

Resilience of Traffic Networks: From Perturbation to Recovery via a Dynamic Restricted Equilibrium Model

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Abstract

When a disruptive event takes place in a traffic network some important questions arise, such as how stressed the traffic network is, whether the system is able to respond to this stressful situation, or how long the system needs to recover a new equilibrium position after suffering this perturbation. Quantifying these aspects allows the comparison of different systems, to scale the degree of damage, to identify traffic network weaknesses, and to analyse the effect of user knowledge about the traffic network state. The indicator that accounts for performance and recovery pattern under disruptive events is known as resilience. This paper presents a methodology to assess the resilience of a traffic network when a given perturbation occurs, from the beginning of the perturbation to the total system recovery. To consider the dynamic nature of the problem, a new dynamic equilibrium-restricted assignment model is presented to simulate the network performance evolution, which takes into consideration important aspects, such as the cost increment due to the perturbation, the system impedance to alter its previous state and the user stress level. Finally, this methodology is used to evaluate the resilience indices of a real network.

Keywords: Dynamic traffic networks; road safety; perturbation resilience; recovery resilience; stress level; policy making.

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Notation

	c_a	Travel cost associated with link a .
	c_{0a}	Free travel time associated with link a .
5	d_{pq}	Demand associated with origin-destination pq .
	h_{pqr}	Flow on route r with origin-destination pq .
	n_{pq}	Number of routes with origin-destination pq .
	t	Time interval.
	t_r	Interval of time at which equilibrium is reached.
10	v_a	Link flow associated with link a .
	v_a^*	Link flow associated with link a corresponding to the UE state.
	v_a^{max}	Link capacity to provide a certain service level.
	C_a	Integral of the travel cost function of link a .
	C_{th}	Cost threshold associated with the system break-down point.
15	C_T	Actual total cost.
	C_0	Initial total cost ($t = 0$).
	R_{pq}	Set of routes with origin-destination pq .
	T_{th}	Reference value associated with the recovery time.
	$\mathcal{A}$	Set of links.
20	$\mathcal{D}$	Subset of origin-destination pairs of nodes.
	$\mathcal{N}$	Set of nodes.
	α	System impedance.
	β_a	Saturation parameter of the cost function.
	γ	Saturation parameter of the cost function.
25	θ_κ	Normalized slope associated with the exhaustion level curve.
	κ	State of perturbation.
	χ_κ^p	Perturbation resilience associated with state of perturbation κ .
	χ_κ^r	Recovery resilience associated with state of perturbation κ .
	ρ_r	Net flow variation among routes within the same origin-destination pair
30		in two consecutive intervals of time.
	σ_κ	Stress level of traffic network associated with state of perturbation κ .

τ_κ Cost level of traffic network associated with state of perturbation κ .
 ψ_κ Exhaustion level of traffic network associated with state of perturbation κ .

1. Introduction

Perturbation of a traffic network is the temporal modification of the physical characteristics of the traffic network, or the conditions to which users are subjected, or both of them, resulting in an overall deterioration of the traffic quality indices, such as travel times, polluting emissions, etc.

When a perturbation takes place in a traffic network, two main effects occur:

(i) user travel costs (generally time) increase and (ii) users become aware of these greater costs and try to reduce them by selecting routes less affected, generating a certain stress level in the network. Then, the new selected routes become more saturated, leading to an increase in the corresponding travel costs. This process continues until an eventual equilibrium is achieved.

The described process can clearly be divided into two different phases, the perturbation stage and the recovery stage. The former implies a modification in the initial network conditions, resulting in a cost increment and a certain degree of user stress. In this phase, it is important to analyse the network capacity to absorb the impact and to adapt to changes. In the recovery stage, when the perturbation has stopped, the system reaches a new equilibrium state compatible with the final network conditions. The critical parameter in this phase is the time necessary to achieve this equilibrium state.

The indicator that accounts for performance and recovery pattern under disruptive events is known as resilience. Resilience was defined by [1] as “the ability for [sic] a transportation network to absorb disruptive events gracefully, maintaining its demonstrated level of service, or to return itself to a level of service equal to or greater than the pre-disruption level of service within a reasonable timeframe”. This definition highlights the crucial factors to be taken into account when analysing the traffic network performance.

Due to the complexity of estimating resilience of a traffic network, some

authors have broken down the most relevant features of this concept with the aim of simplifying its evaluation. According to [2], resilience consists of four parameters; robustness, redundancy, resourcefulness, and rapidity. In a similar way, [3] asserts that resilience is defined in ten dimensions; redundancy, diversity, efficiency, autonomous components, strength, collaboration, adaptability, mobility, safety, ability to recover quickly. Eight resilient design methods containing diversity, adaptability, cohesion, and other characteristics are proposed by [4], and [5] define several qualitative heuristic methods for enhancing the system resilience, considering redundancy, reorganization, adaptation, and other features.

In many cases, the evaluation of these aspects leads to a qualitative characterisation of the system rather than a quantitative analysis. Other authors propose numerical models to assess some of these resilience parameters. For instance, [6] study reliability, vulnerability, survivability, and recoverability; [7] carry out a measure of resilience through the restoration of system performance and the required resource expenditures, and [8] analyse the vulnerability. The variable time when assessing the resilience of complex systems is efficiently introduced by [9], who proposes three time-related metrics, by [10], through the comparison of the system performance over time, and by [11] who state resilience as a ratio of recovery to loss at a given time, showing the recovery as a function of the time. This deterministic formulation is equivalent to that proposed by [12] for the worst-case quantity. Resilience in the restoration process of a transportation system when an earthquake damages a bridge is assessed by [13]. The aim of this method is to establish the bridge restoration activities which maximize the system resilience and minimize costs and time in the restoration activities. Finally [14] propose an optimization program to evaluate the reliability associated with a intermodal freight transportation networks and [15] presents a review of the interactions between congested networks, and safety and security elements under the game theory perspective.

Nevertheless, few approaches have been developed that provide a comprehensive assessment of the resilience of a traffic network including its ability to

prepare and to adapt to changes and its capacity of recovery, over time.

Within this context, this paper presents a novel methodology to assess the resilience of a traffic network suffering from a progressive impact. It is based on
95 a macroscopic traffic model that simulates the dynamic response of the network when suffering a disruption, from the beginning of the perturbation to the total recovery of the system.

The model provides information about the effect of the perturbation upon the travel costs and upon the stress level of users. This information is combined
100 to evaluate the perturbation resilience and the recovery resilience.

It is highlighted that the proposed approach takes into account the system impedance to alter its previous state, given that the actual capacity of adaptation of the system to the new situations determines the traffic network behaviour.

105 The methodology presented in this paper to numerically evaluate the resilience of a traffic network, involves the majority of the qualitative concepts studied by previous authors such as the redundancy, adaptability, ability to recover quickly, etc.

The paper is organized as follows; Section 2 presents a new traffic assignment
110 model, defined as “dynamic equilibrium-restricted assignment model”. Section 3 discusses the important role of the variable ρ to measure the stress level of the system. Section 4 describes how to estimate the resilience associated with the perturbation stage and recovery. Section 5 analyses the influence of the system impedance in the evaluation of the resilience. Section 6 gives a real example of
115 application to illustrate the performance of the proposed method. Finally, in Section 7 some conclusions and future research lines are drawn.

2. A dynamic equilibrium-restricted assignment model

This section is devoted to presenting a new dynamic assignment model, which is used to analyse the system performance before, during and after a per-
120 turbation. This model is based on the following assumptions; (a) The global

behaviour of users is analysed in a day-to-day basis, that is, the problem of within-day dynamics (see [16], [17]) is neglected. (b) Only negative perturbations are taken into account, i.e., those perturbations which imply a travel cost increment. (c) Users select their route choices that reduce their individual travel costs. This selection is based on the complete information about the past day's travel costs. (d) The capacity of adaptation of the users to the changes, the lack of knowledge of the new situation and the lack of information of the behaviour of other users impede the immediate response and recovery of the system.

According to the well-known Wardrop principle, an User Equilibrium (UE) state is reached when, for each origin-destination pair, the actual route travel cost experienced by travelers within a traffic network is equal and minimal (see [18]). Consequently, a “equilibrium-restricted” state can be obtained when, for each origin-destination pair, the actual route travel cost experienced by travelers within a traffic network **tends to** be equal to the Wardropian route travel cost. Nevertheless the system could be unable to reach this state under the existing conditions in the time analysed.

This last definition seems to be more adjusted to reality. The reason that the system does not reach the Wardropian minimum cost state is because the traffic network behaviour is restricted by a system impedance to alter its previous state. This impedance is due to the actual capacity of adaptation to the changes and the lack of knowledge of the current situation.

To introduce this new approach, defined as “dynamic equilibrium-restricted assignment model”, let us consider a connected traffic network with set of nodes $\mathcal{N}$ and set of links $\mathcal{A}$. For certain origin-destination (OD) pairs of nodes, $pq \in \mathcal{D}$, where $\mathcal{D}$ is a subset of $\mathcal{N} \times \mathcal{N}$, connected by a set of routes R_{pq} , there are given positive demands d_{pq} which give rise to a link flow pattern $\mathbf{v} = \{(v_a)_{a \in \mathcal{A}}\}$, and a route flow pattern $\mathbf{h} = \{(h_{pqr})_{r \in R_{pq}, pq \in \mathcal{D}}\}$, when distributed through the network. Furthermore, for each link a there is a positive and strictly increasing travel cost function c_a .

Mathematically, this equilibrium-restricted state can be expressed as an op-

timization problem for each time interval t , that is:

$$\underset{\mathbf{h}, \mathbf{v}, \boldsymbol{\rho}}{\text{Minimize}} \sum_{a \in \mathcal{A}} C_a(v_a(t)), \quad (1)$$

subject to:

$$\sum_{r \in R_{pq}} h_{pqr}(t) = d_{pq}(t) \quad : \lambda_{pq}, \quad \forall pq \in \mathcal{D} \quad (2)$$

$$\sum_{pq \in \mathcal{D}} \sum_{r \in R_{pq}} \delta_{apqr} h_{pqr}(t) = v_a(t) \quad : \lambda_a, \quad \forall a \in \mathcal{A} \quad (3)$$

$$h_{pqr}(t) = \rho_r(t) h_{pqr}(t - \Delta t) \quad : \lambda_r, \quad \forall r \in R_{pq}, \forall pq \in \mathcal{D} \quad (4)$$

$$|\rho_r(t) - 1| \leq \alpha \quad : \mu_{r,1}, \mu_{r,2}, \quad \forall r \in R_{pq} \quad (5)$$

$$h_{pqr}(t) \geq 0 \quad : \mu_{pqr}, \quad \forall r \in R_{pq}, \forall pq \in \mathcal{D} \quad (6)$$

with

$$\delta_{apqr} = \begin{cases} 1, & \text{if route } r \text{ from node } p \text{ to node } q \text{ contains arc } a; \\ 0, & \text{otherwise,} \end{cases}$$

where $C_a(\cdot)$ is the integral of the travel cost function. Furthermore, $\boldsymbol{\rho} = \{\rho_r \in \mathbb{R}^+\}$ measures the variation of route flows in two consecutive intervals of time, $t - \Delta t$ and t , and α represents the system impedance to alter its previous equilibrium state. The lower bound of α is zero, which implies the system is unable to reach a Wardropian equilibrium state and, the upper bound of α is infinite, which means the system reaches the Wardropian equilibrium immediately. It is noted that the system impedance in Eq. (5) can be replaced by a route impedance (α_r) or an OD impedance (α_{pq}). λ_{pq} , λ_a , λ_r , $\mu_{r,1}$, $\mu_{r,2}$ and μ_{pqr} are the dual variables associated to the optimization problem, which are addressed in detail in the Appendix.

The objective function Eq. (1) provides the set of link flows that minimises the sum of the integrals of the link costs subjected to the flow conservation conditions (Eqs. (2) and (3)) and the restriction of adaptation capacity in each time interval analysed. That is, users select those routes which minimize their individual travel costs; Eq. (4) establishes that the route flow in a given time interval differs $|\rho_r - 1|$ w.r.t. the previous time interval. This restriction provides

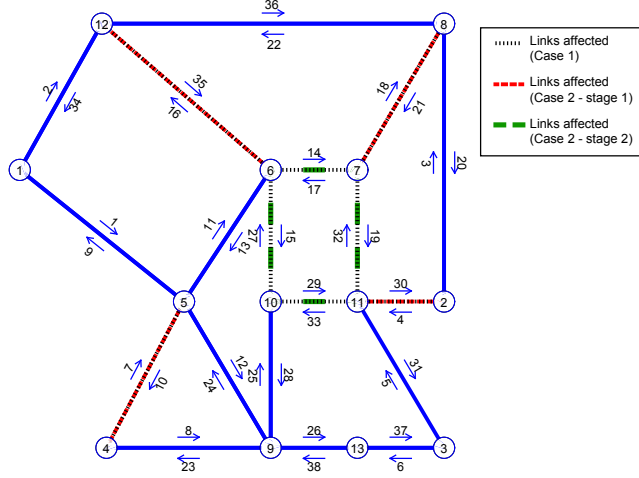


Figure 1: Nguyen-Dupuis network showing the affected links.

temporal continuity for the model. The variation of route flows between two consecutive intervals is limited by the impedance in Eq. (5). Finally, Eq. (6) forces the non-negativity of route flows.

Given that both, the objective function and the feasible region of the optimization problem, are convex, the solution of the model (1)–(6) for each interval of time analysed $t = \tau$, is guaranteed and presents uniqueness in terms of link flow. The optimal link flow pattern v_a^* implies an optimal set of route travel cost for each interval of time.

A detailed explanation of the dynamic equilibrium-restricted assignment model is presented in the Appendix.

2.1. Illustrative example

To facilitate the understanding of all the concepts presented, the Nguyen-Dupuis traffic network is considered, which consists of 13 nodes, 38 links, 66 routes and 34 OD pairs (see Figure 1).

For this example the well-known BPR function is proposed as the cost func-

tion, which is expressed as follows:

$$c_a(t) = c_{0a}(t) \left[1 + \beta_a \left(\frac{v_a(t)}{v_a^{max}(t)} \right)^\gamma \right], \quad (7)$$

and its integral, which is used in Eq. (1), is $c_{0a}(t)v_a(t) \left[1 + \frac{\beta_a}{1+\gamma} \left(\frac{v_a(t)}{v_a^{max}(t)} \right)^\gamma \right]$. c_{0a} , β_a and γ are the free travel time and saturation parameters, respectively, and v_a^{max} is the link capacity to provide a certain service level. Subindex a implies association with link a . In this example $\beta_a(t) = 0.7$, $\gamma = 2.1$, $v_a^{max}(t_0) = 80$ users, and link free travel times are shown in Table 1. The demand for different OD pairs is assumed to be constant [for each time increment](#).

Table 1: Link characteristics at UE state.

Link	c_{0a} (hours)	v_a^* (users)	λ_a^* (hours)	Link	c_{0a} (hours)	v_a^* (users)	λ_a^* (hours)
1	1.86	186.44	9.56	20	2.46	125.00	6.85
2	1.48	163.56	6.14	21	1.50	129.38	4.39
3	2.46	150.00	8.90	22	3.02	170.62	13.41
4	0.77	75.00	1.23	23	1.45	125.00	4.05
5	1.50	120.30	3.98	24	1.50	154.70	5.70
6	0.77	154.70	2.91	25	1.29	75.00	2.08
7	1.46	50.00	1.85	26	0.77	184.32	3.86
8	1.45	125.00	4.05	27	1.16	74.04	1.86
9	1.86	180.12	9.02	28	1.29	100.00	2.74
10	1.46	50.00	1.85	29	0.77	154.83	2.91
11	1.39	102.12	3.02	30	0.77	75.00	1.23
12	1.50	109.32	3.53	31	1.50	115.68	3.79
13	1.39	100.43	2.97	32	1.16	71.26	1.80
14	0.77	162.11	3.13	33	0.77	174.04	3.51
15	1.16	54.83	1.53	34	1.48	169.88	6.53
16	1.97	124.26	5.46	35	1.97	139.82	6.44
17	0.77	175.64	3.56	36	3.02	173.74	13.81
18	1.50	126.26	4.25	37	0.77	184.32	3.86
19	1.16	35.85	1.31	38	0.77	179.70	3.70

Initially the system is in equilibrium. Given that a UE approach provides uniqueness in terms of link flows but not in terms of route flows, the selection of

the route flows, h_{pqr} , is carried out according the criteria of maximum entropy,
 195 which is expressed as follows,

$$\underset{\mathbf{h}}{\text{Maximize}} Z = - \sum_{pq \in D, r \in R_{pq}} h_{pqr} \log(h_{pqr}) \quad (8)$$

subject to

$$\sum_{pq \in D} \sum_{r \in R_{pq}} \delta_{apqr} h_{pqr} = v_a^*, \quad \forall a \in \mathcal{A} \quad (9)$$

$$\sum_{r \in R_{pq}} h_{pqr} = d_{pq}, \quad \forall pq \in \mathcal{D} \quad (10)$$

where $v_a^* = v_a(t=0)$ is the link flow pattern corresponding to the UE state.

In Tables 1 and 2, the initial (equilibrium) state of the proposed example
 is shown. It is noted that, as expected, all routes sharing an OD pair have
 200 the same minimum route travel time, otherwise they do not have any flow.
 When the initial conditions change, the system tries to reach a new equilibrium
 state. However, due to the system impedance, α , this equilibrium is not reached
 immediately. In this case, α is assumed to be 0.10.

To destabilize the traffic network, the capacity of some links are reduced to
 205 50% as a result of road maintenance, which last 15 days, starting at day 10.
 The works affect links [4, 7, 10, 14, 15, 16, 17, 18, 19, 21, 27, 29, 30, 32, 33, 35] (see
 Figure 1).

Figure 2 shows the evolution of link flow and link capacity before, during and
 after the perturbation, for some selected links. With the exception of link 7, all
 210 links seem to suffer important changes in their link flows over time, even when
 their capacities have not directly altered. In parallel, the evolution of route
 flows with time for some selected routes is exhibited in Figure 4. When the
 perturbation stops, the system needs 25 days to recover the equilibrium state
 again. For a better understanding of the effects of the perturbation, Figure 3
 215 provides the evolution of link flow and link capacity for all links. Note that
 the first two sub-figures show the information of those links whose capacity
 has been reduced and the right sub-figure corresponds with those links whose
 capacity keeps constant. It can be appreciated that the links affected by the

Table 2: Characteristics of some selected routes at UE state.

Route	Links	h_{pqr} (users)	λ_{pq}^* (hours)
1	1 - 11 - 14 - 18 - 20	15.01	26.80
2	2 - 35 - 14 - 18 - 20	12.61	26.80
3	2 - 36 - 20	47.38	26.80
OD 1 - 2		75.00	
4	1 - 11 - 14 - 19 - 31	8.74	20.81
5	1 - 11 - 15 - 29 - 31	13.36	20.81
6	1 - 12 - 25 - 29 - 31	0.00	21.87
7	1 - 12 - 26 - 37	109.32	20.81
8	2 - 35 - 14 - 19 - 31	7.34	20.81
9	2 - 35 - 15 - 29 - 31	11.23	20.81
OD 1 - 3		150.00	
10	1 - 11 - 14 - 18	15.01	19.96
11	2 - 35 - 14 - 18	12.61	19.96
12	2 - 36	47.38	19.96
OD 1 - 8		75.00	
13	3 - 21 - 17 - 13 - 9	13.96	28.84
14	3 - 21 - 17 - 16 - 34	14.14	28.84
15	3 - 22 - 34	46.90	28.84
OD 2 - 1		75.00	
16	3 - 21 - 17 - 13 - 10	0.00	21.66
17	3 - 21 - 19 - 33 - 28 - 23	0.00	24.90
18	4 - 33 - 28 - 23	75.00	11.53
OD 2 - 4		75.00	
21	5 - 32 - 17 - 13 - 9	10.92	21.33
22	5 - 32 - 17 - 16 - 34	11.07	21.33
23	5 - 33 - 27 - 13 - 9	11.58	21.33
24	5 - 33 - 27 - 16 - 34	11.73	21.33
25	5 - 33 - 28 - 24 - 9	0.00	24.96
26	6 - 38 - 24 - 9	104.70	21.33
OD 3 - 1		150.00	
31	7 - 11 - 14 - 18 - 20	0.00	19.09
32	8 - 25 - 29 - 30	75.00	10.28
33	8 - 25 - 29 - 32 - 18 - 20	0.00	21.94
OD 4 - 2		75.00	
38	21 - 17 - 13 - 9	13.96	19.94
39	21 - 17 - 16 - 34	14.14	19.94
40	22 - 34 11	46.90	19.94
OD 8 - 1		75.00	

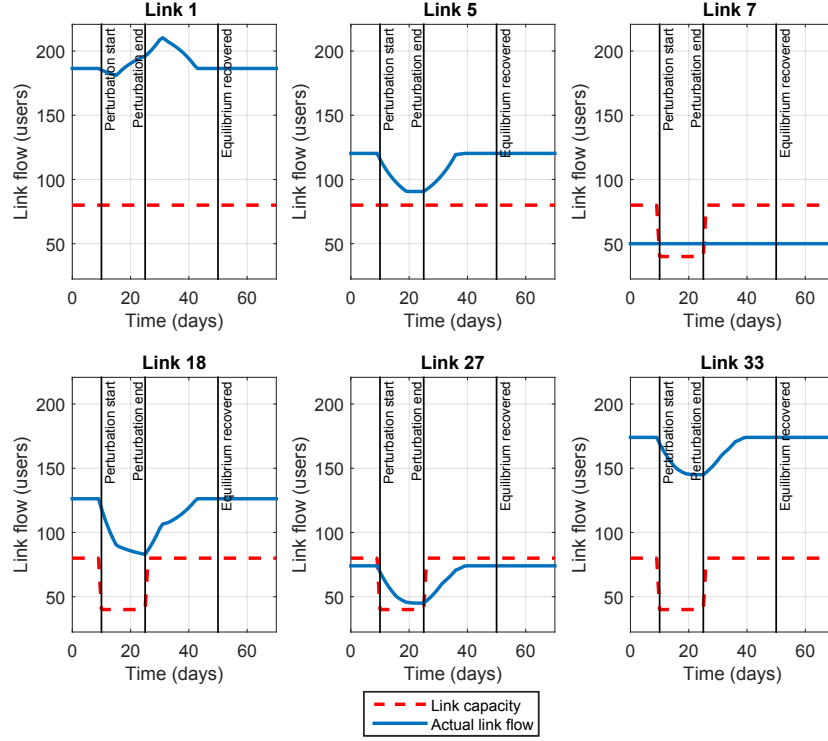


Figure 2: Evolution of link flow and link capacity for some selected links ($\alpha = 0.10$).

220 maintenance works generally tend to suffer reduced flow. When analysing those
 225 links non-affected by the maintenance works, the tendency is not so clear. For
 instance links 6 and 12 gain flow and other links, such as links 5 and 13, suffer a
 flow reduction. This is due to links 5 and 13 belonging to highly affected routes
 (routes 13, 16, 21-25), therefore, their users prefer to choose less impacted routes,
 such as 26 passing throughout link 6. The variation of route flow in routes 13
 and 26 is shown in Figure 4.

To develop this example, the optimization system (1)–(6) has been iteratively
 solved for each time interval, using the Optimization Toolbox of Matlab 2013b.
 The route flow pattern obtained in each iteration has been used as input of the
 following iteration.

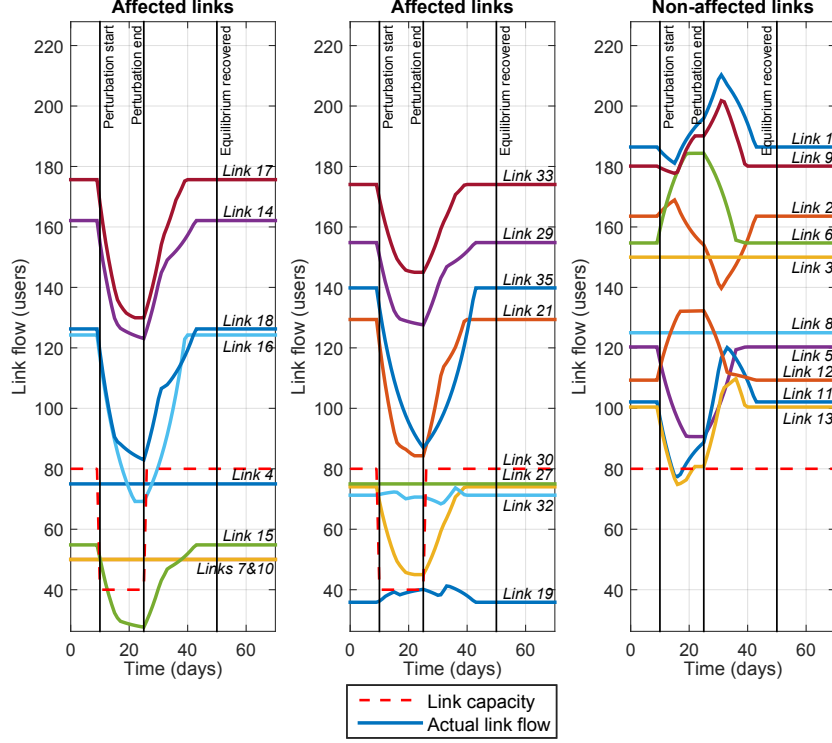


Figure 3: Evolution of link flow and link capacity for all links ($\alpha = 0.10$).

3. Evaluation of the network stress level by means of the ρ variable

Since users are responsible for the traffic network performance when a perturbation occurs, analysing user response is essential to evaluate the system resilience.

In the previous section, variable ρ_r was presented in a mathematical context. Nevertheless, it is worthwhile to pay attention to this informative variable because of its relation with the stress level of the system.

Variable ρ_r , which is limited by the system impedance α , measures the net flow variation among routes within the same OD pair in two consecutive intervals of time. Hence $\rho_r = 1$ implies that route r does not modify its state,

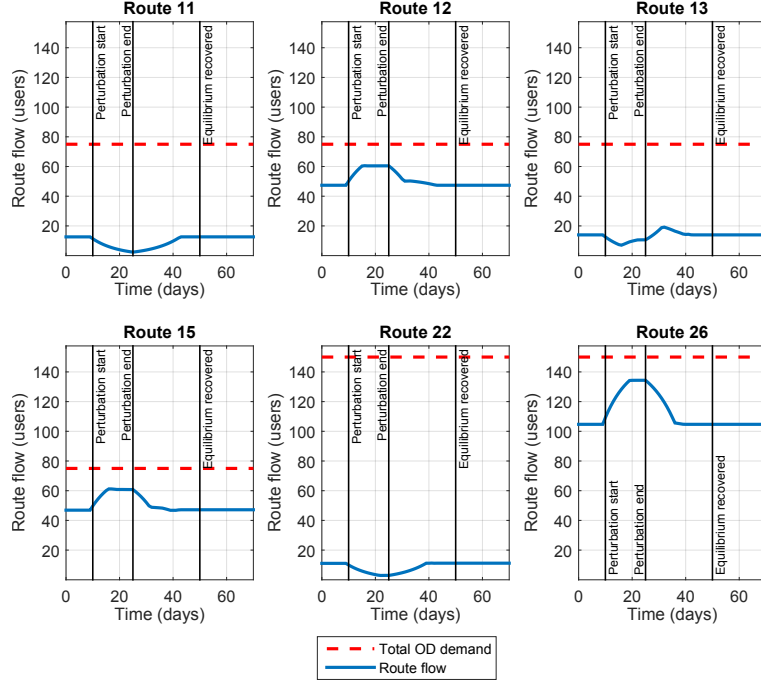


Figure 4: Evolution of route flows with time for some selected routes ($\alpha = 0.10$).

240 i.e., that route has reached an equilibrium position compatible with the existing
 conditions. When $|\rho_r - 1| = \alpha$, the associated route r is changing its state as
 much as its capacity of adaptation allows. In such a case that route presents
 the maximum stress state. The rest of the values within this interval implies
 the route is trying to reach the equilibrium state by losing net flow ($\rho_r < 1$), or
 245 gaining net flow ($\rho_r > 1$).

The system impedance value, α , can be calibrated for a real network by
 means of surveys of the network users. The aim of these surveys is to know the
 dissuasive effect of the delays on their route choices. Finally, the results of the
 surveys have to be quantified by means of real past cases, obtaining α .

250 To illustrate these concepts Figure 5 shows the ρ_r evolution bounded by α
 for the previous example. The first conclusion obtained from this figure is that

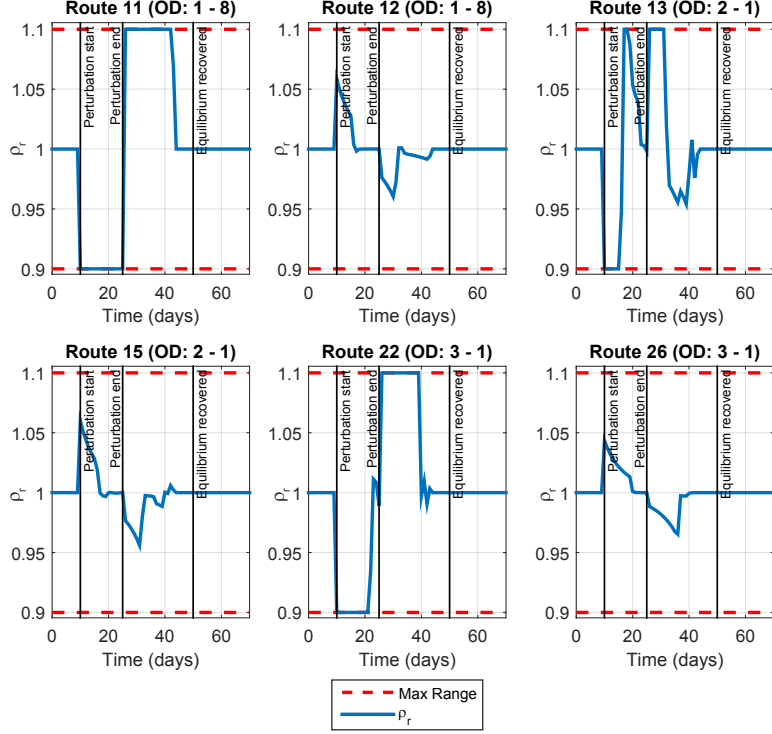


Figure 5: Evolution of ρ_r with time for some selected routes ($\alpha = 0.10$).

routes 11, 13 and 22 are hardly affected by the perturbation as they reach their maximum capacity of adaptation at different times of the performance. On the contrary, links 12, 15 and 26 exhibit a better performance. It is also noted that even when the perturbation has finished, the process of recovering involves high levels of stress.

Assuming that a traffic network is not longer serviceable when at least one OD pair reaches its maximum adaptation capacity, the stress level of a traffic network is defined by the most vulnerable OD pair. Consequently, for a given state of perturbation κ , a normalized stress level σ_κ associated with the whole

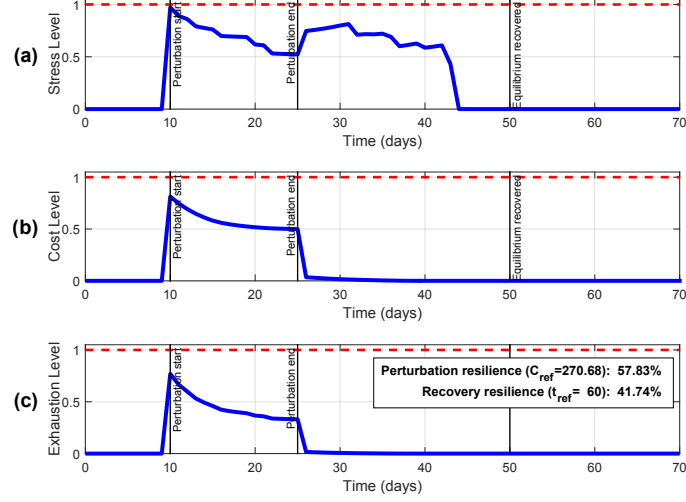


Figure 6: Resilience analysis for perturbation due to maintenance works during 15 days. ($\alpha = 0.10$).

traffic network, is established as follows:

$$\sigma_{\kappa}(t) = \max_{pq \in D} \left(\frac{1}{\alpha} \frac{\sum_{r \in R_{pq}} |\rho_r(t) - 1|}{n_{pq}} \right). \quad (11)$$

where n_{pq} is the number of routes with OD pair pq . We note that σ_{κ} is taking into account the worst states at each time interval. The range of $\sigma_{\kappa}(t)$ is $[0, 1]$, where zero means that the system has achieved an equilibrium state and 1 implies that the adaptation degree is exhausted.

Plot (a) in Figure 6 shows the stress level evolution associated with the illustrative example.

4. Assessment of resilience

In accordance with the definition provided by [1], two types of resilience are considered; the perturbation resilience and the recovery resilience. In order to

assess these resilience indices, the following important points have to be taken into account;

- When a perturbation takes place, users try to adapt to the new situation. Whenever the system adapts, travel costs become minimized, but if the system is unable to absorb the impact, costs rise rapidly (queues, traffic jams, etc.). Therefore, the cost level has to be included in the perturbation resilience estimation.

- If the perturbation is really strong, the system breaks down. As a consequence, a threshold associated with the travel cost is established to define this system break-down point¹. Let C_{th} denote this cost threshold.

Consequently, a normalized cost level, $\tau_\kappa(t)$, is established as follows:

$$\tau_\kappa(t) = \frac{C_T(t) - C_0}{C_{th} - C_0}, \quad (12)$$

where $C_T(t)$ and C_0 are the actual total cost and the initial total cost ($t = 0$), respectively, and they can be computed as:

$$C_T(t) = \sum_{a \in A} c_a(t). \quad (13)$$

Although the normal range of τ_κ is $[0, 1]$, it is highlighted that τ_κ is upper unbounded. This fact allows the consideration of brief rush situations. Plot (b) in Figure 6 shows the cost level evolution associated with the illustrative example.

- When a disturbance takes place, travel costs only remain in low values when the stress level of the traffic network increases. Although the travel costs do not increase, the global situation of the network has deteriorated, therefore the system resilience should decrease. Consequently the formulation should include aspects such as the user response.

¹The system break-down point is the limit-state associated with the failure of the traffic network due to the extreme overcost generated by a strong perturbation. Although the system could theoretically recover, it would imply an unacceptable effort by the system.

With this aim ψ_κ , the exhaustion level associated with a given state of perturbation κ , is introduced. This function includes the cost and stress level, and can be obtained as follows:

$$\psi_\kappa(t) = \frac{1}{2} (1 + \sigma_\kappa(t)) (\tau_\kappa(t))^b, \quad b \geq 1 \quad (14)$$

where b is a coefficient to penalize the cost level when it is larger than the cost threshold. Considering the possible ranges of the variables σ_κ and τ_κ , in normal situations ψ_κ is bounded between $[0, 1]$. Due to possible rush points in the cost level function, ψ_κ can take values larger than 1. In these cases, the system is assumed to break down. It is noted that when a traffic network has a low redundancy, the users will be unable to improve their situation by changing their routes, thus the stress level, σ_κ will take very low values, close to zero, and the cost level will reflect the impact. Eq. (14) has been chosen to guarantee that low stress levels do not result in low (or null) exhaustion levels in those cases. Plot (c) in Figure 6 shows the exhaustion level evolution associated with the illustrative example.

- Following the concept of resilience as the capacity of the network to absorb a shock, the perturbation resilience does not evaluate the damage but rather how far the system is from complete exhaustion. For that reason, the perturbation resilience can be evaluated as the normalized area over the exhaustion curve, as indicated:

$$\chi_\kappa^p = \frac{\int_{t_{p0}}^{t_{p1}} (1 - \psi_\kappa(t)) dt}{t_{p1} - t_{p0}} 100, \quad (15)$$

where t_{p0} and t_{p1} denote the initial and the final time intervals of the disruptive event, respectively. Since $0 \leq \psi_k(t) \leq 1$, the perturbation resilience is defined between $[0, 100]\%$. The perturbation resilience can be understood as the average distance from the exhaustion level to the complete exhaustion within the perturbation period.

- When the perturbation stops, the system tries to reach a new equilibrium position. The longer this recovery process, the worse the recovery

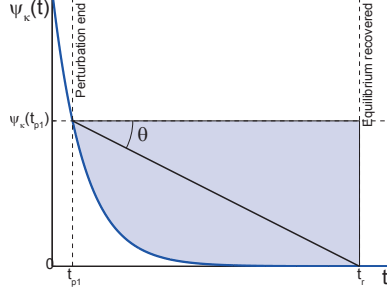


Figure 7: Exhaustion function in the recovery stage.

resilience is. To include this aspect, a normalized slope associated with the exhaustion level curve between the perturbation end interval of time and the equilibrium recovery time interval, is considered. The value of this slope can be calculated as follows:

$$\theta_\kappa = \frac{2}{\pi} \arctan \left(\frac{\psi_\kappa(t_{p1}) T_{th}}{t_r - t_{p1}} \right). \quad (16)$$

With the aim of normalizing, a reference value associated with the recovery time, T_{th} , is established. The range of θ_κ is $[0, 1]$. In Figure 7, where the exhaustion function in the recovery stage is represented, this angle is shown.

- It is important to consider the system pattern when trying to reach a new equilibrium state. A traffic network could partially recover the equilibrium, but could need a long time to reach its final equilibrium. On the contrary, this process could be more continual. In order to take this issue into consideration, the following recovery resilience formulation is proposed:

$$\chi_\kappa^r = \theta_\kappa \frac{\int_{t_{p1}}^{t_r} (\psi_\kappa(t_{p1}) - \psi_\kappa(t)) dt}{\psi_\kappa(t_{p1})(t_r - t_{p1})} 100, \quad (17)$$

where the integral over the curve includes the effect of the different curvatures of the perturbation function. The instant when complete equilibrium is reached is denoted by t_r . It is noted that χ_κ^r takes values between

[0, 100]%. The lower bound implies that the system is enabled to reach an equilibrium state, i.e., the traffic network has a null recovery resilience. The upper bound implies that system recovers its equilibrium immediately. In this case, unlike in Eq. (15), the normalization is carried out in both axis, horizontal (time) and vertical (exhaustion level) axis (see Figure 7) to guarantee that χ_{κ}^r takes values between [0, 100].

For the sake of clarifying these concepts, Figure 6 exhibits the resilience indices associated with the perturbation for the proposed example. In the example $C_{th} = 270.68$ hours (5 times the cost associated with the free flow speed), $T_{th} = 60$ days and $b = 1.2$.

Figure 8 represents the stress level, cost level and exhaustion level curves for a different perturbation. In this case, the reduction of the capacity proposed in the previous example is analysed in two consecutive stages of 15 + 15 days. Only 8 out of the 16 links affected in the previous example, that is, links [4, 7, 10, 16, 18, 21, 30, 35], are affected in the first stage, and afterwards, the rest of them, i.e., links [14, 15, 17, 19, 27, 29, 32, 33] (see Figure 1). Note that this allows the comparison with the previous scheme, where the 16 links were affected at the same time for only 15 days. Comparing both examples, the scheme of the second example (15 + 15 days) provides more resilience for the system during the perturbation than the first scheme (reduction of all links at the same time during 15 days). Nevertheless, due to the higher stress level and longer recovery time, the recovery resilience of the second case is smaller.

5. Influence of the system impedance α

The resilience of a traffic network mainly depends on the following aspects; (i) the perturbation degree, (ii) the network saturation, (iii) the topology of the traffic network, i.e. its redundancy level, and (iv) the impedance α of the users to adapt to the changes.

Figure 9 exhibits the influence of the demand and the α parameter on the resilience index in the presented example. Two different perturbation degrees

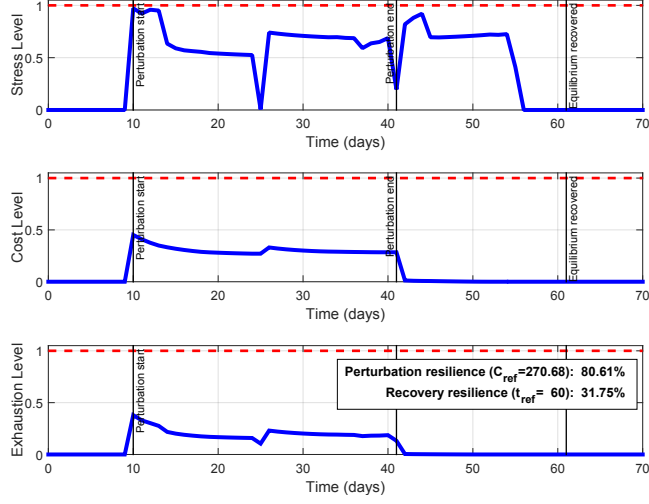


Figure 8: Resilience analysis for perturbation due to works during 15 + 15 days ($\alpha = 0.10$).

have been considered, 25% and 63% of the initial capacity of the affected links during the perturbation.

The influence of α on the resilience index is larger when users play an active role, that is, when they can improve their situation by changing their routes. High congestions show that users with low capacity of adaptation lead to less resilient systems (lower figure). When the intensity of the perturbation and the network saturation take low values, the influence of α is low (see upper figure).

Finally, regarding the traffic network topology, it is noted that when users do not have any option to improve their travel costs due to a low redundancy level, even very high values of α are completely useless to improve the resilience. Therefore, the higher the redundancy, the larger the influence of α because users can play a more active role.

6. Cuenca Example

In this section the applicability of the proposed method on a real network is presented. The Cuenca network (Spain) has been considered, which is shown in

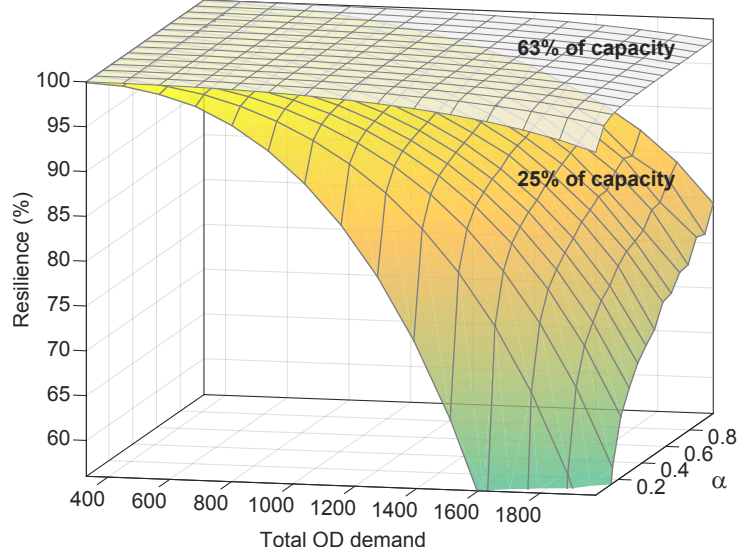


Figure 9: Influence of α in the perturbation resilience for different demand levels and perturbation degrees (25% and 63% of capacity of the affected links during perturbation).

380 Figure 10. This network, which has been previously used by [19, 20], consists of 232 nodes, 672 links, 207 routes and 197 OD pairs, and a total demand of 12420 users. The parameters assumed to carry out the resilience analysis are summarized in Table 3.

385 Due to works for widening the pedestrian pavement, the capacity of 22 central links have been abruptly reduced by 50% for a month. One month later these links recover 90% of their initial conditions. As a result, Figure 11 shows the stress level, the cost level and the exhaustion level for the proposed perturbation. It is noted that a new equilibrium state is completely obtained in 33 days.

390 It is highlighted that the resilience analysis, which has been implemented in Matlab 2013b, has required 34 sec. on a computer with processor Intel(R) Core(TM) i7-3770 CPU@3.40GHz. This fact demonstrates that the proposed methodology is a really useful tool to evaluate the perturbation and recovery

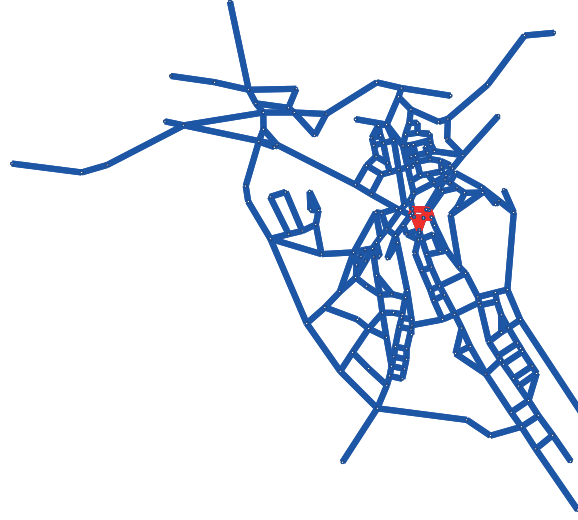


Figure 10: Cuenca network (Spain). The affected links are highlight in red colour.

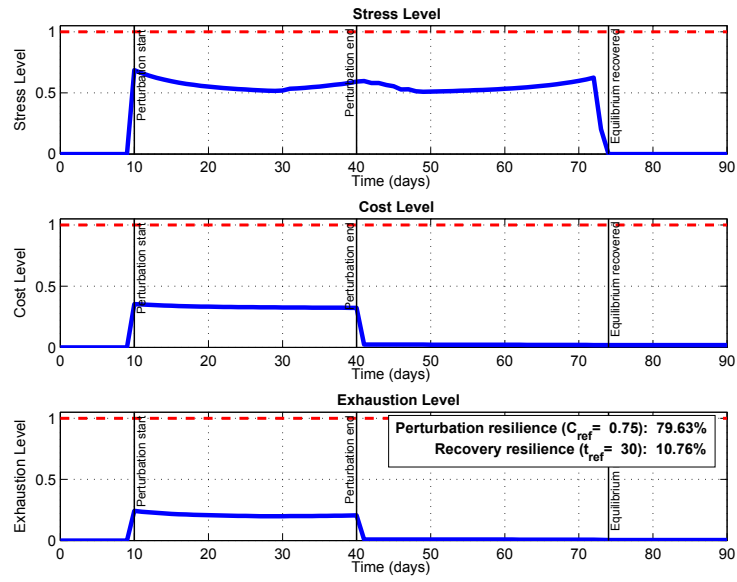


Figure 11: Resilience analysis of Cuenca network.

Table 3: Parameters assumed in Cuenca Example.

BPR parameters			
$\tau_{0a}(t)$	$\beta_a(t)$	γ	$v_a^{max}(t_0)$
30 km/h	1	3	100 users/km
Perturbation characteristics			
Perturbation type	Residual Capacity	Affected Links	
Constant	50 users/km	See highlighted links in Fig. 10	
Performance parameters			
α	Study period	Time interval	
0.10	90 days	one day	
Resilience analysis parameters			
b	C_{th}	T_{th}	
1.2	0.75 hours	30 days	

resilience indices of a real traffic network.

395 7. Conclusions

Based on the discussions presented in this paper in this paper, the following conclusions can be drawn:

- A dynamic approach has been presented in order to evaluate the traffic network resilience. This model is defined as a dynamic equilibrium-restricted assignment model because network performance is restricted by a systems
400 impedance to alter its previous state. This impedance is due to the actual capacity of adaptation to the changes, the lack of knowledge of the new situation and the lack of knowledge of the behaviour of other users.
- The proposed dynamic approach provides the new system equilibrium
405 state reached when the initial state cannot be recovered. Moreover, it allows the identification of a possible collapse of the network in time.

- This model provides a very informative variable, ρ , which allows the analysis of the user response and of the stress level in a network. Knowledge of these aspects is essential to evaluate the system resilience.
- 410 • The proposed straightforward formulation for assessing the traffic network resilience during and after suffering a disruptive event allows the quantification of the perturbation resilience, associated with a given disruptive event in a traffic network, and the recovery resilience, when the perturbation stops.
- 415 • The remaining capacity of the system to absorb the impact and to adapt to changes over time is determined by means of the exhaustion level of the traffic network, which is defined by combining the extra cost generated by the hazard and the system stress level.
- The perturbation resilience formulation takes into account the evolution of the exhaustion level.
- 420 • Important aspects such as the recovery time and the pattern followed in the restore process are included in the recovery resilience formulation.
- A resilience evaluation has been conducted on a real network. Results show that the proposed methodology is a useful tool to assess and compare the resilience capacity of traffic networks.
- 425

Finally, it is noted that more complex dynamic equilibrium-restricted assignment models can be defined, including other important issues of the dynamic assignment approaches. Nevertheless, since the aim of this paper is to present a novel methodology to numerically assess the system resilience involving the dynamic nature of the problem, the assumptions of the proposed model permit this evaluation by capturing the most relevant features of the network (redundancy, connectivity, adaptability, etc.).

Future works will consider the stochastic nature of the problem and the sensitivity analysis of the parameters involved in the model. Furthermore, this

435 paper can be a starting point to develop a methodology for the identification of
the most vulnerable links of a traffic network.

Acknowledgement

This project (RAIN project) has received funding from the European Union's
Seventh Framework Programme for research, technological development and
440 demonstration under grant agreement no 608166.

Appendix

This section is devoted to the mathematical explanation of the dynamic
equilibrium-restricted assignment model, by means of the associated Karush-
Kunn-Tuker (KKT) conditions. These include the primal feasibility conditions
445 corresponding with Eqs. (2)–(6) at the time interval $t = \tau$, and

$$c_a(v_a(\tau)) - \lambda_a = 0; \quad \forall a \in \mathcal{A} \quad (18)$$

$$-\lambda_{pq} + \sum_a \lambda_a \delta_{apqr} + \lambda_r - \mu_{pqr} = 0; \quad \forall (p, q) \in \mathcal{D}, \forall r \in \mathcal{R}_{pq} \quad (19)$$

$$-\lambda_r H_r(\tau) + \mu_{r,1} - \mu_{r,2} = 0; \quad r \in \mathcal{R}_{pq} \quad (20)$$

$$\mu_{pqr} h_{pqr}(\tau) = 0; \quad \forall (p, q) \in \mathcal{D}, \forall r \in \mathcal{R}_{pq} \quad (21)$$

$$\mu_{r,1} (\rho_r(\tau) - 1 - \alpha) = 0; \quad \forall (p, q) \in \mathcal{D}, \forall r \in \mathcal{R}_{pq} \quad (22)$$

$$\mu_{r,2} (\rho_r(\tau) - 1 + \alpha) = 0; \quad \forall (p, q) \in \mathcal{D}, \forall r \in \mathcal{R}_{pq} \quad (23)$$

$$\mu_{pqr} \geq 0; \quad \forall (p, q) \in \mathcal{D}, \forall r \in \mathcal{R}_{pq} \quad (24)$$

$$\mu_{r,1} \geq 0; \quad \forall r \in \mathcal{R}_{pq} \quad (25)$$

$$\mu_{r,2} \geq 0; \quad \forall r \in \mathcal{R}_{pq} \quad (26)$$

where λ_{pq} , λ_a , λ_r , μ_{pqr} , $\mu_{r,1}$ and $\mu_{r,2}$ are the KKT multipliers associated to the
constraints (2)–(6) at $t = \tau$ respectively, and $H_r(\tau) = h_{pqr}(\tau - \Delta t)$.

From Eqs. (18) and (19) it is known that

$$c_{pqr}(\tau) = \lambda_r + \lambda_{pq} + \mu_{pqr}; \quad \forall (p, q) \in \mathcal{D}, \forall r \in \mathcal{R}_{pq}. \quad (27)$$

For those routes whose condition (5) is not active, $\mathcal{R}_{pq}^1 \in \mathcal{R}_{pq}$, it is immediate
 450 from Eqs.(20), (22) and (23) that $\mu_{r,1} = \mu_{r,2} = \lambda_r = 0$. Therefore, the following
 conclusions are derived using Eqs.(21) and (27),

$$c_{pqr}(\tau) = \lambda_{pq} \quad \text{if } h_{pqr}(\tau) > 0; \quad \forall(p, q) \in \mathcal{D}, \forall r \in \mathcal{R}_{pq}^1 \quad (28)$$

$$c_{pqr}(\tau) > \lambda_{pq} \quad \text{if } h_{pqr}(\tau) = 0. \quad \forall(p, q) \in \mathcal{D}, \forall r \in \mathcal{R}_{pq}^1 \quad (29)$$

In other words, λ_{pq} is the optimal route travel cost from origin p to destination
 q , which is independent of the route considered. Equation (29) shows that users
 only select those routes that minimise their route travel costs. Moreover, this
 455 cost is reached when the capacity of adaptation α is large enough to impede the
 system from reaching this state (recall that condition (5) is not active). In the
 case that $\mathcal{R}_{pq}^1 \equiv \mathcal{R}_{pq}$, the solution corresponds to the Wardropian equilibrium,
 that is, $\lambda_{pq} = c_{pq}^*$.

For those routes in $\mathcal{R}_{pq}^2 \in \mathcal{R}_{pq}$, whose condition (5) is active, being $\rho_r(t) =$
 460 $1 - \alpha$, it is known that $\mu_{r,2} > 0$ through the complementary slackness conditions
 (23) and (26). Besides, using (20), (22) and (25) it is obtained $\mu_{r,1} = 0$ and
 $\lambda_r = -\frac{\mu_{r,2}}{H_r(\tau)}$, and consequently,

$$c_{pqr}(\tau) = \lambda_{pq} + \frac{\mu_{r,2}}{H_r(\tau)}, \quad \text{if } h_{pqr}(\tau) > 0; \quad \forall(p, q) \in \mathcal{D}, \forall r \in \mathcal{R}_{pq}^2 \quad (30)$$

$$c_{pqr}(\tau) > \lambda_{pq} + \frac{\mu_{r,2}}{H_r(\tau)} + \mu_{pqr}, \quad \text{if } h_{pqr}(\tau) = 0. \quad \forall(p, q) \in \mathcal{D}, \forall r \in \mathcal{R}_{pq}^2 \quad (31)$$

These equations imply that users selecting routes in $\mathcal{R}_{pq}^2$ will experience route
 travel costs larger than the optimal route travel cost because some users still
 465 have not adapted to the new situation, resulting in an additional cost of $\frac{\mu_{r,2}}{H_r(\tau)}$.
 Therefore, the term $\mu_{r,2}$ represents the cost per user in route $r \in \mathcal{R}_{pq}^2$ who has
 not selected the optimal route pattern.

Parallel, for those routes in $\mathcal{R}_{pq}^3 \in \mathcal{R}_{pq}$ which fulfill $\rho_r(t) = 1 + \alpha$, the
 following expressions are obtained,

$$c_{pqr}(\tau) = \lambda_{pq} - \frac{\mu_{r,1}}{H_r(\tau)} \quad \text{if } h_{pqr}(\tau) > 0; \quad \forall(p, q) \in \mathcal{D}, \forall r \in \mathcal{R}_{pq}^3 \quad (32)$$

$$c_{pqr}(\tau) > \lambda_{pq} - \frac{\mu_{r,1}}{H_r(\tau)} + \mu_{pqr} \quad \text{if } h_{pqr}(\tau) = 0. \quad \forall(p, q) \in \mathcal{D}, \forall r \in \mathcal{R}_{pq}^3 \quad (33)$$

470 That is, users selecting routes in $\mathcal{R}_{pq}^3$ will experience route travel costs smaller than the optimal route travel cost. This is due to those users who are actually in routes in $\mathcal{R}_{pq}^2$ when they should be in $\mathcal{R}_{pq}^3$. Correspondingly $\mu_{r,1}$ represents the cost per user in route $r \in \mathcal{R}_{pq}^1$ who has not selected the optimal route pattern.

Finally, Equations (31) and (33) imply that users only select those routes
475 that minimise their route travel costs within their adaptation capacity.

Considering the Wardropian assignment problem, all users experience the same minimum travel cost, c_{pq}^* , for each OD pair when UE is reached. Thus, the total travel cost of all users travelling from p towards q is $d_{pq}c_{pq}^*$. In the case of the restricted user equilibrium, the total travel cost of all users with the
480 same OD pair in time τ is,

$$d_{pq}\lambda_{pq} - \sum_{r \in \mathcal{R}_{pq}^2} \mu_{r,2}(\alpha - 1) - \sum_{r \in \mathcal{R}_{pq}^3} \mu_{r,1}(\alpha + 1) \geq d_{pq}c_{pq}^*, \quad (34)$$

and consequently,

$$\lambda_{pq} \geq c_{pq}^* + \frac{\sum_{r \in \mathcal{R}_{pq}^2} \mu_{r,2}(\alpha - 1) + \sum_{r \in \mathcal{R}_{pq}^3} \mu_{r,1}(\alpha + 1)}{d_{pq}}. \quad (35)$$

In other words, for each time interval analysed, the optimal route travel cost of the restricted user equilibrium depends on (a) the capacity of adaptation, (b) the OD demand, and (c) the actual number of routes affected by the impedance.

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