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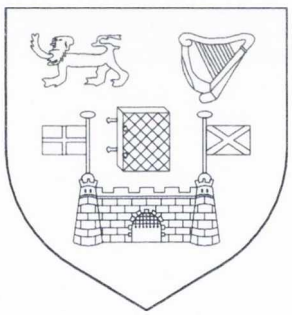
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RANGES OF BIMODULE PROJECTIONS AND CONDITIONAL EXPECTATIONS

BY

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Abstract

The algebraic theory of corner subrings introduced by Lam (as an abstraction of the properties of Peirce corners eRe of a ring R associated with an idempotent $e \in R$) are investigated here in the context of Banach and C^*-algebras. We propose a general algebraic approach which includes the notion of ranges of (completely) contractive conditional expectations on C^*-algebras and on ternary rings of operators and we investigate when topological properties are consequences of the algebraic assumptions. For commutative C^*-algebras we show that dense corners cannot be proper and that self-adjoint corners must be closed and always have closed complements (and may also have non-closed complements). For C^*-algebras we show that Peirce corners and some more general corners are similar to self-adjoint corners. We show uniqueness of complements for certain classes of corners in general C^*-algebras, and establish that a primitive C^*-algebra must be prime if it has a prime Peirce corner.

Further we consider corners in ternary rings of operators (TROs) and characterise corners of Hilbertian TROs as closed subspaces. Additional results concern a definition we introduce for AW^*-TROs which generalize AW^*-algebras in a similar way as W^*-TROs generalize W^*-algebras. In particular, we show that the decomposition of AW^*-algebras into type I, II, and III can be used to give a similar decomposition for AW^*-TROs.

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Chapter 1

Introduction

In ring theory and in the context of algebras, idempotents have well-established uses. In particular, if $e \in R$ is an idempotent of a ring R, then the subring eRe has unit e and there is an eRe-bimodule projection $x \mapsto exe$ from R onto eRe. The kernel $eR(1-e)+(1-e)Re+(1-e)R(1-e)$ of the projection is a complementary eRe-submodule of R.

In probability theory and in the theory of von Neumann algebras the notion of conditional expectation (as a completely positive map $\mathcal{E}\colon M \to M$ on a von Neumann algebra M, with M commutative in the case of probability theory) satisfies similar algebraic properties as the Peirce projections on a ring R or on an algebra A. A well-known result of J. Tomiyama [71] states that a unital and bounded projection $\mathcal{E}\colon A \to A$ with range $S = \mathcal{E}(A)$ a C^*-subalgebra of A must have norm one, must be positive, must satisfy the conditional expectation property $\mathcal{E}(s_1 x s_2) = s_1 \mathcal{E}(x) s_2$ (for $s_1, s_2 \in S$, $x \in A$) and also the Schwarz type inequality $(\mathcal{E}(x))^* \mathcal{E}(x) \leq \mathcal{E}(x^* x)$. In one of the themes of recent research, the notion of injective operator space, a similar algebraic 'conditional expectation' property plays a significant role, interacting with the notion of a ternary ring of operators (TRO).

In [48] T. Y. Lam proposed abstracting the algebraic properties of the Peirce projection $\mathcal{E}_e\colon R \to R$ associated with an idempotent e in a ring R, which is given by $\mathcal{E}_e(x) = exe$ ($x \in R$), and investigating algebraic properties that hold in this more general context. His proposal is to consider (additive) maps $\mathcal{E}\colon R \to R$

with $\mathcal{E} \circ \mathcal{E} = \mathcal{E}$, $S = \mathcal{E}(R)$ a subring of R under the assumption that $\mathcal{E}$ is an S-bimodule map (which means that it satisfies the conditional expectation property $\mathcal{E}(s_1 x s_2) = s_1 \mathcal{E}(x) s_2$ for $s_1, s_2 \in S$, $x \in R$). Lam refers to such subrings S as 'corners'.

We consider this notion principally in the context of a (complex) Banach or C^*-algebra A in place of a ring R and with the assumption that the corner $S = \mathcal{E}(A)$ is a complex subalgebra. Our aim is to characterise such corners as fully as we can, ideally by establishing that they are related to the ranges of the more well-known completely positive (unital) conditional expectations.

Each chapter below begins with a summary of the results in it, and here is an overview of the structure of the chapters.

Chapter 2 outlines the general approach of Lam (in the context of rings) and includes some general facts that will be useful to us later on in the context of algebras. For instance, although a ring-theoretic Lam corner S of a unital algebra A need not be a subalgebra, if S is a subalgebra then $\mathcal{E}$ must be linear, which justifies the definition of corner algebra we use from Chapter 3 on (Definition 3.1.1). We also note that corner rings of unital rings must be inverse closed and point out that graphs of ring homomorphisms are corners, an idea we use later as a source of counterexamples. Chapter 2 also includes some results on properties inherited by corner rings (regularity, π-regularity and others).

In Chapter 3 we move to algebras, our main theme, and adopt a definition modified from the ring-theoretic one (which insists that we deal with corners that are subalgebras and have vector space complements, or equivalently we deal only with linear Lam conditional expectations $\mathcal{E}$). In fact we consider normed algebras, usually Banach algebras and often C^*-algebras A. We first observe that in general a Lam conditional expectation $\mathcal{E}\colon A \to A$ need not be continuous but the initial examples leave open the possibility that there always exists a continuous Lam conditional expectation $\tilde{\mathcal{E}}$ with the same range $S = \mathcal{E}(A) = \tilde{\mathcal{E}}(A)$. While simple examples show that Lam corners S in C^*-algebras need not be self-adjoint subalgebras, Peirce corners in C^*-algebras and certain 'generalised' Peirce corners are similar to self-adjoint corners. We characterise corners in $M_n(\mathbb{C})$ that contain

the diagonal and use that to characterise self-adjoint corners of $\mathcal{B}(H)$ that contain the diagonal (in some basis for H). The latter result generalises to certain corners in purely atomic von Neumann algebras. A final result in this chapter is that if p is a projection in a primitive C^*-algebra, then A is prime if and only if the Peirce corner pAp is prime.

In Chapter 4 we propose a definition of a 'ternary' corner of a TRO T, which we consider as a generalisation of injective operator spaces. An injective operator space is completely isometric to a ternary corner pAq of an injective C^*-algebra A (with $p, q \in A$ projections). Hilbertian TROs are completely characterised as row or column Hilbert spaces and we establish that the ternary corners are precisely the closed subspaces in those cases.

Chapter 5 focusses largely on commutative C^*-algebras and we seek first to relate Lam conditional expectations to retracts on the spectrum of the algebra. Retractions $\tau\colon X \to X$ on a (locally compact) topological space X give rise to Lam conditional expectations $\mathcal{E}_\tau\colon C_0(X) \to C_0(X)$ via $\mathcal{E}_\tau(f) = f \circ \tau$. Such $\mathcal{E}_\tau$ maps have the additional property of being algebra *-homomorphisms. In fact we establish a converse, that a Lam conditional expectation $\mathcal{E}\colon C_0(X) \to C_0(X)$ which is also an algebra homomorphisms must be given by a retraction of the one-point compactification $X^* = X \cup \{\omega\}$ of X fixing ω (Corollary 5.1.12). For commutative C^*-algebras we show that dense corners cannot be proper and that self-adjoint corners must be closed and always have closed complements (and may also have non-closed complements). Two final results in this chapter are that self-adjoint corners S of a C^*-algebra A are square-root closed, and if $x = x^* \in S$ then the C^*-subalgebra of A generated by x must be contained in S.

In Chapter 6 we first examine the concept of norming algebras and its application to the automatic complete boundedness problem. The basic issue here is to determine some algebraic conditions which ensure that a given bounded homomorphism between operator algebras is automatically completely bounded. In Theorem 6.1.5 we give one such condition. We then introduce AW^*-TROs which admit a type decomposition (type I, II, and III), and which generalize AW^*-algebras in a similar way as W^*-TROs generalize W^*-algebras.

Chapter 2

General Theory of Corner Rings

All rings to be consider are associative, but not necessarily unital.

In the first section of this chapter we will review the necessary background material from *Corner Ring Theory* [48]. In the succeeding sections, we will prove some new results, in particular:

- New proofs of uniqueness of complements (§2.2).
- Graphs of ring-homomorphisms are corner rings (§2.4). In particular, one can always find a Lam conditional expectation onto the graph of a ring homomorphism (between two rings); this expectation is multiplicative.
- If $\mathcal{E}\colon R \to R$ is an $\mathcal{E}(R)$-bimodule projection (or a Lam conditional expectation), then $\mathcal{E}$ is multiplicative if and only if $\ker \mathcal{E}$ is an ideal (Proposition 2.4.2).
- Let A be a unital (associative) algebra over a field F, and S a corner of A regarded as a ring with complement M. If S is also a subalgebra of A then then the group M must be a vector space over F (Proposition 2.5.2).
- Every unital corner is inverse closed (Proposition 2.6.1).
- Hereditary properties of corners in regular and $*$-rings. In particular, every self-adjoint corner of a unital Baer $*$-ring is a Baer $*$-ring (Theorem 2.8.5).

2.1 Definitions and Main Theorems

The general theory of *corner rings* was first systematically established in T. Y. Lam's 2006 paper [48], but the concept – without explicitly mentioning the name 'corner rings' – had appeared earlier in the 1999 paper [46] authored by H. Kraft, L. W. Small, and N. R. Wallach.

One of the main objectives of the theory of *corner rings* is to provide an axiomatic foundation for the classical theory of Peirce decompositions. Recall that the Peirce decomposition of an associative ring R with respect to an idempotent $e^2 = e \in R$ is

$$R = eRe \oplus eR(1-e) \oplus (1-e)Re \oplus (1-e)R(1-e).$$

If R has no identity, $R(1-e) := \{x - xe : x \in R\}$ and $(1-e)R := \{x - ex : x \in R\}$. It is important to note that

$$S := eRe \quad \text{and} \quad M := eR(1-e) \oplus (1-e)Re \oplus (1-e)R(1-e)$$

are respectively a subring of R and a subgroup of the additive group of R, and

$$R = S \oplus M, \quad SM \subseteq M, \quad MS \subseteq M.$$

This leads to the following definition of *corners* which allows one to study some features of Peirce decompositions in arbitrary rings, including those with no proper idempotents.

Definition 2.1.1 (Lam [48]). Let R be a ring. A subring S of R is called a *corner ring* (or simply a *corner*) of R if there exists a subgroup $M \subseteq R$ of the additive group of R such that

$$R = S \oplus M, \quad SM \subseteq M, \quad MS \subseteq M. \tag{2.1}$$

Such M is called a *complement* of S in R.

A useful characterization of corner rings, which was proved in [48], asserts that there is a one-to-one correspondence between corners of R and bimodular projections $\mathcal{E}\colon R \to R$ from R onto subrings of R. Specifically, this characterization is as follows.

Proposition 2.1.2 ([48, Proposition 2.1]). Let R be a ring, $S \subseteq R$ a subring. Then S is a corner ring of R if and only if there exists an S-bimodule map $\mathcal{E}\colon R \to R$ (which means that $\mathcal{E}$ is additive, and both left and right S-homogeneous, i.e., $\mathcal{E}(sx) = s\mathcal{E}(x)$ and $\mathcal{E}(xs) = \mathcal{E}(x)s$ for $s \in S$, $x \in R$) such that $\mathcal{E}(R) = S$ and $\mathcal{E} \circ \mathcal{E} = \mathcal{E}$. (We will refer to $\mathcal{E}$ as a *Lam conditional expectation.*)

Proof. If $\mathcal{E}\colon R \to R$ exists, one check easily that $M := \ker \mathcal{E}$ satisfies (2.1), so S is a corner of R. Conversely, if $S \subseteq R$ is a corner of R, choose some complement M so that $R = S \oplus M$, $SM \subseteq M$, $MS \subseteq M$. Then define $\mathcal{E}\colon R \to R$ by $\mathcal{E}(s+m) = s$ for $s \in S$ and $m \in M$. Clearly, $\mathcal{E}$ is additive. If $s_0 \in S$, we have

$$\mathcal{E}(s_0(s+m)) = \mathcal{E}(s_0 s + s_0 m) = s_0 s = s_0 \mathcal{E}(s+m)$$

since $s_0 s \in S$ and $s_0 m \in M$. Thus, $\mathcal{E}$ is left S-homogeneous, and a similar check shows that $\mathcal{E}$ is also right S-homogeneous. It is straightforward to verify that $\mathcal{E}(R) = S$ and $\mathcal{E} \circ \mathcal{E} = \mathcal{E}$. □

The following proposition will be useful for further considerations and we will make reference to it many times. It states that if S is a corner of a unital ring R, then S always has an identity element, although this may not be the identity of R. In particular, if $\mathcal{E}\colon R \to R$ is an S-bimodule projection onto S as in Proposition 2.1.2, then $\mathcal{E}(1)$ is an identity of the ring S.

Proposition 2.1.3 ([48, Proposition 2.2]). Let R be a ring with unit 1, $S \subseteq R$ a corner of R with a complement M so that $R = S \oplus M$, and let $1 = e + f$ where $e \in S$ and $f \in M$. Then e is an identity of the ring S. In particular, the decomposition $1 = e + f$ is independent of the choice of the complement M, e and f are complementary idempotents in R, and if $\mathcal{E}\colon R \to R$ is an S-bimodule map as in Proposition 2.1.2 (with $\mathcal{E}(R) = S$ and $\mathcal{E} \circ \mathcal{E} = \mathcal{E}$) then $\mathcal{E}(1) = e$.

Proof. For any $s \in S$, $s = s1 = s(e+f) = se + sf$. Since $s, se \in S$ and $sf \in M$, we have $s = se$. Similarly, $s = es$, so e is a multiplicative identity element of S. Since the multiplicative identity element of S is unique, the remaining statements in the Proposition follow immediately. In particular, $e^2 = e$ and so $ef = e(1-e) = e - e^2 = 0$. If $\mathcal{E}\colon R \to R$ is an S-bimodule map as in Proposition 2.1.2, then for any $s \in S$, $s\mathcal{E}(1) = \mathcal{E}(s1) = \mathcal{E}(s) = s$. Similarly, $\mathcal{E}(1)s = \mathcal{E}(1s) = \mathcal{E}(s) = s$, so $\mathcal{E}(1)$ is also a multiplicative identity element of S. It follows that $\mathcal{E}(1) = e$ because the multiplicative identity element of a ring with identity is unique. □

We will use the following lemma to prove Proposition 2.5.2.

Lemma 2.1.4. Let R be a ring with unit 1. If $S \subseteq R$ is a corner with a complement M and $1 = e + f$ for $e \in S$, $f \in M$, then $fR + Rf \subseteq M$.

Proof. For $x \in R$ we have $\mathcal{E}(fx) = e\mathcal{E}(fx) = \mathcal{E}(efx) = \mathcal{E}(0) = 0$ since e is the identity for S. Thus $fR \subseteq M$. Similarly, $\mathcal{E}(xf) = \mathcal{E}(xf)e = \mathcal{E}(xfe) = \mathcal{E}(0) = 0$, and so $Rf \subseteq M$.

□

There are four special classes of corners which are of particular importance to the theory. Those are *Peirce corners*, *unital corners*, *rigid* and *split corners*.

Definition 2.1.5 (Lam [48])**.** Let R be a ring, $S \subseteq R$ a corner of R.

1. S is called a *Peirce corner* if there exists $e \in R$ such that $e^2 = e$ and $S = eRe$.

2. When R has an identity element, then (so does S by Proposition 2.1.3 and) S is called a *unital corner* if the identity element of R belongs to S.

3. S is called a *rigid corner* if there exists a unique complement of S in R (two complements M and M' are considered identical if $M = M'$).

4. S is called a *split corner* if there exists a complement of S in R which is an ideal in R (equivalently, if there exists a complement of S in R which is a subring of R; in this case we have a unital ring isomorphism $S \cong R/M$).

Furthermore, if R is a ring and $e_1, e_2, \ldots, e_n \in R$ are idempotents with $e_i e_j = 0$ for $i \neq j$, then $\bigoplus_{i=1}^{n} e_i R e_i$ is a corner of R, which will be called a *generalised Peirce corner*.

We note the following fact.

Fact 2.1.6 (Unitalization)**.** If R is a non-unital ring, its unitalization is the abelian group $R^\sharp = R \oplus \mathbb{Z}$. With the multiplication $(x, m)(y, n) = (xy + nx + my, mn)$ $(x, y \in R;\ m, n \in \mathbb{Z})$, $R^\sharp$ is a unital ring with unit $(0, 1)$, and $R \subseteq R^\sharp$ is an ideal. If $S \subseteq R$ is a corner and $\mathcal{E}\colon R \to R$ is a corresponding Lam conditional expectation with $\mathcal{E}(R) = S$ (see Proposition 2.1.2), then

$$\mathcal{E}^\sharp\colon R^\sharp \to R^\sharp \quad \text{defined by} \quad \mathcal{E}^\sharp(x, m) = (\mathcal{E}(x), m) \quad (x \in R,\ m \in \mathbb{Z})$$

is a Lam conditional expectation from $R^\sharp$ onto $\mathcal{E}^\sharp(R^\sharp) = S^\sharp = S \oplus \mathbb{Z}$. In other words, if S is a corner of R then $S^\sharp$ is a corner of $R^\sharp$. Furthermore, if $\mathcal{E}$ is multiplicative then $\mathcal{E}^\sharp$ is also multiplicative. (Compare this fact with Proposition 3.3.1.)

In light of the following theorem which was proved in [48], the theory of general corners may be reduced to Peirce corners and unital corners in many ways.

Theorem 2.1.7. If R is a unital ring, then:

1. Any corner of R is a unital corner of some Peirce corner of R; and
2. Any corner of R is a Peirce corner of some unital corner of R.

For a detailed proof of these results we refer the reader to [48, Section 5]. Here we briefly outline the key steps. As regards the first result, if S is a corner of a unital ring R, then $R = S \oplus M$ for some group $M \subseteq R$. Therefore the identity 1 of R can be uniquely written as $1 = e + f$ for $e \in S$, $f \in M$. By Proposition 2.1.3, the summand e is an identity element of S and eRe. Since $S \subseteq eRe$, it follows that S is a unital corner of the Peirce corner eRe of R.

As regards the second result, that any corner of R is a Peirce corner of some unital corner of R, assume again that S is a corner of a unital ring R. Then

$R = S \oplus M$ for some group $M \subseteq R$. Since the identity 1 of R can be uniquely written as $1 = e + f$ for $e \in S$, $f \in M$ with $ef = fe = 0$ (Proposition 2.1.3), it follows that $S \cap fRf = 0$. Indeed, if $x \in S \cap fRf$ then $x = ex = f(ex)$ because e is an identity of S and f is an identity of fRf. Thus $x = 0$ because R is an associative ring and $fe = 0$. Now, $S' = S \oplus fRf$ is a unital corner of R and $S = eS'e$ is a Peirce corner of S'.

We shall close this section by noting that H. Kraft, L. W. Small, and N. R. Wallach have proved in [46] a number of important results regarding corner rings (but in [46] the the term 'corner rings' has not been explicitly used). In particular, the authors have showed that when S is a subring of a ring R with $R = S \oplus V$, and V has certain invariance properties, then some properties of R are inherited by S. For instance, if S is a subring of a semisimple ring R, and if S is also a bimodule direct summand of R then S is also semisimple i.e., a corner of a semisimple ring is semisimple (we will use this in Corollary 3.9.2 and in Lemma 3.12.2).

We shall also point out that there is a different definition and approach to 'corners rings' which was proposed in [4] by S. A. Amitsur and L. W. Small. Here is the definition:

Definition 2.1.8. Let R be a ring. A subring S of R is called an *Amitsur-Small corner* of R if $SRS \subseteq S$.

Amitsur-Small corners also generalize subrings of the form eRe to the idea of 'corners'. In other words, every Peirce corner eRe is an Amitsur-Small corner. But there exist corners that are not Amitsur-Small corners and vice versa. For example, the algebra of finite-rank operators $\mathcal{F}(H)$ on an infinite-dimensional Hilbert space H is an Amitsur-Small corner in $\mathcal{B}(H)$ – the algebra of all bounded linear operators on H. Also its norm-closure, the algebra of compact operators $\mathcal{K}(H)$, is an Amitsur-Small corner in $\mathcal{B}(H)$. But neither $\mathcal{F}(H)$ nor $\mathcal{K}(H)$ are corners of $\mathcal{B}(H)$ in the sense of Definition 2.1.1 because they are not unital. On the other hand, if R is any ring with a unit-element 1 and S is a unital proper corner of R, then $SRS \nsubseteq S$ and so S is not an Amitsur-Small corner.

We also note that an unital corner need not be a Peirce corner in general.

For example, let $C([-1,1])$ denotes the C^*-algebra of continuous complex-valued functions on the interval $[-1,1]$. Since $C([-1,1]) = C_{\text{even}}([-1,1]) \oplus C_{\text{odd}}([-1,1])$, the direct sum of even and odd functions, and the product of an even function and an odd function is an odd function, it follows that $C_{\text{even}}([-1,1])$ is a unital corner of $C([-1,1])$ (in the sense of Definition 2.1.1). But $C_{\text{even}}([-1,1])$ is not a Peirce corner of $C([-1,1])$ because there are no nontrivial idempotents in $C([-1,1])$ (the only idempotents of $C([-1,1])$ are the '0' and '1' functions). (Note that $C_{\text{even}}([-1,1])$ is not an Amitsur-Small corner of $C([-1,1])$.) Note also that the space $L^2(\mathbb{R})$ of square-integrable functions with respect to the Lebesgue measure on the real line $\mathbb{R}$ and with the inner product $\langle f,g\rangle = \int_{\mathbb{R}} f(x)\overline{g(x)}dx$, for f and g in $L^2(\mathbb{R})$, is the orthogonal direct sum of the space S of even functions and the space M of odd functions. The orthogonal projection $\mathcal{E}$ of $L^2(\mathbb{R})$ onto the corner S along M is given by $\mathcal{E}(f)(x) = (f(x) + f(-x))/2$ for $f \in L^2(\mathbb{R})$, $x \in \mathbb{R}$. (We could also consider the orthogonal projection $\mathcal{E}_h(f)(x) = \overline{h(x)}[h(x)f(x) \pm h(-x)f(-x)]$ where $|h(x)|^2 + |h(-x)|^2 = 1$.)

2.2 Uniqueness of Complements

In this section we present new proofs of T. Y. Lam's results [48, (2.8)(2), (5.11)] on uniqueness of complements in arbitrary (not necessarily unital) rings.

Theorem 2.2.1. A Peirce corner $S = eRe$ in a ring R (where $e \in R$ is an idempotent) has a unique complement $M = R(1-e) + (1-e)R = eR(1-e) \oplus (1-e)Re \oplus (1-e)R(1-e)$.

Proof. Let M be any complement for S. Then, for $x \in R$ we have $x = s + m$ with $s \in S$, $m \in M$. So $exe = ese + eme = s + eme$. But $eme \in M$ since $e \in S$ and so $eme = exe - s \in S \cap M = \{0\}$. Hence $s = exe$ and $m = x - exe = ex(1-e) + (1-e)x = ex(1-e) + (1-e)xe + (1-e)x(1-e)$.

The summands $eR(1-e)$, $(1-e)Re$ and $(1-e)R(1-e)$ are easily seen to each have trivial intersection with the sum of the other two. □

Theorem 2.2.2. If R is a ring and $e_1, e_2, \ldots, e_n \in R$ are idempotents with $e_i e_j = 0$ for $i \neq j$, then the generalised Peirce corner $S = \bigoplus_{i=1}^{n} e_i R e_i$ has a unique complement.

The unique Lam conditional expectation $\mathcal{E}\colon R \to R$ with range S is given by $\mathcal{E}(x) = \sum_{i=1}^{n} e_i x e_i$.

Proof. Let M be a complement for S with corresponding Lam conditional expectation $\mathcal{E}_0$. The idempotent $e = \sum_{i=1}^{n} e_i$ is the identity element for S and it follows from $R \ni x = s + m$ ($s \in S$, $m \in M$) that $exe = s + eme$ with $eme \in M$. So $\mathcal{E}_0(x) = s = \mathcal{E}_0(exe)$.

Note that for $z \in S$ we have $z = \sum_{k=1}^{n} e_k z e_k$.

For $y \in eRe$ we have $y = eye = \sum_{i,j=1}^{n} e_i y e_j = \sum_{i=1}^{n} e_i y e_i + \sum_{i \neq j} e_i y e_j$. For $i \neq j$ we have $\mathcal{E}_0(e_i y e_j) = \sum_{k=1}^{n} e_k \mathcal{E}_0(e_i y e_j) e_k = \sum_{k=1}^{n} \mathcal{E}_0(e_k e_i x e_j e_k) = 0$. Hence $e_i y e_j \in M$ for $i \neq j$ and $\mathcal{E}_0(y) = \sum_{i=1}^{n} e_i y e_i$.

It follows that for $x \in R$,

$$\mathcal{E}_0(x) = \mathcal{E}_0(exe) = \sum_{i=1}^{n} e_i exee_i = \sum_{i=1}^{n} e_i x e_i. \qquad \square$$

2.3 Annihilators and Split Corners

For a nonempty subset S of a ring R we define the *annihilator* of S as follows $S^{\perp} := \{x \in R : xs = 0 = sx \text{ for all } s \in S\}$. Note that $S^{\perp}$ is a subgroup of the additive group of R, and if S is a two-sided ideal of R, then so is $S^{\perp}$.

Proposition 2.3.1. Let R be a ring and S a two-sided ideal of R such that $S \cap S^{\perp} = \{0\}$. Then S is a corner of R if and only if $R = S \oplus S^{\perp}$.

Proof. Suppose S is a corner of R. Then $R = S \oplus M$ for some subgroup $M \subseteq R$ with $SM \subseteq M$, $MS \subseteq M$. Since S is a two-sided ideal of R we also have that $SM \subseteq S$, $MS \subseteq S$. Therefore $SM \subseteq S \cap M = \{0\}$ and $MS \subseteq S \cap M = \{0\}$, so $R = S + S^{\perp}$ because $M \subseteq S^{\perp}$. Finally, the assumption $S \cap S^{\perp} = \{0\}$ yields that $R = S \oplus S^{\perp}$.

For the converse, if $R = S \oplus S^{\perp}$, then S is a corner of R since $SS^{\perp} = S^{\perp}S = \{0\} \subseteq S^{\perp}$. □

We note that if a corner S in R is complemented by a subring, say M, of R, then M is automatically a two-sided ideal of R. Indeed, suppose $R = S \oplus M$ with S a corner of R and the complement M a subring of R. Then $RM = (S \oplus M)M \subseteq SM + MM \subseteq M + M = M$, so M is a left ideal of R. We also have that $MR = M(S \oplus M) \subseteq MS + MM \subseteq M + M = M$, so M is a right ideal of R. Hence M is a two-sided ideal of R.

Corollary 2.3.2. Assume that S is a corner of a ring R. If S has a complement M that is a ring, then S is a split corner and $R/M \cong S$.

2.4 Graphs of Ring Homomorphisms

The new idea that we present in this section, and which we will develop further in future sections (§5.5, §4.3), is to consider graphs of ring homomorphisms $\phi\colon R_1 \to R_2$ as corner subrings of $R_1 \oplus R_2$ (here R_1 and R_2 are rings). We will show that one can always construct a Lam conditional expectation $\mathcal{E}\colon R_1 \oplus R_2 \to R_1 \oplus R_2$ onto the graph of ϕ and that this $\mathcal{E}$ must be multiplicative. In a similar way, we show that one can also construct a Lam conditional expectation $\mathcal{E}\colon R \oplus M \to R \oplus M$ onto the graph of a derivation $\delta\colon R \to M$ (here R is a ring and M is an R-bimodule). Similar constructions are valid in the context of Lie rings.

Suppose we have a ring homomorphism $\phi\colon R_1 \to R_2$ where R_1 and R_2 are rings.

Let $R = R_1 \oplus R_2$, a ring with the summands as ideals. The subring $R_1 \times \{0\}$ of $R_1 \oplus R_2$ is isomorphic to R_1 and so is identified with R_1; similarly for $\{0\} \times R_2$.

Denote by S the graph of ϕ, that is, $S = \{(x, \phi(x)) : x \in R_1\}$, and define $\mathcal{E}\colon R \to R$ by $\mathcal{E}(a, b) = (a, \phi(a))$ $(a \in R_1, b \in R_2)$. Note the following:

1. S is a subring of R. Indeed, if $x, y \in R_1$ then $(x, \phi(x)) + (y, \phi(y)) = (x + y, \phi(x) + \phi(y)) = (x + y, \phi(x + y)) \in S$, and $(x, \phi(x))(y, \phi(y)) = (xy, \phi(x)\phi(y)) = (xy, \phi(xy)) \in S$.

2. $\mathcal{E} \circ \mathcal{E} = \mathcal{E}$. Indeed, if $a \in R_1$ and $b \in R_2$ then $\mathcal{E} \circ \mathcal{E}(a,b) = \mathcal{E}(a, \phi(a)) = (a, \phi(a)) = \mathcal{E}(a,b)$.

3. $\mathcal{E}$ is an S-bimodule map. Indeed, if $x, a \in R_1$ and $b \in R_2$ then $\mathcal{E}((x,\phi(x))(a,b)) = \mathcal{E}(xa, \phi(x)b) = (xa, \phi(xa)) = (xa, \phi(x)\phi(a)) = (x,\phi(x))(a,\phi(a)) = (x,\phi(x))\mathcal{E}(a,b)$. Similarly, $\mathcal{E}((a,b)(x,\phi(x))) = \mathcal{E}(a,b)(x,\phi(x))$.

4. $\mathcal{E}(R) = S$ (a consequence of the definition of $\mathcal{E}$).

5. $\mathcal{E}$ is multiplicative. Indeed, if $a, c \in R_1$ and $b, d \in R_2$ then $\mathcal{E}((a,b)(c,d)) = \mathcal{E}(ac, bd) = (ac, \phi(ac)) = (ac, \phi(a)\phi(c)) = (a,\phi(a))(c,\phi(c)) = \mathcal{E}(a,b)\mathcal{E}(c,d)$.

6. If R_1 and R_2 are rings with involution, then $R = R_1 \oplus R_2$ becomes a ring with involution $(a,b) \mapsto (a^*, b^*)$ $(a \in R_1, b \in R_2)$. Moreover, if ϕ is *-preserving then $\mathcal{E}$ is *-preserving and S is self-adjoint. Indeed, if $a \in R_1$ and $b \in R_2$ then $\mathcal{E}((a,b)^*) = \mathcal{E}(a^*, b^*) = (a^*, \phi(a^*)) = (a^*, \phi(a)^*) = (a, \phi(a))^* = (\mathcal{E}(a,b))^*$. If $x \in S$ then $(x, \phi(x)) \in S$ and ϕ is *-preserving then $(x,\phi(x))^* = (x^*, \phi(x)^*) = (x^*, \phi(x^*)) \in S$, so S is self-adjoint.

We have shown:

Proposition 2.4.1 (Lam conditional expectations onto graphs of ring homomorphisms)**.** If $\phi\colon R_1 \to R_2$ is a ring homomorphism, then the graph of ϕ, $S = \{(x,\phi(x)) : x \in R_1\}$, is a corner of $R := R_1 \oplus R_2$.

If R_1 and R_2 are rings with involutions and ϕ is involution-preserving, then the corner S is self-adjoint.

Proposition 2.4.1 asserts in particular that one can always find a Lam conditional expectation onto the graph of a ring homomorphism (between two rings); moreover, this expectation is multiplicative. In fact, we will show (Proposition 2.4.2) that a Lam conditional expectation is multiplicative if and only if its kernel is an ideal.

Proposition 2.4.2. Let R be a ring, $\mathcal{E}\colon R \to R$ an additive map with $\mathcal{E} \circ \mathcal{E} = \mathcal{E}$ and

$$\mathcal{E}(\mathcal{E}(x)y) = \mathcal{E}(x)\mathcal{E}(y), \qquad \mathcal{E}(x\mathcal{E}(y)) = \mathcal{E}(x)\mathcal{E}(y)$$

for $x, y \in R$. Then $\mathcal{E}$ is multiplicative if and only if $\ker \mathcal{E}$ is an ideal.

Proof. If $\ker \mathcal{E}$ is an ideal in R, then for $x, y \in R$ we have $\mathcal{E}(m) = 0$, where $m = x - \mathcal{E}(x)$, and $\mathcal{E}(xy) = \mathcal{E}((m+\mathcal{E}(x))y) = \mathcal{E}(my + \mathcal{E}(x)y) = \mathcal{E}(my) + \mathcal{E}(\mathcal{E}(x)y) = \mathcal{E}(x)\mathcal{E}(y)$.

Conversely, if $\mathcal{E}$ is multiplicative, then for $x \in R$, $m \in \ker \mathcal{E}$, we have $\mathcal{E}(xm) = \mathcal{E}(mx) = 0$ thus $xm, mx \in \ker \mathcal{E}$. □

Question. Is every corner the graph of a ring homomorphism if the corresponding Lam conditional expectation $\mathcal{E}$ is multiplicative?

Proposition 2.4.3 (Lam conditional expectations onto graphs of derivations)**.** Let R be a ring and let M be an R-bimodule. Then $R \oplus M$ is a ring with the operations: $(x, m) + (y, n) = (x + y, m + n)$ and $(x, m) \cdot (y, n) = (xy, xn + my)$ for $x, y \in R$, $m, n \in M$. Let $\delta \colon R \to M$ be a derivation (that is: a function satisfying $\delta(x + y) = \delta(x) + \delta(y)$ and $\delta(xy) = \delta(x)y + x\delta(y)$ for $x, y \in R$). Then the graph of δ, i.e. the set $S = \{(x, \delta(x)) : x \in R\}$, is a (corner) subring of $R \oplus M$, and the map $\mathcal{E} \colon R \oplus M \to R \oplus M$ defined by $\mathcal{E}(x, m) = (x, \delta(x))$ for $x \in R$, $m \in M$, is a Lam conditional expectation onto S. Moreover, $\mathcal{E}$ is a ring homomorphism.

Here is another model for corners and corresponding 'expectations' in the context of Lie rings: If R is a ring, write $[x, y] = xy - yx$ for the Lie product of two elements $x, y \in R$ (or consider a 'general' Lie bracket on R), and consider $R = (R, [\cdot, \cdot])$ as a Lie ring. Then a Lie subring, i.e. a subgroup $L \subseteq R$ of the additive group of R with $[L, L] \subseteq L$, may be called a *Lie corner* if $R = L \oplus M$ and $[L, M] \subseteq M$ for some subgroup $M \subseteq R$. Note that L is a Lie corner if and only if there exists an additive group homomorphism $\mathcal{E} \colon R \to R$ such that $\mathcal{E} \circ \mathcal{E} = \mathcal{E}$ and $\mathcal{E}([l, x]) = [l, \mathcal{E}(x)]$, $\mathcal{E}([x, l]) = [\mathcal{E}(x), l]$ for $l \in L$, $x \in R$; such $\mathcal{E}$ may be called a *Lie conditional expectation.*

An example of a Lie corner might be the space Skew_n of n-by-n skew-symmetric matrices ($A^T = -A$). Indeed, Skew_n is a Lie subring of Mat_n, the space of all n-by-n matrices, and $\mathrm{Mat}_n = \mathrm{Skew}_n \oplus \mathrm{Sym}_n$ [since $A = \frac{1}{2}(A - A^T) + \frac{1}{2}(A + A^T)$], where Sym_n denotes the space of symmetric matrices ($A^T = A$). The corresponding Lie conditional expectation is the map $\mathcal{E} \colon \mathrm{Mat}_n \to \mathrm{Mat}_n$ given by $\mathcal{E}(A) = \frac{1}{2}(A - A^T)$.

Proposition 2.4.4 (Lie conditional expectations onto graphs of Lie homomorphisms). Let R_1, R_2 be rings considered as Lie rings, and let $\phi\colon R_1 \to R_2$ be a Lie homomorphism (i.e. an additive group homomorphism with $\phi([x,y]) = [\phi(x), \phi(y)]$ for $x, y \in R_1$). Consider $R = R_1 \oplus R_2$, a Lie ring with $[(x_1, x_2), (y_1, y_2)] = ([x_1, y_1], [x_2, y_2])$ for $x_1, y_1 \in R_1$, $x_2, y_2 \in R_2$. Then the graph of ϕ, i.e the set $L = \{(x, \phi(x)) : x \in R_1\}$, is a Lie corner of R, and the map $\mathcal{E}\colon R \to R$ defined by $\mathcal{E}(x_1, x_2) = (x_1, \phi(x_1))$ ($x_1 \in R_1$, $x_2 \in R_2$) is a Lie conditional expectation onto L. Moreover $\mathcal{E}$ is a Lie homomorphism, i.e. $\mathcal{E}([(x_1, x_2), (y_1, y_2)]) = [\mathcal{E}(x_1, x_2), \mathcal{E}(y_1, y_2)]$ for $x_1, y_1 \in R_1$, $x_2, y_2 \in R_2$.

2.5 $\mathbb{Q}$ is a Dense Corner in $\mathbb{R}$

Consider $\mathbb{R}$, the set of real number, as a vector space over the rationals $\mathbb{Q}$. Let B be a Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$ including 1. Let M be the $\mathbb{Q}$-linear span of $B \setminus \{1\}$. Then $\mathbb{Q}1 = \mathbb{Q}$ and

$$\mathbb{R} = \mathbb{Q} \oplus M.$$

Since $\mathbb{Q}M = M$, $\mathbb{Q}$ is a unital corner ring of $\mathbb{R}$ (but not a vector subspace over $\mathbb{R}$). In Section 3.5 we will show that all dense corners (in unital normed algebras) are automatically unital.

Remark 2.5.1. Note that the corner $\mathbb{Q}$ is not closed under the square root operation, because for example $2 \in \mathbb{Q}$ but $\sqrt{2} \notin \mathbb{Q}$. (More generally, if $k > 1$ and n are positive integers and n is not a k^{th} power, then $\sqrt[k]{n} \notin \mathbb{Q}$, so $\mathbb{Q}$ is not closed under $\sqrt[k]{\cdot}$.) So while unital corners are inverse closed (see Proposition 2.6.1), they need not be closed under 'square roots' in general. But self-adjoint corners of C^*-algebras are closed under square roots of positive elements (see Corollary 5.8.3).

Proposition 2.5.2. Let A be a unital (associative) algebra over a field F, and S a corner of A regarded as a ring with complement M. If S is also a subalgebra of A then the group M must be a vector space over F.

Proof. Since M is closed under addition by definition of a complement we need to prove that if $\alpha \in F$ and $m \in M$, then $\alpha m \in M$.

Let $\mathcal{E}\colon A \to A$ be an S-bimodule idempotent ($\mathcal{E} \circ \mathcal{E} = \mathcal{E}$) with $\mathcal{E}(A) = S$ (see Proposition 2.1.2). Let $e = \mathcal{E}(1_A)$ where 1_A is the identity element of A, $f = 1_A - e$. Then $\alpha m = \alpha(1_A)m = \alpha(e+f)m = (\alpha e)m + f(\alpha m)$.

Now $(\alpha e)m \in M$ since $\alpha e \in S$. Also $f(\alpha m) \in M$ by Lemma 2.1.4. So $\alpha m \in M$. □

Example 2.5.3. $\mathbb{R} = \mathbb{Q} \oplus M$, $\mathbb{Q}$ is a corner and M a vector space over $\mathbb{Q}$.

Remark 2.5.4. In the later chapters we will be considering corners in algebras rather than rings and we will assume that the complement M is a vector space.

2.6 Necessary Condition: Inverse Closure

Let R be a ring with unit 1, and let S be a subring of R with $1 \in S$. We say that S is *inverse closed* in R if whenever $s \in S$ has an inverse s^{-1} in R then $s^{-1} \in S$.

It is known that if A is a unital C^*-algebra and S is a C^*-subalgebra of A with $1 \in S$, then $\sigma_S(x) = \sigma_A(x)$ for all $x \in S$ where $\sigma_S(x)$ denotes the spectrum of x relative S; equivalently, S is inverse closed in A.

In this section we will show that every unital corner of R is inverse closed in R.

Proposition 2.6.1. Let R be a unital ring. If S is a unital corner of R, then S is inverse closed – that is if $s \in S$ is invertible in R , then $s^{-1} \in S$.

Proof. Let M be a complement of S in R. Since $R = S \oplus M$, every element of R can be written uniquely as the sum of an element of S with an element of M. In particular if $s \in S$ is invertible in R, then $s^{-1} = s_0 + m_0$ for a unique $s_0 \in S$ and $m_0 \in M$. Then $1 = ss^{-1} = s(s_0+m_0) = ss_0+sm_0$. Now $sm_0 \in M$ and $1-ss_0 \in S$. Hence $sm_0 = 1 - ss_0 \in S \cap M = \{0\}$, and so $s^{-1}(sm_0) = 0$, $s^{-1} = s_0 + 0 \in S$. □

As an application of the above Proposition, we note that the ring of integers $\mathbb{Z}$ is not a corner of $\mathbb{R}$ (the ring of real numbers) since if $n \in \mathbb{Z} \setminus \{-1, 0, 1\}$ then $\frac{1}{n} \in \mathbb{R}\setminus\mathbb{Z}$; in other words $\mathbb{Z}$ is not inverse closed in $\mathbb{R}$. Also the space of polynomials is not a corner of the ring of continuous functions (on an infinite subset of $\mathbb{R}$ or $\mathbb{C}$) since the inverse of a polynomial need not necessarily be a polynomial.

2.7 Corners in Regular Rings

In this section we will prove certain statements about corners in *regular* rings. To state our results, we need to first recall some definitions.

A ring R is called *von Neumann regular*, or simply *regular*, if for every x in R there exists y in R such that $x = xyx$; y is called a *quasi-inverse* for x. This notion was first introduced by J. von Neumann in [72]; see also K. R. Goodearl [31]. For example, the ring of all linear transformation on an infinite-dimensional vector space is regular (for a proof cf. [36]).

It is known that a Peirce corner of a regular ring is regular, i.e., if R is regular and $e^2 = e \in R$ then eRe is also regular ([8, 2.4]). We can prove a more general version of this result, that every corner of a regular ring is regular.

Proposition 2.7.1. Every corner of a regular ring is regular.

Proof. Let R be a regular ring, $S \subseteq R$ a corner of R. It follows from Proposition 2.1.2 that we may choose an S-bimodule idempotent $\mathcal{E}\colon R \to R$ ($\mathcal{E} \circ \mathcal{E} = \mathcal{E}$) with $\mathcal{E}(R) = S$. Now, if $s \in S$, there is $y \in R$ with $s = sys$ since R is regular. Thus, $s = \mathcal{E}(s) = \mathcal{E}(sys) = s\mathcal{E}(y)s$, and $\mathcal{E}(y) \in S$ is a quasi-inverse for s. Therefore S is regular. □

An important special case of a regular ring is a *strongly regular* ring (first introduced by R. F. Arens and I. Kaplansky in [5]). A ring R is called *strongly regular* if for every x in R there exists an y in R such that $x = x^2y$. By a similar argument to Proposition 2.7.1 we have that every corner of a strongly regular ring is strongly regular.

We prove a somewhat more general result, that every corner of a *π-regular* ring is *π-regular*. Recall that a ring R is called *π-regular* if for every x in R there exist y in R and a positive integer n such that $x^n = x^n y x^n$ (this notion was first introduced by N. H. McCoy in [53]). If $n = 1$ for all $x \in R$ then R is regular.

Proposition 2.7.2. Every corner of a *π-regular* ring is *π-regular*.

Proof. The proof relies on a similar argument used in the proof of Proposition 2.7.1. □

An important special case of a π-regular ring is a *strongly π-regular* ring. A ring R is called *strongly π-regular* if for every x in R there exist y in R and a positive integer n such that $x^n = x^{n+1}y$. We can show similarly that every corner of a strongly π-regular ring is strongly π-regular.

2.8 Corners in Rings with Involution

In the early 1950's, I. Kaplansky initiated a series of studies aimed at developing algebraic models for von Neumann algebras. Beside *AW^*-algebras* ([41, 42, 43]), he introduced more general *Baer $*$-rings* ([44, 45], see also [7, 8]), which ignore much detail about von Neumann algebras, yet still can be divided into types I, II, and III in the same way as von Neumann algebras. *Baer $*$-rings* as well as *Rickart $*$-rings* are special classes of $*$-rings – rings with involutions.

Recall that an involution on an associative ring R is an injective and surjective map $R \to R$, $x \mapsto x^*$, such that $(x+y)^* = x^* + y^*$, $(xy)^* = y^*x^*$, $x^{**} = x$, and $1 = 1^*$ if $1 \in R$. Then a $*$-ring is an associative ring with involution $*$ on the ring. The condition $x = x^2$ defines idempotents, and $x^* = x$ self-adjoint elements.

In a ring R, any two of the following statements imply the third (cf. [44, Theorem 1]):

B1. Every left annihilator is left principal generated by an idempotent, that is, for any set $Z \subseteq R$ there exists an idempotent $e \in R$ such that

$$\operatorname{ann}_l^R(Z) := \{x \in R : xZ = 0\} = Re.$$

B2. Every right annihilator is right principal generated by an idempotent, that is, for any set $Z \subseteq R$ there exists an idempotent $e \in R$ such that

$$\operatorname{ann}_r^R(Z) := \{x \in R : Zx = 0\} = eR.$$

B3. R has a unit 1.

A *Baer ring* is a ring satisfying any two (and hence all three) of the above conditions ([44, Definition 1]).

Examples of Baer rings include: the C^*-algebra of all bounded sequences of complex numbers, the ring of bounded linear operators on a Hilbert space, a von Neumann algebra.

I. Kaplansky has proved that a Peirce corner of a Baer ring is a Baer ring, i.e., if R is a Baer ring, $e \in R$ an idempotent, then eRe is a Baer ring ([44, Theorem 2]). We can generalize this result and prove that every corner of a Baer ring is a Baer ring.

Lemma 2.8.1. Let R be a ring. Let $Z \subseteq S \subseteq R$, where Z is a set, S a corner of R.

1. If $\operatorname{ann}_l^R(Z) = Re$ for an idempotent $e \in R$, then there exists an idempotent $g \in S$ so that $\operatorname{ann}_l^S(Z) = Sg$.

2. If $\operatorname{ann}_r^R(Z) = eR$ for an idempotent $e \in R$, then there exists an idempotent $g \in S$ so that $\operatorname{ann}_r^S(Z) = gS$.

(Here $\operatorname{ann}_r^S(Z) = \operatorname{ann}_r^R(Z) \cap S = \{s \in S : Zs = 0\}$.)

Proof. Since S is a corner ring of R, it follows from Proposition 2.1.2 that we may choose an S-bimodule idempotent $\mathcal{E}\colon R \to R$ ($\mathcal{E} \circ \mathcal{E} = \mathcal{E}$) with $\mathcal{E}(R) = S$.

1. We claim that $\operatorname{ann}_l^S(Z) = Sg$ where $g = \mathcal{E}(e)$. Indeed, for all $z \in Z$ we have $\mathcal{E}(e)z = \mathcal{E}(ez) = \mathcal{E}(eez) = \mathcal{E}(0) = 0$. Hence $\mathcal{E}(e)Z = 0$, or $\mathcal{E}(e) \in \operatorname{ann}_l^R(Z)$. But $\operatorname{ann}_l^R(Z) = Re$, so there exists $r \in R$ with $\mathcal{E}(e) = re$. Right-multiplying both sides of this equation by e yields $\mathcal{E}(e)e = ree = re = \mathcal{E}(e)$. This implies that $\mathcal{E}(e)^2 = \mathcal{E}(e)\mathcal{E}(e) = \mathcal{E}(\mathcal{E}(e)e) = \mathcal{E}(\mathcal{E}(e)) = \mathcal{E}(e)$. Set $g = \mathcal{E}(e) \in S$; g is an idempotent ($g^2 = g$).

If $x \in Sg$, then $x = sg = s\mathcal{E}(e)$ for some $s \in S$. Hence, for $z \in Z$,

$$xz = s\mathcal{E}(e)z = \mathcal{E}(sez) = \mathcal{E}(0) = 0, \quad \text{so } x \in \operatorname{ann}_l^S(Z).$$

Therefore $Sg \subseteq \operatorname{ann}_l^S(Z)$.

For the converse inclusion, if $x \in \operatorname{ann}_l^S(Z) \subseteq \operatorname{ann}_l^R(Z) = Re$, then $x = re$ for some $r \in R$. Right-multiplying both sides of this equation by e yields $xe = ree = re = x$. So $x = \mathcal{E}(x) = \mathcal{E}(xe) = x\mathcal{E}(e) = xg \in Sg$. Therefore $\operatorname{ann}_l^S(Z) \subseteq gS$.

2. We claim that $\operatorname{ann}_r^S(Z) = gS$ where $g = \mathcal{E}(e)$. Indeed, for all $z \in Z$ we have $z\mathcal{E}(e) = \mathcal{E}(ze) = \mathcal{E}(zee) = \mathcal{E}(0) = 0$. Hence $Z\mathcal{E}(e) = 0$, which means $\mathcal{E}(e) \in \operatorname{ann}_r^R(Z)$. But $\operatorname{ann}_r^R(Z) = eR$, so there exists $r \in R$ with $\mathcal{E}(e) = er$. Left-multiplying both sides of this equation by e yields $e\mathcal{E}(e) = eer = er = \mathcal{E}(e)$. This implies that $\mathcal{E}(e)^2 = \mathcal{E}(e)\mathcal{E}(e) = \mathcal{E}(e\mathcal{E}(e)) = \mathcal{E}(\mathcal{E}(e)) = \mathcal{E}(e)$. Set $g = \mathcal{E}(e) \in S$; g is an idempotent ($g^2 = g$).

If $x \in gS$, then $x = gs = \mathcal{E}(e)s$ for some $s \in S$. Hence, for $z \in Z$,

$$zx = z\mathcal{E}(e)s = \mathcal{E}(zes) = \mathcal{E}(0) = 0, \quad \text{so } x \in \operatorname{ann}_r^S(Z).$$

Therefore $gS \subseteq \operatorname{ann}_r^S(Z)$.

For the converse inclusion, if $x \in \operatorname{ann}_r^S(Z) \subseteq \operatorname{ann}_r^R(Z) = eR$, then $x = er$ for some $r \in R$. Left-multiplying both sides of this equation by e yields $ex = eer = er = x$. So $x = \mathcal{E}(x) = \mathcal{E}(ex) = \mathcal{E}(e)x = gx \in gS$. Therefore $\operatorname{ann}_r^S(Z) \subseteq gS$. □

Theorem 2.8.2. Every corner of a Baer ring is a Baer ring.

Proof. Let R be a Baer ring, $S \subseteq R$ a corner of R. If $Z \subseteq S$ is a set, $\operatorname{ann}_l^R(Z) = Re$ for some idempotent $e \in R$. By Lemma 2.8.1 (1), there exists an idempotent $g \in S$ so that $\operatorname{ann}_l^S(Z) = Sg$. So S satisfies **B1**. Proposition 2.1.3 implies that S is unital, and so satisfies **B3**. Hence S is a Baer ring. □

We will now prove that a corner of a *left (right) Rickart* ring is *left (right) Rickart*, respectively.

A ring R is called *left Rickart* if for all z in R there exists an idempotent $e \in R$ such that $\operatorname{ann}_l^R(\{z\}) = \{x \in R : xz = 0\} = Re$; and is called *right Rickart* if for all z in R there exists an idempotent $e \in R$ such that $\operatorname{ann}_r^R(\{z\}) = \{x \in R : zx = 0\} = eR$. Note the left Rickart is condition **B1** (for singleton set $Z = \{z\}$); and the right Rickart is condition **B2** (for singleton set $Z = \{z\}$).

In general, a left Rickart ring need not be right Rickart. An example of a left Rickart ring that is not right Rickart can be found in [47, §2F]. In unital rings, however, left and right Rickart rings are identical.

Proposition 2.8.3. 1. Every corner of a left Rickart ring is left Rickart.

2. Every corner of a right Rickart ring is right Rickart.

Proof. We will prove only the first statement; the proof of the second statements is similar. Let R be a left Rickart ring, $S \subseteq R$ a corner of R. If $z \in S$, there exists an idempotent $e \in R$ such that $\operatorname{ann}_l^R(\{z\}) = Re$. By Lemma 2.8.1 (1), there exists an idempotent $g \in S$ so that $\operatorname{ann}_l^S(\{z\}) = Sg$. So S is a left Rickart ring. □

In $*$-rings, an analogous concept to Baer rings and Rickart rings is that of *Baer $*$-rings* and *Rickart $*$-rings*, respectively. A *Baer $*$-ring* is a $*$-ring R such that for any set $Z \subseteq R$ there exists a self-adjoint projection $p \in R$ ($p^* = p = p^2$) with $\operatorname{ann}_r^R(Z) = pR$ ([44, p. 17]). *A Rickart $*$-ring* is a $*$-ring R such that for any element $z \in R$ there exists a self-adjoint projection $p \in R$ with $\operatorname{ann}_r^R(\{z\}) = pR$ ([51]).

S. K. Berberian has proved that a self-adjoint Peirce corner of a Rickart *-ring is a Rickart *-ring, i.e., if R is a Rickart *-ring, $p \in R$ a self-adjoint projection, then pRp is a Rickart *-ring ([8, Theorem 2.5]). We can generalize this result and prove that every self-adjoint corner of a Rickart *-ring is a Rickart *-ring.

Theorem 2.8.4. If R is a Rickart $*$-ring and S is a self-adjoint corner ring of R, then S is a Rickart $*$-ring.

Proof. Let $z \in S$. We must show that $\operatorname{ann}_r^S(\{z\}) = pS$ for some self-adjoint projection $p \in S$.

Since S is a corner ring of R, it follows from Proposition 2.1.2 that we may choose an S-bimodule idempotent $\mathcal{E}\colon R \to R$ ($\mathcal{E} \circ \mathcal{E} = \mathcal{E}$) with $\mathcal{E}(R) = S$. By hypothesis $\operatorname{ann}_r^R(\{z\}) = qR$ for some self-adjoint projection $q \in R$. Therefore, since $z\mathcal{E}(q) = \mathcal{E}(zq) = \mathcal{E}(zqq) = \mathcal{E}(0) = 0$, $\mathcal{E}(q) \in \operatorname{ann}_r^R(\{z\})$ and so $q\mathcal{E}(q) = \mathcal{E}(q)$. This yields $\mathcal{E}(q)^2 = \mathcal{E}(q\mathcal{E}(q)) = \mathcal{E}(q)$. As before $z\mathcal{E}(q)^* = (\mathcal{E}(q)z^*)^* = \mathcal{E}(qz^*)^* =$

$\mathcal{E}((zqq)^*)^* = 0$, so $\mathcal{E}(q)^* \in \operatorname{ann}_r^R(\{z\})$ and $q\mathcal{E}(q)^* = \mathcal{E}(q)^*$. It now follows that $\mathcal{E}(q)^*\mathcal{E}(q) = \mathcal{E}(\mathcal{E}(q)^*q) = \mathcal{E}((q\mathcal{E}(q))^*) = \mathcal{E}(\mathcal{E}(q)^*) = \mathcal{E}(q\mathcal{E}(q)^*) = \mathcal{E}(q)\mathcal{E}(q)^*$. Set $p = \mathcal{E}(q)^*\mathcal{E}(q) \in S$; p is a self-adjoint projection. It is straightforward to verify that $\operatorname{ann}_r^S(\{z\}) = pS$. □

As a final result of this section we obtain a generalization of S. K. Berberian's theorem ([8, Proposition 2.6]) which asserts that if R is a Baer $*$-ring, $p \in R$ is a self-adjoint projection, then the Peirce corner pRp is a Baer $*$-ring.

Theorem 2.8.5. If R is a unital Baer $*$-ring and S is a self-adjoint corner ring of R, then S is a Baer $*$-ring.

Proof. S is a Baer ring by Theorem 2.8.2, and a Rickart $*$-ring by Theorem 2.8.4. Therefore S a Baer $*$-ring by [8, Proposition 1.24]. □

In the above proof we have used S. K. Berberian's result [8, Proposition 1.24] which asserts that the following conditions are equivalent:

(a) R is a Baer $*$-ring,

(b) R is a Baer ring and a Rickart $*$-ring.

Chapter 3

Corner Algebras

All algebras to be consider are associative, but not necessarily unital.

In this chapter, we establish new results by adopting a novel approach to conditional expectations, generalising the notion in an algebraic way and linking algebraic and topological properties in the settings of Banach algebras and C^*-algebras. This opens up a new range of problems, only some of which are completely solved. For example, we have a good understanding of Lam conditional expectations

$$\mathcal{E}\colon \mathcal{B}(H) \to \mathcal{B}(H)$$

on the algebra $\mathcal{B}(H)$ of bounded operators on a Hilbert space H with an orthonormal basis $(e_i)_{i\in I}$ (which may be countable or uncountable). Theorem 3.12.5, the main result of this chapter, states that if $\mathcal{E}(x^*) = \mathcal{E}(x)^*$ for $x \in \mathcal{B}(H)$ and $e_i \otimes e_i^* \in \mathcal{E}(\mathcal{B}(H))$ for $i \in I$, then $\mathcal{E}$ is of the form

$$\mathcal{E}(x) = \sum_{j\in J} p_j x p_j$$

for some 'diagonal' orthogonal projections $p_j \in \mathcal{B}(H)$. This result is generalized to purely atomic von Neumann algebras in Theorem 3.13.4.

We begin with definitions and preliminary results; this is the content of §3.1.

3.1 Definitions and Characterization

Lam's theory of corner rings [48] leads us to the following definition, a variation of Definition 2.1.1, which we consider to be appropriate one for algebras. In view of the discussion in §2.5 (and Proposition 2.5.2 in particular) the extra condition we propose seems justified.

Definition 3.1.1. Let A be an algebra. A subalgebra S of A is called a *corner algebra* (or simply a *corner*) of A if there exists a vector space $M \subseteq A$ such that

$$A = S \oplus M, \quad SM \subseteq M, \quad MS \subseteq M. \tag{3.1}$$

Such M is called a *complement* of S in A.

The simplest examples of corner algebras can be constructed by taking subalgebras of block diagonal form in the algebra of n-by-n complex matrices $M_n(\mathbb{C})$ generated by the matrix units $e_{ij} \in M_n(\mathbb{C})$ (where e_{ij} is the matrix unit with 1 in the (i, j) position, 0 elsewhere).

Example 3.1.2. For the sake of simplicity, let us consider first the case when $n = 3$. Setting $A = M_3(\mathbb{C})$ and $S = \text{linear span}\{e_{11}, e_{12}, e_{21}, e_{22}\} \subseteq A$, we have $A = S \oplus M$, where $M = \text{linear span}\{e_{13}, e_{23}, e_{31}, e_{32}, e_{33}\} \subseteq A$. Since $SM \subseteq M$ and $MS \subseteq M$, it follows that S is a corner of A, and M a complement of S in A.

Example 3.1.3. Let n be a positive integer and let $A = M_n(\mathbb{C})$, $S = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C}) \subseteq A$ where $n_1 + \cdots + n_k = n$. So S is the algebra of block diagonal matrices with k blocks of sizes $n_i \times n_i$. Then S is a corner of A.

We remark that the corner algebra S in Example 3.1.2 is of the form $S = pAp$ for the projection $p = e_{11} + e_{22} \in A = M_3(\mathbb{C})$, whereas the complement M is given by $M = pAq \oplus qAp \oplus qAq$, where $q = e_{33} = (1 - p)$ is the complementary projection. This remark leads us to the following generalization.

Example 3.1.4. Let A be a unital algebra and let $p \in A$ be an idempotent (that is $p^2 = p$; note that we do not require $p^* = p$ even if A is a C^*-algebra). Set

$S = pAp$. Then $A = S \oplus M$ where $M = pA(1-p) \oplus (1-p)Ap \oplus (1-p)A(1-p)$. Since $SM = pA(1-p) \subseteq M$, $MS = (1-p)Ap \subseteq M$, it follows that S is a corner of A, and M a complement of S in A.

Corners of the form pAp considered in Example 3.1.4 are referred to as *Peirce corners* (see [48, Theorem and Definition 2.8] or Definition 2.1.5). They are always unital subalgebras of A since p is an identity element of pAp. However, the identity element of A does not belong to pAp as long as $p \neq 1$. Corners of A that contain the identity element of A are referred to as *unital corners* (see [48, Definition 2.13] or Definition 2.1.5). *Generalised Peirce corners* are defined as on page 8 (except that here $A = R$ and $S \subseteq A$ are associative algebras rather than just rings).

We now turn to the following characterization of corner algebras (see also [48, Proposition 2.1] or 2.1.2).

Proposition 3.1.5 (Characterization)**.** Let A be an algebra, and let $S \subseteq A$ be a subalgebra of A. Then the following are equivalent:

1. S is a corner algebra of A.

2. There exists a linear S-bimodule map $\mathcal{E}\colon A \to A$ (which means that $\mathcal{E}$ is both left and right S-homogeneous) such that $\mathcal{E}(A) = S$ and $\mathcal{E} \circ \mathcal{E} = \mathcal{E}$. (One may view $\mathcal{E}$ as an 'algebraic' conditional expectation from A onto S [9, II.6.10] since $\mathcal{E}$ is not assumed to be continuous even if A has a topology. We will refer to $\mathcal{E}$ as a *Lam conditional expectation.*)

3. There exists a linear S-bimodule map $\kappa\colon A \to A$ such that $\ker \kappa = S$, $\kappa \circ \kappa = \kappa$, and the following short sequence of S-bimodule maps

$$0 \longrightarrow S \xrightarrow{\iota \text{ inclusion}} A \xrightarrow{\kappa} \kappa(A) \longrightarrow 0$$

   is exact.

Proof. $1 \leftrightarrow 2$: This is Lam's theorem [48, Proposition 2.1].
$2 \to 3$: If $\mathcal{E}$ exists, then $\kappa = 1 - \mathcal{E}$ has the required properties.

$3 \to 1$: If κ exists, then $A = S \oplus M$ for $M = \kappa(A)$. Moreover $SM \subseteq M$ and $MS \subseteq M$ since κ is an S-bimodule map. Thus, S is a corner of A. □

Definition 3.1.6. If A is an algebra we define a *Lam conditional expectation* $\mathcal{E}\colon A \to A$ to be a linear map with range $S = \mathcal{E}(A)$ a subalgebra such that $\mathcal{E} \circ \mathcal{E} = \mathcal{E}$ and $\mathcal{E}$ is an S-bimodule map. If A has an involution * we call $\mathcal{E}$ a *Lam *-conditional expectation* if $\mathcal{E}(x^*) = \mathcal{E}(x)^*$ holds for $x \in A$.

We finish this section by pointing out that obviously not every subalgebra S of an algebra A is a corner of A. Equivalently – by Proposition 3.1.5 – there is not generally a Lam conditional expectation from A onto S even if A and S are C^*-algebras. For example, let $X = \mathbb{Z} \cup \{\pm\infty\}$. Then $A = C(X)$ is the C^*-algebra of sequences $(x_n)_{n\in\mathbb{Z}}$ such that $\lim_{n\to+\infty} x_n$ and $\lim_{n\to-\infty} x_n$ exist. Let S be the C^*-subalgebra of sequences $(x_n)_{n\in\mathbb{Z}} \in A$ such that $\lim_{n\to+\infty} x_n = \lim_{n\to-\infty} x_n$. Then S is not a corner of A. To see this, assume otherwise, and choose some complement $M \subseteq A$ so that $A = S \oplus M$ and $SM \subseteq M$, $MS \subseteq M$. Now, if $f \in M$ and $f \neq 0$, there is $m \in \mathbb{Z}$ with $f(m) \neq 0$. But $\chi_{\{m\}} \in S$ (here $\chi_{\{m\}}$ is the characteristic function which, for every $n \in X$, has value 1 if $n = m$ and 0 if $n \neq m$) and $\chi_{\{m\}} f = f(m)\chi_{\{m\}} \in S \cap M = \{0\}$. Since this is a contradiction, S is not a corner of A. In particular, there is no conditional expectation (in the sense of [9, II.6.10.1]) from A onto S (see [9, p. 135]).

3.2 Closed and Non-closed Complements

Let A be an infinite-dimensional unital complex Banach algebra with an unit 1. It is well known that there are discontinuous linear functionals on A (in fact on any infinite-dimensional space normed space). We will discuss briefly some methods for manufacturing discontinuous linear functionals which do not vanish at 1.

To construct a discontinuous linear functional from A to $\mathbb{C}$ which takes a non-zero value at 1, consider $B \subseteq A$, a Hamel basis of A containing 1. Since A is complete, the Baire category theorem implies that B is uncountable – has at least the cardinality of the continuum. Choose an infinite sequence of distinct

elements of B, say $b_n \in B$ $(n = 1, 2, \dots)$, with $b_1 = 1$. Define $\lambda\colon B \to \mathbb{C}$ by $\lambda(b_n) = n\,\|b_n\|$, and $\lambda(b) = 0$ for $b \in B \setminus \{b_1, b_2, \dots\}$. Extend λ to a linear transformation $\lambda\colon A \to \mathbb{C}$ (which is denoted by the same symbol). Then λ is unbounded (because $\frac{\lambda(b_n)}{\|b_n\|} = n \to \infty$), and so λ is not continuous. Moreover, we have $\lambda(1) \neq 0$.

Another way to construct a discontinuous linear functional $\lambda\colon A \to \mathbb{C}$ with $\lambda(1) \neq 0$ would be to start with a discontinuous linear functional $\kappa\colon A \to \mathbb{C}$ in the first place. We could then take $\lambda = \frac{1}{\kappa(1)}\kappa$ if $\kappa(1) \neq 0$. Otherwise, if $\kappa(1) = 0$, by the Hahn-Banach Theorem there exists a continuous linear functional $\kappa_0\colon A \to \mathbb{C}$ with $\kappa_0(1) = 1$ (indeed, extend the bounded linear map $\mathbb{C}1 \ni z1 \mapsto z \in \mathbb{C}$ to κ_0). Take $\lambda = \kappa_0 + \kappa$. Then λ is not continuous since κ is not continuous, and $\lambda(1) = 1$.

Now, to show that there is a corner S in A which has a not closed complement and also a closed complement, let $\lambda\colon A \to \mathbb{C}$ be a discontinuous linear functional with $\lambda(1) \neq 0$. Set $S = \mathbb{C}1$, $M = \ker\lambda$. Then $A = S \oplus M$, S is a corner in A, and M is a not closed complement of S. On the other hand, by the Hahn-Banach theorem, there is a continuous functional $\psi\colon A \to \mathbb{C}$ with $\psi(1) = 1$, and so the same corner, $S = \mathbb{C}1$, has another complement – namely $\ker\psi$ – which is closed.

Corollary 3.2.1. Let A be an infinite-dimensional unital Banach algebra with an unit 1. Then there is a corner S in A (namely $S = \mathbb{C}1$) which has a closed complement and also a not closed complement.

Remark 3.2.2. If A is an infinite dimensional unital C^*-algebra, the proof of Corollary 3.2.1 provides examples of Lam conditional expectations $\mathcal{E}\colon A \to A$ with $\mathcal{E}(A) = \mathbb{C}1$ and $\mathcal{E}$ not positive. (Every positive linear map between C^*-algebras must be bounded; cf. [49, Lemma 5.1].)

It is maybe worthwhile to reformulate the above corollary in therms of Lam conditional expectations: It states that if $\mathcal{E}\colon A \to A$ is a Lam conditional expectation (not necessarily continuous) from A onto $\mathcal{E}(A) = \mathbb{C}1$, then there exists another Lam conditional expectation $\bar{\mathcal{E}}\colon A \to A$ with the same range ($\bar{\mathcal{E}}(A) = \mathcal{E}(A) = \mathbb{C}1$) which is continuous. Thus in the set of all Lam conditional expectations from A onto $S = \mathbb{C}1$, there always exists a continuous one. In other words, in the set of

all complements of $S = \mathbb{C}1$ in A, one can always find a closed complement (but there are also not closed complements).

In particular, if A is a unital C^*-algebra with an unit 1, we can choose a discontinuous Lam conditional expectation $\mathcal{E}\colon A \to A$ from A onto $S = \mathbb{C}1$. Moreover, as the example below shows, $\mathcal{E}$ can be chosen to be self-adjoint ($\mathcal{E}(x^*) = \mathcal{E}(x)^*$ for $x \in A$). This is in contrast with self-adjoint algebra homomorphisms between C^*-algebras which are always continuous.

Example 3.2.3. Let A be an infinite dimensional unital C^*-algebra, and let A_{sa} be the space of self-adjoint elements of A. Then A_{sa} is an infinite dimensional real Banach space containing the identity element. Choose an unbounded (discontinuous) real-valued linear functional $f\colon A_{sa} \to \mathbb{R}$ such that $f(1) = 1$. Extend f to $g\colon A \to \mathbb{R}$ by $g(a + ib) = f(a)$ for $a, b \in A_{sa}$. Consider the complex-valued linear functional $h\colon A \to \mathbb{C}$ defined by $h(x) = g(x) - ig(ix)$ for $x \in A$. Then h is $\mathbb{C}$-linear, h is not continuous since $h_{|A_{sa}} = f$, and $h(x^*) = \overline{h(x)}$ for $x \in A$. Define $\mathcal{E}\colon A \to A$ by $\mathcal{E}(x) = h(x)1$ for $x \in A$. Then $\mathcal{E}$ is a self-adjoint Lam conditional expectation which is not continuous (and has range $\mathbb{C}1$).

In §5.5 we will discuss further Lam conditional expectations which are not continuous; in particular we will propose a general method of manufacturing such maps. The general idea behind our proposal is described in §2.4.

Remark 3.2.4. If A is a unital C^*-algebra, and $S = \mathbb{C}1$, a conditional expectation of A onto S in the sense of [9, II.6.10.1] is just $\mathcal{E}(x) = \psi(x)1$ with ψ a state on A ([9, II.6.10.4 (ii)]).

3.3 Unitalization

If A is a non-unital algebra, we set $A^\sharp = A \oplus \mathbb{C}$. This is a vector space with the ordinary coordinate-wise vector addition, and a unital algebra with the multiplication defined as follows:

$$(a, \alpha)(b, \beta) = (ab + \beta a + \alpha b, \alpha\beta).$$

The element $(0,1)$ is a unit for this multiplication, and the algebra $A^\sharp$ is called the *unitization* of A. We often write $a+\alpha 1$ for (a,α).

If $\|\ \|$ is a norm on A, we make $A^\sharp$ into a normed algebra by setting

$$\|(a,\alpha)\| = \|a\| + |\alpha|\,. \tag{3.2}$$

Then A can be identified with the closed ideal of codimension one $\{(a,0) : a \in A\}$ in $A^\sharp$ via the isometric isomorphism $a \mapsto (a,0)$, and $A^\sharp/A \cong \mathbb{C}$.

If A is a Banach *-algebra, we make $A^\sharp$ into a (unital) Banach *-algebra with the norm (3.2) by setting $(a,\alpha)^* = a^* + \bar{\alpha}$. With the norm (3.2) $A^\sharp$ is not a C^*-algebra; but if A is a C^*-algebra the operator norm on $A^\sharp$ as left multiplication operators on A, i.e.

$$\|(a,\alpha)\| = \sup\{\|ax+\alpha x\| : x \in A, \quad \|x\| \le 1\}\,, \tag{3.3}$$

is an equivalent C^*-norm on $A^\sharp$ (cf. [9, II.1.2.1]). Then A can be identified with the closed ideal of codimension one $\{(a,0) : a \in A\}$ in $A^\sharp$ via the isometric isomorphism $a \mapsto (a,0)$, and $A^\sharp/A \cong \mathbb{C}$.

Now, we will show that a Lam conditional expectation $\mathcal{E}$ on A extends to a Lam conditional expectation $\mathcal{E}^\sharp$ on $A^\sharp$.

Proposition 3.3.1. Assume S is a corner in a non-unital algebra A, and let $\mathcal{E}\colon A \to A$ be a Lam conditional expectation with $\mathcal{E}(A) = S$. Define $\mathcal{E}^\sharp$ as follows:

$$\mathcal{E}^\sharp\colon A^\sharp \to A^\sharp, \qquad \mathcal{E}^\sharp(a+\alpha 1) = \mathcal{E}(a) + \alpha 1.$$

Then $\mathcal{E}^\sharp$ is a Lam conditional expectation with $\mathcal{E}^\sharp(A^\sharp) = S^\sharp = S \oplus \mathbb{C}$. Moreover, if A is a *-algebra and $\mathcal{E}(a^*) = \mathcal{E}(a)^*$ for $a \in A$, then also $\mathcal{E}^\sharp(x^*) = \mathcal{E}^\sharp(x)^*$ for $x \in A^\sharp$. If $\mathcal{E}$ is multiplicative (equivalently, by Proposition 2.4.2, if $\ker\mathcal{E}$ is an ideal in A) then $\mathcal{E}^\sharp$ is also multiplicative. (Compare this proposition with Fact 2.1.6.)

Proof. $\mathcal{E}^\sharp$ is well defined and linear. If $a + \alpha 1$ is an element of $A^\sharp$, then

$$\begin{aligned}
\mathcal{E}^\sharp \circ \mathcal{E}^\sharp(a + \alpha 1) &= \mathcal{E}^\sharp(\mathcal{E}^\sharp(a + \alpha 1)) \\
&= \mathcal{E}^\sharp(\mathcal{E}(a) + \alpha 1) \\
&= \mathcal{E}((\mathcal{E}(a)) + \alpha 1 \\
&= \mathcal{E}(a) + \alpha 1 \\
&= \mathcal{E}^\sharp(a + \alpha 1),
\end{aligned}$$

thus $\mathcal{E}^\sharp \circ \mathcal{E}^\sharp = \mathcal{E}^\sharp$.

If $s + \lambda 1$ is an element of $S^\sharp = S \oplus \mathbb{C}$, and $a + \alpha 1$ an element of $A^\sharp$, then

$$\begin{aligned}
\mathcal{E}^\sharp((s + \lambda 1)(a + \alpha 1)) &= \mathcal{E}^\sharp((sa + \alpha s + \lambda a) + (\lambda\alpha 1)) \\
&= \mathcal{E}(sa + \alpha s + \lambda a) + \lambda\alpha 1 \\
&= (s\mathcal{E}(a) + \alpha s + \lambda\mathcal{E}(a)) + \lambda\alpha 1 \\
&= (s + \lambda 1)(\mathcal{E}(a) + \alpha 1) \\
&= (s + \lambda 1)\mathcal{E}^\sharp(a + \alpha 1),
\end{aligned}$$

thus $\mathcal{E}^\sharp$ is a left $S^\sharp$-module map. Similarly, it can be shown that $\mathcal{E}^\sharp$ is a right $S^\sharp$-module map.

If $\mathcal{E}$ is self-adjoint and $x = a + \alpha 1$ is an element of $A^\sharp$ then

$$\begin{aligned}
\mathcal{E}^\sharp(x^*) &= \mathcal{E}^\sharp(a^* + \bar{\alpha} 1) \\
&= \mathcal{E}(a^*) + \bar{\alpha} 1 \\
&= (\mathcal{E}(a) + \alpha 1)^* \\
&= \mathcal{E}^\sharp(a + \alpha 1)^* \\
&= \mathcal{E}^\sharp(x)^*,
\end{aligned}$$

thus $\mathcal{E}^\sharp$ is self-adjoint.

Finally, if $\mathcal{E}$ is multiplicative, $a + \alpha 1$ and $b + \beta 1$ are elements of $A^\sharp$, then

$$\begin{aligned}
\mathcal{E}^\sharp((a+\alpha 1)(b+\beta 1)) &= \mathcal{E}^\sharp((ab+\beta a+\alpha b)+(\alpha\beta 1)) \\
&= \mathcal{E}(ab+\beta a+\alpha b)+\alpha\beta 1 \\
&= \mathcal{E}(ab)+\beta\mathcal{E}(a)+\alpha\mathcal{E}(b)+\alpha\beta 1 \\
&= \mathcal{E}(a)\mathcal{E}(b)+\beta\mathcal{E}(a)+\alpha\mathcal{E}(b)+\alpha\beta 1 \\
&= (\mathcal{E}(a)+\alpha 1)(\mathcal{E}(b)+\beta 1) \\
&= \mathcal{E}^\sharp(a+\alpha 1)\mathcal{E}^\sharp(b+\beta 1),
\end{aligned}$$

thus $\mathcal{E}^\sharp$ is multiplicative. □

Remark 3.3.2. With the notation of Proposition 3.3.1, it can be proved that if $\mathcal{E}\colon A \to A$ is a Lam conditional expectation and a positive contraction, then $\mathcal{E}^\sharp\colon A^\sharp \to A^\sharp$ is positive. The proof is essentially the same as the proof of [9, II.6.9.5].

Example 3.3.3. If H is an infinite-dimensional separable Hilbert space, the algebra of compact operators $\mathcal{K}(H)$ is not a corner of $\mathcal{B}(H)$ because it is not unital. But if $A = \mathcal{K}(H) \oplus \mathbb{C}1_H$ then $\mathbb{C}1_H$ is a corner of A complemented by an ideal $\mathcal{K}(H)$ (here 1_H denotes the identity operator on H).

3.4 Invertibility of $1 + \mathcal{E}(x^*x)$

Let A be a C^*-algebra with unit 1.

Recall that an element x in A is said to be *positive* if $x = x^*$ and $\sigma(x) \subseteq [0, \infty)$, where $\sigma(x)$ is the spectrum of x. We write $x \geq 0$. Recall that $x \geq 0 \Leftrightarrow x = y^*y$ for some $y \in A$, and also $x \geq 0$ implies $1 + x^*x$ is invertible.

We will prove the following positivity criterion.

Proposition 3.4.1. If $\mathcal{E}\colon A \to A$ is a Lam *-conditional expectation on a C^*-algebra A (it is enough that $\mathcal{E}$ is linear and a $*$-homomorphism), then the following are equivalent:

(i) $\mathcal{E}(x^*x) \geq 0$ for every $x \in A$;

(ii) $1 + \mathcal{E}(x^*x)$ is invertible in A for every $x \in A$,

are equivalent.

Proof. If $\mathcal{E}(x^*x) \geq 0$ for every $x \in A$, then $1 + \mathcal{E}(x^*x)$ is invertible in A for every $x \in A$ since A is a C^*-algebra.

Conversely, assume that $1 + \mathcal{E}(x^*x)$ is invertible in A for every $x \in A$. Then $\sigma(\mathcal{E}(x^*x)) \subseteq \mathbb{R}$ since $\mathcal{E}$ is self-adjoint. More specifically, for any $\lambda > 0$ we have:

$$\begin{aligned} \lambda 1 + \mathcal{E}(x^*x) &= \lambda[1 + \frac{1}{\lambda}\mathcal{E}(x^*x)] \\ &= \lambda[1 + \mathcal{E}((\frac{x}{\sqrt{\lambda}})^*(\frac{x}{\sqrt{\lambda}}))]. \end{aligned}$$

Thus we see that $\lambda 1 + \mathcal{E}(x^*x)$ is invertible in A. Consequently $-\lambda \notin \sigma(\mathcal{E}(x^*x))$. Therefore $\sigma(\mathcal{E}(x^*x)) \subseteq [0, \infty)$. This shows that $\mathcal{E}(x^*x) \geq 0$ for every $x \in A$. □

3.5 Dense Corners are Unital

Let A be an algebra with identity element 1. It is known ([48, Proposition 2.2], also Proposition 2.1.3) that every corner of A has its own identity – possibly different from 1.

Recall that a corner S of A is referred to as unital if $1 \in S$. We show that any dense corner of A is unital if A is normed.

Proposition 3.5.1. Let A be a normed algebra with identity element 1. If S is a dense corner of A, then S is unital ($1 \in S$).

Proof. Choose a complement M of S. Then $A = S \oplus M$ and $1 = e + f$ for unique $e \in S$, $f \in M$, with e being an identity of S. The algebra eAe is closed because if $x \in \overline{eAe}$ and a sequence $(x_n) \subseteq eAe$ converges to x, then $x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} (ex_ne) = exe$, so x in eAe (because for all x in A, $x \in eAe \Leftrightarrow x = exe$). Since eAe is a closed space containing S, and $\overline{S}$ is the smallest closed space containing S,

we have $S \subseteq \overline{S} \subseteq eAe \subseteq A$. Using $\overline{S} = A$ yields first $eAe = A$, and then $e = 1 \in S$ because e is also an identity for eAe and the identity element of a unital algebra is unique. This completes the proof of the proposition. □

3.6 On the Self-adjointness of Corners

A corner of a C^*-algebra need not be self-adjoint. For instance, if we let A be the C^*-algebra of 2-by-2 matrices $M_2(\mathbb{C})$, then for $e = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ we have $e = e^2$ and $eAe = \left\{\left(\begin{smallmatrix} \alpha & \alpha \\ 0 & 0 \end{smallmatrix}\right) : \alpha \in \mathbb{C}\right\}$ is a Peirce corner of A. However eAe is not self-adjoint since, for example, $e \in eAe$ but $e^* \notin eAe$. Nonetheless, e is similar or conjugate to $p = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ (since $ueu^{-1} = p$ for $u = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$) and $pAp = \left\{\left(\begin{smallmatrix} \alpha & 0 \\ 0 & 0 \end{smallmatrix}\right) : \alpha \in \mathbb{C}\right\}$ is a self-adjoint Peirce corner of A. Thus, although the corner eAe is not self-adjoint itself, there is a sense in which eAe is still 'similar' to a self-adjoint corner of A.

This observation can be generalized, and the following result can be reached.

If H is a Hilbert space and $e \in \mathcal{B}(H)$ has $e^2 = e$, then H is the (non-orthogonal) direct sum $H = eH \oplus (1-e)H$ and eH is the 1-eigenspace of e, $(1-e)H$ is the kernel of e (the 0-eigenspace). It follows, since eH and $(1-e)H$ are closed, that the continuous vector space isomorphism $u\colon K = eH \oplus_2 (1-e)H \to H$ (from the Hilbert space direct sum $eH \oplus_2 (1-e)H$ to H) is an isomorphism. Then $p = u^{-1}eu \in \mathcal{B}(K)$ is an orthogonal self-adjoint projection similar (or conjugate) to e and the isomorphism $\theta\colon \mathcal{B}(H) \to \mathcal{B}(K)$ maps $e\mathcal{B}(H)e$ to the $*$-subalgebra $p\mathcal{B}(K)p$ of $\mathcal{B}(K)$ which is a self-adjoint Peirce corner of $\mathcal{B}(K)$.

Remark/question. If A is an abstract C^*-algebra and $e \in A$ has $e^2 = e$, can we similarly argue that eAe is a C^*-algebra under some equivalent norm and some modified involution? Note that if A is a commutative C^*-algebra, then A must be ($*$-isomorphic to) $C_0(X)$ for some locally compact Hausdorff space X. Notice that if $e \in C_0(X)$ has $e^2 = e$, then $e(x) \in \{0, 1\}$ for each $x \in X$ and so $e^* = e$ holds automatically. In general we have the following theorem of Kaplansky [44, p. 20], which answers this question in the affirmative.

Theorem 3.6.1 (Kaplansky)**.** Let A be a unital C^*-algebra. Any idempotent $e \in A$ is similar to a self-adjoint projection $p \in A$.

This result is known, but since we refer to some of the constructions below (in the proof of Corollary 3.6.2) we prefer to give a detailed proof.

Proof of Theorem 3.6.1. Set $z = 1 + (e^* - e)^*(e^* - e)$. (A unital $*$-ring R is called *symmetric* if $1 + r^*r$ is invertible in R for all $r \in R$. Any unital C^*-algebra is symmetric.) Let $t \in A$ be an inverse of z, that is $tz = zt = 1$. First, note that since e is an idempotent ($e^2 = e$), we have $(e^*)^2 = (e^2)^* = e^*$, and so e^* is also an idempotent. Therefore:

$$\begin{aligned} z &= 1 + (e^* - e)^*(e^* - e) \\ &= 1 + ee^* - e - e^* + e^*e, \end{aligned}$$

and consequently:

$$\begin{aligned} ez &= e(1 + ee^* - e - e^* + e^*e) \\ &= e + ee^* - e - ee^* + ee^*e \\ &= ee^*e, \end{aligned} \qquad \begin{aligned} ze &= (1 + ee^* - e - e^* + e^*e)e \\ &= e + ee^*e - e - e^*e + e^*e, \\ &= ee^*e, \end{aligned}$$

thus $ez = ze = ee^*e$.

Second, since $z^* = z$, also $t^* = t$. Since $ez = ze$, it follows that $et = 1et = tzet = tezt = te1 = te$, so we also have $et = te$. Taking $*$ of both sides of the former equation yields $te^* = e^*t$.

Third, set $p = ee^*t$. Then $p^* = (ee^*t)^* = tee^* = ete^* = ee^*t = p$, and

$$\begin{aligned} p^2 &= e(e^*t)ee^*t \\ &= e(te^*)ee^*t \\ &= t(ee^*e)e^*t \\ &= t(ze)e^*t \\ &= 1ee^*t \\ &= p, \end{aligned}$$

so p is a self-adjoint projection. Note further that $p = ep$ and $pe = epe$. But since

$epe = eee^*te = ete^*e = tee^*e = tze = e$, we have $pe = e$.

Finally, set $u = e + (1 - p)$. Then $u^{-1} = p + (1 - e)$ since $u(p + (1 - e)) = (p + (1 - e))u = 1$, and $ueu^{-1} = p$. Thus e is similar to p. □

Theorem 3.6.1 and constructions in the proof above lead us the following results (Corollary 3.6.2 and Corollary 3.6.3).

Corollary 3.6.2. If A is a C^*-algebra and $e \in A$ is an idempotent, there there is an algebra automorphism $\phi\colon A \to A$ such that $\phi(eAe) = pAp$ for a projection $p = p^* = p^2 \in A$.

Proof. This follows from the proof above of Kaplansky's Theorem 3.6.1 in the case where A is unital, because we can take ϕ to be given by $\phi(x) = uxu^{-1}$ with u as in the proof above.

In the case when A is not unital we consider the unitization $A^\sharp$, which we recall contains A as an ideal (Section 3.3). Then $eAe = eA^\sharp e$ by the ideal property and there is an algebra automorphism $\Phi\colon A^\sharp \to A^\sharp$ given by a conjugation $\Phi(x) = uxu^{-1}$ (with $u \in A^\sharp$) such that $\Phi(e) = p = p^* = p^2$. By the ideal property $\Phi(A) \subseteq A$ and also $\Phi^{-1}(A) \subset A$ since $\Phi^{-1}(x) = u^{-1}xu$. Thus $\phi = \Phi\,|_A$ defines an algebra automorphism of A with $\phi(e) = p$ a projection, $\phi(eAe) = pAp$. □

Corollary 3.6.3. If A is a C^*-algebra, $n \in \mathbb{N}$ and $e_1, e_2, \ldots, e_n \in A$ are idempotents such that $e_ie_j = 0$ for $i \neq j$, there there is an algebra automorphism $\phi\colon A \to A$ and orthogonal projections $p_1, p_2, \ldots, p_n \in A$ such that $\phi(e_i) = p_i$ $(1 \leq i \leq n)$ and $\phi\left(\bigoplus_{i=1}^{n} e_iAe_i\right) = \bigoplus_{i=1}^{n} p_iAp_i$.

Hence each generalised Peirce corner $\bigoplus_{i=1}^{n} e_iAe_i$ of A is isomorphic (or "similar") as a Banach algebra to a (unital) C^*-algebra $\bigoplus_{i=1}^{n} p_iAp_i$.

Proof. Assume first that A itself has an identity. We show by induction on n that there is $u \in A$ invertible so that $ue_iu^{-1} = p_i = p_i^* = p_i^2$ is a projection for $1 \leq i \leq n$.

By Kaplansky's Theorem 3.6.1 there is $u_1 \in A$ invertible with

$$u_1 e_1 u_1^{-1} = p_1 = p_1^* = p_1^2$$

a projection. This is the case $n = 1$. For $n > 1$ we have for $2 \leq i \leq n$, $f_i = u_1 e_i u_1^{-1}$ idempotents in the unital C^* subalgebra $(1 - p_1)A(1 - p_1)$. Moreover $f_i f_j = 0$ for $i \neq j$. By the inductive hypothesis there is $v_2 \in (1 - p_1)A(1 - p_1)$ invertible in the subalgebra with $v_2 f_i w_2 = p_i$ projections for $2 \leq i \leq n$, where w_2 denotes the inverse of $v_2 \in (1 - p_1)A(1 - p_1)$ (so that $v_2 w_2 = w_2 v_2 = 1 - p_1$). Then $u_2 = p_1 + v_2$ is invertible in A (with inverse $u_2^{-1} = p_1 + w_2$) and $u = u_2 u_1$ satisfies $u e_1 u^{-1} = u_2 u_1 e_1 u_1^{-1} u_2^{-1} = u_2 p_1 u_2^{-1} = p_1$ and

$$u e_i u^{-1} = u_2 u_1 e_i u_1^{-1} u_2^{-1} = u_2 f_i u_2^{-1} = v_2 f_i w_2 = p_i \quad (2 \leq i \leq n).$$

Thus, in the unital case we have the automorphism $\phi\colon A \to A$ given by $\phi(x) = uxu^{-1}$ and the conclusions follow.

For the general case of non-unital A, we can find $u \in A^\sharp$ (the unitization; Section 3.3) and restrict the conjugation automorphism of $A^\sharp$ to get the desired automorphism of A (as in the proof of Corollary 3.6.2). □

Remark 3.6.4. It is important to point out that there is a different version of Theorem 3.6.1 (see [12, Lemma 3.2]). It states that if H is a Hilbert space, $F \subseteq \mathcal{B}(H)$ a set of mutually commuting idempotents such that $e + f - ef \in F$ for all $e, f \in F$ and $\|2e - 1\| \leq C$ for all $e \in F$, then there exists $r \in \mathcal{B}(H)$, positive and invertible with $\|r\| \, \|r^{-1}\| \leq C^2$, such that rer^{-1} is self-adjoint for each $e \in F$. Moreover, r lies in the von Neumann algebra generated by 1 and F.

Remark 3.6.5. In Chapter 5, we will show that self-adjoint corners in commutative C^*-algebras are necessarily closed, and that although complements need not be unique in general, there exists a norm one (and positive) conditional expectation onto each such corner.

In Chapter 5, we will show that assuming CH (Continuum Hypotheses) there are corners in noncommutative C^*-algebras that fail to be closed (and are hence

not isomorphic to C^*-algebras).

3.7 Symmetrised Lam Conditional Expectations

Let A be a $*$-algebra, and S a self-adjoint corner of A ($S^* = S$). From Proposition 3.1.5 it follows that there exists a Lam conditional expectation

$$\mathcal{E}\colon A \to A \tag{3.4}$$

from A onto S with $S = \mathcal{E}(A)$. However $\mathcal{E}$ itself need not be self-adjoint, that is $\mathcal{E}$ need not be a Lam *-conditional expectation, $\mathcal{E}(x^*) = \mathcal{E}(x)^*$ need not necessarily hold for all $x \in A$ (see the construction in § 3.2).

We will show that there exists another Lam conditional expectation $\boldsymbol{\delta}\colon A \to A$ from A onto S (with the same image as $\mathcal{E}$; $\boldsymbol{\delta}(A) = \mathcal{E}(A) = S$) which is self-adjoint, and moreover $\ker \boldsymbol{\delta} = \{x \in A : \boldsymbol{\delta}(x) = 0\}$ is $*$-closed (but $\ker \mathcal{E}$ and $\ker \boldsymbol{\delta}$ may be different).

First, define

$$\mathcal{E}^*\colon A \to A \tag{3.5}$$

by $\mathcal{E}^*(x) = [\mathcal{E}(x^*)]^*$ for $x \in A$. Note the following:

1. $\mathcal{E}^*$ is an S-bimodule map. Indeed, for $s \in S$, $x \in A$, we have:

$$\begin{aligned} \mathcal{E}^*(sx) &= \mathcal{E}((sx)^*)^* \\ &= \mathcal{E}(x^*s^*)^* \\ &= \big(\mathcal{E}(x^*)s^*\big)^* \\ &= s\mathcal{E}(x^*)^* \\ &= s\mathcal{E}^*(x), \end{aligned} \qquad \begin{aligned} \mathcal{E}^*(xs) &= \mathcal{E}((xs)^*)^* \\ &= \mathcal{E}(s^*x^*)^* \\ &= \big(s^*\mathcal{E}(x^*)\big)^* \\ &= \mathcal{E}(x^*)^*s \\ &= \mathcal{E}^*(x)s. \end{aligned}$$

2. $\mathcal{E}^* \circ \mathcal{E}^* = \mathcal{E}^*$. Indeed, for $x \in A$, we have:

$$\begin{aligned} \mathcal{E}^* \circ \mathcal{E}^*(x) &= \mathcal{E}^*(\mathcal{E}(x^*)^*) \\ &= \mathcal{E}(\mathcal{E}(x^*))^* \\ &= \mathcal{E}(x^*)^* \\ &= \mathcal{E}^*(x). \end{aligned}$$

3. $\mathcal{E}^*(A) = S$, and $\mathcal{E}^*(s) = s$ for $s \in S$. Indeed, if $s \in S$ then $\mathcal{E}^*(s) = \mathcal{E}(s^*)^* = (s^*)^* = s$, so $s \in \mathcal{E}^*(A)$. For the converse, if $b \in \mathcal{E}^*(A)$ then $b = \mathcal{E}^*(a)$ for some $a \in A$, so $b = \mathcal{E}(a^*)^* \in S$ since S a $*$-subalgebra of A.

4. If $\mathcal{E}$ is multiplicative, so is $\mathcal{E}^*$. Indeed, for $x, y \in A$ we have:

$$\begin{aligned} \mathcal{E}^*(xy) &= \mathcal{E}((xy)^*)^* \\ &= \mathcal{E}(y^*x^*)^* \\ &= \big(\mathcal{E}(y^*)\mathcal{E}(x^*)\big)^* \\ &= \mathcal{E}(x^*)^*\mathcal{E}(y^*)^* \\ &= \mathcal{E}^*(x)\mathcal{E}^*(y). \end{aligned}$$

5. $\mathcal{E} = \mathcal{E}^*$ if and only if $\mathcal{E}$ is self-adjoint (i.e. if and only if $\mathcal{E}(x^*) = \mathcal{E}(x)^*$ for all $x \in A$).

Second, define

$$\boldsymbol{\delta} \colon A \to A, \quad \boldsymbol{\delta} = \frac{\mathcal{E} + \mathcal{E}^*}{2}, \tag{3.6}$$

by $\boldsymbol{\delta}(x) = \frac{\mathcal{E}(x)+\mathcal{E}^*(x)}{2}$ for $x \in A$. Note the following:

1. $\boldsymbol{\delta}$ is an S-bimodule map.

2. $\boldsymbol{\delta} \circ \boldsymbol{\delta} = \boldsymbol{\delta}$. Indeed, for $x \in A$ we have:

$$\begin{aligned}
\boldsymbol{\delta} \circ \boldsymbol{\delta}(x) &= \boldsymbol{\delta}(\boldsymbol{\delta}(x)) \\
&= \boldsymbol{\delta}\left(\frac{1}{2}\Big(\mathcal{E}(x) + \mathcal{E}^*(x)\Big)\right) \\
&= \frac{1}{2}\boldsymbol{\delta}\Big(\mathcal{E}(x) + \mathcal{E}^*(x)\Big) \\
&= \frac{1}{2}\boldsymbol{\delta}\Big(\mathcal{E}(x)\Big) + \frac{1}{2}\boldsymbol{\delta}\Big(\mathcal{E}^*(x)\Big) \\
&= \frac{1}{2}\cdot\frac{1}{2}\left(\mathcal{E}\Big(\mathcal{E}(x)\Big) + \mathcal{E}^*\Big(\mathcal{E}(x)\Big)\right) + \frac{1}{2}\cdot\frac{1}{2}\left(\mathcal{E}\Big(\mathcal{E}^*(x)\Big) + \mathcal{E}^*\Big(\mathcal{E}^*(x)\Big)\right) \\
&= \frac{1}{4}\Big(\mathcal{E}(x) + \mathcal{E}(x)\Big) + \frac{1}{4}\Big(\mathcal{E}^*(x) + \mathcal{E}^*(x)\Big) \text{ (conclusion 3, page 38)} \\
&= \frac{1}{2}\Big(\mathcal{E}(x)\Big) + \frac{1}{2}\Big(\mathcal{E}^*(x)\Big) \\
&= \frac{1}{2}\Big(\mathcal{E}(x) + \mathcal{E}^*(x)\Big) \\
&= \boldsymbol{\delta}(x).
\end{aligned}$$

3. $\boldsymbol{\delta}(A) = S$, and $\boldsymbol{\delta}(s) = s$ for $s \in S$.

4. $\boldsymbol{\delta}$ is self-adjoint. Indeed, for $x \in A$ we have:

$$\begin{aligned}
\boldsymbol{\delta}(x^*) &= \frac{\mathcal{E}(x^*) + \mathcal{E}^*(x^*)}{2} \\
&= \frac{\mathcal{E}(x^*) + \mathcal{E}(x)^*}{2} \\
&= \left(\frac{\mathcal{E}(x) + \mathcal{E}(x^*)^*}{2}\right)^* \\
&= \left(\frac{\mathcal{E}(x) + \mathcal{E}^*(x)}{2}\right)^* \\
&= \boldsymbol{\delta}(x)^*.
\end{aligned}$$

5. $\ker \boldsymbol{\delta}$ is $*$-closed. Indeed, if $x \in \ker \boldsymbol{\delta}$, then $\boldsymbol{\delta}(x) = 0$ and:

$$\begin{aligned} \frac{\mathcal{E}(x) + \mathcal{E}^*(x)}{2} &= 0 \\ \frac{\mathcal{E}(x) + \mathcal{E}(x^*)^*}{2} &= 0 \\ \frac{\mathcal{E}(x^*) + \mathcal{E}(x)^*}{2} &= 0 \\ \frac{\mathcal{E}(x^*) + \mathcal{E}^*(x^*)}{2} &= 0 \\ \boldsymbol{\delta}(x^*) &= 0; \qquad \text{so } x^* \in \ker \boldsymbol{\delta}. \end{aligned}$$

We have shown:

Proposition 3.7.1. If S a self-adjoint corner of a $*$-algebra A, then there exists a Lam *-conditional expectation from A onto S (namely $\boldsymbol{\delta}\colon A \to A$ defined in (3.6)) whose kernel is $*$-closed.

Definition 3.7.2. The map $\boldsymbol{\delta}\colon A \to A$ defined in (3.6) is called the *symmetrised Lam conditional expectation* associated with $\mathcal{E}$.

Remark 3.7.3. Under the above assumptions, note that

$$\mathcal{E} = \frac{\mathcal{E} + \mathcal{E}^*}{2} + i\frac{\mathcal{E} - \mathcal{E}^*}{2i}$$

where $\mathcal{E}^*$ is defined as in (3.5). But $(\mathcal{E} - \mathcal{E}^*)/2i$ is not in general a conditional expectation.

Remark 3.7.4. Let H be a Hilbert space, $\mathcal{B}(H)$ the C^*-algebra of bounded operators on H. If $S \subseteq A \subseteq \mathcal{B}(H)$ are $*$-algebras, and $\mathcal{E}\colon A \to A$ is a Lam condition expectation from A onto S, then $\mathcal{E}^*$ defined in (3.5) can be viewed as an 'adjoint' of $\mathcal{E}$ since

$$\langle \mathcal{E}x, y\rangle = \langle x, \mathcal{E}^* y\rangle\,, \qquad \forall x, y \in A,$$

where $\langle \cdot, \cdot \rangle : A \times A \to S$ is defined by $\langle x, y\rangle = \mathcal{E}(y^* x)$.

3.8 Hereditary C^*-subalgebras

A C^*-subalgebra S of a C^*-algebra A is called *hereditary* (or A_+-*saturated*) if for all $0 \leq a \leq s$, where $a \in A$ and $s \in S$, we have that $a \in S$.

It is known that Peirce corners are hereditary C^*-subalgebras. In general, however, corners fail to be hereditary. For instance, the corner S containing precisely the constant complex-valued functions on a compact Hausdorff space X (with at least two points) is not hereditary in $C(X)$. The reason being that if $a \in C(X)$ is a positive non-constant function, then for $s \in C(X)$ given by $s(x) = \|a\|$, $x \in X$, we have that $0 \leq a \leq s$ and $s \in S$, but $a \notin S$.

The following proposition establishes a relationship between self-adjoint Peirce corners and hereditary subalgebras.

Proposition 3.8.1. Let S be a C^*-subalgebra of a C^*-algebra A.

1. If S is a self-adjoint Peirce corner of A (i.e., if there exists a self-adjoint projection $p \in A$ with $S = pAp$), then S is a hereditary C^*-subalgebra of A.

2. If S is a hereditary C^*-subalgebra with an identity element $e = 1_S$ of its own ($e \in S$), then S is a self-adjoint Peirce corner of A; specifically $S = eAe$.

Proof. 1. This is a well known fact (see, for example, [54, Example 3.2.1] or [68, page 274]).

2. Note that e^* is also an identity for S since $e^*s = (s^*e)^* = (s^*)^* = s$, and similarly $se^* = s$ (for $s \in S$). Therefore $e^* = e$ since identity elements are unique. (Alternatively, we may also note that $e^* \in S$, so $e^* = ee^* = (ee^*)^* = (e^*)^* = e$.) Using [54, Theorem 3.2.2] yields that $SAS \subseteq S$, so in particular $eAe \subseteq S$. Also $S = eSe \subseteq eAe$ because e is an identity element of S. Hence $S = eAe$ is a self-adjoint Peirce corner of A. □

Remark 3.8.2. A hereditary C^*-subalgebra without an identity element need not be a Peirce corner. For instance, the algebra of compact operators $\mathcal{K}(H)$ on an infinite dimensional Hilbert space H is hereditary in $\mathcal{B}(H)$, but $\mathcal{K}(H)$ is not a corner of $\mathcal{B}(H)$ because $\mathcal{K}(H)$ is not unital (see also [11, page 336]).

3.9 Semisimplicity of Corners in C^*-algebras

In this section we cite an important result from [46] which states that every corner of a *semisimple* unital ring is *semisimple*.

Let us first recall that a left (or right) module M over a ring R is called *simple* if M has no non-zero proper submodules; that is, the only submodules of M are 0 and M. A ring R is called *semisimple* if the only element $r \in R$ that satisfies $Mr = 0$ for each simple right R-module M is $r = 0$.

Theorem 3.9.1 ([46, Theorem 1])**.** Let A be a subring of a semisimple unital ring R such that $R = A \oplus V$ as an A-bimodule. Then A is also semisimple.

Since every C^*-algebra is semisimple [9, II.1.6.4], we get the following corollary:

Corollary 3.9.2. Every corner of a C^*-algebra is semisimple.

Remark 3.9.3. A semisimple subalgebra of a semisimple algebra need not be a corner. For example, If H an infinite dimensional Hilbert space, both $\mathcal{B}(H)$, the algebra of bounded operators on H, and $\mathcal{K}(H)$, the algebra of compact operators on H, are semisimple, but $\mathcal{K}(H)$ is not a corner of $\mathcal{B}(H)$.

3.10 Ideals and Some Application of Annihilators

Let A be a normed algebra. The *annihilator* of a non-empty set $S \subseteq A$ is defined as $S^\perp = \{a \in A : as = sa = 0 \text{ for all } s \in S\}$. It is a closed subspace of A (and a two-sided ideal if S is a two-sided ideal), and $S^\perp = (\overline{S})^\perp$, where $\overline{S}$ is the closure of S in A.

Proposition 3.10.1. Let $S \subseteq A$ be a two-sided *-ideal of a C^*-algebra A (not necessarily closed, but if S is closed, it is automatically a *-ideal). Then S is a corner if and only if $A = S \oplus S^\perp$.

Proof. Let M be a complement of S in A. Then $A = S \oplus M$ and $SM \subseteq M$, $MS \subseteq M$. Since S is a two-sided ideal, we also have that $SM \subseteq S$, $MS \subseteq S$. Thus $SM \subseteq S \cap M = \{0\}$ and $MS \subseteq S \cap M = \{0\}$, which means that $M \subseteq S^{\perp}$, and so $A = S + S^{\perp}$. If $s \in S \cap S^{\perp}$, then $s^* \in S$ (since S is *-closed) and so $ss^* = s^*s = 0$. Consequently we have the identities: $\|s\|^2 = \|s^*s\| = 0$, $s = 0$. This shows that $S \cap S^{\perp} = \{0\}$ and hence $A = S \oplus S^{\perp}$.

Conversely, if $A = S \oplus S^{\perp}$, then of course S is a corner of A. □

Corollary 3.10.2. If $S \subseteq A$ is a dense corner of a C^*-algebra A and a two-sided *-ideal, then $S = A$.

Proof. By Proposition 3.10.1, $A = S \oplus S^{\perp}$. But $S^{\perp} = (\overline{S})^{\perp} = A^{\perp} = \{0\}$. Thus $A = S \oplus \{0\}$, and so $S = A$. □

Proposition 3.10.3. If S is a corner of a normed algebra A, then S is also a corner of its own closure $\overline{S}$.

Proof. Choose a complement M of S. Then $A = S \oplus M$, and so for all $a \in \overline{S} \subseteq A$ we have $a = s + m$ for $s \in S$, $m \in M$. Also $m = a - s \in \overline{S}$. It follows that $\overline{S} = S + (M \cap \overline{S})$. Since $S \cap (M \cap \overline{S}) \subseteq S \cap M = \{0\}$, we obtain that $\overline{S} = S \oplus (M \cap \overline{S})$. It is readily checked that $S(M \cap \overline{S}) \subseteq (M \cap \overline{S})$ and $(M \cap \overline{S})S \subseteq (M \cap \overline{S})$. So $(M \cap \overline{S})$ is an (S, S)-bimodule, which means that S is a corner of $\overline{S}$. □

Corollary 3.10.4. Let S be a corner of a C^*-algebra A. If S is a two-sided *-ideal in its own closure $\overline{S}$, then S is closed and $S = \overline{S}$.

Proof. Proposition 3.10.3 implies that S is a corner of its own closure $\overline{S}$. Since S is a dense two-sided *-ideal in $\overline{S}$, we have that $S = \overline{S}$ by Corollary 3.10.2. □

Consider now a non-unital C^*-algebra A and the multiplier algebra $M(A)$ of A (see B. Blackadar's monograph [9, II.7.3]). Then $A \subsetneq M(A)$ since $M(A)$ is unital, and A is a two-sided essential *-ideal in $M(A)$. We remark, however, that A is not a corner of $M(A)$, as the opposite would imply that $M(A) = A \oplus A^{\perp}$, which is impossible since $A^{\perp} = \{0\}$ (provided $A \neq 0$). This also follows from the fact that A is not unital (Proposition 2.1.3). In particular, we have the following negative results.

1. Suppose X is a locally compact (non-compact) Hausdorff space, and let βX be the Čech-Stone compactification of X. Then the C^*-algebra $C_0(X)$ of continuous complex-valued functions on X vanishing at infinity is not a corner of $C(\beta X)$ since $M(C_0(X)) = C(\beta X)$. This is the prototype for the theory of multiplier algebras.

2. Suppose H is a separable infinite-dimensional Hilbert space. Then the C^*-algebra $\mathcal{K}(H)$ of compact operators on H is not a corner of $\mathcal{B}(H)$ since $M(\mathcal{K}(H)) = \mathcal{B}(H)$. In fact, for any infinite-dimensional Hilbert space H, $\mathcal{K}(H)$ is not a corner of $\mathcal{B}(H)$ since $\mathcal{B}(H) \neq \mathcal{K}(H) \oplus \mathcal{K}(H)^\perp = \mathcal{K}(H) \oplus 0$. This also follows from the fact that $\mathcal{K}(H)$ is not unital (Proposition 2.1.3). There is an important result from 1982 due to M. Feder [30] which states that $\mathcal{K}(H)$ is not complemented in $\mathcal{B}(H)$ by a bounded projection provided $\mathcal{K}(H) \subsetneq \mathcal{B}(H)$. However this is not applicable in our case.

3. As a special case of 1, consider $\ell^\infty = \ell^\infty(\mathbb{N})$, the C^*-algebra of all bounded sequences of complex numbers, and $c_0 = c_0(\mathbb{N})$, the C^*-subalgebra of all sequences whose limit is zero. Then again, Proposition 3.10.1 implies that c_0 is not a corner of ℓ^∞ since $(c_0)^\perp = 0$ and $c_0 \subsetneq \ell^\infty$ (or because c_0 is not unital). An important result from 1940 due to R. S. Phillips [59] (see also R. Whitley [74]) states that c_0 is non-complemented in ℓ^∞ by a bounded projection. However this is not applicable in our case.

The structure of two-sided weak*-closed ideals in von Neumann algebras is relatively simple. The two-sided weak*-closed ideals in a von Neumann algebra A are precisely the principal ideals Ap generated by central projections p. Specifically, if $p \in \mathcal{Z}(A) = \{a \in A : ax = xa \text{ for all } x \in A\}$ is a self-adjoint projection, then Ap is a weak*-closed two-sided ideal in A. Conversely, if I a weak*-closed two-sided ideal in A, then there exists a self-adjoint projection $p \in \mathcal{Z}(A)$ such that $I = Ap$ and

$$A = Ap \oplus A(1-p),$$

where $A(1-p)$ is the complementary ideal of Ap. It follows that $Ap = App = pAp$

is a weakly closed corner of A (in fact a Peirce corner).

Corollary 3.10.5. Every two-sided weak*-closed ideal in a von Neumann algebra is a corner.

Remark 3.10.6. A two-sided norm-closed (thus automatically $*$-closed) ideal in a von Neumann algebra need not be a corner – for instance $\mathcal{K}(H)$ is not a corner of the von Neumann algebra $\mathcal{B}(H)$ when H has infinite dimension.

3.11 Separating Spaces

Recall that if A and B are Banach spaces, and $\phi\colon A \to B$ a linear mapping, the *separating space* ([69, p. 3]) of ϕ is defined as

$$\mathfrak{S}(\phi) = \left\{ b \in B \,:\, \exists (a_n)_{n=1}^{\infty} \in A \text{ with } \lim_{n\to\infty} a_n = 0, \lim_{n\to\infty} \phi(a_n) = b \right\}.$$

By linearity, $\mathfrak{S}(\phi) = \{0\}$ if and only if ϕ has closed graph and by completeness, this is equivalent to continuity of ϕ. An important result in automatic continuity theory, known as the *stability lemma*, asserts that if $A_0 = A, A_1, A_2, \ldots$ is a sequence of Banach spaces and $\theta_i\colon A_i \to A_{i-1}$ is a continuous linear operator for $i = 1, 2, \ldots$, then $(\mathfrak{S}(\phi \circ \theta_1 \circ \cdots \circ \theta_n) \,:\, n \in \mathbb{N})$ is a nest[1] in B that stabilizes – i.e., there is a positive integer n_0 such that $\mathfrak{S}(\phi \circ \theta_1 \circ \cdots \circ \theta_{n_0}) = \mathfrak{S}(\phi \circ \theta_1 \circ \cdots \circ \theta_n)$ for all $n \geq n_0$. This is used to prove that all derivations from a semi-simple Banach algebra to itself are automatically continuous ([35]). Although the stability lemma seemed to be an interesting result, we have not been able to applay it to Lam conditional expectation.

We prove the following results.

Proposition 3.11.1. Let A be a Banach algebra and S a corner of A with a corresponding Lam conditional expectation $\mathcal{E}\colon A \to A$. Then $\mathfrak{S}(\mathcal{E})$ is a closed ideal of $\bar{S}$ (the closure of S).

[1] A nest in B is a set of subspaces of B which is totally ordered by inclusion and closed under arbitrary intersections and closed linear spans (it is a complete lattice).

Proof. From the definition of $\mathfrak{S}(\mathcal{E})$ we clearly have $\mathfrak{S}(\mathcal{E}) \subseteq \bar{S}$. It is known that $\mathfrak{S}(\mathcal{E})$ is a closed linear subspace of A ([69, Lemma 1.2], [18, Proposition 5.1.2]).

If $a \in \mathfrak{S}(\mathcal{E})$ and $s \in S$, let $(x_n)_{n=1}^{\infty}$ be a sequence in A with $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} \mathcal{E}(x_n) = a$. Observe that $\lim_{n\to\infty} sx_n = 0$ and $\lim_{n\to\infty} \mathcal{E}(sx_n) = \lim_{n\to\infty} s\mathcal{E}(x_n) = sa$ and so $sa \in \mathfrak{S}(\mathcal{E})$. Similarly $as \in \mathfrak{S}(\mathcal{E})$ and so $\mathfrak{S}(\mathcal{E})$ is a sub S-bimodule of A.

For $a \in \mathfrak{S}(\mathcal{E})$ and $b \in \bar{S}$, density of S implies there is a sequence $(s_n)_{n=1}^{\infty}$ in S with $\lim_{n\to\infty} s_n = b$. So then

$$ba = \lim_{n\to\infty} s_n a \in \mathfrak{S}(\mathcal{E})$$

since $\mathfrak{S}(\mathcal{E})$ is closed. Similarly $ab \in \mathfrak{S}(\mathcal{E})$ and so $\mathfrak{S}(\mathcal{E})$ is a (closed) ideal of $\bar{S}$. □

Proposition 3.11.2. For a Banach algebra A and a Lam conditional expectation $\mathcal{E}\colon A \to A$, let $\mathfrak{I}(\mathcal{E}) = \{a \in A : a\,\mathfrak{S}(\mathcal{E}) = \mathfrak{S}(\mathcal{E})a = \{0\}\}$ (the annihilator ideal for $\mathfrak{S}(\mathcal{E})$, known as the 'continuity ideal' of $\mathcal{E}$ [18, Definition 5.3.1]). Then,

$$b\mathcal{E}(a) = ba \text{ and } \mathcal{E}(a)b = ab \qquad (a \in \bar{S}, b \in \mathfrak{I}(\mathcal{E})).$$

Proof. Fix $b \in \mathfrak{I}(\mathcal{E})$ and consider the map $\mathcal{E}_b\colon A \to A$ given by $\mathcal{E}_b(a) = b\mathcal{E}(a)$. For the bounded left multiplication map $L_b\colon A \to A$, we have $L_b\,\mathfrak{S}(\mathcal{E}) = \{0\}$ and so by [69, Lemma 1.2], $\mathcal{E}_b = L_b\mathcal{E}$ is continuous. For $s \in S$ we have $\mathcal{E}_b(s) = b\mathcal{E}(s) = bs\mathcal{E}(1) = bs1 = bs$. By continuity of $\mathcal{E}_b$ and density of S in $\bar{S}$ we have $\mathcal{E}_b(a) = b\mathcal{E}(a) = ba$ for all $a \in \bar{S}$.

Similarly $\mathcal{E}(a)b = ab$ holds. □

Corollary 3.11.3. If S is a dense corner of a Banach algebra A and $\mathcal{E}\colon A \to A$ a corresponding Lam conditional expectation, then $\mathfrak{S}(\mathcal{E})$ is an ideal of A and

$$b\mathcal{E}(a) = ba \text{ and } \mathcal{E}(a)b = ab \qquad (a \in A, b \in \mathfrak{I}(\mathcal{E})).$$

3.12 Corners Containing Diagonals

Let $\mathcal{B}(H)$ be the algebra of bounded linear operators on a Hilbert space H with an orthonormal basis $(e_i)_{i\in I}$ (which may be countable or uncountable). For any $i \in I$, consider the *diagonal operator* $e_i \otimes e_i^* \in \mathcal{B}(H)$ defined by $(e_i \otimes e_i^*)(\xi) = \langle \xi, e_i \rangle e_i$ for $\xi \in H$, which is the orthogonal self-adjoint projection of H onto the one-dimensional subspace of H spanned by e_i. This terminology for such operators $e_i \otimes e_i^*$ recalls the notion of 'diagonal matrices' $e_{ii} \in M_I(\mathbb{C})$ with 1 on the (i,i) position and 0 elsewhere.

The objects of study in this section are self-adjoint corner algebras S of $\mathcal{B}(H)$ containing the diagonal operators $e_i \otimes e_i^*$. We may assume $\mathcal{E}\colon \mathcal{B}(H) \to \mathcal{B}(H)$ is a Lam *-conditional expectation with range S.

We will need the following result, which follows from Wedderburn's theorem.

Proposition 3.12.1 ([15, Proposition 5.2.6])**.** Any semisimple finite-dimensional algebra R over an algebraically closed field k is a direct product of full matrix rings over k.

Lemma 3.12.2. If $\mathcal{E}\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a Lam conditional expectation and $S = \mathcal{E}(M_n(\mathbb{C}))$ (the range of $\mathcal{E}$) contains the diagonal subalgebra, then S is a generalised Peirce corner and $\|\mathcal{E}\| = 1$.

Moreover there are orthogonal diagonal projections $p_1, \ldots, p_k$ such that $\mathcal{E}(x) = \sum_{j=1}^k p_j x p_j$.

Proof. Since $S = \mathcal{E}(M_n(\mathbb{C}))$ is a finite dimensional algebra over $\mathbb{C}$ and semisimple by Corollary 3.9.2, it must be isomorphic to a finite direct sum of full matrix algebras over $\mathbb{C}$ by Proposition 3.12.1. Let e_{ii} denotes the n-by-n matrix with entry 1 in the (i,i) position and 0 elsewhere, and let

$$\phi\colon S \to M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$$

be an isomorphism. Since $e_{ii} \in S$ $(1 \leq i \leq n)$, it follows that $\phi(e_{ii})$ is an idempotent in $M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$, and so $\phi(e_{ii}) = f_{i1} \oplus \cdots \oplus f_{ik}$ with $f_{ij} \in M_{n_j}(\mathbb{C})$ $(1 \leq j \leq k)$ an idempotent. Since e_{ii} is minimal in S, we must have $f_{ij} \neq 0$ for

just one j; $\phi(e_{ii}) = f_{ij}$ for some j. Moreover $\phi(e_{ii}) = f_{ij}$ must be a rank one idempotent in $M_{n_j}(\mathbb{C})$, and we can partition $\{1, \ldots, n\}$ into k classes where the j^{th} class is $C_j = \{i : \phi(e_{ii}) \in M_{n_j}(\mathbb{C})\}$. Put $p_j = \sum_{i \in C_j} e_{ii}$, and let 1_n denote the n-by-n identity matrix. Then $\phi(1_n) = \sum_{i=1}^{n} \phi(e_{ii}) = \sum_{j=1}^{k} \phi(p_j)$ is the identity of $\phi(S)$. Hence $\phi(p_j) \in M_{n_j}(\mathbb{C})$ is the identity. It follows that C_j must have n_j members and $\sum_{j=1}^{k} n_j = n$. Therefore, if $x \in S$, then $\phi(x) \in M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$ and $\phi(x) = \sum_{j=1}^{k} \phi(p_j)\phi(x)\phi(p_j) = \phi(\sum_{j=1}^{k} p_j x p_j)$, so $S \subseteq \bigoplus_{j=1}^{k} p_j M_n(\mathbb{C}) p_j$. But $\bigoplus_{j=1}^{k} p_j M_n(\mathbb{C}) p_j$ is a generalized Peirce corner of dimension $\sum_{j=1}^{k} \operatorname{rank}(p_j)^2 = \sum_{j=1}^{k} n_j^2 = \dim(\phi(S)) = \dim S$. Therefore $S = \bigoplus_{j=1}^{k} p_j M_n(\mathbb{C}) p_j$.

Hence $\mathcal{E}(x) = \sum_{j=1}^{k} p_j x p_j$ by Theorem 2.2.2. Thus $\mathcal{E}$ is a positive unital map and so $\|\mathcal{E}\| = 1$. (That $\|\mathcal{E}\| = 1$ follows also from [9, II.3.2.16].) □

Lemma 3.12.3. If $(p_i)_{i \in I}$ are orthogonal projections $p_i \in \mathcal{B}(H)$ (so that $p_i = p_i^* = p_i^2$ for $i \in I$ and $p_i p_j = 0$ for $i \neq j$) then we can define a Lam *-conditional expectation $\mathcal{E} \colon \mathcal{B}(H) \to \mathcal{B}(H)$ by

$$\mathcal{E}(x) = \sum_{i \in I} p_i x p_i = \lim_{\alpha \in \mathcal{F}(I)} \sum_{i \in \alpha} p_i x p_i$$

where the limit is taken in the strong operator topology of $\mathcal{B}(H)$ and $\mathcal{F}(I)$ denotes the collection of finite subsets $\alpha \subseteq I$ (ordered by inclusion).

Proof. First we check that the sum converges in the strong operator topology (though this is well known). If $\xi \in H$ and $\alpha \in \mathcal{F}(I)$ then let $\xi_\alpha = \sum_{i \in \alpha} p_i \xi$. Since $\sum_{i \in \alpha} p_i$ is a projection we have $\|\xi_\alpha\| \leq \|\xi\|$. So $\sup_\alpha \|\xi_\alpha\| < \infty$.

Also

$$\|\xi_\alpha\|^2 = \sum_{i \in \alpha} \|p_i \xi\|^2$$

and it follows that for each n there can be at most finitely many $i \in I$ (in fact no more that $n\|\xi\|^2$) t follows that ifwith $\|p_i \xi\|^2 > 1/n$. Hence at most countable many $i \in I$ with $p_i \xi \neq 0$, which we may list as a finite or infinite list $i_1, i_2, \ldots$ and

(assuming the list is infinite)

$$\sum_{j=1}^{\infty} \|p_{i_j}\xi\|^2 = \sup_{n\geq 1} \sum_{j=1}^{n} \|p_{i_j}\xi\|^2 \leq \|\xi\|^2 < \infty.$$

It follows that if n is large enough and $\alpha \in \mathcal{F}(I)$ has $\alpha \supset \alpha_n = \{i_1, i_2, \ldots, i_n\}$ then $\|\xi_\alpha - \xi_{\alpha_n}\|$ must be small. Thus

$$\begin{aligned} \left\| \sum_{i\in\alpha} p_i x p_i \xi - \sum_{i\in\alpha_n} p_i x p_i \xi \right\|^2 &= \sum_{i\in\alpha\setminus\alpha_n} \|p_i x p_i \xi\|^2 \\ &\leq \sum_{i\in\alpha\setminus\alpha_n} \|x\|^2 \|p_i\xi\|^2 \\ &= \|x\|^2 \|\xi_\alpha - \xi_{\alpha_n}\|^2 \end{aligned}$$

must also be small. It follows that $\left(\sum_{i\in\alpha} p_i x p_i \xi\right)_\alpha$ is a Cauchy net in H and so it has a limit in H by completeness of H. We can check then that $T\xi = \lim_\alpha \sum_{i\in\alpha} p_i x p_i \xi$ defines a linear operator on H and that $\|T\xi\| \leq \|x\|\|\xi\|$ which shows that $T \in \mathcal{B}(H)$. Thus $\mathcal{E}x$ is defined.

To show it is a bimodule map

$$\begin{aligned} \mathcal{E}(\mathcal{E}(x)y) &= \sum_i p_i \left(\sum_j p_j x p_j \right) y p_i = \sum_i \left(\sum_j p_i p_j x p_j \right) y p_i \\ &= \sum_i p_i x p_i y p_i \end{aligned}$$

while

$$\begin{aligned} \mathcal{E}(x)\mathcal{E}(y) &= \sum_i p_i x p_i \left(\sum_j p_j y p_j \right) = \sum_i \left(\sum_j p_i x p_i p_j y p_j \right) \\ &= \sum_i p_i x p_i y p_i \\ &= \mathcal{E}(\mathcal{E}(x)y). \end{aligned}$$

Similarly $\mathcal{E}(x\mathcal{E}(y)) = \mathcal{E}(x)\mathcal{E}(y)$ and $\mathcal{E}(\mathcal{E}(x)) = \mathcal{E}(x)$ is straightforward also. □

Definition 3.12.4. We refer to the conditional expectations in Lemma 3.12.3 as *Generalised Peirce Lam *-conditional expectations* and to the ranges as *Generalised Peirce *-corners.*

Theorem 3.12.5. Let $\mathcal{E}\colon \mathcal{B}(H) \to \mathcal{B}(H)$ be a Lam conditional expectation such that $\mathcal{E}(x^*) = \mathcal{E}(x)^*$ holds for $x \in \mathcal{B}(H)$ and $e_i \otimes e_i^* \in \mathcal{E}(\mathcal{B}(H))$ for all $i \in I$. Then $\mathcal{E}$ is a Generalised Peirce Lam *-conditional expectation of the form

$$\mathcal{E}(x) = \sum_{j \in J} p_j x p_j$$

for 'diagonal' orthogonal projections p_j (that is projections with $p_j e_i \in \mathbb{C}e_i$ for each $i \in I$).

Proof. If $\alpha \subset I$ is a finite set, we write $\pi = \pi_\alpha$ for the orthogonal projection of H onto the linear span$\{e_i : i \in \alpha\}$, which is $\pi_\alpha = \sum_{i\in\alpha} e_i \otimes e_i^*$ and is in the range of $\mathcal{E}$. Let $A_\alpha = \{x \in \mathcal{B}(H) : x = \pi x \pi\} = \pi\mathcal{B}(H)\pi$, a C^*-subalgebra (in fact a self-adjoint Peirce corner) of $\mathcal{B}(H)$. Note that if $x \in A_\alpha$ then $\mathcal{E}(x) = \mathcal{E}(\pi x \pi) = \pi\mathcal{E}(x)\pi \in A_\alpha$. We now define $\mathcal{E}_\alpha\colon A_\alpha \to A_\alpha$ by $\mathcal{E}_\alpha = \mathcal{E}|_{A_\alpha}$ (restriction of $\mathcal{E}$) and we can check easily that $\mathcal{E}_\alpha$ is a Lam *-conditional expectation on A_α.

If we write $\alpha = \{i_1, i_2, \ldots, i_n\}$ there is a *-isomorphism $\theta\colon M_n(\mathbb{C}) \to A_\alpha$ given by $\theta((x_{r,s})_{r,s=1}^n) = \sum_{r,s=1}^n x_{r,s} e_r \otimes e_s^*$. Transferring $\mathcal{E}_\alpha$ to $M_n(\mathbb{C})$ via this isomorphism we get a Lam *-conditional expectation $\mathcal{E}'\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by $\mathcal{E}'(x) = \theta^{-1}\mathcal{E}_\alpha(\theta(x))$. Moreover the range of $\mathcal{E}'$ contains the diagonal and so by Lemma 3.12.2 there are diagonal projections $p_1', \ldots, p_k'$ in so that $\mathcal{E}'(x) = \sum_{j=1}^k p_j' x p_j'$.

Define a relation on I by $i_1 \sim i_2$ if $\mathcal{E}(e_{i_1} \otimes e_{i_2}^*) = e_{i_1} \otimes e_{i_2}^*$. Since the range of $\mathcal{E}$ contains the diagonal $i \sim i$ for all $i \in I$. From the fact that $(e_{i_1} \otimes e_{i_2}^*)^* = (e_{i_2} \otimes e_{i_1}^*)$ we have that if $i_1 \sim i_2$ then $i_2 \sim i_1$. To show transitivity of $\sim$, assuming $i_1 \sim i_2$ and $i_2 \sim i_3$, the remarks in the previous paragraph applied to $\alpha = \{i_1, i_2, i_3\}$ show that $\mathcal{E}_\alpha$ is the identity and $i_1 \sim i_3$. So we have an equivalence relation $\sim$ on I.

Take J to be the set of equivalence classes and for $j \in J$ define

$$p_j = \sum_{i \in j} e_i \otimes e_i^*$$

(sum converging in strong operator topology).

Observe that $\mathcal{E}(e_{i_1} \otimes e_{i_2}^*) = \sum_j p_j(e_{i_1} \otimes e_{i_2}^*)p_j$ for all $i_1, i_2 \in I$ because if $i_1 \sim i_2$ then $p_j(e_{i_1} \otimes e_{i_2}^*)p_j$ is zero for all equivalence classes j other than the one containing i_1, while $p_j(e_{i_1} \otimes e_{i_2}^*)p_j = e_{i_1} \otimes e_{i_2}^*$ when j is the equivalence class of i_1.

It follows that if $\xi, \eta \in H$ are finite linear combinations of the basis vectors, say $\xi = \sum_{i \in \alpha} \xi_i e_i$ and $\eta = \sum_{i \in \alpha} \eta_i e_i$, and $x \in \mathcal{B}(H)$ then

$$\begin{aligned} \langle \mathcal{E}(x)\xi, \eta \rangle &= \langle \mathcal{E}(x)\pi_\alpha \xi, \pi_\alpha \eta \rangle = \langle \pi_\alpha \mathcal{E}(x)\pi_\alpha \xi, \pi_\alpha \eta \rangle \\ &= \langle \mathcal{E}(\pi_\alpha x \pi_\alpha)\xi, \pi_\alpha \eta \rangle \\ &= \left\langle \sum_j p_j x p_j \xi, \eta \right\rangle \end{aligned}$$

(since $\pi_\alpha x \pi_\alpha$ is a finite linear combination of terms $e_{i_1} \otimes e_{i_2}^*$ with $i_1, i_2 \in \alpha$).

Because such finite linear combinations are dense in H, we conclude that for $x \in \mathcal{B}(H)$ we have

$$\langle \mathcal{E}(x)\xi, \eta \rangle = \left\langle \sum_j p_j x p_j \xi, \eta \right\rangle$$

for all $\xi, \eta \in H$ and so

$$\mathcal{E}(x) = \sum_j p_j x p_j \quad (\text{for } x \in \mathcal{B}(H)). \qquad \square$$

Corollary 3.12.6. Let $\mathcal{E} \colon \mathcal{B}(H) \to \mathcal{B}(H)$ be a Lam conditional expectation such that $\mathcal{E}(x^*) = \mathcal{E}(x)^*$ holds for $x \in \mathcal{B}(H)$ and $e_i \otimes e_i^* \in \mathcal{E}(\mathcal{B}(H))$ for all $i \in I$. Then (using Tomiyama's Theorem) $\mathcal{E}$ is completely positive, $\|\mathcal{E}\|_{cb} = 1$, and $\mathcal{E}(1) = 1$.

Remark 3.12.7. The maximal abelian $*$-subalgebra associated with the orthonormal basis is more than just the linear span of the diagonal rank one operators $e_i \otimes e_i^*$. The maximal abelian $*$-subalgebra would be the weak*-closure of that span.

3.13 Corners Containing Diagonals II

Recall the following facts. A projection p ($p = p^* = p^2$) in a von Neumann algebra R is called:

- *abelian* if pRp is commutative;
- *minimal* (or an *atom*) if $p \neq 0$, and $(q \leq p) \rightarrow (q = 0 \text{ or } q = p)$ for all projections $q \in R$. (Equivalently: p is minimal in R if and only if $pRp = \mathbb{C}p$.)

A von Neumann algebra R is called:

- a *Type I factor* if R is *-algebraically isomorphic to all bounded operators on some Hilbert space;
- *purely atomic* if R contains a set $\{e_i\}_{i \in I}$ (I is an index set which may be countable or uncountable) of minimal projections with $\sum_{i \in I} e_i = 1$. (Equivalently: R is generated by its minimal projections.)
- *atom free* if R has no minimal projections.

Since a minimal projection is abelian, a purely atomic von Neumann algebra is Type I, but many Type I von Neumann algebras are not purely atomic. In fact, a von Neumann algebra is purely atomic if and only if it is a product of Type I factors [9, IV.2.2.1]. It is known that any von Neumann algebra may be uniquely decomposed into a direct sum of a purely atomic von Neumann algebra and an atom free von Neumann algebra. More specifically (see, for example, [39, Theorem 1.15]):

Theorem 3.13.1. Let R be a von Neumann algebra. Then

$$R = (\bigoplus_{i \in I} R_i) \oplus R_0, \tag{3.7}$$

where R_0 is the atom free portion of R and $\bigoplus_{i \in I} R_i$ is purely atomic (each R_i is a Type I factor).

Statement 3.13.2. In the decomposition (3.7), the purely atomic von Neumann algebra $\bigoplus_{i\in I} R_i$ is a corner of R because it is a direct summand.

In the above, as is conventional, for a family R_i of von Neumann algebras, $\bigoplus_{i\in I} R_i$ denotes $\{(x_i)_{i\in I} : x_i \in R_i \text{ for all } i \in I \text{ and } \sup_{i\in I} \|x_i\| < \infty\}$. Note that $\bigoplus_{i\in I} R_i$ is a von Neumann algebra with coordinatewise definitions of the algebra operations and the *, and norm $\|(x_i)_{i\in I}\| = \sup_{i\in I} \|x_i\|$.

Lemma 3.13.3. Let $R = \bigoplus_{i\in I} R_i$ be the ℓ_I^∞ direct product of type I von Neumann algebra factors $R_i \cong \mathcal{B}(H_i)$ (with H_i Hilbert spaces).

(a) if $e \in R$ is a minimal projection then there is $i_0 \in I$ so that $e \in R_{i_0}$ (or more formally $e = (e_i)_{i\in I}$ satisfies $e_i = 0$ for all $i \neq i_0$ and $e_{i_0} \in R_{i_0} = \mathcal{B}(H_{i_0})$ is a rank one projection).

(b) If $\pi = \sum_{k=1}^n e_k$ is a finite sum of orthogonal minimal projections $e_k \in R$, then $\pi R \pi$ is finite dimensional.

Proof. (a) From $e = (e_i)_{i\in I} = e^2 = e^*$ we see immediately that $e_i = e_i^2 = e_i^*$ for each $i \in I$ and so each e_i is a projection. If there were two or more values of $i \in I$ with $e_i \neq 0$, e would not be minimal.

(b) By the first part we can find $i_k \in I$ with $e_k \in R_{i_k}$ for $k = 1, 2, \ldots, n$. We proceed by induction on the number of distinct i_k. If $i_k = i_1$ for all k, then $\pi \in \mathcal{B}(H_{i_k})$ is a projection of rank n and $\pi R \pi = \pi \mathcal{B}(H_{i_k}) \pi \cong \mathcal{B}(H_{i_k})$ is of dimension $n^2 < \infty$.

If there are more than two values of the i_k, assuming the inductive hypothesis that we know the result for fewer distinct i_k, write $\pi = \pi_1 + \pi_1'$ where $\pi_1 = \sum_{i_k = i_1} e_k \in R_{i_1}$ and $\pi_1' = \pi - \pi_1 = \sum_{i_k \neq i_1} e_k \in R' = \bigoplus_{i\neq i_1} R_i$. Then $\pi R \pi = \pi_1 R_{i_1} \pi_1 \oplus \pi' R' \pi'$ has finite dimension by the inductive hypothesis.

□

Theorem 3.13.4. Let R be a purely atomic von Neumann algebra containing a set $\{e_i\}_{i\in I}$ of minimal self-adjoint projections with $\sum_{i\in I} e_i = 1$. Let $\mathcal{E}\colon R \to R$

be a Lam conditional expectation such that $\mathcal{E}(x^*) = \mathcal{E}(x)^*$ holds for $x \in R$ and $e_i \in \mathcal{E}(R)$ for all $i \in I$. Then $\mathcal{E}$ is a Generalised Peirce Lam *-conditional expectation of the form

$$\mathcal{E}(x) = \sum_{j \in J} p_j x p_j$$

for 'diagonal' orthogonal projections p_j (that is projections with $p_j f_i = f_i p_j \in \mathbb{C} f_i$ for each $i \in I$).

Proof. If $\alpha \subset I$ is a finite set, we define $\pi = \pi_\alpha = \sum_{i \in \alpha} e_i$ which is in the range of $\mathcal{E}$. Let $R_\alpha = \{x \in R : x = \pi x \pi\} = \pi R \pi$, a C^*-subalgebra (in fact a self-adjoint Peirce corner) of R. From Lemma 3.13.3 we have that $\pi R \pi$ is finite dimensional. (For a proof that $\pi R \pi$ is a von Neumann algebra see, for example, [24, Proposition 1 (i)].) Note that if $x \in R_\alpha$ then $\mathcal{E}(x) = \mathcal{E}(\pi x \pi) = \pi \mathcal{E}(x) \pi \in R_\alpha$. We now define $\mathcal{E}_\alpha \colon R_\alpha \to R_\alpha$ by $\mathcal{E}_\alpha = \mathcal{E}|_{R_\alpha}$ (restriction of $\mathcal{E}$) and we can check easily that $\mathcal{E}_\alpha$ is a Lam *-conditional expectation on R_α.

If we write $\alpha = \{i_1, \ldots, i_n\}$ there is a *-isomorphism

$$\theta \colon M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C}) \to R_\alpha \qquad (n_1 + \cdots + n_k = n = |\alpha|)$$

such that $\theta(M_{n_1}(\mathbb{C})) \oplus \cdots \oplus \theta(M_{n_k}(\mathbb{C})) = R_\alpha$ and each of the summands is contained in some Type I factor sitting in R; here $M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$ is regarded as a subalgebra of block-diagonal matrices naturally embedded into $M_n(\mathbb{C})$. Let $\mathcal{E}_n \colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be the unique Lam *-conditional expectation with the image $M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$. Transferring $\mathcal{E}_\alpha$ to $M_n(\mathbb{C})$ via θ we get a Lam *-conditional expectation $\mathcal{E}' \colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by $\mathcal{E}'(x) = \theta^{-1} \mathcal{E}_\alpha \theta \mathcal{E}_n(x)$. Moreover the range of $\mathcal{E}'$ contains the diagonal and so by Lemma 3.12.2 there are diagonal projections $p'_1, \ldots, p'_l$ in so that $\mathcal{E}'(x) = \sum_{j=1}^{l} p'_j x p'_j$.

Since R is a product of Type I factors, $e_{i_1} R e_{i_2}$ is of dimension at most 1 (for all $i_1, i_2 \in I$), and so we can define a relation on I by $i_1 \sim i_2$ if $0 \neq e_{i_1} R e_{i_2} \subseteq \mathcal{E}(R)$. Since the range of $\mathcal{E}$ contains the diagonal $i \sim i$ for all $i \in I$. From the fact that $(e_{i_1} R e_{i_2})^* = e_{i_2} R e_{i_1}$ we have that if $i_1 \sim i_2$ then $i_2 \sim i_1$. To show transitivity of $\sim$, assuming $i_1 \sim i_2$ and $i_2 \sim i_3$, the remarks in the previous paragraph applied to

$\alpha = \{i_1, i_2, i_3\}$ show that $\mathcal{E}_\alpha$ is the identity and $i_1 \sim i_3$. So we have an equivalence relation $\sim$ on I.

Take J to be the set of equivalence classes and for $j \in J$ define

$$p_j = \sum_{i \in j} e_i$$

(sum converging in strong operator topology).

Because R is a product of Type I factors, Theorem 3.12.5 implies that

$$\mathcal{E}(x) = \sum_j p_j x p_j \quad (\text{for } x \in R). \qquad \square$$

Example 3.13.5. For a positive integer n, let $M_n(\mathbb{C})$ denotes the factor of full n-by-n-matrices over the field of complex numbers $\mathbb{C}$. Then $\bigoplus_{n=1}^{\infty} M_n(\mathbb{C})$ is a Type I finite von Neumann algebra with centre isomorphic to ℓ^∞. There are many ways to write the identity as a sum of projections $\sum_{i \in I} p_i$ such that each p_i is of the form $p_i = (p_{i,n})_{n=1}^{\infty}$ with $p_{i,n} \in M_n(\mathbb{C})$ diagonal. Such a sum $\sum_{i \in I} p_i$ gives rise to a Generalized Peirce Lam *-conditional expectation on $\bigoplus_{n=1}^{\infty} M_n(\mathbb{C})$.

3.14 Spectral Radius

We begin this section by recalling the following definitions and facts. For an element x in a unital algebra A, the spectrum of x is the subset $\sigma(x)$ of $\mathbb{C}$ given by

$$\sigma(x) = \{\lambda \in \mathbb{C} : x - \lambda 1 \text{ is not invertible in } A\}.$$

The spectral radius $r(x)$ of an element $x \in A$ is defined as

$$r(x) = \{|\lambda| : \lambda \in \sigma(x)\}$$

provided $\sigma(x)$ is not empty. If A is a Banach algebra, the spectral radius formula is

$$r(x) = \lim_{n\to\infty} \|x^n\|^{1/n} = \inf \|x^n\|^{1/n}.$$

If A is a C^*-algebra and $x \in A$ is self-adjoint ($x = x^*$), then $\|x\| = r(x)$.

The above facts have an interesting application when combined with the following proposition, which states that if a self-adjoint Lam conditional expectation $\mathcal{E}$ contracts the spectral radius ($r(\mathcal{E}x) \leq r(x)$), then $\|\mathcal{E}\| \leq 1$.

Proposition 3.14.1. Let A be a unital C^*-algebra. If $\mathcal{E}\colon A \to A$ is a Lam *-conditional expectation such that $r(\mathcal{E}x) \leq r(x)$ hold for $x \in A$, then $\|\mathcal{E}\| \leq 1$.

Proof. For $x \in A$ we have $\|\mathcal{E}x\|^2 = \|(\mathcal{E}x)^*\mathcal{E}x\| = \|\mathcal{E}(x^*)\mathcal{E}x\| = \|\mathcal{E}(x^*\mathcal{E}x)\|$. Since $\mathcal{E}(x^*\mathcal{E}x)$ is self-adjoint,

$$\|\mathcal{E}(x^*\mathcal{E}x)\| = r(\mathcal{E}(x^*\mathcal{E}x)).$$

Therefore $\|\mathcal{E}x\|^2 = r(\mathcal{E}(x^*\mathcal{E}x)) \leq r(x^*\mathcal{E}x)$. This finally implies that $\|\mathcal{E}x\|^2 \leq r(x^*\mathcal{E}x) \leq \|x^*\mathcal{E}x\| \leq \|x\|\,\|\mathcal{E}x\|$, and $\|\mathcal{E}x\| \leq \|x\|$. □

Remark 3.14.2. In general, a Lam conditional expectation need not contract the spectral radius (see Example 3.16.2).

The following is a corollary to Proposition 3.14.1.

Corollary 3.14.3. Let A be a unital C^*-algebra. If $\mathcal{E}\colon A \to A$ is a Lam *-conditional expectation such that $\sigma(\mathcal{E}x) \subseteq \sigma(x)$ hold for $x \in A$ then $\|\mathcal{E}\| \leq 1$.

3.15 Peirce Corners in Prime Algebras

Recall that a *representation* of a C^*-algebra A on a Hilbert space H is any $*$-algebra homomorphism $\pi\colon A \to \mathcal{B}(H)$. (It follows that $\|\pi\| \leq 1$, and the range of π is always closed). If A is unital, it is required that $\pi(1) = 1$. The representation π is called *faithful* if $\ker \pi = \{0\}$, and is called:

- *algebraically reducible* if there is a non-zero proper (not necessarily closed) subspace of H invariant for each operator in $\pi(A)$,

- *reducible* if there is a non-zero proper closed subspace of H invariant for each operator in $\pi(A)$,
- *algebraically irreducible* if it is not algebraically reducible,
- *irreducible* if it is not reducible.

Obviously, an algebraically irreducible representation is irreducible. An important result asserts that the converse also holds, that is, an irreducible representation is algebraically irreducible. This is an immediate corollary of the following Kadison transitivity theorem [37]; cf. [9, II.6.1.12].

Theorem 3.15.1 (R. V. Kadison (1957))**.** Let π be an irreducible representation of a C^*-algebra A on a Hilbert space H and $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n \in H$ with $\xi_1, \ldots, \xi_n$ linearly independent. Then there exists $a \in A$ such that $\pi(a)\xi_k = \eta_k$ $(1 \leq k \leq n)$.

Corollary 3.15.2. Let π be a representation of a C^*-algebra A on a Hilbert space H. Then π is algebraically irreducible if and only if it is irreducible.

Proof. Assume π is irreducible. If E is a non-zero proper (not necessarily closed) subspace of H, and $\xi \in E$, $\eta \in H \setminus E$, with $\xi \neq 0$, then by Theorem 3.15.1, there exists $a \in A$ such that $\pi(a)\xi = \eta$. This shows that E is not an invariant subspace of $\pi(a)$, and consequently that π is algebraically irreducible. □

A C^*-algebra A is called *prime* if it has no ideal divisors of 0, that is, whenever I and J are closed non-zero two-sided ideals of A, the product

$$IJ = \{ab : a \in I, b \in J\}$$

is non-zero. ('Prime' is defined also for any algebra.) A C^*-algebra is called *primitive* if it has a faithful irreducible representation. Any primitive C^*-algebra is prime, but the converse holds only for separable C^*-algebras – it is a result of J. Dixmier [22]. It has been shown by N. Weaver [73] and M. J. Crabb [16] that there are nonseparable C^*-algebras that are prime but not primitive.

We shall be concerned now with primeness and primitivity of C^*-algebras when they are considered purely algebraically as algebras, forgetting the topology.

Definition 3.15.3 ((see [63, §II.1])). A left ideal L of an algebra A is called *modular* if $L \neq A$ and there is $u \in A$ so that $xu - x \in L$ for each $x \in A$.

By a *representation* of an algebra A (over a field $\mathbb{K}$ which will always be $\mathbb{K} = \mathbb{C}$ for us) on a vector space V we mean an algebra homomorphism $\pi\colon A \to \mathrm{End}(V)$ from A to the algebra $\mathrm{End}(V)$ of all linear transformations on V (with V a vector space over $\mathbb{K}$). We apply this concept in a purely algebraic context without any topological restrictions and we will not assume immediately that $\pi(1)$ is the identity when A has a unit. Then a representation corresponds to a left A-module structure on V, that is a bilinear multiplication $(a, v) \mapsto av\colon A \times V \to V$ with the associativity property $(ab)v = a(bv)$ (for $a, b \in A$, $v \in V$).

Definition 3.15.4. A representation $\pi\colon A \to \mathrm{End}(V)$ of an algebra a is said to have a *cyclic vector* $v \in V$ if $\pi(A)(v) = V$. In module terminology this means $Av = V$.

A representation is called *cyclic* if a cyclic vector exists for the representation. We apply the same terminology to modules.

Suppose we are given a modular left ideal L of an algebra A (so that L is a left submodule of A where the multiplication $A \times A \to A$ is the algebra product). Assume that we fix $u \in A$ so that $xu - x \in L$ for each $x \in L$. We can check that the quotient (left) A-module A/L is a cyclic (left) A-module with cyclic vector $v = u + L$.

Definition 3.15.5. Given two representations $\pi_1\colon A \to \mathrm{End}(V_1)$ and $\pi_2\colon A \to \mathrm{End}(V_2)$, we say that they are *equivalent* if there is a vector space isomorphism $\theta\colon V_1 \to V_2$ such that $\theta \circ \pi_1(a) \circ \theta^{-1} = \pi_2(a)$ holds for each $a \in A$.

If $v_i \in V_i$ is a cyclic vector for π_i $(i = 1, 2)$ then we say that the pair (π_1, v_1) is equivalent to (π_2, v_2) if there we can find such a θ with $\theta(v_1) = v_2$.

Note that we could rephrase the notion of equivalence to say that $\theta\colon V_1 \to V_2$ is an isomorphism of (left) A-modules (with the added condition $\theta(v_1) = v_2$ when we fix cyclic vectors).

Lemma 3.15.6. Let A be an algebra. Given a pair (V, v) consisting of a cyclic (left) A-module V and a cyclic vector $v \in V$, there is a modular left ideal $L = \{a \in A : av = 0\}$ in A and a cyclic vector $u + L \in A/L$ such that (V, v) is equivalent to $(A/L, u + L)$.

Proof. (See for example [63, §II.1].) Since $Av = V$ there is $u \in A$ with $uv = v$. It is easy to check that L is a left ideal, that $xu - x \in L$ for all $x \in A$, and that the map $a + L \mapsto av \colon A/L \to V$ is an isomorphism of modules sending $u + L$ to v. (Hence $\theta^{-1}(v) = u + L$ is a cyclic vector for the module A/L.) □

Lemma 3.15.7 ([23, Corollary to Theorem 2.9.5])**.** Each irreducible representation of a C^*-algebra A on a complex vector space V is algebraically equivalent to a *-representation $\pi \colon A \to \mathcal{B}(H)$ on a Hilbert space H.

Corollary 3.15.8. Let A be a C^*-algebra.

Given a maximal modular left ideal $L \subseteq A$ there is an irreducible *-representation $\pi \colon A \to \mathcal{B}(H)$ of A on a Hilbert space H and a nonzero (cyclic) vector $\xi_0 \in H$ such that $L = \{x \in A : \pi(x)(\xi_0) = 0\}$.

Moreover, π is a faithful representation if $\{x \in A : xA \subseteq L\} = \{0\}$.

Conversely, given an irreducible *-representation $\pi' \colon A \to \mathcal{B}(H)$ (with $H \neq \{0\}$) every $\xi \in H \setminus \{0\}$ is cyclic and $L_\xi = \{x \in A : \pi'(x)(\xi) = 0\}$ is a maximal modular left ideal of A.

Proof. From the above the representation of A on the vector space A/L is cyclic, with cyclic vector $u + L$ where u is chosen so that $xu - x \in L$ for all $x \in A$.

The module A/L has no proper submodules since if L'/L is a proper submodule then L' is a proper left ideal of A with $xu - x \in L \subseteq L'$ for all $x \in A$. Hence L' is a modular left ideal containing L and so $L = L'$ by maximality of L. (See for example [63, Theorem 2.2.1] for this.)

From Lemma 3.15.7 there must be an equivalence $\theta \colon A/L \to H$ from the left A-module A/L to a left module H that is a Hilbert space and where the module action obeys $\langle a\xi, \eta \rangle = \langle \xi, a^*\eta \rangle$ for all $\xi, \eta \in H$. Let $\xi_0 = \theta(u + L)$. The *-representation π is $\pi(x)(\xi) = a\xi$ and $\xi_0 \in H$ is cyclic.

Note that $\ker \pi$ is an ideal of A (in fact a 2-sided *-ideal) and if $x \in \ker \pi$ then $\pi(x)(\xi_0) = 0$ so $x \in L$ and therefore $\ker \pi \subseteq L$. For $x \in \ker \pi$ and $a \in A$ we have $ax \in \ker \pi \subseteq L$ and so $xA \subseteq L$. The condition $\{x \in A : xA \subseteq L\} = \{0\}$ clearly implies $\ker \pi = 0$ which means that π is faithful. (See for example [63, Theorem 2.2.9] for this.)

For the converse assertion, if $\pi' \colon A \to \mathcal{B}(H)$ is an irreducible *-representation and $\eta \in H \setminus \{0\}$, then we have $\pi'(A)(\eta) = H$ by the Kadison transitivity theorem (Theorem 3.15.1). So, as an A-module, H has no proper submodules. So if $\xi \in H \setminus \{0\}$, A/L_ξ is equivalent to H as an A-module and L_ξ must be a maximal proper left ideal of A (because if $L' \supsetneq L_\xi$ is a proper left ideal of A, then L'/L_ξ would be a proper submodule). By Lemma 3.15.6, L_ξ is modular. □

Theorem 3.15.9. Let A be a prime C^*-algebra containing a nonzero self-adjoint projection p. Then A is a primitive C^*-algebra if and only if pAp is a primitive C^*-algebra.

Proof. Suppose A is primitive. Choose a Hilbert space H and a faithful irreducible representation $\pi \colon A \to \mathcal{B}(H)$. Define $\rho \colon pAp \to \mathcal{B}(\pi(p)H)$ by $\rho(pap)\xi := \pi(p)\pi(a)\pi(p)\xi = \pi(p)\pi(a)\xi$ for all $a \in A$, $\xi \in \pi(p)H$. Then ρ is a faithful irreducible representation of pAp on $\pi(p)H$. Indeed, ρ is clearly an algebra homomorphism. Note that for all $\xi, \eta \in \pi(p)H$ we have $\langle \rho(pap)\xi, \eta\rangle = \langle \pi(p)\pi(a)\pi(p)\xi, \eta\rangle = \langle \xi, \pi(p)\pi(a)^*\pi(p)\eta\rangle = \langle \xi, \pi(p)\pi(a^*)\pi(p)\eta\rangle = \langle \xi, \rho(pa^*p)\eta\rangle = \langle \xi, \rho((pap)^*)\eta\rangle$, and so $\rho((pap)^*) = \rho(pap)^*$; thus ρ is a $*$-algebra homomorphism. Now, if E is a nontrivial closed subspace of $\pi(p)H$ and $\xi \in E$, $\eta \in \pi(p)H \setminus E$ with $\xi \neq 0$, then Theorem 3.15.1 implies that there exists $a \in A$ such that $\pi(a)\xi = \eta$. Since $\pi(p)\xi = \xi$ and $\pi(p)\eta = \eta$, we have that $\rho(pap)\xi = \eta$. Thus E is not an invariant subspace of $\rho(pap)$, and so ρ is irreducible. It easily follows that ρ is also faithful. Hence pAp is a primitive C^*-algebra.

For the converse, suppose pAp is a primitive. Choose a nonzero Hilbert space H and a faithful irreducible representation $\pi \colon pAp \to \mathcal{B}(H)$. By Lemma 3.15.7, π can be assumed to be a *-representation. By Corollary 3.15.8, every $\xi \in H \setminus \{0\}$ is cyclic and $L_\xi = \{x \in pAp \ : \ \pi(x)(\xi) = 0\}$ is a maximal modular left ideal of pAp.

Define $I = AL_\xi + A(1-p)$. Then I is a left ideal of A, $pIp = pAL_\xi p + 0 = L_\xi$ because p is an identity element of pAp (and $\pi(p)$ is the identity operator on H), and $I \subsetneq A$ because $p \notin I$ (if $p \in I$ then we would get $p = \sum a_i l_i + a(1-p)$ and $p = \sum pa_i l_i p = \sum pa_i l_i \in L_\xi$ – a contradiction). Consider

$$\mathcal{J} = \{J \subseteq A : J \text{ is a left ideal of } A,\ I \subseteq J, \text{ and } pJp = L_\xi\}.$$

Apply Zorn's lemma to $\mathcal{J}$ to get a maximal such J. Now J is modular because if $x \in A$ then $xp - x = -x(1-p) \in A(1-p) \subseteq I \subseteq J$ (and we also have $p \notin J$). Furthermore, we claim that J is a maximal left ideal of A.

Indeed; if $J \subsetneq K \subseteq A$ with K a left ideal of A, then $I \subsetneq K$ and $pIp = L_\xi = pJp \subseteq pKp$. By maximality of J we must have $L_\xi \neq pKp$, so we can choose $x \in pKp \setminus L_\xi$. Then $L_\xi + (pAp)x$ is a left ideal of pAp and contains x (since $x = pap$ for some $a \in K$ and $x = pppx \in (pAp)x$). In fact, $L_\xi + (pAp)x$ is a modular left ideal of pAp because if $y \in pAp$ then $yp - y = 0 \in L_\xi + (pAp)x$. By maximality of L_ξ in pAp we must have $L_\xi + (pAp)x = pAp$. Therefore, since $p \in pAp$, we have $p = l + (pap)x$ for some $l \in L_\xi$, $a \in A$. This implies first that $p \in pKp$ (since $l \in L_\xi \subseteq pKp$ and $papx = paxp \in pAKp \subseteq pKp$), so $p \in pKp \subseteq Kp$; and then that $p \in K$ because $Kp \subseteq K$ (indeed, if $x \in K$ then $x = xp + x(1-p)$ and $x(1-p) \in A(1-p) \subseteq I \subseteq K$, so $x - x(1-p) = xp \in K$). Consequently, since $p \in K$, we have $A = Ap + A(1-p) \subseteq AK + K \subseteq K$; thus $K = A$.

By Corollary 3.15.8, there is an irreducible *-representation $\pi\colon A \to \mathcal{B}(H)$ of A on a Hilbert space H and a nonzero (cyclic) vector $\xi_0 \in H$ such that $J = \{x \in A : \pi(x)(\xi_0) = 0\}$; and since $\{x \in A : xA \subseteq J\} = \{0\}$, it follows that A is a primitive. This completes the proof of the theorem. □

3.16 Examples of Conditional Expectations

In this section we give examples of expectations (Example 3.16.2, Example 3.16.3, Example 3.16.5) which have not been discussed in this fore elsewhere.

Let $M_n(\mathbb{C})$ be the algebra of n-by-n matrices over the field of complex numbers

$\mathbb{C}$. The trace of a matrix $x = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}$ in $M_n(\mathbb{C})$ is $\operatorname{trace}(x) = x_{11} + \cdots + x_{nn}$. eThe identity matrix in $M_n(\mathbb{C})$ is denoted 1_n, or 1 if the size can be determined by the context.

If A is a C^*-algebra, $M_n(A)$ denotes the $*$-algebra of n-by-n square matrices with entries in A. The conjugate of a matrix $x = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \in M_n(A)$ is given by $x^* = \begin{pmatrix} x_{11}^* & \cdots & x_{n1}^* \\ \vdots & & \vdots \\ x_{1n}^* & \cdots & x_{nn}^* \end{pmatrix}$. Note that if $\pi\colon A \hookrightarrow \mathcal{B}(H)$ is a faithful representation of A on a complex Hilbert space H, then $\pi^{(n)}\colon M_n(A) \hookrightarrow M_n(\mathcal{B}(H)) \cong \mathcal{B}(H^n)$ defined by $\pi^{(n)}\begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} = \begin{pmatrix} \pi(x_{11}) & \cdots & \pi(x_{1n}) \\ \vdots & & \vdots \\ \pi(x_{n1}) & \cdots & \pi(x_{nn}) \end{pmatrix}$ is a faithful representation of $M_n(A)$ on $H^n = H \oplus \cdots \oplus H$. Thus $M_n(A)$ is a C^*-algebra with the operator norm inherited from $\mathcal{B}(H^n)$ (the structure of $M_n(A)$ is, in fact, independent of the representation). This operator norm is the unique C^*-norm on $M_n(A)$ with respect to the involution defined above, but there is no explicit formula for the norm of a matrix x in $M_n(A)$ in terms of the entries of x. Nevertheless, for $x = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \in M_n(A)$ we have the following estimation:

$$\max_{i,j} \|x_{ij}\| \leq \|x\| \leq \sum_{i,j} \|x_{ij}\| .$$

Example 3.16.1. Define $\mathcal{E}_{\frac{1}{2}}\colon M_2(\mathbb{C}) \to M_2(\mathbb{C})$ by

$$\mathcal{E}_{\frac{1}{2}}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \frac{\operatorname{trace}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}}{2} \cdot 1 = (\frac{1}{2}x_{11} + \frac{1}{2}x_{22}) \cdot 1 = \begin{pmatrix} \frac{x_{11}+x_{22}}{2} & 0 \\ 0 & \frac{x_{11}+x_{22}}{2} \end{pmatrix}.$$

Then $\mathcal{E}_{\frac{1}{2}}$ is a conditional expectation from $M_2(\mathbb{C})$ onto $\mathbb{C}1 = \left\{\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} : \alpha \in \mathbb{C}\right\}$. In particular, $\mathcal{E}_{\frac{1}{2}}$ is positive, $\ker \mathcal{E}_{\frac{1}{2}} = \{x \in M_2(\mathbb{C}) : \operatorname{trace}(x) = 0\}$, and $\mathbb{C}1$ is a corner of $M_2(\mathbb{C})$. Furthermore $\left\|\mathcal{E}_{\frac{1}{2}}\right\| = 1$ because if $\left\|\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\right\| \leq 1$ then $\max\{|x_{11}|, |x_{22}|\} \leq 1$, and so $\left\|\mathcal{E}_{\frac{1}{2}}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\right\| = \left\|\begin{pmatrix} \frac{x_{11}+x_{22}}{2} & 0 \\ 0 & \frac{x_{11}+x_{22}}{2} \end{pmatrix}\right\| = \left|\frac{x_{11}+x_{22}}{2}\right| \leq \frac{1}{2}|x_{11}| + \frac{1}{2}|x_{22}| \leq \frac{1}{2} + \frac{1}{2} = 1$.

Example 3.16.2. Let $\alpha \in \mathbb{C}$, $\alpha \neq 1$. Define $\mathcal{E}_\alpha\colon M_2(\mathbb{C}) \to M_2(\mathbb{C})$ by

$$\mathcal{E}_\alpha\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \frac{\alpha x_{11} - x_{22}}{\alpha - 1} \cdot 1 = (\frac{\alpha}{\alpha - 1}x_{11} - \frac{1}{\alpha - 1}x_{22}) \cdot 1.$$

Then $\mathcal{E}_\alpha$ is a conditional expectation from $M_2(\mathbb{C})$ onto $\mathbb{C}1 = \left\{ \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) : a \in \mathbb{C} \right\}$, and $\|\mathcal{E}_\alpha\| = \frac{|\alpha|+1}{|\alpha-1|}$ because if $\left\| \left(\begin{smallmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{smallmatrix}\right) \right\| \leq 1$ then $\max\{|x_{11}|, |x_{22}|\} \leq 1$, and so

$$\begin{aligned} \left\| \mathcal{E}_\alpha \left(\begin{smallmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{smallmatrix}\right) \right\| &= \left\| \left(\begin{smallmatrix} \frac{\alpha}{\alpha-1}x_{11} - \frac{1}{\alpha-1}x_{22} & 0 \\ 0 & \frac{\alpha}{\alpha-1}x_{11} - \frac{1}{\alpha-1}x_{22} \end{smallmatrix}\right) \right\| \\ &= \left| \tfrac{\alpha}{\alpha-1}x_{11} - \tfrac{1}{\alpha-1}x_{22} \right| \\ &\leq \tfrac{|\alpha|}{|\alpha-1|}|x_{11}| + \tfrac{1}{|\alpha-1|}|x_{22}| \\ &\leq \tfrac{|\alpha|}{|\alpha-1|} + \tfrac{1}{|\alpha-1|} \\ &= \tfrac{|\alpha|+1}{|\alpha-1|}. \end{aligned}$$

On the other hand, for $x = \left(\begin{smallmatrix} \frac{\overline{\alpha}}{|\alpha|} & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $\alpha \neq 0$, we have

$$\mathcal{E}_\alpha(x) = \left(\begin{smallmatrix} \frac{|\alpha|+1}{\alpha-1} & 0 \\ 0 & \frac{|\alpha|+1}{\alpha-1} \end{smallmatrix}\right),$$

and so

$$\|\mathcal{E}_\alpha(x)\| = \left\| \frac{|\alpha|+1}{\alpha-1} \cdot 1 \right\| = \frac{|\alpha|+1}{|\alpha-1|} \cdot \|1\| = \frac{|\alpha|+1}{|\alpha-1|}.$$

In particular, $\mathcal{E}_\alpha$ is a Lam conditional expectation, and $\mathbb{C}1 = \mathcal{E}_\alpha(M_2(\mathbb{C}))$ is a corner of $M_2(\mathbb{C})$. Note, however, that $\mathbb{C}1$ is not a Peirce corner of $M_2(\mathbb{C})$ since it is not true that $\mathbb{C}1$ has a unique complement (see Theorem 2.2.1). In fact every three-dimensional subspace M of $M_2(\mathbb{C})$ such that $1 \notin M$ is a complement of $\mathbb{C}1$. Note also that if α is real then we have $\mathcal{E}_\alpha(x^*) = \mathcal{E}_\alpha(x)^*$ for all $x \in M_2(\mathbb{C})$. if $1 \neq \alpha > 0$ then the spectral radius of $\mathcal{E}_\alpha\left(\begin{smallmatrix} \frac{\overline{\alpha}}{|\alpha|} & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is $\frac{|\alpha|+1}{|\alpha-1|}$ which is greater than 1, whereas the spectral radius of $\left(\begin{smallmatrix} \frac{\overline{\alpha}}{|\alpha|} & 0 \\ 0 & 1 \end{smallmatrix}\right)$ which is 1; so $\mathcal{E}_\alpha$ does not contract the the spectral radius.

Example 3.16.3. Assume $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$, $\alpha_1 + \alpha_2 + \cdots + \alpha_n \neq 0$. Define $\mathcal{E}\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by

$$\mathcal{E}\left(\begin{smallmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{smallmatrix}\right) = \tfrac{\alpha_1 x_{11} + \alpha_2 x_{22} + \cdots + \alpha_n x_{nn}}{\alpha_1 + \alpha_2 + \cdots + \alpha_n} \cdot 1.$$

Then $\mathcal{E}$ is a conditional expectation from $M_n(\mathbb{C})$ onto $\mathbb{C}1_n = \{a1_n : a \in \mathbb{C}\}$, and

$\|\mathcal{E}\| = \frac{|\alpha_1|+|\alpha_2|+\cdots+|\alpha_n|}{|\alpha_1+\alpha_2+\cdots+\alpha_n|}$ because if $\left\|\begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}\right\| \leq 1$ then $\max_{i,j} |x_{ij}| \leq 1$, and so

$$\begin{aligned} \left\|\mathcal{E}\begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}\right\| &= \left\|\tfrac{\alpha_1 x_{11}+\cdots+\alpha_n x_{nn}}{\alpha_1+\cdots+\alpha_n} \cdot 1\right\| \\ &= \left|\tfrac{\alpha_1 x_{11}+\cdots+\alpha_n x_{nn}}{\alpha_1+\cdots+\alpha_n}\right| \\ &\leq \tfrac{|\alpha_1|}{|\alpha_1+\cdots+\alpha_n|}|x_{11}|+\cdots+\tfrac{|\alpha_n|}{|\alpha_1+\cdots+\alpha_n|}|x_{nn}| \\ &\leq \tfrac{|\alpha_1|}{|\alpha_1+\cdots+\alpha_n|}+\cdots+\tfrac{|\alpha_n|}{|\alpha_1+\cdots+\alpha_n|} \\ &= \tfrac{|\alpha_1|+\cdots+|\alpha_n|}{|\alpha_1+\cdots+\alpha_n|}. \end{aligned}$$

On the other hand, for $x = \begin{pmatrix} \frac{\overline{\alpha_1}}{|\alpha_1|} & & 0 \\ & \ddots & \\ 0 & & \frac{\overline{\alpha_n}}{|\alpha_n|} \end{pmatrix}$, $\alpha_i \neq 0$, we have

$$\mathcal{E}(x) = \frac{|\alpha_1| + \cdots + |\alpha_n|}{\alpha_1 + \cdots + \alpha_n} \cdot 1,$$

and so $\|\mathcal{E}(x)\| = \frac{|\alpha_1|+\cdots+|\alpha_n|}{|\alpha_1+\cdots+\alpha_n|}$. In particular, $\mathcal{E}$ is a Lam conditional expectation, and $\mathbb{C}1_n = \mathcal{E}(M_n(\mathbb{C}))$ is a corner of $M_n(\mathbb{C})$. Note, however, that $\mathbb{C}1_n$ is not a Peirce corner of $M_n(\mathbb{C})$ since it is not true that $\mathbb{C}1_n$ has a unique complement (see Theorem 2.2.1). In fact every (n^2-1)-dimensional subspace M of $M_n(\mathbb{C})$ such that $1_n \notin M$ is a complement of $\mathbb{C}1_n$.

Example 3.16.4. Recall that a *centre-valued trace* on a C^*-algebra A with centre Z is a positive linear mapping $\tau\colon A \to A$ such that

0. $\tau(A) \subseteq Z$,
1. $\tau(xy) = \tau(yx)$,
2. $\tau(zx) = z\tau(x)$,
3. $\tau(z) = z$,

whenever $x, y \in A$ and $z \in Z$. Such a mapping has norm one unless $Z = \{0\}$. Suppose that A is a C^*-algebra with unit element 1, centre Z, and a centre-valued trace $\tau\colon A \to A$. Furthermore, suppose that we have $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$,

$\alpha_1 + \alpha_2 + \cdots + \alpha_n \neq 0$. Then $\mathcal{E}_\tau \colon M_n(A) \to M_n(A)$ defined by

$$\mathcal{E}_\tau \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} = \tau\left(\frac{\alpha_1 x_{11} + \cdots + \alpha_n x_{nn}}{\alpha_1 + \cdots + \alpha_n}\right) \cdot 1_n$$

is a Lam conditional expectation.

Example 3.16.5. We can modify the preceding example as follows. Suppose that A is a C^*-algebra with unit element 1, and a subalgebra S. Furthermore, suppose that $\mathcal{E} \colon A \to A$ is a Lam conditional expectation from A onto S (so $\mathcal{E}$ is an S-bimodule map with $\mathcal{E}(A) = S$ and $\mathcal{E} \circ \mathcal{E} = \mathcal{E}$), and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$, $\alpha_1 + \alpha_2 + \cdots + \alpha_n \neq 0$. Then $\mathcal{E}_\mathcal{E} \colon M_n(A) \to M_n(A)$ defined by

$$\mathcal{E}_\mathcal{E} \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} = \mathcal{E}\left(\frac{\alpha_1 x_{11} + \cdots + \alpha_n x_{nn}}{\alpha_1 + \cdots + \alpha_n}\right) \cdot 1_n$$

is a Lam conditional expectation onto $S1_n \subseteq M_n(A)$.

Chapter 4

Ternary Corners: A Generalisation of Injective Operator Spaces

All algebras to be consider are associative, but not necessarily unital.

An injective operator space is completely isometric to a 'ternary' corner pAq of an injective C^*-algebra A with $p, q \in A$ projections (see [65]). Abstracting the algebraic properties of pAq, we propose a definition of a general 'ternary' corner of a TRO (Definition 4.1.1), which we consider as a generalisation of injective operator spaces, and we investigate when topological properties are consequences of the algebraic assumptions. The main results of this chapter include:

- Graphs of TRO-homomorphisms are ternary corners of TROs (Corollary 4.3.1).
- Every ternary corner of a column Hilbert space H_c is closed (Theorem 4.5.2).
- Characterization of row and column Hilbert spaces: Let H be a Hilbert space. Assume $T \subseteq \mathcal{B}(H)$ is a TRO with the property that every closed subspace of T is a ternary corner in T. Then T is TRO-isomorphic to row or column Hilbert space (Theorem 4.6.1).

4.1 Definitions and Characterization

For a C^*-algebra A, a norm-closed vector space $S \subseteq A$ is referred to as a *TRO* (the 'TRO' stands for 'ternary ring of operators') if for all $a, b, c \in S$ the ternary product ab^*c belongs to S (see H. Zettl [75]). In a more general setting, if A is an associative $*$-algebra, a vector space $S \subseteq A$ is referred to as a *pre-TRO* if $ab^*c \in S$ for all $a, b, c \in S$. Two pre-TROs S and T are consider to be equivalent if there exists a *TRO-isomorphism* from S into T, that is, a bijective linear map $\phi\colon S \to T$ such that $\phi(ab^*c) = \phi(a)\phi(b)^*\phi(c)$ for all $a, b, c \in S$. Any pre-TRO contained in a C^*-algebra can be completed in a unique way up to TRO-isomorphism to a TRO (in a similar way as any pre-Hilbert space can be completed to a Hilbert).

There are three important subspaces of an associative $*$-algebra A which are naturally associated with a pre-TRO $S \subseteq A$:

1. The pre-TRO $S^* = \{s^* \in A : s \in S\}$ which is called the *adjoint space* of S.
2. The $*$-algebra $SS^* =$ linear span $\{ab^* \in A : a, b \in S\}$ which is called the *left* $*$-algebra of S.
3. The $*$-algebra $S^*S =$ linear span $\{b^*c \in A : b, c \in S\}$ which is called the *right* $*$-algebra of S.

In addition, there is a $*$-subalgebra of $\mathcal{S}$ of $M_2(A)$ and projections $p, q \in \mathcal{S}$ such that $S \simeq p\mathcal{S}q$. Thus, in addition to being (SS^*, S^*S)-bimodule, S has the property of being representable as the 'top-off-diagonal corner' of a $*$-algebra $\mathcal{S}$. Indeed, $\mathcal{S}$ may be taken as the *linking* $*$*-algebra* $\mathcal{S} := \left(\begin{smallmatrix} SS^* & S \\ S^* & S^*S \end{smallmatrix}\right) \subseteq M_2(A)$. Then, adjoining (if necessary) identity elements to SS^* and S^*S and setting $p = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $q = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, we have $\mathcal{S} = p\mathcal{S}p \oplus p\mathcal{S}q \oplus q\mathcal{S}p \oplus q\mathcal{S}q$ and $S \simeq p\mathcal{S}q$. This leads to the following definition.

Definition 4.1.1. Let A be an associative $*$-algebra. Let $S \subseteq A$ and $T \subseteq A$ be pre-TROs such that $S \subseteq T$. Then S is called a *ternary corner algebra* (or simply a *ternary corner*) of T if there exists a vector space $M \subseteq T$ such that

$T = S \oplus M$ and $SS^*M \subseteq M$, $MS^*S \subseteq M$. Such M is called a *complement* of S in T.

We will now show some examples of ternary corner algebras. Perhaps the simplest ones can be constructed in the algebra of n-by-n complex matrices $M_n(\mathbb{C})$ generated by the matrix units $e_{ij} \in M_n(\mathbb{C})$, where e_{ij} is the matrix unit with 1 in the (i, j) position, 0 elsewhere. For the sake of simplicity, let us consider the case when $n = 3$. Setting $A = M_3(\mathbb{C})$,

$$S = \text{linear span}\,\{e_{11}, e_{12}\} = \left\{ \begin{pmatrix} \alpha & \beta & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \alpha, \beta \in \mathbb{C} \right\},$$

and $T = \text{linear span}\,\{e_{11}, e_{12}, e_{13}, e_{21}, e_{22}, e_{23}\} \subseteq A$, we see that $T = S \oplus M$, where $M = \text{linear span}\,\{e_{13}, e_{21}, e_{22}, e_{23}\} \subseteq T$. Furthermore,

$$SS^* = \left\{ \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \alpha \in \mathbb{C} \right\}, \qquad S^*S = \left\{ \begin{pmatrix} \alpha & \beta & 0 \\ \gamma & \delta & 0 \\ 0 & 0 & 0 \end{pmatrix} : \alpha, \beta, \gamma, \delta \in \mathbb{C} \right\},$$

and

$$SS^*M = \left\{ \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \alpha \in \mathbb{C} \right\} \subset M, \qquad MS^*S = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ \alpha & \beta & 0 \\ 0 & 0 & 0 \end{pmatrix} : \alpha, \beta \in \mathbb{C} \right\} \subset M.$$

Thus S is a ternary corner of T and M is a complement of S in T. Note that S is a subalgebra of A, but S is not a corner of A since it does not have an identity element (and A does; $1_A = e_{11} + e_{22} + e_{33}$). Note also that $SM \nsubseteq M$.

Remark 4.1.2. In general, a ternary corner of a $*$-algebra A which is also a subalgebra of A need not be a corner of A. For example, $S = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} : \alpha, \beta \in \mathbb{C} \right\}$ is a subalgebra and a ternary corner of $A = M_2(\mathbb{C})$ but not a corner since it has no (right) identity. This example can be extended to n-by-n matrices, or to matrices in $\mathcal{B}(\ell^2)$ where ℓ^2 is the Hilbert space of complex square-summable sequences and $\mathcal{B}(\ell^2)$ is the C^*-algebra of bounded operators on ℓ^2.

We now state and prove a theorem that characterizes ternary corner algebras and establishes a one-to-one correspondence between ternary corner algebras and bimodular projections – also known as conditional expectations.

Theorem 4.1.3 (Characterization). Let A be a C^*-algebra. Let $S \subseteq A$ and $T \subseteq A$ be pre-TROs such that $S \subseteq T$. Then the following are equivalent:

1. S is a ternary-corner algebra of T.

2. There exists an (SS^*, S^*S)-bimodule map $\mathcal{E}\colon T \to T$ (which means that $\mathcal{E}$ is linear, left SS^*-homogeneous, and right S^*S-homogeneous) such that $\mathcal{E}(T) = S$ and $\mathcal{E} \circ \mathcal{E} = \mathcal{E}$.

3. There exists an (SS^*, S^*S)-bimodule map $\kappa\colon T \to T$ such that $\ker \kappa = S$, $\kappa \circ \kappa = \kappa$, and the following short sequence of (SS^*, S^*S)-bimodule maps

$$0 \longrightarrow S \xrightarrow{\iota \text{ inclusion}} T \xrightarrow{\kappa} \kappa(T) \longrightarrow 0$$

is exact.

Proof. $1 \to 2$: If S is a ternary-corner algebra of T, there exists a vector space $M \subseteq T$ such that $T = S \oplus M$ and $SS^*M \subseteq M$, $MS^*S \subseteq M$. This specifies a map $\mathcal{E}\colon T \to T$ by $\mathcal{E}(s + m) = s$ for $s \in S$ and $m \in M$. It is easy to check that $\mathcal{E}$ is an (SS^*, S^*S)-bimodule map and satisfies the required properties.

$2 \to 1$: If $\mathcal{E}\colon T \to T$ exists, then clearly $T = S \oplus M$ for $M = \ker \mathcal{E}$. We wish to show that $SS^*M \subseteq M$. So for this, pick $s_1, s_2 \in S$ and $m \in M$. Since M is the kernel of $\mathcal{E}$, we have that $\mathcal{E}(m) = 0$. Hence $\mathcal{E}(s_1 s_2^* m) = s_1 s_2^* \mathcal{E}(m) = 0$, and so $s_1 s_2^* m \in M$. This shows the desired inclusion $SS^*M \subseteq M$. We may argue in a similar way to show that $MS^*S \subseteq M$. Thus we see that M is a complement of S in T, so S is a ternary-corner algebra of T.

$2 \to 3$: If $\mathcal{E}$ exists, then $\kappa = 1 - \mathcal{E}$ has the required properties.

$3 \to 1$: If κ exists, then $A = S \oplus M$ for $M = \kappa(T)$. Moreover $SS^*M \subseteq M$ and $MS^*S \subseteq M$ since κ is an (SS^*, S^*S)-bimodule map. Thus, S is a ternary-corner algebra of T. □

4.2 Descent and Transitivity

In this section we study some properties of ternary corners which have already been proved by T. Y. Lam for corner rings. We begin with a property, called *descent*, which asserts that for a given corner S of a ring R, and a complement M of S in R, the following holds (cf. [48, Proposition 2.3 (Descent)]).

1. If S' is a subring of R with $S \subseteq S'$, then S is a corner of S' with a complement $M \cap S'$.

2. If R' is a subring of R with $M \subseteq R'$, then $S \cap R'$ is a corner of R' with a complement M.

We show that the same descent property holds for ternary corners.

Proposition 4.2.1 (Descent)**.** Let A be an associative $*$-algebra. Let $S \subseteq A$ and $T \subseteq A$ be pre-TROs such that $S \subseteq T$. Assume that S is a ternary corner of T with a complement M so that $T = S \oplus M$. Then:

1. If S' is a sub-pre-TRO of T with $S \subseteq S'$, then S is a ternary corner of S' with a complement $M \cap S'$.

2. If T' is a sub-pre-TRO of T with $M \subseteq T'$, then $S \cap T'$ is a ternary corner of T' with a complement M.

Proof. 1. Let $M_0 = M \cap S'$. Then M_0 is a vector subspace of S' and $S' = S \oplus M_0$. Furthermore, we have that $SS^*M_0 \subseteq SS^*M \subseteq M$, and also $SS^*M_0 \subseteq SS^*S' \subseteq S'(S')^*S' \subseteq S'$. Therefore $SS^*M_0 \subseteq M \cap S' = M_0$. The inclusion $M_0S^*S \subseteq M_0$ is proved similarly. This proves that S is a ternary corner of S' with a complement $M \cap S'$.

2. Let $S_0 = S \cap T'$. Then S_0 a sub-pre-TRO of T' and $T' = S_0 \oplus M$. Furthermore, we have that $S_0(S_0)^*M \subseteq SS^*M \subseteq M$, and also $M(S_0)^*S_0 \subseteq MS^*S \subseteq M$. This proves that $S_0 = S \cap T'$ is a ternary corner of T' a complement M. □

Transitivity for corner rings has been proved by T. Y. Lam in [48, Proposition 2.4 (Transitivity)]. It asserts that whenever S is a corner ring of S', and S' in turn is a corner ring of a ring R, then S is a corner of R. Transitivity holds also for ternary corners:

Proposition 4.2.2 (Transitivity)**.** Let A be an associative $*$-algebra. Let $S \subseteq A$, $S' \subseteq A$, and $T \subseteq A$ be pre-TROs such that $S \subseteq S' \subseteq T$. If S is a ternary corner of S' and S' is a ternary corner of T, then S is a ternary corner of T.

Proof. Assume that M_0 is a complement of S in S', and M' is a complement of S' in T. Then $S' = S \oplus M_0$ and $T = S' \oplus M'$. For $M := M_0 + M'$ we have $T = S \oplus M$. Furthermore, $SS^*M \subseteq SS^*M_0 + SS^*M' \subseteq M_0 + S'(S')^*M' \subseteq M_0 + M' = M$, and similarly $MS^*S \subseteq M_0S^*S + M'S^*S \subseteq M_0 + M'(S')^*S' \subseteq M_0 + M' = M$. This proves that S is a ternary corner of T with a complement $M = M_0 + M'$. □

Remark 4.2.3. A ternary corner of a sub-TRO of a TRO T need not in general be a ternary corner of T. For example, if H is an infinite-dimensional Hilbert space, then $\mathcal{K}(H)$, the compact operators on H, is a ternary corner of itself since $\mathcal{K}(H) = \mathcal{K}(H) \oplus \{0\}$, but $\mathcal{K}(H)$ is not a ternary corner of $\mathcal{B}(H)$.

Corollary 4.2.4 (Complement extension)**.** Let A be an associative $*$-algebra, and let $T \subseteq A$ be a pre-TRO. If $S \subseteq S'$ are both ternary corner rings of T, then every complement M' of S' in T can be enlarged to one of S.

Proof. Proposition 4.2.1 (1) implies that S is a ternary corner of S', so $S' = S \oplus M_0$ where M_0 is a complement of S in S'. Since $T = S' \oplus M'$, we have $T = S \oplus (M_0 \oplus M')$ and M' is enlarged to a complement $M_0 \oplus M'$ of S in T. □

4.3 Graphs of TRO Homomorphisms

Let A_1 and A_2 be C^*-algebras, and let T_1 and T_2 be TROs contained in A_1 and A_2, respectively. This means that T_i $(i = 1, 2)$ is a norm-closed subspace of A_i which is algebraically closed under the ternary product

$$T_i \times T_i \times T_i \longrightarrow T_i, \qquad (x, y, z) \longmapsto xy^*z.$$

Consider the C^*-algebra $A = A_1 \oplus A_2$ equipped with pointwise operations and the maximum norm which are defined as follows:

$$\begin{aligned}(a_1, a_2) + (a_1', a_2') &= (a_1 + a_1', a_2 + a_2') \\ \alpha(a_1, a_2) &= (\alpha a_1, \alpha a_2) \\ (a_1, a_2)^* &= (a_1^*, a_2^*) \\ \|(a_1, a_2)\| &= \max\{\|a_1\|, \|a_2\|\}\end{aligned}$$

for $a_i, a_i' \in A_i$, $\alpha \in \mathbb{C}$. Put $T = T_1 \oplus T_2$, and consider T as a TRO contained in A.

Suppose we have a TRO-homomorphism $\phi\colon T_1 \to T_2$, which means that ϕ is a vector space homomorphism preserving the ternary product in the sense that

$$\phi(xy^*z) = \phi(x)\phi(y)^*\phi(z)$$

holds for all elements $x, y, z \in T_1$. Denote by S the graph of ϕ, that is

$$S = \{(x, \phi(x)) : x \in T_1\},$$

and define $\mathcal{E}\colon T \to T$ by $\mathcal{E}(a, b) = (a, \phi(a))$ for all $(a, b) \in T$. Note the following:

1. S is a sub-TRO of T. Indeed, for $x, y, z \in T_1$ we have:

$$\begin{aligned}(x, \phi(x))(y, \phi(y))^*(z, \phi(z)) &= (x, \phi(x))(y^*, \phi(y)^*)(z, \phi(z)) \\ &= (xy^*z, \phi(x)\phi(y)^*\phi(z)) \\ &= (xy^*z, \phi(xy^*z)) \in S.\end{aligned}$$

   Since every TRO-homomorphism ϕ is automatically completely contractive ([33, 26]), it follows that S is norm-closed in T.

2. $\mathcal{E} \circ \mathcal{E} = \mathcal{E}$. Indeed, for $(a, b) \in T$ we have:

   $\mathcal{E} \circ \mathcal{E}(a, b) = \mathcal{E}(a, \phi(a)) = (a, \phi(a)) = \mathcal{E}(a, b)$

3. $\mathcal{E}$ is an (SS^*, S^*S)-bimodule map. Indeed, for $x, y, a, b \in T_1$ we have:

$$\begin{aligned}
\mathcal{E}((x,\phi(x))(y,\phi(y))^*(a,b)) &= \mathcal{E}((x,\phi(x))(y^*,\phi(y)^*)(a,b)) \\
&= \mathcal{E}(xy^*a, \phi(x)\phi(y)^*b) \\
&= (xy^*a, \phi(xy^*a)) \\
&= (xy^*a, \phi(x)\phi(y)^*\phi(a)) \\
&= (x,\phi(x))(y^*,\phi(y)^*)(a,\phi(a)) \\
&= (x,\phi(x))(y,\phi(y))^*\mathcal{E}(a,b),
\end{aligned}$$

and similarly $\mathcal{E}((a,b)(x,\phi(x))^*(y,\phi(y))) = \mathcal{E}(a,b)(x,\phi(x))^*(y,\phi(y))$.

4. $\mathcal{E}(T) = S$ (so $\mathcal{E}$ is a ternary Lam conditional expectation). The inclusion $\mathcal{E}(T) \subseteq S$ follows from the definition of $\mathcal{E}$. The converse inclusion $S \subseteq \mathcal{E}(T)$ follows immediately since $\mathcal{E} \circ \mathcal{E} = \mathcal{E}$.

5. $\mathcal{E}$ is a TRO-homomorphism. Indeed if $(a_1, b_1), (a_2, b_2), (a_3, b_3) \in T$ then we have:

$$\begin{aligned}
\mathcal{E}((a_1,b_1)(a_2,b_2)^*(a_3,b_3)) &= \mathcal{E}((a_1,b_1)(a_2^*,b_2^*)(a_3,b_3)) \\
&= \mathcal{E}(a_1a_2^*a_3, b_1b_2^*b_3) \\
&= (a_1a_2^*a_3, \phi(a_1a_2^*a_3)) \\
&= (a_1a_2^*a_3, \phi(a_1)\phi(a_2)^*\phi(a_3)) \\
&= (a_1,\phi(a_1))(a_2^*,\phi(a_2)^*)(a_3,\phi(a_3)) \\
&= (a_1,\phi(a_1))(a_2,\phi(a_2))^*(a_3,\phi(a_3)) \\
&= \mathcal{E}(a_1,b_1)\mathcal{E}(a_2,b_2)^*\mathcal{E}(a_3,b_3).
\end{aligned}$$

6. It follows that $\mathcal{E}$ is completely contractive ([26, Proposition 2.1]) and, in particular, $\|\mathcal{E}\| = 1$.

7. $\mathcal{E}$ is *-preserving if ϕ is *-preserving.

Corollary 4.3.1. The graph of any TRO-homomorphism appears as a ternary corner of some TRO. More specifically, if $\phi\colon T_1 \to T_2$ is a TRO-homomorphism, then the graph of ϕ, that is the set $S = \{(x, \phi(x)) : x \in T_1\}$, is a norm-closed ternary corner of $T_1 \oplus T_2$.

4.4 The Finite-dimensional Case and Injectivity

We begin this section by stating useful and well-known results.

- Every non-zero finite-dimensional TRO can be identified (is completely isometric) with a direct sum of rectangular matrix spaces ([26] and [70]); i.e., it has the form
$$M_{m_1,n_1}(\mathbb{C}) \oplus \cdots \oplus M_{m_l,n_l}(\mathbb{C}).$$

- Every non-zero finite-dimensional C^*-algebra can be identified (is completely isometric) with a direct sum of full square matrix algebras ([54, Theorem 6.3.8]); i.e., it has the form
$$M_{m_1}(\mathbb{C}) \oplus \cdots \oplus M_{m_l}(\mathbb{C}).$$

- Every finite-dimensional TRO is injective ([26, Proposition A.1]).

- Every finite-dimensional C^*-algebra is injective ([9, IV.2.1.2 (i)]).

- An operator space E is injective if and only if there exists an injective C^*-algebra A and two projections p and q in A such that E is completely isometric to pAq ([65, Theorem 4.5]).

- A finite dimensional operator space E is injective if and only there exists a finite dimensional C^*-algebra A and two projections p and q in A such that E is completely isometric to pAq ([70]).

As an immediate corollary to the above results we get the following proposition:

Proposition 4.4.1. 1. Every finite-dimensional TRO is a ternary corner of any C^*-algebra that contains it.

2. Every finite-dimensional C^*-algebra is a corner of any C^*-algebra that contains it.

3. Every injective operator space is completely isometric to a ternary corner of some injective C^*-algebra.

4. Every finite dimensional injective operator space is completely isometric to a ternary corner of some finite dimensional C^*-algebra.

4.5 Row and Column Hilbert Spaces

Throughout this section H is an arbitrary non-zero complex Hilbert space. We will be concerned with the following two operator space structures on H. One structure is the column operator space structure on H, and is inherited from $\mathcal{B}(H)$ via any self-adjoint rank-one projection $u \in \mathcal{B}(H)$ by the identifications:

$$H \cong \mathcal{B}(\mathbb{C}, H) \cong \mathcal{B}(H)u \subseteq \mathcal{B}(H).$$

When H is considered with this structure, we refer to it as the column Hilbert space, but we deal with $\mathcal{B}(H)u$ rather that with H (for some self-adjoint rank-one projection $u \in \mathcal{B}(H)$).

The other structure on H is the row operator space structure. This is inherited from $\mathcal{B}(H)$ via any self-adjoint rank-one projection $u \in \mathcal{B}(H)$ by the identifications:

$$H \cong \overline{H} \cong \mathcal{B}(H, \mathbb{C}) \cong u\mathcal{B}(H) \subseteq \mathcal{B}(H)$$

where $\overline{H}$ is the complex conjugate space. ($\overline{H}$ is identical to H as an additive group, but has multiplication by scalars given by $(\alpha, \xi) \mapsto \overline{\alpha}\xi$ and inner product $(\xi, \eta) \mapsto \overline{\langle \xi, \eta \rangle} = \langle \eta, \xi \rangle$ for $\alpha \in \mathbb{C}$, $\xi, \eta \in H$.) When H is considered with this

structure, we refer to it as the row Hilbert space, but we deal with $u\mathcal{B}(H)$ rather that with H (for some self-adjoint rank-one projection $u \in \mathcal{B}(H)$).

In order to more explicitly describe the column and row Hilbert spaces, we make a few observations regarding rank-one operators in $\mathcal{B}(H)$. First, every rank-one operator in $\mathcal{B}(H)$ is of the form $\xi \otimes \eta^*$ for non-zero vectors $\xi, \eta \in H$, where

$$\xi \otimes \eta^* \colon H \to H, \qquad (\xi \otimes \eta^*)(\zeta) = \langle \zeta, \eta \rangle \xi \qquad \text{for all } \zeta \in H.$$

Its norm is $\|\xi \otimes \eta^*\| = \|\xi\| \, \|\eta\|$, its adjoint operator is given by $(\xi \otimes \eta^*)^* = (\eta \otimes \xi^*)$, and its trace is given by $\mathrm{tr}(\xi \otimes \eta^*) = \langle \xi, \eta \rangle$ (cf. [1, Lemma 2.61]). Second, for any $\xi, \zeta, \eta, \gamma \in H$ and $a \in \mathcal{B}(H)$ we have the following multiplication rules ([54, p. 55]):

$$\begin{aligned} (\xi \otimes \eta^*)(\zeta \otimes \gamma^*) &= \langle \zeta, \eta \rangle (\xi \otimes \gamma^*) \\ a(\xi \otimes \eta^*) &= a(\xi) \otimes \eta^* \\ (\xi \otimes \eta^*)a &= \xi \otimes (a^*(\eta))^* \end{aligned}$$

Finally, the operator $\xi \otimes \xi^*$ is a rank-one self-adjoint projection if and only if $\langle \xi, \xi \rangle = \|\xi\|^2 = 1$, that is, ξ is a unit vector. Conversely, every rank-one self-adjoint projection is of the form $\xi \otimes \xi^*$ for some unit vector $\xi \in H$.

Now, let $\eta \otimes \eta^*$ be a rank-one self-adjoint projection with range $\mathbb{C} \cdot \eta$ given by some unit vector $\eta \in H$ ($\langle \eta, \eta \rangle = \|\eta\|^2 = 1$). We shall consider the column Hilbert space H_c in $\mathcal{B}(H)$ as

$$\begin{aligned} H_c &= \mathcal{B}(H)(\eta \otimes \eta^*) \\ &= \{a(\eta \otimes \eta^*) \,:\, a \in \mathcal{B}(H)\} \\ &= \{a(\eta) \otimes \eta^* \,:\, a \in \mathcal{B}(H)\} \\ &= \{\xi \otimes \eta^* \,:\, \xi \in H\} \end{aligned}$$

with the inner product $\langle \xi \otimes \eta^*, \zeta \otimes \eta^* \rangle = \langle \xi, \zeta \rangle$ for $\xi, \zeta \in H$. Clearly, H_c is a norm

closed subspace of $\mathcal{B}(H)$, and closed under the ternary product since

$$(\xi \otimes \eta^*)(\zeta \otimes \eta^*)^*(\gamma \otimes \eta^*) = \langle \gamma, \zeta \rangle (\xi \otimes \eta^*)$$

whenever $\xi, \zeta, \gamma \in H$; thus H_c is a TRO contained in $\mathcal{B}(H)$.

Similarly, we shall consider the row Hilbert space H_r in $\mathcal{B}(H)$ as

$$\begin{aligned} H_r &= (\eta \otimes \eta^*)\mathcal{B}(H) \\ &= \{(\eta \otimes \eta^*)a \,:\, a \in \mathcal{B}(H)\} \\ &= \{\eta \otimes (a^*(\eta))^* \,:\, a \in \mathcal{B}(H)\} \\ &= \{\eta \otimes \xi^* \,:\, \xi \in H\} \end{aligned}$$

with the inner product $\langle \eta \otimes \xi^*, \eta \otimes \zeta^* \rangle = \langle \xi, \zeta \rangle$ for $\xi, \zeta \in H$. It is readily seen that H_r is a TRO contained in $\mathcal{B}(H)$.

We will now prove some results regarding H_c (analogous results hold for H_r).

Proposition 4.5.1. If S is a dense ternary corner of H_c then $S = H_c$ (that is, there does not exist a dense proper ternary corner of H_c).

Proof. Assume that $S \subseteq H_c$ is a dense ternary corner of H_c, M a complement; so that $H_c = S \oplus M$ and $SS^*M \subseteq M$, $MS^*S \subseteq M$. First, we show that

$$s_1 s_2^* m = 0 \tag{4.1}$$

for all $s_1, s_2 \in S$, and all $m \in M$. Indeed, if $s_1 = \xi_1 \otimes \eta^* \in S$, $s_2 = \xi_2 \otimes \eta^* \in S$, and $m = \theta \otimes \eta^* \in M$, for some $\xi_1, \xi_2, \theta \in H$, then $s_1 s_2^* m = \langle \theta, \eta \rangle \langle \xi_2, \eta \rangle (\xi_1 \otimes \eta^*) = \langle \theta, \eta \rangle \langle \xi_2, \eta \rangle s_1 \in S$. On the other hand $s_1 s_2^* m \in M$ since $SS^*M \subseteq M$. Therefore $s_1 s_2^* m \in S \cap M = \{0\}$, so $s_1 s_2^* m = 0$.

Second, we show that

$$s^* m = 0 \tag{4.2}$$

for all $s \in S$, and all $m \in M$. Indeed, if there were $s = \xi \otimes \eta^* \in S$ and $m = \theta \otimes \eta^* \in M$, for some $\xi, \theta \in H$, with $s^* m = \langle \theta, \xi \rangle (\eta \otimes \eta^*) \neq 0$, we would have

$\langle \theta, \xi \rangle \neq 0$. But from (4.1) we would also have that $s_1 s^* m = \langle \theta, \xi \rangle s_1 = 0$ for all $s_1 \in S$, which is impossible since $S \neq \{0\}$.

Finally, since S is dense in H_c, for all $m \in M$ we get

$$s^* m = 0 \quad \text{for all } s \in S \qquad \text{if and only if} \qquad m = 0.$$

This and (4.2) imply that $M = \{0\}$, and so $H_c = S \oplus M = S \oplus \{0\} = S$. □

Theorem 4.5.2. Every ternary corner of H_c is closed.

Proof. Let S be a ternary corner of H_c. Then by Proposition 4.2.1 (1), S is a ternary corner of $\overline{S}$, where $\overline{S}$ is the norm-closure of S in H_c. Since S is dense in $\overline{S}$, Proposition 4.5.1 implies that $S = \overline{S}$, and so S is closed. □

Theorem 4.5.3. Every closed subspace S of H_c is a ternary corner of H_c (with complement $S^\perp$).

Proof. Let S be a closed subspace of H_c. If $s_1 = \xi_1 \otimes \eta^* \in S$, $s_2 = \xi_2 \otimes \eta^* \in S$, and $s_3 = \xi_3 \otimes \eta^* \in S$, then $s_1 s_2^* s_3 = \langle \xi_3, \xi_2 \rangle (\xi_1 \otimes \eta^*) = \langle \xi_3, \xi_2 \rangle s_1 \in S$; thus S is a sub-TRO of H_c.

Let M be the orthogonal complement of S in H_c with respect to the inner product on H_c, that is $M = S^\perp = \{m \in H_c : \langle m, s \rangle = s^* m = 0 \text{ for all } s \in S\}$. Then $H_c = S \oplus M$. Since $S^* S \cong \mathbb{C}$, it follows that $MS^*S \subseteq M$, and so M is a right S^*S-module. Also, if $s_1, s_2 \in S$, $m \in M$, then $s_2^* m = 0$ so $s_1 s_2^* m = 0 \in M$. This implies that $SS^*M \subseteq M$, and so M is a left SS^*-module.

This shows that S is a ternary corner of H_c (with complement $S^\perp$). □

Corollary 4.5.4. The ternary corners in H_c are precisely the closed subspaces of H_c.

We may sum up this whole discussion on automatic closure of ternary corners in row and column Hilbert spaces as follows.

1. Every ternary corner of a row Hilbert space is closed.

2. Every ternary corner of a column Hilbert space is closed.

3. There is one-one correspondence between the ternary corner of a row Hilbert space R and the closed subspaces of R.

4. There is one-one correspondence between the ternary corner of a column Hilbert space C and the closed subspaces of C.

For the sake of example, consider the matrix representation for operators in $\mathcal{B}(\ell_2)$, where $\ell_2 = \{(\xi_i) : \sum_{i=1}^{\infty} |\xi_i|^2 < \infty\}$. In particular, $\{e_{ij}\}$ represents the system of unit operators, where for all $i, j \geq 1$, e_{ij} is the matrix with a 1 in the (i, j)-entry and zeros elsewhere. The row Hilbert space determined by e_{11} is

$$R = \overline{\text{linear span}}\,\{e_{1j} : j \geq 1\} = e_{11}\mathcal{B}(\ell_2) \subseteq \mathcal{B}(\ell_2)$$

with their finite-dimensional version $R_n = \text{linear span}\,\{e_{11}, \ldots, e_{1n}\} \subseteq M_n(\mathbb{C})$. The column Hilbert space determined by e_{11} is

$$C = \overline{\text{linear span}}\,\{e_{i1} : i \geq 1\} = \mathcal{B}(\ell_2)e_{11} \subseteq \mathcal{B}(\ell_2)$$

with their finite-dimensional version $C_n = \text{linear span}\,\{e_{11}, \ldots, e_{n1}\} \subseteq M_n(\mathbb{C})$. The elements of R are complex infinite matrices $\begin{pmatrix} \xi_1 & \xi_2 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \vdots \end{pmatrix}$ where $(\xi_i) \in \ell_2$, and the elements of C are complex infinite matrices of the form $\begin{pmatrix} \xi_1 & 0 & \cdots \\ \xi_2 & 0 & \cdots \\ \vdots & \vdots & \vdots \end{pmatrix}$ where $(\xi_i) \in \ell_2$. (R and C are Banach isometric - both Banach isometric to ℓ_2 - but R and C are not completely isomorphic (see [64]). The finite-dimensional versions R_n and C_n are completely isomorphic but not completely isometric.)

Keeping this notation, it easy to see that if S is a dense ternary corner of R then $S = R$ (which is a particular cases of our general result). Indeed, if M is a complement of S in R, then $R = S \oplus M$ and $SS^*M \subseteq M$, $MS^*S \subseteq M$. For $\xi \in S$, $\zeta \in S$, and $\theta \in M$ we have

$$\theta\zeta^*\xi = \begin{pmatrix} \theta_1 & \theta_2 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} \overline{\zeta_1} & 0 & \cdots \\ \overline{\zeta_1}_2 & 0 & \cdots \\ \vdots & \vdots & \end{pmatrix} \begin{pmatrix} \xi_1 & \xi_2 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \vdots \end{pmatrix} = \alpha \begin{pmatrix} \xi_1 & \xi_2 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \vdots \end{pmatrix} \in S$$

where $\alpha = \sum_{i=1}^{\infty} \theta_i\overline{\zeta_i} \in \mathbb{C}$. On the other hand $\theta\zeta^*\xi \in M$ since $MS^*S \subseteq M$.

Therefore $\theta\zeta^*\xi \in S \cap M = \{0\}$, $\theta\zeta^*\xi = 0$ for all $\xi \in S$, all $\zeta \in S$, and all $\theta \in M$. Also, note that $\langle\theta, \zeta\rangle = \sum_{i=1}^{\infty} \theta_i\overline{\zeta_i} = 0$ for all $\vartheta = \begin{pmatrix} \theta_1 & \theta_2 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \vdots \end{pmatrix} \in M, \zeta = \begin{pmatrix} \zeta_1 & \zeta_2 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \vdots \end{pmatrix} \in S$ (if there were $\theta \in M$ and $\zeta \in S$ with $\alpha := \langle\theta, \zeta\rangle \neq 0$, we would have that $\alpha\xi = 0$ for all $\xi \in S$, which is impossible since $S \neq \{0\}$). Finally, since S is dense in R, for all $\theta \in M$ we get $\langle\theta, \zeta\rangle = 0$ for all $\zeta \in S$ if and only if $\theta = 0$. This imply that $M = \{0\}$, and so $R = S \oplus M = S \oplus \{0\} = S$.

4.6 Characterization of Row and Column Hilbert Spaces

The main result we prove in this section is the following characterization theorem.

Theorem 4.6.1. Let H be an arbitrary complex Hilbert space. Assume $T \subseteq \mathcal{B}(H)$ is a TRO with the property that every closed subspace of T is a ternary corner in T. Then T is TRO-isomorphic row or column Hilbert space.

Note. Hypothesis implies each closed subspace of T is a sub-TRO of T.

Proof of Theorem 4.6.1. The proof of the theorem proceeds in three steps. The first step is to show that for all $x \in T$ we have

$$xx^*x = \|x\|^2 x.$$

To do this, consider $x \in T$, $x \neq 0$. The assumption that every closed subspace of T is a ternary corner in T implies that $S = \mathbb{C}x$ is a ternary corner in T. In particular, S is a one-dimensional sub-TRO of T. Therefore since S is closed under the ternary product, there exists a complex number λ such that $xx^*x = \lambda x$. Multiplying by x^* on the left-hand side of the equation yields:

$$\begin{aligned} x^*(xx^*x) &= x^*\lambda x, \\ (x^*x)(x^*x) &= \lambda(x^*x), \end{aligned}$$

which shows that $\lambda > 0$ because x^*x and $(x^*x)^2 = (x^*x)^*(x^*x)$ are both positive. Now, since $\frac{x}{\|x\|}\left(\frac{x}{\|x\|}\right)^*\frac{x}{\|x\|} = \left(\frac{\lambda}{\|x\|^2}\right)\frac{x}{\|x\|}$, for $y = \frac{x}{\|x\|}$ we have $yy^*y = \alpha y$ with $\alpha = \frac{\lambda}{\|x\|^2} > 0$. This implies that $\|\alpha y\| = \|yy^*y\| = \|y\|^3 = 1$. So $\alpha\|y\| = 1$, $\alpha = 1$, and consequently we get $xx^*x = \|x\|^2 x$.

The second step is to show that all two-dimensional subspaces of T are Hilbert spaces. To do this, let $S \subseteq T$ is a two-dimensional subspaces of T. Then S is a TRO of dimension two, and arguments we have had before show that $xx^*x = \|x\|^2 x$ for each $x \in S$. But S is TRO-isomorphic to one of the following three spaces (see [34]):

$$M_{1,1}(\mathbb{C}) \oplus M_{1,1}(\mathbb{C}), \qquad M_{2,1}(\mathbb{C}), \qquad M_{1,2}(\mathbb{C})$$

where $M_{m,n}(\mathbb{C})$ denotes the space of rectangular matrices of size m-by-n whose entries are complex numbers. Notice that both $M_{2,1}(\mathbb{C})$ and $M_{1,2}(\mathbb{C})$ are Hilbert spaces, and that S cannot be TRO-isomorphic to $M_{1,1}(\mathbb{C}) \oplus M_{1,1}(\mathbb{C})$ since for $x = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{smallmatrix}\right) \in M_{1,1}(\mathbb{C}) \oplus M_{1,1}(\mathbb{C})$ we have $xx^*x \neq \|x\|^2 x$.

The third step is a straightforward application of the parallelogram identity. Indeed, since all two-dimensional subspaces of T are Hilbert spaces, the parallelogram identity holds in T. Therefore T is isometric to a Hilbert space. We know that T must be (TRO-isomorphic to) row or column Hilbert space (see [66] or [55, Lemma 2.9]). □

For different conditions that characterize row and column Hilbert space we refer the reader to [52].

Note 1: There is a one-to-one correspondence between the set of all ternary corners of a row/column Hilbert space and the set of all bounded conditional expectations.

Note 2: If T is itself an injective TRO and each sub-TRO $S \subseteq T$ is a ternary corner with a closed complement then there exists a bounded conditional expectation $\mathcal{E}: T \to T$ with range S. Does it follows that $\|\mathcal{E}\| = 1$ and $\|\mathcal{E}\|_{cb} = 1$?

Note 2: Similar types of question arise in the context of 1-*complemented* subspaces. A closed subspace S of a normed spaces X is called 1-*complemented* if S is the range of a norm one projection on X. It is known that If X is a Banach space of dimension more than 2, $2 \leq k < \dim X$ and every k-dimensional subspace

of X is 1-complemented, then X is linearly isometric to a Hilbert space (see [57]). The operator space version of this problem has been studied in [57]. The main result is the following: If every finite-dimensional subspace of an infinite-dimensional operator space X is completely 1-complemented in it, then X is 1-Hilbertian and 1-homogeneous (this is not true for finite-dimensional operator spaces).

Theorem 4.6.2. [56, Theorem 1.1] Let S be a subspace of the predual of a TRO. Then, S is completely 1-complemented if and only if S is completely isometric to the predual of some TRO.

In 2009, Le Merdy, Ricard, and Roydor [50] determine all the completely 1-complemented subspaces of the Schatten class $S^p(H, K)$.

4.7 Tripotents and Peirce Spaces

We begin with a simple observation. Let $T = M_{2,3}(\mathbb{C})$. Then T, the space of all complex 2-by-3 rectangular matrices, is a TRO because $xy^*z \in T$ whenever $x, y, z \in T$. Consider $e = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right) \in T$; e is a tripotent because $ee^*e = e$. Associated to e, we have the Peirce decomposition of T into vector space direct summands:

$$T = T_2(e) \oplus T_1(e) \oplus T_0(e)$$

where $T_\lambda(e) = \{x \in T : ee^*x + xe^*e = \lambda x\}$, $\lambda \in \{0, 1, 2\}$ ($T_\lambda(e)$ is called the Peirce λ-space). Specifically we have:

$$\begin{aligned} T_2(e) &= \left\{ \left(\begin{smallmatrix} x_{1,1} & x_{1,2} & 0 \\ x_{2,1} & x_{2,2} & 0 \end{smallmatrix}\right) : x_{i,j} \in \mathbb{C} \right\}, \\ T_1(e) &= \left\{ \left(\begin{smallmatrix} 0 & 0 & x_{1,3} \\ 0 & 0 & x_{2,3} \end{smallmatrix}\right) : x_{i,j} \in \mathbb{C} \right\}, \\ T_0(e) &= \{0\}. \end{aligned}$$

Moreover, by the standard Peirce theory we have the following decompositions:

$$\begin{aligned} T &= ee^*T \oplus (1-ee^*)T \\ &= Te^*e \oplus T(1-e^*e) \\ &= ee^*Te^*e \oplus ee^*T(1-e^*e) \oplus (1-ee^*)Te^*e \oplus (1-ee^*)T(1-e^*e), \end{aligned}$$

and a straightforward calculations show that:

$$\begin{aligned} T_2(e) &= ee^*Te^*e, \\ T_1(e) &= ee^*T(1-e^*e) \oplus (1-ee^*)Te^*e, \\ T_0(e) &= (1-ee^*)T(1-e^*e). \end{aligned}$$

The Peirce 2-space $T_2(e)$ has a particular significance because it is a ternary corner ring of T (also a TRO) complemented by $M := T_1(e) \oplus T_0(e)$, and $e \in T_2(e)$.

Now, put $S := T_2(e)$, and define $\mathcal{E}_e \colon T \to T$ by $\mathcal{E}_e(x) = ee^*xe^*e$ (for $x \in T$). Then $\mathcal{E}_e(T) = S$, $\mathcal{E}_e(s) = s$ for all $s \in S$, $\mathcal{E}_e \circ \mathcal{E}_e = \mathcal{E}_e$, and $\mathcal{E}_e$ is an (SS^*, S^*S)-bimodule map.

In the following example, we generalize this observation.

Example 4.7.1. Let A be a C^*-algebra, and $T \subseteq A$ a TRO. If $e \in T$ is a tripotent ($ee^*e = e$), then the Peirce decomposition of T (into vector space direct summands) associated with e is

$$T = T_2(e) \oplus T_1(e) \oplus T_0(e)$$

where $T_\lambda(e) = \{x \in T : ee^*x + xe^*e = \lambda x\}$, $\lambda \in \{0, 1, 2\}$ ($T_\lambda(e)$ is called the Peirce λ-space). The space $S := T_2(e)$ is a sub-TRO of T and a ternary corner of T. The corresponding 'ternary conditional expectation' associated with S is the map $\mathcal{E}_e \colon T \to T$ defined by $\mathcal{E}_e(x) = ee^*xe^*e$ (for $x \in T$).

Chapter 5

Corners in $C(K)$

Here we focus largely on commutative C^*-algebras; cthe main results include:

- Good understanding of the relationship between retracts of topological spaces K and corners in $C(K)$.
- Neighbourhood retracts: If K is a compact Hausdorff space and $\mathcal{E}\colon C(K) \to C(K)$ a Lam conditional expectation which is also an algebra homomorphism, then there is a clopen subset $L \subseteq K$ and retraction $\tau\colon L \to L$ such that $\mathcal{E}$ is given by $\mathcal{E}(f)(t) = \begin{cases} f(\tau(t)) & \text{if } t \in L \\ 0 & \text{for } t \in K \setminus L \end{cases}$ (for $f \in C(K)$, $t \in K$). Moreover $\mathcal{E}(\bar{f}) = \overline{\mathcal{E}f}$ for $f \in C(K)$ (Corollary 5.1.10).
- Let K be a compact Hausdorff space, and $S \subseteq C(K)$ a unital C^* subalgebra and a corner with a bounded Lam conditional expectation $\mathcal{E}\colon C(K) \to C(K)$ onto S. Then there exists a compact Hausdorff space L, a *-isomorphism $\phi\colon S \to C(L)$, and a continuous $\tau\colon K \to L$ such that $\phi^{-1}(g) = g \circ \tau$ for $g \in C(L)$. Moreover there exists a complex measure μ_s on K for each $s \in L$ so that μ_s is supported on $\tau^{-1}(s)$ and

$$\mathcal{E}(f)(t) = \int_K f \, d\mu_{\tau(t)} \quad (f \in C(K)).$$

Further, we have $\mathcal{E}^* = \mathcal{E}$ if and only if μ_s is a real-valued measure for each $s \in L$. We have $\mathcal{E} \geq 0$ if and only if $\mu_s \geq 0$ for each $s \in S$. Finally, $\mathcal{E}$ is a homomorphism if and only if each μ_s is a point mass. In that case $\rho \circ \tau \colon K \to K$ is a retraction, $\mathcal{E}(f) = f \circ \rho \circ \tau$ and $\ker \mathcal{E}$ is an ideal of $C(K)$ (Theorem 5.3.3).

- Assume $C(K) = S \oplus M$, where S is a corner, M a complement (and K a compact Hausdorff space). Let $\mathcal{Z}(M)$ denote the set of common zeroes of all the functions in M. Then M separates the points of $K \setminus \mathcal{Z}(M)$, and if $\mathcal{Z}(M)$ has at most one point then M separates the points of K (Theorem 5.4.1).

- A method of constructing discontinuous conditional expectations (p. 104).

- Assuming the Continuum Hypothesis (CH), there is a discontinuous algebra homomorphism $\phi \colon C([0,1]) \to \mathcal{B}(H)$ for some Hilbert space H. Moreover, there exists a commutative corner in the C^*-algebra $C([0,1]) \oplus \mathcal{B}(H)$ which is not closed (Statement 5.5.10). But this corner is not self-adjoint (Remark 5.5.13).

- Assuming the Continuum Hypothesis (CH), if K is an infinite compact Hausdorff space then there exists a commutative corner algebra of $C(K) \oplus \mathcal{B}(H)$ (for some Hilbert space H) that is not closed (Corollary 5.5.11).

- Assuming the Continuum Hypothesis (CH), there exists a Hilbert space H and a commutative corner algebra of $\mathcal{B}(H)$ that is not closed (Corollary 5.5.12).

- Let $S \subseteq A$ be a dense corner in $A = C(K)$. Then $S = A$ (Theorem 5.6.7).

- If S is a $*$-corner in a commutative C^*-algebra $A = C(K)$, then S is closed (Corollary 5.6.8).

- If $S \subseteq A$ is a dense corner in a commutative C^*-algebra A, then $S = A$ (Corollary 5.6.9).

- If S is a $*$-corner in a commutative C^*-algebra A, then S is closed (Corollary 5.6.10).

- Let K be a compact Hausdorff space, $\mathcal{E}\colon C(K) \to C(K)$ a unital Lam $*$-conditional expectation with range $S := \mathcal{E}(C(K))$. As S is a closed (by Corollary 5.6.8) unital $*$-subalgebra we know that there is a $*$-isomorphism $\pi\colon S \to C(L)$ for some compact Hausdorff space L. Associated with the inverse map, the $*$-homomorphism $\pi^{-1}\colon C(L) \to S \hookrightarrow C(K)$, there is a continuous $\phi\colon K \to L$ such that $\pi^{-1}(h) = h \circ \phi$ for $h \in C(L)$. Denote by $\mathfrak{S}(\mathcal{E})$ the separating space of $\mathcal{E}$. Then:

  - $\mathfrak{S}(\mathcal{E}) \subseteq S$ and $\mathfrak{S}(\mathcal{E})$ is a closed ideal of S (Lemma 5.7.1).

  - There is a finite set F of isolated points of L so that $\pi(\mathfrak{S}(\mathcal{E})) = \{h \in C(L) : L \setminus F \subseteq Z(h)\}$ and $\phi^{-1}(t)$ is an infinite clopen subset of K for each $t \in K$. Moreover, there exists a bounded conditional expectation operator $\mathcal{E}'\colon C(K) \to C(K)$ with $\mathcal{E}'(C(K)) = S = \mathcal{E}(C(K))$. If K is connected and $\dim S > 1$, or if L has no isolated points, then $\mathfrak{S}(\mathcal{E}) = \{0\}$ and $\mathcal{E}$ is bounded (Theorem 5.7.3).

- Let A be a unital (not necessarily commutative) C^*-algebra and $S \subseteq A$ a $*$-corner. If $x = x^* \in S$ then $\langle x \rangle \subseteq S$, where $\langle x \rangle$ denotes the smallest unital C^*-subalgebra of A containing x (Statement 5.8.1).

- Let A be a C^*-algebra (not necessarily commutative/unital) and $S \subseteq A$ a $*$-corner of A. If $x = x^* \in S$ then $\langle x \rangle_0 \subseteq S$, where $\langle x \rangle_0$ denotes the smallest C^*-subalgebra of A containing x (Corollary 5.8.2).

- Let A be a C^*-algebra and $S \subseteq A$ a $*$-corner of A. If $x = x^* \in S$, $x \geq 0$, and α is a positive real number, then $x^\alpha \in S$ (in particular $x^{1/2} = \sqrt{x} \in S$ so S is square-root closed) (Corollary 5.8.3).

5.1 Retracts in Compact and Locally Compact Spaces

For a compact Hausdorff space K, we let $C(K)$ denote the unital C^*-algebra of complex-valued continuous functions on K with the supremum norm.

We show that every retract of K gives rise to a corner in $C(K)$. It is important to point out that the C^*-algebra $C(K)$ has no proper projections if K is connected, and therefore $C(K)$ contains no proper Peirce corners.

Proposition 5.1.1. Let K be a compact Hausdorff space and let $\tau\colon K \to K$ be a retraction of K onto $\tau(K)$ (which means that τ is continuous and $\tau \circ \tau = \tau$). Let $\mathcal{E}_\tau\colon C(K) \to C(K)$ be defined by $\mathcal{E}_\tau(f) = f \circ \tau$ for all $f \in C(K)$. Then

- $C(K) = \operatorname{ran}\mathcal{E}_\tau \oplus \ker\mathcal{E}_\tau$,
- $\operatorname{ran}\mathcal{E}_\tau = \mathcal{E}_\tau(C(K))$ is a corner of $C(K)$,
- $\ker\mathcal{E}_\tau$ is an ideal in $C(K)$,
- $\ker\mathcal{E}_\tau = \{f \in C(K) \,:\, f_{|\tau(K)} = 0\}$,

Proof. Setting $A = C(K)$ and $S = \operatorname{ran}\mathcal{E}_\tau = \mathcal{E}_\tau(C(K))$ we see that S is a subalgebra of A. Also, $\mathcal{E}_\tau$ is an S-bimodule map such that $\operatorname{ran}\mathcal{E}_\tau = S$ and $\mathcal{E}_\tau \circ \mathcal{E}_\tau = \mathcal{E}_\tau$. Thus, Proposition 3.1.5 $(2 \to 1)$ yields that S is a corner algebra of A. Straightforward calculations show that $\ker\mathcal{E}_\tau$ is an ideal in $C(K)$. □

Remark 5.1.2. Observe that if K is a compact Hausdorff space and $\tau\colon K \to K$ is a retraction then, by Proposition 5.1.1, there is a corresponding Lam conditional expectation

$$\begin{aligned} \mathcal{E}_\tau\colon C(K) &\to C(K) \\ \mathcal{E}_\tau(f) &= f \circ \tau. \end{aligned}$$

We have $\mathcal{E}_\tau \circ \mathcal{E}_\tau = \mathcal{E}_\tau$ and $\mathcal{E}_\tau$ is a *-algebra homomorphism with kernel

$$M_\tau = \{f \in C(K) : f\,|_{\tau(K)} = 0\}$$

a closed two sided ideal in $C(K)$. Note that $\mathcal{E}_\tau$ is a conditional expectation in the usual sense (see [9, II.6.10]) because it is a positive unital idempotent map on $C(K)$, hence completely positive, and it is then certainly a Lam $*$-conditional expectation.

Example 5.1.3. Consider $K = [0,1]$, and the retraction $\tau\colon K \to K$ given by $\tau(x) = 0$ for all $x \in K$. Then τ retracts $[0,1]$ into $\{0\}$ and $\mathcal{E}_\tau\colon C(K) \to C(K)$, given by $\mathcal{E}_\tau(f) = f \circ \tau$ for all $f \in C(K)$, projects $C(K)$ onto the space of constant functions. Thus the corner algebra $\operatorname{ran}\mathcal{E}_\tau = \mathcal{E}_\tau(C(K))$ consists precisely of the constant functions in $C(K)$.

Example 5.1.4. Consider $K = [0,1] \times [0,1]$, and the retraction $\tau\colon K \to K$ given by $\tau(x,y) = (x,0)$ for all $(x,y) \in K$. We see that τ retracts K into $\{(x,0) : 0 \leq x \leq 1\}$ and $\mathcal{E}_\tau\colon C(K) \to C(K)$, given by $\mathcal{E}_\tau(f) = f \circ \tau$ for all $f \in C(K)$, projects $C(K)$ onto $\{f(\cdot\,,0) : f \in C(K)\}$. Thus the corner algebra $\operatorname{ran}\mathcal{E}_\tau = \mathcal{E}_\tau(C(K))$ consists precisely of those functions in $C(K)$ which depend on the first variable only.

Example 5.1.5. Let X is a noncompact locally compact Hausdorff space, $C_0(X)$ the C^*-algebra of continuous complex-valued functions on X that vanish at infinity.

The Alexandroff one-point compactification of X is the space $X \cup \{\omega\}$, where $\omega \notin X$. The open sets of the space $X \cup \{\omega\}$ are the open sets of X together with the sets of the form $G \cup \{\omega\}$, where G is an open subset of X such that $X \setminus G$ is (closed and) compact. In other words, for a point x in X, basic neighbourhoods in $X \cup \{\omega\}$ are the neighbourhoods of x in the original topology on X. Basic neighbourhoods of ω in $X \cup \{\omega\}$ are the neighbourhoods of the form $(X \setminus F) \cup \{\omega\}$, where $F \subseteq X$ is compact. The space $X \cup \{\omega\}$ is compact and Hausdorff, and X is dense and open in $X \cup \{\omega\}$.

Notice that the C^*-algebra $A = C_0(X)$ is not unital. However, for the natural unitization $A^\sharp = A \oplus \mathbb{C}$ of A we have that $A^\sharp \cong C(X \cup \{\omega\})$, where $1 = (0,1)$ in $A^\sharp$ is identified with $1_{X \cup \{\omega\}}$.

A Generalization of the Alexandroff one-point compactification of X is known as Alexandroff-type compactification and it is constructed as follows: Let $h\colon X \to$

K be a continuous map, K a compact Hausdorff with $X \cap K = \emptyset$. The open sets of the space $X \cup K$ (relative to h) are:

- the open sets of X together with
- the sets of the form $(h^{-1}(U) \setminus F) \cup U$, where $U \subseteq K$ is open and $F \subseteq X$ is compact.

We always assume that h and $X \cup K$ are as above. The space $X \cup K$ is compact and Hausdorff, X is open in $X \cup K$, K is closed in $X \cup K$, and both X and K inherit their original topology as subspaces of $X \cup K$ ([20, Lemma 3.2]). It is not hard to see that the map $\tau\colon X \cup K \to K$ defined by

$$\tau(t) = \begin{cases} h(t) & \text{for } t \in X \\ 1_K(t) = t & \text{for } t \in K \end{cases}$$

is a retraction (i.e., continuous with $\tau \circ \tau = \tau$) of $X \cup K$ onto K. Now, define $\mathcal{E}_\tau\colon C(X \cup K) \to C(X \cup K)$, by $\mathcal{E}_\tau(f) = f \circ \tau$ for all $f \in C(X \cup K)$. Then $\mathcal{E}_\tau(C(X \cup K))$ is a corner of $C(X \cup K)$ given by the retraction τ.

From Proposition 5.1.1 we know that every retraction $\tau\colon K \to K$ gives rise to a corner in $C(K)$. A natural question is when the converse is true. That is, does every corner in $C(K)$ arise from some retraction $\tau\colon K \to K$? In general, the answer is no (Proposition 5.1.6), but in a special case the answer is yes (Theorem 5.1.9).

Proposition 5.1.6. Let $K = S^1 = \{e^{i\theta} : 0 \leq \theta \leq 2\pi\}$ and define

$$\mathcal{E}\colon C(K) \to C(K) \text{ by } \mathcal{E}(f)(\zeta) = \frac{f(\zeta) + f(-\zeta)}{2}.$$

Then $\mathcal{E} \circ \mathcal{E} = \mathcal{E}$, $S = \{f \in C(K) : \mathcal{E}(f) = f\}$ is a closed *-subalgebra of $C(K)$, $\mathcal{E}$ is an S-module map and S is a corner in $C(K)$.

There is no retraction $\tau\colon K \to K$ (continuous map with $\tau \circ \tau = \tau$) with $S = \{f \in C(K) : f \circ \tau = f\}$.

Proof. If $g \in C(K)$ then $f(\zeta) = g(\zeta^2)$ is in S. If $\zeta_0 \in S^1$ has $\tau(\zeta_0)^2 \neq \zeta_0^2$, there is $g \in C(K)$ with $g(\tau(\zeta_0)^2) \neq g(\zeta_0^2)$ and so $f(\zeta) = g(\zeta^2)$ has $f \in S$ but $f \circ \tau \neq f$. Thus we must have $\tau(\zeta)^2 = \zeta^2$ for all ζ and

$$\frac{\tau(\zeta)}{\zeta} = \pm 1.$$

By connectedness of S^1 this continuous function is constant and neither $\tau(\zeta) = \zeta$ nor $\tau(\zeta) = -\zeta$ will work as a retract such that $\mathcal{E}(f) = f \circ \tau$.

Note that for the same S there are other complements, or equivalently other Lam conditional expectation operators. Given a function $h\colon S^1 \to \mathbb{C}$ such that $h(\zeta) + h(-\zeta) = 1$ we can define $\mathcal{E}_h\colon C(S^1) \to C(S^1)$ by

$$\mathcal{E}_h(f)(\zeta) = h(\zeta)f(\zeta) + h(-\zeta)f(-\zeta).$$

The original $\mathcal{E}$ arises from $h(\zeta) = 1/2$.

To get other examples, we could for instance take

$$h(e^{i\theta}) = \frac{1}{2} + \sin\theta \qquad (0 \leq \theta \leq \pi)$$

and $h(-e^{i\theta}) = h(e^{i(\theta+\pi)}) = 1 - h(e^{i\theta})$ (for $0 \leq \theta \leq \pi$)). This defines a continuous $h\colon S^1 \to \mathbb{R}$ since $h(1) = h(-1) = 1/2$.

We can check that in this example $(\mathcal{E}_h)^* = \mathcal{E}_h$ if and only if h is real valued, and $\mathcal{E}_h \geq 0$ if and only if $h \geq 0$. In this example, there are no extreme points in $\{\mathcal{E}_h : h \geq 0\}$. □

Lemma 5.1.7. If A is an algebra and $\mathcal{E}\colon A \to A$ is a Lam conditional expectation, then $\mathcal{E}$ is an algebra homomorphism if and only if $\ker \mathcal{E}$ is an ideal of A.

Proof. This follows by Proposition 2.4.2 and linearity of $\mathcal{E}$, which we assume in the algebra context. □

Theorem 5.1.8 ([63, Theorem 2.5.17])**.** Let $\theta\colon C(K) \to C(L)$ be an algebra homomorphism, where K and L are compact Hausdorff spaces. Then $\|\theta\| \leq 1$.

Proof. For each $t \in L$, consider the character $\pi_t \colon C(L) \to \mathbb{C}$ given by $\pi_t(g) = g(t)$ for $g \in C(L)$. The composition $\pi_t \circ \theta \colon C(K) \to \mathbb{C}$ is a character of $C(K)$ with $(\pi_t \circ \theta)(1_{C(K)}) = \pi_t(1_{C(L)}) = 1 \neq 0$. It follows (see for instance [38, Corollary 3.4.2]) that there is $s \in K$ so that $(\pi_t \circ \theta)(f) = f(s)$ for $f \in C(K)$. Hence $|(\theta(f))(t)| = |(\pi_t \circ \theta)(f)| = |f(s)| \leq \|f\|$ for $f \in C(K)$. As this holds for each $s \in L$ we get $\|\theta(f)\| \leq \|f\|$ and so $\|\theta\| \leq 1$, θ must be continuous. □

Theorem 5.1.9. Let K be a compact Hausdorff space and $\mathcal{E} \colon C(K) \to C(K)$ a unital Lam conditional expectation which is also an algebra homomorphism. Then there is a retraction $\tau \colon K \to K$ such that $\mathcal{E} = \mathcal{E}_\tau$ (where $\mathcal{E}_\tau(f) = f \circ \tau$ for $f \in C(K)$).

Proof. Note that $\mathcal{E}$ is continuous since it is a homomorphism (see 5.1.8, [63, Theorem (2.5.17)]). By Lemma 5.1.7, $\ker \mathcal{E}$ is an ideal in $C(K)$, in fact a closed ideal and hence there is a closed $K_1 \subseteq K$ so that

$$\ker \mathcal{E} = \{f \in C(K) : f \mid_{K_1} = 0\}$$

(see [63, Theorem (4.2.4)]). If we let $S_\mathcal{E} = \mathcal{E}(C(K))$, then $S_\mathcal{E}$ is a closed subalgebra of $C(K)$ (containing the constants) and $\mathcal{E}$ induces an algebra isomorphism

$$\tilde{\mathcal{E}} \colon C(K)/\ker \mathcal{E} \to S_\mathcal{E}$$

given by $\tilde{\mathcal{E}}(f + \ker \mathcal{E}) = \mathcal{E}(f)$.

We also have an isomorphism

$$\pi \colon C(K)/\ker \mathcal{E} \to C(K_1)$$

given by $\pi(f + \ker \mathcal{E}) = f \mid_{K_1}$ (see [63, Theorem (4.2.4)]).

Now $\tilde{\mathcal{E}} \circ \pi^{-1} \colon C(K_1) \to S_\mathcal{E} \subseteq C(K)$ is a unital algebra homomorphism and so there exists a continuous map $\phi \colon K \to K_1$ such that $(\tilde{\mathcal{E}} \circ \pi^{-1})(h) = h \circ \phi$ for $h \in C(K_1)$ [9, II.2.2.5].

Let $\tau \colon K \to K$ be given by $\tau(t) = \phi(t)$ (for $t \in K$), so that τ has the same

values as ϕ but with a different co-domain. Note that τ is continuous (since ϕ is). We claim that $\mathcal{E}(f) = f \circ \tau$ for $f \in C(K)$.

For $f \in C(K)$,

$$\begin{aligned} \pi(f + \ker \mathcal{E}) = f \mid_{K_1} &\Rightarrow \pi^{-1}(f \mid_{K_1}) = f + \ker \mathcal{E} \\ &\Rightarrow (\tilde{\mathcal{E}} \circ \pi^{-1})(f \mid_{K_1}) = \mathcal{E}(f). \end{aligned}$$

But also $(\tilde{\mathcal{E}} \circ \pi^{-1})(f \mid_{K_1}) = (f \mid_{K_1}) \circ \phi = f \circ \tau$. Hence we have $\mathcal{E}(f) = f \circ \tau$ for $f \in C(K)$.

Since $\mathcal{E}(\mathcal{E}(f)) = \mathcal{E}(f)$ we must have $f \circ \tau \circ \tau = f \circ \tau$ for each $f \in C(K)$. Because the functions in $C(K)$ separate the points of K it follows that $\tau \circ \tau = \tau$ so that τ is a retraction. □

Corollary 5.1.10. Let K be a compact Hausdorff space and $\mathcal{E} \colon C(K) \to C(K)$ a Lam conditional expectation which is also an algebra homomorphism. Then there is a clopen subset $L \subseteq K$ and retraction $\tau \colon L \to L$ such that $\mathcal{E}$ is given by

$$\mathcal{E}(f)(t) = \begin{cases} f(\tau(t)) & \text{if } t \in L \\ 0 & \text{for } t \in K \setminus L \end{cases}$$

(for $f \in C(K)$, $t \in K$). Moreover $\mathcal{E}(\bar{f}) = \overline{\mathcal{E}f}$ for $f \in C(K)$.

Proof. Let 1_K denote the constant function 1 in $C(K)$. Then we have $(\mathcal{E}(1_K))^2 = \mathcal{E}(1_K)\mathcal{E}(1_K) = \mathcal{E}(1_K\mathcal{E}(1_K)) = \mathcal{E}(1_K)$ and so there is $L = \{t \in K : \mathcal{E}(1_K)(t) = 1\}$ so that $\mathcal{E}(1_K) = 1_L$ (the characteristic function of L). Moreover since $1_L \in C(K)$, $L \subseteq K$ is a clopen subset. Since $\mathcal{E}(1_K - 1_L) = \mathcal{E}(1_K) - \mathcal{E}(1_L) = \mathcal{E}(1_K) - \mathcal{E}(\mathcal{E}(1_K)) = 0$, we have that $1_K - 1_L \in \ker \mathcal{E}$ (which is an ideal). So if $f \in C(K)$ then $\mathcal{E}(f) = \mathcal{E}(1_L f + (1_K - 1_L)f) = \mathcal{E}(1_L f) = 1_L \mathcal{E}(1_L f)$.

Identifying $C(L)$ with $\{f \in C(K) : f = 1_L f\}$ via $g \in C(L) \mapsto \tilde{g} \in C(K)$ (where $\tilde{g} \mid_L = g$ and $\tilde{g}(t) = 0$ for $t \in K \setminus L$) we see that $\mathcal{E}$ induces a unital Lam conditional expectation $\mathcal{E}_L \colon C(L) \to C(L)$ by $\mathcal{E}_L(g) = \mathcal{E}(\tilde{g}) \mid_L$ such that $\mathcal{E}_L$ is a homomorphism. The result follows by applying Theorem 5.1.9 to $\mathcal{E}_L$. □

We will now consider locally compact Hausdorff spaces X along with $C_0(X)$, the space of continuous complex-valued functions on X that vanish at infinity.

Example 5.1.11. Let X be a locally compact Hausdorff space and $X^* = X \cup \{\omega\}$ the space X with one point adjoined according to the recipe for the one point compactification. We use this notation here even when X is already compact, in which case $\{\omega\}$ is open (and closed) in X^*. As in Example 5.1.5 subsets U of X^* are open if $U \cap X$ is open in X and if $\omega \in U$ we insist that $X^* \setminus U$ be a compact subset of X.

We consider $C_0(X)$ as embedded in $C(X^*)$ via

$$f \mapsto \tilde{f} \colon C_0(X) \to C(X^*)$$

where

$$\tilde{f}(t) = \begin{cases} f(t) & \text{if } t \in X \\ 0 & \text{for } t = \omega \end{cases}$$

Note then that this identifies $C_0(X)$ with $\{g \in C(X^*) : g(\omega) = 0\}$ (the maximal ideal of $C(X^*)$ consisting of functions which take the value zero at ω) and $f \mapsto \tilde{f}$ is a *-algebra isomorphism onto its range.

If $\tau \colon X^* \to X^*$ is a retraction such that $\tau(\omega) = \omega$ then we can define a Lam conditional expectation

$$\mathcal{E}_{\tau,*} \colon C_0(X) \to C_0(X)$$

by

$$\mathcal{E}_{\tau,*}(f) = (\tilde{f} \circ \tau)|_X.$$

Corollary 5.1.12. If X is a locally compact Hausdorff space and $\mathcal{E} \colon C_0(X) \to C_0(X)$ is a Lam conditional expectation which is also an algebra homomorphism, then there is a retraction $\tau \colon X^* \to X^*$ ($X^* = X \cup \{\omega\}$) with $\tau(\omega) = \omega$ such that $\mathcal{E} = \mathcal{E}_{\tau,*}$ in the notation of Example 5.1.11 above.

Proof. First consider the case when X is compact. We apply Corollary 5.1.10

above to get $L \subseteq X$ compact and clopen and $\tau\colon L \to L$ a retraction with

$$\mathcal{E}(f)(t) = \begin{cases} f(\tau(t)) & \text{if } t \in L \\ 0 & \text{if } t \in X \setminus L \end{cases}.$$

Define a retraction $\rho\colon X^* \to X^*$ by $\rho(t) = \tau(t)$ for $t \in L$ and $\rho(t) = \omega$ for $t \in (X \setminus L) \cup \{\omega\}$. Since L is clopen and so is $\{\omega\}$, ρ is continuous. We can verify that $\rho \circ \rho = \rho$ and $\mathcal{E} = \mathcal{E}_{\rho,*}$.

In the case that X is not compact, note that $C(X^*)$ is isomorphic as a *-algebra to the unitisation $C_0(X)^\sharp$. The isomorphism is given by $g \mapsto \phi(g) := (g|X - g(\omega), g(\omega))$. Indeed, if $\phi(g_1) = \phi(g_2)$ then $g_1(\omega) = g_2(\omega)$ and $g_1|X = g_2|X$, thus $g_1 = g_2$. On the other hand, if $(h, \alpha) \in C_0(X)^\sharp$ then $\phi(g) = (h, \alpha)$ where

$$g(x) = \begin{cases} h(x) + \alpha & \text{if } x \in X \\ \alpha & \text{if } x = \omega \end{cases}.$$

Regard $\mathcal{E}^\sharp\colon C_0(X) \oplus \mathbb{C} \to C_0(X) \oplus \mathbb{C}$ as $\mathcal{E}^\sharp\colon C(X^*) \to C(X^*)$. By Proposition 3.3.1, $\mathcal{E}^\sharp$ is an algebra homomorphism and a Lam conditional expectation.

We apply Corollary 5.1.10 above to get $L \subset X^*$ clopen and $\tau\colon L \to L$ a retraction so that

$$\mathcal{E}^\sharp(f)(t) = \begin{cases} f(\tau(t)) & \text{if } t \in L \\ 0 & \text{if } t \in X^* \setminus L \end{cases}, \qquad \text{for } f \in C(X^*). \tag{5.1}$$

Since $C_0(X)$ can be identified with the maximal ideal of $C(X^*)$ consisting of functions which take the value zero at ω, i.e., $C_0(X) = \{g \in C(X^*) : g(\omega) = 0\}$, we have

$$\mathcal{E}^\sharp(C_0(X)) = \mathcal{E}^\sharp(\{g \in C(X^*) : g(\omega) = 0\}) \subset C_0(X),$$

and, therefore, if $\omega \in L$ then $\tau(\omega) = \omega$. (Indeed: if $\omega \in L$ and $\tau(\omega) = t \in X$, there is $f \in C_0(X)$ with $f(t) = 1$ and then we have a contradiction from $0 = \mathcal{E}^\sharp(f)(\omega) = \tilde{f}(\tau(\omega)) = f(t) \neq 0$.) If $\omega \notin L$ then $\omega \in X^* \setminus L$, $L \subseteq X$ is compact, and $\mathcal{E}^\sharp$ is given by (5.1).

Thus we can extend τ to a retraction $\rho\colon X^* \to X^*$ by $\rho(t) = \tau(t)$ for $t \in L$ and $\rho(t) = \omega$ for $t \in X^* \setminus L$. Since L is clopen ρ is continuous. We can verify that $\rho \circ \rho = \rho$ and $\mathcal{E} = \mathcal{E}_{\rho,*}$.

□

5.2 Sigma-algebra of Sets and Commutative Algebras

Let (X, Σ, μ) be a probability space. The relationship between sub-sigma-algebras of Σ and corners in the von Neumann algebra $L^\infty(X, \Sigma, \mu)$ is analogous to that between retracts of a compact Hausdorff space K and corners in the C^*-algebra $C(K)$. Every sub-σ-algebra of Σ gives rise to a corner in $L^\infty(X, \Sigma, \mu)$, but not conversely. Indeed, if Γ is a sub-sigma-algebra of Σ, there exists an ordinary in the sense of probability theory conditional expectation $\mathcal{E}_\Gamma$ acting on $L^2(X, \Sigma, \mu)$ such that $\mathcal{E}_\Gamma(L^\infty(X, \Sigma, \mu)) = L^\infty(X, \Gamma, \mu)$. Therefore $L^\infty(X, \Gamma, \mu)$ is a corner in $L^\infty(X, \Sigma, \mu)$ (cf. [9, II.6.10.4 (i)]).

However the converse, that all corners S in $L^\infty(X, \Sigma, \mu)$ are of the form $S = L^\infty(X, \Gamma, \mu)$ for some sub-sigma-algebra Γ of Σ, is not true in general. In other words, a corner of a commutative von Neumann algebra need not be a von Neumann algebra. Here is an example:

Example 5.2.1. Consider the algebra

$$A = B^\infty([0,1])/N([0,1]),$$

where $B^\infty([0,1])$ is the C^*-algebra of complex-valued bounded Borel functions on $[0,1]$ equipped with the supremum norm, and $N([0,1])$ is the ideal of meagre Borel functions; $N([0,1]) = \{f \in B^\infty([0,1]) : \{x \in [0,1] : f(x) \neq 0\}$ is of 1st category$\}$. Since A is a unital C^*-algebra[1], there is an injective $*$-homomorphism (embedding)

[1]The fact that A is not a von Neumann algebra is in the famous paper [21]. (A is a von Neumann algebra iff it is a dual space iff the union ot the supports of the normal measures is dense. But there are no non-zero normal measures by a theorem of Horn-Tarski.)

$\pi\colon A \to \mathcal{B}(H)$, for some Hilbert space H, with $\pi(1) = 1$. Now, $\pi^{-1}\colon \pi(A) \to A$ extends to a completely positive map $\phi\colon \mathcal{B}(H) \to A$, and $\pi \circ \phi\colon \mathcal{B}(H) \to \pi(A)$ is a unital completely positive projection (so it is an ordinary conditional expectation). But $\pi(A)$ is not a von Neumann algebra (it is a commutative AW^*-algebra). Indeed, if M is the weak $*$-algebra generated by $\pi(A) \subseteq \mathcal{B}(H)$ (actually the double commutant $\pi(A)''$), then $(\pi \circ \phi)_{|M}\colon M \to \pi(A)$ is an ordinary conditional expectation. Since M is a von Neumann algebra, $M \cong L^\infty(X, \Sigma, \mu)$ for some measure space (X, Σ, μ). But $\pi(A)$ is not of the form $L^\infty(X, \Gamma, \nu)$ because it is not weak $*$-closed algebra. So $\pi(A)$ is a corner in some von Neumann algebra $\mathcal{B}(H)$ and $\pi(A)$ is not a von Neumann algebra.

Example 5.2.2. Let $X = \{0, 1\}$, and consider the σ-algebra $\Sigma = \mathcal{P}(X) = \{\emptyset, \{0\}, \{1\}, X\}$ of subsets of X. Then $\Gamma = \{\emptyset, X\}$ is a sub-σ-algebra of Σ, and $\mu\colon \sigma \to [0, 1]$, defined by $\mu(\{0\}) = \frac{1}{2} = \mu(\{1\})$, is a probability measure. Γ gives rise to a corner in $L^\infty(X, \Sigma, \mu)$ via the standard conditional expectation $\mathcal{E}_\Gamma\colon L^\infty(X, \Sigma, \mu) \to L^\infty(X, \Sigma, \mu)$ defined by $\mathcal{E}_\Gamma(f) = \frac{f(0)+f(1)}{2} 1_{\{0,1\}}$, where $1_{\{0,1\}}$ is the characteristic function of $\{0, 1\}$. Note that $\mathcal{E}_\Gamma$ is not multiplicative, for example $\mathcal{E}_\Gamma(1_{\{0\}} \cdot 1_{\{1\}}) \neq \mathcal{E}_\Gamma(1_{\{0\}}) \cdot \mathcal{E}_\Gamma(1_{\{1\}})$.

5.3 Algebras of Continuous Functions and Measures

We briefly recall some well-known and important facts about algebras of continuous functions, their first duals, and Borel measures.

There is a contravariant functor from the category of compact Hausdorff spaces to the category of unital commutative C^*-algebras. It takes a compact Hausdorff space K to the C^*-algebra $C(K)$ of continuous complex valued functions of K, and takes a continuous map $\tau\colon K \to L$ between compact Hausdorff spaces K and L to the map $\tau^\circ\colon C(L) \to C(K)$ defined by $\tau^\circ(f) = f \circ \tau$ for $f \in C(L)$. The diagram

below illustrates this.

$$\begin{array}{ccc} L & \longrightarrow & C(L) \\ {\scriptstyle\tau}\uparrow & & \downarrow{\scriptstyle\tau^\circ} \\ K & \longrightarrow & C(K) \end{array}$$

The map τ° is a *-homomorphism with $\|\tau^\circ\| = 1$. Further, τ° is an injection/a surjection if and only if τ is a surjection/an injection, respectively.

The 'converse' functor goes the other way taking a commutative unital C^*-algebra onto its space of *characters*. If $\phi\colon C(L) \to C(K)$ is a *-homomorphism then ϕ is continuous with $\|\phi\| = 1$, and there exists a continuous map $\tau\colon K \to L$ with $\phi = \tau^\circ$; indeed $\tau = \phi'_{|K}$ [9, II.2.2.5].

A *character* on a unital commutative normed algebra A is a non-zero multiplicative linear functional from A onto the field $\mathbb{C}$. The set of all characters on A, denoted by $\Omega(A)$, is a subset of the closed unit ball of the topological dual A' of A. More information about characters is recorded in the following propositions.

Proposition 5.3.1. Let A be a unital commutative Banach algebra.

1. The set $\Omega(A)$ is non-empty.
2. If $\tau \in \Omega(A)$, then $\|\tau\| = 1 = \tau(1)$.
3. $\Omega(A)$ with the subspace topology inherited from the weak* topology on A' is a compact Hausdorff space. ($\Omega(A)$ is always taken to be equipped with this topology.)
4. If $\tau \in \Omega(A)$, then $\ker\tau$ is a maximal two-sided ideal of A.
5. The map $\tau \mapsto \ker\tau$ defines a bijection from $\Omega(A)$ onto the set of all maximal two-sided ideals of A.
6. An element $x \in A$ is invertible in A if and only if $\tau(x) \neq 0$ for all $\tau \in \Omega(A)$.
7. If $x \in A$, then $\sigma(x) = \{\tau(x) : \tau \in \Omega(A)\}$, $r(x) = \sup\{|\tau(x)| : \tau \in \Omega(A)\}$, where $\sigma(x)$ and $r(x)$ denote respectively the ordinary spectrum of x and the spectral radius of x.

Proposition 5.3.2. Let K be a non-empty compact Hausdorff space.

1. $\Omega(C(K)) = \{\delta_x : x \in K\}$, where the point evaluation functionals $\delta_x : C(K) \to \mathbb{C}$ are defined by $\delta_x(f) = f(x)$ for $x \in K$.

2. The map $K \to \Omega(C(K))$, $x \mapsto \delta_x$ is a homeomorphism, thus we have $\Omega(C(K)) = \{\delta_x : x \in K\} \cong K$.

3. $K \cong \{I \subsetneq C(K) : I \text{ is a maximal norm-closed ideal of } C(K)\}$.

4. Every unital commutative C^*-algebra A can be identified with $C(\Omega(A))$.

We now turn our attention to the concept of representation of elements in $C(K)'$ by measures on K. Here, as throughout, $C(K)'$ denotes the continuous dual space of $C(K)$, that is the space of all continuous linear functionals from $C(K)$ onto $\mathbb{C}$, and K is a non-empty compact Hausdorff space.

Denote by $\mathscr{B}(K)$ the smallest σ-algebra of subsets of K containing all of the open sets of K. The elements of $\mathscr{B}(K)$ are called Borel sets. If $B \in \mathscr{B}(K)$, call a family $\{B_i\}_{i=1}^{\infty}$ of Borel sets a partition of B if $B = \cup_{i=1}^{\infty} B_i$ and $B_i \cap B_j = \emptyset$ for $i \neq j$.

A complex Borel measure is any function $\mu : \mathscr{B}(K) \to \mathbb{C}$ such that

$$\mu(\emptyset) = 0, \qquad \mu(B) = \sum_{i=1}^{\infty} \mu(B_i)$$

for all $B \in \mathscr{B}(K)$ and all partitions $\{B_i\}_{i=1}^{\infty}$ of B. The total variation of a complex Borel measure μ is the non-negative, real-valued, countably additive Borel measure $|\mu|$ defined by

$$|\mu|(B) = \sup\left\{\sum_{i=1}^{\infty} |\mu(B_i)| : \{B_i\}_{i=1}^{\infty} \text{ is a partition of } B\right\}$$

for all $B \in \mathscr{B}(K)$ (cf. [67, Section 6.1]).

A complex Borel measure μ is called

- inner regular if for each Borel set B

$$\mu(B) = \sup\{\mu(F) : F \subseteq B, \quad F \text{ compact}\},$$

- outer regular if for each Borel set B

$$\mu(B) = \inf\{\mu(U) : B \subseteq U, \quad U \text{ open}\},$$

- regular if it is both inner regular and outer regular.

Let $\mathcal{M}(K)$ denote the set of all complex-valued, countably additive, regular Borel measures on K. It is known that $\mathcal{M}(K)$ is a Banach space. The vector space operations on $\mathcal{M}(K)$ are defined as:

$$(\mu + \nu) = \mu(B) + \nu(B), \quad (\alpha\mu)(B) = \alpha\mu(B), \qquad \text{for } \mu, \nu \in \mathcal{M}(K), \alpha \in \mathbb{C},$$

for all Borel sets $B \subseteq K$, and the norm is $\|\mu\| = |\mu|(K)$ for $\mu \in \mathcal{M}(K)$.

From the Riesz representation theorem ([32, pp. 247-248], [25, §IV.6.3]) we know that $\mathcal{M}(K)$ is isometrically isomorphic to $C(K)'$ via the mapping $\mathcal{M}(K) \ni \mu \mapsto I_\mu \in C(K)'$ where

$$I_\mu(f) = \int_K f d\mu, \qquad f \in C(K)$$

(see also [67, Theorem 6.19] for the version of the Riesz representation theorem here).

By the *support* $\operatorname{supp}\mu$ of a complex measure μ we mean the support of $|\mu|$. To define $\operatorname{supp}|\mu|$ we say that $x \in K \setminus \operatorname{supp}|\mu|$ if there is an open set $U \subseteq K$ with $x \in U$ and $|\mu|(U) = 0$. Since $U \subset K \setminus \operatorname{supp}|\mu|$ for any such U, $K \setminus \operatorname{supp}|\mu|$ is open, $\operatorname{supp}|\mu|$ is closed. If K is second countable there are countably many open sets U_n with $\bigcup_{n=1}^{\infty} U_n = K \setminus \operatorname{supp}|\mu|$ so that $|\mu|(K \setminus \operatorname{supp}|\mu|) \leq \sum_{n=1}^{\infty} |\mu(U_n)| = 0$. Thus (for K second countable) $\mu(E) = \mu(E \cap \operatorname{supp}|\mu|)$ for each $E \in \mathscr{B}(K)$ and $\int_K f\, d\mu = \int_{\operatorname{supp}\mu} f\, d\mu$ for $f \in C(K)$.

We are now in the position to state and prove the main result of this section.

Theorem 5.3.3. Let $A = C(K)$ for K a compact Hausdorff space, $S \subseteq A$ a unital norm closed *-subalgebra of A and $\mathcal{E}\colon A \to A$ a bounded linear operator such that $\mathcal{E}(A) = S$, $\mathcal{E} \circ \mathcal{E} = \mathcal{E}$ and $\mathcal{E}$ an S-module map.

Then there exists a compact Hausdorff space L, a *-isomorphism $\phi\colon S \to C(L)$, and a continuous $\tau\colon K \to L$ such that $\phi^{-1}(g) = g \circ \tau$ for $g \in C(L)$.

Moreover there exists a complex measure $\mu_s \in \mathcal{M}(K)$ for each $s \in L$ so that μ_s is supported on $\tau^{-1}(s)$,

$$\mathcal{E}(f)(t) = \int_K f \, d\mu_{\tau(t)} \quad (f \in C(K)),$$

$$\int_K 1 \, d\mu_s = 1 \quad (s \in L)$$

and

$$s \mapsto \mu_s\colon L \to C(K)'$$

is continuous for the weak*-topology on the dual space $C(K)' \cong \mathcal{M}(K)$.

We have $\mathcal{E}^* = \mathcal{E}$ if and only if μ_s is a real-valued measure for each $s \in L$. We have $\mathcal{E} \geq 0$ if and only if $\mu_s \geq 0$ for each $s \in S$.

Finally, $\mathcal{E}$ is a homomorphism if and only if each μ_s is a point mass, that is

$$\mu_s = \delta_{\rho(s)}$$

(where $\delta_t \in C(K)' \cong \mathcal{M}(K)$ is defined by $\delta_t(f) = f(t)$ for $f \in C(K)$). In that case $\rho \circ \tau\colon K \to K$ is a retraction, $\mathcal{E}(f) = f \circ \rho \circ \tau$ and $\ker \mathcal{E}$ is an ideal of $C(K)$.

Proof. Since S is a unital C^*-algebra, by the Gelfand representation theorem, there exists a compact Hausdorff space L and an isometric *-isomorphism ϕ from S onto $C(L)$. Specifically, $L = \Omega(S)$ and $S \cong C(\Omega(S))$. Now, $\phi^{-1}\colon C(L) \to C(K)$ is an injective *-homomorphism, therefore, there exists a continuous surjective map $\tau\colon K \to L$ with $\tau^\circ = \phi^{-1}$, where $\phi^{-1}(g) = \tau^\circ(g) = g \circ \tau$ for $g \in C(L)$ [9, II.2.2.5].

For $s \in L$ define $\alpha_s \colon C(K) \to \mathbb{C}$ by

$$\alpha_s = \delta_s \circ \phi \circ \mathcal{E} \qquad (\text{or } \alpha_s(f) = (\phi(\mathcal{E}(f)))(s) \text{ for } f \in C(K)).$$

Then $\alpha_s \in C(K)'$ and so there is a unique $\mu_s \in \mathcal{M}(K)$ with $\alpha_s(f) = \int_K f \, d\mu_s$ for $f \in C(K)$.

If $y \in K \setminus \tau^{-1}(s)$, then $\tau(y) \neq s$. By the Hausdorff property of L, there is an open $U \subseteq L$ with $\tau(y) \in U$ but $s \notin \bar{U}$ (the closure of U) and so there is $g \in C(L)$ with $g(s) = 0$ and $g(u) \equiv 1$ for $u \in \bar{U}$ (by Urysohn's lemma). Then $\phi^{-1}(g) = g \circ \tau \in S$, $g \circ \tau$ is identically 1 on $\tau^{-1}(U)$ and so if $f \in C(K)$ vanishes on $K \setminus \tau^{-1}(U)$, then

$$\begin{aligned} f = (g \circ \tau) f \Rightarrow \mathcal{E}((g \circ \tau) f) &= (g \circ \tau)\mathcal{E}(f) \\ \Rightarrow \phi(\mathcal{E}((g \circ \tau) f)) &= \phi(g \circ \tau)\phi(\mathcal{E}(f)) \\ &= g\phi(\mathcal{E}(f)) \\ \Rightarrow \alpha_s(f) &= \alpha_s((g \circ \tau) f) \\ &= \big(\phi(\mathcal{E}((g \circ \tau) f))\big)(s) \\ &= g(s)\phi(\mathcal{E}(f)))(s) = 0 \end{aligned}$$

It follows that $\mu_s|_V = 0$ for $V = \tau^{-1}(U)$ since if $h \in C_0(V)$ we may define $f \in C(K)$ by $f(z) = h(z)$ for $z \in V$ and $f(z) = 0$ for $z \in K \setminus V$. Then we have

$$0 = \alpha_s(f) = \int_K f \, d\mu_s = \int_V f \, d\mu_s|_V = \int_V h d\mu_s|_V$$

and so $\mu_s|V = 0$ by [67, Theorem 6.19]. Hence $|\mu_s|(V) = 0$ and $y \notin \operatorname{supp} \mu_s$. So $K \setminus \tau^{-1}(s) \subseteq \operatorname{supp} \mu_s$, or $\operatorname{supp} \mu_s \subseteq \tau^{-1}(s)$.

Since $\mathcal{E}(1_K)$ is the identity for S (where 1_K means the constant function 1 on K), $\phi(\mathcal{E}(1_K)) = 1_L$ and so $\alpha_s(1) = 1$ for each $s \in L$. That means that $\int_K 1 \, d\mu_s = 1$.

To show weak*-continuity of $s \mapsto \mu_s$ (from L to $C(K)'$ with the weak*-topology) we need to show that $s \mapsto \mu_s(f) = \int_K f \, d\mu_s$ is continuous for each $f \in C(K)$. But that is true since $\phi(\mathcal{E}(f))(s) = \int_K f \, d\mu_s$ and $\phi(\mathcal{E}(f)) \in C(L)$.

If μ_s is real-valued then for $f \in C(K)$ we have $\alpha(\bar{f}) = \int_K \bar{f}\, d\mu_s = \overline{\int_K f\, d\mu_s}$ for $s \in K$. Thus $\phi(\mathcal{E}(\bar{f})) = \overline{\phi(\mathcal{E}(f))}$ and so $\mathcal{E}(\bar{f}) = \overline{\mathcal{E}(\bar{f})}$ (since ϕ is a *-isomorphism) showing that $\mathcal{E}^*(f) = \overline{\mathcal{E}(\bar{f})} = \mathcal{E}(f)$. On the other hand, if μ_s is not real valued for some s, then when we express $\mu_s = \mu_s^R + i\mu_s^I$ as a linear combination of real (signed) measures μ_s^R and μ_s^I we have $\mu_s^I \neq 0$. By [67, Theorem 6.19] there is $f\colon K \to \mathbb{R}$ continuous with $\int_K f\, d\mu_s^I \neq 0$ and so $\alpha_s(f) \notin \mathbb{R}$. Then $\mathcal{E}^*(f) \neq \mathcal{E}(f)$.

If we now assume that $\mathcal{E}$ is a homomorphism (of algebras) we see that α_s is a multiplicative linear functional on $C(K)$ (for each $s \in L$), hence given by a point mass measure (or point evaluation functional) $\delta_{\rho(s)}$. That is $\alpha_s(f) = \delta_{\rho(s)}(f) = f(\rho(s))$ for $f \in C(K)$ (with $\rho(s) \in K$ uniquely determined by s). So $\mu_s = \delta_{\rho(s)}$ and since $s \mapsto \mu_s$ is continuous for the weak*-topology and the map $t \mapsto \delta_t\colon K \to C(K)'$ is a homeomorphism onto its range, we have that $\rho\colon L \to K$ is continuous.

We have

$$(\mathcal{E}(f))(t) = \phi(\mathcal{E}(f))(\tau(t)) = \alpha_{\tau(t)}(f) = \delta_{\rho(\tau(t))}(f) = f(\rho(\tau(t)))$$

and so $\mathcal{E}(f) = f \circ \rho \circ \tau$.

As $\mathcal{E}(\mathcal{E}(f)) = \mathcal{E}(f)$ we have

$$f\big((\rho \circ \tau)\big((\rho \circ \tau)(t)\big)\big) = f\big((\rho \circ \tau)(t)\big)$$

whenever $t \in K$ and $f \in C(K)$. This implies

$$(\rho \circ \tau)\big((\rho \circ \tau)(t)\big) = (\rho \circ \tau)(t)$$

for each $t \in K$ and so $\rho \circ \tau$ is a retraction.

Clearly $\ker \mathcal{E}$ must be an ideal if $\mathcal{E}$ is a homomorphism. □

Remark 5.3.4. Suppose A is a C^*-algebra, $S \subset A$ is a *-subalgebra and

- $\mathcal{E}\colon A \to A$ is linear
- $\mathcal{E} \circ \mathcal{E} = \mathcal{E}$
- $\mathcal{E}^* = \mathcal{E}$,

- $S = \mathcal{E}(A)$, and
- $\mathcal{E}$ is an S-module map.

Then $\mathcal{E}$ is an algebra homomorphism (multiplicative) if and only if $\ker \mathcal{E}$ is an ideal of A, and in that case $\mathcal{E}$ is continuous, and S is closed.

5.4 Common Zeros

Let K be a compact Hausdorff space, $C(K)$ the C^*-algebra of all continuous complex-valued functions on K.

Theorem 5.4.1. Let $C(K) = S \oplus M$, where S is a corner, and M a complement of S in $C(K)$. Consider

$$\mathcal{Z}(M) = \{x \in K : f(x) = 0 \text{ for all } f \in M\} = \bigcap_{f \in M} Z(f),$$

the set of common zeroes of all the functions f in M. Then M separates the points of $K \setminus \mathcal{Z}(M)$ (and if $\mathcal{Z}(M)$ has at most one point then M separates the points of K).

Proof. Introduce an equivalence relation on K by setting $x \sim_M y$ if and only if $f(x) = f(y)$ for all $f \in M$, $x, y \in K$.

Assume that there exist two different points x and y in K with $x \sim_M y$.
Claim: $f(x) = f(y) = 0$ for all $f \in M$; that is $x, y \in \mathcal{Z}(M)$ $(x \neq y)$.
Verification: Let $h \in C(K)$ with $h(x) \neq h(y)$. Since $C(K) = S \oplus M$, there exists a unique $s \in S$ and $m \in M$ with $h = s + m$. Thus

$$s(x) + m(x) = h(x) \neq h(y) = s(y) + m(y).$$

But $m(x) = m(y)$, so $s(x) \neq s(y)$. This implies that $f(x) = 0$ for all $f \in M$, as the opposite, that there exists $f \in M$ with $f(x) \neq 0$, would imply that $s(x)f(x) = s(y)f(y)$ (since $SM \subseteq M$) and consequently $s(x) = s(y)$.

Now it follows that M separates the "points" of $K/\mathcal{Z}(M) = \{[x]_{\sim_M} : x \in K\}$ or the equivalence classes for the equivalence relation $\sim_M$ are $\mathcal{Z}(M)$ and the singleton subsets of $K \setminus \mathcal{Z}(M)$. □

Example 5.4.2. Consider $K = [-1, 1]$, and $C(K) = S \oplus M$ where S is the algebra of even functions, M the set of odd functions. Then $\mathcal{Z}(M) = \{0\}$ and all equivalence classes for the equivalence relation $\sim_M$ are points; M separates the points of K.

5.5 Discontinuous Conditional Expectations

One method of constructing discontinuous conditional expectations is as follows:

- Suppose we can find a discontinuous homomorphism $\phi\colon A_1 \to A_2$ where A_1 and A_2 are C^*-algebras.
- Let $A = A_1 \oplus A_2$, a C^*-algebra with the summands as ideals (where we take the max norm).
- Let S be the graph of ϕ, i.e., $S = \{(x, \phi(x)) : x \in A_1\}$. Define $\mathcal{E}\colon A \to A$ by $\mathcal{E}(a, b) = (a, \phi(a))$.

Then, since ϕ is discontinuous, $\mathcal{E}$ is a discontinuous conditional expectation of A onto S, and S is not a closed corner in A. (Corners which appear as graphs of ring homomorphisms have also been described in section 2.4.)

So we are left with the question of constructing a suitable discontinuous homomorphism between C^*-algebras. In the following subsection we gather some information on automatic continuity and conclude that if the Continuum Hypothesis (CH) is assumed, there exists a discontinuous conditional expectation $\mathcal{E}\colon A \to A$, where $A = C([0, 1]) \oplus \mathcal{B}(H)$ is a C^*-algebra (see Statement 5.5.10).

5.5.1 Review of Results on Automatic Continuity

The first result concerning the continuity of homomorphisms from a C^*-algebra $C(K)$ (K is a compact Hausdorff space) into a Banach algebra is Kaplansky's

theorem [40] (see also [63]) which states that if $|||\ |||$ is any norm with respect to which $C(K)$ is a normed algebra, then $\|f\| \leq |||f|||$ for all f in $C(K)$ (where $\|\ \|$ is the usual supremum C^*-norm); equivalently: if ϕ is any monomorphism from $C(K)$ into a Banach algebra, then $\|f\| \leq \|\phi(f)\|$ for all f in $C(K)$. Kaplansky's theorem implies that there is an incomplete algebra norm on $C(K)$ if and only if there is a discontinuous monomorphism from $C(K)$ into a Banach algebra.

A construction of a discontinuous homomorphism from $A = C(K)$ into a Banach algebra was obtained independently in [17] and in [27], [29], [28]. See also [19].

Certainly there is no discontinuous *-homomorphism in the category of C^*-algebras. Indeed, if A and B are C^*-algebras and $\phi\colon A \to B$ is a *-homomorphism, then $\|\phi\| \leq 1$. (Reason: for $x = x^* \in A$ and A has an identity we know that the norm coincides with the spectral radius: $\|x\| = \rho(x) = \sup\{|\lambda| : \lambda - x \text{ is not invertible in } A\}$. Since the spectrum of $\phi(x)$ with respect to the unital C^*-algebra $\phi(1)B\phi(1)$ is smaller than the spectrum of x, it follows that $\|\phi(x)\| \leq \|x\|$ when $x = x^*$. For general $a \in A$ use $\|a\|^2 = \|a^*a\|$. For the case when A has no identity, some modification of the argument is needed. One way is to express the spectrum in terms of quasi-inverses.)

Consequently the method described at the start of the section could never produce a self-adjoint corner where $\mathcal{E}$ is not bounded.

Conjecture: If A is a C^*-algebra and S is both a corner in A and self-adjoint, then S must be closed (this is true if A is a commutative unital C^*-algebra; see Corollary 5.6.8).

Now G. Allan [2, Corollary 2 of Theorem 2] shows that there is a discontinuous injective algebra homomorphism from the disc algebra

$$A(\mathbb{D}) = \{f \in C(\bar{\mathbb{D}}) : f|_{\mathbb{D}} \text{ analytic}\}$$

to another Banach algebra V. The algebra V is the unitization of the algebra $(L^1[0,1], *)$ where the product $*$ is convolution.

Allan's approach involves considering the the algebra of $\mathfrak{F} = \mathbb{C}[[X]]$ of all formal

power series in a single unknown X. So elements of $\mathfrak{F}$ are sequences $\alpha = (\alpha_n)_{n=0}^{\infty}$ of scalars $\alpha_n \in \mathbb{C}$ and we could say that $\mathfrak{F}$ is $\mathbb{C}^{\mathbb{N}\cup\{0\}}$, all functions $\alpha\colon \mathbb{N}\cup\{0\} \to \mathbb{C}$ with the product topology on $\mathbb{C}^{\mathbb{N}\cup\{0\}}$ or the topology of pointwise convergence of functions (which is a Fréchet topology). However representing elements as $\alpha = \sum_{n=0}^{\infty} \alpha_n X^n$ suggests the multiplication rule

$$\left(\sum_{n=0}^{\infty} \alpha_n X^n\right)\left(\sum_{n=0}^{\infty} \beta_n X^n\right) = \left(\sum_{n=0}^{\infty} \gamma_n X^n\right), \qquad \gamma_n = \sum_{j=0}^{n} \alpha_j \beta_{n-j}$$

(and the other algebra operations). Allan shows that this topology is the unique Fréchet-algebra topology on $\mathfrak{F}$ (that is the unique Fréchet topology on the vector space $\mathfrak{F}$ in which multiplication is continuous).

Allan investigates conditions on a commutative Banach algebra A so that there is an injective algebra homomorphism $\theta\colon \mathfrak{F} \to A$ with a given value a for $\theta(X) = a \in A$.

Definition 5.5.1. Let A be a commutative Banach algebra and $a \in A$. Then a is said to have *finite closed descent* if there is $m \geq 0$ so that

$$\overline{a^{m+1}A} \supseteq a^m A.$$

(Here $a^m A$ means A when $m = 0$.) The smallest such m is denoted $\delta_A(a)$ in [3] and Allan [3] writes $\delta_A(a) = \infty$ if there is no such m.

Note that Dales [18, Definition 2.2.10] adds the extra condition $a^m \notin a^{m+1}A$, which rules out nilpotent a ($a^m = 0$ or $a^{m-1} = 0$ in [3] and invertible a which have $\delta_A(a) = 0$ in [3].)

Theorem 5.5.2 ([2, Theorem 1])**.** Let A be a commutative Banach algebra with 1, let $\theta\colon \mathfrak{F} \to A$ be a unital homomorphism and put $a = \theta(X)$. Then $a \in \operatorname{rad} A$ and there is an integer $m \geq 1$ so that $\overline{a^{m+1}A} \supseteq a^m A \neq \{0\}$.

Theorem 5.5.3 ([2, Theorem 2])**.** Let a be a commutative Banach algebra with 1 and lets $a \in \operatorname{rad} A$ be such that $\overline{a^{m+1}A} \supseteq a^m A \neq \{0\}$ for some m (≥ 1). Then there is a unital homomorphisms $\theta\colon \mathfrak{F} \to A$ with $a = \theta(X)$.

Moreover, if $f \in \mathfrak{F}$ is transcendental over the polynomials, and if $j \in \bigcap_{n=1}^{\infty} a^n A$ then there is another unital homomorphism $\theta' \colon \mathfrak{F} \to A$ with $\theta'(X) = a$ but $\theta'(f) = \theta(f) + j$; also the ideal $\bigcap_{n=1}^{\infty} a^n A$ is dense in $a^m A$.

Corollary 5.5.4 ([2, Corollary 2])**.** There is a discontinuous monomorphism from $A(\mathbb{D})$ (in its uniform norm) into V. Thus there is an incomplete algebra norm on $A(\mathbb{D})$ that dominates the uniform norm.

The idea of the proof is to apply the previous theorem to produce $\theta, \theta' \colon \mathfrak{F} \to V$ with $\theta(X) = \theta'(X)$ but $\theta(e^X) \neq \theta'(e^X)$. This requires a proof that there is an element $h \in L^1[0,1] \subset V$ such that Vh^2 is dense in Vh and in $L^1[0,1]$ (and the fact that $\operatorname{rad} V = L^1[0,1]$). Then 'restrict' θ and θ' to $A(\mathbb{D})$ (which can be considered as an algebra of power series by mapping $g \in A(\mathbb{D})$ to $\sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} X^n$). Since the $\theta(p) = \theta'(p) = \sum_{n=0}^{\infty} \frac{p^{(n)}(0)}{n!} a^n$ for polynomials p (when the sum is finitely nonzero) and polynomials are dense in $A(\mathbb{D})$, we cannot have both θ and θ' continuous on $A(\mathbb{D})$.

Remarks 5.5.5. As Allan noted in 1972, this method of producing discontinuous homomorphisms does not apply when the domain is a C^*-algebra. Moreover, Allan considered the question of finding discontinuous homomorphisms with values in an operator algebra, specifically with values in the (commutative) closed unital subalgebra A of $\mathcal{B}(H)$ generated by a single operator T. He did not show that this was not possible, but he ruled out certain choices for T. He showed [2, Example 2] that if $T \in \mathcal{B}(H)$ is a unilateral weighted shift on a separable Hilbert space H given on an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ of H by $Te_n = w_n e_{n+1}$ where $\lim_{n\to\infty} w_n = 0$, and A is the closed unital algebra generated by T, then there is no $a \in A$ satisfying the hypotheses $\overline{a^{m+1}A} \supseteq a^m A \neq \{0\}$ (for any $m \geq 1$). By the way, this kind of example has $\operatorname{rad} A$ big (contains the non-unital algebra generated by T).

Remarks 5.5.6. Dales [18, §5.7] does discuss results where the domain is a commutative Banach algebra $C(K)$ for K a compact Hausdorff space.

Notice that if $\theta \colon C(K) \to B$ is a unital algebra homomorphism with values in a (unital) C^*-algebra B, then the range $\theta(C(K))$ is a commutative unital subalgebra

of B and if we take A to be the closure in B of $\theta(C(K))$, then A must be a commutative operator algebra (with identity). We can consider θ as a homomorphism $\theta\colon C(K) \to A$.

If A was self-adjoint, it would be a commutative unital C^*-algebra. Hence we could take $A = C(L)$ for some compact Hausdorff space L. However, there are no discontinuous homomorphisms in that setting.

Theorem 5.5.7 ([18, Theorem 5.7.3 (i)]). Let K be a compact Hausdorff space and let A be a commutative, unital Banach algebra. If there is a discontinuous homomorphism from $C(K)$ into A, then there is a closed ideal $\mathcal{K}$ in $\operatorname{rad} A$ such that $(\operatorname{rad} A)/\mathcal{K}$ is a universal, radical Banach algebra.

Theorem 5.5.8 ([18, Theorem 5.7.20 (iii)]). Let K be an infinite, compact space. Assuming CH, there is a discontinuous unital homomorphism from $C(K)$ into some Banach algebra, and there is an algebra norm on $C(K)$ which is not equivalent to the uniform norm.

Theorem 5.5.9. Assume CH. Let A be a commutative, unital Banach algebra. Then the following conditions on A are equivalent:

(i) A is a universal algebra (universal algebras have been introduced by H. G. Dales in [18, Definition 5.7.8]);

(ii) there is a unital embedding of $\mathfrak{F}$ into A;

(iii) there is a discontinuous homomorphism from $C(K)$ into A for some compact space K;

(iv) there is a discontinuous homomorphism from $C(K)$ into A for each infinite, compact space K.

This is (a) $\iff$ (c) $\iff$ (d) $\iff$ (e) of [18, Theorem 5.7.30].

Dales [18, page 771] states that M/f_0M is a universal, commutative, radical Banach algebra where

$$M = \{f \in A(\mathbb{D}) : f(1) = 0\}$$

(a maximal ideal in the disc algebra) and

$$f_0(\zeta) = \exp\left(\frac{\zeta+1}{\zeta-1}\right) \qquad (\zeta \in \bar{\mathbb{D}} \setminus \{1\}).$$

(Since $(\zeta+1)/(\zeta-1)$ maps the unit disc to the left half plane, $|f_0(\zeta)| \leq 1$. It follows that $f_0 M \subseteq M$ even though f_0 is not in $A(\mathbb{D})$.) So the unitization of M/f_0M, denoted here by $M/f_0M + \mathbb{C}1$, is the target for a discontinuous homomorphism[2] on, say, $C([0,1])$.

Combining the above with [10, Proposition 2.3.4] we can conclude the following:

Statement 5.5.10. Assuming the Continuum Hypothesis (CH), there is a discontinuous algebra homomorphism $\phi\colon C([0,1]) \to \mathcal{B}(H)$ for some Hilbert space H. Moreover, there exists a commutative corner in the C^*-algebra $C([0,1]) \oplus \mathcal{B}(H)$ which is not closed.

Proof. Assuming the Continuum Hypothesis (CH), there exists a discontinuous homomorphism $\phi\colon C([0,1]) \to M/f_0M + \mathbb{C}1$. By [10, Proposition 2.3.4], there exist a Hilbert space H and an isometric homomorphism $\pi\colon M/f_0M + \mathbb{C}1 \to \mathcal{B}(H)$.

Therefore, assuming the Continuum Hypothesis (CH), there exists a discontinuous homomorphism between C^*-algebras; namely $\pi \circ \phi\colon C([0,1]) \to \mathcal{B}(H)$.

Moreover, if the Continuum Hypothesis (CH) is assumed, then

$$\mathcal{E}\colon C([0,1]) \oplus \mathcal{B}(H) \to C([0,1]) \oplus \mathcal{B}(H),$$

defined by

$$\mathcal{E}(f,x) = (f, \pi \circ \phi(f)) \tag{5.2}$$

for $f \in C([0,1])$, $x \in \mathcal{B}(H)$, is a discontinuous Lam conditional expectation. Hence the graph of ϕ which coincides with $\mathcal{E}(C([0,1]) \oplus \mathcal{B}(H))$ (see § 2.4) is a commutative corner that is not closed. □

[2]All this last depends on the Continuum Hypothesis (CH). There is a model of ZFC + MA + NDH, where ZFC = Zermelo Fraenkel set theory, MA = Martin's axiom and NDH = the statement that "for each compact K, each homomorphism from $C(K)$ to a Banach algebra is continuous ".

Corollary 5.5.11. Assuming the Continuum Hypothesis (CH), if K is an infinite compact Hausdorff space then there exists a commutative corner algebra of $C(K) \oplus \mathcal{B}(H)$ that is not closed (for some Hilbert space H).

Proof. Same as for Statement 5.5.10. □

Corollary 5.5.12. Assuming the Continuum Hypothesis (CH), there exists a Hilbert space H and a commutative corner algebra of $\mathcal{B}(H)$ that is not closed.

Proof. Let (X, μ) be an infinite, locally finite measure space. Since $L^\infty = L^\infty(X, \mu)$ acts naturally as a von Neumann algebra of operators on $L^2 = L^2(X, \mu)$, and $L^\infty = (L^\infty)'$ (the commutant of L^∞), we can consider L^∞ as a maximal abelian subalgebra of $\mathcal{B}(L^2)$. Furthermore, L^∞ is a corner of $\mathcal{B}(L^2)$ because there exists a conditional expectation from $\mathcal{B}(L^2)$ onto L^∞ (see [9, IV.2.2.3]). By the the Gelfand-Naimark theorem (see [9, II.2.2.4]), $L^\infty \cong C(K)$ for some compact Hausdorff space K, and K is infinite because X is infinite. By Corollary 5.5.11, there exists a commutative corner algebra S in $C(K) \oplus \mathcal{B}(H_0)$ that is not closed (for some Hilbert space H_0). Since

$$S \subseteq C(K) \oplus \mathcal{B}(H_0) \cong \begin{pmatrix} L^\infty & 0 \\ 0 & \mathcal{B}(H_0) \end{pmatrix} \subseteq \begin{pmatrix} \mathcal{B}(L^2) & 0 \\ 0 & \mathcal{B}(H_0) \end{pmatrix} \subseteq \mathcal{B}(L^2 \oplus H_0),$$

it follows that S is a commutative corner algebra of $\mathcal{B}(H)$ (where $H = L^2 \oplus H_0$) that is not closed. □

Remark 5.5.13. The not closed corner $\mathcal{E}(C([0,1]) \oplus \mathcal{B}(H))$ constructed in the proof of Statement 5.5.10 is not self-adjoint since ϕ is not self-adjoint.

Questions: 1. Can the Hilbert space H in the proof of Statement 5.5.10 be separable? **2.** Is every self-adjoint corner of a C^*-algebra closed?

We are grateful to H. G. Dales for providing the following two alternative arguments that there are discontinuous homomorphisms between C^*-algebras.

Statement 5.5.14. Set $V = L^1[0,1]$, with cut-off convolution multiplication, so that V is a universal radical Banach algebra. We can regard V as a subalgebra of $\mathcal{B}(H)$, where H is the Hilbert space $L^2[0,1]$ by the injective mapping $T\colon f \mapsto T_f$,

where $T_f(g) = f \star g$ $(g \in H)$. Note that $\exp T_f = T_{\exp f}$ for $f \in V$. By [18, 5.7.33], there is $\theta\colon C(K) \to V^\sharp$ witn $\theta(\exp F) \neq \exp\theta(F)$ for suitable $F \in C(K)$. Thus $(T \circ \theta)(\exp F) \neq \exp(T \circ \theta)(F)$, and so $(T \circ \theta)$ is discontinuous homomorphism into $\mathcal{B}(H)$.

Alternatively, let A be the closure of V in $\mathcal{B}(H)$. Then the radical $\operatorname{rad} A$ contains all the quasi-nilpotents of A, and hence contains V; but $\operatorname{rad} A$ is closed and so is all of A. Thus A satisfies the conditions in [18, Theorem 5.7.28] - and so there is a discontinuous homomorphism from suitable $C(K)$ into $A^\sharp$, and hence into $\mathcal{B}(H)$.

5.6 Closure Question - Commutative Case

In this section we show that self-adjoint corners in commutative C^*-algebras are closed.

We assume that A is a unital C^*-algebra and that S is a Lam $*$-corner in A, which means that:

- $S \subseteq A$ is a subalgebra of A (not necessarily closed);
- there is an S-bimodule $M \subseteq A$ so that $A = S \oplus M$ (direct sum of S-bimodules);
- if $x \in S$ then $x^* \in S$.

We denote the projection on A with range S and kernel M by $\mathcal{E}\colon A \to A$ (we refer to $\mathcal{E}$ as a Lam conditional expectation).

Note that $\mathcal{E}$ is a homomorphism of S-bimodules, meaning

$$\mathcal{E}(sa) = s\mathcal{E}(a) \text{ and } \mathcal{E}(as) = \mathcal{E}(a)s \qquad (s \in S, a \in A).$$

We also assume that

- S contains the identity of A;

- S is dense in A.

Question 0. Are Lam $*$-corners S in a C^*-algebra necessarily closed?

We know that corners S in a normed algebra A are corners in their closure $\bar{S}$. Since we are assuming $S^* = S$ and A a C^*-algebra, we know that $\bar{S}$ is a C^*-subalgebra of A. Hence the reduction to the case we have assumed above that $\bar{S} = A$.

Question 1. Must $S = A$, $M = 0$?

In this section, we will deal with the commutative case, and we will assume from now on that $A = C(K)$ with K a compact Hausdorff space.

Proposition 5.6.1. Assume $\mathcal{E}\colon C(K) \to C(K)$ is a Lam conditional expectation with dense range and $S = \mathcal{E}(C(K))$. Let $f \in C(K)$ such that the zero set

$$Z(f) = f^{-1}(0) = \{t \in K : f(t) = 0\}$$

has nonempty interior $(Z(f))^\circ$. Then $\mathcal{E}(f)(t) = 0$ for each $t \in (Z(f))^\circ$.

Proof. Let $t_0 \in (Z(f))^\circ$. Because K is normal, there is $V \subseteq K$ open with $t_0 \in V \subseteq \bar{V} \subseteq (Z(f))^\circ$. By Urysohn's lemma, there is $g\colon K \to [0,1]$ continuous with $g(t_0) = 0$ and $g(t) = 1$ for $t \in K \setminus V$.

It follows that there is $h_1 \in S$ with $\|g - h_1\| < 1/4$. Then $|g(t_0) - h_1(t_0)| = |h_1(t_0)| < 1/4$ and so $h = h_1 - h_1(t_0) \in S$ and satisfies $\|g - h\| < 1/2$ and $h(t_0) = 0$. Note that for $t \in K \setminus \bar{V}$ we have $|h(t)| > |g(t)| - 1/2 > 1/2$ and so $h(t) \neq 0$. Thus $f(t)/h(t)$ is continuous on (the open set) $K \setminus \bar{V}$. Defining

$$f_1(t) = \begin{cases} f(t)/h(t) & \text{if } t \in K \setminus \bar{V} \\ 0 & \text{if } t \in Z(f) \end{cases}$$

we see that f_1 is well-defined on K and continuous on both open sets $K \setminus \bar{V}$ and $(Z(f))^\circ$, thus continuous on $K = (K \setminus \bar{V}) \cup (Z(f))^\circ$. Moreover $f = hf_1$.

Hence we have that if $\mathcal{E}(f) = h\mathcal{E}(f_1)$ then $\mathcal{E}(f)(t_0) = h(t_0)\mathcal{E}(f_1)(t_0) = 0$. □

Corollary 5.6.2. Assume $\mathcal{E}\colon C(K) \to C(K)$ is a Lam conditional expectation with dense range. For each open set $V \subset K$ define

$$J_V = \{f \in C(K) : V \subset Z(f)\}. \tag{5.3}$$

Then J_V is a closed subspace of $C(K)$ and $\mathcal{E}(J_V) \subseteq J_V$.

Proof. It is clear that J_V is closed and $\mathcal{E}(J_V) \subseteq J_V$ by Proposition 5.6.1. □

Lemma 5.6.3. Assume $\mathcal{E}\colon C(K) \to C(K)$ is a Lam conditional expectation with dense range, and J_V is defined as in (5.3). If $V_1, V_2, \ldots, V_n \subseteq K$ are open with $\bar{V}_i \cap \bar{V}_j = \emptyset$ for $i \neq j$, then

$$(J_{V_1} \cap \cdots \cap J_{V_{n-1}}) + J_{V_n} = C(K).$$

Proof. By Urysohn there is $g\colon K \to [0,1]$ continuous with $g(t) = 0$ for $t \in \bigcup_{j=1}^{n-1} \bar{V}_j$ and $g(t) = 1$ for $t \in \bar{V}_n$. If $f \in C(K)$ then $gf \in J_{V_1} \cap \cdots \cap J_{V_{n-1}}$, $(1-g)f \in J_{V_n}$ and $f = gf + (1-g)f$. □

Definition 5.6.4. Let $\mathcal{E}\colon C(K) \to C(K)$ be a Lam conditional expectation with dense range. For $V \subset K$ open define $\mathcal{E}_V\colon C(K) \to C(K)/J_V$ by $\mathcal{E}_V(f) = \mathcal{E}(f) + J_V$.

We say that $t \in K$ is a "discontinuity point for $\mathcal{E}$" if $\mathcal{E}_V$ is discontinuous (unbounded with respect to the quotient norm on $C(K)/J_V$) for each open $V \subseteq K$ such that $t \in V$.

Lemma 5.6.5. Let $\mathcal{E}\colon C(K) \to C(K)$ be a Lam conditional expectation with dense range. Then, there is at most a finite number of discontinuity points for $\mathcal{E}$ in K.

Proof. This follows from the Lemma 5.6.3 and [69, Theorem 2.3]. □

Proposition 5.6.6. Let $\mathcal{E}\colon C(K) \to C(K)$ be a Lam conditional expectation with dense range. If $t \in K$ is not a discontinuity point for $\mathcal{E}$, then $\mathcal{E}(f)(t) = f(t)$ for each $f \in C(K)$.

Proof. Since t is not a discontinuity point for $\mathcal{E}$, there exists $V \subseteq K$ open so that $\mathcal{E}_V$ is bounded. For $h \in S$ we have $\mathcal{E}_V(h) = \mathcal{E}(h) + J_V = h + J_V$ and hence (by density of S and continuity of $\mathcal{E}_V$) $\mathcal{E}_V(f) = \mathcal{E}(f) + J_V = f + J_V$ for each $f \in C(K)$. That is $\mathcal{E}(f) - f \in J_V$, which implies $\mathcal{E}(f)(t) - f(t) = 0$. □

Theorem 5.6.7. Let $S \subseteq A$ be a dense corner in $A = C(K)$. Then $S = A$.

Proof. From Lemma 5.6.5, we have a finite set $F \subset K$ of discontinuity points for $\mathcal{E}$ and $\mathcal{E}(f)(t) = f(t)$ for each $f \in C(K)$ and $t \in K \setminus F$. If $t_0 \in F$ is not isolated in K, then $\mathcal{E}(f)(t_0) = f(t_0)$ follows from continuity of f (for each $f \in C(K)$). On the other hand if $t_0 \in F$ is isolated in K, then $V = \{t_0\}$ is open in K, and if $f \in C(K)$, $g = f - f(t_0) \in J_V$ and $\mathcal{E}(g) = \mathcal{E}(f) - f(t_0) \in J_V$ (by Corollary 5.6.2), and so $\mathcal{E}(f)(t_0) - f(t_0) = 0$.

Hence $\mathcal{E}(f) = f$ for each $f \in C(K)$. □

Corollary 5.6.8. If S is a $*$-corner in a commutative C^*-algebra $A = C(K)$, then S is closed.

Proof. As noted earlier it is sufficient to consider the case when S contains the identity of A. Moreover S is a corner in the unital C^*-subalgebra $\bar{S}$ (the closure of S) and so by Theorem 5.6.7 we have $S = \bar{S}$. □

Corollary 5.6.9. If $S \subseteq A$ is a dense corner in a commutative C^*-algebra A, then $S = A$.

Proof. If A is unital, then $A \cong C(K)$ for some compact Hausdorff space K, and $S = A$ by Theorem 5.6.7.

If A is not unital, consider $\mathcal{E}^\sharp : A^\sharp \to A^\sharp$, $\mathcal{E}^\sharp(A^\sharp) = S^\sharp$ (see Proposition 3.3.1), and by the unital case $S^\sharp = A^\sharp$. Hence $S = A$. □

Corollary 5.6.10. If S is a $*$-corner in a commutative C^*-algebra A, then S is closed.

Proof. In the case when A is unital this follows from Corollary 5.6.8.

If A is not unital, consider the Lam $*$-conditional expectation $\mathcal{E}^\sharp : A^\sharp \to A^\sharp$ which has range $S^\sharp$ (see Proposition 3.3.1). As $S^\sharp$ must be closed from the unital case, S is closed also. □

5.7 Existence of Bounded Conditional Expectations – Commutative Case

Suppose for this section that $A = C(K)$ and that $\mathcal{E}\colon A \to A$ is a unital Lam $*$-conditional expectation with range S. As S is a closed (by Corollary 5.6.8) unital $*$-subalgebra we know that there is a $*$-isomorphism $\pi\colon S \to C(L)$ for some compact Hausdorff space L. Associated with the inverse map, the $*$-homomorphism $\pi^{-1}\colon C(L) \to S \hookrightarrow C(K)$, there is a continuous $\phi\colon K \to L$ such that $\pi^{-1}(h) = h \circ \phi$ for $h \in C(L)$.

We first consider the separating space

$$\mathfrak{S}(\mathcal{E}) = \{a \in A : \exists (x_n)_{n=1}^{\infty} \in A \text{ with } \lim_{n\to\infty} x_n = 0, \lim_{n\to\infty} \mathcal{E}(x_n) = a\}.$$

$\mathfrak{S}(\mathcal{E})$ is known to be a closed linear subspace of A ([69, Lemma 1.2], [18, Proposition 5.1.2]).

Lemma 5.7.1. $\mathfrak{S}(\mathcal{E}) \subseteq S$ and $\mathfrak{S}(\mathcal{E})$ is a closed ideal of S. (See also Proposition 3.11.1.)

Proof. Since S is closed we know that $\mathfrak{S}(\mathcal{E}) \subseteq S$ from the definition of $\mathfrak{S}(\mathcal{E})$.

If $a \in \mathfrak{S}(\mathcal{E})$ and $s \in S$, let $(x_n)_{n=1}^{\infty}$ be a sequence in A with $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} \mathcal{E}(x_n) = a$. Observe that $\lim_{n\to\infty} sx_n = 0$ and $\lim_{n\to\infty} \mathcal{E}(sx_n) = \lim_{n\to\infty} s\mathcal{E}(x_n) = sa$ and so $sa \in \mathfrak{S}(\mathcal{E})$. Hence $\mathfrak{S}(\mathcal{E})$ is an ideal in S. □

Lemma 5.7.2. For $W \subset L$ open, let $I_W = \{f \in C(K) : \phi^{-1}(W) \subseteq Z(f)\}$. Then

(i) I_W is a closed linear subspace of A (for each $W \subset L$ open);

(ii) $\mathcal{E}(I_W) \subset I_W \cap S$;

(iii) if $W_1, W_2, \ldots, W_n \subset L$ are open and $\bar{W}_i \cap \bar{W}_j = \emptyset$ for $i \neq j$, then

$$(I_{W_1} \cap \cdots \cap I_{W_{n-1}}) + I_{W_n} = C(K).$$

Proof. (i) This is straightforward.

(ii) Let $f \in I_W$ and fix $t_0 \in W$ By Urysohn's lemma there is $g \in C(L)$ with $g(t_0) = 0$ but $g(t) = 1$ for $t \in L \setminus W$. Now $g \circ \phi \in S$ and $(g \circ \phi)f = f$. Hence $\mathcal{E}(f) = \mathcal{E}(g \circ \phi)f) = (g \circ \phi)\mathcal{E}(f) \Rightarrow \mathcal{E}(f)(u) = (g \circ \phi)(u)\mathcal{E}(f)(u) = g(t_0)\mathcal{E}(f)(u)$ for $u \in \phi^{-1}(t_0)$. As $t_0 \in W$ is arbitrary , $\mathcal{E}(f) \in I_W$.

(iii) Since $\phi^{-1}(\bar{W}_i)$ $(i = 1, 2, \ldots, n)$ are disjoint closed subsets of K, an application of Urysohn's lemma (as in the proof of Lemma 5.6.3) shows that $(I_{W_1} \cap \cdots \cap I_{W_{n-1}}) + I_{W_n} = C(K)$. □

Theorem 5.7.3. With the notation introduced earlier, there is a finite set F of isolated points of L so that $\pi(\mathfrak{S}(\mathcal{E})) = \{h \in C(L) : L \setminus F \subseteq Z(h)\}$ and $\phi^{-1}(t)$ is an infinite clopen subset of K for each $t \in F$.

Moreover, there exists a bounded conditional expectation operator $\mathcal{E}' \colon C(K) \to C(K)$ with $\mathcal{E}'(C(K)) = S = \mathcal{E}(C(K))$.

If K is connected and $\dim S > 1$, or if L has no isolated points, then $\mathfrak{S}(\mathcal{E}) = \{0\}$ and $\mathcal{E}$ is bounded.

Proof. For $W \subset L$ open, define $\mathcal{E}_W \colon C(K) \to C(K)/I_W$ by $\mathcal{E}_W(f) = \mathcal{E}(f) + I_W$. We call $t_0 \in L$ a 'continuity point' if there is an open $W \subset L$ satisfying $t_0 \in W$ and $\mathcal{E}_W$ bounded.

It follows from Lemma 5.7.2 and [69, Theorem 2.3] that there are at most finitely many points of L that fail to be continuity points. Write F_0 for that finite subset of L.

Now consider $t_0 \in L \setminus F_0$. Then there is an open neighbourhood W of t_0 in L so that $\mathcal{E}_W$ is bounded. Consider any sequence $(f_n)_{n=1}^{\infty}$ in $C(K)$ converging to 0 in norm. Since $\mathcal{E}_W$ is bounded, we have $\mathcal{E}(f_n) + I_W \to 0$. Thus if $u \in \phi^{-1}(W)$,

$$|\mathcal{E}(f_n)(u)| \leq \|\mathcal{E}(f_n) + I_W\| \Rightarrow \lim_{n \to \infty} \mathcal{E}(f_n)(u) = 0. \tag{5.4}$$

It follows that for $f \in \mathfrak{S}(\mathcal{E})$ we have $f(u) = 0$, because we can chose a null sequence $(f_n)_{n=1}^{\infty}$ so that $\lim_{n\to\infty} \mathcal{E}(f_n) = f$, implying $|\mathcal{E}(f_n)(u) - f(u)| \leq \|\mathcal{E}(f_n) - f\| \to 0$ and so we have $f(u) = \lim_{n\to\infty} \mathcal{E}(f_n)(u) = 0$. In particular, for $u \in \phi^{-1}(t_0)$ we have $f(u) = 0$. As this holds for all $f \in \mathfrak{S}(\mathcal{E})$ and $t_0 \in L \setminus F_0$, we have

$\phi^{-1}(L \setminus F_0) \subseteq \bigcap\{Z(f) : f \in \mathfrak{S}(\mathcal{E})\}$.

As a closed ideal of $C(L)$, $\pi(\mathfrak{S}(\mathcal{E}))$ is determined by its zero set

$$L_0 = \bigcap\{Z(\pi(f)) = Z(f \circ \phi) : f \in \mathfrak{S}(\mathcal{E})\}.$$

That is $\pi(\mathfrak{S}(\mathcal{E})) = \{h \in C(L) : h(t) = 0 \forall t \in L_0\}$. We have $L \setminus F_0 \subset L_0$ and hence $F = L \setminus L_0 \subseteq F_0$ finite.

For each $t_0 \in F$, there is $h \in \pi(\mathfrak{S}(\mathcal{E}))$ with $h(t_0) \neq 0$ and then it follows from the ideal property together with the fact that there is a function in $C(L)$ that is 1 at t_0 and zero on $F \setminus \{t_0\}$ that the characteristic function $1_{\{t_0\}} \colon L \to \{0, 1\}$ is continuous on L. Thus t_0 must be isolated in L, and so $\phi^{-1}(t_0)$ must be a clopen subset of K.

It follows that the characteristic function $1_F \in C(L)$ and that $p_F = 1_F \circ \phi$ is an idempotent in $S \subset C(K)$. If we define $\mathcal{E}_0 \colon C(K) \to C(K)$ by $\mathcal{E}_0(f) = (1 - p_F)\mathcal{E}(f)$ we get a Lam $*$-conditional expectation with range $(1 - p_F)S$ and $\mathfrak{S}(\mathcal{E}_0) = \{0\}$ (use (5.4) to check that). Thus $\mathcal{E}_0$ is bounded.

If $F \neq \emptyset$ and $t \in F$, then for $f \in C(K)$ we have that $(\mathcal{E}(f) \circ \phi)(t) = 1_{\{t\}}(t)(\mathcal{E}(f) \circ \phi)(t)$. Applying π^{-1} we deduce that for $u \in \phi^{-1}(t)$, $p_t = \pi^{-1}(1_{\{t\}}) = 1_{\phi^{-1}(t)} \in S$, we have

$$\mathcal{E}(f)(u) = p_t(u)\mathcal{E}(f)(u) = \mathcal{E}(p_t f)(u).$$

Observe then that $\phi^{-1}(t)$ cannot be a finite set (for $t \in F$) as finiteness would imply finite dimensionality of $p_t C(K)$ and so continuity of $f \mapsto \mathcal{E}(p_t f)$. Then if $(f_n)_{n=1}^{\infty}$ converges to 0 in $C(K)$, we would have $\lim_{n\to\infty} \mathcal{E}(f_n)(u) = \lim_{n\to\infty} \mathcal{E}(p_t f_n)(u) = 0$ for $u \in \phi^{-1}(t)$, so that $f(u) = 0$ for all $f \in \mathfrak{S}(\mathcal{E})$, contradicting $t \in F = L \setminus L_0$. Thus $\phi^{-1}(t)$ must be an infinite clopen subset of K for each $t \in F$. If S is not one dimensional, then L has more than one point and $\phi^{-1}(t)$ must be a proper nonempty clopen subset of K for each $t \in F$. If K is connected, this forces F to be empty (unless S is one dimensional and $F = L$ a singleton), hence $\mathfrak{S}(\mathcal{E}) = \{0\}$ and $\mathcal{E}$ bounded. Similarly if L has no isolated points then $F = \emptyset$ and $\mathcal{E}$ bounded, establishing the last part of the statement.

For $F \neq \emptyset$ we can express $K = \phi^{-1}(L_0) \cup \bigcup_{t\in F} \phi^{-1}(t)$, a finite disjoint union of clopen subsets. Write $p_0 = 1_{\phi^{-1}(L_0)} = 1 - p_F$ and $p_t = 1_{\phi^{-1}(t)}$ for $t \in F$. We have a decomposition of the identity in $C(K)$ as $p_0 + \sum_{t\in F} p_t$, a sum of projections in S. Also $S = p_0 S \oplus \bigoplus_{t\in F} p_t S$ with $p_t S$ one dimensional for each $t \in F$. We have $\mathcal{E}_0(f) = p_0\mathcal{E}(f) = \mathcal{E}(p_0 f)$ and so $p_0 S$ is a corner in $p_0 C(K)$ (and in $C(K)$) with an associated bounded Lam $*$-conditional expectation. To get a bounded conditional expectation $\mathcal{E}'$ on $C(K)$ with range S, choose one point $u_t \in \phi^{-1}(t)$ for each $t \in F$ and define

$$\mathcal{E}'(f) = \mathcal{E}(p_0 f) + \sum_{t\in F} f(u_t)p_t$$

(for $f \in C(K)$). □

5.8 Remarks on the Non-commutative Case

For A a unital C^*-algebra and $x \in A$ let $\langle x\rangle$ denote the smallest unital C^*-subalgebra of A containing x.

Statement 5.8.1. Let A be a unital C^*-algebra and $S \subseteq A$ a $*$-corner. If $x = x^* \in S$ then $\langle x\rangle \subseteq S$.

Proof. Let $\mathcal{E}\colon A \to A$ be a Lam conditional expectation with $\mathcal{E}(A) = S$. We can replace $\mathcal{E}$ by the symmetrised Lam $*$-conditional expectation $x \mapsto (\mathcal{E}(x)+\mathcal{E}(x^*)^*)/2$ and suppose that $\mathcal{E}$ satisfies $\mathcal{E}(x^*) = \mathcal{E}(x)^*$ for all $x \in A$ (see § 3.7).

Let $\mathcal{C}_x$ denote a maximal abelian $*$-subalgebra of S containing x. (The existence of such a $\mathcal{C}_x$ can be shown via Zorn's lemma.)

Write $\mathcal{C}'_x = \{z \in A : zy = yz \forall y \in \mathcal{C}_x\}$ for the commutant of $\mathcal{C}_x$ and $\mathcal{C}''_x = \{w \in A : wz = zw \forall z \in \mathcal{C}'_x\}$ for the bicommutant.

Since $\mathcal{C}_x \subseteq \mathcal{C}'_x$ we have that if $\mathcal{C}''_x \subseteq \mathcal{C}'_x$ then $\mathcal{C}''_x$ is commutative. Also $\mathcal{C}''_x$ is a closed $*$-subalgebra of A. Similarly if $\mathcal{C}_x \subseteq \mathcal{C}'_x \cap S$ then $(\mathcal{C}'_x \cap S)' \cap S$ is commutative and $(\mathcal{C}'_x \cap S)' \cap S = \mathcal{C}_x$ by maximality of $\mathcal{C}_x$.

For $z \in \mathcal{C}'_x \cap S$, $w \in \mathcal{C}''_x$ we have

$$\mathcal{E}(w)z = \mathcal{E}(wz) = \mathcal{E}(zw) = z\mathcal{E}(w) \Rightarrow \mathcal{E}(w) \in (\mathcal{C}'_x \cap S)' \cap S = \mathcal{C}_x.$$

It follows that $\mathcal{E}(\mathcal{C}_x'') \subset \mathcal{C}_x$ and in fact that the restriction of $\mathcal{E}$ to $\mathcal{C}_x''$ is a (unital) Lam $*$-conditional expectation with range $\mathcal{C}_x$. Restricting $\mathcal{E}$ further to the closure of $\mathcal{C}_x$ in $\mathcal{C}_x''$ we have a unital Lam$*$-conditional expectation on a commutative C^*-algebra with dense range. Hence $\mathcal{C}_x$ is closed (complete) by Theorem 5.6.7 and thus contains $\langle x \rangle$.

□

For A a C^*-algebra and $x \in A$, let $\langle x \rangle_0$ denote the smallest C^*-subalgebra of A containing x.

Corollary 5.8.2. Let A be a C^*-algebra and $S \subseteq A$ a $*$-corner of A. If $x = x^* \in S$ then $\langle x \rangle_0 \subseteq S$.

Proof. If A has an identity then this follows from Statement 5.8.1 because $\langle x \rangle_0 \subseteq \langle x \rangle \subseteq S$.

In the non-unital case we know $S^\sharp$ is a Lam $*$-corner of $A^\sharp$ (see Proposition 3.3.1). So if $x = x^* \in S$, Statement 5.8.1 implies that $\langle x \rangle \subseteq S^\sharp$. Hence $\langle x \rangle_0 \subseteq \langle x \rangle \cap A \subseteq S^\sharp \cap A = S$. □

Corollary 5.8.3. Let A be a C^*-algebra and $S \subseteq A$ a $*$-corner of A. If $x = x^* \in S$, $x \geq 0$, and α is a positive real number, then $x^\alpha \in S$ (in particular $x^{1/2} = \sqrt{x} \in S$ so S is square-root closed).

Proof. Using [9, II.3.1.2 (vii)] and Corollary 5.8.2, we have $x^\alpha \in \langle x \rangle_0 \subseteq S$. □

Chapter 6

Addendum

In this final chapter, we first examine the concept of norming algebras and its application to the automatic complete boundedness problem. The basic issue here is to determine some algebraic conditions which ensure that a given bounded homomorphism between operator algebras is automatically completely bounded. For example, D. Pitts [61] has proved that a bounded epimorphism $\phi\colon A \to B$ from an operator algebra A onto an operator algebra B is automatically completely bounded provided B contains a norming C^*-subalgebra (Theorem 6.1.4). In Theorem 6.1.5, we give an algebraic condition which forces a bounded bimodule map to be automatically completely bounded.

We then turn to a consideration of ternary rings of operators (TROs) and we define AW^*-TROs. This is a new 'subclass' of TROs which generalizes AW^*-algebras in a similar way as TROs/W^*-TROs generalize C^*-algebras/W^*-algebras. In particular it is shown that every AW^*-TRO can be uniquely decomposed into the direct sum of AW^*-TROs of type I, type II, and type III (Theorem 6.3.2).

6.1 Complete Boundedness of Bimodule Maps

For the most part, our notation and terminology in this section follows that of D. R. Pitts [61]. Thus, H and K are complex Hilbert spaces, and $\mathcal{B}(H, K)$ is the space of all bounded linear operators from H into K. Put $\mathcal{B}(H) = \mathcal{B}(H, H)$, and

consider $H^n = H \oplus \overset{n}{\cdots} \oplus H$ (the direct sum of n copies of H) as the space of single-column vectors of height n with entries in H. This allows us to represent operators from $\mathcal{B}(H^n, H^m)$ as m-by-n matrices with entries in $\mathcal{B}(H)$; thus

$$M_{m,n}(\mathcal{B}(H)) \cong \mathcal{B}(H^n, H^m).$$

Moreover, if X is a complex-vector subspace of $\mathcal{B}(H)$, the space of all m-by-n matrices with entries in X – denoted by $M_{m,n}(X)$ – is not only a subspace of $M_{m,n}(\mathcal{B}(H))$, but also can be identified as a subspace of $B(H^n, H^m)$. Thus, putting it all together we have:

$$M_{m,n}(X) \subseteq M_{m,n}(\mathcal{B}(H)) \cong \mathcal{B}(H^n, H^m)$$

and the vector-space structure, as well as the topological one, on $M_{m,n}(X)$ is inherited from $\mathcal{B}(H^n, H^m)$. In particular, if $n = m$, put $M_n(X) = M_{n,n}(X)$, and since $M_n(X) \subseteq M_n(\mathcal{B}(H)) \cong \mathcal{B}(H^n)$, consider $M_n(X)$ as a subspace of $\mathcal{B}(H^n)$ together with the (unique C^*-algebra) norm of $\mathcal{B}(H^n)$ restricted to $M_n(X)$. In what follows, it will always be assumed that $M_n(X)$ is endowed with this norm.

Recall that an *operator space* is a complex-vector subspace of $\mathcal{B}(H)$ that is closed in the norm of $\mathcal{B}(H)$. Similarly, an *operator algebra* is a complex subalgebra of $\mathcal{B}(H)$ that is closed in the norm of $\mathcal{B}(H)$. Equivalently, since every C^*-algebra can be consider as a closed $*$-subalgebra of some $\mathcal{B}(H)$, we can consider operator spaces as closed subspaces of C^*-algebras. Similarly, we can consider operator algebras as closed subalgebras of C^*-algebras.

Morphisms between operator spaces are *completely bounded* operators, and they are defined as follows. If $\phi\colon X \to Y$ is a linear map between operator spaces $X \subseteq \mathcal{B}(H)$ and $Y \subseteq \mathcal{B}(K)$, define $\phi^{(m,n)}\colon M_{m,n}(X) \to M_{m,n}(Y)$ by $\phi^{(m,n)}([x_{ij}]) = [\phi x_{ij}]$ for all $[x_{ij}] \in M_{m,n}(X)$ $(1 \leq i \leq m,\ 1 \leq j \leq n)$. In particular, if $n = m$, put $\phi^{(n)} = \phi^{(n,n)}$. Then, $\phi\colon X \to Y$ is called a *completely bounded* operator if there exists a non-negative real number c ("constant") such that for all integers $n \geq 1$

and all matrices $[x_{ij}] \in M_n(X)$ we have

$$\left\|\phi^{(n)}([x_{ij}])\right\| \leq c \left\|[x_{ij}]\right\|.$$

In this case, the smallest such constant c is a norm of ϕ, which is referred to as the *completely bounded norm* (or the *cb-norm*) of ϕ and is denoted by $\|\phi\|_{cb}$. Thus

$$\|\phi\|_{cb} = \inf\left\{c \geq 0 \ : \ \forall_{n\in\mathbb{N}} \ \forall_{[x_{ij}]\in M_n(X)} \ \left\|[\phi x_{ij}]\right\|_n \leq c \left\|[x_{ij}]\right\|_n\right\}.$$

A completely bounded operator $\phi\colon X \to Y$ between operator spaces $X \subseteq \mathcal{B}(H)$ and $Y \subseteq \mathcal{B}(K)$ is called *completely contractive* if $\|\phi\|_{cb} \leq 1$; and it is called a *complete isometry* if ϕ is an invertible and $\|\phi\|_{cb} = \|\phi^{-1}\|_{cb} = 1$. Thus, two operator spaces $X \subseteq \mathcal{B}(H)$ and $Y \subseteq \mathcal{B}(K)$ are consider to be equivalent if there exists a complete isometry $\phi\colon X \to Y$. Consequently, an operator space should really be considered as an equivalence class.

An important new tool in analysis of completely bounded operators is the notion of *norming algebras* which have been introduced in the 2000 paper [62] authored by F. Pop, A. Sinclair, and R. Smith. The authors of [62] were able to give a simple condition for a bounded operator to be automatically completely bounded. In order to quote this result, we need the following definition.

Definition 6.1.1 (Norming Algebra)**.** Let $C \subseteq A$ be C^*-algebras. Then C is said to be *norming* for A (or A is said to be *normed* by C) if for all integers $n \geq 1$ and all matrices $[a_{ij}] \in M_n(A)$ (with entries in A) the following condition holds

$$\left\|[a_{ij}]\right\| = \sup\left\{\left\|\sum_{i,j=1}^{n} x_i a_{ij} y_j\right\| \ : \ x_i, y_j \in C, \left\|\sum_{i=1}^{n} x_i x_i^*\right\| \leq 1, \left\|\sum_{j=1}^{n} y_j^* y_j\right\| \leq 1\right\}. \tag{6.1}$$

Note that the condition (6.1) can be rewritten as:

$$\left\|[a_{ij}]\right\| = \sup\left\|[x_1 \cdots x_n][a_{ij}]\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}\right\|,$$

where the supremum runs over all contractive row and column matrices over C,

i.e., $[x_1 \cdots x_n] \in M_{1,n}(C)$ and $\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in M_{n,1}(C)$ with $\|[x_1 \cdots x_n]\| \leq 1$, $\left\| \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right\| \leq 1$. In other words, condition (6.1) asserts that the C^*-algebra-norm $\|[a_{ij}]\|$ of $a = [a_{ij}] \in M_n(A)$ agrees with the supremum norm of the bilinear operator

$$f_a \colon M_{1,n}(C) \times M_{n,1}(C) \to A, \qquad f_a(x, y) = xay$$

defined for all $x = [x_1 \cdots x_n] \in M_{1,n}(C)$, $y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in M_{n,1}(C)$. That is

$$\|a\| = \|[a_{ij}]\| = \sup \|f_a(x, y)\| = \|f_a\|,$$

and the supremum above is taken over all $x = [x_1 \cdots x_n] \in M_{1,n}(C)$ and $y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in M_{n,1}(C)$ with $\|x\| \leq 1$, $\|y\| \leq 1$.

Example 6.1.2. Consider the C^*-algebra $A = C(K)$ of continuous complex-valued functions on a compact Hausdorff space K, and denote by C the C^*-subalgebra of A consisting of all constant functions. Then C is a corner of A, and C is norming for A (see [62, Example 2.5], or [58, p. 107]). In fact, if A is a C^*-algebra with unit 1, then $\mathbb{C}1$ is norming for A if and only if A is commutative (see [62, Example 2.8]).

Norming algebras have applications to the bounded projection problem which can be formulated as follows: If M is a von Neumann subalgebra of some $\mathcal{B}(H)$, does the existence of any bounded linear projection from $\mathcal{B}(H)$ onto M imply the injectivity of M? (see [60], and [13, 14]). As mentioned earlier, norming algebras also have applications in proving automatic complete boundedness of linear maps on C^*-algebras. Here is an important result which indicates some significant correlations between norming algebras and automatic complete boundedness of bounded operators on C^*-algebras.

Theorem 6.1.3 ([62, Theorem 2.10])**.** Let A, B, and C be C^*-algebras with C a subalgebra of both A and B. If C is norming for B and $\phi \colon A \to B$ is a bounded C-modular map, then $\|\phi\| = \|\phi\|_{cb}$.

The definition of norming algebra can be modified as follows: Assume A, B are operator algebras, and M is an operator space as well as an (A, B)-bimodule

satisfying

$$\|[a_{ij}][x_{ij}][b_{ij}]\|_{M_n(M)} \leq \|[a_{ij}]\|_{M_{n,k}(A)} \|[x_{ij}]\|_{M_k(M)} \|[b_{ij}]\|_{M_{k,n}(B)} \tag{6.2}$$

for all $[a_{ij}] \in M_{n,k}(A)$, $[x_{ij}] \in M_k(M)$, and $[b_{ij}] \in M_{k,n}(B)$; M will be referred to as a *completely contractive* (A, B)*-bimodule.* Then

$$\eta_n([x_{ij}]) = \sup \left\| [a_1 \cdots a_n][x_{ij}] \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \right\|, \tag{6.3}$$

where the supremum runs over all contractive row and column matrices over A and B, i.e., $[a_1 \cdots a_n] \in M_{1,n}(A)$ and $\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \in M_{n,1}(B)$ with $\|[a_1 \cdots a_n]\| \leq 1$, $\left\| \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \right\| \leq 1$, is a norm on $M_n(M)$ and $\eta_n(x) \leq \|x\|_{M_n(M)}$ for all $x = [x_{ij}] \in M_n(M)$. If there is some $\lambda > 0$ with

$$\lambda \|x\|_{M_n(M)} \leq \eta_n(x) \quad (\text{for all } n \in \mathbb{N},\ x = [x_{ij}] \in M_n(M))$$

then M is said to be λ-normed by (A, B). When $\lambda = 1$, M is said to be normed by (A, B); and when $A = B$, M is said to be normed by A. (Note that, if A and B are pre-C^*-algebras, then M is λ-normed by (A, B) if and only if M is λ-normed by $(\bar{A}, \bar{B})$.)

The following is the main result of [61].

Theorem 6.1.4 (D. Pitts [61, Theorem 1.4.]). Let B be a norm-closed operator algebra containing a C^*-algebra D such that B is λ-normed by D for some $\lambda > 0$. If A is a norm-closed operator algebra, $\phi\colon A \to B$ is a bounded epimorphism and $\tilde{\phi}\colon A/\ker\phi \to B$ is the induced isomorphism, then ϕ is completely bounded and $\|\phi\|_{cb} \leq \lambda^{-1} \|\phi\| \left\|\tilde{\phi}^{-1}\right\|^4$.

We will prove the following theorem:

Theorem 6.1.5. Let H be a Hilbert space. Let A, B be C^*-subalgebras of $B(H)$, and let $M \subseteq B(H)$ be an operator space and a completely contractive (A, B)-bimodule. If M is λ-normed by (A, B) for some $0 < \lambda \leq 1$ and $\phi\colon B(H) \to M$

is a bounded (A, B)-bimodule map, then ϕ is automatically completely bounded and $\|\phi\|_{cb} \leq \lambda^{-1} \|\phi\|$.

Proof. Let $[x_{ij}] \in M_n(\mathcal{B}(H))$, and consider $\phi^{(n)}([x_{ij}]) = [\phi(x_{ij})] \in M_n(M)$. Since M is λ-normed by (A, B), we have

$$\lambda \, \|[\phi(x_{ij})]\|_{M_n(M)} \leq \eta_n([\phi(x_{ij})]).$$

Thus, in particular, if $[\phi(x_{ij})] \neq 0$,

$$(\lambda - \epsilon) \, \|[\phi(x_{ij})]\|_{M_n(M)} < \lambda \, \|[\phi(x_{ij})]\|_{M_n(M)} \leq \eta_n([\phi(x_{ij})]) \qquad \text{(for all } \epsilon > 0).$$

Since η_n is defined as the supremum of some quantities (see (6.3)), it follows that for all $\epsilon > 0$ there exist $[a_1 \cdots a_n] \in M_{1,n}(A)$ and $\begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \in M_{n,1}(B)$ with $\|[a_1 \cdots a_n]\| \leq 1$, $\left\| \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \right\| \leq 1$, and

$$\begin{aligned} (\lambda - \epsilon) \, \|[\phi(x_{ij})]\|_{M_n(M)} &\leq \left\| [a_1 \cdots a_n][\phi(x_{ij})] \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \right\| \\ &= \left\| \sum_{i,j=1}^{n} a_i \phi(x_{ij}) b_j \right\| \\ &= \left\| \phi\Big(\sum_{i,j=1}^{n} a_i x_{ij} b_j \Big) \right\|. \end{aligned}$$

Moreover, since ϕ is bounded and M is a completely contractive (A, B)-bimodule (satisfying (6.2)), we have

$$\begin{aligned} \left\| \phi\Big(\sum_{i,j=1}^{n} a_i x_{ij} b_j \Big) \right\| &\leq \|\phi\| \left\| \sum_{i,j=1}^{n} a_i x_{ij} b_j \right\| \\ &= \|\phi\| \left\| [a_1 \cdots a_n][x_{ij}] \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \right\| \\ &\leq \|\phi\| \, \|[a_1 \cdots a_n]\| \, \|[x_{ij}]\| \left\| \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \right\| \\ &\leq \|\phi\| \, \|[x_{ij}]\|. \end{aligned}$$

Hence $(\lambda - \epsilon)\,\|[\phi(x_{ij})]\|_{M_n(M)} \leq \|\phi\|\,\|[x_{ij}]\|$, or

$$\left\|\phi^{(n)}([x_{ij}])\right\|_{M_n(M)} \leq \frac{\|\phi\|}{\lambda - \epsilon}\,\|[x_{ij}]\| \qquad (\text{if } \lambda \neq \epsilon),$$

which implies that

$$\left\|\phi^{(n)}\right\| \leq \frac{\|\phi\|}{\lambda - \epsilon} \qquad (\text{if } \lambda \neq \epsilon).$$

Letting $\epsilon \to 0$ we see that $\left\|\phi^{(n)}\right\| \leq \frac{\|\phi\|}{\lambda}$. Therefore $\|\phi\|_{cb} = \sup_n \left\|\phi^{(n)}\right\| \leq \lambda^{-1}\,\|\phi\|$. □

6.2 AW^*-TROs

AW^*-algebras were introduced by I. Kaplansky [41] as those C^*-algebras whose order structure is identical to that of von Neumann algebras. Specifically, an *AW^*-algebra* is a unital C^*-algebra A such that each maximal commutative *-subalgebra of A is generated by projections and the set of all self-adjoint projections in A is a complete lattice. The class of AW^*-algebras is strictly larger than the class of von Neumann algebras, but is smaller than the class of all C^*-algebras. This is schematically illustrated in the diagram below:

$$\{\text{von Neumann algebras}\} \subsetneq \{AW^*\text{-algebras}\} \subsetneq \{C^*\text{-algebras}\}.$$

The theory of C^*-algebras has found generalization in what is now called the theory of *TROs* (the '*TRO*' stands for 'ternary ring of operators'). Recall that given a C^*-algebra A, a *TRO* is defined to be a norm-closed subspace T of A with the property that $xy^*z \in T$ for all $x, y, z \in T$. Then, the adjoint space $T^* = \{x^* \in A : x \in T\} \subseteq A$ is also a TRO, and

$$TT^* = \text{closure}\left(\text{linear span}\,\{xy^* \in A : x, y \in T\}\right),$$

$$T^*T = \text{closure}\left(\text{linear span}\,\{x^*y \in A : x, y \in T\}\right),$$

are C^*-subalgebras of A (the closure above is taken in the unique C^*-algebra norm of A); TT^* is called the *left C^*-algebra* of T, and T^*T is called the *right C^*-algebra* of T. Note that $\mathscr{L}_T = \left(\begin{smallmatrix} TT^* & T \\ T^* & T^*T \end{smallmatrix}\right) \subseteq M_2(A)$ is a C^*-subalgebra of $M_2(A)$; $\mathscr{L}_T$ is called the *linking C^*-algebra* of T. A TRO $T \subseteq A$ is called a *W^*-TRO* if it is weak* closed in A. *AW^*-TROs* are defined as follows:

Definition 6.2.1. Let A be a C^*-algebra and let $T \subseteq A$ be a TRO. Then T is called an *AW^*-TRO* if $\mathscr{L}_T = \left(\begin{smallmatrix} TT^* & T \\ T^* & T^*T \end{smallmatrix}\right)$ is an AW^*-algebra.

The following statement shows that, if T is an AW^*-TRO then TT^* (the left C^*-algebra of T) and T^*T (the right C^*-algebra of T) are both unital.

Statement 6.2.2. Let A be a C^*-algebra and let $T \subseteq A$ be an AW^*-TRO. Then each of the algebras TT^* and T^*T has a (multiplicative) unit element of its own.

Proof. Since $\mathscr{L}_T = \left(\begin{smallmatrix} TT^* & T \\ T^* & T^*T \end{smallmatrix}\right)$ is an AW^*-algebra, it has a unit element – denote it by $1_{\mathscr{L}_T} = \left(\begin{smallmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{smallmatrix}\right)$. Then since for all $\left(\begin{smallmatrix} x & 0 \\ 0 & 0 \end{smallmatrix}\right) \in \mathscr{L}_T$ we have

$$\left(\begin{smallmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{smallmatrix}\right)\left(\begin{smallmatrix} x & 0 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} x & 0 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{smallmatrix}\right) = \left(\begin{smallmatrix} x & 0 \\ 0 & 0 \end{smallmatrix}\right),$$

it follows that $e_{11}x = xe_{11} = x$ (for all $x \in TT^*$). Consequently e_{11} is a unit element of TT^*.

A similar argument also shows that $e_{22}x = xe_{22} = x$ for all $x \in T^*T$; so e_{22} is a unit element of T^*T. □

We will need the following results:

Theorem 6.2.3 (I. Kaplansky [41, Theorem 2.4])**.** If A is an AW^*-algebra, so is the centre of A, any maximal commutative self-adjoint subalgebra of A, and any subalgebra of the form pAp, where p is a self-adjoint projection in A ($p = p^* = p^2$).

Theorem 6.2.4 (S. K. Berberian [6])**.** If A is an AW^*-algebra, then (for all positive integers n) the matrix algebra $M_n(A)$ is also an AW^*-algebra.

Corollary 6.2.5. If T is an AW^*-TRO, then (for all positive integers m, n) the matrix space $M_{m,n}(T)$ is also an AW^*-TRO.

Proof. Assume $m \leq n$, and T is contained in a unital C^*-algebra A (if $m > n$, the proof proceeds analogously). Since T is an AW^*-TRO, the linking algebra of T, $\mathscr{L}_T = \begin{pmatrix} TT^* & T \\ T^* & T^*T \end{pmatrix}$, is an AW^*-algebra. By Theorem 6.2.4, $M_n(\mathscr{L}_T)$ is also an AW^*-algebra and we have

$$M_n(\mathscr{L}_T) = \left(\begin{array}{cc|c|cc} TT^* & T & \cdots & TT^* & T \\ T^* & T^*T & & T^* & T^*T \\ \hline \vdots & & & & \vdots \\ \hline TT^* & T & & TT^* & T \\ T^* & T^*T & \cdots & T^* & T^*T \end{array}\right) \subseteq M_{2n}(A).$$

Reorder rows and columns of $M_n(\mathscr{L}_T)$ to get

$$M_n(\mathscr{L}_T)' = \left(\begin{array}{ccc|ccc} TT^* & \cdots & TT^* & T & \cdots & T \\ \vdots & {\scriptstyle n \times n} & \vdots & \vdots & {\scriptstyle n \times n} & \vdots \\ TT^* & \cdots & TT^* & T & \cdots & T \\ \hline T^* & \cdots & T^* & T^*T & \cdots & T^*T \\ \vdots & {\scriptstyle n \times n} & \vdots & \vdots & {\scriptstyle n \times n} & \vdots \\ T^* & \cdots & T^* & T^*T & \cdots & T^*T \end{array}\right) \subseteq M_{2n}(A).$$

More specifically, $M_n(\mathscr{L}_T)' = uM_n(\mathscr{L}_T)u^*$ where $u \in M_{2n}(A)$ is defined as

$$u = \begin{cases} \text{the matrix of the permutation } (2,\ 2n-1)(4,\ 2n-3)\cdots(n,\ n+1) & \text{if } n \text{ is even;} \\ \text{the matrix of the permutation } (2,\ 2n-1)(4,\ 2n-3)\cdots(n-1,\ n+2) & \text{if } n \text{ is odd.} \end{cases}$$

Since the mapping $M_{2n}(A) \ni x \mapsto uxu^* \in M_{2n}(A)$ ($uu^* = u^*u = 1$) is an inner automorphism, it follows that $M_n(\mathscr{L}_T)' = uM_n(\mathscr{L}_T)u^*$ is an AW^*-algebra. By Theorem 6.2.3, $pM_n(\mathscr{L}_T)'p$ is also an AW^*-algebra, where $p \in M_{2n}(A)$ is a diagonal matrix (the entries outside the main diagonal are all 0s) whose the first 'top' $n-m$ diagonal entries are 0s and the remaining diagonal entries are 1s. Finally, we have

$$pM_n(\mathscr{L}_T)'p = \mathscr{L}_{M_{m,n}(T)}$$

where $\mathscr{L}_{M_{m,n}(T)}$ is viewed as the 'bottom right hand $(m+n)$-by-$(m+n)$ corner' of $M_{2n}(A)$. Therefore $M_{m,n}(T)$ is an AW^*-TRO. □

Corollary 6.2.6. Let A be a C^*-algebra and let $T \subseteq A$ be an AW^*-TRO. Then TT^* and T^*T are both AW^*-algebras.

Proof. By Statement 6.2.2, the algebras TT^* and T^*T are unital; we use 1 to denote the unit element of TT^* as well as the unit element of TT^* (although these unit elements may be different). Since $p = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is a self-adjoint projection in the AW^*-algebra $\mathscr{L}_T = \left(\begin{smallmatrix} TT^* & T \\ T^* & T^*T \end{smallmatrix}\right)$, and $p\mathscr{L}_T p = \left(\begin{smallmatrix} TT^* & 0 \\ 0 & 0 \end{smallmatrix}\right)$, it follows by a result of I. Kaplansky (Theorem 6.2.3) that TT^* is an AW^*-algebra.

A similar argument also shows that T^*T is an AW^*-algebra because $q\mathscr{L}_T q = \left(\begin{smallmatrix} 0 & 0 \\ 0 & T^*T \end{smallmatrix}\right)$, where $q = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathscr{L}_T$. □

Corollary 6.2.7. Every AW^*-algebra is an AW^*-TRO.

Proof. Let A be an AW^*-algebra. Then, in particular, A is a unital C^*-algebra and a TRO ($A \subseteq A$). Since $\mathscr{L}_A = \left(\begin{smallmatrix} AA^* & A \\ A^* & A^*A \end{smallmatrix}\right) = \left(\begin{smallmatrix} A & A \\ A & A \end{smallmatrix}\right) = M_2(A)$, and $M_2(A)$ an AW^*-algebra (by Theorem 6.2.4), it follows that A is an AW^*-TRO. □

Lemma 6.2.8. Let A be a C^*-algebra and let $T \subseteq A$ be a TRO. If e is a tripotent in T ($ee^*e = e \in T$), then:

$$ee^*Te^* \subseteq (ee^*Te^*e)(ee^*Te^*e)^*, \tag{6.4}$$

$$eT^*ee^* \subseteq (ee^*Te^*e)(ee^*Te^*e)^*, \tag{6.5}$$

$$e^*Te^*e \subseteq (ee^*Te^*e)^*(ee^*Te^*e), \tag{6.6}$$

$$e^*eT^*e \subseteq (ee^*Te^*e)^*(ee^*Te^*e). \tag{6.7}$$

Proof. We will only prove (6.4) and (6.5) ((6.6) and (6.7) are proved similarly).

Since $e = ee^*e$, we have $e^* = e^*ee^*$ and

$$\begin{aligned} ee^*Te^* &= ee^*Te^*ee^* \\ &= ee^*T(e^*e)^3e^* \\ &= (ee^*Te^*e)e^*ee^*ee^* \\ &= (ee^*Te^*e)(ee^*ee^*e)^* \\ &\subseteq (ee^*Te^*e)(ee^*Te^*e)^*. \end{aligned}$$

Take adjoints to get $eT^*ee^* = (ee^*Te^*)^* \subseteq (ee^*Te^*e)(ee^*Te^*e)^*$. □

Lemma 6.2.9. Let A be a C^*-algebra and let $T \subseteq A$ be a TRO. If e is a tripotent in T ($ee^*e = e \in T$), then $\hat{e} = \frac{1}{2}\left(\begin{smallmatrix} ee^* & e \\ e^* & e^*e \end{smallmatrix}\right)$ is a self-adjoint projection in $\mathscr{L}_T = \left(\begin{smallmatrix} TT^* & T \\ T^* & T^*T \end{smallmatrix}\right)$, and $\mathscr{L}_{ee^*Te^*e} = \hat{e}\mathscr{L}_T\hat{e}$.

Proof. By the definition of linking algebra we have:

$$\mathscr{L}_{ee^*Te^*e} = \begin{pmatrix} (ee^*Te^*e)(ee^*Te^*e)^* & ee^*Te^*e \\ (ee^*Te^*e)^* & (ee^*Te^*e)^*(ee^*Te^*e) \end{pmatrix},$$

$$\begin{aligned} \hat{e}\mathscr{L}_T\hat{e} &= \tfrac{1}{4}\left(\begin{smallmatrix} ee^* & e \\ e^* & e^*e \end{smallmatrix}\right)\left(\begin{smallmatrix} TT^* & T \\ T^* & T^*T \end{smallmatrix}\right)\left(\begin{smallmatrix} ee^* & e \\ e^* & e^*e \end{smallmatrix}\right) \\ &= \begin{pmatrix} ee^*TT^*ee^*+eT^*ee^*+ee^*Te^*+eT^*Te^* & ee^*TT^*e+eT^*e+ee^*Te^*e+eT^*Te^*e \\ e^*TT^*ee^*+e^*eT^*ee^*+e^*Te^*+e^*eT^*Te^* & e^*TT^*e+e^*eT^*e+e^*Te^*e+e^*eT^*Te^*e \end{pmatrix}. \end{aligned}$$

Then, Lemma 6.2.8 ((6.4) and (6.5)) implies that the $(1,1)$ entry of $\hat{e}\mathscr{L}_T\hat{e}$ is contained in the $(1,1)$ entry of $\mathscr{L}_{ee^*Te^*e}$. Indeed; since

$$\begin{aligned} ee^*TT^*ee^* &= ee^*(TT^*e)e^* \subseteq ee^*Te^* \subseteq (ee^*Te^*e)(ee^*Te^*e)^*, \\ eT^*ee^* &\subseteq (ee^*Te^*e)(ee^*Te^*e)^*, \\ ee^*Te^* &\subseteq (ee^*Te^*e)(ee^*Te^*e)^*, \\ eT^*Te^* &= ee^*(eT^*T)e^* \subseteq ee^*Te^* \subseteq (ee^*Te^*e)(ee^*Te^*e)^*, \end{aligned}$$

it follows that $ee^*TT^*ee^* + eT^*ee^* + ee^*Te^* + eT^*Te^* \subseteq (ee^*Te^*e)(ee^*Te^*e)^*$.

Similarly, it can be shown that the entries $(1,2)$, $(2,1)$, and $(2,2)$ of $\hat{e}\mathscr{L}_T\hat{e}$ are contained respectively in the entries $(1,2)$, $(2,1)$, and $(2,2)$ of $\mathscr{L}_{ee^*Te^*e}$ (use (6.6) and (6.7) to prove that the $(2,2)$ entry of $\hat{e}\mathscr{L}_T\hat{e}$ is contained in the $(2,2)$ entry of $\mathscr{L}_{ee^*Te^*e}$). Therefore $\hat{e}\mathscr{L}_T\hat{e} \subseteq \mathscr{L}_{ee^*Te^*e}$. The inclusion $\mathscr{L}_{ee^*Te^*e} \subseteq \hat{e}\mathscr{L}_T\hat{e}$ is easy to verify. □

Theorem 6.2.10. Let A be a C^*-algebra. If $T \subseteq A$ is an AW^*-TRO, so is any sub-TRO of the form ee^*Te^*e (the Peirce 2-space; see § 4.7), where e a tripotent in T ($ee^*e = e \in T$).

Proof. Since $\mathscr{L}_T = \left(\begin{smallmatrix} TT^* & T \\ T^* & T^*T \end{smallmatrix}\right)$ is an AW^*-algebra, it follows that $\hat{e}\mathscr{L}_T\hat{e}$ is also an AW^*-algebra (by Theorem 6.2.3). But $\mathscr{L}_{ee^*Te^*e} = \hat{e}\mathscr{L}_T\hat{e}$ (by Lemma 6.2.9), so ee^*Te^*e is an AW^*-TRO. □

6.3 Type Decomposition of AW^*-TROs

In 2004, Ruan [66] showed that the decomposition of von Neumann algebras into type I, II, and III can be used to give a similar decomposition for W^*-TROs. A W^*-TRO is said to be of *type I, II or III* if its linking von Neumann algebra is of type I, II or III, respectively. It has been shown in [66] that every W^*-TRO T can be uniquely decomposed into the direct sum of W^*-TROs of type I, type II, and type III ($T = T_I \oplus T_{II} \oplus T_{III}$). In this section we establish that, in fact, this decompositions is applicable to AW^*-TROs.

Definition 6.3.1. An AW^*-TRO T is said to be of *type I, type II* or *type III* if its linking AW^*-algebra $\mathscr{L}_T = \left(\begin{smallmatrix} TT^* & T \\ T^* & T^*T \end{smallmatrix}\right)$ is of type I, type II or type III, respectively.

Theorem 6.3.2. Every AW^*-TRO T can be uniquely decomposed into the direct sum of AW^*-TROs of type I, type II, and type III. More specifically: if T is an AW^*-TRO, then there exist unique self-adjoint projections $p_I^l, p_{II}^l, p_{III}^l \in \mathcal{Z}(TT^*)$ – the centre of the AW^*-algebra TT^* – with sum 1, such that

$$T = p_I^l T \oplus p_{II}^l T \oplus p_{III}^l T$$

and $p_i^l T$ is a TRO ideal as well as an AW^*-TRO of type i (i = I, II, III).

Before we prove Theorem 6.3.2, let us first recall the type-decomposition theorem for AW^*-algebras (which follows, for example, from [44, Proposition 4, p. 11] or [7, Theorem 2, p. 94]):

Theorem 6.3.3. Let A be an AW^*-algebra. Then there exist unique, mutually orthogonal, central, self-adjoint projections $p_I, p_{II}, p_{III} \in A$ with sum 1, such that $p_i A$ is an AW^*-algebra of type i (i = I, II, III) and $A = p_I A \oplus p_{II} A \oplus p_{III} A$.

We also need to establish some auxiliary results. The first of which characterizes the central elements in the linking algebra of a TRO. Following standard notation, we let $\mathcal{Z}(A)$ denote the centre of a given algebra A.

Lemma 6.3.4. Let T be a TRO. Then the central elements of the linking C^*-algebra $\mathscr{L}_T = \left(\begin{smallmatrix} TT^* & T \\ T^* & T^*T \end{smallmatrix}\right)$ have the following characterization: $z \in \mathcal{Z}(\mathscr{L}_T)$ if and only if $z = \left(\begin{smallmatrix} z^l & 0 \\ 0 & z^r \end{smallmatrix}\right)$ where $z^l \in \mathcal{Z}(TT^*)$, $z^r \in \mathcal{Z}(T^*T)$, and $z^l x = x z^r$ for all $x \in T$.

Proof. Let $z = \left(\begin{smallmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{smallmatrix}\right) \in \mathcal{Z}(\mathscr{L}_T)$. Then $z\left(\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right) z$ for all $x \in T$. In particular, it follows that $z_{11} x = x z_{22}$ for all $x \in T$. Similarly $z\left(\begin{smallmatrix} y & 0 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} y & 0 \\ 0 & 0 \end{smallmatrix}\right) z$ for all $y \in TT^*$. Hence $y z_{12} = 0$ for all $y \in TT^*$, and in particular $(z_{12} z_{12}^*) z_{12} = 0$. Therefore $z_{12} = 0$ by [9, Proposition II. 3.2.3]. It is proved similarly that $z_{21} = 0$. Since $\left(\begin{smallmatrix} z_{11} & 0 \\ 0 & z_{22} \end{smallmatrix}\right)\left(\begin{smallmatrix} y & 0 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} y & 0 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} z_{11} & 0 \\ 0 & z_{22} \end{smallmatrix}\right)$ for all $y \in TT^*$, it follows that $z_{11} \in \mathcal{Z}(TT^*)$. Since $\left(\begin{smallmatrix} z_{11} & 0 \\ 0 & z_{22} \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 \\ 0 & w \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & w \end{smallmatrix}\right)\left(\begin{smallmatrix} z_{11} & 0 \\ 0 & z_{22} \end{smallmatrix}\right)$ for all $w \in T^*T$, it follows that $z_{22} \in \mathcal{Z}(T^*T)$.

The converse of the lemma is easy to verify. □

Recall that a norm-closed subspace I of a TRO T is called a *TRO ideal* of T if $IT^*T \subseteq I$ and $TT^*I \subseteq I$ (it follows that $TI^*T \subseteq I$; cf. [26]). Clearly TRO ideals are sub-TROs of T. Below, we show that the decomposition of projections $p = \left(\begin{smallmatrix} p^l & 0 \\ 0 & p^r \end{smallmatrix}\right) \in \mathcal{Z}(\mathscr{L}_T)$ (as in Lemma 6.3.4) gives rise to TRO ideals of T, and consequently to a decomposition of T into a direct sum of TRO ideals.

Lemma 6.3.5. Let T be a TRO, $p \in \mathcal{Z}(\mathscr{L}_T)$ a self-adjoint projection. Apply Lemma 6.3.4 to p to get $p = \left(\begin{smallmatrix} p^l & 0 \\ 0 & p^r \end{smallmatrix}\right)$ for some $p^l \in \mathcal{Z}(TT^*)$, $p^r \in \mathcal{Z}(T^*T)$, with $p^l x = x p^r$ for all $x \in T$. Then:

1. p^l and p^r are self-adjoint projections;
2. $p^l T$ is a TRO ideal of T;
3. $\mathscr{L}_{p^l T} = p \mathscr{L}_T$.

Proof. 2. Let $a \in p^lT$ and $x, y \in T$. Then $ax^*y = p^l(ax^*y) \in p^lT$, so it follows that $p^lT^*T \subseteq p^lT$. Similarly, since $xy^*a = (xy^*)p^la = p^l(xy^*)a \in p^lT$, we conclude that $TT^*p^lT \subseteq p^lT$.

3. Since p is in the centre of $\mathscr{L}_T$, we have $\left(\begin{smallmatrix} p^l & 0 \\ 0 & p^r \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 \\ T^* & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ T^* & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} p^l & 0 \\ 0 & p^r \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} p^l & 0 \\ 0 & p^r \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & T \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & T \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} p^l & 0 \\ 0 & p^r \end{smallmatrix}\right)$. Therefore $p^rT^* = T^*p^l$ and $p^lT = Tp^r$. Consequently

$$\begin{aligned}
\mathscr{L}_{p^lT} &= \begin{pmatrix} (p^lT)(p^lT)^* & p^lT \\ (p^lT)^* & (p^lT)^*(p^lT) \end{pmatrix} \\
&= \begin{pmatrix} p^lTT^*p^l & p^lT \\ (Tp^r)^* & (Tp^r)^*(Tp^r) \end{pmatrix} \\
&= \begin{pmatrix} p^lTT^* & p^lT \\ p^rT^* & p^rT^*Tp^r \end{pmatrix} \\
&= \begin{pmatrix} p^lTT^* & p^lT \\ p^rT^* & p^rT^*T \end{pmatrix} \\
&= \begin{pmatrix} p^l & 0 \\ 0 & p^r \end{pmatrix}\begin{pmatrix} TT^* & T \\ T^* & T^*T \end{pmatrix} \\
&= p\mathscr{L}_T,
\end{aligned}$$

which establishes the required equality and completes the proof of the lemma. □

We now have enough information to prove Theorem 6.3.2.

Proof of Theorem 6.3.2. Let T be an AW^*-TRO. Then $\mathscr{L}_T = \left(\begin{smallmatrix} TT^* & T \\ T^* & T^*T \end{smallmatrix}\right)$ is an AW^*-algebra, and (by Theorem 6.3.3) there exist unique, mutually orthogonal, central, self-adjoint projections $p_I, p_{II}, p_{III} \in \mathscr{L}_T$ with sum 1, such that $p_i\mathscr{L}_T$ is an AW^*-algebra of type i (i = I, II, III) and

$$\mathscr{L}_T = p_I\mathscr{L}_T \oplus p_{II}\mathscr{L}_T \oplus p_{III}\mathscr{L}_T.$$

In fact from Lemma 6.3.4 (for each i = I, II, III) we have $p_i = \left(\begin{smallmatrix} p_i^l & 0 \\ 0 & p_i^r \end{smallmatrix}\right)$ where $p_i^l \in \mathcal{Z}(TT^*)$, $p_i^r \in \mathcal{Z}(T^*T)$, and $p_i^lx = xp_i^r$ for all $x \in T$. Therefore from Lemma 6.3.5 it follows that

$$\mathscr{L}_T = \mathscr{L}_{p_I^lT} \oplus \mathscr{L}_{p_{II}^lT} \oplus \mathscr{L}_{p_{III}^lT}.$$

Setting $e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in \mathscr{L}_T$, identifying T with $e\mathscr{L}_T(1-e)$ and p_i^lT with $e\mathscr{L}_{p_i^lT}(1-e)$

(i = I, II, III), we get

$$\begin{aligned} T = e\mathscr{L}_T(1-e) &= e\Big[\mathscr{L}_{p_I^l T} \oplus \mathscr{L}_{p_{II}^l T} \oplus \mathscr{L}_{p_{III}^l T}\Big](1-e) \\ &= e\mathscr{L}_{p_I^l T}(1-e) \oplus e\mathscr{L}_{p_{II}^l T}(1-e) \oplus e\mathscr{L}_{p_{III}^l T}(1-e) \\ &= p_I^l T \oplus p_{II}^l T \oplus p_{III}^l T. \end{aligned}$$

Thus, we have shown that T is a direct sum TRO ideals $p_i^l T$, and $p_i^l T$ is an AW^*-TRO of type i since $\mathscr{L}_{p_i^l T} = p_i \mathscr{L}_T$ is an AW^*-algebra of type i (i = I, II, III). $\square$

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