BOUNDARY PROBLEM FOR LEVI FLAT GRAPHS

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Abstract. In [DTZ2] the authors provided general conditions on a real codimension 2 submanifold $S \subset \mathbb{C}^n$, $n \geq 3$, such that there exists a possibly singular Levi-flat hypersurface M bounded by S.

In this paper we consider the case when S is a graph of a smooth function over the boundary of a bounded strongly convex domain $\Omega \subset \mathbb{C}^{n-1} \times \mathbb{R}$ and show that in this case M is necessarily a graph of a smooth function over Ω. In particular, M is non-singular.

1. Introduction

The problem of finding a Levi-flat hypersurface $M \subset \mathbb{C}^n$ with prescribed boundary S (the complex analogue of the real Plateau's problem), has been extensively studied for $n = 2$ (cf. [Bi, BeG, BeK, Kr, CS, Sh, SlT, ShT]). In [DTZ2] (announced in [DTZ1]) we addressed this problem for $n \geq 3$, where the situation is substantially different. In contrast to the case $n = 2$, for $n \geq 3$ the boundary S has to satisfy certain compatibility conditions. Assuming those necessary conditions as well as the existence of complex points, their ellipticity and non-existence of complex subvarieties in S, we have constructed in [DTZ2] a (unique but possibly singular) solution to the above problem. An example was also provided in [DTZ2] showing that one may not always expect a smooth solution M in general.

The purpose of this paper is to show that the solution M is smooth if the given boundary has certain "graph form". More precisely, in the coordinates $(z, u + iv) \in \mathbb{C}^{n-1} \times \mathbb{C}$, we assume that S is the graph of a smooth function $g \colon b\Omega \to \mathbb{R}_v$, where $b\Omega$ is the smooth boundary of a strongly convex bounded domain Ω in $\mathbb{C}^{n-1}_z \times \mathbb{R}_u$ and S satisfies the assumptions of [DTZ2] mentioned above. Let M be the solution given by these theorems. Recall that it is obtained as a projection to $\mathbb{C}^n$ of a Levi-flat subvariety with negligible singularities in $[0, 1] \times \mathbb{C}^n$. Let $q_1, q_2 \in b\Omega$ be the projections of the complex points p_1, p_2 of S. Using

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a theorem of Shcherbina on the polynomial envelope of a graph in $\mathbb{C}^2$ (cf. [Sh]) we here prove (cf. Theorem 3.1) that

i) the solution M is the graph of a Lipschitz function $f\colon \overline{\Omega} \to \mathbb{R}_v$ with $f|_{\mathrm{b}\Omega} = g$ which is smooth on $\overline{\Omega} \setminus \{q_1, q_2\}$;
ii) $M_0 = \mathrm{graph}(f) \smallsetminus S$ is a Levi flat hypersurface in $\mathbb{C}^n$.

The regularity of f at q_1 and q_2 remains an interesting open problem closely related to the work of Kenig and Webster [KW1, KW2].

2. Preliminaries

In this section we collect some facts that will be used in the sequel.

2.1. **Remarks about Harvey-Lawson theorem.** Let D be a strongly pseudoconvex bounded domain in $\mathbb{C}^n$, $n \geq 3$, with boundary $\mathrm{b}D$, $\Sigma \subset \mathrm{b}D$ a compact connected maximally complex $(2d-1)$-submanifold with $d > 1$. Then, in view of the theorem of Harvey and Lawson in [HL1, Theorem 10.4] (see also [HL2]), Σ is the boundary of a uniquely determined relatively compact subset $V \subset \overline{D}$ such that: $\overline{V} \setminus \Sigma$ is a complex analytic subset of D with finitely many singularities of pure dimension d and, near Σ, $\overline{V}$ is a d-dimensional complex manifold with boundary. We refer to $V = V_\Sigma$ as the *solution of the boundary problem corresponding to* Σ. A simple consequence is the following:

Lemma 2.1. *Let $D \subset \mathbb{C}^n$ be as above and Σ_1, Σ_2 connected, maximally complex $(2d-1)$-submanifolds of* $\mathrm{b}D$. *Let V_1, V_2 be the corresponding solutions of the boundary problem. If $d > 1$, $2d > n$ and $\Sigma_1 \cap \Sigma_2 = \varnothing$, then $V_1 \cap V_2 = \varnothing$.*

Proof. Suppose $V_1 \cap V_2 \neq \varnothing$. Then $2d > n$ implies $\dim V_1 \cap V_2 \geq 1$. Since $V_1 \cap V_2$ is an analytic subset of D, its closure $\overline{V_1 \cap V_2}$ must intersect $\mathrm{b}D$ and hence also $\Sigma_1 \cap \Sigma_2 \neq \varnothing$, which contradicts the assumption. □

2.2. **Known results.** First, we have the following: a real 2-codimensional submanifold S of $\mathbb{C}^n$, $n \geq 3$, which locally bounds a Levi flat hypersurface must be nowhere minimal near a CR point, i.e. all local CR orbits must be of positive codimension (cf. [DTZ2, Section 2]). If $p \in S$ is a complex point, consider local holomorphic coordinates $(z, w) \in \mathbb{C}^{n-1}_z \times \mathbb{C}_w$, vanishing at p, such that S is locally given by the equation

$$w = Q(z) + O(|z|^3), \tag{2.1}$$

where $Q(z)$ is a complex valued quadratic form in the real coordinates $(\mathsf{Re}\, z, \mathsf{Im}\, z) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$. Observing that not all quadratic forms Q can appear when S bounds a Levi flat hypersurface one comes to the condition that p must be *flat*, i.e. $Q(z) \in \mathbb{R}$ in suitable coordinates. A natural stronger condition is that of *ellipticity* which means by definition that $Q(z) \in \mathbb{R}_+$ for every $z \neq 0$ in suitable coordinates.

Assume that:

(1) S is compact, connected and nowhere minimal at its CR points;
(2) S has at least one complex point and every such point of is flat and elliptic;
(3) S does not contain complex manifold of dimension $(n-2)$.

Then in [DTZ2, Proposition 3.1] it was proved that

a) S is diffeomorphic to the unit sphere with two complex points p_1, p_2;
b) the CR orbits of S are topological $(2n-3)$-spheres that can be represented as level sets of a smooth function $\nu : S \to \mathbb{R}$, inducing on $S_0 = S \setminus \{p_1, p_2\}$ a foliation $\mathcal{F}$ of class C^∞ with 1-codimensional compact leaves.

Next, by applying a parameter version of Harvey-Lawson's theorem [HL1, Theorem 8.1], we obtained in [DTZ2, Theorem 1.3] a solution to the boundary problem as follows:

Theorem 2.2. *Let $S \subset \mathbb{C}^n$, $n \geq 3$ satisfy the above conditions. Then there exist a smooth submanifold $\widetilde{S}$ and a Levi flat $(2n-1)$-subvariety $\widetilde{M}$ in $\mathbb{C}^n \times [0,1]$ (i.e. $\widetilde{M}$ is Levi flat in $\mathbb{C}^n \times \mathbb{C}$) such that $\widetilde{S} = d\widetilde{M}$ in the sense of currents and the natural projection $\pi \colon \mathbb{C}^n \times [0,1] \to \mathbb{C}^n$ restricts to a diffeomorphism between $\widetilde{S}$ and S.*

As for the singularities of $\widetilde{M}$ we have the following results [DTZ2, Theorems 1.4]:

Theorem 2.3. *The Levi-flat $(2n-1)$-subvariety $\widetilde{M}$ can be chosen with the following properties:*

(i) *$\widetilde{S}$ has two complex points $\tilde{p}_0$ and $\tilde{p}_1$ with $\widetilde{S} \cap (\mathbb{C}^n \times \{j\}) = \{\tilde{p}_j\}$ for $j = 0, 1$; every other slice $\mathbb{C}^n \times \{x\}$ with $x \in (0,1)$, intersects $\widetilde{S}$ transversally along a submanifold diffeomorphic to a sphere that bounds (in the sense of currents) the (possibly singular) irreducible complex-analytic hypersurface $(\widetilde{M} \setminus \widetilde{S}) \cap (\mathbb{C}^n \times \{x\})$;*
(ii) *the singular set $\operatorname{Sing} \widetilde{M}$ is the union of $\widetilde{S}$ and a closed subset of $\widetilde{M} \setminus \widetilde{S}$ of Hausdorff dimension at most $2n-3$; moreover each slice $(\operatorname{Sing} \widetilde{M} \setminus \widetilde{S}) \cap (\mathbb{C}^n \times \{x\})$ is of Hausdorff dimension at most $2n-4$;*
(iii) *there exists a closed subset $\widetilde{A} \subset \widetilde{S}$ of Hausdorff $(2n-2)$-dimensional measure zero such that away from $\widetilde{A}$, $\widetilde{M}$ is a smooth*

submanifold with boundary $\widetilde{S}$ near $\widetilde{S}$; moreover $\widetilde{A}$ can be chosen such that each slice $\widetilde{A} \cap (\mathbb{C}^n \times \{x\})$ is of Hausdorff $(2n-3)$-dimensional measure zero.

3. The case of graph

From now on we assume that $S \subset \mathbb{C}^n$, $n \geq 3$, is a graph. Consider $\mathbb{C}^n = \mathbb{C}^{n-1}_z \times \mathbb{C}_w$ with complex coordinates $z = (z_1, \ldots, z_{n-1})$ and w where $z_\alpha = x_\alpha + iy_\alpha$, $1 \leq \alpha \leq n-1$, $w = u + iv$. We also write $\mathbb{C}^n = \big(\mathbb{C}^{n-1}_z \times \mathbb{R}_u\big) \times i\mathbb{R}_v$. Accordingly, a point of $\mathbb{C}^n$ will be denoted by $(z, u, v) = (z, u + iv)$.

Let Ω be a bounded strongly convex domain of $\mathbb{C}^{n-1}_z \times \mathbb{R}_u$ with smooth boundary $\mathrm{b}\Omega$. By strong convexity here we mean that the second fundamental form of the boundary $\mathrm{b}\Omega$ of Ω is everywhere positive definite. In particular, $\Omega \times i\mathbb{R}_v$ is a strongly pseudoconvex domain in $\mathbb{C}^n$.

Let $g : \mathrm{b}\Omega \to \mathbb{R}_v$ be a smooth function, and $S \subset \mathbb{C}^n$ the graph of g. We assume that S satisfies the conditions of [DTZ2, Theorem 1.3] and denote $q_1, q_2 \in \mathrm{b}\Omega$ the natural projections of the complex points p_1, p_2 of S, respectively.

Our goal is to prove the following:

Theorem 3.1. *Let $q_1, q_2 \in \mathrm{b}\Omega$ be the projections of the complex points p_1, p_2 of S, respectively. Then, there exists a Lipschitz function $f : \overline{\Omega} \to \mathbb{R}_v$ which is smooth on $\overline{\Omega} \setminus \{q_1, q_2\}$ and such that $f_{|\mathrm{b}\Omega} = g$ and $M_0 = \mathrm{graph}(f) \smallsetminus S$ is a Levi flat hypersurface of $\mathbb{C}^n$. Moreover, each complex leaf of M_0 is the graph of a holomorphic function $\phi \colon \Omega' \to \mathbb{C}$ where $\Omega' \subset \mathbb{C}^{n-1}$ is a domain with smooth boundary (that depends on the leaf) and ϕ is smooth on $\overline{\Omega}'$.*

The natural candidate to be the graph M of f is $\pi\big(\widetilde{M}\big)$ where $\widetilde{M}$ and π are as in Theorem 2.2. We prove that this is the case proceeding in several steps.

3.1. **Behaviour near** S**.** Set $m_1 = \min\limits_S g$, $m_2 = \max\limits_S g$ and $r \gg 0$ such that
$$D = \Omega \times [m_1, m_2] \Subset \mathbb{B}(r) \cap \big(\Omega \times i\mathbb{R}_v\big)$$
where $\mathbb{B}(r)$ is the ball $\{|(z, w)| < r\}$.

Let Σ be a CR-orbit of the foliation of $S \setminus \{p_1, p_2\}$. Then, Σ is a compact maximally complex $(2n-3)$-dimensional real submanifold of $\mathbb{C}^n$, which is contained in the boundary of the strongly pseudoconvex domain $\Omega \times i\mathbb{R}_v$ of $\mathbb{C}^n$. Let V be the solution to the boundary problem corresponding to Σ, i.e. the complex-analytic subvariety of $\Omega \times i\mathbb{R}_v$

bounded by Σ. We refer to V as the *leaf* bounded by Σ. From Theorems 2.2 and 2.3 it follows that V is obtained as projection $\pi(\widetilde{V})$, where $\widetilde{V} = (\widetilde{M} \setminus \widetilde{S}) \cap (\mathbb{C}^n \times \{x\})$ for suitable $x \in (0,1)$. In particular, if M denotes $\pi(\widetilde{M})$, $\pi_{|\widetilde{V}}$ defines a biholomorphism $\widetilde{V} \simeq V$ and $M \setminus S \subset D$.

Now let Σ_1 and Σ_2 be two distinct CR orbits of the foliation of $S \setminus \{p_1, p_2\}$, and let $\overline{V}_1$, $\overline{V}_2$ be the corresponding leaves bounded by them. Then $\overline{V}_1$, $\overline{V}_2$ do not intersect by Lemma 2.1.

Remark 3.1. In the previous discussion, we only employed the fact that $\Omega \times \mathbb{R}_v$ is a strongly pseudoconvex domain and S is contained in its boundary, without regarding the graph nature of S. It can happen that the leaves have isolated singularities. We shall show that this cannot happen in our case.

Lemma 3.2. *Let $p \in S$ be a CR point. Then, near p, M is the graph of a function ϕ on a domain $U \subset \mathbb{C}_z^{n-1} \times \mathbb{R}_u$, which is smooth up to the boundary of U.*

Proof. Near p, S is foliated by local CR orbits. As a consequence of Theorem 2.2, each local CR orbit extends to a compact global CR orbit Σ that bounds a complex codimension 1 subvariety $V_\Sigma \subset \Omega \times i\mathbb{R}_v$. Furthermore, near p, each Σ is smooth and can be represented as the graph of a CR function over a strongly pseudoconvex hypersurface and V_Σ as the graph of the local holomorphic extension of this function. It follows from the Hopf Lemma that V is transversal to the strongly pseudoconvex hypersurface $b\Omega \times i\mathbb{R}_v$ near p. Hence the family of V_Σ near p forms a smooth real hypersurface with boundary on S that can be seen as the graph of a smooth function ϕ from a relative open neighbourhood U of p in $\overline{\Omega}$ into $\mathbb{R}_v$. Finally, Lemma 2.1 guarantees that this family does not intersect any other leaf V from M. This completes the proof. $\square$

Corollary 3.3. *If $p \in S$ is a CR point, each complex leaf V of M, near p, is the graph of a holomorphic function on a domain $\Omega_V \subset \mathbb{C}_z^{n-1}$, which is smooth up to the boundary of Ω_V.*

Proof. Since M is the graph of a smooth function near p, its tangent space at every point near p is transversal to $i\mathbb{R}_v$. Hence the complex tangent space of M at every point near p is transversal to $\mathbb{C}_w$. Since the tangent spaces of the complex leaves of M coincide with the complex tangent spaces of M, it follows that each leaf V projects immersively to $\mathbb{C}_z^{n-1}$ and the conclusion follows. $\square$

3.2. M **is the graph of a Lipschitz function.** Assume as before that Ω is strongly convex. We have the following

Proposition 3.4. *M is the graph of a Lipschitz function $f : \overline{\Omega} \to \mathbb{R}_v$.*

Proof. We fix a nonzero vector $a \in \mathbb{C}_z^{n-1}$ and for a given point $(\zeta, \xi) \in \Omega$ denote by $H_{(\zeta,\xi)} \subset \mathbb{C}_z^{n-1} \times \{\xi\}$ the complex line through (ζ, ξ) in the direction of $(a, 0)$. Furthermore, we set

$$L_{(\zeta,\xi)} = H_{(\zeta,\xi)} + \mathbb{R}(0,1), \quad \Omega_{(\zeta,\xi)} = L_{(\zeta,\xi)} \cap \Omega, \quad S_{(\zeta,\xi)} = \big(H_{(\zeta,\xi)} + \mathbb{C}(0,1)\big) \cap S$$

Then $S_{(\zeta,\xi)}$ is contained in the strongly convex cylinder

$$\big(H_{(\zeta,\xi)} + \mathbb{C}(0,1)\big) \cap \big(\mathrm{b}\Omega \times i\mathbb{R}_v\big)$$

over $H_{(\zeta,\xi)} + \mathbb{C}(0,1) \simeq \mathbb{C}^2$ and it is the graph of $g_{|\mathrm{b}\Omega_{(\zeta,\xi)}}$.

Since $\Omega_{(\zeta,\xi)} = \Omega \cap L_{(\zeta,\xi)}$, in view of the main theorem of [Sh], the polynomial hull $\widehat{S}_{(\zeta,\xi)}$ of $S_{(\zeta,\xi)}$ is a continuous graph over $\overline{\Omega}_{(\zeta,\xi)}$. Consider $M = \pi(\widetilde{M})$ and set

$$M_{(\zeta,\xi)} = (H_{(\zeta,\xi)} + \mathbb{C}(0,1)) \cap M.$$

Since M is a union of irreducible analytic subvarieties of codimension 1 in $\mathbb{C}^n$ with boundary in the graph S, each intersection $M_{(\zeta,\xi)}$ is the union of a family $\mathcal{A}$ of 1-dimensional analytic subsets. Clearly, the boundary of a connected component of any such analytic set is contained in $S_{(\zeta,\xi)}$. It follows that $M_{(\zeta,\xi)}$ is contained in the polynomial hull $\widehat{S}_{(\zeta',\xi)}$ of $S_{(\zeta,\xi)}$. In view of the main theorem of Shcherbina [Sh], $\widehat{S}_{(\zeta,\xi)}$ is a graph over $\overline{\Omega}_{(\zeta,\xi)} = \overline{\Omega} \cap L_{(\zeta,\xi)}$, foliated by analytic discs, so $M_{(\zeta,\xi)}$ is a graph over a subset U of $\overline{\Omega}_{(\zeta,\xi)}$.

On the other hand, every analytic disc Δ of $\widehat{S}_{(\zeta,\xi)}$ has its boundary on $S_{(\zeta,\xi)} \subset S$. Since all elliptic complex points are isolated, the boundary of Δ contains a CR point p of S. In view of Lemma 3.2, near p, $M_{(\zeta,\xi)}$ is also a graph over $\overline{\Omega}_{(\zeta,\xi)}$. Thus, near p, we must have $M_{(\zeta,\xi)} = \widehat{S}_{(\zeta,\xi)}$. In particular, near p, Δ is contained in $M_{(\zeta,\xi)}$, and therefore in a leaf V_Σ of M. Since V_Σ is a closed analytic subset in $\mathbb{C}^n \setminus S$, the whole disc Δ is contained in V_Σ and hence in M. Moreover, $\Delta \subset H_{(\zeta,\xi)} + \mathbb{C}(0,1)$ thus we conclude that $\Delta \subset M_{(\zeta,\xi)}$. Therefore, every analytic disc of $\widehat{S}_{(\zeta,\xi)}$ is contained in $M_{(\zeta,\xi)}$, consequently $M_{(\zeta,\xi)}$ and $\widehat{S}_{(\zeta,\xi)}$ coincide. It follows that M is the graph of a function $f : \overline{\Omega} \to \mathbb{R}_u$.

Let us prove that f is a continuous function. Choose $(\zeta, \xi) \in \Omega$ and a complex line $H_{(\zeta,\xi)}$ as before. Consider a neighborhood U of (ζ, ξ) in $\mathbb{C}_z^{n-1} \times \mathbb{R}_u$. For $q \in U$, let H_q be the translated of $H_{(\zeta,\xi)}$ which passes through q. With the notation corresponding to the one employed above, we can state the following. For a small enough neighborhood

$V \subset U$ of p in $\mathbb{C}^{n-1}_z \times \mathbb{R}_u$, let $\widehat{S}_q$ be the polynomial hull of S_q in $H_q + \mathbb{C}(0,1)$, and let

$$\mathcal{S}_U = \bigcup_{q \in U} \widehat{S}_q;$$

then $\mathcal{S}_U$ is the graph of a continuous function. Indeed let $\overline{q}$ be a point in V, and let $\{q_m\}_{m \in \mathbb{N}}$ be a sequence of points such that $q_n \to \overline{q}$. Then, obviously, the sets S_{q_m} converge to the set $S_{\overline{q}}$ in the Hausdorff metric as $n \to \infty$. Moreover, it is also clear that $\widetilde{\Omega}_{q_n} \to \widetilde{\Omega}_{\overline{q}}$ for $n \to \infty$. Then, by [Sh, Lemma 2.4] it follows that $\widehat{S}_{q_m} \to \widehat{S}_{\overline{q}}$ as $m \to \infty$. Since every $\widehat{S}_q$ is a continuous graph, this allows to prove easily that $\mathcal{S}_U$ is a continuous graph as a whole.

Thus, f is continuous on Ω, whence on $\overline{\Omega} \smallsetminus \{q_1, q_2\}$ in view of Lemma 3.2. Continuity at q_1 is proved as follows. Let let $\{a_m\}_{m \in \mathbb{N}} \subset \Omega$ be a sequence of points which converges to q_1. Each point $\big(a_m, f(a_m)\big)$ belongs to a complex leaf V_{Σ_m} of M which is bounded by a compact CR orbit Σ_m of the foliation of $S \smallsetminus \{p_1, p_2\}$ (cf. Section 2). By the maximum principle, for every $m \in \mathbb{N}$ there exists a point $(b_m, g(b_m))$ in Σ_m such that

$$|(q_1, g(q_1)) - (a_m, f(a_m))| \leq |(q_1, g(q_1)) - (b_m, g(b_m))|.$$

We claim that

$$|(q_1, g(q_1)) - (b_m, g(b_m))| \to 0$$

as $m \to \infty$. If not there exists an open $B = B(q_1, r)\Omega \times \mathbb{R}_u$ centered at q_1 such that $b_m \notin \overline{B}$ for all m. It follows that

$$\Sigma_m \cap \pi^{-1}(\overline{B}) = \varnothing$$

for all m and

$$V_{\Sigma_m} \cap \pi^{-1}(B) \neq \varnothing$$

for $m \gg 0$. This violates the Kontinuitätsatz since $\Omega \times i\mathbb{R}_v$ is a domain of holomorphy.

Continuity at q_2 is proved in a similar way.

Thus f is continuous on $\overline{\Omega}$ and smooth near $\mathrm{b}\Omega \smallsetminus \{q_1, q_2\}$.

In order to show that f is Lipschitz we now observe that, as it is easily proved, $f_{|\Omega}$ is a weak solution of the *Levi-Monge-Ampère* operator defined in [SlT] with smooth boundary value, so, in view of [SlT, Theorems 2.4, 4.4, 4.6], it is Lipschitz. This concludes the proof of Proposition 3.4. □

Remark 3.2. M is the envelope of holomorphy of S.

3.3. **Regularity.** In order to prove that $M \setminus \{p_1, p_2\}$ is a smooth manifold with boundary we need the following:

Lemma 3.5. *Let U be a domain in $\mathbb{C}_z^{n-1} \times \mathbb{R}_u$, $n \geq 2$, $f : U \to \mathbb{R}_v$ a continuous function. Let $A \subset \operatorname{graph}(f)$ be a germ of complex analytic set of codimension* 1*. Then A is a germ of a complex manifold, which is a graph over $\mathbb{C}_z^{n-1}$.*

Proof. The idea of the proof (here is slightly modified) is due to Jean-Marie Lion cfr. [L].

Let us denote by $z_1, \ldots, z_{n-1}, w = u + iv$, the complex coordinates in $\mathbb{C}_z^{n-1} \times \mathbb{C}_w$. We may suppose that A is a germ at 0. Let $h \in \mathcal{O}_{n+1}$ be a non identically vanishing germ of holomorphic function such that $A = \{h = 0\}$. Let $\mathbb{D}_\varepsilon$ be the disc $\{z = 0\} \cap \{|w| < \varepsilon\}$. Then, for $\varepsilon << 1$, we have either $A \cap \mathbb{D}_\varepsilon = \{0\}$ or $A \cap \mathbb{D}_\varepsilon = \mathbb{D}_\varepsilon$. The latter is not possible since $\mathbb{D}_\varepsilon$ is not contained in any graph over $\mathbb{C}^{n-1} \times \mathbb{R}_u$. It follows that $A \cap \mathbb{D}_\varepsilon = \{0\}$, i.e. A is w-regular. Let us denote by π the projection $\mathbb{C}_z^n \to \mathbb{C}_z^{n-1}$. Then, by the local parametrization theorem for analytic sets there exists $d \in \mathbb{N}$ such that

- for some neighborhood U of 0 in $\mathbb{C}_z^{n-1}$, there exists an analytic set $\Delta \subset U$ such that $A_\Delta = A \cap ((U \setminus \Delta) \times \mathbb{D}_\varepsilon)$ is a manifold;
- $\pi : A_\Delta \to U \setminus \Delta$ is a d-sheeted covering.

We claim that the covering $\pi : H_\Delta \to U \setminus \Delta$ is trivial. Otherwise, there would exist a closed loop $\gamma : [0,1] \to U \setminus \Delta$ whose lift $\widetilde{\gamma}$ to A_Δ is not closed. We extend γ to $\mathbb{R}$ by periodicity and extend $\widetilde{\gamma}$ to $\mathbb{R}$ as lift of γ. Define $\alpha = u \circ \widetilde{\gamma} = u \circ \gamma$, $\beta = v \circ \widetilde{\gamma}$. Since α is continuous and bounded, there exists $\theta \in \mathbb{R}$ such that $\alpha(\theta) = \alpha(\theta + 1)$. But then $\beta(\theta) = \beta(\theta + 1)$ since by the assumption, $\beta(\theta) = f(\gamma(\theta), \alpha(\theta))$. Hence $\widetilde{\gamma}(\theta) = \widetilde{\gamma}(\theta + 1)$, a contradiction with the assumption that $\widetilde{\gamma}$ is not closed.

Since $\pi : A_\Delta \to U \setminus \Delta$ is a trivial covering, we may define d holomorphic functions $\tau_1, \ldots, \tau_d : U \setminus \Delta \to \mathbb{C}$ such that A_Δ is a union of the graphs of the τ_j's. By Riemann's extension theorem, the functions τ_j extend as holomorphic functions $\tau_j \in \mathcal{O}(U)$. The desired conclusion will follow from the fact that all the τ_j coincide. Indeed, suppose, by contradiction, $\tau_1 \neq \tau_2$; then for some disc $\mathbb{D} \subset U$ centered at 0 we have $\tau_1|_\mathbb{D} \neq \tau_2|_\mathbb{D}$ and then, after shrinking $\mathbb{D}$, $(\tau_1 - \tau_2)|_\mathbb{D}$ vanishes only at 0. But, by virtue of the hypothesis, $\{\mathsf{Re}\,(\tau_1 - \tau_2) = 0\} \subset \{\tau_1 - \tau_2 = 0\} = \{0\}$, when restricted to $\mathbb{D}$. The latter is not possible since $(\tau_1 - \tau_2)|_\mathbb{D} \neq 0$ is holomorphic and thus an open map (whose image must include a segment of the imaginary axis). $\square$

Proof of Theorem 3.1. Consider the foliation on $S \smallsetminus \{p_1, p_2\}$ given by the level sets of the smooth function $\nu\colon S \to [0,1]$ as in Section 2 and set $L_t = \{\nu = t\}$ for $t \in (0,1)$. Let $V_t \subset \overline{\Omega} \times i\mathbb{R}_v \subset \mathbb{C}^n$ be the complex leaf of M bounded by L_t and $\pi\colon \mathbb{C}_z^{n-1} \times \mathbb{C}_w \to \mathbb{C}_z^{n-1}$ denote the natural projection. We have:

- by Proposition 3.4, M is the graph of a continuous function over Ω and by Lemma 3.5, each leaf V_t is a complex hypersurface and $\pi|_{V_t}$ is a submersion.
- Since Ω is strongly convex, an argument completely analogous to that of [Sh, Lemma 3.2] shows that $\pi_{|V_t}$ is one-to-one, then, by Corollary 3.3, π sends V_t onto a domain $\Omega_t \subset \mathbb{C}_z^{n-1}$ with smooth boundary.

  If
  $$\pi_u : \left(\mathbb{C}_z^{n-1} \times \mathbb{R}_u\right) \times i\mathbb{R}_v \to \mathbb{R}_u,$$
  $$\pi_v : \left(\mathbb{C}_z^{n-1} \times \mathbb{R}_u\right) \times i\mathbb{R}_v \to \mathbb{R}_v$$
  denote the natural projections then $\pi_{u|L_t} = a_t \circ \pi_{|L_t}$ and $\pi_{v|L_t} = b_t \circ \pi_{|L_t}$, where a_t and b_t are smooth functions in $\mathrm{b}\Omega_t$. Furthermore, the boundary $\mathrm{b}\Omega_t$ and a_t, b_t depend smoothly on t for $t \in (0,1)$. The latter property means that one has a local parametrization of $\mathrm{b}\Omega_t$ smoothly depending on t and such that the functions a_t, b_t also depend smoothly on t when composed with this parametrization. It follows that
- if $(z_t, w_t) \in M$, then $w_t = u_t + iv_t$ is varying in V_t, so $u_t + iv_t$ is the holomorphic extension to Ω_t of $a_t + ib_t$. In particular, u_t and v_t are smooth functions in (z,t), e.g. as a consequence of the Martinelli-Bochner formula.
- The derivative $\partial u_t/\partial t$ is defined and harmonic in Ω_t for each t, and has a smooth extension to the boundary $\mathrm{b}\Omega_t$. Moreover, it follows from Lemma 3.2 and Corollary 3.3 that $\partial u_t/\partial t$ does not vanish on $\mathrm{b}\Omega_t$. Since the CR orbits L_t are connected in view of Theorem 2.2, the boundary $\mathrm{b}\Omega_t$ is also connected and hence $\partial u_t/\partial t$ has constant sign on $\mathrm{b}\Omega_t$. Then, by the maximum principle, $\partial u_t/\partial t$ has constant sign in Ω_t and, in particular, does not vanish. The latter implies the $M \setminus S$ is the graph of a smooth function over Ω, which extends smoothly to $\overline{\Omega} \setminus \{q_1, q_2\}$.
- It furthermore follows from Proposition 3.4 that M is the graph of a Lipschitz function over $\overline{\Omega}$. This completes the proof of Theorem 3.1.

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