho}]), \quad (\text{D.11})$$

where $\mathcal{L}_U^+$ denotes the Drazin inverse of $\mathcal{L}_U$, to connect our results to closed system adiabatic perturbation theory. First, observe that

$$\partial_g \mathcal{L}_U[\hat{\rho}] = -i[\partial_g \hat{H}_g, \hat{\rho}]. \quad (\text{D.12})$$

We then apply $\mathcal{L}_U^+$ to the commutator. In the eigenbasis $\{|n\rangle\}$ of $\hat{H}_g$, the operators $|n\rangle\langle m|$ form a complete basis of the operator space and satisfy

$$\mathcal{L}_U^+[|n\rangle\langle m|] = \begin{cases} \frac{i}{E_n - E_m} |n\rangle\langle m| & \text{if } n \neq m, \\ 0 & \text{if } n = m. \end{cases} \quad (\text{D.13})$$

Expanding the commutator,

$$[\partial_g \hat{H}_g, \hat{\rho}] = \sum_{n \neq 0} (\langle n | \partial_g \hat{H}_g | 0 \rangle |n\rangle\langle 0| - \langle 0 | \partial_g \hat{H}_g | n \rangle |0\rangle\langle n|), \quad (\text{D.14})$$

and applying $\mathcal{L}_U^+$ yields

$$\begin{aligned}
-\mathcal{L}_U^+ (\partial_g \mathcal{L}_U [\hat{\rho}]) &= i \mathcal{L}_U^+ ([\partial_g \hat{H}_g, \hat{\rho}]) \\
&= i \sum_{n \neq 0} (\langle n | \partial_g \hat{H}_g | 0 \rangle \frac{i}{E_n - E_0} |n\rangle \langle 0| - \langle 0 | \partial_g \hat{H}_g | n \rangle \frac{i}{E_0 - E_n} |0\rangle \langle n|) \\
&= \sum_{n \neq 0} \frac{1}{E_n - E_0} (\langle n | \partial_g \hat{H}_g | 0 \rangle |n\rangle \langle 0| + \langle 0 | \partial_g \hat{H}_g | n \rangle |0\rangle \langle n|) .
\end{aligned} \tag{D.15}$$

The quasistatic gauge potential superoperator thus generalises the adiabatic gauge potential operator to open quantum systems evolving under GKSL dynamics.

Bibliography

- [1] H. B. Callen, *Thermodynamics and an Introduction to Thermostatistics*, 2nd ed. (John Wiley & Sons, Nashville, TN, 1985).
- [2] R. Clausius, *Annalen der Physik* **169**, 481 (1854).
- [3] R. Clausius, *Annalen der Physik* **201**, 353 (1865).
- [4] E. Broda, *Berichte zur Wissenschaftsgeschichte* **6**, 103 (1983).
- [5] U. Seifert, *Reports on Progress in Physics* **75**, 126001 (2012).
- [6] C. Jarzynski, *Annual Review of Condensed Matter Physics* **2**, 329 (2011).
- [7] J. Goold, M. Huber, A. Riera, L. d. Rio, and P. Skrzypczyk, *Journal of Physics A: Mathematical and Theoretical* **49**, 143001 (2016).
- [8] S. Vinjanampathy and J. Anders, *Contemporary Physics* **57**, 545 (2016).
- [9] F. Binder, L. A. Correa, C. Gogolin, J. Anders, and G. Adesso, eds., *Thermodynamics in the quantum regime*, 2018th ed., Fundamental Theories of Physics (Springer International Publishing, Cham, Switzerland, 2019).
- [10] S. Campbell, I. D’Amico, M. A. Ciampini, J. Anders, N. Ares, S. Artini, A. Auffeves, L. B. Otfelie, L. P. Bettmann, M. V. Bonança, T. Busch, M. Campisi, M. F. Cavalcante, L. A. Correa, E. Cuestas, C. B. Dag, D. Salambô, S. Deffner, A. del Campo, A. Deutschmann-Olek, S. Donadi, E. Doucet, C. Elouard, K. Ensslin, P. Erker, N. Fabbri, F. Fedele, G. Fiusa, T. Fogarty, J. A. Folk, G. Guarnieri, A. S. Hegde, S. Hernández-Gómez, C.-K. Hu, F. Iemini, B. Karimi, N. Kiesel, G. Landi, A. Lasek, S. Lemziakov, G. Lo Monaco, E. Lutz, D. Lvov, O. Maillet, M. Mehboudi, T. M. Mendonça, H. J. D. Miller, A. K. Mitchell, M. Mitchison, V. Mukherjee, M. Paternostro, J. P. Pekola, M. Perarnau-Llobet, U. G. Poschinger, A. Rolandi, D. Rosa, R. Sanchez, A. C. Santos, R. S. Sarthour, E. Sela, A. Solfanelli, A. M. de Souza, J. Splettstoesser, D. Tan, L. Tesser, T. V. Vu, A. Widera, N. Yunger Halpern, and K. Zawadzki, *Quantum Science and Technology* (2025).
- [11] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, 2007).
- [12] R. Kosloff and A. Levy, *Annual Review of Physical Chemistry* **65**, 365 (2014).

- [13] G. Benenti, G. Casati, K. Saito, and R. S. Whitney, [Physics Reports](#) **694**, 1 (2017).
- [14] N. M. Myers, O. Abah, and S. Deffner, [AVS Quantum Science](#) **4** (2022).
- [15] J. Baugh, O. Moussa, C. A. Ryan, A. Nayak, and R. Laflamme, [Nature](#) **438**, 470 (2005).
- [16] A. Solfanelli, A. Santini, and M. Campisi, [AVS Quantum Science](#) **4** (2022).
- [17] M. A. Aamir, P. Jamet Suria, J. A. Marín Guzmán, C. Castillo-Moreno, J. M. Epstein, N. Yunger Halpern, and S. Gasparinetti, [Nature Physics](#) **21**, 318 (2025).
- [18] P. Lipka-Bartosik, M. Perarnau-Llobet, and N. Brunner, [Science Advances](#) **10** (2024).
- [19] M. T. Mitchison, [Contemporary Physics](#) **60**, 164 (2019).
- [20] J. Antonio Marín Guzmán, P. Erker, S. Gasparinetti, M. Huber, and N. Yunger Halpern, [Reports on Progress in Physics](#) **87**, 122001 (2024).
- [21] A. Levy and R. Kosloff, [Physical Review Letters](#) **108** (2012).
- [22] J. Ding, Y. Li, H. Wang, G. Xue, T. Su, C. Wang, W. Sun, F. Li, Y. Zhang, Y. Gao, J. Peng, Z. H. Jiang, Y. Yu, H. Yu, and F. Yan, [Physical Review Applied](#) **23** (2025).
- [23] B. Bhandari and A. N. Jordan, [Physical Review Research](#) **4** (2022).
- [24] Y. Levinson, [Europhysics Letters \(EPL\)](#) **39**, 299 (1997).
- [25] I. L. Aleiner, N. S. Wingreen, and Y. Meir, [Physical Review Letters](#) **79**, 3740 (1997).
- [26] L. Stodolsky, [Physics Letters B](#) **459**, 193 (1999).
- [27] M. Büttiker and A. M. Martin, [Physical Review B](#) **61**, 2737 (2000).
- [28] V. Wadhia, F. Meier, F. Fedele, R. Silva, N. Nurgalieva, D. L. Craig, D. Jirovec, J. Saez-Mollejo, A. Ballabio, D. Chrastina, G. Isella, M. Huber, M. T. Mitchison, P. Erker, and N. Ares, [Physical Review Letters](#) **135** (2025).
- [29] L. P. Bettmann, M. J. Kewming, and J. Goold, [Physical Review E](#) **107** (2023).
- [30] L. Buffoni, A. Solfanelli, P. Verrucchi, A. Cuccoli, and M. Campisi, [Physical Review Letters](#) **122** (2019).
- [31] F. Barra, [Scientific Reports](#) **5** (2015).
- [32] F. Ciccarello, S. Lorenzo, V. Giovannetti, and G. M. Palma, [Physics Reports](#) **954**, 1 (2022).
- [33] G. Lindblad, [Communications in Mathematical Physics](#) **48**, 119 (1976).
- [34] V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, [Journal of Mathematical Physics](#) **17**, 821 (1976).

- [35] P. Talkner and P. Hänggi, [Reviews of Modern Physics](#) **92** (2020).
- [36] M. Esposito, U. Harbola, and S. Mukamel, [Reviews of Modern Physics](#) **81**, 1665 (2009).
- [37] Y. Blanter and M. Büttiker, [Physics Reports](#) **336**, 1 (2000).
- [38] B. K. Agarwalla, B. Li, and J.-S. Wang, [Phys. Rev. E](#) **85**, 051142 (2012).
- [39] M. Esposito, M. A. Ochoa, and M. Galperin, [Phys. Rev. B](#) **91**, 115417 (2015).
- [40] M. Moskalets and M. Büttiker, [Phys. Rev. B](#) **70**, 245305 (2004).
- [41] M. Moskalets, [Phys. Rev. Lett.](#) **112**, 206801 (2014).
- [42] E. Potanina, C. Flindt, M. Moskalets, and K. Brandner, [Phys. Rev. X](#) **11**, 021013 (2021).
- [43] E. Aurell, [Phys. Rev. E](#) **97**, 062117 (2018).
- [44] K. Funo and H. T. Quan, [Phys. Rev. E](#) **98**, 012113 (2018).
- [45] M. Kilgour and D. Segal, [The Journal of Chemical Physics](#) **143** (2015).
- [46] M. Popovic, M. T. Mitchison, A. Strathearn, B. W. Lovett, J. Goold, and P. R. Eastham, [PRX Quantum](#) **2**, 020338 (2021).
- [47] A. Imamoglu, [Physical Review A](#) **50**, 3650 (1994).
- [48] B. M. Garraway, [Physical Review A](#) **55**, 4636 (1997).
- [49] B. M. Garraway, [Physical Review A](#) **55**, 2290 (1997).
- [50] M. Brenes, J. J. Mendoza-Arenas, A. Purkayastha, M. T. Mitchison, S. R. Clark, and J. Goold, [Physical Review X](#) **10** (2020).
- [51] H. M. Wiseman and G. J. Milburn, [Quantum Measurement and Control](#) (Cambridge University Press, 2009).
- [52] L. P. Bettmann, M. J. Kewming, G. T. Landi, J. Goold, and M. T. Mitchison, [Phys. Rev. E](#) **112**, 014105 (2025).
- [53] M. Kang, S. Kim, Y. Qian, P. M. Neves, L. Ye, J. Jung, D. Puntel, F. Mazzola, S. Fang, C. Jozwiak, A. Bostwick, E. Rotenberg, J. Fuji, I. Vobornik, J.-H. Park, J. G. Checkelsky, B.-J. Yang, and R. Comin, [Nature Physics](#) **21**, 110 (2024).
- [54] S. Kim, Y. Chung, Y. Qian, S. Park, C. Jozwiak, E. Rotenberg, A. Bostwick, K. S. Kim, and B.-J. Yang, [Science](#) **388**, 1050 (2025).
- [55] J. Yu, B. A. Bernevig, R. Queiroz, E. Rossi, P. Törmä, and B.-J. Yang, [npj Quantum Materials](#) **10** (2025).
- [56] C.-K. Chiu, J. C. Teo, A. P. Schnyder, and S. Ryu, [Reviews of Modern Physics](#) **88** (2016).
- [57] M. G. A. Paris, [International Journal of Quantum Information](#) **07**, 125 (2009).

- [58] W. Janke, D. A. Johnston, and R. Kenna, *Physica A: Statistical Mechanics and its Applications Proceedings of the XVIII Max Born Symposium "Statistical Physics outside Physics"*, **336**, 181 (2004).
- [59] S. Deffner and S. Campbell, *Journal of Physics A: Mathematical and Theoretical* **50**, 453001 (2017).
- [60] T. Nishiyama and Y. Hasegawa, *Unified speed limits in classical and quantum dynamics via temporal fisher information* (2025).
- [61] L. Mandelstam and I. Tamm, The uncertainty relation between energy and time in non-relativistic quantum mechanics, in *Selected Papers*, edited by B. M. Bolotovskii, V. Y. Frenkel, and R. Peierls (Springer Berlin Heidelberg, Berlin, Heidelberg, 1991) pp. 115–123.
- [62] N. Margolus and L. B. Levitin, *Physica D: Nonlinear Phenomena* **120**, 188 (1998).
- [63] M. M. Taddei, B. M. Escher, L. Davidovich, and R. L. de Matos Filho, *Physical Review Letters* **110** (2013).
- [64] S. Ito and A. Dechant, *Physical Review X* **10** (2020).
- [65] D. P. Pires, M. Cianciaruso, L. C. Céleri, G. Adesso, and D. O. Soares-Pinto, *Phys. Rev. X* **6**, 021031 (2016).
- [66] L. P. García-Pintos, S. B. Nicholson, J. R. Green, A. del Campo, and A. V. Gorshkov, *Physical Review X* **12** (2022).
- [67] D. P. Pires, E. R. deAzevedo, D. O. Soares-Pinto, F. Brito, and J. G. Filgueiras, *Communications Physics* **7** (2024).
- [68] J. Bringewatt, Z. Steffen, M. A. Ritter, A. Ehrenberg, H. Wang, B. S. Palmer, A. J. Kollár, A. V. Gorshkov, and L. P. García-Pintos, *Physical Review Research* **7** (2025).
- [69] S. B. Nicholson, A. del Campo, and J. R. Green, *Physical Review E* **98** (2018).
- [70] S. Ito, *Phys. Rev. Lett.* **121**, 030605 (2018).
- [71] L. P. Bettmann and J. Goold, *Phys. Rev. E* **111**, 014133 (2025).
- [72] G. E. Crooks, *Physical Review Letters* **99** (2007).
- [73] P. Salamon and R. S. Berry, *Physical Review Letters* **51**, 1127 (1983).
- [74] D. A. Sivak and G. E. Crooks, *Physical Review Letters* **108** (2012).
- [75] D. Mandal and C. Jarzynski, *Journal of Statistical Mechanics: Theory and Experiment* **2016**, 063204 (2016).
- [76] M. Scandi and M. Perarnau-Llobet, *Quantum* **3**, 197 (2019).
- [77] P. Abiuso, H. J. D. Miller, M. Perarnau-Llobet, and M. Scandi, *Entropy* **22**, 1076 (2020).

- [78] A. M. Lacerda, L. P. Bettmann, and J. Goold, [Phys. Rev. E **112**, L022101 \(2025\)](#).
- [79] T. Hatano and S. Sasa, [Phys. Rev. Lett. **86**, 3463 \(2001\)](#).
- [80] M. Esposito and C. Van den Broeck, [Physical Review E **82** \(2010\)](#).
- [81] L. M. Sieberer, M. Buchhold, J. Marino, and S. Diehl, [Rev. Mod. Phys. **97**, 025004 \(2025\)](#).
- [82] B. Öztop, M. Bordyuh, O. E. Müstecaplioglu, and H. E. Türeci, [New Journal of Physics **14**, 085011 \(2012\)](#).
- [83] H. Carmichael, [Physical Review X **5** \(2015\)](#).
- [84] H. Weimer, [Phys. Rev. Lett. **114**, 040402 \(2015\)](#).
- [85] M. Benito, C. Sánchez Muñoz, and C. Navarrete-Benlloch, [Physical Review A **93** \(2016\)](#).
- [86] J. J. Mendoza-Arenas, S. R. Clark, S. Felicetti, G. Romero, E. Solano, D. G. Angelakis, and D. Jaksch, [Physical Review A **93** \(2016\)](#).
- [87] W. Casteels, R. Fazio, and C. Ciuti, [Physical Review A **95**, 012128 \(2017\)](#).
- [88] W. Casteels, F. Storme, A. Le Boité, and C. Ciuti, [Physical Review A **93**, 033824 \(2016\)](#).
- [89] W. Casteels and C. Ciuti, [Physical Review A **95**, 013812 \(2017\)](#).
- [90] N. Bartolo, F. Minganti, W. Casteels, and C. Ciuti, [Physical Review A **94** \(2016\)](#).
- [91] M. Foss-Feig, K. R. A. Hazzard, J. J. Bollinger, and A. M. Rey, [Phys. Rev. A **87**, 042101 \(2013\)](#).
- [92] M. Biondi, G. Blatter, H. E. Türeci, and S. Schmidt, [Physical Review A **96** \(2017\)](#).
- [93] A. Biella, F. Storme, J. Lebreuilly, D. Rossini, R. Fazio, I. Carusotto, and C. Ciuti, [Physical Review A **96** \(2017\)](#).
- [94] V. Savona, [Physical Review A **96** \(2017\)](#).
- [95] C. Sánchez Muñoz, A. Lara, J. Puebla, and F. Nori, [Physical Review Letters **121** \(2018\)](#).
- [96] L. M. Sieberer, S. D. Huber, E. Altman, and S. Diehl, [Physical Review Letters **110**, 195301 \(2013\)](#).
- [97] L. M. Sieberer, S. D. Huber, E. Altman, and S. Diehl, [Physical Review B **89** \(2014\)](#).
- [98] E. Altman, L. M. Sieberer, L. Chen, S. Diehl, and J. Toner, [Physical Review X **5** \(2015\)](#).

- [99] T. E. Lee, S. Gopalakrishnan, and M. D. Lukin, [Physical Review Letters](#) **110** (2013).
- [100] J. Jin, A. Biella, O. Viyuela, L. Mazza, J. Keeling, R. Fazio, and D. Rossini, [Physical Review X](#) **6** (2016).
- [101] E. M. Kessler, G. Giedke, A. Imamoglu, S. F. Yelin, M. D. Lukin, and J. I. Cirac, [Physical Review A](#) **86**, 012116 (2012).
- [102] T. E. Lee, H. Häffner, and M. C. Cross, [Physical Review A](#) **84** (2011).
- [103] C.-K. Chan, T. E. Lee, and S. Gopalakrishnan, [Physical Review A](#) **91** (2015).
- [104] M. F. Maghrebi and A. V. Gorshkov, [Physical Review B](#) **93** (2016).
- [105] R. Rota, F. Storme, N. Bartolo, R. Fazio, and C. Ciuti, [Physical Review B](#) **95** (2017).
- [106] D. Roscher, S. Diehl, and M. Buchhold, [Physical Review A](#) **98** (2018).
- [107] V. R. Overbeck, M. F. Maghrebi, A. V. Gorshkov, and H. Weimer, [Physical Review A](#) **95** (2017).
- [108] K. Baumann, C. Guerlin, F. Brennecke, and T. Esslinger, [Nature](#) **464**, 1301 (2010).
- [109] F. Ferri, R. Rosa-Medina, F. Finger, N. Dogra, M. Soriente, O. Zilberberg, T. Donner, and T. Esslinger, [Physical Review X](#) **11**, 041046 (2021).
- [110] M. Fitzpatrick, N. M. Sundaresan, A. C. Y. Li, J. Koch, and A. A. Houck, [Physical Review X](#) **7**, 011016 (2017).
- [111] S. R. K. Rodriguez, W. Casteels, F. Storme, N. Carlon Zambon, I. Sagnes, L. Le Gratiet, E. Galopin, A. Lemaître, A. Amo, C. Ciuti, and J. Bloch, [Physical Review Letters](#) **118**, 247402 (2017).
- [112] T. Fink, A. Schade, S. Höfling, C. Schneider, and A. Imamoglu, [Nature Physics](#) **14**, 365 (2018).
- [113] G. Beaulieu, F. Minganti, S. Frasca, V. Savona, S. Felicetti, R. Di Candia, and P. Scarlino, [Nature Communications](#) **16**, 1954 (2025).
- [114] P. Kirton, M. M. Roses, J. Keeling, and E. G. D. Torre, [Advanced Quantum Technologies](#) **2**, 1800043 (2019).
- [115] M. Brunelli, L. Fusco, R. Landig, W. Wieczorek, J. Hoelscher-Obermaier, G. Landi, F. Semião, A. Ferraro, N. Kiesel, T. Donner, G. De Chiara, and M. Paternostro, [Physical Review Letters](#) **121** (2018).
- [116] A. del Campo and W. H. Zurek, [Int. J. Mod. Phys. A](#) **29**, 1430018 (2014).
- [117] W. H. Zurek, U. Dorner, and P. Zoller, [Physical Review Letters](#) **95**, 105701 (2005).
- [118] A. Polkovnikov, [Phys. Rev. B](#) **72**, 161201 (2005).

- [119] S. Deffner, [Physical Review E](#) **96**, 052125 (2017).
- [120] Z. Fei, N. Freitas, V. Cavina, H. T. Quan, and M. Esposito, [Physical Review Letters](#) **124**, 170603 (2020).
- [121] P. Nazé, M. V. S. Bonança, and S. Deffner, [Entropy](#) **24**, 666 (2022).
- [122] Y. Shen, Z. Jiang, Y. Huang, B. M. Cleary, Y. Jiang, G. M. Rotskoff, and A. M. Lindenberg, [Quantifying and minimizing dissipation in a non-equilibrium phase transition](#) (2025).
- [123] Z. Ma, A. K. Mitchell, and E. Sela, [Phys. Rev. Lett.](#) **135**, 130402 (2025).
- [124] A. Carollo, D. Valenti, and B. Spagnolo, [Physics Reports](#) **838**, 1 (2020).
- [125] L. P. Bettmann, A. M. Lacerda, M. T. Mitchison, and J. Goold, [Thermodynamic universality across dissipative quantum phase transitions](#) (2025).
- [126] J. Preskill, [Quantum](#) **2**, 79 (2018).
- [127] L. M. Cangemi, C. Bhadra, and A. Levy, [Physics Reports](#) **1087**, 1 (2024), quantum engines and refrigerators.
- [128] P. P. Potts, [Quantum Thermodynamics](#) (2024).
- [129] Y. V. Nazarov and Y. M. Blanter, *Quantum Transport* (Cambridge University Press, Cambridge, England, 2009).
- [130] K. Kraus, *States, effects, and operations*, 1983rd ed., edited by A. Bohm, J. D. Dollard, and W. H. Wootters, Lecture Notes in Physics (Springer, Berlin, Germany, 1983).
- [131] H.-P. Breuer, E.-M. Laine, J. Piilo, and B. Vacchini, [Reviews of Modern Physics](#) **88** (2016).
- [132] K. Mølmer, Y. Castin, and J. Dalibard, [Journal of the Optical Society of America B](#) **10**, 524 (1993).
- [133] H. Carmichael, *An Open Systems Approach to Quantum Optics: Lectures Presented at the Université Libre de Bruxelles October 28 to November 4, 1991* (Springer Berlin Heidelberg, 1993).
- [134] A. Purkayastha, [Physical Review A](#) **105** (2022).
- [135] K. Itô, [Nagoya Mathematical Journal](#) **3**, 55 (1951).
- [136] R. Alicki, [Reports on Mathematical Physics](#) **10**, 249 (1976).
- [137] J. J. Koliha and I. Straškraba, [Applications of Mathematics](#) **44**, 289 (1999).
- [138] G. De Chiara, G. Landi, A. Hewgill, B. Reid, A. Ferraro, A. J. Roncaglia, and M. Antezza, [New Journal of Physics](#) **20**, 113024 (2018).
- [139] M. Cattaneo, G. L. Giorgi, S. Maniscalco, and R. Zambrini, [New Journal of Physics](#) **21**, 113045 (2019).

- [140] P. P. Hofer, M. Perarnau-Llobet, L. D. M. Miranda, G. Haack, R. Silva, J. B. Brask, and N. Brunner, [New Journal of Physics](#) **19**, 123037 (2017).
- [141] H. Wichterich, M. J. Henrich, H.-P. Breuer, J. Gemmer, and M. Michel, [Physical Review E](#) **76** (2007).
- [142] A. Levy and R. Kosloff, [EPL \(Europhysics Letters\)](#) **107**, 20004 (2014).
- [143] A. S. Trushechkin and I. V. Volovich, [EPL \(Europhysics Letters\)](#) **113**, 30005 (2016).
- [144] A. M. Lacerda, A. Purkayastha, M. Kewming, G. T. Landi, and J. Goold, [Phys. Rev. B](#) **107**, 195117 (2023).
- [145] H. Spohn, [J. Math. Phys.](#) **19**, 1227 (1978).
- [146] M. Esposito, K. Lindenberg, and C. Van den Broeck, [New Journal of Physics](#) **12**, 013013 (2010).
- [147] D. Reeb and M. M. Wolf, [New Journal of Physics](#) **16**, 103011 (2014).
- [148] G. T. Landi and M. Paternostro, [Reviews of Modern Physics](#) **93** (2021).
- [149] P. Strasberg and A. Winter, [PRX Quantum](#) **2** (2021).
- [150] P. Strasberg, M. G. Díaz, and A. Riera-Campenya, [Phys. Rev. E](#) **104**, L022103 (2021).
- [151] S. Carnot, *Réflexions sur la puissance motrice du feu et sur les machines propres à développer cette puissance* (1824).
- [152] H. Haug and A.-P. Jauho, *Quantum kinetics in transport and optics of semiconductors*, 2nd ed., Springer Series in Solid-State Sciences (Springer, Berlin, Germany, 2007).
- [153] Y. Imry and R. Landauer, [Rev. Mod. Phys.](#) **71**, S306 (1999).
- [154] M. Büttiker, [Phys. Rev. Lett.](#) **57**, 1761 (1986).
- [155] M. Büttiker, [Phys. Rev. B](#) **38**, 9375 (1988).
- [156] M. F. Ludovico, M. Moskalets, D. Sánchez, and L. Arrachea, [Physical Review B](#) **94** (2016).
- [157] A. Riera-Campenya, M. Mehboudi, M. Pons, and A. Sanpera, [Phys. Rev. E](#) **99**, 032126 (2019).
- [158] K. Brandner, [Zeitschrift für Naturforschung A](#) **75**, 483 (2020).
- [159] A. Nazir and G. Schaller, The reaction coordinate mapping in quantum thermodynamics, in *Thermodynamics in the Quantum Regime* (Springer International Publishing, 2018) pp. 551–577.
- [160] I. de Vega, U. Schollwöck, and F. A. Wolf, [Phys. Rev. B](#) **92**, 155126 (2015).
- [161] A. W. Chin, A. Rivas, S. F. Huelga, and M. B. Plenio, [Journal of Mathematical Physics](#) **51**, 092109 (2010).

- [162] P. Strasberg, G. Schaller, T. L. Schmidt, and M. Esposito, [Phys. Rev. B](#) **97**, 205405 (2018).
- [163] J. Prior, A. W. Chin, S. F. Huelga, and M. B. Plenio, [Phys. Rev. Lett.](#) **105**, 050404 (2010).
- [164] M. P. Woods, M. Cramer, and M. B. Plenio, [Phys. Rev. Lett.](#) **115**, 130401 (2015).
- [165] M. P. Woods and M. B. Plenio, [Journal of Mathematical Physics](#) **57**, 022105 (2016).
- [166] F. Mascherpa, A. Smirne, S. F. Huelga, and M. B. Plenio, [Phys. Rev. Lett.](#) **118**, 100401 (2017).
- [167] A. Garg, J. N. Onuchic, and V. Ambegaokar, [The Journal of Chemical Physics](#) **83**, 4491 (1985).
- [168] M. T. Mitchison and M. B. Plenio, [New Journal of Physics](#) **20**, 033005 (2018).
- [169] R. Alicki, [Journal of Physics A: Mathematical and General](#) **12**, L103 (1979).
- [170] A. C. Barato and U. Seifert, [Physical Review Letters](#) **114** (2015).
- [171] P. Pietzonka, A. C. Barato, and U. Seifert, [Physical Review E](#) **93** (2016).
- [172] T. Van Vu and K. Saito, [Physical Review X](#) **13** (2023).
- [173] D. Reiche, J.-T. Hsiang, and B.-L. Hu, [Entropy](#) **24**, 1016 (2022).
- [174] R. Hanson, L. P. Kouwenhoven, J. R. Petta, S. Tarucha, and L. M. K. Vandersypen, [Reviews of Modern Physics](#) **79**, 1217 (2007).
- [175] Y. S. Liu, X. F. Yang, X. K. Hong, M. S. Si, F. Chi, and Y. Guo, [Applied Physics Letters](#) **103** (2013).
- [176] F. L. Curzon and B. Ahlborn, [American Journal of Physics](#) **43**, 22 (1975).
- [177] M. Josefsson, A. Svilans, A. M. Burke, E. A. Hoffmann, S. Fahlvik, C. Thelander, M. Leijnse, and H. Linke, [Nature Nanotechnology](#) **13**, 920 (2018).
- [178] J. O. González, L. A. Correa, G. Nocerino, J. P. Palao, D. Alonso, and G. Adesso, [Open Systems & Information Dynamics](#) **24**, 1740010 (2017).
- [179] P. Strasberg, [Physical Review E](#) **100** (2019).
- [180] A. Soret, V. Cavina, and M. Esposito, [Physical Review A](#) **106** (2022).
- [181] A. Hewgill, G. De Chiara, and A. Imparato, [Physical Review Research](#) **3** (2021).
- [182] P. P. Potts, A. A. S. Kalaei, and A. Wacker, [New Journal of Physics](#) **23**, 123013 (2021).
- [183] P. Schindler, M. Müller, D. Nigg, J. T. Barreiro, E. A. Martinez, M. Hennrich, T. Monz, S. Diehl, P. Zoller, and R. Blatt, [Nature Physics](#) **9**, 361 (2013).

- [184] S. R. de Groot and P. Mazur, *Non-Equilibrium Thermodynamics* (Dover Publications, New York, 1984).
- [185] M. Esposito and C. Van den Broeck, *Phys. Rev. Lett.* **104**, 090601 (2010).
- [186] G. Manzano, J. M. Horowitz, and J. M. R. Parrondo, *Physical Review X* **8** (2018).
- [187] Y. Oono and M. Paniconi, *Progress of Theoretical Physics Supplement* **130**, 29 (1998).
- [188] H. Ge, *Phys. Rev. E* **80**, 021137 (2009).
- [189] H. Ge and H. Qian, *Physical Review E* **81**, 051133 (2010).
- [190] G. Manzano and R. Zambrini, *AVS Quantum Science* **4** (2022).
- [191] D. J. Evans and D. J. Searles, *Physical Review E* **50**, 1645 (1994).
- [192] D. J. Evans and D. J. Searles, *Advances in Physics* **51**, 1529 (2002).
- [193] G. Gallavotti and E. G. D. Cohen, *Physical Review Letters* **74**, 2694 (1995).
- [194] C. Jarzynski, *Physical Review Letters* **78**, 2690 (1997).
- [195] G. E. Crooks, *Physical Review E* **60**, 2721 (1999).
- [196] M. Campisi, P. Hänggi, and P. Talkner, *Reviews of Modern Physics* **83**, 771 (2011).
- [197] T. B. Batalhão, A. M. Souza, L. Mazzola, R. Auccaise, R. S. Sarthour, I. S. Oliveira, J. Goold, G. De Chiara, M. Paternostro, and R. M. Serra, *Physical Review Letters* **113** (2014).
- [198] A. Solfanelli, A. Santini, and M. Campisi, *PRX Quantum* **2** (2021).
- [199] H. Li, J. Xie, H. Kwon, Y. Zhao, M. S. Kim, and L. Zhang, *Science Advances* **11** (2025).
- [200] J. M. Horowitz and J. M. R. Parrondo, *New Journal of Physics* **15**, 085028 (2013).
- [201] U. Seifert, *Physical Review Letters* **95** (2005).
- [202] S. Ciliberto, *Physical Review X* **7** (2017).
- [203] M. Campisi, *Journal of Physics A: Mathematical and Theoretical* **47**, 245001 (2014).
- [204] J. L. W. V. Jensen, *Acta Math.* **30**, 175 (1906).
- [205] G. Manzano, R. Fazio, and e. Roldán, *Physical Review Letters* **122** (2019).
- [206] J. M. R. Parrondo, J. M. Horowitz, and T. Sagawa, *Nature Physics* **11**, 131 (2015).

- [207] K. Maruyama, F. Nori, and V. Vedral, [Reviews of Modern Physics](#) **81**, 1 (2009).
- [208] G. Manzano, D. Subero, O. Maillet, R. Fazio, J. P. Pekola, and E. Roldán, [Physical Review Letters](#) **126** (2021).
- [209] D. Williams, [Probability with Martingales](#) (Cambridge University Press, 1991).
- [210] S. R. Pliska, *Introduction to mathematical finance* (Blackwell, London, England, 1997).
- [211] A. Serafini, *Quantum continuous variables*, 2nd ed. (CRC Press, London, England, 2023).
- [212] J. B. Brask, [Gaussian states and operations – a quick reference](#) (2022).
- [213] A. Ferraro, S. Olivares, and M. Paris, *Gaussian states in quantum information* (Bibliopolis, 2005).
- [214] X. Wang, T. Hiroshima, A. Tomita, and M. Hayashi, [Physics Reports](#) **448**, 1 (2007).
- [215] R. J. Glauber, [Physical Review](#) **131**, 2766 (1963).
- [216] J. R. Klauder and B.-S. Skagerstam, *Coherent states: Applications in physics and mathematical physics* (World Scientific Publishing, Singapore, Singapore, 1985).
- [217] G. T. Landi, M. J. Kewming, M. T. Mitchison, and P. P. Potts, [PRX Quantum](#) **5**, 020201 (2024).
- [218] J. Surace and L. Tagliacozzo, [SciPost Physics Lecture Notes](#) (2022).
- [219] H. P. Robertson, [Physical Review](#) **34**, 163 (1929).
- [220] G. C. Wick, [Physical Review](#) **80**, 268 (1950).
- [221] T. Prosen and T. H. Seligman, [Journal of Physics A: Mathematical and Theoretical](#) **43**, 392004 (2010).
- [222] T. Barthel and Y. Zhang, [Journal of Statistical Mechanics: Theory and Experiment](#) **2022**, 113101 (2022).
- [223] H. van Houten and C. Beenakker, [Physics Today](#) **49**, 22 (1996).
- [224] W. G. van der Wiel, S. De Franceschi, J. M. Elzerman, T. Fujisawa, S. Tarucha, and L. P. Kouwenhoven, [Reviews of Modern Physics](#) **75**, 1 (2002).
- [225] G. B. Cuetara and M. Esposito, [New Journal of Physics](#) **17**, 095005 (2015).
- [226] T. Fujisawa, T. Hayashi, R. Tomita, and Y. Hirayama, [Science](#) **312**, 1634 (2006).
- [227] B. Küng, C. Rössler, M. Beck, M. Marthaler, D. S. Golubev, Y. Utsumi, T. Ihn, and K. Ensslin, [Physical Review X](#) **2** (2012).
- [228] Y. Utsumi, D. S. Golubev, M. Marthaler, K. Saito, T. Fujisawa, and G. Schön, [Physical Review B](#) **81** (2010).

- [229] H.-S. Goan, G. J. Milburn, H. M. Wiseman, and H. Bi Sun, [Physical Review B](#) **63** (2001).
- [230] H.-P. Breuer, [Physical Review A](#) **68** (2003).
- [231] D. J. Griffiths, *Introduction to electrodynamics*, 5th ed. (Cambridge University Press, Cambridge, England, 2023).
- [232] W. M. Itano, D. J. Heinzen, J. J. Bollinger, and D. J. Wineland, [Physical Review A](#) **41**, 2295 (1990).
- [233] L. D. Contreras-Pulido, M. Bruderer, S. F. Huelga, and M. B. Plenio, [New Journal of Physics](#) **16**, 113061 (2014).
- [234] M. B. Plenio and S. F. Huelga, [New Journal of Physics](#) **10**, 113019 (2008).
- [235] P. Rebentrost, M. Mohseni, I. Kassal, S. Lloyd, and A. a. Aspuru-Guzik, [New Journal of Physics](#) **11**, 033003 (2009).
- [236] A. M. Lacerda, J. Goold, and G. T. Landi, [Physical Review B](#) **104** (2021).
- [237] J. J. Mendoza-Arenas, T. Grujic, D. Jaksch, and S. R. Clark, [Physical Review B](#) **87** (2013).
- [238] G. S. Engel, T. R. Calhoun, E. L. Read, T.-K. Ahn, T. Mančal, Y.-C. Cheng, R. E. Blankenship, and G. R. Fleming, [Nature](#) **446**, 782 (2007).
- [239] E. Collini, C. Y. Wong, K. E. Wilk, P. M. G. Curmi, P. Brumer, and G. D. Scholes, [Nature](#) **463**, 644 (2010).
- [240] P. Pietzonka and U. Seifert, [Physical Review Letters](#) **120** (2018).
- [241] P. Pietzonka, F. Ritort, and U. Seifert, [Physical Review E](#) **96** (2017).
- [242] J. M. Horowitz and T. R. Gingrich, [Physical Review E](#) **96** (2017).
- [243] M. J. Kewming, M. T. Mitchison, and G. T. Landi, [Physical Review A](#) **106** (2022).
- [244] G. Schaller, *Open Quantum Systems Far from Equilibrium* (Springer International Publishing, 2014).
- [245] A. Purkayastha, G. Guarnieri, S. Campbell, J. Prior, and J. Goold, [Physical Review B](#) **104** (2021).
- [246] A. Purkayastha, G. Guarnieri, S. Campbell, J. Prior, and J. Goold, [Quantum](#) **6**, 801 (2022).
- [247] G. Wójtowicz, A. Purkayastha, M. Zwolak, and M. M. Rams, [Physical Review B](#) **107** (2023).
- [248] M. Brenes, G. Guarnieri, A. Purkayastha, J. Eisert, D. Segal, and G. Landi, [Physical Review B](#) **108** (2023).
- [249] A. M. Lacerda, M. J. Kewming, M. Brenes, C. Jackson, S. R. Clark, M. T. Mitchison, and J. Goold, [Phys. Rev. E](#) **110**, 014125 (2024).

- [250] A. Ronzani, B. Karimi, J. Senior, Y.-C. Chang, J. T. Peltonen, C. Chen, and J. P. Pekola, [Nature Physics](#) **14**, 991 (2018).
- [251] A. Purkayastha, A. Dhar, and M. Kulkarni, [Physical Review A](#) **93** (2016).
- [252] A. Schnell, A. Eckardt, and S. Denisov, [Physical Review B](#) **101** (2020).
- [253] A. Schnell, S. Denisov, and A. Eckardt, [Physical Review B](#) **104** (2021).
- [254] G. Stefanucci and R. van Leeuwen, *Nonequilibrium Many-Body Theory of Quantum Systems: A Modern Introduction* (Cambridge University Press, 2013).
- [255] A. Kamenev, *Field Theory of Non-Equilibrium Systems* (Cambridge University Press, 2011).
- [256] D. Gruss, K. A. Velizhanin, and M. Zwolak, [Scientific Reports](#) **6** (2016).
- [257] J. E. Elenewski, D. Gruss, and M. Zwolak, [The Journal of Chemical Physics](#) **147** (2017).
- [258] D. Gruss, A. Smolyanitsky, and M. Zwolak, [The Journal of Chemical Physics](#) **147** (2017).
- [259] G. Wójtowicz, J. E. Elenewski, M. M. Rams, and M. Zwolak, [Physical Review A](#) **101** (2020).
- [260] J. E. Elenewski, G. Wójtowicz, M. M. Rams, and M. Zwolak, [The Journal of Chemical Physics](#) **155** (2021).
- [261] G. Wójtowicz, J. E. Elenewski, M. M. Rams, and M. Zwolak, [Physical Review B](#) **104** (2021).
- [262] W. Alford, L. P. Bettmann, and G. T. Landi, [Subtleties in the pseudomodes formalism](#) (2025).
- [263] A. M. Lyapunov and J. A. Walker, [Journal of Applied Mechanics](#) **61**, 226 (1994).
- [264] K. Funo and H. T. Quan, [Phys. Rev. Lett.](#) **121**, 040602 (2018).
- [265] M. Kilgour, B. K. Agarwalla, and D. Segal, [The Journal of Chemical Physics](#) **150**, 084111 (2019).
- [266] G. E. Fux, D. Kilda, B. W. Lovett, and J. Keeling, [Phys. Rev. Res.](#) **5**, 033078 (2023).
- [267] S. Bravyi, [Quantum Information & Computation](#) **5**, 216 (2005).
- [268] M. Coppola, D. Karevski, and G. T. Landi, [Physical Review B](#) **110** (2024).
- [269] J. M. Horowitz, [Physical Review E](#) **85** (2012).
- [270] G. Manzano, J. M. Horowitz, and J. M. R. Parrondo, [Physical Review E](#) **92** (2015).
- [271] C. Elouard, D. A. Herrera-Martí, M. Clusel, and A. Auffèves, [npj Quantum Information](#) **3** (2017).

- [272] D. A. Abanin and L. S. Levitov, [Physical Review Letters](#) **94** (2005).
- [273] I. Klich, [Full counting statistics: An elementary derivation of levitov's formula](#) (2002).
- [274] H. J. D. Miller, G. Guarnieri, M. T. Mitchison, and J. Goold, [Physical Review Letters](#) **125** (2020).
- [275] Y.-Z. Zhen, D. Egloff, K. Modi, and O. Dahlsten, [Phys. Rev. Lett.](#) **127**, 190602 (2021).
- [276] T. Van Vu and K. Saito, [Phys. Rev. Lett.](#) **128**, 010602 (2022).
- [277] S. Dago and L. Bellon, [Phys. Rev. Lett.](#) **128**, 070604 (2022).
- [278] A. Rolandi and M. Perarnau-Llobet, [Quantum](#) **7**, 1161 (2023).
- [279] L. Banchi, P. Giorda, and P. Zanardi, [Physical Review E](#) **89** (2014).
- [280] I. Bengtsson and K. Życzkowski, [Geometry of Quantum States: An Introduction to Quantum Entanglement](#) (Cambridge University Press, Cambridge, 2006).
- [281] R. A. Fisher, [Philosophical Transactions of the Royal Society of London, Series A: Containing Papers of a Mathematical or Physical Character](#) **222**, 309 (1922).
- [282] C. R. Rao, Information and the accuracy attainable in the estimation of statistical parameters, in [Breakthroughs in Statistics](#) (Springer New York, 1992) pp. 235–247.
- [283] H. Cramer, [Mathematical methods of statistics \(PMS-9\), volume 9](#), Princeton Landmarks in Mathematics and Physics (Princeton University Press, Princeton, NJ, 1999).
- [284] J. Heseltine and E.-j. Kim, [Journal of Physics A: Mathematical and Theoretical](#) **49**, 175002 (2016).
- [285] S. Nicholson and E.-j. Kim, [Physics Letters A](#) **379**, 83 (2015).
- [286] S. B. Nicholson, L. P. García-Pintos, A. del Campo, and J. R. Green, [Nature Physics](#) **16**, 1211 (2020).
- [287] N. N. Cencov, [Statistical Decision Rules and Optimal Inference](#), Translations of Mathematical Monographs, Vol. 53 (American Mathematical Society, Providence, R.I., 1982).
- [288] S.-i. Amari and H. Nagaoka, [Methods of Information Geometry](#), edited by D. Harada (American Mathematical Society, 2007).
- [289] J. Burbea and C. Rao, [Journal of Multivariate Analysis](#) **12**, 575 (1982).
- [290] S. M. Kay, [Fundamentals of statistical processing, volume I](#) (Prentice Hall, Philadelphia, PA, 1993).
- [291] D. Petz, [Linear Algebra and its Applications](#) **244**, 81 (1996).

- [292] E. A. Morozova and N. N. Chentsov, [Journal of Soviet Mathematics](#) **56**, 2648 (1991).
- [293] A. Uhlmann, [Reports on Mathematical Physics](#) **33**, 253 (1993).
- [294] P. Gibilisco and T. Isola, [Journal of Mathematical Physics](#) **44**, 3752 (2003).
- [295] F. Hiai and D. Petz, *Introduction to Matrix Analysis and Applications*, Universitext (Springer, Cham, Switzerland, 2014).
- [296] J. E. Avron, R. Seiler, and L. G. Yaffe, [Communications in Mathematical Physics](#) **110**, 33 (1987).
- [297] D. Girolami, [Physical Review Letters](#) **122** (2019).
- [298] S. Alipour, A. Chenu, A. T. Rezakhani, and A. del Campo, [Quantum](#) **4**, 336 (2020).
- [299] M. R. Frey, [Quantum Information Processing](#) **15**, 3919 (2016).
- [300] Z. Gong and R. Hamazaki, [International Journal of Modern Physics B](#) **36** (2022).
- [301] D. Petz, [Journal of Physics A: Mathematical and General](#) **35**, 929 (2002).
- [302] C. E. Shannon, [Bell System Technical Journal](#) **27**, 379 (1948).
- [303] J. Schnakenberg, [Reviews of Modern Physics](#) **48**, 571 (1976).
- [304] T. R. Gingrich, J. M. Horowitz, N. Perunov, and J. L. England, [Physical Review Letters](#) **116** (2016).
- [305] D. Brody and N. Rivier, [Physical Review E](#) **51**, 1006 (1995).
- [306] R. Kubo, [Reports on Progress in Physics](#) **29**, 255 (1966).
- [307] T. Shitara and M. Ueda, [Physical Review A](#) **94**, 10.1103/physreva.94.062316 (2016).
- [308] F. Weinhold, [The Journal of Chemical Physics](#) **63**, 2479 (1975).
- [309] F. Schlögl, [Zeitschrift für Physik B Condensed Matter](#) **59**, 449 (1985).
- [310] S. Blaber and D. A. Sivak, [Journal of Physics Communications](#) **7**, 033001 (2023).
- [311] P. Abiuso and M. Perarnau-Llobet, [Physical Review Letters](#) **124** (2020).
- [312] K. Brandner and K. Saito, [Physical Review Letters](#) **124** (2020).
- [313] H. J. D. Miller and M. Mehboudi, [Physical Review Letters](#) **125** (2020).
- [314] V. Cavina, A. Mari, and V. Giovannetti, [Physical Review Letters](#) **119** (2017).
- [315] J. C. Gower, T. L. Boullion, and P. L. Odell, [Technometrics](#) **14**, 806 (1972).
- [316] D. A. Garanin, [Physical Review E](#) **54**, 3250 (1996).

- [317] K. Funo, N. Shiraishi, and K. Saito, *New J. Phys.* **21**, 013006 (2019).
- [318] M. Moroder, O. Culhane, K. Zawadzki, and J. Goold, *Physical Review Letters* **133** (2024).
- [319] E. B. Mpemba and D. G. Osborne, *Physics Education* **4**, 172 (1969).
- [320] G. S. Kell, *American Journal of Physics* **37**, 564 (1969).
- [321] C. Hu, J. Li, S. Huang, H. Li, C. Luo, J. Chen, S. Jiang, and L. An, *Crystal Growth & Design* **18**, 5757 (2018).
- [322] Y.-H. Ahn, H. Kang, D.-Y. Koh, and H. Lee, *Korean Journal of Chemical Engineering* **33**, 1903 (2016).
- [323] P. Chaddah, S. Dash, K. Kumar, and A. Banerjee, *Overtaking while approaching equilibrium* (2010).
- [324] P. A. Greaney, G. Lani, G. Cicero, and J. C. Grossman, *Metallurgical and Materials Transactions A* **42**, 3907 (2011).
- [325] A. Lasanta, F. Vega Reyes, A. Prados, and A. Santos, *Physical Review Letters* **119** (2017).
- [326] T. Keller, V. Torggler, S. B. Jäger, S. Schütz, H. Ritsch, and G. Morigi, *New Journal of Physics* **20**, 025004 (2018).
- [327] X. Turkeshi, P. Calabrese, and A. De Luca, *Physical Review Letters* **135** (2025).
- [328] F. Ares, S. Murciano, and P. Calabrese, *Nature Communications* **14** (2023).
- [329] S. Murciano, F. Ares, I. Klich, and P. Calabrese, *Journal of Statistical Mechanics: Theory and Experiment* **2024**, 013103 (2024).
- [330] S. Yamashika, F. Ares, and P. Calabrese, *Physical Review B* **110** (2024).
- [331] S. Liu, H.-K. Zhang, S. Yin, and S.-X. Zhang, *Physical Review Letters* **133** (2024).
- [332] L. K. Joshi, J. Franke, A. Rath, F. Ares, S. Murciano, F. Kranzl, R. Blatt, P. Zoller, B. a. Vermersch, P. Calabrese, C. F. Roos, and M. K. Joshi, *Physical Review Letters* **133** (2024).
- [333] A. Summer, M. Moroder, L. P. Bettmann, X. Turkeshi, I. Marvian, and J. Goold, *A resource theoretical unification of mpemba effects: classical and quantum* (2025).
- [334] I. Medina, O. Culhane, F. C. Binder, G. T. Landi, and J. Goold, *Phys. Rev. Lett.* **134**, 220402 (2025).
- [335] E. Davies, *Journal of Functional Analysis* **34**, 421 (1979).
- [336] W. Roga, M. Fannes, and K. Zyczkowski, *Reports on Mathematical Physics* **66**, 311 (2010).

- [337] E. P. Wigner and M. M. Yanase, [Proceedings of the National Academy of Sciences](#) **49**, 910 (1963).
- [338] F. Minganti, A. Biella, N. Bartolo, and C. Ciuti, [Physical Review A](#) **98**, 042118 (2018).
- [339] R. Botet, R. Jullien, and P. Pfeuty, [Phys. Rev. Lett.](#) **49**, 478 (1982).
- [340] S. Sachdev, *Quantum Phase Transitions*, 2nd ed. (Cambridge University Press, Cambridge, England, 2011).
- [341] N. Goldenfeld, *Lectures on phase transitions and the renormalization group* (CRC Press, London, England, 2019).
- [342] J. Cardy, *Scaling and Renormalization in Statistical Physics* (Cambridge University Press, 1996).
- [343] M. P. A. Fisher, P. B. Weichman, G. Grinstein, and D. S. Fisher, [Physical Review B](#) **40**, 546 (1989).
- [344] O. Castaños, R. López-Peña, J. G. Hirsch, and E. López-Moreno, [Physical Review B](#) **74** (2006).
- [345] F. Brennecke, R. Mottl, K. Baumann, R. Landig, T. Donner, and T. Esslinger, [Proceedings of the National Academy of Sciences](#) **110**, 11763 (2013).
- [346] H. Pichler, A. J. Daley, and P. Zoller, [Physical Review A](#) **82** (2010).
- [347] T. W. B. Kibble, [J. Phys. A: Math. Gen.](#) **9**, 1387 (1976).
- [348] W. H. Zurek, [Nature](#) **317**, 505 (1985).
- [349] J. Dziarmaga, [Adv. Phys.](#) **59**, 1063 (2010).
- [350] J. Klinder, H. Kessler, M. Wolke, L. Mathey, and A. Hemmerich, [Proceedings of the National Academy of Sciences](#) **112**, 3290 (2015).
- [351] A. Zamora, G. Dagvadorj, P. Comaron, I. Carusotto, N. P. Proukakis, and M. H. Szymańska, [Physical Review Letters](#) **125**, 095301 (2020).
- [352] P. Hedvall and J. Larson, [Dynamics of non-equilibrium steady state quantum phase transitions](#) (2017).
- [353] A. Polkovnikov, K. Sengupta, A. Silva, and M. Vengalattore, [Rev. Mod. Phys.](#) **83**, 863 (2011).
- [354] Y. Shen, Z. Jiang, Y. Huang, B. M. Cleary, Y. Jiang, G. M. Rotskoff, and A. M. Lindenberg, Quantifying and minimizing dissipation in a non-equilibrium phase transition (2025).
- [355] B. Debecker, J. Martin, and F. Damanet, [Physical Review A](#) **110** (2024).
- [356] T. Kato, *Perturbation Theory for Linear Operators* (Springer Berlin Heidelberg, 1995).
- [357] B. Horstmann, J. I. Cirac, and G. Giedke, [Physical Review A](#) **87** (2013).

- [358] X. H. H. Zhang and H. U. Baranger, [Physical Review A](#) **103**, 033711 (2021).
- [359] R. H. Dicke, [Physical Review](#) **93**, 99 (1954).
- [360] K. Hepp and E. H. Lieb, [Physical Review A](#) **8**, 2517 (1973).
- [361] Y. K. Wang and F. T. Hioe, [Physical Review A](#) **7**, 831 (1973).
- [362] F. T. Hioe, [Physical Review A](#) **8**, 1440 (1973).
- [363] F. Dimer, B. Estienne, A. S. Parkins, and H. J. Carmichael, [Physical Review A](#) **75**, 013804 (2007).
- [364] J. Lang and F. Piazza, [Physical Review A](#) **94**, 033628 (2016).
- [365] D. A. Paz and M. F. Maghrebi, [Europhysics Letters](#) **136**, 10002 (2022).
- [366] D. Nagy, G. Szirmai, and P. Domokos, [Physical Review A](#) **84**, 043637 (2011).
- [367] J. A. Mlynek, A. A. Abdumalikov, C. Eichler, and A. Wallraff, [Nature Communications](#) **5** (2014).
- [368] A. Safavi-Naini, R. Lewis-Swan, J. Bohnet, M. Gärtner, K. Gilmore, J. Jordan, J. Cohn, J. Freericks, A. Rey, and J. Bollinger, [Physical Review Letters](#) **121** (2018).
- [369] T. Holstein and H. Primakoff, [Physical Review](#) **58**, 1098 (1940).
- [370] P. D. Drummond and D. F. Walls, [Journal of Physics A: Mathematical and General](#) **13**, 725 (1980).
- [371] S. Puri, S. Boutin, and A. Blais, [npj Quantum Information](#) **3** (2017).
- [372] C. H. Meaney, H. Nha, T. Duty, and G. J. Milburn, [EPJ Quantum Technology](#) **1** (2014).
- [373] S. Lieu, R. Belyansky, J. T. Young, R. Lundgren, V. V. Albert, and A. V. Gorshkov, [Physical Review Letters](#) **125**, 240405 (2020).
- [374] P. Brookes, G. Tancredi, A. D. Patterson, J. Rahamim, M. Esposito, T. K. Mavrogordatos, P. J. Leek, E. Ginossar, and M. H. Szymanska, [Science Advances](#) **7** (2021).
- [375] Z. Leghtas, S. Touzard, I. M. Pop, A. Kou, B. Vlastakis, A. Petrenko, K. M. Sliwa, A. Narla, S. Shankar, M. J. Hatridge, M. Reagor, L. Frunzio, R. J. Schoelkopf, M. Mirrahimi, and M. H. Devoret, [Science](#) **347**, 853 (2015).
- [376] M. E. Fisher, [Reviews of Modern Physics](#) **46**, 597 (1974).
- [377] H. E. Stanley, *Introduction to phase transitions and critical phenomena*, International Series of Monographs on Physics (Oxford University Press, New York, NY, 1987).
- [378] M.-J. Hwang, P. Rabl, and M. B. Plenio, [Phys. Rev. A](#) **97**, 013825 (2018).
- [379] W. Verstraelen and M. Wouters, [Physical Review A](#) **101**, 043826 (2020).

- [380] E. G. D. Torre, S. Diehl, M. D. Lukin, S. Sachdev, and P. Strack, [Physical Review A](#) **87**, 023831 (2013).
- [381] C. De Grandi, V. Gritsev, and A. Polkovnikov, [Physical Review B](#) **81**, 012303 (2010).
- [382] R. Barankov and A. Polkovnikov, [Phys. Rev. Lett.](#) **101**, 076801 (2008).
- [383] R. Puebla, O. Marty, and M. B. Plenio, [Physical Review A](#) **100**, 032115 (2019).
- [384] J. M. Horowitz and T. Sagawa, [Journal of Statistical Physics](#) **156**, 55 (2014).
- [385] R. Kubo, M. Toda, and N. Hashitsume, *Statistical Physics II: Nonequilibrium Statistical Mechanics* (Springer Science & Business Media, 2012).
- [386] H. E. Stanley, [Rev. Mod. Phys.](#) **71**, S358 (1999).
- [387] J. a. S. Ferreira and P. Ribeiro, [Phys. Rev. B](#) **100**, 184422 (2019).
- [388] J.-Y. Liu, L.-L. Nian, N. Zhou, L. Xiong, J.-T. Lü, and B. Zheng, [Phys. Rev. A](#) **111**, L040201 (2025).
- [389] O. L. Acevedo, L. Quiroga, F. J. Rodríguez, and N. F. Johnson, [Phys. Rev. Lett.](#) **112**, 030403 (2014).
- [390] M.-J. Hwang, R. Puebla, and M. B. Plenio, [Physical Review Letters](#) **115**, 180404 (2015).
- [391] L. C. Venuti and P. Zanardi, [Physical Review Letters](#) **99**, 095701 (2007).
- [392] P. Zanardi and N. Paunković, [Physical Review E](#) **74**, 031123 (2006).
- [393] P. Zanardi, P. Giorda, and M. Cozzini, [Physical Review Letters](#) **99**, 100603 (2007).
- [394] T. L. Wang, L. N. Wu, W. Yang, G. R. Jin, N. Lambert, and F. Nori, [New Journal of Physics](#) **16**, 063039 (2014).
- [395] U. Marzolino and T. Prosen, [Physical Review B](#) **96** (2017).
- [396] W.-L. You, Y.-W. Li, and S.-J. Gu, [Physical Review E](#) **76**, 022101 (2007).
- [397] G. Ruppreiter, [Rev. Mod. Phys.](#) **67**, 605 (1995).
- [398] E. H. Trepagnier, C. Jarzynski, F. Ritort, G. E. Crooks, C. J. Bustamante, and J. Liphardt, [Proceedings of the National Academy of Sciences](#) **101**, 15038 (2004).
- [399] A. Mounier and A. Naert, [EPL \(Europhysics Letters\)](#) **100**, 30002 (2012).
- [400] L. Granger, J. Mehlis, E. Roldán, S. Ciliberto, and H. Kantz, [New Journal of Physics](#) **17**, 065005 (2015).
- [401] S. Joubaud, N. B. Garnier, and S. Ciliberto, [EPL \(Europhysics Letters\)](#) **82**, 30007 (2008).

- [402] G. M. Wang, E. M. Sevick, E. Mittag, D. J. Searles, and D. J. Evans, [Physical Review Letters](#) **89** (2002).
- [403] G. H. Aguilar, T. L. Silva, T. E. Guimarães, R. S. Piera, L. C. Céleri, and G. T. Landi, [Physical Review A](#) **106** (2022).
- [404] J. V. Koski, T. Sagawa, O.-P. Saira, Y. Yoon, A. Kutvonen, P. Solinas, M. Möttönen, T. Ala-Nissila, and J. P. Pekola, [Nature Physics](#) **9**, 644 (2013).
- [405] F. W. J. Hekking and J. P. Pekola, *Phys. Rev. Lett.* **111**, 093602 (2013).
- [406] J. P. Pekola, [Nature Physics](#) **11**, 118 (2015).
- [407] M. Rossi, L. Mancino, G. T. Landi, M. Paternostro, A. Schliesser, and A. Belenchia, [Physical Review Letters](#) **125** (2020).
- [408] B. O. Goes, C. E. Fiore, and G. T. Landi, [Physical Review Research](#) **2**, 013136 (2020).
- [409] J. P. Garrahan and I. Lesanovsky, [Physical Review Letters](#) **104** (2010).
- [410] J. P. Garrahan, A. D. Armour, and I. Lesanovsky, [Physical Review E](#) **84** (2011).
- [411] F. Carollo, R. L. Jack, and J. P. Garrahan, [Physical Review Letters](#) **122** (2019).
- [412] C. Ates, B. Olmos, J. P. Garrahan, and I. Lesanovsky, [Physical Review A](#) **85** (2012).
- [413] F. Ares, P. Calabrese, and S. Murciano, [Nature Reviews Physics](#) **7**, 451 (2025).
- [414] P. B. Melo, F. Iemini, D. O. Soares-Pinto, S. M. D. Queirós, and W. A. M. Morgado, [Physical Review E](#) **112** (2025).
- [415] P. B. Melo, P. V. Paraguassú, S. M. D. Queirós, F. Iemini, M. Paternostro, and W. A. M. Morgado, [Stochastic quantum information geometry and speed limits at the trajectory level](#) (2026).
- [416] O. M. Basri and O. Raz, [Thermodynamic geometry through second order phase transitions](#) (2025).
- [417] R. Puebla, S. Deffner, and S. Campbell, [Physical Review Research](#) **2** (2020).
- [418] P. Orlov, G. V. Shlyapnikov, and D. V. Kurllov, [Physical Review B](#) **111**, L081105 (2025).
- [419] P. W. Claeys, M. Pandey, D. Sels, and A. Polkovnikov, [Physical Review Letters](#) **123** (2019).
- [420] A. Carollo, B. Spagnolo, and D. Valenti, [Scientific Reports](#) **8** (2018).
- [421] F. Bascone, L. Leonforte, D. Valenti, B. Spagnolo, and A. Carollo, [Physical Review B](#) **99** (2019).
- [422] B. Buča and T. Prosen, [New Journal of Physics](#) **14**, 073007 (2012).

-
- [423] T. Prosen and I. Pižorn, [Physical Review Letters](#) **101** (2008).
- [424] P. Jordan and E. Wigner, [Zeitschrift für Physik](#) **47**, 631 (1928).
- [425] M. Mehboudi and H. J. D. Miller, [Physical Review A](#) **105** (2022).
- [426] H. Abou-Kandil, G. Freiling, V. Ionescu, and G. Jank, *Matrix Riccati equations in control and systems theory*, 2003rd ed., Systems & Control: Foundations & Applications (Birkhauser Verlag AG, Basel, Switzerland, 2003).
- [427] H. Lipkin, N. Meshkov, and A. Glick, [Nuclear Physics](#) **62**, 188 (1965).
- [428] S. Morrison and A. S. Parkins, [Journal of Physics B: Atomic, Molecular and Optical Physics](#) **41**, 195502 (2008).
- [429] S. Kothe and P. Kirton, [PT-symmetry breaking phase transitions in an LMG dimer](#) (2025).
- [430] F. Vicentini, F. Minganti, R. Rota, G. Orso, and C. Ciuti, [Physical Review A](#) **97**, 013853 (2018).