Higher Spin Theories in Twistor Space

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This thesis is based on the publications [1] and [2] and on unpublished work [3].

The content of [1] can be found in section 4.2, chapters 5 and 6, and sections 7.3 and 8.4.1. The content of [2] is mainly used for chapter 8. Unpublished work can be found in sections 6.3.1, 7.1, 7.2 and 8.4.2.

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Summary

In this thesis we formulate an action principle for conformal higher spin theory on twistor space and construct an MHV amplitude expansion by considering anti-self-dual fluctuations moving in a self-dual background. Higher spin theories appear in the tensionless limit of String Theory, which is thought to be the most promising model for a consistent description of quantum gravity. The higher-spin modes in String Theory are massive and could be thought of arising through some spontaneous symmetry breaking of a massless higher spin theory. Those massless, unitary higher spin theories have been extensively studied, most notably by Vasiliev, and have been constructed at the level of the equations of motion to all orders in the interactions. Extracting specific interaction terms at higher than cubic order is however very complicated and little is known.

The existence of higher spin theories is tightly constrained by the no-go theorem of Coleman and Mandula and its generalisations [4, 5], and it is generally agreed that the only two well-defined theories circumventing these theorems are Vasiliev Theory [6, 7, 8] and conformal higher spin theory [9]. The latter occurs in the holographic description as the boundary theory of unitary higher spin theories on an AdS background [10, 11, 12] but can be considered in its own right as a non-unitary theory containing ghost modes, which, through truncation, can be restricted to a unitary sub-sector analogous to Maldacena's argument that conformal gravity can be classically truncated to Einstein gravity [13]. We make this explicit by identifying a ghost-free, unitary sub-sector of the conformal higher spin theory in twistor space

and demonstrating that it generates the unique three-point anti-MHV amplitudes for arbitrary spin consistent with Poincaré invariance and helicity constraints [14].

To arrive there, we review some elements of gauge theory for the Yang-Mills and gravity case and its higher-spin generalisations. The formalism which we choose to construct the higher spin theories is twistor theory, which was invented by Penrose [15]. It took however until Witten's [16] and Mason's [17] work to realise that twistor (string) theory is a useful formalism that allows the construction of an action which serves as a generating functional for scattering amplitudes in theories of particles with spin ≤ 2, see [18] for a review. We extend the action on twistor space to include an infinite tower of higher-spin fields, which truncates to the known cases, and propose interaction terms that generate all n-point MHV amplitudes.

The twistor space construction relies on the non-linear graviton construction by Penrose and Ward [19], which associates to self-dual manifolds an integrable deformation of the complex structure on twistor space. In all known cases, a conformal space-time symmetry has been useful to realise this association, even though Maldacena's truncation suggests that it may not be necessary. This however motivates the use of conformal higher spin theory in the twistor space construction. Using our proposal for the twistor action of conformal higher spin theory, we find expressions for all three-point anti-MHV amplitudes and all MHV amplitudes involving positive helicity conformal gravity particles and two negative helicity higher spins, which provides the on-shell analogue for the covariant coupling of conformal higher-spin fields to a conformal gravity background. We study the flat-space limit and show that the restricted amplitudes vanish, supporting the conjecture that in the unitary sector the S-matrix of conformal higher spin theories is trivial. However, by appropriately rescaling the amplitudes we find non-vanishing results which we compare with chiral flat-space higher spin theories.

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Chapter 1

Introduction

A central problem in theoretical physics is how to reconcile quantum mechanics and our modern understanding of gravity based on Einstein's theory of general relativity. This has important consequences for a wide range of fundamental questions such as those regarding the earliest moments of the big bang or the nature of black holes.

The standard model of elementary particle physics is a collection of models for all types of interactions we observe in nature, except gravity. The underlying mathematical framework for the standard model is Yang-Mills theory, which was very successful in unifying the formalism to describe most of the forces we observe in nature. Electrodynamics as well as the weak and strong interactions can be understood through viewing the effect of the classically described forces as the effect of an exchange of 'force-carrier particles', the gauge particles, between matter fields.

All observed particles can be classified by their charges under symmetry transformations in our (modelled) world. Electrodynamics, for example, describes interactions with respect to a $U(1)$ symmetry group, so there is a classification with respect to one charge, the electric charge. The weak interactions are explained through a spontaneous breaking of the symmetry group $SU(2)$, also leaving only one charge, the weak isospin, whereas the strong interactions, having an $SU(3)$ symmetry, have three charges, the color charges. There are also charges associated to the symmetries

of space-time itself, the Poincaré group $ISO(1,3)$, whose unitary irreducible representations are labelled by the spin number. The standard model describes particles of spin ≤ 1, while gravity describes spin 2.

The most promising model so far for a consistent description of quantum gravity is String Theory, which is hoped to also provide a unification of gravity with the standard model at high energies. Amongst the many problems of String Theory, one is that it predicts an infinite number of massive modes, which in the tensionless limit correspond to particles with all possible values of spins, especially > 2, which have not been observed yet. By viewing this limit as describing an effective field theory, we can study its structure and hopefully understand more about String Theory itself.

An interesting method to study any quantum field theory, at least perturbatively, is by analysing so-called scattering amplitudes. These quantities are important physically as they describe the interaction of the fundamental excitations of physical fields, but perhaps even more importantly, as we have learned in recent years, scattering amplitudes can often reveal hidden structures and profound new ways of understanding the theory. Higher-spin theories have been extensively studied, most notably by Vasiliev, and display an enormous amount of complexity. They have been constructed at the level of the equations of motion for the higher-spin fields to all orders in the interactions, however extracting specific interaction terms at higher than cubic order, i.e. scattering amplitudes of interactions with more than three particles, is very complicated and little is known.

Scattering amplitudes are probing the dynamics of a theory. Given a theory with field content ϕ and action $S[\phi]$, we can use the perturbiner formalism [20] to compute n-point tree-level amplitudes by schematically evaluating the action on asymptotic solutions of the classical equations of motion. Concretely, given asymptotic states $\phi_1, \ldots, \phi_n$ with positive frequency on the light-cone at past infinity if incoming, and negative frequency on the light-cone at future infinity if outgoing, a classical solution ϕ_{cl} can be constructed such that $\phi_{\text{cl}} - \sum_{i=1}^{n} \epsilon_i \phi_i$ has the opposite frequencies. The

n-point amplitude is then given by

$$\mathcal{M}_n(\phi_1,\ldots,\phi_n) = \frac{\partial^n S[\phi - \sum_{i=1}^n \epsilon_i \phi_i]}{\partial\epsilon_1 \cdots \partial\epsilon_n}\Big|_{\epsilon_1=\cdots=\epsilon_n=0} . \tag{1.1}$$

The existence of interacting field theories with higher spins in $d > 3$ is tightly constrained by a variety of no-go theorems. Indeed, it seems that there are only two such higher-spin theories which are generally agreed to be well-defined: Vasiliev's theory in space-times with a non-vanishing cosmological constant [6, 21], and *conformal higher spin* (CHS) theories [9]. CHS theories can be thought of as higher-spin generalizations of conformal gravity and as such, they are non-unitary theories whose equations of motion involve higher derivatives of the underlying gauge fields. Despite this obvious lack of unitarity, there are several reasons why these theories are interesting.

Over the last thirty years, CHS theory has been extended to an interacting theory involving single copies of conformal fields at *all* integer spins $s \geq 1$ [22, 11]. Both conformal gravity and CHS theories play an interesting role in the study of other CFTs, particularly in the context of the AdS/CFT correspondence. Here the higher-spin fields act as external sources and, after integrating out the CFT fields, the effective action, W_{eff}, is a functional of the higher-spin fields. For example, in the case of $\mathcal{N} = 4$ super-Yang-Mills (SYM) on a curved background [23] the effective action, after integrating out the SYM fields, consists of a logarithmically divergent piece W_{div} and a finite piece W_{fin}. W_{div} is a functional of the fields forming the $\mathcal{N} = 4$ conformal supergravity multiplet, collectively denoted G, and is exactly the action of $\mathcal{N} = 4$ conformal supergravity

$$W_{\text{div}}[G] = \frac{N^2}{4(4\pi)^2} \ln\Lambda\, S_{\text{CSG}}[G] \ , \tag{1.2}$$

where N is the rank of the SYM gauge group, and Λ is the UV cutoff. In the limit where the $\mathcal{N} = 4$ SYM is taken to be free there are infinitely many conserved traceless bilinear currents which can be coupled to conformal higher-spin fields, collectively denoted ϕ. Expanding the resulting divergent part of the effective action, $W_{\text{div}}[\phi]$,

to quadratic order in higher-spin fields one finds, see [10], a sum over the free CHS actions of each spin. Those actions are gauge invariant quadratic actions in terms of a differential operator $P^{(\mu_1\ldots\mu_s)(\nu_1\ldots\nu_s)}(\partial)$ of order-$2s$ in derivatives

$$S_s[\phi] = \int \mathrm{d}^4x\, \phi_{\mu_1\ldots\mu_s} P^{(\mu_1\ldots\mu_s)(\nu_1\ldots\nu_s)}(\partial)\, \phi_{\nu_1\ldots\nu_s} \tag{1.3}$$

where $P^{(\mu_1\ldots\mu_s)(\nu_1\ldots\nu_s)} = P^{(\nu_1\ldots\nu_s)(\mu_1\ldots\mu_s)}$, $P^{\mu_1}{}_{\mu_1}{}^{\cdots} = 0$ and $P^{\mu_1\cdots}\partial_{\mu_1} = 0$.

A related case is that of the free $O(N)$ vector model consisting of N massless complex scalar fields, χ^i. This model is conjectured [24, 25] to be dual to Vasiliev's higher spin theory [6, 8, 21] on AdS space-time. One can minimally couple the free theory to an infinite set of symmetric, traceless Noether currents, $J^{\mu_1\cdots\mu_s} \sim \chi_i^*\partial^{\mu_1}\ldots\partial^{\mu_s}\chi^i$, and the corresponding effective action, which depends on the infinite tower of higher-spin source fields, acts as the generating functional for connected correlation functions of the currents

$$W[\phi] = N \log\det(-\partial^2 + \sum_s \phi_{\mu_1\ldots\mu_s} J^{\mu_1\cdots\mu_s})\,. \tag{1.4}$$

The UV divergent part of this effective action can be taken as defining a consistent interacting theory of conformal higher-spin fields [10, 11, 12]. CHS theory similarly appears in the UV divergent part of the effective action found by minimally coupling the free $O(N)$ vector model in $d = 4$ to an infinite set of higher spin symmetry currents [10, 11, 12]; in fact this is often the most practical way of computing the non-linear CHS interactions. The free $O(N)$ vector model in 4-dimensions is itself the conjectured holographic dual [25] to Vasiliev's higher spin theory in AdS_5 [21].

The infinite-dimensional higher-spin conformal symmetry should constrain CHS theory to be renormalizable and even UV-finite at the quantum level. In fact it has been shown that the free, one-loop partition function on a Minkowski [26] and 4-sphere [27] backgrounds vanishes, which indicates that, for a suitable regularisation, there is a remarkable cancellation between physical degrees of freedom. Despite these intriguing features and the various non-perturbative calculations that have

been done, relatively little is known about the perturbative observables such as scattering amplitudes in CHS theory. At a practical level, this is due to the difficulty of determining interacting terms in the Lagrangian, defined as an induced theory.

Nevertheless, recent progress has been made in calculating the four-point tree-level scattering amplitudes of CHS theory with external scalars [28], gluons or gravitons [29]. In these calculations, external states are chosen to be solutions of the linearized two derivative – or *unitary* – equations of motion, which form a consistent subset of the linearized CHS equations of motion. By unitary we mean that the spectrum does not contain any ghost modes. When all CHS intermediary states are summed over, the resulting four-point amplitudes vanish. It has been conjectured that this vanishing should extend to any number of unitary external states, order-by-order in perturbation theory: in other words, the S-matrix of CHS theory (defined in this way) is trivial.

Given the complexity of the space-time action, it seems a difficult task to prove this conjecture even at tree-level, however. An alternative approach is offered by *twistor theory*, which is a natural formalism for studying any four-dimensional theory with conformal symmetry. Starting from the work of Witten [16] on twistor string theory the cases of spin-one, Yang-Mills, and spin-two, Weyl gravity, have been well-studied. Of particular relevance to our considerations are the twistor space actions for these theories, which will provide a model for the higher-spin case. Self-dual $\mathcal{N} = 4$ super-Yang-Mills theory [30, 31] was reformulated by Witten [16] as a holomorphic Chern-Simons theory on the super-twistor space $\mathbb{CP}^{3|4}$. This action was also studied by Sokatchev [32], though from a slightly different perspective.

The extension of the twistor space action to the full theory was studied by Mason in [17] and found by Boels, Mason and Skinner in [33]. Based on considerations from twistor string theory, twistor actions for the self-dual sector of conformal gravity were proposed by Berkovits and Witten in [34], and the extension to the full theory was also proposed by Mason in [17] and further studied by Adamo and Mason [35, 36].

A related twistor action for Einstein gravity was studied in [37], though it is not proven if the twistor action is actually fully equivalent to the space-time action [38], although the fact that it gives the right MHV and anti-MHV amplitudes is non-trivial evidence in its favour. Further extensions to higher spin theories have been made as part of this thesis [1, 2].

Twistor space exists in any dimension, however, four dimensional manifolds are special as in that case the Ward correspondence relates holomorphic vector bundles to self-dual Yang-Mills fields, whereas in any other dimension the relation goes just to trivially flat connections, see e.g. [39, 40, 41]. For dimensions three, six and ten the twistor space construction is related to the representations of the division rings $\mathbb{R}, \mathbb{H}$ and $\mathbb{O}$ respectively. Focussing on four dimensions, the twistor space actions are in large part motivated by the non-linear graviton construction [19], which relates self-dual Weyl curvatures on space-time to complex deformations on twistor space. Constructing the CHS theory via the self-dual sector like this, we can easily recover the full theory by introducing the missing anti-self-dual interaction terms in twistor space.

At first, this may just seem like an entertaining exercise until we realise that this reformulation gives us the full action directly in terms of an MHV-amplitude expansion for multi-particle scattering processes of the involved gauge fields. There are many more interesting connections between twistors and higher spin theory. For example, in [42] a twistor-like interpretation of the $Sp(8)$ invariant formulation of massless fields in four-dimensions was given. It would be very interesting to better understand the relation between this formulation and that discussed in this work.

We show that linear CHS theory can be described by action functionals in twistor space. As in the spin-two case for conformal gravity, the on-shell representation of the Poincaré algebra is not diagonalisable which is a manifestation of the failure of the theory to be unitary. Maldacena [13] has argued that conformal gravity with appropriate boundary conditions is classically equivalent to Einstein gravity

with a non-zero cosmological constant, $\Lambda \neq 0$. This implies [35] that Einstein gravity amplitudes can be calculated in conformal gravity by restricting to the asymptotic states of Einstein gravity and accounting for the appropriate powers of the cosmological constant. Adamo and Mason [43, 36] studied supergravity scattering amplitudes by performing such a truncation of conformal gravity in the twistor space description and were able to show that the resulting determinant formula was directly related to Hodges' formula [44]. We perform an analogous truncation on the higher-spin action to identify a unitary sub-sector of CHS theory [1, 2]. At the quadratic level this sector has the usual Fronsdal [45, 46, 47] spectrum of massless higher spins and we show that the $\overline{\text{MHV}}$ three-point amplitude agrees with the constraints from Poincaré invariance [14].

In order to interpret the higher-spin deformation arising from the non-linear graviton construction in a geometric sense, we must include an infinite number of interacting higher-spin fields. This is because once we go beyond the spin-two case the Maurer-Cartan equations for a single spin can no longer be interpreted as the integrability condition for a holomorphic structure on a vector bundle. This can be rectified by including an infinite tower of interacting spins and by interpreting the deformations as acting on the corresponding infinite jet bundle of the space of symmetric products of the (co)tangent bundle. Equivalently, this can also be formulated using the Weyl-Moyal star-product.

The structure of the thesis is as follows:

We start by reviewing elements of gauge theory for the Yang-Mills case in chapter 2, where we define the Chalmers-Siegel action and the spinor-helicity formalism.

In chapter 3 we use the gauge theoretical language to describe Einstein-Cartan theory, conformal gravity, and discuss its classical truncation to Einstein gravity.

We continue in chapter 4 with an outline of linearised massless higher spin theory and the higher-derivative conformal theory.

Chapter 5 is giving a short overview of twistor theory by defining twistor space

for flat space-time and curved ones through the non-linear graviton construction, and introduces the Penrose transform, which identifies massless fields on space-time with cohomology classes on twistor space. We finish this chapter by outlining the basic ingredients of the twistor construction for Yang-Mills theory and gravity.

The generalisation of those actions can be found in chapter 6, where we start off with the spin-3 example, before presenting the twistor space action for the self-dual sector of CHS theory. We identify a truncation to a unitary sub-sector, which reproduces the spectrum of massless Fronsdal fields.

In order to make contact with existing higher-spin formulations, we use chapter 7 to rewrite the holomorphicity conditions using the Weyl-Moyal star-product and the jet-bundle language, and propose additional terms to be added to describe the interactions of the anti-self-dual fields extending the action to the full higher spin theory.

The objective of chapter 8 is to apply that twistor action to the computation of some tree-level scattering amplitudes in the CHS theory and its unitary sub-sector.

We finish the presentation by leaving some remarks and conclusions in chapter 9.

A detailed summary of the mathematical aspects of gauge field theory can be found in appendix A, which we will assume the reader is mostly familiar with.

1.1 Notation

Let us summarise our notation that we use throughout the thesis.

Fields and Forms: A – Yang-Mills connection form; D_A – covariant derivative w.r.t. A; F – Yang-Mills curvature form; ω – spin-2 or higher-spin Lorentz connection form; e – coframe field; T – torsion form; R – spin-2 or higher-spin curvature form; $\mathscr{R}$ – Riemann tensor; $\mathscr{C}$ – Weyl tensor; $\Psi/\tilde{\Psi}$ – anti-self-dual/self-dual part of the Weyl tensor; G – Lagrangian multiplier in the Chalmers-Siegel-like action; η – flat

metric (Minkowski or AdS); T – generators of some algebra.

Starting with chapter 6: f – self-dual modes on twistor space; g – anti-self-dual modes on twistor space.

Coordinates and Spaces: M – real semi-Riemannian manifold; $\mathbb{M}$ – complexified, conformally compactified semi-Riemannian manifold; x – coordinates on $\mathbb{M}$; Λ – cosmological constant; $\mathbb{T}$ – twistor space; Z – coordinates on $\mathbb{T}$; $\mathscr{T}$ – deformed twistor space; $\mathscr{Z}$ – coordinates on $\mathscr{T}$.

Indices: $\mu, \nu, \tau, \ldots$ – four-component space-time indices; $A, B, C, \ldots$ – complex two-component Weyl spinor indices of $\mathbb{S}^+$; $A', B', C', \ldots$ – complex two-component Weyl spinor indices of $\mathbb{S}^-$; $\alpha, \beta, \gamma, \ldots$ – (usually) four-component holomorphic twistor indices; $\underline{\alpha}, \underline{\beta}, \underline{\gamma}, \ldots$ – four-component holomorphic local twistor indices; $\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \ldots$ – four-component anti-holomorphic twistor indices;

$\mu(s) = (\mu \ldots \mu)$ – symmetrisation over s space-time indices; $A(s) = (A \ldots A)$ – symmetrisation over s spinor indices when they are associated to space-time indices; $\alpha_I = \alpha_{i_1} \ldots \alpha_{i_{s-1}}$ – multi-index notation for twistor indices; $A_I = A_{i_1} \ldots A_{i_{s-1}}$ – multi-index notation for the unprimed part of twistor indices.

We agree on the convention that we always sum over indices that appear twice, even if there is an index-dependent coefficient, e.g. if $\alpha_I = (\alpha_1, \ldots, \alpha_n)$, then

$$|I|!\, g_{\alpha_I} \wedge f^{\alpha_I} := \sum_{n=0}^{\infty} \sum_{\alpha_1,\ldots,\alpha_n=0}^{3} n!\, g_{\alpha_1 \ldots \alpha_n} \wedge f^{\alpha_1 \ldots \alpha_n}\,.$$

Miscellaneous: $\mathcal{M}$ – amplitude in conformal theory; $\widetilde{\mathcal{M}}$ – amplitude in unitary theory; $\star$ – Weyl-Moyal star-product; $\mathbf{f} = j^\infty(f)$ – infinite jet prolongation of f.

Chapter 2

Spin-1: Yang-Mills Theory

In this chapter we will begin by laying out some of the basic concepts and structures that are essential in the discussion throughout the thesis. A detailed mathematical summary can be found in appendix A.

We start off with reminding ourselves that the gauge field A of a Yang-Mills theory is described by a connection form on a principal $SU(N)$-bundle, or more generally, a principal G-bundle for some compact Lie group G, over our space-time manifold M. To describe A as a field on M, we need to consider it in a gauge, which is just a chosen section s of the bundle. In this gauge, the gauge field is now a map

$$\Omega^1(M, \mathfrak{g}) \ni A^s = s^* A : TM \to \mathfrak{g}\,, \tag{2.1}$$

where $\mathfrak{g}$ is the Lie algebra associated to G. A priori, A^s has $\dim(M) \cdot \dim(\mathfrak{g})$ degrees of freedom, though not all of them are physical. The inherent redundancy of this formulation becomes apparent once we realise that physical quantities are invariant under gauge transformations. A gauge transformation allows us to change between two gauges: consider a second gauge t, which is related to s by some group element $t = gs$, then the gauge fields in those two gauges are related through

$$A^t = g^{-1} A^s g + g^{-1}\,\mathrm{d}g\,. \tag{2.2}$$

The actual gauge we choose will only ever appear implicitly, and so we will usually drop the superscript $A^s \to A$. In those cases we also just write

$$\delta A = g^{-1}\,\mathrm{d}g = \mathrm{d}\chi \tag{2.3}$$

for some function $\chi : M \to \mathfrak{g}$.

The connection form A defines a covariant derivative

$$D_A = \mathrm{d} + A : \Omega^k(M, \mathfrak{g}) \to \Omega^{k+1}(M, \mathfrak{g})\,, \tag{2.4}$$

through which we can define the curvature of the bundle associated to A by

$$\Omega^2(M, \mathfrak{g}) \ni F = D_A A = \mathrm{d}A + \frac{1}{2}[A, A]\,. \tag{2.5}$$

The length of the curvature in turn defines the Yang-Mills functional, which is the action of the theory

$$S[A] = \frac{1}{\lambda}\int_M \mathrm{tr}_{\mathfrak{g}}(F \wedge *F)\,, \tag{2.6}$$

where λ is some possibly dimensional parameter that defines the strength of the coupling of the theory. The action is a gauge-invariant object, i.e. invariant under gauge transformations. By variation with respect to A we can find the equations of motion

$$D_A F = 0\,, \qquad *\,D_A * F = 0\,. \tag{2.7}$$

The first equation is automatically satisfied by definition of F and is known as the (second) Bianchi identity. Since the action is gauge-invariant, so are the equations of motion, and thus there are $\dim(\mathfrak{g})$ redundant degrees of freedom in this formulation. The equations of motion themselves pose $\dim(\mathfrak{g})$ constraints on A, which leaves the gauge field with $(d-2)\dim(\mathfrak{g})$ physical degrees of freedom. If a gauge field satisfies the equations of motion, we call it on-shell. We can then think of each generator of $\mathfrak{g}$ carrying $d-2$ degrees of freedom.

In $d = 4$ dimensions, each of those generators has thus two remaining degrees of freedom and can be identified with a massless particle. Examples are

1. Electrodynamics with gauge group $G = U(1)$ has only one massless gauge field, the photon.

2. Electro-weak theory with gauge group $G = SU(2) \times U(1)$ leads to $3 + 1$ gauge fields, however the $SU(2)$ symmetry is broken through the interaction with an additional scalar particle, the Higgs particle, so three of them have three, not two degrees of freedom and are thus massive particles: the Z^0, the W^+, and the W^- bosons.

3. Quantum chromodynamics with gauge group $G = SU(3)$ has eight massless gauge fields, the gluons.

2.1 Chalmers-Siegel Action

In four dimensions, another interesting thing happens: the Hodge operator becomes a bijection on the space of two-forms, and its square is proportional to the identity $*^2 = (-1)^q \,\mathrm{id}_{\Lambda^2}$, where q is the number of minus signs in signature of the metric. We can thus decompose any two form in terms of the eigenspaces of this operator, which we call self-dual and anti-self-dual respectively

$$\Lambda^2(T^*M) = \Lambda^2_+(T^*M) \oplus \Lambda^2_-(T^*M)\,. \tag{2.8}$$

This is especially interesting for the curvature, as it thus has a decomposition in a self-dual and an anti-self-dual part

$$F = F^+ + F^-\,, \tag{2.9}$$

where $*\,F^+ = F^+$ and $*\,F^- = -F^-$ if M has Euclidean or split signature, or $*\,F^+ = iF^+$ and $*\,F^- = -iF^-$ if M has Lorentzian signature. Since $D_A(F^+ - F^-) = -2D_AF^-$, the equations of motion now become

$$D_AF^- = 0\,. \tag{2.10}$$

The corresponding action principle is equivalent to the Yang-Mills action up to a topological term, which vanishes in the equations of motion and is thus irrelevant for perturbation theory. It appears when considering the action (in Riemannian or split signature)

$$\begin{aligned}\int_M \mathrm{tr}_{\mathfrak{g}}(F \wedge *F) &= \int_M \mathrm{tr}_{\mathfrak{g}}(F^+ \wedge F^+ - F^- \wedge F^-) \\ &= -2\int_M \mathrm{tr}_{\mathfrak{g}}(F^- \wedge F^-) + \int_M \mathrm{tr}_{\mathfrak{g}}(F^+ \wedge F^+ + F^- \wedge F^-)\,. \end{aligned} \tag{2.11}$$

The second term is of a total derivative and thus of topological nature, and the first term is equivalent to the Chalmers-Siegel action [48, 31]

$$S[A,G] = -\int_M \mathrm{tr}_{\mathfrak{g}}\left(F^- \wedge G - \frac{\lambda}{2} G \wedge G\right)\,, \tag{2.12}$$

which gives the equations of motion

$$F^- = \lambda G\,, \qquad D_A G = 0\,. \tag{2.13}$$

Taking the limit $\lambda \to 0$ sets the anti-self-dual part of the curvature to zero. For $\lambda \neq 0$ we recover the full non-self-dual theory, in which the new field G is treated as an anti-self-dual mode on the self-dual background. In this sense, λ can be interpreted as the parameter of a perturbative expansion.

This form of the action will be very important throughout the thesis, and it will be serving us as a guide in formulating an action principle on twistor space.

2.2 Spinors and Spinor-Helicity Formalism

Our goal is to study the (self-)interactions of gauge fields, that is to compute scattering amplitudes. For massless particles, those amplitudes have a particularly simple form using the spinor-helicity formalism, which we will review shortly.

As we have already seen when discussing the self-duality condition, certain factors and signs depend on which signature we chose for our manifold in the beginning. In

order to avoid this case by case study and treat all signatures equally, we consider the complexification of our manifold and then, at the end, impose the appropriate reality conditions which identifies the signature we want. We come back to this discussion in a moment.

Complexification of $M \to M^{\mathbb{C}} = M \otimes_{\mathbb{R}} \mathbb{C}$ is useful since the group $O(p,q;\mathbb{C})$ is isomorphic to the usual orthogonal group $O(p+q;\mathbb{C})$, and as we have discussed in the introduction, the spin is governed by a representation under $\mathfrak{so}(p,q)$, which is naturally associated to the tangent bundle TM. This means that independently of the signature of M we can study the complexified algebra $\mathfrak{so}(4,\mathbb{C})$, which in turn is isomorphic to $\mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$.

The algebra $\mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$ admits the complex two-component Weyl spinor representations $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ respectively. Spinors transforming in $(\frac{1}{2},0)$ are said to have positive chirality, and spinors transforming in $(0,\frac{1}{2})$ are said to have negative chirality, and we refer to $\mathbb{S}^{\pm}M^{\mathbb{C}}$ as the spaces of the corresponding two-component Weyl spinors. Sections of the spin bundle $\mathbb{S}M^{\mathbb{C}}$ are Dirac spinors, which transform in the $(\frac{1}{2},0) \oplus (0,\frac{1}{2})$ representation, which means they transform under a reducible representation of the Lorentz group.

There is an isomorphism $\sigma : TM^{\mathbb{C}} \to \mathbb{S}M^{\mathbb{C}}$ which associates to each vector field $v = v^{\mu}\partial_{\mu} \in TM^{\mathbb{C}}$ a spinor field on $\mathbb{S}M^{\mathbb{C}}$ by

$$v^{AA'} = (\sigma_{\mu})^{AA'}\, v^{\mu}\,, \tag{2.14}$$

whose matrix representation is given by

$$(v^{AA'}) = \frac{1}{\sqrt{2}} \begin{pmatrix} v^0 + v^1 & v^2 + iv^3 \\ v^2 - iv^3 & v^0 - v^1 \end{pmatrix}. \tag{2.15}$$

This translates into the statement that vectors on $M^{\mathbb{C}}$ transform in the $(\frac{1}{2},\frac{1}{2})$ representation. The metric on $\mathbb{S}M^{\mathbb{C}}$ acts independently on each of the two copies of $\mathfrak{sl}(2,\mathbb{C})$, on which it is respectively given by the totally antisymmetric tensor ϵ,

whose matrix representation reads

$$(\epsilon_{AB}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = (\epsilon_{A'B'})\,. \tag{2.16}$$

We define the $\mathfrak{sl}(2,\mathbb{C})$-invariant inner products as

$$\forall \lambda, \mu \in \mathbb{S}^+M^{\mathbb{C}} : \langle vw \rangle := \lambda^A \mu^B \epsilon_{AB}\,, \tag{2.17}$$

$$\forall \lambda, \mu \in \mathbb{S}^-M^{\mathbb{C}} : [vw] := \lambda^{A'} \mu^{B'} \epsilon_{A'B'}\,. \tag{2.18}$$

Naturally, these inner products define an isomorphism between the spin bundle and its dual $\mathbb{S}^{\pm}M^{\mathbb{C}} \to \mathbb{S}^{\pm *}M^{\mathbb{C}}$, which allows us to raise and lower indices through

$$\lambda^A = \epsilon^{AB}\lambda_B\,, \qquad \lambda_A = \lambda^B \epsilon_{BA}\,, \tag{2.19}$$

and analogously for spinors with dotted indices.

The complexified Minkowski metric η on $\mathbb{C}^4$, on which $\mathfrak{so}(4,\mathbb{C})$ naturally acts, can be mapped to a metric on $\mathbb{S}M^{\mathbb{C}}$ via the inverse of σ for which we will use the same symbol and differentiate it from $\sigma : TM^{\mathbb{C}} \to \mathbb{S}M^{\mathbb{C}}$ only by the position of its indices:①

$$(\sigma^\mu)_{AA'}(\sigma^\nu)_{BB'}\,\eta_{\mu\nu} = \epsilon_{AB}\epsilon_{A'B'}\,. \tag{2.20}$$

Similarly, we can use this isomorphism to transform vector indices into spinor indices and vice versa. Given a tensor with components $T^{\mu_1\ldots\mu_r}_{\nu_1\ldots\nu_s}$, its spinor representation is

$$T^{M_1\ldots M_r M'_1\ldots M'_r}_{N_1\ldots N_s N'_1\ldots N'_s} = (\sigma_{\mu_1})^{M_1 M'_1}\cdots(\sigma^{\nu_s})_{N_s N'_s} T^{\mu_1\ldots\mu_r}_{\nu_1\ldots\nu_s}\,. \tag{2.21}$$

This identification lets us informally write

$$MM' \equiv \mu\,, \tag{2.22}$$

① Now it becomes obvious that indeed $\sigma : \mathbb{C}^4 \to \mathfrak{psl}(2,\mathbb{C})$ and its action onto $TM^{\mathbb{C}}$ should be via the introduction of some frame field $\hat{e} : \mathbb{C}^4 \to TM^{\mathbb{C}}$, but we left this implicit in the formulation as for Minkowski space this frame field would just be the identity.

and we will make extensive use of it throughout the thesis.

Let us remark on the effects of choosing different reality structures. The signature of the real manifold we started out with determines the action of the conjugation map after the complexification. Thus, choosing the conjugation map selects the reality structure.

- $\mathbb{R}^4$: suppose the manifold was Euclidean, then the conjugation map acts as [49, 50]

$$\begin{aligned} \lambda^A = (\lambda^1, \lambda^2) \mapsto \hat{\lambda}^A = (\overline{\lambda}^2, -\overline{\lambda}^1) \\ \text{and} \qquad \mu^{A'} = (\mu^1, \mu^2) \mapsto \hat{\mu}^{A'} = (-\overline{\mu}^2, \overline{\mu}^1)\,. \end{aligned} \tag{2.23}$$

- $\mathbb{R}^{1,3}$: suppose the manifold was Lorentzian, then the conjugation map acts as

$$\begin{aligned} \lambda^A = (\lambda^1, \lambda^2) \mapsto \hat{\lambda}^A = \overline{\lambda}^{A'} = (\overline{\lambda}^1, \overline{\lambda}^2) \\ \text{and} \qquad \mu^{A'} = (\mu^1, \mu^2) \mapsto \hat{\mu}^{A'} = \overline{\mu}^A = (\overline{\mu}^1, \overline{\mu}^2)\,. \end{aligned} \tag{2.24}$$

  This maps the representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ into each other.

- $\mathbb{R}^{2,2}$: since $\mathfrak{so}(2,2) \simeq \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R})$, the Weyl spinor representations are real. Thus, choosing real spinor coordinates selects this split signature.

Another consequence of the spinor-helicity formulation is that given a (complex) anti-symmetric tensor with components $F_{\alpha\beta}$, it can be uniquely expressed as

$$F_{\alpha\beta} \equiv \phi_{AB}\epsilon_{A'B'} + \epsilon_{AB}\psi_{A'B'}\,, \tag{2.25}$$

where $\phi_{AB} = \phi_{(AB)}$ and $\psi_{A'B'} = \psi_{(A'B')}$ are symmetric. If $F_{\alpha\beta}$ are real, this means that $\psi_{A'B'} = \overline{\phi_{AB}} = \overline{\phi}_{A'B'}$. The other benefit is that the statements about (anti-)self-duality can now be restated as

- F is self-dual iff $F_{\alpha\beta} \equiv \epsilon_{AB}\psi_{A'B'}$;

- F is anti-self-dual iff $F_{\alpha\beta} \equiv \phi_{AB}\epsilon_{A'B'}$.

The spinor formalism is particularly well-adapted to study null vectors. Suppose $(v^{AA'})$ corresponds to a null vector in $TM^{\mathbb{C}}$. Then,

$$0 = v^{\mu}v_{\mu} = v^{AA'}v_{AA'} = \det(v)\,. \tag{2.26}$$

Since $(v^{AA'})$ corresponds to a 2×2 matrix, and the rank of a 2×2 matrix is less than two iff its determinant vanishes, this implies that we can write

$$v^{AA'}_{\text{null}} = \lambda^{A}\tilde{\lambda}^{A'} \tag{2.27}$$

for $(\lambda^{A}) \in \mathbb{S}^{+}M^{\mathbb{C}}$ and $(\tilde{\lambda}^{A'}) \in \mathbb{S}^{-}M^{\mathbb{C}}$. This formalism is known as spinor-helicity formalism and very powerful in the study of on-shell objects, since on-shell gauge fields are massless, which means that their four-momentum vector is null. Foreshadowing chapter 5, we note that the identification (2.27) is ambiguous, since any transformation $(\lambda, \tilde{\lambda}) \mapsto (t\lambda, t^{-1}\tilde{\lambda})$ for $t \in \mathbb{C}^{*}$ leaves v_{null} unchanged. We will come back to this observation in chapter 5 when we define twistors.

2.2.1 MHV Amplitudes

The asymptotic behaviour of states at infinity is determined by the dynamics in the limit of weak interactions, that is for the linearised theory. In this limit, we drop the term $\frac{1}{2}[A, A]$ in the curvature, and the equations of motion reduce to the plane wave equations

$$\Box A_{\mu} = 0\,. \tag{2.28}$$

As we counted before, there are only two propagating degrees of freedom, which identifies it as a massless field, for which the four-momentum is null. This means that there is a decomposition

$$k^{AA'} = p^{A}\tilde{p}^{A'}\,. \tag{2.29}$$

It is also useful to think about the propagating degrees of freedom as curvature fluctuations instead of fluctuations of the connection form. For self-dual theories, the equations of motion then read

$$\begin{cases} \nabla^{AA'}\phi_{AB} = 0 \\ \nabla^{AA'}\psi_{A'B'} = 0 \end{cases} \quad \Longleftrightarrow \quad \begin{cases} k^{AA'}\phi_{AB} = 0 \\ k^{AA'}\psi_{A'B'} = 0 \end{cases} \tag{2.30}$$

by Fourier transform, for which we find the solutions

$$\phi_{AB} = p_A p_B e^{ix\cdot p}\,, \qquad \psi_{A'B'} = \tilde{p}_{A'}\tilde{p}_{B'}e^{ix\cdot p}\,. \tag{2.31}$$

These states are asymptotic plane waves. Evaluating an n^{th} order interaction vertex of the action on those states gives an n-point amplitude. For Yang-Mills theory, those amplitudes include a trace over the product of generators of $\mathfrak{g}$, which is cyclic. Through the commutation relations of $\mathfrak{g}$, we can relate all possible permutations of the product to a single, ordered one. This allows us to color-strip the amplitude

$$\mathcal{M}_n = \mathcal{M}_n^{(a_1,\ldots,a_n)}\,\mathrm{tr}(\mathrm{T}^{a_1}\cdots\mathrm{T}^{a_n}) \tag{2.32}$$

for some specific ordering $a_1,\ldots,a_n$. It is common to also strip off a momentum-conserving delta function, which we will not do in order to keep consistency with the notation of the rest of the thesis. It is now always those color-stripped amplitudes that we are considering.

We generally denote an n-point amplitude of particles with helicities $h_1,\ldots,h_n$ of which $k+2$ are negative as $\mathcal{M}_{n,k}(h_1,\ldots,h_n)$. For later chapters, if all particles have the same spin, we also simply write $\mathcal{M}_{n,k}^{(s)}(\sigma_1,\ldots,\sigma_n)$, where $\sigma_i = \mathrm{sign}(h_i)$. MHV amplitudes are those for which $k=0$, i.e. $\mathcal{M}_{n,0}$, and $\overline{\text{MHV}}$ amplitudes are $\mathcal{M}_{n,n-4}$.

For Yang-Mills theories, evaluating the cubic interaction term on the plane wave states (2.31) gives the three-point amplitudes, which take the remarkably simple

form

$$\mathcal{M}^{(1)}_{3,-1}(-,+,+) = \frac{[23]^3}{[12]\,[31]}\delta^4\left(\sum_i k_i\right) \tag{2.33}$$

$$\text{and} \qquad \mathcal{M}^{(1)}_{3,0}(-,-,+) = \frac{\langle 12\rangle^3}{\langle 31\rangle\,\langle 23\rangle}\delta^4\left(\sum_i k_i\right)\,, \tag{2.34}$$

where $[ij] = \tilde{p}_i^{A'}\tilde{p}_j^{B'}\epsilon_{A'B'}$ and $\langle ij\rangle = p_i^A p_j^B \epsilon_{AB}$. We chose this labelling of the particles to increase the recognizability of their higher-spin analogues later in this thesis.

The general n-point MHV amplitude has been found by Parke and Taylor [51] to be

$$\mathcal{M}^{(1)}_{n,0}(+,\ldots,+,\underset{i}{-},+,\ldots,+,\underset{j}{-},+,\cdots+) = \frac{\langle ij\rangle^4}{\langle 12\rangle\,\langle 23\rangle\cdots\langle n1\rangle}\delta^4\left(\sum_i k_i\right)\,. \tag{2.35}$$

This is a remarkably simple expression and suggests an underlying, hidden symmetry, which is related to the emergent space-time proposals like the Grassmannian formulation of [52]. However, for spin higher than one, the n-point amplitude looses its simplicity, and over the course of the thesis we will mainly be making contact with the three-point amplitudes, which remain in a recognisable pattern.

Chapter 3

Spin-2: Gravity

Gravitational interactions are qualitatively different than those described through a Yang-Mills theory. This is because mainly we think about the 'force-carrier particle' as having spin-2, that is the metric field, instead of spin-1. This chapter highlights the gauge-field-theoretical formulation of gravity and shows in which way it is different from the previously discussed spin-1 case. The quintessence is that, even though we can formulate Einstein's theory as a gauge theory, now we have two connection forms, the spin connection ω and the coframe field e, and arriving at Einstein's equations of motion demands that we have to set the theory half on-shell, that is using the equations for ω to express it in terms of e. This is also exactly the requirement for having vanishing torsion. The physical degrees of freedom of the metric field are then built out of the two propagating degrees of freedom of e.

After that we continue in discussing conformal gravity (CG), which is not unitary, but thought to have a much better behaviour in the quantum regime than Einstein's gravity (EG). It also allows for a Chalmers-Siegel-like action, which can be easily generalized to the higher spin case next chapter. But this is not all. Interestingly, EG sits as a unitary sub-sector inside CG, a relation which we will rediscover and utilize for their higher-spin generalizations as well.

3.1 Einstein-Cartan Theory

Gravity is basically the physical interpretation of the structures in differential geometry on a smooth, four-dimensional manifold of Lorentzian signature. The mathematical details of this can be found in the appendix in section A.5, and we follow roughly the presentation of [53]. Gauge-theoretically, we start by taking the gauge group to be the Poincaré group, which is the holonomy group of a Lorentzian manifold $\mathrm{Hol}(M,g) = SO^+(1,3) \ltimes \mathbb{R}^{1,3}$. It encodes local Lorentz rotations and infinitesimal translation. The rotations are generated by the so-called spin connection ω, and the translations are generated by the coframe field e.

Depending on the asymptotic behaviour of M, parametrized by the cosmological constant Λ, we consider an extension $G \supset \mathrm{Hol}(M,g)$ with

$$G = \begin{cases} SO(1,4) & \Lambda > 0 \ \ (\mathrm{dS}) \\ ISO(1,3) & \Lambda = 0 \ \ (\mathrm{Minkowski}) \\ SO(2,3) & \Lambda < 0 \ \ (\mathrm{AdS}) \end{cases} \tag{3.1}$$

such that its algebra $\mathfrak{g}$ has an orthogonal splitting

$$\mathfrak{g} \simeq \mathfrak{so}(1,3) \oplus \mathbb{R}^{1,3} \tag{3.2}$$

as vector spaces. Let $(\mathrm{L}^a{}_b)$ be a basis of the matrix representation of $\mathfrak{gl}(1,3)$ restricted to $\mathfrak{so}(1,3)$ and (P_a) a basis of the vector representation of $\mathbb{R}^{1,3}$, and let $(\mathrm{d}x^\mu)$ be a (holonomic) coordinate basis of T^*M. The connection form of the principal G-bundle has then as components the spin connection form ω and the coframe field e

$$\Omega^1(M,\mathbb{R}^{1,3}) \ni e = e_\mu{}^a \mathrm{P}_a \, \mathrm{d}x^\mu \,, \tag{3.3}$$

$$\Omega^1(M,\mathfrak{so}(1,3)) \ni \omega = (\omega_\mu)^a{}_b \mathrm{L}^b{}_a \, \mathrm{d}x^\mu \,, \tag{3.4}$$

$$\Omega^1(M,\mathfrak{g}) \ni A = \omega + \frac{1}{l} e \,, \tag{3.5}$$

where $l^2\Lambda = 3\,\mathrm{sign}(\Lambda)$ defines a length scale, where Λ is the radius of AdS or dS (we take $\Lambda \neq 0$), which is proportional to the cosmological constant. Using the

isomorphism $\Lambda^2\mathbb{R}^{1,3} \simeq \mathfrak{so}(1,3)$ to think about $e \wedge e$ as an $\mathfrak{so}(1,3)$-valued 2-form, the curvature associated to this connection form is given by its component curvatures associated to the translational part $R^{(\mathrm{P})}$ and the $\mathfrak{so}(1,3)$ part $R^{(\mathrm{L})}$ by

$$F = D_A A = R^{(\mathrm{P})} + R^{(\mathrm{L})} = \frac{1}{l} D_\omega e + D_\omega \omega - \frac{1}{l^2} e \wedge e \,. \tag{3.6}$$

Setting $F = 0$ requires $R^{(\mathrm{P})}$ and $R^{(\mathrm{L})}$ to vanish independently and hence identifies the conditions for (M, g) to be an Einstein space ($R = D_\omega \omega = \frac{1}{l^2} e \wedge e$) and torsion-free ($T = D_\omega e = 0$).

Since we took Λ to be non-vanishing, we need to introduce a spontaneous symmetry breaking of our enlarged group $G \to ISO(1,3)$ to the Poincaré group if we want to discuss $\Lambda = 0$, in order to obtain the theory of Einstein gravity. This can either be done by hand by restricting $F \to R^{(\mathrm{L})} = R - \frac{1}{l^2} e \wedge e$ (which is manually setting the torsion to zero) and formulating the (MacDowell-Mansouri) action in terms of this restricted curvature [54]

$$\begin{aligned} S_{\mathrm{MM}} &= -\frac{3}{2G\Lambda} \int \mathrm{tr}\left(R^{(\mathrm{L})} \wedge * R^{(\mathrm{L})}\right) \\ &= \frac{1}{G} \int \mathrm{tr}\left(-\frac{3}{2\Lambda} R \wedge * R + e \wedge e \wedge * R - \frac{\Lambda}{6} e \wedge e \wedge *(e \wedge e)\right) . \end{aligned} \tag{3.7}$$

Alternatively, in a more rigorous way, which we will not present here, we could introduce the AdS analogues of the coframe fields as non-propagating fields, whose mass determines the scale of the coframe field e [55]. By integrating out this auxiliary field, we obtain the MacDowell-Mansouri action above, which is equivalent to the Palatini action (with cosmological constant) up to a (Chern-Gauß-Bonnet) topological term

$$S_{\mathrm{MM}} = S_{\mathrm{Pal},\Lambda} - \frac{3}{2G\Lambda} \int \mathrm{tr}\,(R \wedge * R) \,. \tag{3.8}$$

The Chern-Gauß-Bonnet term is dynamically irrelevant as it drops out of the equations of motion. This leaves us with the Palatini action, first without cosmological

constant considered in [56],

$$S_{\mathrm{Pal},\Lambda}[\omega, e] = \frac{1}{G}\int \mathrm{tr}\left(e \wedge e \wedge * R - \frac{\Lambda}{6} e \wedge e \wedge *(e \wedge e)\right) . \tag{3.9}$$

The variation of $S_{\mathrm{Pal},\Lambda}$ with respect to ω gives $D_\omega(e \wedge e) = 0$, which is equivalent to the zero-torsion constraint $D_\omega e = 0$, which in turn can be used to express the spin connection as a function of the coframe field $\omega = \omega(e)$. The other variation with respect to the coframe field e gives Einstein's equations of motion

$$e \wedge R - \frac{\Lambda}{3} e \wedge e \wedge e = 0 , \tag{3.10}$$

which, using the equations of motion for ω, are equations for e and thus the metric g only. In this sense, Einstein's equations of motion are half on-shell.

In order to count degrees of freedom, we realize that the second Bianchi identity $D_\omega R = 0$ translates to $\nabla_{[\sigma}\mathscr{R}_{\tau\nu]\alpha\mu} = 0$ for the Riemann tensor (see section A.5 for more details on their definition), which reduces the number of independent components of $(e_\mu{}^a)$ to ten, in the same way as it sets the metric to be a symmetric tensor. Furthermore, under gauge transformations, the coframe field transforms as

$$\delta e_\mu{}^a = \partial_\mu \xi^a \tag{3.11}$$

which means for the metric

$$\delta g_{\mu\nu} = 2 e_{(\mu}{}^a \partial_{\nu)} \xi_a = 2 \partial_{(\mu} \xi_{\nu)} . \tag{3.12}$$

From those now ten independent components, four of can be fixed through the choice of ξ, and further four by setting it on-shell, leaving only two propagating degrees of freedom, which makes it to be a massless particle.

3.1.1 Linearised Gravity

It is interesting to study the interactive behaviour in the weakly interactive limit, in which we can think of the graviton particle propagating on a flat background. As

such, a small perturbation can be expressed through an expansion of the metric to linear order

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \tag{3.13}$$

where η is the flat metric of Minkowski space. All indices are raised and lowered with η. Neglecting all terms quadratic in h in this perturbation, the Riemann tensor reduces to

$$\mathscr{R}_{\tau\nu\sigma\mu}[h] = \frac{1}{2}(\partial_\nu\partial_\sigma h_{\tau\mu} - \partial_\tau\partial_\sigma h_{\nu\mu} + \partial_\tau\partial_\mu h_{\nu\sigma} - \partial_\nu\partial_\mu h_{\tau\sigma})\,. \tag{3.14}$$

Choosing a particular gauge, the de Donder gauge, $\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h^\mu{}_\mu = 0$, the equations of motion (3.10) now assume the simple form of the wave equations

$$\Box h_{\mu\nu} = 0\,. \tag{3.15}$$

These equations are not gauge-invariant, however it is possible to write them in a gauge-invariant form by considering (3.14) as fluctuations of the curvature. For the linearisation, the curvature allows for a decomposition

$$\mathscr{R}_{\alpha\beta\gamma\delta}[h] \equiv H_{ABCD}\,\epsilon_{A'B'}\epsilon_{C'D'} + H_{A'B'C'D'}\,\epsilon_{AB}\epsilon_{CD}\,, \tag{3.16}$$

for which the equations of motion can be equivalently written as

$$\begin{cases} \nabla^{AA'} H_{ABCD}(x) = 0 \\ \nabla^{AA'} H_{A'B'C'D'}(x) = 0 \end{cases} \quad\Longleftrightarrow\quad \begin{cases} k^{AA'} H_{ABCD}(k) = 0 \\ k^{AA'} H_{A'B'C'D'}(k) = 0 \end{cases} \tag{3.17}$$

by Fourier transform, for which $k^{AA'} = p^A \tilde{p}^{A'}$ is the (null) momentum of the particle. The solutions are easily determined to be plane wave solutions in momentum space

$$H_{ABCD} = p_A p_B p_C p_D e^{ix\cdot k} \qquad \text{and} \qquad H_{A'B'C'D'} = \tilde{p}_{A'}\tilde{p}_{B'}\tilde{p}_{C'}\tilde{p}_{D'} e^{ix\cdot k}\,, \tag{3.18}$$

encoding the negative and positive helicity states of the graviton respectively.

The n-point MHV amplitudes of gravitons on an asymptotically Minkowski background have been computed in [57], using KLT relations from string theory [58],

which relate gravity and Yang-Mills amplitudes. The results have been recalculated using BCFW recursion relations [59], which have been adapted to the gravity setting in [60]. For example, the three-point $\overline{\text{MHV}}$ and MHV amplitudes are just the squares of the corresponding Yang-Mills amplitudes

$$\widetilde{\mathcal{M}}^{(2)}_{3,-1}(-,+,+) = \left(\frac{[23]^3}{[12]\,[31]}\right)^2 \delta^4\left(\sum_i k_i\right) \tag{3.19}$$

$$\text{and} \qquad \widetilde{\mathcal{M}}^{(2)}_{3,0}(-,-,+) = \left(\frac{\langle 12\rangle^3}{\langle 31\rangle\,\langle 23\rangle}\right)^2 \delta^4\left(\sum_i k_i\right) . \tag{3.20}$$

It was argued [61] that all amplitudes are generated through MHV vertices and scalar propagators, which is hopeless to see from the space-time point of view, however the procedure fails starting from 12 points on [62]. A different approach using twistor theory is able to construct the MHV expansion to all orders [63], and it is in this spirit that we will be considering the amplitudes construction for the higher spin theory later on.

3.2 Conformal Gravity

We can ask what changes in our analysis when we allow general conformal transformations

$$g(x) \mapsto \Omega(x)^2 g(x) \tag{3.21}$$

for some arbitrary, non-vanishing function $\Omega : M \to \mathbb{R}$ to generate the gauge symmetries of our theory. This equation includes in addition to translations and rotations also scalings, which means that we would like to gauge the inhomogeneous Weyl group. From the knowledge of the global conformal group however it is tempting to simply declare the group $C(1,3)$ the gauge group of our theory. Comparing global translations and global conformal transformations (accelerations) though,

$$x^\mu \to x^\mu + a^\mu\,, \qquad x^\mu \to \frac{x^\mu + c^\mu x^2}{1 + c\cdot x + c^2 x^2}\,, \tag{3.22}$$

we see that, a and c being arbitrary vectors on M, those transformations differ in character. As local transformations however, a and c being arbitrary functions on M, the second case is already included in the first, and it would be wrong to include a separate gauge field for that as it would amount to gauging the same symmetry twice. Thus, indeed the correct gauge group is taken to be the inhomogeneous Weyl group $IW(1,3)$ [64]. We can however, much like we did for Einstein-Cartan theory, enlarge the gauge group $C(1,3) \supset IW(1,3)$, but treat the special conformal transformations as *auxiliary* gauge transformations whose corresponding gauge field is not propagating and still discuss a gauge theory of the full conformal group $C(1,3)$ [65]. Its connection form

$$\Omega^1(M, \mathfrak{c}(1,3)) \ni \mathbb{A} = (e^a_\mu \, \mathrm{P}_a \, + (\omega_\mu)^a{}_b \mathrm{L}^b{}_a + b_\mu \, \mathrm{D} \, + f_\mu{}^a \, \mathrm{K}_a) \, \mathrm{d}x^\mu \tag{3.23}$$

now not only contains translations P_a and rotations $\mathrm{L}^b{}_a$, but also dilatations D and special conformal transformations K_a. Setting dilatations and special conformal transformations to zero yields the Poincaré algebra $\mathfrak{iso}(1,3) \subset \mathfrak{c}(1,3)$.

The curvature form of the full bundle decomposes into the components associated to the generators

$$\Omega^2(M, \mathbb{R}^{1,3}) \ni R^{(\mathrm{P})} = D_\omega e - e \wedge b \,, \tag{3.24}$$

$$\Omega^2(M, \mathfrak{so}(1,3)) \ni R^{(\mathrm{L})} = D_\omega \omega - e \wedge f \,, \tag{3.25}$$

$$\Omega^2(M, \mathbb{R}) \ni R^{(\mathrm{D})} = \mathrm{d}b + \mathrm{tr}(e \wedge f) \,, \tag{3.26}$$

$$\Omega^2(M, \mathbb{R}^{1,3}) \ni R^{(\mathrm{K})} = D_\omega f + f \wedge b \,. \tag{3.27}$$

Similar to the argument we gave for the action of Einstein-Cartan theory, it turns out that the proper action for the $IW(1,3)$ gauge theory is only given by the Yang-Mills action for the Lorentz curvature $R^{(\mathrm{L})}$. In principle, we would also expect the translational and dilatational curvature to appear, but they can essentially be set to zero through their corresponding equations of motion. The resulting action thus

reads

$$S[e,\omega,b] = \frac{1}{\varepsilon^2}\int_M \mathrm{tr}(R^{(\mathrm{L})} \wedge * R^{(\mathrm{L})})\,, \tag{3.28}$$

which is also the only action we can construct from the curvatures that is parity conserving and has a dimensionless parameter ε as coupling constant.

Variation with respect to ω does not directly lead to the on-shell vanishing of the torsion, but it can be shown that the induced gauge transformations from imposing $R^{(\mathrm{P})} = 0$ (as on-shell conditon) vanish or cancel [65]. This condition is also necessary for the action to be invariant under proper conformal gauge transformations. Thus, we can again think about ω as $\omega(e, b(e))$, enforced through this condition. We also note that the field equations of the non-propagating field f are

$$\mathrm{tr}(e \wedge f) = \frac{1}{2}\left(\mathrm{Ric} - \frac{1}{6}\mathcal{R}\, g\right) = \mathscr{P}\,, \tag{3.29}$$

where Ric is the Ricci tensor and $\mathcal{R}$ is the Ricci scalar. (See section A.5 for their definitions.) This defines the Schouten tensor $\mathscr{P}$, which appears in the Ricci decomposition of the Riemann tensor

$$\mathscr{R} = \mathscr{C} + \mathscr{P} \owedge g\,, \tag{3.30}$$

where $\mathscr{C}$ is the Weyl tensor, and $\owedge$ is the Kulkarni–Nomizu product (see appendix A.5). This decomposition can possibly be better understood by realizing that the Weyl tensor is just the trace-free part of the Riemann tensor

$$\mathscr{C}_{\tau\nu\alpha\mu} = \mathscr{R}_{\tau\nu\alpha\mu} - (g_{\tau[\alpha}\mathrm{Ric}_{\mu]\nu} - g_{\nu[\alpha}\mathrm{Ric}_{\mu]\tau}) + \frac{1}{3}\mathcal{R}\, g_{\tau[\alpha}g_{\mu]\nu}\,. \tag{3.31}$$

It thus has the same symmetries as the Riemann tensor with an additional tracelessness condition

$$\mathscr{C}_{\tau\nu\alpha\mu} = \mathscr{C}_{[\tau\nu][\alpha\mu]} = \mathscr{C}_{\alpha\mu\tau\nu} \qquad \text{and} \qquad \mathscr{C}^{\tau}{}_{\nu\tau\mu} = 0\,. \tag{3.32}$$

While the Riemann tensor is invariant under $\mathfrak{iso}(1,3)$-transformations, the Weyl tensor is invariant under conformal transformations.

Using (3.29), the Lorentz curvature takes the form $e^{-1}R^{(\mathrm{L})}e = \mathscr{R} - \mathscr{P} \owedge g = \mathscr{C}$, and we obtain the well-known action of conformal gravity

$$S[e] = \frac{1}{\varepsilon^2}\int_M \mathscr{C} \wedge *\mathscr{C}\,. \tag{3.33}$$

Indeed we notice that the final action is independent of b, as it should be, since a coupling to b could be interpreted as a coupling to an electromagnetic potential and would spoil the purely gravitational character of this theory.

The equations of motion for the metric are the Bach equations

$$(2\nabla_\nu\nabla_\mu + \mathrm{Ric}_{\nu\mu})\mathscr{C}^{\tau\nu\alpha\mu} = 0 \tag{3.34}$$

and solutions are conformal classes of the metric.

Going back to the beginning of our analysis, we started with the conformal group instead of the Poincaré group, which means there are a priori 15 degrees of freedom. Under Weyl-gauge transformations, the coframe field transforms as

$$\delta e_\mu{}^a = \partial_\mu\xi^a + \alpha\, e_\mu{}^a \tag{3.35}$$

which means for the metric ($\Omega(x)^2 = e^{2\alpha(x)}$ from (3.21))

$$\delta g_{\mu\nu} = 2\partial_{(\mu}\xi_{\nu)} + 2\alpha\, g_{\mu\nu}\,. \tag{3.36}$$

Therefore, four (from ξ) plus one (from α) of those 15 components can be fixed through choosing a gauge, and further four are fixed through the equations of motion for e. This leaves in total six propagating degrees of freedom, whose corresponding states we will identify shortly.

3.2.1 Self-Dual Sector

In a similar fashion to Yang-Mills theory, we can define a self-dual sector of gravitational theories as well. For the Riemann tensor there are two ways of taking its dual: one with respect to the $\Lambda^2(M)$ space, and one with respect to the $\mathfrak{so}(1,3)$ space,

which is inherited from its isomorphism to $\Lambda^2(\mathbb{R}^{1,3})$. This leads to four terms in the spinor decomposition of the curvature:

$$\begin{aligned}\mathscr{R}_{\alpha\beta\gamma\delta} \equiv X_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD} \\ + \widetilde{\Phi}_{A'B'CD}\epsilon_{AB}\epsilon_{C'D'} + \widetilde{X}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD}\,.\end{aligned} \tag{3.37}$$

As in the Yang-Mills case, for Lorenzian signature, $\overline{\Phi_{ABC'D'}} = \widetilde{\Phi}_{A'C'CD}$ and $\overline{X_{ABCD}} = \widetilde{X}_{A'B'C'D'}$. Now, X_{ABCD} inherits the symmetry properties of $\mathscr{R}$, and since $X_{ABCD} = X_{CDAB}$, the two different ways of dualizing act the same on X. We can define its trace to be

$$X_{AB}{}^{AB} = 6\Lambda\,, \tag{3.38}$$

which leads to the decomposition

$$X_{ABCD} = \Psi_{ABCD} + \Lambda(\epsilon_{AC}\epsilon_{BD} + \epsilon_{AD}\epsilon_{BC}) \tag{3.39}$$

into a trace part and a trace-less symmetric part $\Psi_{ABCD} = \Psi_{(ABCD)}$. The symmetric part forms the Weyl tensor

$$\mathscr{C}_{\alpha\beta\gamma\delta} \equiv \Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \widetilde{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD}\,. \tag{3.40}$$

This allows us to rewrite the equations of motion of conformal gravity in the spinor form as

$$0 = 2(\nabla^{C}{}_{A'}\nabla^{D}{}_{B'} + \Phi^{CD}{}_{A'B'})\Psi_{ABCD} = 2(\nabla_{C}{}^{A'}\nabla_{D}{}^{B'} + \Phi_{CD}{}^{A'B'})\widetilde{\Psi}_{A'B'C'D'}\,. \tag{3.41}$$

If we substitute the spinor form of the curvature into the action for conformal gravity, we obtain

$$\begin{aligned}S[g] &= \frac{1}{\varepsilon^2}\int_M \mathscr{C}\wedge *\mathscr{C} = \frac{1}{\varepsilon^2}\int_M \mathrm{d}^4x\sqrt{|g|}\left(\widetilde{\Psi}^{A'B'C'D'}\widetilde{\Psi}_{A'B'C'D'} + \Psi^{ABCD}\Psi_{ABCD}\right) \\ &= \frac{2}{\varepsilon^2}\int_M \mathrm{d}^4x\sqrt{|g|}\,\Psi^{ABCD}\Psi_{ABCD} + \mathrm{top}\,,\end{aligned} \tag{3.42}$$

where the topological term is equal to $\frac{12\pi^2}{\varepsilon^2}(\tau(M) - \eta(\partial M))$, which is the signature of M minus the η-invariant of its boundary [66]. Analogous to the Yang-Mills case, this is equivalent to

$$S[g, G] = \int_M \mathrm{d}^4x\sqrt{|g|}\left(G^{ABCD}\Psi_{ABCD} - \frac{\varepsilon^2}{4}G^{ABCD}G_{ABCD}\right), \tag{3.43}$$

for which we find the equations of motion

$$\Psi_{ABCD} = \frac{\varepsilon^2}{2}G_{ABCD} \qquad \text{and} \qquad (\nabla^C{}_{A'}\nabla^D{}_{B'} + \Phi^{CD}{}_{A'B'})G_{ABCD} = 0\,. \tag{3.44}$$

Again, taking the limit $\varepsilon \to 0$ sets the anti-self-dual part of the curvature to zero. For $\varepsilon \neq 0$ we recover the full non-self-dual theory, in which the new field G is treated as an anti-self-dual mode on the self-dual background.

For the linearised approximation, we again rather consider the curvature fluctuations than the metric fluctuations, which are related through

$$\Psi_{ABCD} = \nabla_{(C}{}^{C'}\nabla_D{}^{D'}h_{AB)C'D'}\,, \tag{3.45}$$

and similarly for the primed part of the curvature. In [67] it was shown that for flat asymptotic boundary conditions the six on-shell degrees of freedom in this theory correspond to two massless spin-2 particles, one conformal graviton and one conformal ghost, and one spin-1 particle. This can be easily seen from the equations of motion in spinor form (3.41), which for the curvature fluctuations are simply

$$\begin{cases} \nabla^{AA'}\nabla^{BB'}\Psi_{ABCD}(x) = 0 \\ \nabla^{AA'}\nabla^{BB'}\widetilde{\Psi}_{A'B'C'D'}(x) = 0 \end{cases} \iff \begin{cases} k^{AA'}k^{BB'}\Psi_{ABCD}(k) = 0 \\ k^{AA'}k^{BB'}\widetilde{\Psi}_{A'B'C'D'}(k) = 0 \end{cases} \tag{3.46}$$

by Fourier transform, for which $k^{AA'} = p^A\tilde{p}^{A'}$ is the (null) momentum of the particle. The solutions are easily given [68] by

$$\begin{aligned} \Psi_{ABCD} &= p_Ap_Bp_Cp_De^{ix\cdot k}\,, & \widetilde{\Psi}_{A'B'C'D'} &= \tilde{p}_{A'}\tilde{p}_{B'}\tilde{p}_{C'}\tilde{p}_{D'}e^{ix\cdot k}\,, \\ \Psi'_{ABCD} &= x^2p_Ap_Bp_Cp_De^{ix\cdot k}\,, & \widetilde{\Psi}'_{A'B'C'D'} &= x^2\tilde{p}_{A'}\tilde{p}_{B'}\tilde{p}_{C'}\tilde{p}_{D'}e^{ix\cdot k}\,, \\ \Psi^{(1)}_{ABCD} &= \alpha_{(A}p_Bp_Cp_{D)}e^{ix\cdot k}\,, & \widetilde{\Psi}^{(1)}_{A'B'C'D'} &= \overline{\alpha}_{(A'}\tilde{p}_{B'}\tilde{p}_{C'}\tilde{p}_{D')}e^{ix\cdot k}\,, \end{aligned} \tag{3.47}$$

for some arbitrary constant spinor α. Here, Ψ is the conformal graviton, Ψ' is the conformal ghost and $\Psi^{(1)}$ is the spin-1 particle. For asymptotically AdS or dS boundary conditions, this spectrum transforms into one massless and one partially massless spin-2 particle [69].

Interestingly, most amplitudes of conformal gravity states on an asymptotically Minkowski background vanish. The reason for that will become clear in chapter 8, but it essentially turns out that some of them scale proportionally to the cosmological constant Λ, which is zero for Minkowski space.

3.3 From Conformal to Einstein Gravity

Following the work of Anderson [70], Maldacena [13] argued that classically, conformal gravity supplemented with appropriate boundary conditions is equivalent to Einstein gravity with a non-vanishing cosmological constant.

The argument is based on two points:

1. The on-shell action of Einstein gravity for an Einstein space can be computed in terms of the action of conformal gravity.

2. Any metric that is conformal to a solution of Einstein's equations of motion is a solution to the Bach equations.

1. Starting from (3.9), we can rewrite the action in Einstein-Hilbert form

$$S_{\mathrm{EH}}[g] = \frac{1}{G}\int_M \mathrm{d}^4x\sqrt{|g|}(\mathcal{R} - 2\Lambda)\,. \tag{3.48}$$

On a de Sitter space solution to the equations of motion, $\mathrm{Ric}_{\mu\nu} = \Lambda g_{\mu\nu}$, it is

$$S_{\mathrm{EH}}[\mathrm{dS}] = \frac{2\Lambda}{G}\int_{\mathrm{dS}} \mathrm{d}^4x\sqrt{|g|} = \frac{2\Lambda}{G}\mathrm{vol}(\mathrm{dS})\,. \tag{3.49}$$

This volume is however infinite for any space that is asymptotically dS, and so we need to renormalize it by taking certain boundary terms into account,

$$S^{\mathrm{ren}}_{\mathrm{EH}}[\mathrm{dS}] = \frac{2\Lambda}{G}\mathrm{vol}^{\mathrm{ren}}(\mathrm{dS})\,. \tag{3.50}$$

Starting from the conformal gravity action (3.33), the decomposition in terms of the Riemann tensor leads to the (classically) equivalent formulation

$$S_{\mathrm{CG}}[g] = \frac{1}{\varepsilon^2}\int_M \mathrm{d}^4x\sqrt{|g|}\left(\mathrm{Ric}_{\mu\nu}\mathrm{Ric}^{\mu\nu} - 2\mathcal{R}^2\right) + \frac{1}{\varepsilon^2}\chi(M) \tag{3.51}$$

up to the Chern-Gauß-Bonnet term. For de Sitter space, $\mathrm{Ric}_{\mu\nu} = \Lambda g_{\mu\nu}$, this action reduces to

$$S_{\mathrm{CG}}[\mathrm{dS}] = -\frac{2\Lambda^2}{3\varepsilon^2}\mathrm{vol}(\mathrm{dS}) + \frac{1}{\varepsilon^2}\chi(M) \tag{3.52}$$

This relationship is invariant under renormalization of the volume and the Euler characterstic by taking boundery terms into consideration as above [70]. Thus, perturbatively ($\chi = 0$), we have the relation

$$S_{\mathrm{CG}}[\mathrm{dS}] = -\frac{\Lambda G}{3\varepsilon^2}S^{\mathrm{ren}}_{\mathrm{EH}}[\mathrm{dS}]\,. \tag{3.53}$$

2. It is easy to see that for every Einstein manifold, for which $\mathrm{Ric}_{\mu\nu} \propto g_{\mu\nu}$, the Bach equations are trivially satisfied. Conformal gravity however has other modes that do not satisfy Einstein's equations. In order to select the Einstein modes, we can impose certain boundary conditions, which project out the ghost and spin-1 mode. For this we consider the Fefferman-Graham expansion of an Einstein space near the boundary $z = 0$ that is asymptotically dS or hyperbolic

$$\mathrm{d}s^2 = \frac{1}{z^2}\left(\mathrm{d}z^2 + \left(g^{(0)}_{ij}(x) + z^2 g^{(2)}_{ij}(x) + z^3 g^{(3)}_{ij}(x)\right)\mathrm{d}x^i \otimes \mathrm{d}x^j\right) \tag{3.54}$$

and note that $g^{(1)}$ is absent, which it is not for a general solution of Bach's equations. This means that solutions to Einstein's equations of motion are conformal to solutions of Bach's equations which at $z = 0$ obeys $\partial_z g\Big|_{z=0} = 0$. We can select the Einstein mode by restricting the solutions to positive frequency modes and then imposing the Neumann boundary condition.

Concretely, in terms of the spinor solutions for Einstein and conformal gravity, the graviton mode is a linear combination of the conformal graviton and the ghost

mode [68]. On a conformally flat background with coordinates for which the metric takes the form $\mathrm{d}s^2 = (1-\Lambda x^2)^{-2}\,\eta_{\mu\nu}\,\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$, the Einstein modes can be selected through

$$H_{ABCD} = \Psi_{ABCD} - \Lambda\Psi'_{ABCD} = (1-\Lambda x^2)p_A p_B p_C p_D e^{ix\cdot k}\,, \tag{3.55}$$

and for the primed components respectively. We will see in section 6.3 how the Neumann boundary condition is being replaced by an algebraic condition in twistor space through explicitly restricting the spectrum, and how it is generalized to the higher-spin case.

Chapter 4

To Infinity and Beyond

The purpose of this chapter is to give a brief introduction into the ingredients of higher-spin theory. We present the higher-spin generalisation of the Lorentz connection forms ω, discuss the linearised action, and introduce the basics of higher-spin algebras. The algebraic presentation leads to the definition of the Weyl-Moyal product, which we will rediscover in chapter 7 in the twistor space context. Discussing the unfolded formulation should give an idea of the complexity and difficulties in formulating the dynamical content of the higher-spin theories and why it is advantageous to find an alternative formulation that is more attuned to the calculation of amplitudes.

4.1 Higher Spin Theory at the Linear Level

At the heart of higher spin theory lies the representation theory of unitary irreducible representations of the Poincaré group. As for the spin-2 case, it is again useful and because of no-go theorems [4, 5] for higher-spin theories on flat space necessary to consider an extension, e.g. $\mathfrak{iso}(1,3) \subset \mathfrak{so}(2,3)$. We thus study a perturbation around an AdS background, which does not allow for a flat limit to Minkowski space as long as higher-spin modes, $s > 2$, are present. Furthermore, there is no consistent truncation to just a finite number of higher-spin fields with $s > 2$. Considering those

constraints circumvents the conditions of the no-go theorems for a fully non-linear theory.

The connection forms ω and coframe fields e describing spin-2 fields in the last chapter transform in a specific representation of $\mathfrak{iso}(1,3)$, labelled by $\omega_a{}^b$ and e^a. Choosing different representations corresponding to different spin extends the set of coframe fields to

$$e_\mu{}^a\,, \quad e_\mu{}^{a_2a_3}\,, \quad \dots\,, \quad e_\mu{}^{a_2\dots a_s}\,, \quad \dots \tag{4.1}$$

and the Lorentz connection forms to

$$(\omega_\mu)^a{}_b\,, \quad (\omega_\mu)^{a_2a_3}{}_b\,, \quad \dots\,, \quad (\omega_\mu)^{a_2\dots a_s}{}_b\,, \quad \dots\,. \tag{4.2}$$

All the components are symmetric in the indices $^{a_2\dots a_s}$ and traceless. We are only discussing the linearised theory as the non-linear dynamics are implicitly described in the all-order formulation at the level of the equations of motion, and it is not necessary for us to lay out all the details of this formulation. The cubic interactions were consistently constructed in [71], and then later extended to all orders at the level of the equations of motion in [6, 8, 7]. It remains a difficult problem to extract the explicit interaction vertices from this formulation.

The presentation in this section mainly follows [72].

4.1.1 Linearised Higher-Spin Action

A linearisation of the theory can be carried out by choosing a vacuum solution of Einstein's equations of motion, e_0 and $\omega_0(e_0)$, and considering the higher-spin field fluctuations propagating on this background. For this we choose an AdS background. In the same manner as we could describe the graviton in terms of metric fluctuations, we can construct higher valence tensor fields associated to the propagating coframe fields (4.1) by

$$h_{\mu_1\dots\mu_s} = e_{(\mu_1}{}^{a_2\dots a_s} e_{0\mu_2}{}^{b_2} \cdots e_{0\mu_s)}{}^{b_s}\, \eta_{a_2b_2} \cdots \eta_{a_sb_s}\,, \tag{4.3}$$

where η is the flat metric of AdS space. Generalised higher-spin diffeomorphisms are generated through the gauge transformations

$$\delta h_{\mu_1\ldots\mu_s} = \partial_{(\mu_1}\xi_{\mu_2\ldots\mu_s)}\,, \tag{4.4}$$

and because of the tracelessness of the coframe fields, the metric-like field h is double traceless

$$h^{\mu\nu}{}_{\mu\nu\mu_5\ldots\mu_s} = 0 \qquad \text{and} \qquad \xi^{\mu}{}_{\mu\mu_3\ldots\mu_{s-1}} = 0\,. \tag{4.5}$$

The fields h are Fronsdal fields and have been originally formulated in [45, 47], together with a classification of the higher-spin analogues of the Christoffel symbols. The double tracelessness of the Fronsdal field and the tracelessness of the gauge parameter is a feature of this formulation and needs to be imposed in order for the Lagrangian to be gauge invariant. The corresponding Fronsdal action [45] is just the higher-spin generalisation of the Fierz-Pauli action [73], which in turn can be obtained from the linearisation of the Einstein-Hilbert action.

It is convenient to combine all coframe fields and connection forms into a single connection form [74], which takes values in an irreducible $\mathfrak{so}(2,3)$-module characterized by a two-row traceless rectangular Young tableau of length $s-1$, that is we construct a connection form with components

$$\begin{gathered}(\omega_\mu)^{\hat a(s-1)}{}_{\hat b(s-1)} = (\omega_\mu)^{\hat a_1\ldots\hat a_{s-1}}{}_{\hat b_1\ldots\hat b_{s-1}} = (\omega_\mu)^{(\hat a_1\ldots\hat a_{s-1})}{}_{\hat b_1\ldots\hat b_{s-1}} = (\omega_\mu)^{\hat a_1\ldots\hat a_{s-1}}{}_{(\hat b_1\ldots\hat b_{s-1})}\\ (\omega_\mu)^{(\hat a_1\ldots\hat a_{s-1}\,\hat b_1)}{}_{\hat b_2\ldots\hat b_{s-1}} = 0 \qquad \text{and} \qquad (\omega_\mu)^{\hat a(s-3)\hat c}{}_{\hat c\,\hat b(s-1)} = 0\,,\end{gathered} \tag{4.6}$$

where $\hat a, \hat b, \hat c = 0, \ldots, 4$, and indices are raised and lowered with the flat metric $\eta_{\hat a\hat b}$ of $\mathbb{R}^{2,3}$. We also employ the notation

$$A_{\mu(s)} = A_{(\mu_1\ldots\mu_s)} \tag{4.7}$$

from now on. The coframe fields (4.1) and Lorentz-like connection forms (4.2) are packaged as the components

$$e^{a(s-1)} = \omega^{a(s-1)}{}_{4\ldots4} \qquad \text{and} \qquad \omega^{a(s-1)}{}_{b(t)} = \Pi(\omega^{a(s-1)}{}_{b(t)4\ldots4})\,, \tag{4.8}$$

for $t \leq s-1$, where Π projects onto the Lorentz-traceless part, which is necessary for $t \geq 2$. At the linearised level, the auxiliary fields $\omega^{a(s-1)}{}_{b(t)}$ are related to the coframe field $e^{a(s-1)}$ by taking derivatives t times. For the Lorentz-like connection $\omega^{a(s-1)}{}_{b}$ this is just the vanishing torsion constraint: $T = D_\omega e = \mathrm{d}e + \omega \wedge e = 0$, and for $t \geq 2$ they are the higher-spin analogues.

The linearised curvature associated to this combined connection form ω on an AdS background is given by taking the covariant derivative with respect the background connection form $\omega_0{}^{\hat{a}}{}_{\hat{b}}$,

$$R_1{}^{\hat{a}(s-1)}{}_{\hat{b}(s-1)} = D_0 \omega^{\hat{a}(s-1)}{}_{\hat{b}(s-1)} \,. \tag{4.9}$$

Since $D_0^2 = R_0 = 0$, the linearised curvature R_1 has the gauge symmetry

$$\delta_0 \omega^{\hat{a}(s-1)}{}_{\hat{b}(s-1)} = D_0 \hat{\xi}^{\hat{a}(s-1)}{}_{\hat{b}(s-1)} \,, \tag{4.10}$$

where the gauge parameter $\hat{\xi}^{\hat{a}(s-1)}{}_{\hat{b}(s-1)}$ is traceless.

Given these curvatures, we can write down an action which is $\mathfrak{so}(2,3)$-invariant and quadratic in R_1. The most general action with these properties is given by

$$S^{(s)}[\omega] = \frac{1}{2} \sum_{p=0}^{s-2} a^{(s,p)} \int_M \epsilon_{\hat{a}_1 \ldots \hat{a}_4 4} \, R_1^{\hat{a}_1 \hat{b}(s-2)\, \hat{a}_2 4 \ldots 4 \, \hat{d}(p)} \wedge R_1^{\hat{a}_3}{}_{\hat{b}(s-2)}{}^{\hat{a}_4 4 \ldots 4}{}_{\hat{d}(p)} \,, \tag{4.11}$$

where $a^{(s,p)}$ are arbitrary coefficients. They can be fixed up to an arbitrary function only depending on s by demanding that the equations of motion are non-trivial only for the coframe fields (4.1) and the Lorentz-like connection forms (4.2). All other fields are of auxiliary nature and should appear solely through total derivatives. This action is in fact equivalent to the Fronsdal action since it was constructed satisfying the same constraints of being gauge invariant and second order in the derivatives of the fields [75].

Constructing higher-order terms in the dynamic perturbation around AdS is becoming exceedingly difficult, and not actually leading towards the aim of finding a full theory that includes all the non-linear dynamics. This theory, Vasiliev Theory,

has been formulated on the level of the equations of motion, using an algebraic approach.

4.1.2 Higher-Spin Algebras

The starting point is to consider the generators T of

$$\omega = \omega^{\hat{a}(s-1)}{}_{\hat{b}(s-1)}\, \mathrm{T}^{\hat{b}(s-1)}{}_{\hat{a}(s-1)}\,, \tag{4.12}$$

with the symmetries (4.6), forming a basis of some non-Abelian algebra $\mathfrak{h} \supset \mathfrak{so}(2,3)$. If such an algebra exists, then we can regard R_1 as the linearisation of the curvature $R = D_W W$ associated to the connection form $W = \omega_0 + \omega$ on $\mathfrak{h}$. Those algebras have been constructed in various dimensions, but most relevantly to us in four dimensions with AdS background [76]. This construction was later extended to conformal higher-spin algebras [77], whose realisation as a quotient of the universal enveloping algebra of $\mathfrak{so}(2,d)$ in any dimension d was given by Eastwood in [78].

The higher spin algebra in four dimensions is just the Heisenberg algebra A_4 consisting of two pairs of canonical variables, obeying

$$\left[\hat{Y}_i^{\hat{a}}, \hat{Y}_j^{\hat{b}}\right] = \epsilon_{ij}\eta^{\hat{a}\hat{b}}\,, \tag{4.13}$$

where $i,j = 1,2$. A particular realisation of this algebra is

$$\hat{Y}_1^{\hat{a}} = \frac{\partial}{\partial x_{\hat{a}}} \qquad \text{and} \qquad \hat{Y}_2^{\hat{a}} = x^{\hat{a}}\,, \tag{4.14}$$

where $x^{\hat{a}} \in \mathbb{R}^{2,3}$. We will encounter this later in the thesis again. The generators

$$\mathrm{T}^{\hat{a}\hat{b}} = \frac{1}{4}\left\{\hat{Y}^{\hat{a}i}, \hat{Y}_i^{\hat{b}}\right\} \tag{4.15}$$

obey the $\mathfrak{so}(2,3)$ algebra relations

$$\left[\mathrm{T}^{\hat{a}\hat{b}}, \mathrm{T}^{\hat{c}\hat{d}}\right] = \frac{1}{2}\left(\mathrm{T}^{\hat{a}\hat{d}}\,\eta^{\hat{b}\hat{c}} - \mathrm{T}^{\hat{b}\hat{d}}\,\eta^{\hat{a}\hat{c}} - \mathrm{T}^{\hat{a}\hat{c}}\,\eta^{\hat{b}\hat{d}} + \mathrm{T}^{\hat{b}\hat{c}}\,\eta^{\hat{a}\hat{d}}\right) \tag{4.16}$$

and generate $\mathbb{R}^{2,3}$-rotations through the realisation (4.14): $\mathrm{T}^{\hat{a}\hat{b}} = x^{[\hat{a}}\partial^{\hat{b}]}$.

The higher spin algebra is now spanned by all functions that are polynomial in the variables $\hat{Y}_i^{\hat{a}}$, that is

$$f(\hat{Y},x)=\sum_{n=0}^{\infty} f^{i_1\dots i_n}_{\hat{a}_1\dots\hat{a}_n}(x)\,\hat{Y}_{i_1}^{\hat{a}_1}\cdots\hat{Y}_{i_n}^{\hat{a}_n}\,. \tag{4.17}$$

Using the commutation relations (4.13), we can equivalently write this in a symmetric basis by symmetrising over all $\hat{Y}_1^{\hat{a}}$'s and $\hat{Y}_2^{\hat{a}}$'s. This ordering is called Weyl ordering and allows us to simply use commuting variables $Y_1^{\hat{a}}$ and $Y_2^{\hat{a}}$ instead for the expansion

$$f(Y,x)=\sum_{p,q=0}^{\infty} f_{\hat{a}(p)\,\hat{b}(q)}(x)\,Y_1^{\hat{a}(p)}Y_2^{\hat{b}(q)}\,. \tag{4.18}$$

The trade-off is that the product between two elements f and g of the higher-spin algebra is given by the Weyl-Moyal star-product

$$(f\star g)(Y)=f(Y)\,\exp\left(\frac{1}{2}\epsilon_{ij}\eta^{\hat{a}\hat{b}}\frac{\overleftarrow{\partial}}{\partial Y_i^{\hat{a}}}\frac{\overrightarrow{\partial}}{\partial Y_j^{\hat{b}}}\right)g(Y)\,. \tag{4.19}$$

In terms of these variables, we can now formulate the connection form as

$$\omega(Y,x)=\sum_{s\geq 1}\omega^{\hat{a}(s-1)}{}_{\hat{b}(s-1)}(x)\,Y_{1\hat{a}(s-1)}Y_2^{\hat{b}(s-1)} \tag{4.20}$$

and the curvature as

$$R(Y,x)=\mathrm{d}\omega(Y,x)-(\omega\star\omega)(Y,x)\,. \tag{4.21}$$

Having pinned down the algebra, the next step is to find a way to formulate the dynamics between the higher-spin fields. This is done using the unfolded formulation.

4.1.3 Unfolded Formalism

Every partial differential equation of any degree can be rewritten as one of first degree of the form

$$\mathrm{d}W^{\mathcal{A}}=F^{\mathcal{A}}(W)\qquad\text{and}\qquad F^{\mathcal{A}}(W)=\sum_k F^{\mathcal{A}}_{\mathcal{B}_1\dots\mathcal{B}_k}W^{\mathcal{B}_1}\wedge\cdots\wedge W^{\mathcal{B}_k}\,, \tag{4.22}$$

where the fields $W^{\mathcal{A}}$ are sets of differential forms of various form degrees, labelled by $\mathcal{A}$, and the coefficients $F^{\mathcal{A}}_{\mathcal{B}_1\ldots\mathcal{B}_k}$ are the structure constants of the unfolded system, and space-time independent. This is always possible by introducing enough auxiliary fields. In particular, the integrability condition $\mathrm{d}^2 = 0$ translates to the quadratic condition

$$\mathrm{d}^2 W^{\mathcal{A}} = \mathrm{d}F^{\mathcal{A}}(W) = \mathrm{d}W^{\mathcal{B}} \wedge \frac{\partial}{\partial W^{\mathcal{B}}} F^{\mathcal{A}}(W) = F^{\mathcal{B}}(W) \wedge \frac{\partial}{\partial W^{\mathcal{B}}} F^{\mathcal{A}}(W) = 0\,, \quad (4.23)$$

which possesses the gauge symmetry

$$\delta W^{\mathcal{A}} = \mathrm{d}\xi^{\mathcal{A}} + \xi^{\mathcal{B}} \wedge \frac{\partial}{\partial W^{\mathcal{B}}} F^{\mathcal{A}}(W) \quad (4.24)$$

where $\xi^{\mathcal{A}}$ has form degree one less than $W^{\mathcal{A}}$.

Spin-1 Consider the simple spin-1 example for some connection form $A \in \Omega^1(M, \mathfrak{g})$ of some algebra $\mathfrak{g}$ to some gauge group. The label $\mathcal{A}$ labels the connection form, and $W^A = A$. The unfolded equations (4.22) are just the flatness condition

$$\mathrm{d}W^A = \mathrm{d}A = \frac{1}{2}[A, A]\,, \quad (4.25)$$

and the integrability constraint is the Jacobi identity for the generators of $\mathfrak{g}$.

Now consider that $\mathcal{A}$ also labels some scalar field C and that, to lowest order, $F(W)$ is linear in A and C, that is for some matrix representation $(\mathrm{T}^a)^i{}_j$ of $\mathfrak{g}$ it is

$$F(A, C)^i = A^a (\mathrm{T}^a)^i{}_j C^j\,. \quad (4.26)$$

Then the unfolded equations imply $D_A C = 0$.

Spin-2 To make the unfolding strategy more explicit, we quickly outline the general idea for the spin-2 case of gravity. We start with considering the linearised theory around a Minkowski background, with flat connection ω_0 giving a covariant derivative D_0, and flat coframe field e_0. On-shell, the Riemann tensor is equal to the Weyl tensor, and thus the full equations of motion can also be written as $R^{ab} = C^{ac\,bd} e_c \wedge e_d$,

where $C^{ac\,bd}$ is a zero-form which has the same symmetries as the Weyl tensor. In the linearised theory this is

$$R_1^{ab} = C^{ac\,bd} e_{0c} \wedge e_{0d}\,. \tag{4.27}$$

The second Bianchi identity $D_0 R_1 = 0$ translates to

$$e_{0c} \wedge e_{0d} \wedge D_0 C^{ab\,cd} = 0 \tag{4.28}$$

and since the coframe field e_0 is a basis of the tangent bundle, we can decompose the covariant derivative as

$$D_0 C^{ac\,bd} = e_{0f}(2C^{acf\,bd} + C^{acb\,df} + C^{acd\,bf})\,. \tag{4.29}$$

Now, since $D_0^2 = 0$, by taking higher covariant derivatives, this procedure continues infinitely, where at each step we have the decomposition

$$D_0 C^{a(k)\,bd} = e_{0f}(kC^{a(k)f\,bd} + C^{a(k)b\,df} + C^{a(k)d\,bf})\,. \tag{4.30}$$

Each $C^{a(k)\,bd}$ transforms in an irreducible representation of $\mathfrak{so}(1,3)$, symmetric in its indices $C^{a(k)\,bd} = C^{a(k)\,(bd)}$ and traceless in the first k indices. They form a basis in the space of all non-trivial combinations of the derivatives of the spin-2 gauge field that are gauge-invariant and on-shell. This concludes the unfolding procedure for linearised gravity. For the full theory, decomposing the covariant derivative of the right hand side of (4.29) introduces correction terms quadratic in C, which lead to correction terms with even higher powers at higher iterations of the unfolding. So far, the problem of unfolding the full non-linear theory at all orders remains unsolved.

Higher-Spin Coming back now to the higher-spin case, for the linearised theory, the unfolded system takes the form

$$\mathrm{d}\omega = F_0^\omega(\omega,\omega) + F_1^\omega(\omega,\omega,C) + \ldots \tag{4.31}$$

$$\mathrm{d}C = F_0^C(\omega,C) + F_1^C(\omega,C,C) + \ldots \tag{4.32}$$

where, analogous to the example of the spin-2 case, the structure functions have an expansion in powers of C with the initial conditions

$$F_0^{\omega}(\omega,\omega) = \omega \star \omega \qquad \text{and} \qquad F_0^{C}(\omega, C) = \omega \star C - C \star \pi(\omega)\,. \tag{4.33}$$

The map π is induced by the reflection of AdS translations.

The connection form ω is the one introduced above in (4.6), and the scalar field C has, in extension of the spin-2 example, the symmetries of the higher-spin Weyl tensor, which we will discuss in more detail in the next section.

4.2 Conformal Higher Spin Theory

As the higher-spin theory introduced in the last chapter is a generalisation of Einstein's theory of gravity, there is equally a higher-spin generalisation of conformal gravity. It was first introduced by Fradkin and Tseytlin [9] as a linearised theory and later non-linearly extended by Segal [11]. We will introduce the linearised Weyl curvatures, mainly in the spinor language, as we are interested in the the self-dual sector, in which we can again find a Chalmers-Siegel-like action. This will be useful later on as our twistor space description is based on the self-dual description of the conformal higher spin theory.

In the linearised spin-2 theory the relationship with the metric fluctuation, $h_{ABC'D'}$, is given by

$$\Psi_{ABCD} = \nabla_{(C}{}^{C'} \nabla_{D}{}^{D'} h_{AB)C'D'}\,. \tag{4.34}$$

In the higher-spin case we will have fields analogous to the metric fluctuations $h_{A(s)B'(s)}$, which satisfy a condition $\Psi_{A(s)B(s)} = 0$, where $\Psi_{A(s)B(s)}$ is the higher-spin analogue of the Weyl spinor, with higher numbers of derivatives

$$\Psi_{A(s)B(s)} = \nabla_{(A(s)}{}^{A'(s)} h_{B(s))A'(s)}\,, \tag{4.35}$$

where

$$\nabla^{A'(s)}_{A(s)} = \underbrace{\nabla_A{}^{A'} \cdots \nabla_A{}^{A'}}_{s \text{ times}} . \tag{4.36}$$

We further define the self-dual spinor

$$\widetilde{\Psi}_{A'(s)B'(s)} = \nabla^{A(s)}{}_{(A'(s)} h_{B'(s))A(s)} \tag{4.37}$$

in terms of the same potential fields.

4.2.1 Linearised Action

We consider the combination of self-dual and anti-self-dual spinor fields $\Psi_{A(s)B(s)}$ and $\widetilde{\Psi}_{A'(s)B'(s)}$, and we define the field strength

$$\mathscr{C}_{\alpha_1\beta_1\ldots\alpha_s\beta_s} = \epsilon_{A'_1B'_1} \cdots \epsilon_{A'_sB'_s} \Psi_{A_1\ldots A_sB_1\ldots B_s} + \epsilon_{A_1B_1} \cdots \epsilon_{A_sB_s} \widetilde{\Psi}_{A'_1\ldots A'_sB'_1\ldots B'_s} . \tag{4.38}$$

This tensor is anti-symmetric in each pair (α_i, β_i) for $i = 1, 2, 3$ and symmetric between pairs $(\alpha_i, \beta_i) \leftrightarrow (\alpha_j, \beta_j)$ for $i \neq j$. Moreover, if we contract over a pair of α or β indices we find zero, for example

$$\mathscr{C}^{\alpha_1}{}_{\beta_1\alpha_1\beta_2\ldots\alpha_s\beta_s} = 0 . \tag{4.39}$$

This is essentially the spin-s case of the higher-spin curvatures introduced by Weinberg [79]. Such generalised Weyl curvatures can be viewed as following from higher-spin analogues of Riemann curvatures by removing all traces. The symmetries of the indices are different than those of the curvatures introduced by de Wit and Freedman [47], however they are trivially related; for the explicit relation between these formulations and further relevant discussion see [80]. Given such curvatures, we can naturally form a Lagrangian density quadratic in the field strength

$$\mathcal{L}_{\text{spin}-s} = \frac{1}{2^{2s}} \mathscr{C}^{\alpha_1\beta_1\ldots\alpha_s\beta_s} \mathscr{C}_{\alpha_1\beta_1\ldots\alpha_s\beta_s} . \tag{4.40}$$

We can reformulate this in a notation closer to that used by Fradkin and Tseytlin in their discussion of quadratic higher spin theory [9] by introducing a potential for this field strength, $\phi_{\beta(s)}$, which is symmetric and pairwise traceless in its indices. The field strength

$$\mathscr{C}_{\alpha_1\beta_1\ldots\alpha_s\beta_s} = \partial^s_{\alpha(s)}\phi_{\beta(s)} \pm \text{permutations} \tag{4.41}$$

is found by anti-symmetrising on pairs of indices α_i, β_i. Rewriting this in terms of the potential and neglecting total derivative terms, we have e.g. for the spin-3 case explicitly

$$\frac{1}{8}\phi^{\gamma_1\gamma_2\gamma_3}P^{\beta_1\beta_2\beta_3}_{\gamma_1\gamma_2\gamma_3}\Big(\Box^3\delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_2}\delta^{\alpha_3}_{\beta_3} - 3\Box^2\partial^{\alpha_1}\partial_{\beta_1}\delta^{\alpha_2}_{\beta_2}\delta^{\alpha_3}_{\beta_3} + 3\Box\partial^{\alpha_1}\partial^{\alpha_2}\partial_{\beta_1}\partial_{\beta_2}\delta^{\alpha_3}_{\beta_3} - \partial^{\alpha_1}\partial^{\alpha_2}\partial^{\alpha_3}\partial_{\beta_1}\partial_{\beta_2}\partial_{\beta_3}\Big)P^{\delta_1\delta_2\delta_3}_{\alpha_1\alpha_2\alpha_3}\phi_{\delta_1\delta_2\delta_3}\,,$$

where $P^{\alpha_1\alpha_2\alpha_3}_{\beta_1\beta_2\beta_3}$ projects onto symmetric, pairwise traceless tensors, and for the spin-s case generally the Lagrangian

$$\mathcal{L}_{\text{spin}-s} = \frac{1}{2^s}\phi^{\beta(s)}D^{\alpha(s)}_{\beta(s)}\phi_{\alpha(s)} \tag{4.42}$$

in terms of a kinetic operator $D^{\alpha(s)}_{\beta(s)}$ which is symmetric, pairwise traceless and transverse, that is satisfying

$$D^{\alpha(s)}_{\beta(s)}\partial_\alpha = 0\,. \tag{4.43}$$

The resulting Lagrangian has the form of the conformal higher spin theory described in [9]. In terms of the spinor fields, the quadratic action for spin-s fields is thus

$$S_{\text{spin}-\text{s}}[\phi] = \frac{1}{2^s\lambda}\int \mathrm{d}^4x\sqrt{|g|}\,\left(\Psi^{A(s)B(s)}\Psi_{A(s)B(s)} + \widetilde{\Psi}^{A'(s)B'(s)}\widetilde{\Psi}_{A'(s)B'(s)}\right)\,, \tag{4.44}$$

where λ is a dimensionless parameter.

4.2.2 Self-Dual Sector

Given the symmetries of the field strength $\mathscr{C}_{\alpha_1\beta_1\ldots\alpha_s\beta_s}$, we can naturally define a dual field strength by dualizing in the $\Lambda^2(M)$ part

$$*\mathscr{C}_{\alpha_1\beta_1\ldots\alpha_s\beta_s} = \epsilon_{\alpha_1\beta_1\gamma\delta}\mathscr{C}^{\gamma\delta}{}_{\alpha_2\beta_2\ldots\alpha_s\beta_s} \,. \tag{4.45}$$

In terms of spinor quantities the Levi-Civita tensor is

$$\epsilon_{\alpha\beta\gamma\delta} = i\epsilon_{AC}\epsilon_{BD}\epsilon_{A'D'}\epsilon_{B'C'} - i\epsilon_{AD}\epsilon_{BC}\epsilon_{A'C'}\epsilon_{B'D'} \tag{4.46}$$

such that, by construction, the anti-self-dual part is

$$\begin{aligned}\mathscr{C}^{-}_{\alpha_1\beta_1\ldots\alpha_s\beta_s} &= \frac{1}{2}(\mathscr{C}_{\alpha_1\beta_1\ldots\alpha_s\beta_s} + i*\mathscr{C}_{\alpha_1\beta_1\ldots\alpha_s\beta_s}) \\ &= \epsilon_{A'_1B'_1}\cdots\epsilon_{A'_sB'_s}\Psi_{A_1\ldots A_sB_1\ldots B_s} \,,\end{aligned}$$

and the self-dual part is

$$\begin{aligned}\mathscr{C}^{+}_{\alpha_1\beta_1\ldots\alpha_s\beta_s} &= \frac{1}{2}(\mathscr{C}_{\alpha_1\beta_1\ldots\alpha_s\beta_s} - i*\mathscr{C}_{\alpha_1\beta_1\ldots\alpha_s\beta_s}) \\ &= \epsilon_{A_1B_1}\cdots\epsilon_{A_sB_s}\widetilde{\Psi}_{A'_1\ldots A'_sB'_1\ldots B'_s} \,.\end{aligned} \tag{4.47}$$

It is straightforward to show that the term

$$i*\mathscr{C}^{\alpha_1\beta_1\ldots\alpha_s\beta_s}\mathscr{C}_{\alpha_1\beta_1\ldots\alpha_s\beta_s} = \Psi^{A(s)B(s)}\Psi_{A(s)B(s)} - \widetilde{\Psi}^{A'(s)B'(s)}\widetilde{\Psi}_{A'(s)B'(s)} \tag{4.48}$$

is a total derivative and so does not affect any perturbative calculations. The argument goes by considering the field strength written in terms of the potentials

$$\begin{aligned}\Psi^{A(s)B(s)}\Psi_{A(s)B(s)} &= \nabla^{A_1}{}_{A'_1}\cdots\nabla^{A_s}{}_{A'_s}\phi^{B(s)A'(s)}\,\nabla_{A_1}{}^{B'_1}\cdots\nabla_{A_s}{}^{B'_s}\phi_{B(s)B'(s)} \\ &= (-1)^s\phi^{B(s)A'(s)}\nabla^{A_s}{}_{A'_s}\cdots\nabla^{A_1}{}_{A'_1}\nabla_{A_1}{}^{B'_1}\cdots\nabla_{A_s}{}^{B'_s}\phi_{B(s)B'(s)}\end{aligned}$$

where we again drop total derivatives. As we can take derivatives to commute, we have

$$\Psi^{A(s)B(s)}\Psi_{A(s)B(s)} = \frac{1}{2^s}\phi^{B(s)B'(s)}(\nabla^2)^s\phi_{B(s)B'(s)} \,, \tag{4.49}$$

but repeating the same arguments leads to

$$\widetilde{\Psi}^{A'(s)B'(s)}\widetilde{\Psi}_{A'(s)B'(s)} = \frac{1}{2^s}\phi^{B(s)B'(s)}(\nabla^2)^s\phi_{B(s)B'(s)} \tag{4.50}$$

up to total derivatives, and so

$$\Psi^{A(s)B(s)}\Psi_{A(s)B(s)} - \widetilde{\Psi}^{A'(s)B'(s)}\widetilde{\Psi}_{A'(s)B'(s)} = 0\,. \tag{4.51}$$

Following the construction of the Chalmers-Siegel action for Yang-Mills [31, 48] and its analogue for conformal gravity [34], though of course here we are working only at the linearised level, we can add this term to the action so that, up to boundary terms, we find

$$\begin{aligned} S_{\text{spin}-3} &= \frac{1}{2^{s-1}\lambda}\int \mathrm{d}^4x\sqrt{|g|}\,\Psi^{A(s)B(s)}\Psi_{A(s)B(s)} \\ &= \int \mathrm{d}^4x\sqrt{|g|}\left(G^{A(s)B(s)}\Psi_{A(s)B(s)} - \frac{\lambda}{2}\,G^{A(s)B(s)}G_{A(s)B(s)}\right), \end{aligned} \tag{4.52}$$

where in the last line we have introduced the anti-self-dual Lagrange multiplier field $G^{A(s)B(s)}$, which is symmetric in all indices. This action gives the equations of motion

$$\Psi_{A(s)B(s)} = \lambda G_{A(s)B(s)} \qquad \nabla^{A(s)A'(s)}G_{A(s)B(s)} = 0\,. \tag{4.53}$$

This action will also suggest, much as in the case of conformal gravity, how to extend the twistor action to the full theory beyond the self-dual sector.

Chapter 5

Twistor Theory

As outlined in the introduction, twistor theory is a suitable and powerful formalism to study conformal theories. The twistor string description of the self-dual sector of conformal gravity [34] and the twistor description of the full theory [17, 33] as well as its applications to compute amplitudes in this context [35, 36] have sparked an interest in using this formalism to extend those descriptions to higher spin theories. Twistors are natural objects to the higher spin community, but have only been used as bookkeeping devices of some sort. In this chapter we give a short introduction to twistor theory and outline the most important ideas behind the formalism.

There are several excellent introductory textbooks to twistors, e.g. [81], and reviews e.g. [50, 18, 82] and we refer the reader to those for more details; we will mostly use the notations of [18].

5.1 Twistor Space

Starting from our discussion in section 2.2, we first continue this line of thought and give a less precise, but more pictorial idea of what twistor space is, before going into the more mathematical details of its definition.

In the spinor construction in section 2.2, we realized that σ really is a map

$\sigma : \mathbb{C}^4 \to \mathfrak{psl}(2,\mathbb{C})$, and thus it also serves to give a spinorial representation of the coordinates on complexified space-time $M^{\mathbb{C}}$

$$x^{AA'} = (\sigma_\mu)^{AA'} x^\mu . \tag{5.1}$$

For null coordinates, $x^2 = 0$, we discussed that there is a decomposition

$$x^{AA'} = \tilde{\mu}^A \mu^{A'} , \tag{5.2}$$

which has the ambiguity $(\tilde{\mu}^A, \mu^{A'}) \sim (t\tilde{\mu}^A, t^{-1}\mu^{A'})$ for $t \in \mathbb{C}^*$. We can make this ambiguity manifest by performing a half Fourier transformation

$$\tilde{\mu}^A \equiv \frac{1}{i}\frac{\partial}{\partial \lambda_A} , \tag{5.3}$$

for which we now have

$$i x^{AA'} \lambda_A = \mu^{A'} . \tag{5.4}$$

Thus, a point $x \in M^{\mathbb{C}}$ now corresponds to a line $(\lambda, \mu) = Z$, where $Z \sim tZ , t \in \mathbb{C}^*$, and the space spanned by $(\lambda_A, \mu^{A'})$ is called twistor space.

The mathematically more precise way is as follows. To construct the twistor space corresponding to a flat four-complex-dimensional space-time $M^{\mathbb{C}}$, we consider the total space of the lower index un-primed spinor-bundle with points described by $(x^{AA'}, \sigma_A)$ ①. We can define twistor space by projecting the $\mathbb{S}^{+*}M^{\mathbb{C}}$ onto $\mathbb{T}$ with coordinates $Z^\alpha = (\lambda_A, \mu^{A'})$ by using the incidence relation

$$\mathbb{S}^{+*}M^{\mathbb{C}} \ni (x^{AA'}, \sigma_A) \mapsto (\lambda_A, \mu^{A'}) = (\sigma_A, i x^{AA'} \sigma_A) \in \mathbb{T} . \tag{5.5}$$

Strictly speaking, this does not cover all of $\mathbb{T}$, and we should consider the conformal compactification of $M^{\mathbb{C}}$, which implies a natural action of the conformal group. The conformal group has the double covering

$$C(1,3) \xrightarrow{1:2} O(2,4)$$

① This Weyl spinor σ_A has of course nothing to do with the Pauli spin matrices $(\sigma_\mu)^{AA'}$.

which comes from the fact that the conformal compactification of $M = \mathbb{R}^{1,3}$ is a quadric $N^{1,3} \subset \overline{\mathbb{RP}^{2,4}}$. The second double covering

$$O(2,4) \xrightarrow{1:2} SU(2,2)$$

is obtained by further complexification of $N^{1,3}$. Thus, $SU(2,2)$ naturally acts on $\overline{\mathbb{RP}^{2,4}} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{CP}^3$. We denote the complexification as $\mathbb{M} := N^{1,3} \otimes_{\mathbb{R}} \mathbb{C}$, and we obtain the SU-frame bundle over complexified, compactified Minkowski space. Through this construction, twistor space $\mathbb{T} \subset \mathbb{C}^4$ is the representation space of the complex Weyl spinor representation of $\mathfrak{su}(2,2)$. The same projection for the projectivised spin bundle defines projective twistor space $\mathbb{PT} \cong \mathbb{CP}^3$, which in homogeneous coordinates is $Z^\alpha \sim tZ^\alpha$ for $t \in \mathbb{C}^*$.

Through the pairings $(C(1,3), M) \to (O(2,4), N^{1,3}) \to (SU(2,2), \mathbb{PT})$, twistor space naturally obtains a projective structure $\mathbb{PT} \subset \mathbb{CP}^3$. This is captured in the incidence relations, which identify a point in complexified, compactified Minkowski space $\mathbb{M}$ with a line in $\mathbb{PT}$.

5.2 The Non-Linear Graviton Construction

The construction above only works for flat space-times, but has been extended to curved manifolds [19]. Specifically, the non-linear graviton construction, taken from [83], states

5.2.1 Theorem **(Non-Linear Graviton)**

There is a one-to-one correspondence between: (a.) self-dual space-times $\mathbb{M}$*, and (b.) twistor spaces* $\mathbb{P}\mathscr{T}$*, complex projective 3-manifold obtained as complex deformations of* $\mathbb{PT}$*, containing a rational curve* Ξ_0 *with normal bundle* $\mathcal{N}(\Xi_0) \simeq \mathcal{O}(1) \oplus \mathcal{O}(1)$ *.*

There is a metric in this self-dual conformal class with scalar curvature $\mathcal{R} = 4\Lambda$ *iff* $\mathbb{P}\mathscr{T}$ *is equipped with*

- *a non-degenerate holomorphic contact structure* $\tau \in \Omega^{1,0}(\mathbb{P}\mathscr{T}, \mathcal{O}(2))$ *;*

- *a holomorphic 3-form* $D^3\mathcal{Z} \in \Omega^{3,0}(\mathbb{P}\mathscr{T}, \mathcal{O}(4))$ *obeying* $\tau \wedge \mathrm{d}\tau = \frac{\Lambda}{3} D^3\mathcal{Z}$.

We will explain the relation to complex deformations later, when we interpret the higher spin fields as components of the holomorphic structure on $\mathbb{P}\mathscr{T}$. In this section, we discuss the geometry of $\mathbb{P}\mathscr{T}$ in relation to $\mathbb{PT}$, while only referring to the first part of the theorem. The second part will become clear once we discuss the unitary truncation of a conformal theory.

A curved twistor space $\mathscr{T}$ is a complex four-dimensional manifold with an Euler vector field E and a non-vanishing holomorphic three-form Ω satisfying

$$\pounds_E \Omega = 4\Omega \quad \text{and} \quad \iota(E)\Omega = 0 \ . \tag{5.6}$$

We can choose local homogeneous coordinates, $\mathcal{Z}^\alpha$, $\alpha = 0, 1, 2, 3$, on $\mathscr{T}$ such that

$$E = \mathcal{Z}^\alpha \frac{\partial}{\partial \mathcal{Z}^\alpha} \qquad \text{and} \qquad \Omega = \frac{1}{6} \epsilon_{\alpha\beta\gamma\delta}\, \mathcal{Z}^\alpha \, \mathrm{d}\mathcal{Z}^\beta \wedge \mathrm{d}\mathcal{Z}^\gamma \wedge \mathrm{d}\mathcal{Z}^\delta \ . \tag{5.7}$$

$\mathbb{P}\mathscr{T}$ corresponds to the space of orbits of E in $\mathscr{T}$. Curved projective twistor spaces $\mathbb{P}\mathscr{T}$ contain a four parameter family of compact holomorphic curves, L_x, with the topology of Riemann Spheres each of which has the same normal bundle as a $\mathbb{CP}^1$ in $\mathbb{CP}^3$. One identifies the points in the curved complex space, $x \in \mathbb{M}$, with these curves.

In order to consider real space-times, we must further define a reality structure. If we wish to choose $\mathbb{M}$ to be Lorentzian with signature $(1, 3)$, we restrict to Hermitian $x^{AA'}$, and the primed and un-primed spinor bundles are related by conjugation. However, as this also relates the self-dual part of the Weyl curvature to the anti-self dual part, the non-linear graviton construction can only be carried out in the conformally flat case. For real manifolds of definite signature — for technical reasons the signature is in fact all negative — there is an anti-linear conjugation on the spinors

$$\alpha^A \mapsto \hat{\alpha}^A \, , \qquad \mu_{A'} \mapsto \hat{\mu}_{A'} \, , \tag{5.8}$$

such that $\hat{\hat{\alpha}}^A = -\alpha^A$ and $\hat{\hat{\mu}}^{A'} = -\mu^{A'}$. For bi-spinors this conjugation is however involutative, and we define the real manifold to be the set of points satisfying $\hat{x}^{AA'} = x^{AA'}$. This induces a map on twistor space $\mathcal{Z}^\alpha \mapsto \hat{\mathcal{Z}}^\alpha$ that has no fixed points, however the lines in twistor space corresponding to the fixed space-time points are fixed lines. Consider the flat space-time case, $\mathbb{M} = \mathbb{E}$; given two points on such a line Z^α and $\hat{Z}^\alpha$ we can define the projection $\mathbb{T} \to \mathbb{E}$ using the formula

$$x^{AA'} = -i\frac{\hat{\lambda}^A \mu^{A'} - \lambda^A \hat{\mu}^{A'}}{\langle \lambda \hat{\lambda} \rangle} \ . \tag{5.9}$$

In this case the unprimed spinor bundle, which is now an eight-dimensional real manifold, and twistor space $\mathbb{T}$ can be identified, and similarly for the projective spinor bundle and projective twistor space.

This is true in the general case where the curved twistor space $\mathscr{T}(\mathbb{M})$ is identified with the unprimed spinor bundle over $\mathbb{M}$ with non-holomorphic coordinates $(x^{AA'}, \sigma_A)$, where σ_A is a spinor at $x^{AA'} \in \mathbb{M}$. There is an almost complex structure on $\mathscr{T}$ such that the space of $(0,1)$-tangent vectors at $(x^{AA'}, \sigma_A)$ is spanned by

$$\hat{V}_{A'} = -\frac{i}{\langle \sigma \hat{\sigma} \rangle} \sigma^A \nabla_{AA'} \qquad \text{and} \qquad \hat{V}^A = \hat{\partial}^A = \frac{\partial}{\partial \hat{\sigma}_A} \ . \tag{5.10}$$

Projective twistor space $\mathbb{P}\mathscr{T}(\mathbb{M})$ corresponds to the projective unprimed spinor bundle over $\mathbb{M}$ where σ_A are now the homogeneous coordinates of the $\mathbb{CP}^1$ fibre. The almost complex structure on $\mathscr{T}(\mathbb{M})$ reduces to $\mathbb{P}\mathscr{T}(\mathbb{M})$ as the tangent space of projective twistor space can be found by factoring out the fields $E = \sigma_A \partial^A$ and $\hat{E} = \hat{\sigma}_A \hat{\partial}^A$. As shown by Atiyah *et al*, [84], the almost complex structure is integrable if and only if $\mathbb{M}$ has vanishing anti-self-dual Weyl curvature $\Psi_{ABCD} = 0$. This almost complex structure reduces to a complex structure on $\mathbb{P}\mathscr{T}(\mathbb{M})$ as the vector E is holomorphic.

For a generic manifold we cannot generally define global twistors. However, we can define at each point $x \in \mathbb{M}$ a local twistor $\mathcal{Z}^{\underline{\alpha}}$, which for a given metric g is

represented by a pair of spinors $(\lambda_A, \pi^{A'})$ which transform as

$$\begin{aligned} \tilde{\lambda}_A &= \lambda_A - i\Upsilon_{AA'}\pi^{A'} \\ \text{and} \quad \tilde{\pi}^{A'} &= \pi^{A'} \end{aligned} \tag{5.11}$$

under the Weyl transformation $g \mapsto \Omega^2 g$ with $\Upsilon_{AA'} = \nabla_{AA'} \log \Omega$. We denote the corresponding rank-four local twistor bundle over $\mathbb{M}$ by $\mathbb{LT}$. Pulling back a section of $\mathbb{LT}$, given by the spinor fields $(\lambda_A, \pi^{A'})$, we can define a (1,0)-vector field on $\mathbb{T}(\mathbb{M})$ by

$$T = \lambda_A(x)V^A + \pi^{A'}(x)V_{A'} \ , \tag{5.12}$$

where $V^A = \partial^A$ and $V_{A'} = \frac{-i}{\langle \sigma\hat{\sigma}\rangle}\hat{\sigma}^A \nabla_{A'A}$. One can show, see e.g. [50], that such vectors are holomorphic if and only if they are parallel under local twistor transport. That is to say when they satisfy the conditions

$$\begin{aligned} &\nabla_{BB'}\pi^{A'} + i\lambda_B \epsilon_{B'}{}^{A'} = 0 \\ \text{and} \quad &\nabla_{BB'}\lambda_A - i(\Phi_{ABA'B'} - \Lambda\epsilon_{AB}\epsilon_{A'B'})\pi^{A'} = 0 \, , \end{aligned} \tag{5.13}$$

where $\Phi_{ABA'B'}$ is proportional to the trace-free Ricci tensor and Λ to the Ricci scalar. In the conformally flat space case these equations have a four-complex-parameter family of solutions given by

$$\pi^{A'} = \mu_0^{A'} + i\lambda_A x^{AA'} \tag{5.14}$$

for constant $\mu_0^{A'}$ and λ_A. We can thus identify the solution space of the twistor equation with flat twistor space $\mathbb{T} \simeq \mathbb{C}^4$. Given a holomorphic field, T, of the form (5.12) on $\mathbb{T}$ we can choose linear coordinates Z^α such that

$$T = T^\alpha \frac{\partial}{\partial Z^\alpha} \tag{5.15}$$

with T^α being constant.

5.3 Penrose Transform

In this section we give a brief review of the Penrose transform in the language of Dolbeault cohomology [85], following closely [50], and including the generalisation to homogeneous tensors on twistor space [86, 87]. Given any $\overline{\partial}$-closed $(0,1)$-form α on $\mathbb{T}(\mathbb{M})$ representing a class in $H^{0,1}(\mathbb{P}\mathscr{T}, \mathcal{O}(-n-2))$, $n \geq 0$, there is an associated right-handed field $\Psi_{A_1 \dots A_n}$ on $\mathbb{M}$ which is symmetric in its n indicies

$$\Psi_{A_1 \dots A_n}(x) = \frac{1}{2\pi i} \int_{\Xi} \sigma_{A_1} \dots \sigma_{A_n} \alpha(x,\sigma)\Big|_{\Xi} \wedge D\sigma \tag{5.16}$$

where Ξ is the fibre of $\mathbb{P}\mathscr{T}$ above x and $D\sigma = \sigma_A \, \mathrm{d}\sigma^A$. This integral depends only on the cohomology class of α. In particular, every class in $H^{0,1}(\mathbb{P}\mathscr{T}, \mathcal{O}(-n-2))$, $n \geq 0$, contains a unique harmonic representative $\alpha = \hat{\partial} f$ defined by

$$f(x,\sigma) = \left(\frac{1}{\langle \sigma \hat{\sigma} \rangle^{n+1}} \right) \varphi_{A_1 \dots A_n}(x) \, \hat{\sigma}^{A_1} \dots \hat{\sigma}^{A_n} \ . \tag{5.17}$$

and it can be shown that $\overline{\partial}\hat{\partial} f = \overline{\partial}\alpha = 0$ if and only if

$$\nabla^{A_1' A_1} \Psi_{A_1 \dots A_n} = 0 \,, \quad \text{for } n > 0 \qquad \text{or} \quad (\Box + \tfrac{1}{6} R)\Psi = 0 \,, \quad \text{for } n = 0 \,. \tag{5.18}$$

Thus we have the isomorphism between the cohomology group $H^{0,1}(\mathbb{P}\mathscr{T}, \mathcal{O}(-n-2))$, $n \geq 0$, and the space of solutions of the zero-rest-mass equations of motion on $\mathbb{M}$.

For $n < 0$ we no longer have space-time solutions of equations of motion but rather an identification with potentials modulo gauge transformations. Every class in $H^{0,1}(\mathbb{P}\mathscr{T}, \mathcal{O}(-n-2))$, $n < 0$, contains an element of the form

$$\alpha(x,\sigma) = - \left(\frac{1}{i^{|n|} \langle \sigma \hat{\sigma} \rangle} \right) \varphi_{A'}{}^{A_2 \dots A_{|n|}}(x) \, \sigma_{A_2} \cdots \sigma_{A_{|n|}} \hat{\sigma}_A \, \mathrm{d}x^{A'A} \,, \tag{5.19}$$

where $\varphi_{A'}{}^{A_2 \dots A_{|n|}}$ is symmetric in its $|n|-1$ unprimed indicies. It can be shown that $\overline{\partial}\alpha = 0$ if and only if

$$\nabla_{A'(A_1} \varphi^{A'}{}_{A_2 \dots A_{|n|})} = 0 \,. \tag{5.20}$$

Moreover, in the case $n \leq -2$ the $(0,1)$-form α is trivial, i.e. $[\alpha] = 0$, if and only if φ is pure gauge, i.e. $\varphi_{A'A_2\dots A_{|n|}} = \nabla_{A'(A_2}\xi_{A_3\dots A_{|n|})}$. This gives rise to the isomorphism between $H^{0,1}(\mathbb{P}\mathscr{T}, \mathcal{O}(-n-2))$ for $n < 0$ and the space of solutions of (5.20) modulo the gauge freedom $\delta\varphi_{A'A_2\dots A_{|n|}} = \nabla_{A'(A_2}\xi_{A_3\dots A_{|n|})}$.

These results can be extended to tensor-valued $(0,1)$-forms $g^{\alpha_1\dots}_{\beta_1\dots}$ of homogeneity $n-2$ on $\mathbb{P}\mathscr{T}$. Those can be identified with fields on space-time by introducing a frame $e_{\underline{\alpha}}{}^{\alpha}$ (denoted $\delta_{\underline{\alpha}}{}^{\alpha}$ in the flat case) and then performing the above Penrose transform on each component. We will focus on the case of flat space-time $\mathbb{M}$, and so consider the twistor fields as deformations of flat twistor space, however it is straightforward to generalise to deformations of an arbitrary curved twistor space. We consider the bundle

$$\mathbb{T}^{\alpha\prime} \to \mathbb{P}\mathbb{T}(\mathbb{M}) \tag{5.21}$$

whose local holomorphic sections are represented by vector fields T on $\mathbb{T}(\mathbb{M})$, satisfying

$$[E, T] = -T \quad \text{and} \quad [\overline{E}, T] = 0\,, \tag{5.22}$$

where E is the Euler vector field. This can be considered as the pull-back of the local twistor bundle $\mathbb{T}^{\alpha} \to \mathbb{M}$ with fibre coordinates $(\lambda_A, \mu^{A'})$. We choose a basis on $\mathbb{T}^{\alpha\prime}$, which we denote $\delta^{\alpha}_{\underline{\alpha}}$, $\alpha = 0, 1, 2, 3$, such that

$$\delta^{\alpha}_{\underline{\alpha}} \frac{\partial}{\partial Z^{\alpha}} \tag{5.23}$$

is a global holomorphic frame. We can now perform the Penrose transform for each component separately. This mean that for $n > 0$ we have the field

$$\Gamma^{\underline{\alpha}_1\dots}_{\underline{\beta}_1\dots\ A_1\dots A_{|n|}} \quad \text{satisfying} \quad \nabla^{A'A_1}\Gamma^{\underline{\alpha}_1\dots}_{\underline{\beta}_1\dots\ A_1\dots A_{|n|}} = 0\,, \tag{5.24}$$

where the covariant derivative is understood to act on the twistor indices via the local twistor connection. For $n = 0$ we have

$$\Gamma^{\underline{\alpha}_1\dots}_{\underline{\beta}_1\dots} \quad \text{satisfying} \quad (\Box + \tfrac{1}{6}R)\Gamma^{\underline{\alpha}_1\dots}_{\underline{\beta}_1\dots} = 0\,, \tag{5.25}$$

and finally for $n < 0$ we have potentials

$$\Gamma^{\underline{\alpha}_1\cdots\ A'}_{\underline{\beta}_1\cdots\ A_2\ldots A_n} \quad \text{satisfying} \quad \nabla_{A'(A_1}\Gamma^{\underline{\alpha}_1\cdots A'}_{\underline{\beta}_1\cdots\ A_2\ldots A_n)} = 0 \tag{5.26}$$

modulo the gauge transformations

$$\delta\Gamma^{\underline{\alpha}_1\cdots\ A'}_{\underline{\beta}_1\cdots\ A_2\ldots A_n} = \nabla^{A'}{}_{(A_2}\xi^{\underline{\alpha}_1\cdots}_{\underline{\beta}_1\cdots\ A_3\ldots A_n)}\,. \tag{5.27}$$

For $n \leq 0$ the Penrose transform takes the form

$$\Gamma^{\underline{\alpha}_1\cdots}_{\underline{\beta}_1\cdots\ A_1\ldots A_{|n|}}(x) = \frac{1}{2\pi i}\int_\Xi \sigma_{A_1}\cdots\sigma_{A_n}\alpha^{\underline{\alpha}_1\cdots}_{\underline{\beta}_1\cdots}(x,\sigma)\Big|_\Xi \wedge D\sigma\,. \tag{5.28}$$

In this way we are now able to relate tensor-valued (0,1)-forms on twistor space to tensors on space-time.

5.4 Twistor Space Actions for Yang-Mills and Gravity Theories

As we have already started to describe in the introduction, starting with Witten's work on twistor string theory [16] the spin ≤ 2 cases have been well-studied. We will be focussing on those theories that will provide a model for the higher-spin case, and thus the discussion of Yang-Mills theory is deliberately brief.

Self-dual $\mathcal{N} = 4$ super-Yang-Mills theory [30, 31] was reformulated by Witten [16] as a holomorphic Chern-Simons theory on the super-twistor space $\mathbb{CP}^{3|4}$. Focusing on the non-supersymmetric gauge fields, this involves $(0,1)$-forms A and G on $\mathbb{PT} \subset \mathbb{CP}^3$, taking values in $\mathfrak{gl}(N,\mathbb{C})$ and which are respectively homogeneous of degree zero and -4 in the bosonic $\mathbb{CP}^3$ coordinates.

The self-dual action is given by

$$S_{\text{s.d.}}[A] = \int_{\mathbb{PT}} \Omega \wedge \operatorname{tr}(G \wedge (\overline{\partial}A + A \wedge A))\,, \tag{5.29}$$

where Ω is a holomorphic $(3,0)$-form of degree four, and $\overline{\partial}$ is the Dolbeault operator on $\mathbb{CP}^3$. Adding anti-self-dual modes is a subtler problem, but can be essentially solved by considering

$$S_{\text{a.s.d.}}[A] = \int_{\mathbb{M}_{\mathbb{R}}} \mathrm{d}^4X \log\det(\overline{\partial} + A)\Big|_{\Xi}\,, \tag{5.30}$$

where $\Xi \simeq \mathbb{CP}^1$ in $\mathbb{PT}$ that correspond to the chosen real slice $\mathbb{M}_{\mathbb{R}} \subset \mathbb{M}$, and d^4X is the measure on it. Expanding the $\log\det = \mathrm{tr}\log$ term schematically yields the amplitudes expansion around the self-dual background. This form however is not prescriptive enough for us to extend it to the higher-spin case since the anti-self-dual interactions come in a different form, which is closer related to the gravity case.

The twistor actions for theories including gravity modes are motivated in large part by the non-linear graviton construction of Penrose and Ward [19], which provides a means to identify curved twistor spaces, $\mathbb{P}\mathscr{T}$, with self-dual space-times, that is those for which the anti-self-dual Weyl spinor vanishes $\Psi_{ABCD} = 0$. Curved twistor spaces can be thought of as locally the same as flat twistor space $\mathbb{PT} = \mathbb{CP}^3$ but with the complex structure deformed. A useful way to describe the deformed complex structure [88] is to modify the Dolbeault operator $\overline{\partial}$. Given homogeneous coordinates Z^α, $\alpha = 0,1,2,3$, on some patch, we can define a background, undeformed complex structure by the operators

$$\partial = \mathrm{d}Z^\alpha \frac{\partial}{\partial Z^\alpha}\,, \quad \overline{\partial} = \mathrm{d}\overline{Z}^{\overline{\alpha}} \frac{\partial}{\partial \overline{Z}^{\overline{\alpha}}}\,, \tag{5.31}$$

which naturally defines the splitting of the complex tangent space into $(1,0)$ and $(0,1)$ parts. We can deform the complex structure by adding elements of $T^{(1,0)}$ to $T^{(0,1)}$, that is we define a new, deformed Dolbeault operator

$$\overline{\partial}_f = \overline{\partial} + f\,, \tag{5.32}$$

where f is a $(1,0)$-vector valued $(0,1)$-form, $f = f^{\alpha}_{\overline{\alpha}}\,\mathrm{d}\overline{Z}^{\overline{\alpha}} \otimes \frac{\partial}{\partial Z^\alpha}$.

The integrability condition for this new complex structure is the Kodaira-Spencer

equation:

$$\overline{\partial}_f^2 \equiv N^\alpha(\mathcal{Z})\partial_\alpha = (\overline{\partial} f^\alpha + f^\beta \wedge \partial_\beta f^\alpha)\partial_\alpha = 0 \ . \tag{5.33}$$

It is often convenient to think of our fields as living on the non-projective twistor space, $\mathscr{T}$, in which case we must restrict to deformations which preserve the Euler vector $E = Z^\alpha \partial_\alpha$, in the sense that $\pounds_f E = -\pounds_E f = 0$, or equivalently, f^α is of homogeneity degree one in Z^α and we must make the identification of the vector field f under shifts proportional to the Euler vector field

$$f \to f + \Lambda E \ , \tag{5.34}$$

where Λ is a $(0,1)$-form of homogeneity degree zero in Z^α. We can fix this gauge invariance by demanding that the deformation preserves the volume form $\mathrm{d}\Omega$ on $\mathscr{T}$, which imposes $\partial_\alpha f^\alpha = 0$.

It is possible to write an action for which the corresponding equations of motion imply the integrability of the complex structure [34]. This can be done in a coordinate independent fashion [17] or by introducing a particular background and using explicit coordinates [36]. The action functional on twistor space is given in terms of a Lagrange multiplier field $g \in \Omega^{3,0}(\mathbb{P}\mathscr{T}, \Omega^{1,1}(\mathbb{P}\mathscr{T}))$, imposing the constraint corresponding to integrability of the deformed complex structure.

Introducing an appropriate basis (or working in abstract index notation) such that the components fields are f^α and $g^\Omega_\alpha = g_\alpha \wedge \Omega$, where Ω is the holomorphic volume form, we can write the action as

$$S_{\text{s.d.}}[f,g] = \int_{\mathbb{P}\mathscr{T}} g^\Omega_\alpha \wedge N^\alpha \, . \tag{5.35}$$

The equations of motion following from the action are the integrability conditions for the deformed Dolbeault operator and

$$\overline{\partial}_f g^\Omega_\alpha \equiv \overline{\partial} g^\Omega_\alpha + g^\Omega_\beta \wedge \partial_\alpha f^\beta + \partial_\beta (g^\Omega_\alpha \wedge f^\beta) = 0 \ . \tag{5.36}$$

As the field g must be defined on $\mathbb{P}\mathscr{T}$, we additionally have the constraint $\iota(E)g = 0$, and with this constraint the action has the gauge invariance (5.34).

In general the field g describes an anti-self-dual excitation moving in a self-dual background. Treating the fields as small deformations of the undeformed twistor space, we can focus on the linearised theory. We now have $f^\alpha \in \Omega^{0,1}(\mathbb{PT}, \mathcal{O}(1))$, and $g_\alpha \in \Omega^{0,1}(\mathbb{PT}, \mathcal{O}(-5))$ i.e. a $(0,1)$-form of homogeneity -5. The equations of motion reduce to

$$\overline{\partial} f^\alpha = 0 \quad \text{and} \quad \overline{\partial} g_\alpha = 0 \ . \tag{5.37}$$

Additionally, the gauge invariance of the action demonstrated in [17] becomes $g_\alpha \to g_\alpha + \overline{\partial}\chi$ where χ is a section of $\mathcal{O}(-4)$. Consequently, on-shell, we can interpret g_α as defining an element in the $\overline{\partial}$-cohomology group $H^{0,1}(\mathbb{PT}, \mathcal{O}(-5))$, and so by the Penrose transform g_α corresponds to a helicity -2 particle. To include the self-interactions of the anti-self-dual fields and to consider amplitudes with more than a single helicity -2 particle one must include additional terms in the action [17]. We will leave this discussion for the higher spin case in a later chapter.

Chapter 6

Self-Dual Higher Spin Theories in Twistor Space

We wish to generalise the construction we reviewed in the last chapter to higher spins by considering deformations corresponding to higher-rank symmetric tensors. Starting with the spin-three case we propose Maurer-Cartan-like equations of motion for the deformations, and by introducing an appropriate Lagrange multiplier field we define a twistor space action describing the self-dual sector. The twistor fields can be defined for a general curved twistor space and so define higher-spin fields in an arbitrary self-dual geometry. However to understand the space-time interpretation we focus on the flat twistor space case, in particular showing that the twistor fields give rise, via the Penrose transform, to space-time fields satisfying the zero-rest-mass equations for a spin-three field. To make the identification between twistor fields and space-time fields clearer we construct the space-time action for the self-dual sector of the higher spin theory at the quadratic level. Finally we demonstrate that, after accounting for the gauge freedom, the on-shell spectrum matches with that of [9].

As in the spin-two case, the on-shell representation of the Poincaré algebra is not diagonalisable which is a manifestation of the failure of the theory to be unitary. Maldacena [13] has argued that conformal gravity with appropriate boundary

conditions is classically equivalent to Einstein gravity with a non-zero cosmological constant, $\Lambda \neq 0$. This implies [35] that Einstein gravity amplitudes can be calculated in conformal gravity by restricting to the asymptotic states of Einstein gravity and accounting for the appropriate powers of the cosmological constant. Adamo and Mason [43, 36] studied supergravity scattering amplitudes by performing such a truncation of conformal gravity in the twistor space description and were able to show that the resulting determinant formulae was directly related to Hodges' formula [44]. We perform an analogous truncation on the higher-spin action to identify a "unitary" sub-sector of CHS theory. At the quadratic level this sector has the usual Fronsdal [45, 46, 47] spectrum of massless higher spins and we show in chapter 8 that the $\overline{\text{MHV}}$ three-point amplitude agrees with the constraints from Poincaré invariance [14].

6.1 Spin-Three Fields

We start our study of the higher spin theory with the case of spin-three fields. We again pick some background twistor space $\mathscr{T}$ and its projective version $\mathbb{P}\mathscr{T}$. We will often take $\mathscr{T}$ to be flat twistor space $\mathbb{T}$, however, if we wish to consider higher-spin fields on a general self-dual background, it will be the corresponding curved twistor space. We will take $\mathcal{Z}^\alpha$ to be homogeneous coordinates on the background twistor space, and denote them Z^α if it is $\mathbb{T}$. The complex deformations corresponding to the spin-two case discussed in the introduction are $(0,1)$-forms taking values in the $(1,0)$ part of the tangent space, that is they are elements of $\Omega^{0,1}(\mathbb{P}\mathscr{T}, T^{1,0}(\mathbb{P}\mathscr{T}))$. We now consider deformations corresponding to $(0,1)$-forms taking values in symmetric products of the tangent space, that is they are elements of $\Omega^{0,1}(\mathbb{P}\mathscr{T}, \text{Sym}^2(T^{1,0}(\mathbb{P}\mathscr{T})))$,

$$f^{(3)} = f^{\alpha\beta}\partial_\alpha \otimes \partial_\beta \,. \tag{6.1}$$

This twistor field will be interpreted in space-time as a massless spin-three field and we will refer it as such. As it is in fact defined on projective space, rather than the full twistor space, we must make the identification analogous to (5.34) for the field f

$$f^{(3)} \to f^{(3)} + E \otimes \Lambda + \Lambda \otimes E \,, \tag{6.2}$$

where $\Lambda \in \Omega^{0,1}(\mathbb{P}\mathscr{T}, T^{1,0}(\mathbb{P}\mathscr{T}))$ is now vector-valued, and E is again the background Euler vector field. More particularly, we consider an operator

$$\overline{\partial}_f : \Omega^{p,q}(\mathbb{P}\mathscr{T}) \to \Omega^{p,q+1}(\mathbb{P}\mathscr{T}) \tag{6.3}$$

acting on forms $k \in \Omega^{0,1}(\mathbb{P}\mathscr{T})$ as

$$k \mapsto \overline{\partial} k + f^{\alpha\beta} \wedge \partial_\alpha \partial_\beta k \in \Omega^{0,2}(\mathbb{P}\mathscr{T}) \,. \tag{6.4}$$

Such an operator can be naturally viewed as a deformation of the Dolbeault operator, however, due to the presence of higher derivatives, it is not a derivation.

We will impose the condition that the deformation f satisfies the equation

$$N^{(3)} \equiv \left(\overline{\partial} f^{\alpha\beta} + f^{\gamma\delta} \wedge \partial_\gamma \partial_\delta f^{\alpha\beta}\right) \partial_\alpha \partial_\beta = 0 \,. \tag{6.5}$$

This condition does not imply $\overline{\partial}_f^{\,2} = 0$; however, as we will see later, this failure can be compensated by including an infinite tower of higher-spin fields. The necessity of such an infinite tower is unsurprising given known results in space-time formulations. However, we will postpone a discussion of this, and here only consider the self-interactions of the spin-three fields.

As for the spin-two case, to define an action functional we introduce a corresponding Lagrange multiplier field g which takes values in the dual space, i.e.

$$g^{(3)} \in \Omega^{0,1}(\mathbb{P}\mathscr{T}, \mathrm{Sym}^2(T^{*,1,0}(\mathbb{P}\mathscr{T}))) \otimes \Omega \,. \tag{6.6}$$

As before, Ω is a section of $\Omega^{3,0}(\mathbb{PT}) \otimes \mathcal{O}(4)$, and we thus have

$$g^{(3)} = (g_{\alpha\beta} \wedge \Omega) \otimes \mathrm{d}\mathcal{Z}^\alpha \otimes \mathrm{d}\mathcal{Z}^\beta \,, \tag{6.7}$$

where $g_{\alpha\beta} = g_{(\alpha\beta)} \in \Omega^{0,1}(\mathbb{P}\mathscr{T}, \mathcal{O}(-6))$. To ensure the appropriate gauge invariance we impose the constraint $\mathcal{Z}^\alpha g_{\alpha\beta} = 0$. The twistor space action for the self-dual sector is proposed to be the obvious analogue of the spin-two case

$$S_{\text{s.d.}}\left[f^{(3)}, g^{(3)}\right] = \int_{\mathbb{P}\mathscr{T}} \Omega \wedge g_{\alpha\beta} \wedge \left(\overline{\partial} f^{\alpha\beta} + f^{\gamma\delta} \wedge \partial_\gamma \partial_\delta f^{\alpha\beta}\right) , \tag{6.8}$$

from which follows that $N^{(3),\alpha\beta} = 0$ as required, and also

$$\overline{\partial} g^\Omega_{\alpha\beta} + g^\Omega_{\gamma\delta} \wedge \partial_\alpha \partial_\beta f^{\gamma\delta} - \partial_\gamma \partial_\delta (g^\Omega_{\alpha\beta} \wedge f^{\gamma\delta}) = 0 \,, \tag{6.9}$$

where $g^\Omega_{\gamma\delta} = g_{\gamma\delta} \wedge \Omega$. In local holomorphic coordinates, $\mathcal{Z}^\alpha$, we will often write $\Omega = D^3\mathcal{Z}$, where $D^3\mathcal{Z}$ is the usual weight four holomorphic volume form.

We will next be interested in understanding the spectrum of this action corresponding to its quadratic approximation about a given background. We thus focus on the linearised equations of motion, which are straightforwardly given by

$$\overline{\partial} f^{\alpha\beta} = 0 \qquad \text{and} \qquad \overline{\partial} g_{\alpha\beta} = 0 \,. \tag{6.10}$$

Also at the linearised level, the action has the additional gauge invariance $g_{\alpha\beta} \to g_{\alpha\beta} + \overline{\partial}\chi_{\alpha\beta}$. Consequently, we can think of $g_{\alpha\beta}$ as defining an element in the Dolbeault cohomology $H^{0,1}(\mathbb{P}\mathscr{T}, \mathcal{O}(-6))$, satisfying $\mathcal{Z}^\alpha g_{\alpha\beta} = 0$.

6.1.1 Space-Time Interpretation

In the twistor description of conformal gravity we start with two tensor-valued forms, $f^\alpha(\mathcal{Z})$ and $g_\alpha(\mathcal{Z})$, defined on $\mathbb{P}\mathscr{T}$ of homogeneity degree 1 and -5 respectively. These fields correspond, via the Penrose transform, to the space-time anti-self-dual Weyl spinor Ψ_{ABCD}, which vanishes for self-dual backgrounds, and the Lagrange multiplier field G_{ABCD}, which satisfies a second order equation of motion [17]. In the higher-rank spin-three generalisation above, we have the fields $f^{\alpha_1\alpha_2}(\mathcal{Z})$, of homogeneity degree 2, and $g_{\alpha_1\alpha_2}(\mathcal{Z})$, of homogeneity degree -6. The zero-rest-mass

equations on space-time can be found by using the Penrose transform for such tensors as described earlier or in [87], or see [18] for a recent review.

The components of the tensors corresponding to the spin-two and spin-three fields with respect to the frame (or its dual) are

$$f^{\underline{\alpha}}\,,\ g_{\underline{\alpha}}\,,\ f^{\underline{\alpha}_1\underline{\alpha}_2}\,,\ g_{\underline{\alpha}_1\underline{\alpha}_2}\,,\ \text{etc}\ , \tag{6.11}$$

and for each component we can perform the Penrose transform. Considering the spin-three case we have a $(0,1)$-form-valued tensor $g_{\underline{\alpha}_1\underline{\alpha}_2}$ whose components have homogeneity -6. This corresponds to the case in [87] of a tensor with homogeneity $m-2$, with $m<0$, and so in the Penrose transform when performing the integration over the projective complex line $\Xi\simeq\mathbb{CP}^1$, corresponding to the space-time point x,

$$G_{(\underline{\alpha}_1\underline{\alpha}_2)(B_1\ldots B_4)}=\frac{1}{2\pi i}\int_{\Xi}\lambda_{B_1}\cdots\lambda_{B_4}\,g_{\underline{\alpha}_1\underline{\alpha}_2}\Big|_{\Xi}\wedge D\lambda\ , \tag{6.12}$$

we include four factors of λ_A, the homogeneous coordinates on Ξ, to compensate for the weight of $g_{\underline{\alpha}_1\underline{\alpha}_2}$ and holomorphic measure $D\lambda=\langle\lambda\,\mathrm{d}\lambda\rangle$ on Ξ. The resulting space-time field $G_{(\underline{\alpha}_1\underline{\alpha}_2)(B_1\ldots B_4)}$ satisfies the zero-rest-mass equation

$$\nabla^{B_1'B_1}G_{(\underline{\alpha}_1\underline{\alpha}_2)(B_1\ldots B_4)}=0\ . \tag{6.13}$$

Due to the specific choice of frame we can think of the twistor space tensor indices as local twistor indices. In particular, the covariant derivative acts on the tensors with twistor indices by the local twistor connection, which for flat space gives

$$\begin{aligned}
\nabla^{B_1'B_1}G_{A_1'A_2'B_1\ldots B_4}&=0\ ,\\
\nabla^{B_1'B_1}G^{A_1}{}_{A_2'B_1\ldots B_4}-i\epsilon^{B_1A_1}G^{B_1'}{}_{A_2'B_1\ldots B_4}&=0\ ,\\
\nabla^{B_1'B_1}G^{A_2}{}_{A_1'B_1\ldots B_4}-i\epsilon^{B_1A_2}G^{B_1'}{}_{A_1'B_1\ldots B_4}&=0\ ,\\
\nabla^{B_1'B_1}G^{A_1A_2}{}_{B_1\ldots B_4}-i\epsilon^{B_1A_1}G^{B_1'A_2}{}_{B_1\ldots B_4}-i\epsilon^{B_1A_2}G^{B_1'A_1}{}_{B_1\ldots B_4}&=0\ .
\end{aligned} \tag{6.14}$$

We can use these to determine all the components of $G_{(\underline{\alpha}_1\underline{\alpha}_2)(B_1\ldots B_4)}$ in terms of the derivatives of the fields $G^{A_1A_2}{}_{B_1\ldots B_4}$. Furthermore we must impose the twistor space

constraint on the Lagrange multiplier field $Z^{\underline{\alpha}_1} g_{\underline{\alpha}_1 \underline{\alpha}_2} = 0$, which corresponds to acting with a helicity lowering operator and imposes the space-time condition

$$G^{A_1}{}_{\underline{\alpha}_2 A_1 \dots B_4} = 0 \ , \tag{6.15}$$

namely that the field $G_{A_1 A_2 B_1 \dots B_4}$ is symmetric in all its indices. Using this and (6.14) we find that the anti-self-dual field satisfies the third-order equations

$$\nabla^{A_1' A_1} \nabla^{A_2' A_2} \nabla^{A_3' A_3} G_{A_1 A_2 A_3 A_4 A_5 A_6} = 0 \ , \tag{6.16}$$

which is to say that $G_{A_1 A_2 A_3 A_4 A_5 A_6}$ satisfies the zero-rest-mass equation for a spin-three field.

As the tensor field $f^{\underline{\alpha}_1 \underline{\alpha}_2}$ has homogeneity $m - 2 = 2$, $m > 0$, in a general background it transforms into a potential. This is exactly analogous to the conformal gravity spin-two case where the field f^{α} of homogeneity one describes the chiral Weyl spinor Ψ_{ABCD}, see [89]. In the linearised spin-two theory the relationship with the metric fluctuation, $h_{ABC'D'}$, is given by

$$\Psi_{ABCD} = \nabla^{C'}_{(C} \nabla^{D'}_{D} h_{AB)C'D'} \ . \tag{6.17}$$

This arises by taking the Penrose transform of f^{α} which gives the potential $\Sigma^{\underline{\alpha}}{}_{B_1 B_2 C'}$, which can be decomposed as

$$\Sigma^{\underline{\alpha}}{}_{B_1 B_2 C'} = \left(\widetilde{\Sigma}_{(A B_1 B_2) C'}, i h_{B_1 B_2 C'}^{\ \ \ A'} \right) \ , \tag{6.18}$$

where the symmetry in the unprimed indices follows from using the gauge symmetry of $f^{\alpha} \to f^{\alpha} + Z^{\alpha} \Lambda$. The condition that the solution be self-dual, that is $\Psi_{ABCD} = 0$, follows from the condition on the potential

$$\nabla^{C'}{}_{(C} \Sigma^{\underline{\alpha}}{}_{B_1 B_2) C'} = 0 \ , \tag{6.19}$$

and by using the local twistor connection as above. In the spin-three case we will have a higher-rank potential

$$\Sigma^{\underline{\alpha}_1 \underline{\alpha}_2}{}_{B_1 B_2 B_3 C'} \ , \tag{6.20}$$

which will give rise to fields analogous to the metric fluctuations $\phi_{A_1A_2A_3}{}^{B_1'B_2'B_3'}$, which satisfy a condition $\Psi_{A_1A_2A_3B_1B_2B_3} = 0$, where $\Psi_{A_1A_2A_3B_1B_2B_3}$ is the spin-three analogue of the Weyl spinor with higher numbers of derivatives

$$\Psi_{A_1A_2A_3B_1B_2B_3} = \nabla^{A_1'}_{(A_1}\nabla^{A_2'}_{A_2}\nabla^{A_3'}_{A_3}\phi_{B_1B_2B_3)A_1'A_2'A_3'}\,. \tag{6.21}$$

6.1.2 Minkowski Space-Time Spectrum

Given the matching of the equations of motion, it should be unsurprising that the counting of the on-shell degrees of freedom in both the twistor and space-time descriptions of CHS theory also agrees. Nonetheless it provides a useful check and provides further insight into the twistor description, particularly the appearance of the ghost degrees of freedom, which result in the theory failing to be unitary. To this end we wish to understand the flat-space spectrum ①, corresponding to the twistor fields $f^{\alpha\beta}$ of homogeneity $n = 2$ and $g_{\alpha\beta}$ of homogeneity $n = -6$, while taking into account the gauge invariance and constraint, respectively

$$f^{\alpha\beta} \to f^{\alpha\beta} + \mathcal{Z}^{(\alpha}\Lambda^{\beta)} \quad \text{and} \quad \mathcal{Z}^{\alpha}g_{\alpha\beta} = 0\,. \tag{6.22}$$

In the standard application of the Penrose transform, a function of the homogeneous coordinates $\mathcal{Z}^{\alpha}$ with homogeneity degree n corresponds to a massless state of helicity $s = 1 + n/2$. In the case at hand we must further take into account the indices α, β etc. One way to do this is, following [34], to form invariant functions using the flat space twistor coordinates λ_A and $\mu^{A'}$, which then correspond to definite helicity states. For example there are three homogeneity four functions or $s = 3$ states ②

$$\lambda_A\lambda_B f^{AB}\,, \qquad \mu^{A'}\mu^{B'}f_{A'B'}\,, \qquad \lambda_A\mu^{A'}f^{A}{}_{B'}\,. \tag{6.23}$$

① A conjecture for the CHS spectrum around (A)dS space was made in [90] and computed in [91]. It would be interesting to directly compare the twistor results with these calculations.

② As we are considering flat space we can raise and lower spinor indices using ϵ_{AB} and $\epsilon_{A'B'}$ as required.

Additionally, we may form invariants using derivatives, $\partial_A = \frac{\partial}{\partial \lambda^A}$ and $\partial^{A'} = \frac{\partial}{\partial \mu_{A'}}$, so there are four homogeneity two functions or $s = 2$ states,

$$\lambda_A \partial_B f^{AB}\,, \qquad \mu^{A'} \partial^{B'} f_{A'B'}\,, \qquad \lambda_A \partial^{A'} f^A{}_{B'}\,, \qquad \mu^{A'} \partial_A f^A{}_{B'}\,, \tag{6.24}$$

and finally there are three $s = 1$ states

$$\partial_A \partial_B f^{AB}\,, \qquad \partial^{A'} \partial^{B'} f_{A'B'}\,, \qquad \partial_A \partial^{A'} f^A{}_{B'}\,. \tag{6.25}$$

Hence there are ten on-shell degrees of freedom, however some of these are simply gauge and can be removed by a transformation using Λ^α. In particular we can form the invariants

$$\lambda_A \Lambda^A\,, \qquad \mu^{A'} \Lambda_{A'}\,, \qquad \partial_A \Lambda^A\,, \qquad \partial^{A'} \Lambda_{A'} \tag{6.26}$$

and so remove four degrees of freedom. Specifically, we can use this freedom to set $\partial_\alpha f^{\alpha\beta} = 0$ and so remove two states with $s = 2$ and two corresponding to $s = 1$. Hence we find a total of six on-shell states from the tensor field $f^{\alpha\beta}$. We can repeat this argument for $g_{\alpha\beta}$, for which we have ten invariants $\lambda^A \lambda^B g_{AB}$, $\mu_{A'} \mu_{B'} g^{A'B'}$, $\lambda^A \mu_{A'} g_A{}^{B'}$, $\lambda^A \partial^B g_{AB}$, $\mu_{A'} \partial_{B'} g^{A'B'}$, $\lambda^A \partial_{A'} g_A{}^{B'}$, $\mu_{A'} \partial^A g_A{}^{B'}$, $\partial^A \partial^B g_{AB}$, $\partial_{A'} \partial_{B'} g^{A'B'}$, $\partial^A \partial_{A'} g_A{}^{B'}$, which correspond to three states of $s = -3$, four of $s = -2$ and three of $s = -1$. Four degrees of freedom are removed by the constraint $\mathcal{Z}^\alpha g_{\alpha\beta} = 0$, of which two are $s = -2$ and two are $s = -1$. Hence in total we find twelve on-shell degrees of freedom.

We could alternatively have made use of the duality between $f^{\alpha\beta}$ and $g_{\alpha\beta}$, *c.f.* [34], following from the Fourier-like transform to the dual twistor space described by coordinates $\mathcal{W}_\alpha$:

$$\tilde{g}^{\alpha\beta}(\mathcal{W}) = \int D^3\mathcal{Z}\, \mathrm{d}t\, t^6\, f^{\alpha\beta}(\mathcal{Z}) \exp(t\mathcal{W} \cdot \mathcal{Z})\,. \tag{6.27}$$

The homogeneity in $\mathcal{W}$ of $n = -6$ for $\tilde{g}^{\alpha\beta}$ follows immediately from the homogeneity in $\mathcal{Z}$ of $n = 2$ for $f^{\alpha\beta}$ and the weight, 4, of the measure. Similarly, the gauge

invariance of $f^{\alpha\beta}$ implies $\tilde{g}^{\alpha\beta} \to \tilde{g}^{\alpha\beta} + \partial^{(\alpha}\Lambda^{\beta)}$ which can be fixed by imposing the constraint (6.22). Thus we expect to find the same number of space-time states described by $\tilde{g}^{\alpha\beta}$ as by $f^{\alpha\beta}$, namely six.

6.2 Conformal Higher Spin Twistor Action

There are well-established reasons for believing that there is no consistent interacting theory for a single $s > 2$ field, and in the twistor theory we can see such difficulties emerge when we attempt to give a geometric interpretation to our equations of motion. As we have mentioned, beyond the spin-two case our equation of motion does not imply that the operator $\overline{\partial}_f$ is integrable. This failure can be see quite easily in the spin-three case where

$$\overline{\partial}_f^2 = \left(\overline{\partial} f^{\alpha\beta} + f^{\gamma\delta} \wedge \partial_\gamma \partial_\delta f^{\alpha\beta}\right) \partial_\alpha \partial_\beta + 2 f^{\gamma\delta} \wedge \partial_\gamma f^{\alpha\beta} \partial_\delta \partial_\alpha \partial_\beta \ , \tag{6.28}$$

with the last term remaining even after imposing the spin-three equations of motion previously used. However, the structure of this term involving three derivatives appears in the spin-4 equation, and so one can include this additional term as a source term for spin-4 fields. Of course the spin-4 fields will generate additional terms which will not be cancelled but which in turn source spin-5 fields and so on.

It is not difficult to also include spin-one fields in our description, though they have to be treated somewhat differently as they do not take values in some product of the tangent bundle but in the Lie algebra $\mathfrak{g}$ corresponding to a given Lie (gauge) group G, i.e. $f^{\alpha_\emptyset} \equiv A^+ \in \Omega^{0,1}(\mathbb{P}\mathscr{T}, \mathcal{O}(0) \otimes \mathfrak{g})$ and $g_{\alpha_\emptyset} \equiv A^- \in \Omega^{0,1}(\mathbb{P}\mathscr{T}, \mathcal{O}(-4) \otimes \mathfrak{g})$. To avoid unnecessary clutter like taking traces when necessary, we will just consider the abelian case $G = U(1)$, however a generalisation is straightforward. Thus, whenever we write $\mathfrak{g}$ we just consider $\mathfrak{u}(1)$ for simplicity. We can now define the operator

$$\overline{\partial}_f = \overline{\partial} + \sum_{|J|=0}^{\infty} f^{\beta_J} \partial_{\beta_J} \ . \tag{6.29}$$

We can expand the condition $\overline{\partial}_f^2 = 0$ in powers of derivatives ∂_α and impose the vanishing component by component. Focusing on spin two we have

$$\overline{\partial} f^\alpha + \sum_{|J|=0}^{\infty} f^{\beta_J} \wedge \partial_{\beta_J} f^\alpha = 0 \ , \tag{6.30}$$

where we see the coupling of the spin-two field to all higher-spin fields. In this equation it is of course consistent to set all the f^{α_I} for $|I| > 1$ as well as $|I| = 0$ equal to zero and so recover the spin-two equation of motion for pure conformal gravity. For the spin-three equation of motion we now have

$$\overline{\partial} f^{\alpha_1\alpha_2} + \sum_{|J|=0}^{\infty} f^{\beta_J} \wedge \partial_{\beta_J} f^{\alpha_1\alpha_2} + \sum_{|J|=0}^{\infty} (|J|+1) f^{(\alpha_1\beta_J} \wedge \partial_{\beta_J} f^{\alpha_2)} = 0 \ . \tag{6.31}$$

Because $f^{(\alpha_1} \wedge f^{\alpha_2)} = 0$, the spin-three fields are not sourced by purely spin-two fields, and so truncating to just spin-two is consistent as is expected. The spin-four equations of motion however have spin-two and spin-three source terms,

$$\begin{aligned}\overline{\partial} f^{\alpha_1\alpha_2\alpha_3} + \sum_{|J|=0}^{\infty} f^{\beta_J} \wedge \partial_{\beta_J} f^{\alpha_1\alpha_2\alpha_3} + \sum_{|J|=0}^{\infty} (|J|+1) f^{(\alpha_1\beta_J} \wedge \partial_{\beta_J} f^{\alpha_2\alpha_3)} \\ + \sum_{|J|=0}^{\infty} (|J|+1)(|J|+2) f^{(\alpha_1\alpha_2\beta_J} \wedge \partial_{\beta_J} f^{\alpha_3)} = 0 \ .\end{aligned} \tag{6.32}$$

This means we cannot truncate to just the spin-three sector and the appearance of the spin-four fields is necessary for consistency. For generic spin, and using the multi-index notation, we have the equation

$$N^{\alpha_I} \equiv \overline{\partial} f^{\alpha_I} + \sum_{|J|=0}^{\infty} \sum_{|L|=0}^{|I|} C_{|J||L|} f^{\beta_J(\alpha_L} \wedge \partial_{\beta_J} f^{\alpha_{I-L})} = 0 \ , \tag{6.33}$$

where the multi-index $I - J$ corresponds to the complement of J in I, and the coefficients $C_{|K||J|} = \binom{|K|+|J|}{|J|}$. Here we can see the source terms due to lower-spin fields in higher-spin equations of motion. Hence we see the need for an infinite number of fields, one of each spin, interacting non-trivially with one another.

As before, we can introduce Lagrange multiplier fields $g_{\alpha_I} \in \Omega^{0,1}(\mathbb{P}\mathscr{T}, \mathcal{O}(-4-|I|))$ to impose these conditions to write an action for the self-dual sector

$$S_{\text{s.d.}}\left[f^{(\bullet)}, g^{(\bullet)}\right] = \int_{\mathbb{P}\mathscr{T}} \Omega \wedge \sum_{|I|=0}^{\infty} \left(g_{\alpha_I} \wedge N^{\alpha_I}\right) . \tag{6.34}$$

We also write $g^{\Omega}_{\alpha_I} = g_{\alpha_I} \wedge \Omega$. The fields g_{α_I} can be interpreted as anti-self-dual fields propagating on a self-dual background, and their equation of motion are obtained through variation principle as

$$\overline{\partial} g^{\Omega}_{\alpha_I} + \sum_{|K|=0}^{\infty} \sum_{L=0}^{|I|} C_{|I-L||L|} \partial_{(\alpha_{I-L}} f^{\gamma_K} \wedge g^{\Omega}_{\alpha_L)\gamma_K} \\ - \sum_{|J|,|K|=0}^{\infty} (-1)^{|J|} C_{|J||K|} f^{\beta_J \gamma_K} \wedge \partial_{\beta_J} g^{\Omega}_{\alpha_I \gamma_K} = 0 \,. \tag{6.35}$$

Because of their additional gauge freedom on the linearized level

$$g_{\alpha\beta_I} \to g_{\alpha\beta_I} + \partial_{(\alpha}\gamma_{\beta_I)} + \overline{\partial}_f \chi_{\alpha\beta_I} \,,$$

they can be thought of as cohomology classes $g_{\alpha_I} \in H^{0,1}(\mathbb{P}\mathscr{T}, \mathcal{O}(-4-|I|))$.

The spectrum of this theory coincides with the spectrum of the conformal higher spin theory of Fradkin/Tseytlin, and it can be shown that the self-dual twistor action is equivalent to the self-dual CHS space-time action.

At the linearised level the analysis is as in the previous sections as the individual spins decouple, moreover one can focus on the individual spins to calculate the self-interaction three-point functions, and the results from previous sections will still hold. Nonetheless, this action is significantly more involved, and without a proper geometric understanding of the deformation much of the structure remains unclear.

One natural approach is to attempt to interpret the deformation as defining a new complex structure. For example we can consider the space spanned by the deformed vectors $e_\alpha = \partial_\alpha + \sum_{|I|=0}^{\infty} \overline{f}^{\overline{\alpha}_I}_{\alpha} \overline{\partial}_{\overline{\alpha}_I}$ and ask if it is closed under commutation. That is given $V = V^\alpha e_\alpha$ and $W = W^\alpha e_\alpha$ we calculate

$$[V,W] = \left(V^\alpha (\partial^{\overline{f}}_\alpha W^\beta) - W^\alpha (\partial^{\overline{f}}_\alpha V^\beta)\right) e_\beta \\ + \left(\sum_{|I|=0}^{\infty} \sum_{|J|=0}^{|I|} C_{|I||J|} \left(V^\alpha \overline{\partial}_{\overline{\gamma}_J} W^\beta - W^\alpha \overline{\partial}_{\overline{\gamma}_J} V^\beta \right) \overline{f}^{\overline{\gamma}_J \overline{\gamma}_{I-J}}_{\alpha} \overline{\partial}_{\overline{\gamma}_{I-J}} \right) e_\beta \,. \tag{6.36}$$

The presence of the second line would seem to require at the very least a significant generalisation of the usual notions. In particular the appearance of infinite numbers

of derivatives suggests a non-local formulation. This can also be seen if we consider a C^∞-function on projective twistor space $\phi(\mathcal{Z})$ and attempt to define the notion of holomorphicity with respect to a deformed complex structure by writing

$$\overline{\partial}_f \phi(\mathcal{Z}) = \overline{\partial}\phi(\mathcal{Z}) + \sum_{|I|=0}^{\infty} f^{\alpha_I} \partial_{\alpha_I} \phi(\mathcal{Z}) = 0\,. \tag{6.37}$$

This condition can be written in a more suggestive notation as

$$\overline{\partial}\phi(\mathcal{Z}) + f \cdot \phi(\mathcal{Z}) = 0 \tag{6.38}$$

where

$$f \cdot \phi(\mathcal{Z}) = \int_{\mathbb{PT}} D^3\mathcal{Z}'\, f(\mathcal{Z}, \mathcal{Z}')\phi(\mathcal{Z}') \tag{6.39}$$

with $f(\mathcal{Z}, \mathcal{Z}') = \sum_{|I|=0}^{\infty} f^{\alpha_I}(\mathcal{Z})\, \partial_{\alpha_I}^{(\mathcal{Z})} \overline{\delta}^3(\mathcal{Z}, \mathcal{Z}')$. Here $\overline{\delta}^3(\mathcal{Z}, \mathcal{Z}')$ is the projective delta-function defined on the background projective twistor space such that

$$\phi(\mathcal{Z}) = \int_{\mathbb{PT}} D^3\mathcal{Z}' \wedge \overline{\delta}^3(\mathcal{Z}, \mathcal{Z}')\phi(\mathcal{Z}')\,. \tag{6.40}$$

Such bi-local expressions are common in higher spin theories and have been interpreted in terms of infinite jet bundles [92]. We will make contact with this formulation in the next chapter.

6.2.1 Linearised Spin-s Fields

At the linearised level the equations of motion are simply $\overline{\partial} f^{\alpha_I} = 0$ and $\overline{\partial} g_{\alpha_I} = 0$. Using the Penrose transform, one could find the space-time fields corresponding to the twistors fields f^{α_I} and g_{α_I}, and so show that they satisfy the massless spin-s wave equation in exactly the same fashion as in the spin-two and spin-three cases. Instead, as it also allows us to discuss the truncation to the unitary sector, we will briefly consider the on-shell spectrum of the theory.

Minkowski space-time spectrum: We can repeat the analysis of section 6.1.2 for this general case. The twistor fields $f^{\alpha_1\ldots\alpha_n}(\mathcal{Z})$ have homogeneity n, and the fields $g_{\alpha_1\ldots\alpha_n}(\mathcal{Z})$ are of homogeneity $-n-4$, while the gauge invariance and constraint are now

$$f^{\alpha_1\ldots\alpha_n} \to f^{\alpha_1\ldots\alpha_n} + \mathcal{Z}^{(\alpha_1}\Lambda^{\alpha_2\ldots\alpha_n)} \quad \text{and} \quad \mathcal{Z}^{\alpha_1} g_{\alpha_1\ldots\alpha_n} = 0\,. \tag{6.41}$$

Following the prescription in section 6.1.2 for the tensor $f^{\alpha_1\ldots\alpha_n}(\mathcal{Z})$, we can form

$$\sum_{\ell=0}^{n}(n+1-\ell)(\ell+1) = \frac{1}{6}(n+1)(n+2)(n+3) \tag{6.42}$$

invariants, and so after removing the gauge degrees of freedom, for which we repeat the counting above but with n replaced by $n-1$, we have $\frac{1}{2}(n+2)(n+1)$ on-shell degrees of freedom. As the highest homogeneity is $2n$, the highest helicity state is $s = 1 + (2n)/2 = n+1$, and so we have $\frac{1}{2}s(s+1)$ degrees of freedom. Using the duality argument discussed above in the spin-three case, we find exactly the same number of on-shell states from $g_{\alpha_1\ldots\alpha_n}(Z)$ but with the opposite helicities, and so the total number of on-shell degrees of freedom is

$$\nu_s = s(s+1)\ . \tag{6.43}$$

For $s = 1$ we find the usual number two on-shell vector states while for $s = 2$ we find the six on-shell degrees of freedom of Weyl gravity [67]. More generally, the formula matches with the number of on-shell states in the conformal higher spin theory described by Fradkin and Tseytlin [9].

6.3 Unitary Truncation

As is well known, conformal higher spin theories are not unitary. One symptom of this is the fact that the representation of certain Poincaré generators on on-shell states is not diagonalisable. For example, the generator of space-time translations

on twistor space is given by the vector field

$$P_{AA'} = \lambda_A \frac{\partial}{\partial \mu^{A'}} \ . \tag{6.44}$$

Its action on twistor space contravariant tensors $f^{\alpha_1 \dots \alpha_n}$ can be calculated straightforwardly from the Lie derivative

$$\pounds_{P_{AA'}} f^{\alpha_1 \dots} = \lambda_A \frac{\partial}{\partial \mu^{A'}} f^{\alpha_1 \dots} - \sum_{i=1}^{n} \delta^{\alpha_i}_{A'} f^{\alpha_1 \dots \alpha_{i-1}}{}_{A}{}^{\alpha_{i+1} \dots \alpha_n} \ , \tag{6.45}$$

which can be seen to be non-diagonalisable. For example, in the case of conformal gravity it was shown, [34], that the generator $P_{AA'}$ acting on the pair $\begin{pmatrix} f_B \\ f^{B'} \end{pmatrix}$ is represented by the non-diagonalisable matrix

$$(P_{AA'}) = \begin{pmatrix} \lambda_A \frac{\partial}{\partial \mu^{A'}} & 0 \\ * & \lambda_A \frac{\partial}{\partial \mu^{A'}} \end{pmatrix} \ . \tag{6.46}$$

That such generators are not Hermitian is one aspect of the lack of unitarity of the full theory. One can however truncate to a unitary sector by restricting to space-time fields corresponding to twistor components $f^{A'}$ and g^A. Exactly analogous arguments can be made for higher-rank symmetric twistor tensors where we must restrict to contravariant tensors with only primed indices $f_{A'_1 \dots A'_n}$ and unprimed covariant tensors $g^{A_1 \dots A_n}$. This can be phrased in an alternative manner by making use of the infinity twistor $I^{\alpha\beta}$, which introduces a scale and thus breaks conformal invariance.

By a scale we understand a function $d : \mathbb{M} \times \mathbb{M} \to \mathbb{C}$ such that

$$d(x, y) := \epsilon_{AB} \epsilon_{A'B'} (x - y)^{AA'} (x - y)^{BB'} \ ,$$

written in terms of the spinor representation. This is equivalent to the definition of some ϵ_{AB} and $\epsilon_{A'B'}$, i.e. bilinear forms on $\mathbb{S}^+\mathbb{M}$ and $\mathbb{S}^-\mathbb{M}$. Since $(\mathcal{Z}^\alpha) \equiv (\lambda^A, \mu_{A'})$, a bilinear form on $\mathscr{T}$ such as $\langle \mathcal{Z}, \mathcal{Z}' \rangle := I_{\alpha\beta} \mathcal{Z}^\alpha \mathcal{Z}'^\beta$ only defines ϵ_{AB} and $\epsilon^{A'B'}$, and for a definition of both ϵ_{AB} and $\epsilon_{A'B'}$ we also need to define the inverse to $(I_{\alpha\beta})$. Thus, we need to introduce $I^{\alpha\beta}$ and $I_{\alpha\beta}$ on twistor space, which correspond to two

polarization states in a conformal theory. Demanding that they satisfy the constraint $I^{\alpha\beta}I_{\beta\gamma} \propto \delta^{\alpha}{}_{\gamma}$ fixes a scale between ϵ_{AB} and $\epsilon_{A'B'}$ such that a notion of length is properly defined. (However, there is still the freedom of choice of the scaling of each infinity twistor, since the definition of length on $\mathbb{M}$ is independent under a rescaling $I \mapsto cI$ and $I \mapsto c^{-1}I$.) The restriction to Einstein gravity exactly corresponds to breaking the scaling invariance on $\mathbb{M}$, and we will choose

$$(I^{\alpha\beta})\big|_{\mathrm{EG}} = \begin{pmatrix} \Lambda\epsilon_{AB} & 0 \\ 0 & \epsilon^{A'B'} \end{pmatrix}, \qquad (I_{\alpha\beta})\big|_{\mathrm{EG}} = \begin{pmatrix} \epsilon^{AB} & 0 \\ 0 & \Lambda\epsilon_{A'B'} \end{pmatrix} \tag{6.47}$$

such that $I^{\alpha\beta}I_{\alpha\beta} = 4\Lambda$. In general, different theories (Einstein gravity and conformal gravity) will have different coupling constants corresponding to the same interaction vertices. In fact, the coupling constant of conformal gravity is dimensionless, whereas the one of Einstein gravity is dimensionful, as we have seen in the earlier chapters, and they are related by $S[\mathrm{dS}_4] = -\frac{\Lambda G}{3\varepsilon^2} S^{\mathrm{EG}}_{\mathrm{ren}}[\mathrm{dS}_4]$. This is where the proportionality to Λ comes from.

Introducing the infinity twistor $I^{\alpha\beta}$ and its inverse $I_{\alpha\beta}$, we can define a canonical $\mathfrak{psl}(2)$-invariant measure on $\Xi \simeq \mathbb{CP}^1 \subset \mathbb{P}\mathscr{T}$ by

$$\tau = I_{\alpha\beta}\mathcal{Z}^{\alpha}D\mathcal{Z}^{\beta} = \langle \mathcal{Z}, D\mathcal{Z} \rangle \ . \tag{6.48}$$

This is the contact structure in the second part of the non-linear graviton construction in theorem 5.2.1. The infinity twistor also naturally defines a Poisson structure

$$\{p, q\} := [\partial p, \partial q] := I^{\alpha\beta}\partial_{\alpha}p \wedge \partial_{\beta}q \tag{6.49}$$

which $\overline{\partial}_f$ respects, i.e. each $f^{(s)}$ must be a Hamiltonian field

$$\forall f^{(s)}\, \exists h^{(s)} \in H^{0,1}(\mathbb{P}\mathscr{T}, \mathcal{O}(2s-2)) : f^{(s)} = \{h^{(s)}, \cdot\} \, . \tag{6.50}$$

$h^{(s)}$ is a cohomology class since any exact form can be absorbed in the gauging freedom.

Since $g_{\alpha_I} \in H^{0,1}(\mathbb{P}\mathscr{T}, \mathcal{O}(-4-|I|))$, the Penrose transform of $I^{\alpha_I\beta_I}\partial_{\alpha_I} g_{\beta_I} \in H^{0,1}(\mathbb{P}\mathscr{T}, \mathcal{O}(-4-2\,|I|))$ can be identified with a particle of helicity $-(|I|+1)$. However, for any given $\tilde{h} \in H^{0,1}(\mathbb{P}\mathscr{T}, \mathcal{O}(-4-2\,|I|))$ we can also write $g_{\alpha_I} = I_{\alpha_I\beta_I}\mathcal{Z}^{\beta_I}\tilde{h} \in H^{0,1}(\mathbb{P}\mathscr{T}, \mathcal{O}(-4-|I|))$, which obeys the projectivity constraint $\mathcal{Z}^{\alpha_1} g_{\alpha_1\ldots\alpha_{s-1}} = 0$ for $|I| = s-1$.

This means that we can define the fields $h^{(s)}$ and $\tilde{h}^{(s)}$ corresponding to a sub-sector for which the generators are diagonalisable by the relations

$$f^{\alpha_1\ldots\alpha_{s-1}}(\mathcal{Z}) = I^{\beta_1\alpha_1}\ldots I^{\beta_{s-1}\alpha_{s-1}}\partial_{\beta_1}\ldots\partial_{\beta_{s-1}} h^{(s)}(\mathcal{Z})\,,$$
$$g_{\alpha_1\ldots\alpha_{s-1}}(\mathcal{Z}) = I_{\alpha_1\beta_1}\ldots I_{\alpha_{s-1}\beta_{s-1}} Z^{\beta_1}\ldots Z^{\beta_{s-1}}\tilde{h}^{(s)}(\mathcal{Z})\,,$$

or using the multi-index notation, where $I_{\alpha_I\beta_I} := I_{\alpha_1\beta_1}\cdots I_{\alpha_{s-1}\beta_{s-1}}$ for $|I| = s-1$,

$$f^{\alpha_I}(\mathcal{Z}) = I^{\beta_I\alpha_I}\partial_{\beta_I} h^{(s)}(\mathcal{Z})\,, \qquad g_{\alpha_I}(\mathcal{Z}) = I_{\alpha_I\beta_I} Z^{\beta_I}\tilde{h}^{(s)}(\mathcal{Z})\,, \tag{6.51}$$

as well as $I^{\alpha_I\beta_I} I_{\beta_I\gamma_I} = (-\Lambda)^{|I|}\delta^{\alpha_I}_{\gamma_I}$. In this case, in the linearised approximation about flat twistor space we have that $h^{(s)} \in H^{0,1}(\mathbb{PT}, \mathcal{O}(2s-2))$ and $\tilde{h}^{(s)} \in H^{0,1}(\mathbb{PT}, \mathcal{O}(-2s-2))$ so that $h^{(s)}(\mathcal{Z})$ corresponds to a state of helicity s and $\tilde{h}^{(s)}(\mathcal{Z})$ to a state of helicity $-s$. Taking the limit $\Lambda \to 0$ produces the higher-spin analogue of the truncation of conformal gravity to Einstein gravity at the linearised level around Minkowski space-time. That is for every spin-s we have two on-shell degrees of freedom, $h^{(s)}$ and $\tilde{h}^{(s)}$, which correspond to space-time helicities of $\pm s$. This is the spectrum of massless higher spins found by Fronsdal [45] for the spectrum of the massless limit of the Hagen-Singh theory [93].

The three-point interaction term for definite spins s_1, s_2, and s_3, in the unitary sub-sector is now given by

$$S^{3\mathrm{pt}}_{\mathrm{s.d.}}[\tilde{h}, h] = \int_{\mathbb{P}\mathscr{T}} D^3\mathcal{Z} \wedge I_{\alpha_I\gamma_I}\,\mathcal{Z}^{\gamma_I}\tilde{h}^{(s_1)} \tag{6.52}$$
$$\wedge\, \frac{(|J|+|K|)!}{|J|!\,|K|!} I^{\alpha_J\delta_J} I^{\beta_K\mu_K}\partial_{\delta_J}\partial_{\mu_K} h^{(s_2)} \wedge I^{\alpha_{I-J}\nu_{I-J}}\partial_{\beta_K}\partial_{\nu_{I-J}} h^{(s_3)}\,,$$

where $|I| = s_1 - 1$, $|J| + |K| = s_2 - 1$, and $|I| - |J| = s_3 - 1$. Contracting the infinity twistors, this expression further simplifies to

$$\Lambda^{|I|} \int D^3\mathcal{Z} \wedge \mathcal{Z}^{\gamma_I} \tilde{h}^{(s_1)} \wedge \frac{(|J|+|K|)!}{|J|!\,|K|!} I^{\beta_K \mu_K} \partial_{\mu_K} \partial_{(\gamma_J} h^{(s_2)} \wedge \partial_{\gamma_{I-J})} \partial_{\beta_K} h^{(s_3)} \,. \tag{6.53}$$

The overall spin-dependent power of the cosmological constant is an important feature of all CHS amplitudes restricted to the unitary sub-sector.

Now, the differential operator $\mathcal{Z} \cdot \partial$ is just the Euler vector field on $\mathbb{P}\mathscr{T}$; when acting on a homogeneous function f of weight κ it gives $\mathcal{Z} \cdot \partial f = \kappa f$. Since all wavefunctions appearing in (6.53) have well-defined homogeneities, successive applications of the Euler vector results in:

$$S^{\text{3pt}}_{\text{s.d.}}[\tilde{h}, h] = \Lambda^{s_1 - 1} \widetilde{\mathcal{N}}^{(s_1,s_2,s_3)} \int_{\mathbb{P}\mathscr{T}} D^3Z \wedge \tilde{h}^{(s_1)} \wedge \left\{ h^{(s_2)}, h^{(s_3)} \right\}_{(-s_1+s_2+s_3)} \,, \tag{6.54}$$

where the unitary sub-sector normalization constant is

$$\widetilde{\mathcal{N}}^{(s_1,s_2,s_3)} := \frac{1}{\Gamma(-s_1+s_2+s_3)} \frac{\Gamma(s_1-s_2+s_3)\,\Gamma(s_1+s_2-s_3)}{\Gamma(s_1-s_2+1)\,\Gamma(s_1+1-s_3)} \,, \tag{6.55}$$

and we have defined the bracket

$$\{h, k\}_{(s)} := I^{\gamma_I \sigma_I} \partial_{\gamma_I} h \wedge \partial_{\sigma_I} k \quad \text{for } |I| = s - 1 \,. \tag{6.56}$$

Clearly, $\{h,k\}_{(s)} = (-1)^s \{k,h\}_{(s)}$ and thus $\{h,h\}_{(s)} = 0$ for any odd integer s, which implies that the $\overline{\text{MHV}}$ three-point amplitude is vanishing for any odd spin configuration $s_1 + s_2 + s_3$. This is for higher-spin fields that are not charged under a possibly present gauge group. Having $f^{(n)}$ and $g^{(n)}$ transform in a non-trivial representation of the gauge group, i.e. $f^{(n)} \in \Omega^{0,1}(\mathbb{P}\mathscr{T}, \mathfrak{g} \otimes \text{Sym}^n(T^{1,0}(\mathbb{P}\mathscr{T})))$ and similarly for $g^{(n)}$, leads to non-vanishing amplitudes for differently color-charged spin-s fields, analogous to the spin-1 case. In the following, we will only consider spin-s fields, $s > 1$, in the trivial representation of $\mathfrak{g}$.

After truncating the spectrum, the self-dual action of the unitary sub-sector is

now given as

$$S_{\text{s.d.}}\left[\tilde{h}^{(\bullet)}, h^{(\bullet)}\right] = \sum_{s_1=1}^{\infty} \Lambda^{s_1-1} \int_{\mathbb{PT}} D^3\mathcal{Z} \wedge \tilde{h}^{(s_1)} \\ \wedge \left(\frac{(2s_1-1)!}{2\,s_1!} \overline{\partial} h^{(s_1)} + \sum_{s_2=1}^{\infty} \sum_{s_3=1}^{s_1-1} \widetilde{\mathcal{N}}^{(s_1,s_2,s_3)} \left\{ h^{(s_2)}, h^{(s_3)} \right\}_{(-s_1+s_2+s_3)} \right). \tag{6.57}$$

6.3.1 Space-Time Action

Following the derivation of [82] for the Yang-Mills case, we can analogously integrate out the $\mathbb{CP}^1$ fibre of the self-dual action (6.57).

We choose non-holomorphic coordinates $(x^{AA'}, \sigma_{A'})$ and a corresponding basis for (0,1)-differential forms on $\mathbb{PT}$ such that it splits into horizontal components along $\mathbb{M}$, e^A, with scaling weight -1, and vertical components along the $\mathbb{CP}^1$ fibre, e^0, of scaling weight -2. Their anti-holomorphic part is given by

$$\overline{e}^0 = \frac{[\hat{\sigma}\,\mathrm{d}\hat{\sigma}]}{[\hat{\sigma}\sigma]^2} \qquad \text{and} \qquad \overline{e}^A = -\frac{\mathrm{d}x^{AA'}\hat{\sigma}_{A'}}{[\sigma\hat{\sigma}]}\,. \tag{6.58}$$

We also introduce a dual basis for vector fields $(\overline{V}_A, \overline{V}_0)$ satisfying

$$\begin{aligned} \overline{V}_0 \lrcorner \overline{e}^0 &= 1\,, & \overline{V}_A \lrcorner \overline{e}^0 &= 0\,, \\ \overline{V}_0 \lrcorner \overline{e}^B &= 0\,, & \overline{V}_A \lrcorner \overline{e}^B &= \delta_A^B\,. \end{aligned} \tag{6.59}$$

In this basis we can write the $\overline{\partial}$ operator as

$$\overline{\partial} = \overline{e}^0 \overline{V}_0 + \overline{e}^A \overline{V}_A \tag{6.60}$$

and expand the fields as

$$\tilde{h}_s = \tilde{h}_{s,0}\overline{e}^0 + \tilde{h}_{s,A}\overline{e}^A\,, \qquad h_s = h_{s,0}\overline{e}^0 + h_{s,A}\overline{e}^A\,. \tag{6.61}$$

If $\tilde{h}_s$ has weight $2s-2$, then $\tilde{h}_{s,0}$ has weight $2s$ and $\tilde{h}_{s,A}$ has weight $2s-1$. In a corresponding basis for (1,0) forms it is

$$D^3\mathcal{Z} = \Omega = \frac{1}{4}\,[\sigma\hat{\sigma}]^4\, e^0 \wedge e^A \wedge e_A\,. \tag{6.62}$$

We now have explicitly

$$\overline{V}_\alpha = \left(\sigma^{A'} \frac{\partial}{\partial x^{AA'}} \, , \quad - [\sigma\hat{\sigma}] \, [\sigma \partial_{\hat{\sigma}}] \right) \tag{6.63}$$

and

$$\frac{\Omega \wedge \overline{\Omega}}{[\sigma\hat{\sigma}]^4} = \frac{[\sigma \, \mathrm{d}\sigma] \wedge [\hat{\sigma} \, \mathrm{d}\hat{\sigma}]}{[\sigma\hat{\sigma}]^2} \wedge \mathrm{d}^4 x \, . \tag{6.64}$$

We now consider the kinetic part of the action, which for $s_1 = s$ is

$$\begin{aligned} S[\tilde{h}_s, h_s] &\propto \int_{\mathbb{PT}} \Omega \wedge \tilde{h}_s \wedge \overline{\partial} h_s \\ &= \int_{\mathbb{M}\times\mathbb{CP}^1} \Omega \wedge \left(\tilde{h}_0 \overline{e}^0 + \tilde{h}_A \overline{e}^A \right) \wedge \left(\overline{e}^0 \overline{V}_0 + \overline{e}^B \overline{V}_B \right) \wedge \left(h_0 \overline{e}^0 + h_C \overline{e}^C \right) \\ &= \int_{\mathbb{M}\times\mathbb{CP}^1} \Omega \wedge \Big(\tilde{h}_0 \overline{V}_B h_C \overline{e}^0 \wedge \overline{e}^B \wedge \overline{e}^C + \tilde{h}_A \overline{V}_0 h_C \overline{e}^A \wedge \overline{e}^0 \wedge \overline{e}^C \\ &\qquad\qquad + \tilde{h}_A \overline{V}_B h_0 \overline{e}^A \wedge \overline{e}^B \wedge \overline{e}^0 \Big) \\ &= 2 \int_{\mathbb{M}\times\mathbb{CP}^1} \frac{\Omega \wedge \overline{\Omega}}{[\sigma\hat{\sigma}]^4} \left(\tilde{h}_0 \overline{V}_A h^A - \tilde{h}_A \overline{V}_0 h^A + \tilde{h}_A \overline{V}^A h_0 \right) \\ &= 2 \int\limits_{\mathbb{M}\times\mathbb{CP}^1} \mathrm{d}^4 x \wedge \frac{D\sigma \wedge D\hat{\sigma}}{[\sigma\hat{\sigma}]^2} \left(\tilde{h}_0 \sigma^{A'} \partial_{AA'} h^A + \tilde{h}_A \, [\sigma\hat{\sigma}] \, [\sigma \partial_{\hat{\sigma}}] \, h^A - \tilde{h}_A \sigma_{A'} \partial^{AA'} h_0 \right) . \end{aligned}$$

Choosing space-time gauge $\overline{\partial}^\dagger\big|_{\mathbb{CP}^1}(\overline{e}^0 h_0) = 0$, which also implies $\overline{\partial}\big|_{\mathbb{CP}^1}(\overline{e}^0 h_0) = 0$, makes these components harmonic on the fibres. Since $H^1(\mathbb{CP}^1, \mathcal{O}(2h-2)) = 0$ for $h > 0$ we have that $h_0 = 0$. On the other hand, $\tilde{h}_A$ can be treated as Lagrangian multiplier, which enforces $\overline{V}_0 h^A = 0$. This means that h^A must be holomorphic in $\hat{\sigma}_{A'}$, i.e. $h^A(x,\sigma) = \sigma^{A'(2s-1)} h^A_{A'(2s-1)}(x)$.

Together with the Penrose transform, we obtain

$$\int_{\mathbb{M}\times\mathbb{CP}^1} \mathrm{d}^4 x \wedge \frac{D\sigma \wedge D\hat{\sigma}}{[\sigma\hat{\sigma}]^2} \tilde{h}_0 \sigma^{A'} \sigma^{B'(2s-1)} \partial_{AA'} h^A_{B'(2s-1)} = \int_{\mathbb{M}} \mathrm{d}^4 x \, \tilde{h}^{A'B'(2s-1)} \partial_{AA'} h^A_{B'(2s-1)} \, ,$$

and the free part of the action is now given by

$$S_{\text{free}}[\tilde{h}_s, h_s] = \Lambda^{s-1} \frac{(2s-1)!}{s!} \int_{\mathbb{M}} \mathrm{d}^4 x \, \tilde{h}^{A'B'(2s-1)} \nabla_{AA'} h^A_{B'(2s-1)} \, , \tag{6.65}$$

which reproduces the equations of motion for a massless spin-s particle

$$\nabla_{AA'} h^A_{B'(2s-1)} = 0 \qquad \text{and} \qquad \nabla_{AA'} \tilde{h}^{A'B'(2s-1)} = 0 \, . \tag{6.66}$$

Chapter 7

Stars, Jets and Anti-Self-Dual Interactions

We have seen how to construct the self-dual interactions of a conformal higher spin theory on twistor space and how to truncate this theory to its unitary sub-sector. In order to interpret the higher-spin deformation in a geometric sense we must include an infinite number of interacting higher-spin fields. This is because once we go beyond the spin-two case, the Maurer-Cartan equations for a single spin can no longer be interpreted as the integrability condition for a holomorphic structure on a vector bundle. This can be rectified by including an infinite tower of interacting spins and by interpreting the deformations as acting on the corresponding infinite jet bundle of the space of symmetric products of the (co)tangent bundle. At the end of this chapter we describe additional terms to be added to describe the interactions of the anti-self-dual fields and extend the action to the full higher spin theory.

We will first however make two detours to relate our formulation to two other common formalisms in higher spin theory, that is using $\star$-products and using jet bundles. Those two formalisms are very closely related to each other of course, as both have the same goal: packaging the field content into one single object and

expressing the equations of motion in an elegant form, like

$$(\overline{\partial} + f) \star f = 0 \qquad \text{and} \qquad (\overline{\partial} + f) \star g = 0\,, \tag{7.1}$$

for some action $\star$.

Using $\star$-products is in the spirit of considering generating functions, where products between two functions are taken by some action of differential operators. The jet bundle language represents those actions algebraically, leading to the dictionary

$$\begin{aligned} \text{generating functions} \quad &\longleftrightarrow \quad \text{jets ('vectors')} \\ \star\text{-products} \quad &\longleftrightarrow \quad \text{jet products ('matrix products')}\,. \end{aligned}$$

In the higher-spin literature the language of both $\star$-products and jet bundles is used interchangeably, e.g. [94] for $\star$-products and [95] for a recent discussion using the jet bundle terminology. While we will not make concrete contact with the Vasiliev formalism, we can make a preliminary step by noting that the product (6.39), and so the holomorphicity condition (6.38), can be written by using a star product, though in a somewhat unintuitive way by introducing a projection. Dropping the projection leads to an extension of the theory with a more general interaction vertex, but has the problem that only the equations of motion for f benefit from it. Wanting to consistently write the equations of motion for g in a naturally extended way fixes the coefficients in such a way that the theory becomes free.

Similar troubles arise in the jet bundle language, where we have to employ some projection in order to arrive at the correct equations of motion. Extending the formulation in an intuitive way defines the same generalised self-dual three-point vertices as in the extended $\star$-product formulation, i.e. it gives the same equations of motion for f, but fails to reproduce the corresponding equations of motion for g.

We remind ourselves that in order to construct a higher spin theory on twistor space, we consider a generalised complex deformation of the form

$$\overline{\partial}_f = \overline{\partial} + \sum_{|J|=0}^{\infty} f^{\beta_J} \partial_{\beta_J}\ , \tag{7.2}$$

where $f^{\alpha_I} \in \Omega^{0,1}(\mathbb{P}\mathscr{T}, \mathcal{O}(|I|))$ and especially for spin-one $f^{\alpha_\emptyset} \in \Omega^{0,1}(\mathbb{P}\mathscr{T}, \mathfrak{g} \otimes \mathcal{O}(0))$, where $\mathfrak{g}$ is the Lie algebra corresponding to the preferred gauge group. Since we want to interpret $\overline{\partial}_f$ as a holomorphic structure, it should satisfy the integrability condition $\overline{\partial}_f^2 = 0$. There are essentially three ways of understanding this condition: consider $f = f^a \, \mathrm{T}^a$ in some representation of some generic action with generators T^a, then

1. f^a are scalars with respect to T^a and thus

$$\overline{\partial}_f^2 = \overline{\partial} f + f^a \wedge f^b \, \mathrm{T}^a \, \mathrm{T}^b \ , \tag{7.3}$$

2. f^a transform under action of T^a and the generators act to the right (or left)

$$\overline{\partial}_f^2 = \overline{\partial} f + f^a \wedge \mathrm{T}^a \circ (f^b \, \mathrm{T}^b) \ , \tag{7.4}$$

3. f^a transform under action of T^a and the generators act to the right and left

$$\overline{\partial}_f^2 = \overline{\partial} f + f^a \overleftrightarrow{\mathrm{T}^a} \wedge f^b \overleftrightarrow{\mathrm{T}^b} \ . \tag{7.5}$$

We work in a gauge where f is volume-preserving, which means

$$\partial_\alpha f^{\cdots \alpha \cdots} = 0 = \mathrm{T}^a(f^a) \, . \tag{7.6}$$

Interpretation 1 is not interesting as it results in equations of motion for f that are free, $\overline{\partial} f^{\alpha_I} = 0$, which also leads to free equations of motion for the Lagrange multiplier field g, $\overline{\partial} g_{\alpha_I} = 0$. This theory is therefore trivial.

The second interpretation is exactly what we have been investigating so far, as the generators T are the canonical basis of $\mathrm{Sym}^n(T\mathbb{P}\mathscr{T})$, that is $\mathrm{T} = \partial$, which naturally acts to the right. Interpreting the this structure in terms of $\star$-products and jet-bundles however makes the third case an immediate one to consider.

7.1 Reaching for the Stars

Much of the $\star$-product formulation is about an intelligent packaging of the fields and their derivatives in terms of generating functions. For this we introduce an auxiliary twistor space with holomorphic coordinates U^α, $\alpha = 0, 1, 2, 3$, where $\overline{U}_\alpha \equiv \partial_\alpha$, and define the generating function corresponding to the field f as

$$f(Z, U, \overline{U}) = \sum_{|I|,|J|=0}^{\infty} \frac{1}{|J|!} \partial_{\beta_J} f^{\alpha_I}(Z) \overline{U}_{\alpha_I} U^{\beta_J} \,. \tag{7.7}$$

Considering a scalar field with the same representation

$$\varphi(Z, U) = \sum_{|J|=0}^{\infty} \frac{1}{|J|!} \partial_{\beta_J} \varphi(Z) U^{\beta_J} \,, \tag{7.8}$$

we define the product

$$f(U, \overline{U}) \star \varphi(U) = f(U, \overline{U}) \exp\left(\frac{\overleftarrow{\partial}}{\partial \overline{U}_\gamma} \wedge \frac{\overrightarrow{\partial}}{\partial U^\gamma} \right) \varphi(U) \tag{7.9}$$

for which it is easy to see that

$$\overline{\partial} \varphi(Z) + \sum_{|I|=0}^{\infty} f^{\alpha_I} \wedge \partial_{\alpha_I} \varphi(Z) = 0 \tag{7.10}$$

can be written as

$$\left(\overline{\partial} \varphi(Z, U) + f(Z, U, \overline{U}) \star \varphi(Z, U) \right) \Big|_{U=0, \overline{U}=0} = 0 \,. \tag{7.11}$$

In fact one can relax the condition that $U = 0$ as each term in the expansion simply corresponds to taking Z-derivatives of the original equation. There is an ambiguity in the $\star$-product as ϕ has no dependence on $\overline{U}$ and so we could consider additional terms involving $\frac{\overrightarrow{\partial}}{\partial \overline{U}}$. This form of the $\star$-product is natural, though still not unique from the perspective of twistor quantisation where twistors and their complex conjugates are naturally canonically conjugate variables. This can be seen by noting that it implies

$$\overline{U}_\beta \star U^\alpha - U^\alpha \star \overline{U}_\beta = \delta^\alpha_\beta \,, \tag{7.12}$$

which can be represented on the space of functions with the identification

$$\overline{U}_\beta \to \frac{\partial}{\partial U^\beta} \,. \tag{7.13}$$

An extension of the product (7.9) is

$$f \star g = f \, \exp\Big(\alpha \tfrac{\overleftarrow{\partial}}{\partial \overline{U}_\gamma} \wedge \tfrac{\overrightarrow{\partial}}{\partial U^\gamma} + \tfrac{\overleftarrow{\partial}}{\partial U^\gamma} \wedge \tfrac{\overrightarrow{\partial}}{\partial \overline{U}_\gamma}\Big) \, g \,, \tag{7.14}$$

where we leave the sign $\alpha = \pm 1$ unfixed for now. This product has a (sine) bracket

$$[f, g]_\star = f \star g - g \star f \tag{7.15}$$

for which

$$\Big[\overline{U}_\beta, U^\alpha\Big]_\star = (\alpha - 1)\delta^\alpha_\beta \,, \qquad \Big[\overline{U}_\alpha, \overline{U}_\beta\Big]_\star = 0 \,, \qquad \Big[U^\alpha, U^\beta\Big]_\star = 0 \,. \tag{7.16}$$

For $\alpha = -1$ this is in fact a representation of the Weyl algebra A_1.

In order to include g in this language, we define the generating function

$$g(Z, U, \overline{U}) = \sum_{|I|,|J|=0}^{\infty} \frac{(-1)^{|J|}}{|J|!} \partial_{\beta_J} g_{\alpha_I}(Z) U^{\alpha_I} \frac{\overleftrightarrow{\partial}}{\partial \overline{U}_{\beta_J}} \,. \tag{7.17}$$

As we will see shortly, we need the left and right action in order to reproduce the correct equations of motion.

7.1.1 $\star$-Product with Principal Projection

To obtain the second interpretation (7.4), we have to adjust the definition of the $\star$-product and include a projection onto the principal part of the generating function, i.e.

$$\overline{\star}\, f := \star \left(f\Big|_{U=0}\right) \qquad \text{and} \qquad \overline{\star}\, g := \star \left(g\Big|_{\overline{U}=0}\right) \tag{7.18}$$

For this $\overline{\star}$-product we can similarly define brackets

$$[f, g]_{\overline{\star}} = f \,\overline{\star}\, g - g \,\overline{\star}\, f \qquad \text{and} \qquad \{f, g\}_{\overline{\star}} = f \,\overline{\star}\, g + g \,\overline{\star}\, f \,, \tag{7.19}$$

for which it is

$$\left[\overline{U}_\beta, U^\alpha\right]_{\bar\star} = \left[\overline{U}_\beta, U^\alpha\right]_\star = (\alpha-1)\delta^\alpha_\beta\,, \qquad \text{however} \qquad \left[U^\alpha, \overline{U}_\beta\right]_{\bar\star} = 0\,. \tag{7.20}$$

The projection thus induces an ordering. The star products generating the terms for the equations of motion are:

$$(f \mathbin{\bar\star} f)\Big|_{U=0} = -C_{|J||L|}\, f^{\beta_J(\alpha_L} \wedge \partial_{\beta_J} f^{\alpha_{I-L})}\, \overline{U}_{\alpha_I}\,, \tag{7.21}$$

$$(f \mathbin{\bar\star} g)\Big|_{\overline{U}=0} = \alpha^{|K|} C_{|I-L||L|} \partial_{(\alpha_L} f^{\gamma_K} \wedge g_{\alpha_{I-L})\gamma_K}\, U^{\alpha_I}\,, \tag{7.22}$$

$$(g \mathbin{\bar\star} f)\Big|_{\overline{U}=0} = -(-1)^{|J|} C_{|J||K|} f^{\beta_J\gamma_K} \wedge \partial_{\beta_J} g_{\alpha_I\gamma_K}\, U^{\alpha_I}\,, \tag{7.23}$$

where $C_{|J||K|} = \binom{|K|+|J|}{|J|}$. Consistency with the equations of motion (6.33) and (6.35) demands that $\alpha = 1$, and we thus have a free algebra. With those choices, we can now rewrite the equations of motion as

$$\left(\overline{\partial} f - f \mathbin{\bar\star} f\right)\Big|_{U=0} = 0 \qquad \text{and} \qquad \left(\overline{\partial} g + \{f,g\}_{\bar\star}\right)\Big|_{\overline{U}=0} = 0\,. \tag{7.24}$$

The projection within the $\star$-product seems unintuitive at first, but becomes the natural choice when reminding ourselves of what we wanted to achieve by using this language: we wanted to find a formalism in which we can write the equations of motion in a schematic form like $(\overline{\partial} + f) \circ f^{\alpha_I} = 0$ and $(\overline{\partial} + f) \diamond g_{\alpha_I} = 0$. The right part of the operations $\circ$ and $\diamond$ is always just the first term in the expansion of derivatives on f and g respectively, which is exactly what the projection is selecting out.

7.1.2 $\star$-Product without Projection

Of course we can ask what happens if we allow the full generating function on the right hand side. And indeed, considering the $\star$-product without projection leads us to the third interpretation (7.5). However, for $\alpha = 1$ the theory becomes free and as interesting as the first interpretation.

If we consider the full star product, then we have

$$(f \star f)\big|_{U=0} = -\alpha^{|M|} C_{|J||L|} C_{|I-L||M|}\, \partial_{\mu_M} f^{\beta_J(\alpha_L} \wedge \partial_{\beta_J} f^{\alpha_{I-L})\mu_M}\, \overline{U}_{\alpha_I}\,. \tag{7.25}$$

Note that $f \star f = -f \star f = 0$ if $\alpha = 1$. The generalised equations of motion are now

$$0 = \left(\overline{\partial} f - f \star f\right)\big|_{U=0} = \overline{\partial} f^{\alpha_I} + \alpha^{|M|} C_{|J||L|} C_{|I-L||M|} \partial_{\mu_M} f^{\beta_J(\alpha_L} \wedge \partial_{\beta_J} f^{\alpha_{I-L})\mu_M}\,. \tag{7.26}$$

Formulating this as an action principle like we did in the previous chapter, we obtain the generalised equations of motion for g

$$\overline{\partial} g_{\alpha_I} + \left(\alpha^{|L|} - \alpha^{|J|}\right)(-1)^{|J|} C_{|K||J|} C_{|I-L||L|} \partial_{(\alpha_L|} f^{\beta_J\gamma_K} \wedge \partial_{\beta_J} g_{|\alpha_{I-L})\gamma_K} = 0\,. \tag{7.27}$$

However, we also see that the full $\star$-products are

$$(f \star g)\big|_{\overline{U}=0} = \alpha^{|K|}(-1)^{|J|} C_{|J||K|} C_{|I-L||L|}\, \partial_{(\alpha_L|} f^{\beta_J\gamma_K} \wedge \partial_{\beta_J} g_{|\alpha_{I-L})\gamma_K}\, U^{\alpha_I}\,, \tag{7.28}$$

$$(g \star f)\big|_{\overline{U}=0} = -(-1)^{|J|} C_{|J||K|} C_{|I-L||L|}\, \partial_{(\alpha_L|} f^{\beta_J\gamma_K} \wedge \partial_{\beta_J} g_{|\alpha_{I-L})\gamma_K}\, U^{\alpha_I}\,, \tag{7.29}$$

and rewriting the equations of motion (7.27) as $\left(\overline{\partial} g + \{f, g\}_\star\right)\big|_{\overline{U}=0} = 0$ is only possible for $\alpha = 1$. This, however, means that there are no interactions as the equations of motion both for f and g become free. This could mean that the symmetries of the extended theory might be too restrictive to allow for consistent interactions.

If we set $\alpha = -1$ in the equations of motion (7.27), they assume the form

$$0 = \overline{\partial} g^{(s_1)}_{\alpha_I} + \left((-1)^{s_1+s_2+s_3} - 1\right) C_{|J||K|} C_{|I-L||L|} \partial_{(\alpha_L|} f^{(s_3)\,\beta_J\gamma_K} \wedge \partial_{\beta_J} g^{(s_2)}_{|\alpha_{I-L})\gamma_K}\,. \tag{7.30}$$

because $|J + L| = |I| - |I - L + K| + |J + K| = s_1 - s_2 + s_3$.

For the same choice of α for the equations of motion for f, (7.26), we can again write this condition as an action principle, like we did in the last chapter, for which the unitary truncation takes the same form

$$\hat{S}^{\text{3pt}}_{\text{s.d.}}[\tilde{h}, h] = \Lambda^{s_1-1} \widehat{\widetilde{\mathcal{N}}}^{(s_1,s_2,s_3)} \int_{\mathbb{P}\mathscr{T}} D^3Z \wedge \tilde{h}^{(s_1)} \wedge \left\{h^{(s_2)}, h^{(s_3)}\right\}_{(-s_1+s_2+s_3)}\,, \tag{7.31}$$

but with a different normalisation constant

$$\begin{aligned}\widehat{\widetilde{\mathcal{N}}}^{(s_1,s_2,s_3)} &= \frac{1}{\Gamma(-s_1+s_2+s_3)}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{\Gamma(s_1-s_2+s_3)\Gamma(s_1+s_2-s_3)}{\Gamma(s_1-s_2+k+1)\Gamma(s_1-s_3+k+1)}\\ &\qquad\times\frac{\Gamma(s_2)\Gamma(s_3)}{\Gamma(s_2-k)\Gamma(s_3-k)}\frac{\Gamma(-s_1+s_2+s_3)}{\Gamma(-s_1+s_2+s_3-k)}\\ &= \frac{2^{s_1+s_2+s_3-3}}{\sqrt{\pi}\,\Gamma(s_1)}\frac{\Gamma\left(\frac{1}{2}(s_1-s_2+s_3)\right)\,\Gamma\left(\frac{1}{2}(s_1+s_2-s_3)\right)}{\Gamma(-s_1+s_2+s_3)}\frac{\Gamma\left(\frac{1}{2}(s_1+s_2+s_3-1)\right)}{\Gamma\left(\frac{1}{2}(s_1-s_2-s_3+2)\right)}\,,\end{aligned} \tag{7.32}$$

of which the constant $\widetilde{\mathcal{N}}^{(s_1,s_2,s_3)}$ from (6.55) is the $k=0$ term in the sum. This new constant is neither more nor less revealing than the old one, but could be an interesting starting point for investigating this extended theory, if it consistently exists.

7.2 Grasping Jets

The language of jet bundles, which we briefly review below, will allow us to give a more geometric interpretation of the higher-spin equations of motion as the integrability condition for a holomorphic structure. Given a complex manifold, here we are obviously considering $\mathbb{P}\mathscr{T}$, we can define the corresponding Dolbeault operator $\overline{\partial}$ mapping (p,q)-forms to $(p,q+1)$-forms. This can be naturally generalised to (p,q) forms taking values in sections of some complex vector bundle $\mathcal{B}\to\mathbb{P}\mathscr{T}$, that is elements of $\Omega^{p,q}(\mathbb{P}\mathscr{T},\mathcal{B})$. We will be mostly considering bundles whose sections are symmetric covariant tensors of $\mathrm{Sym}^n(T\mathbb{P}\mathscr{T})$ or contravariant tensors of $\mathrm{Sym}^n(T^*\mathbb{P}\mathscr{T})$ or $\mathfrak{g}$-valued for $n=0$. We wish to define a holomorphic structure on $\mathcal{B}$, this is a sequence of operators

$$\overline{\partial}_{\mathbf{f}}:\Omega^{p,q}(\mathbb{P}\mathscr{T},\mathcal{B})\to\Omega^{p,q+1}(\mathbb{P}\mathscr{T},\mathcal{B}) \tag{7.33}$$

such that

$$\begin{aligned} &\text{i)} \quad \overline{\partial}_{\mathbf{f}} \circ \overline{\partial}_{\mathbf{f}} = 0 \\ &\text{ii)} \quad \overline{\partial}_{\mathbf{f}}(\omega \wedge \mathbf{g}) = \overline{\partial}(\omega) \wedge \mathbf{g} + (-1)^{m+n} \omega \wedge \overline{\partial}_{\mathbf{f}}(\mathbf{g}) \end{aligned} \tag{7.34}$$

where $\omega \in \Omega^{m,n}(\mathbb{P}\mathscr{T})$ and $\mathbf{g} \in \Omega^{p,q}(\mathbb{P}\mathscr{T}, \mathcal{B})$ for any (m,n) and (p,q). The $\mathbf{f}$ and $\mathbf{g}$ are some suitable objects for which those definitions make sense, however we will identify them shortly with infinite jets of some jet bundle.

The holomorphic structure $\overline{\partial}_{\mathbf{f}}$ on $\mathcal{B}$ induces a holomorphic structure, also denoted $\overline{\partial}_{\mathbf{f}}$, on $\mathrm{End}(\mathcal{B})$, obeying

$$\overline{\partial}_{\mathbf{f}}(\mathbf{h}) = \overline{\partial}_{\mathbf{f}} \circ \mathbf{h} - (-1)^{p+q} \mathbf{h} \circ \overline{\partial}_{\mathbf{f}} \ , \quad \mathbf{h} \in \Omega^{p,q}(\mathbb{P}\mathscr{T}, \mathrm{End}(\mathcal{B})) \ . \tag{7.35}$$

Let $\overline{\partial}_{\mathbf{f}'}$ be another holomorphic structure on $\mathcal{B}$, then there exists a section $\tilde{\mathbf{f}} \in \Omega^{(0,1)}(\mathbb{P}\mathscr{T}, \mathrm{End}(\mathcal{B}))$ such that

$$\overline{\partial}_{\mathbf{f}'}(\mathbf{g}) = \overline{\partial}_{\mathbf{f}}(\mathbf{g}) + \tilde{\mathbf{f}} \circ \mathbf{g} \tag{7.36}$$

where $\mathbf{g} \in \Omega^{p,q}(\mathbb{P}\mathscr{T}, \mathcal{B})$ and $\tilde{\mathbf{f}}$ satisfies the Mauer-Cartan equation

$$\overline{\partial}_{\mathbf{f}} \tilde{\mathbf{f}} + \tilde{\mathbf{f}} \circ \tilde{\mathbf{f}} = 0 \ . \tag{7.37}$$

Conversely if $\tilde{\mathbf{f}}$ is a section satisfying (7.37), then defining $\overline{\partial}_{\mathbf{f}'}$ by (7.36) gives another holomorphic structure on $\mathcal{B} \to \mathbb{P}\mathscr{T}$.

Making contact with our previous considerations we see that the deformation in (6.29), f, is to take values in $\Omega^{0,1}(\mathbb{P}\mathscr{T}, \mathrm{Sym}^{(\bullet)}(T^{1,0}\mathbb{P}\mathscr{T}))$, i.e. $f(\mathcal{Z}) = \sum f^{\alpha_I}_{\overline{\alpha}}(\mathcal{Z})\, \mathrm{d}\overline{\mathcal{Z}}^{\overline{\alpha}} \partial_{\alpha_I}$. For $|I| > 1$, this is not a derivation and therefore does not give rise to a holomorphic structure in the usual sense. In order to interpret the higher powers of derivatives as linear operators on some vector space we must think of the deformed operators as acting on an infinite dimensional vector formed from the field ϕ and all of its derivatives

$$(\phi, \partial_\alpha \phi, \partial_{\alpha_1} \partial_{\alpha_2} \phi, \dots) \ . \tag{7.38}$$

Such an object is an infinite jet, called the infinite prolongation $j^\infty\phi$ of ϕ, and and we can now interpret f as taking values in the endomorphisms of the jet bundle. The higher powers of derivatives act as generators of the space of endomorphisms.

Jet bundles: To be slightly more precise, see [96] for a textbook treatment, we wish to consider fields which are sections of some bundle B i.e. the fields will be sections of $\mathrm{Sym}^{(n)}(T^{1,0}\mathbb{P}\mathscr{T})$ or $\mathrm{Sym}^{(n)}(T^{*1,0}\mathbb{P}\mathscr{T})$ or $\mathfrak{g}$. To describe this we choose an appropriate adapted local coordinate system, ψ, on the total space of the bundle B where for a given subspace $W \subset B$ the coordinates can be split into those parametrising the base space and those distinguishing points on the fibre: $\psi = (\mathcal{Z}^\alpha, \psi_{\beta_J})$. Given two such sections, say g and $\tilde{g}$, we say they have the same k-jet at $\mathcal{Z} \in \mathbb{P}\mathscr{T}$ if in any particular coordinate system their first k derivatives coincide, i.e.

$$\partial_{\alpha_I} g_{\beta_J}(\mathcal{Z}) = \partial_{\alpha_I} \tilde{g}_{\beta_J}(\mathcal{Z}) \,, \quad 0 \leq |I| \leq k \,, \tag{7.39}$$

where $g_{\beta_J} = \psi_{\beta_J} \circ g$. This definition is in fact independent of the particular coordinate system. The k-th jet of g at $\mathcal{Z}$, denoted $j^k_{\mathcal{Z}} g$, is the equivalence class of all sections with the same k-jet. The k-th jet manifold, which we denote $J^k(B)$, is the totality of all such jets,

$$J^k(B) = \left\{ j^k_{\mathcal{Z}} g : \forall \mathcal{Z} \in \mathbb{P}\mathscr{T}, g \in \Omega^{0,1}(\mathbb{P}\mathscr{T}, B) \right\} . \tag{7.40}$$

The jet-manifold combined with the so-called source projection $\pi_k : j^k_{\mathcal{Z}} g \mapsto \mathcal{Z}$, can be viewed as a bundle over the base space $\mathbb{P}\mathscr{T}$. The coordinate system ψ on B induces a coordinate system ψ^k on $J^k(B)$: given $W \subset B$ we define $W^k = \{ j^k_{\mathcal{Z}} g : g(\mathcal{Z}) \in W \}$ and

$$\psi^k = (\mathcal{Z}^\alpha, \psi_{\beta_J}, \psi_{\beta_J;\alpha_1}, \ldots, \psi_{\beta_J;\alpha_1 \ldots \alpha_k}) \tag{7.41}$$

are called derivative coordinates where for $j^k_{\mathcal{Z}} g \in W^k$ we define $\mathcal{Z}^\alpha(j^k_{\mathcal{Z}} g) = \mathcal{Z}^\alpha$, $\psi_{\beta_J}(j^k_{\mathcal{Z}} g) = g_{\beta_J}(\mathcal{Z})$, and

$$\psi_{\beta_J;\alpha_I}(j^k_{\mathcal{Z}} g) = \partial_{\alpha_I} g_{\beta_J}(\mathcal{Z}) \,. \tag{7.42}$$

Correspondingly, given a open subset $U \in \mathbb{P}\mathscr{T}$ and local section $g \in \Gamma_U(B)$ we define the k-th prolongation of g as the section $j^k g \in \Gamma_U(J^k(B))$ defined by $j^k g(\mathcal{Z}) = j^k_{\mathcal{Z}} g$ for $\mathcal{Z} \in U$ with coordinate representation

$$(g_{\beta_J}, \partial_{\alpha_1} g_{\beta_J}, \ldots, \partial_{\alpha_1} \cdots \partial_{\alpha_k} g_{\beta_J}) \ . \tag{7.43}$$

It is worthwhile to note that, while these prolongations will be the focus of our interest, they are very non-generic sections of the jet bundle since their adaptive coordinates are strongly related to each other, which generically does not need to be the case. The infinite jet bundle corresponds to the limiting case $k \to \infty$. For $\mathcal{Z} \in \mathbb{P}\mathscr{T}$ the ∞-th jet of g, which we denote $j^\infty_{\mathcal{Z}} g$, is the equivalence class of sections whose derivatives coincide with those of g at all orders and the space $J^\infty(B)$ is an infinite dimensional manifold which can be shown to have the structure of a bundle over $\mathbb{P}\mathscr{T}$.

In order to interpret $\overline{\partial}_f$ from (6.29) as a holomorphic structure $\overline{\partial}_{\mathbf{f}}$, we define linear operators T acting on the jet bundle $\mathcal{B} = J^\infty(B)$ that form a basis of $\mathrm{End}(J^\infty(B))$, where

$$B = \left(\mathfrak{g} \oplus \bigoplus_{r=1}^{\infty} \mathrm{Sym}^{(r)}(T^{*1,0}\mathbb{P}\mathscr{T}) \right) \otimes \Omega^{(3,0)}(\mathbb{P}\mathscr{T}) \otimes \mathcal{O}(4) \ . \tag{7.44}$$

By looking at (6.35) we see that we need operators that raise the degree of differentiation

$$\mathrm{T}_{;\beta_J} : j^k(g^\Omega) \to j^{k+|J|}(g^\Omega) \ , \tag{7.45}$$

and operators that raise and lower the form degree

$$\begin{aligned} \mathrm{T}_{\gamma_K} &: \mathrm{Sym}^{(r)}(T^{*1,0}\mathbb{P}\mathscr{T}) \to \mathrm{Sym}^{(r+|K|)}(T^{*1,0}\mathbb{P}\mathscr{T}) \ , \\ \mathrm{T}^{\delta_L} &: \mathrm{Sym}^{(r)}(T^{*1,0}\mathbb{P}\mathscr{T}) \to \mathrm{Sym}^{(r-|L|)}(T^{*1,0}\mathbb{P}\mathscr{T}) \ . \end{aligned} \tag{7.46}$$

The semicolon notation is inspired from (7.42), denoting derivatives. In the following we will denote the prolongation $j^\infty(g^\Omega_{\alpha_I})$ by $\mathbf{g}^\Omega_{\alpha_I}$ while omitting writing the basis

$D\mathcal{Z}^{\alpha_I}$. The action of the generators can be defined through

$$\mathrm{T}_{;\beta_J}\,\mathbf{g}^{\Omega}_{\alpha_I} = (-1)^{|J|}\partial_{\beta_J}\mathbf{g}^{\Omega}_{\alpha_I}\,, \tag{7.47}$$

$$\mathrm{T}_{\gamma_K}\,\mathbf{g}^{\Omega}_{\alpha_I} = \mathbf{g}^{\Omega}_{(\alpha_I\gamma_K)}\,, \tag{7.48}$$

$$\mathrm{T}^{\delta_L}\,\mathbf{g}^{\Omega}_{\alpha_I} = \begin{cases} 0 \quad \text{for } |L| > |I| \\ C_{|L||I-L|}\delta^{\delta_L}_{(\alpha_L}\mathbf{g}^{\Omega}_{\alpha_{I-L})} \quad \text{else}\,. \end{cases} \tag{7.49}$$

The operator $\mathrm{T}_{;\beta_J}$ carries homogeneity as the action of the Euler operator is given by $\Upsilon(\mathrm{T}_{;\beta_J}) = -\,|J|\,\mathrm{T}_{;\beta_J}$. This way we have that

$$\mathbf{g}^{\Omega} = j^{\infty}(g^{\Omega}) \in \Omega^{(0,1)}(\mathbb{P}\mathscr{T}, J^{\infty}(B)) \quad \text{and} \quad \mathbf{f} \in \Omega^{(0,1)}(\mathbb{P}\mathscr{T}, \mathrm{End}(J^{\infty}(B)))\,. \tag{7.50}$$

We can now write the equations of motion (6.35) as

$$\begin{aligned} 0 = \overline{\partial}_{\mathbf{f}}\mathbf{g}^{\Omega}_{\alpha_I} &= \overline{\partial}\mathbf{g}^{\Omega}_{\alpha_I} + \mathbf{f}\wedge\mathbf{g}^{\Omega}_{\alpha_I} \\ &= \overline{\partial}g^{\Omega}_{\alpha_I} + \mathbf{f}^{\gamma_K}_{\delta_L}\,\mathrm{T}_{\gamma_K}\,\mathrm{T}^{\delta_L}\,\mathbf{g}^{\Omega}_{\alpha_I} - \mathbf{f}^{\gamma_K;\beta_J}\,\mathrm{T}_{;\beta_J}\,\mathrm{T}_{\gamma_K}\,\mathbf{g}^{\Omega}_{\alpha_I} \\ &= \overline{\partial}g^{\Omega}_{\alpha_I} + C_{|L||I-L|}\partial_{(\alpha_L}f^{\gamma_K}\wedge g^{\Omega}_{\alpha_{I-L})\gamma_K} - (-1)^{|J|}C_{|K||J|}f^{\beta_J\gamma_K}\wedge\partial_{\beta_J}g^{\Omega}_{\alpha_I\gamma_K} \end{aligned}$$

where we have introduced the $(0,1)$-forms

$$\begin{aligned} \mathbf{f} = \mathbf{f}_1 - \mathbf{f}_2 &= \mathbf{f}^{\gamma_K}_{\delta_L}\,\mathrm{T}_{\gamma_K}\,\mathrm{T}^{\delta_L} - \mathbf{f}^{\gamma_K;\beta_J}\,\mathrm{T}_{;\beta_J}\,\mathrm{T}_{\gamma_K} \\ &= \partial_{\delta_L}f^{\gamma_K}\,\mathrm{T}_{\gamma_K}\,\mathrm{T}^{\delta_L} - C_{|K||J|}C_{|J-M||M|}\partial_{\beta_{J-M}}f^{\beta_J\gamma_K}\,\mathrm{T}_{;\beta_M}\,\mathrm{T}_{\gamma_K} \end{aligned} \tag{7.51}$$

In the gauge $\partial_{\beta_i}f^{\cdots\beta_i\cdots} = 0$, which we usually employ, this becomes

$$\mathbf{f} = \partial_{\delta_L}f^{\gamma_K}\,\mathrm{T}_{\gamma_K}\,\mathrm{T}^{\delta_L} - C_{|K||J|}f^{\beta_J\gamma_K}\,\mathrm{T}_{;\beta_J}\,\mathrm{T}_{\gamma_K} \tag{7.52}$$

These (0,1)-forms are extending the action of the complex deformation

$$f = f^{\beta_J}\partial_{\beta_J} = f^{\beta_J}\,\mathrm{T}_{;\beta_J} \to \mathbf{f}^{\gamma_K}_{\delta_L}\,\mathrm{T}_{\gamma_K}\,\mathrm{T}^{\delta_L} - \mathbf{f}^{\gamma_K;\beta_J}\,\mathrm{T}_{;\beta_J}\,\mathrm{T}_{\gamma_K} = \mathbf{f}_1 - \mathbf{f}_2 = \mathbf{f}\,. \tag{7.53}$$

Note that $\mathbf{f}$ transforms in a different representation than $\mathbf{g}$, so the action of $\overline{\partial}_{\mathbf{f}}$ on $\mathbf{f}$

is different. We define the action of the generators T on $\mathbf{f}$ as

$$\mathrm{T}_{;\mu_M}\,\mathbf{f}^{;\alpha_I} = \partial_{\mu_M}\mathbf{f}^{;\alpha_I}\,, \tag{7.54}$$

$$\mathrm{T}_{\nu_N}\,\mathbf{f}^{;\alpha_I} = \begin{cases} 0 \quad \text{for } |N| > |I| \\ \delta^{(\alpha_N}_{\nu_N}\mathbf{f}^{;\alpha_{I-N})} \quad \text{else}\,, \end{cases} \tag{7.55}$$

and thus

$$\mathrm{T}_{\nu_N} : \mathrm{Sym}^{(r)}(T^{1,0}\mathbb{P}\mathscr{T}) \to \mathrm{Sym}^{(r-|N|)}(T^{1,0}\mathbb{P}\mathscr{T})\,. \tag{7.56}$$

The corresponding action of T^{ρ_R} is of different nature though, as will we see below. If we define it for now as $\mathrm{T}^{\rho_R}\,\mathbf{f} = 0$, then we can write the equations of motion (6.33) as

$$\begin{aligned} \overline{\partial}\mathbf{f}^{;\alpha_I} - \mathbf{f}\wedge\mathbf{f}^{;\alpha_I} &= \overline{\partial}\mathbf{f}^{;\alpha_I} + \mathbf{f}^{\gamma_J;\beta_K}\,\mathrm{T}_{;\beta_K}\,\mathrm{T}_{\gamma_J}\,\wedge\mathbf{f}^{;\alpha_I} \\ &= \overline{\partial}\mathbf{f}^{;\alpha_I} + \mathbf{f}^{(\alpha_J|;\beta_K}\wedge\partial_{\beta_K}\mathbf{f}^{;|\alpha_{I-J})} \\ &= \overline{\partial}f^{\alpha_I} + C_{|K||J|}f^{\beta_K(\alpha_J}\wedge\partial_{\beta_K}f^{\alpha_{I-J})}\,. \end{aligned} \tag{7.57}$$

We thus have the translation between $\star$-product and jet-bundle formulation

$$\begin{array}{cc} \left(\overline{\partial}f - f\,\overline{\star}\,f\right)\big|_{U=0} = 0 & \\ \left(\overline{\partial}g + \{f,g\}_{\overline{\star}}\right)\big|_{\overline{U}=0} = 0 & \end{array} \quad\Longleftrightarrow\quad \begin{array}{c} \overline{\partial}\mathbf{f}^{;\alpha_I} - \mathbf{f}\wedge\mathbf{f}^{;\alpha_I} = 0 \\ \overline{\partial}\mathbf{g}_{\alpha_I} + \mathbf{f}\wedge\mathbf{g}_{\alpha_I} = 0 \end{array} \tag{7.58}$$

In this way we see that those two formulations are completely equivalent. The projection in the $\star$-product language does not have an analogous statement in the jet language as it is natural enough to select the right components by hand. The condition $\mathrm{T}^{\rho_R}\,\mathbf{f} = 0$ on the other hand is analogous to picking $\alpha = 1$ for the $\overline{\star}$-product. We can understand this by first defining a proper action of T^{ρ_R} on $\mathbf{f}$ and then extending the action of $\mathbf{f}$.

7.2.1 Extension of f

Similar to the duality of the actions of T_{γ_K} and T^{δ_L} on $\mathbf{g}$, we expect

$$\mathrm{T}^{\rho_R}\,\mathbf{f}^{;\alpha_I} = \alpha^{|R|}\mathbf{f}^{(\rho_R;\alpha_I)}\,, \tag{7.59}$$

for some unfixed sign $\alpha = \pm 1$. This action corresponds to the proper dual action of this generator on $\mathbf{g}$ and generates the appropriate numerical factor. It is now straightforward to define

$$\hat{\mathbf{f}} = -\mathbf{f}_{\delta_L}^{\gamma_K;\beta_J}\, \mathrm{T}_{;\beta_J}\, \mathrm{T}_{\gamma_K}\, \mathrm{T}^{\delta_L} = -C_{|K||J|}\partial_{\delta_L} f^{\beta_J\gamma_K}\, \mathrm{T}_{;\beta_J}\, \mathrm{T}_{\gamma_K}\, \mathrm{T}^{\delta_L}\,, \tag{7.60}$$

for which $\mathbf{f}_1 = -\hat{\mathbf{f}}(|J| = 0)$ and $\mathbf{f}_2 = -\hat{\mathbf{f}}(|L| = 0)$. Note that $\mathbf{f}$ is not a limit of $\hat{\mathbf{f}}$ as $\mathbf{f} = \mathbf{f}_1 - \mathbf{f}_2$, whereas $\hat{\mathbf{f}} = -\mathbf{f}_1 - \mathbf{f}_2 + \ldots$. Defining an extended holomorphic structure through $\hat{\mathbf{f}}$, we realise

$$\begin{aligned}\left(\overline{\partial} - \hat{\mathbf{f}}\right) \wedge \mathbf{f}^{;\alpha_I} &= \overline{\partial}\mathbf{f}^{;\alpha_I} + \mathbf{f}_{\mu_M}^{\gamma_L;\beta_J}\, \mathrm{T}_{;\beta_J}\, \mathrm{T}_{\gamma_L}\, \mathrm{T}^{\mu_M} \wedge \mathbf{f}^{;\alpha_I} \\ &= \overline{\partial}\mathbf{f}^{;\alpha_I} + \alpha^{|M|}\mathbf{f}_{\mu_M}^{(\alpha_L|;\beta_J} \wedge \mathbf{f}_{\beta_K}^{\mu_M;|\alpha_{I-L})} \\ &= \overline{\partial} f^{\alpha_I} + \alpha^{|M|} C_{|J||L|} C_{|I-L||M|} \partial_{\mu_M} f^{\beta_J(\alpha_L} \wedge \partial_{\beta_J} f^{\alpha_{I-L})\mu_M}\,, \end{aligned} \tag{7.61}$$

which is the same as the equations of motion from the full $\star$-product (7.26) for $\alpha = -1$. Since setting $\alpha = 1$ has the quadratic term in f vanish, it is practically the same condition as $\mathrm{T}^{\rho_R}\,\mathbf{f} = 0$. The corresponding action of $\hat{\mathbf{f}}$ on $\mathbf{g}$ is

$$\begin{aligned}\left(\overline{\partial} + \hat{\mathbf{f}}\right) \wedge \mathbf{g}_{\alpha_I} &= \overline{\partial}\mathbf{g}_{\alpha_I} - \mathbf{f}_{\delta_L}^{\gamma_K;\beta_J}\, \mathrm{T}_{;\beta_J}\, \mathrm{T}_{\gamma_K}\, \mathrm{T}^{\delta_L} \wedge \mathbf{g}_{\alpha_I} \\ &= \overline{\partial}\mathbf{g}_{\alpha_I} - (-1)^{|J|} C_{|I-L||L|} \mathbf{f}_{(\alpha_L|}^{\gamma_K;\beta_J} \wedge \partial_{\beta_J}\mathbf{g}_{|\alpha_{I-L})\gamma_K} \\ &= \overline{\partial} g_{\alpha_I} - (-1)^{|J|} C_{|J||K|} C_{|I-L||L|} \partial_{(\alpha_L|} f^{\beta_J\gamma_K} \wedge \partial_{\beta_J} g_{|\alpha_{I-L})\gamma_K}\,, \end{aligned} \tag{7.62}$$

which we can recognise from (7.29). Having α appear in the result can be achieved e.g. through a redefinition of $\mathrm{T}_{\gamma_K}\,\mathbf{g}_{\alpha_I} = \alpha^{|K|}\mathbf{g}_{(\alpha_I\gamma_K)}$, which makes (7.62) match (7.28). By redefining the actions of the other generators in this way, it is possible to have α appear with any desired power, and including an anti-commutator in the definition of $\hat{\mathbf{f}}$

$$\hat{\mathbf{f}} = -\mathbf{f}_{\delta_L}^{\gamma_K;\beta_J}\, \mathrm{T}_{;\beta_J} \left\{\mathrm{T}_{\gamma_K}, \mathrm{T}^{\delta_L}\right\} \tag{7.63}$$

would generate the two needed terms for the equations of motion for $\mathbf{g}$, since it leaves the equations of motion for $\mathbf{f}$ unchanged as the generators T commute in their action

on $\mathbf{f}$. However, this still does not recover the correct equations of motion, and it remains unclear how to fix this shortcoming.

Overall we saw that interpretation (7.3) is trivial, and that between interpretations (7.4) and (7.5) only the second one allows the form (7.58) that we were looking for. However, the equations of motion for $\mathbf{f}$ are consistent in the extension and lead to a proper representation of the Weyl algebra by choosing $\alpha = -1$. It is not clear how the equations of motion for $\mathbf{g}$ can be brought to the desired form.

7.2.2 Commutation Relations

It is interesting to see what kind of algebra the generators T span. For that we compute the action of their commutators on $\mathrm{End}(\mathcal{B})$ and $\mathcal{B}$ and realise that, in order to close the algebra, we need to view the generators defined above as first-level generators and extend the algebra by an infinite number of levels. By definition, each level has a decomposition in terms of first-level generators.

Action on $\mathrm{End}(\mathcal{B})$

Since T_{ν_N} and T^{ρ_R} remove and add indices in different positions, all generators commute when acting on $\mathbf{f}$.

Action on $\mathcal{B}$

We assume that all generators acting on $\mathbf{g}$ are defined as in (7.47) to (7.49). Two generators of the same kind commute, especially for T^{ρ_R} and T^{δ_L} because

$$\binom{|I-L|}{|R|}\binom{|I|}{|L|} = \binom{|I-R|}{|L|}\binom{|I|}{|R|} \,.$$

Also, $\mathrm{T}_{;\beta_J}$ commutes with all other generators, $[\mathrm{T}_{;\beta_J}, \cdot] = 0$. Note that the product is ordered in T_{γ_K} and T^{δ_L} because (we assume $|K| \leq |L| \leq |I|$ to avoid a case by

case study)

$$\begin{aligned}
\mathrm{T}^{\delta_L}\,\mathrm{T}_{\gamma_K}\,\mathbf{g}_{\alpha_I} &= \tbinom{|I+K|}{|L|}\delta^{\delta_L}_{(\alpha_L}\mathbf{g}_{\alpha_{I-L}\gamma_K)} \\
&= \tfrac{|I+K|!}{|L|!|I-L+K|!}\tfrac{|L-K|!}{|L-K|!}\tfrac{|I|!}{|I+K|!}\tfrac{|L-M|!|I-L+M|!|I|!}{|I|!|L-M|!|I-L+M|!}\sum_{|M|=0}^{|K|}\delta^{(\delta_{L-M}}_{(\alpha_{L-M}}\mathbf{g}_{\alpha_{I-L+M})(\gamma_{K-M}}\delta^{\delta_M)}_{\gamma_M)} \\
&= \tfrac{|L-K|!}{|L|!}\sum_{|M|=0}^{|K|}\frac{\binom{|I|}{|L-K|}}{\binom{|I|}{|L-M|}}\,\mathrm{T}_{(\gamma_{K-M}}\,\delta^{(\delta_M}_{\gamma_M)}\,\mathrm{T}^{\delta_{L-M})}\,\mathbf{g}_{\alpha_I}\,. \qquad (7.64)
\end{aligned}$$

If $^{\delta_L}$ and $_{\gamma_K}$ are contracted with $\mathbf{f}^{\gamma_K}_{\delta_L}$, then for the usual gauge $\mathbf{f}^{\cdots\gamma_i\cdots}_{\cdots\gamma_i\cdots}=0$ the sum reduces to the term with $|M|=0$ and thus we obtain the simple relation

$$\mathbf{f}^{\gamma_K}_{\delta_L}\,\mathrm{T}^{\delta_L}\,\mathrm{T}_{\gamma_K}\,\mathbf{g}_{\alpha_I} = \tfrac{|I-L|!}{|I-L+K|!}\mathbf{f}^{\gamma_K}_{\delta_L}\,\mathrm{T}_{\gamma_K}\,\mathrm{T}^{\delta_L}\,\mathbf{g}_{\alpha_I}\,. \qquad (7.65)$$

In order to close the algebra spanned by the generators T, i.e. to write down commutation relations, we have to extend the algebra and include a 'new' set of (composite or second-level) generators

$$\mathrm{T}^{\delta_L}_{\gamma_K} : \mathrm{Sym}^{(r)}(T^{*1,0}\mathbb{P}\mathscr{T}) \to \mathrm{Sym}^{(r-|L|+|K|)}(T^{*1,0}\mathbb{P}\mathscr{T}) \qquad (7.66)$$

$$\mathrm{T}^{\delta_L}_{\gamma_K} = \mathrm{T}_{\gamma_K}\,\mathrm{T}^{\delta_L} : \mathbf{g}_{\alpha_I} \mapsto \tbinom{|I|}{|L|}\delta^{\delta_L}_{(\alpha_L}\mathbf{g}_{\alpha_{I-L})\gamma_K}\,. \qquad (7.67)$$

They appear in the commutation relations

$$[\mathrm{T}^{\delta_L},\mathrm{T}_{\gamma_K}]\mathbf{g}_{\alpha_I} = \left(\tfrac{|I-L|!}{|I-L+K|!}-1\right)\mathrm{T}^{\delta_L}_{\gamma_K}\,\mathbf{g}_{\alpha_I} + \tfrac{|L-K|!}{|L|!}\sum_{|M|=1}^{|K|}\frac{\binom{|I|}{|L-K|}}{\binom{|I|}{|L-M|}}\delta^{(\delta_M}_{(\gamma_M}\,\mathrm{T}^{\delta_{L-M})}_{\gamma_{K-M})}\,\mathbf{g}_{\alpha_I}\,. \qquad (7.68)$$

Further commutation relations with these new generators give rise to yet another level of generators

$$[\mathrm{T}^{\rho_R},\mathrm{T}^{\delta_L}_{\gamma_K}]\mathbf{g}_{\alpha_I} = \left(\tfrac{|I-R|!}{|I-R+K|!}-1\right)\mathrm{T}^{\rho_R\delta_L}_{\gamma_K}\,\mathbf{g}_{\alpha_I} + \tfrac{|R-K|!}{|R|!}\sum_{|M|=1}^{|K|}\frac{\binom{|I|}{|R-K|}}{\binom{|I|}{|R-M|}}\delta^{(\rho_M}_{(\gamma_M}\,\mathrm{T}^{\rho_{R-M})\delta_L}_{\gamma_{K-M})}\,\mathbf{g}_{\alpha_I}\,, \qquad (7.69)$$

$$[\mathrm{T}_{\eta_N},\mathrm{T}^{\delta_L}_{\gamma_K}]\mathbf{g}_{\alpha_I} = \left(1-\tfrac{|I-L|!}{|I-L+N|!}\right)\mathrm{T}^{\delta_L}_{\eta_N\gamma_K}\,\mathbf{g}_{\alpha_I} - \tfrac{|L-N|!}{|L|!}\sum_{|M|=1}^{|N|}\frac{\binom{|I|}{|L-N|}}{\binom{|I|}{|L-M|}}\delta^{(\delta_M}_{(\eta_M}\,\mathrm{T}^{\delta_{L-M})}_{\eta_{N-M})\gamma_K}\,\mathbf{g}_{\alpha_I}\,, \qquad (7.70)$$

with third-level generators $\mathrm{T}^{\rho_R\delta_L}_{\gamma_K} = \mathrm{T}_{\gamma_K}\,\mathrm{T}^{\rho_R}\,\mathrm{T}^{\delta_L}$ and $\mathrm{T}^{\delta_L}_{\eta_N\gamma_K} = \mathrm{T}_{\eta_N}\,\mathrm{T}_{\gamma_K}\,\mathrm{T}^{\delta_L}$. Continuing in this manner gives rise to an infinite tower of generators whose commutation relations are nested applications of (7.69) and (7.70).

We end our detour here and take the results gained from those two formalisms as inspirations for what comes next: extending the self-dual action we discussed in the last chapter and include anti-self-dual interactions to recover the full theory. We will mainly continue working with the jet bundle formalism.

7.3 Anti-Self-Dual Interactions

To go beyond the self-dual sector to the full theory we must include interactions of the anti-self-dual fields. There are a number of possible interactions, however we will restrict ourselves to the simplest case by formulating the twistor analogue of interaction term in (4.52). In this we will closely follow the discussion in [17].

Given a curved twistor space, $\mathscr{T}$, with fibre coordinates σ_A over a manifold $\mathcal{M}$ with space-time coordinates $x^{AA'}$, we can choose an adapted vector bundle coordinate system for the cotangent bundle. This defines a set of dual sections in $\Omega^{1,0}(\mathscr{T})$ which we label $e^\alpha = (e_A, e^{A'})$. The one-forms e^A are of homogeneity degree one, and when restricted to constant $x^{AA'}$, that is to the fibres of $\mathscr{T} \to \mathcal{M}$, they are given by $e^A = \mathrm{d}\sigma^A$. The one-form $e_0 = \sigma^A e_A$ is well-defined on $\mathbb{P}\mathscr{T}$ with values in $\mathcal{O}(2)$. The $(1,0)$-forms $e^{A'}$, also of homogeneity one in σ_A, can be defined at each point to be orthogonal to the fibres of $\mathscr{T} \to \mathcal{M}$. We can additionally choose the holomorphic volume form to be

$$\Omega = \frac{1}{2}\epsilon_{A'B'} e^{A'} \wedge e^{B'} \wedge \sigma_A \,\mathrm{d}\sigma^A \ . \tag{7.71}$$

In flat twistor space these forms can be given explicitly as

$$e_A = \mathrm{d}\sigma_A \quad \text{and} \quad e^{A'} = i\sigma_A \,\mathrm{d}x^{AA'} \tag{7.72}$$

as well as

$$\Omega = \frac{1}{2}\epsilon_{A'B'}\sigma_A\sigma_B\,\mathrm{d}x^{AA'}\wedge\mathrm{d}x^{BB'}\wedge\langle\sigma\,\mathrm{d}\sigma\rangle\ . \tag{7.73}$$

This coordinate system, and the dual sections, naturally define a basis for our homogeneous tensors, and we can expand our twistor space tensors in this basis $g = g_{\alpha_I}e^{\alpha_I}$. For example in the spin-three case

$$\begin{aligned} g = \Big(&g^{A_1A_2}\wedge(e_{A_1}\otimes e_{A_2}) + g^{A_1}{}_{A_2'}\wedge(e_{A_1}\otimes e^{A_2'}) \\ &+ g_{A_1'}{}^{A_2}\wedge(e^{A_1'}\otimes e_{A_2}) + g_{A_1'A_2'}\wedge(e^{A_1'}\otimes e^{A_2'})\Big)\otimes\Omega\,, \end{aligned} \tag{7.74}$$

where the $g_{A_1A_2}$ etc. are $(0,1)$-forms of homogeneity -6. By integrating over the fibres of twistor space we can now define space-time 2-forms $G^{A_1A_2}{}_{B_1B_2}$, or more generally $G^{A_I}{}_{B_I}$, via

$$G^{A_I}{}_{B_I}(x) = \int_\Xi \sigma_{B_I}g^{A_I}\wedge\Omega\,. \tag{7.75}$$

Motivated by the form of the anti-self-dual interactions in the linearized space-time action,

$$\int G^{A_I}{}_{B_I}\wedge G^{B_I}{}_{A_I}\,, \tag{7.76}$$

as well as the interactions for conformal gravity and Yang-Mills, we consider the twistor space expression

$$\int_{\mathbb{P}\mathscr{T}\times_{\mathcal{M}}\mathbb{P}\mathscr{T}}\sum_{|I|=0}^{\infty}(\sigma_{1A_I}\sigma_{2B_I})\,g_1^{\Omega B_I}\wedge g_2^{\Omega A_I}\,. \tag{7.77}$$

Here the space $\mathbb{P}\mathscr{T}\times_{\mathcal{M}}\mathbb{P}\mathscr{T}$ is the space whose fibres over the manifold $\mathcal{M}$ are Cartesian products of the fibres of the individual twistor spaces $\mathbb{P}\mathscr{T}\to\mathcal{M}$, namely $\Xi_1\times\Xi_2\simeq\mathbb{CP}^1\times\mathbb{CP}^1$, with homogeneous fibre coordinates $(\sigma_{1A},\sigma_{2B})$. The fields g_1 and g_2 are $(0,1)$-forms depending on the respective fibre coordinates, while Ω_1 and Ω_2 are the respective holomorphic volume forms. This action is constructed using the Penrose transform with respect to the background complex structure, and so

at the linearised level it is obviously invariant under shift of g by $\overline{\partial}\chi$ terms as they result in total derivative terms with respect to the fibre integration. However if we wish to include the effects of the deformation f, the equation satisfied by g, $\overline{\partial}_f g = 0$, is modified.

We account for this deformation by inserting the appropriate Green's function $\overline{\partial}_f^{-1}$ to propagate the fields in deformed twistor space along the fibres

$$S_{\text{a.s.d.}} = \int_{\mathbb{PT}\times_{\mathcal{M}}\mathbb{PT}} \sum_{|I|=0}^{\infty} (\sigma_{1A_I}\sigma_{2B_I}) \left(\overline{\partial}\,\overline{\partial}_{\mathbf{f}}^{-1}\Big|_{\Xi_1}\right) g_1^{\Omega B_I} \wedge \left(\overline{\partial}\,\overline{\partial}_{\mathbf{f}}^{-1}\Big|_{\Xi_2}\right) g_2^{\Omega A_I} \ . \tag{7.78}$$

For small deformations we can expand the Green function to $\overline{\partial}_{\mathbf{f}}$ as

$$\overline{\partial}\,\overline{\partial}_{\mathbf{f}}^{-1}\Big|_{\Xi} \equiv \sum_{n=0}^{\infty} \left(\overline{\partial}^{-1}\mathbf{f}\Big|_{\Xi}\right)^n \ . \tag{7.79}$$

We thus need to know how $\overline{\partial}\Big|_{\Xi}^{-1}$ acts on holomorphic one-forms of homogeneity n, i.e. elements of $\Omega^{0,1}(\mathbb{CP}^1, \mathcal{O}(n))$. This has been previously discussed in context of twistor actions, e.g. [37], where it was shown that it can be expressed in terms of the Cauchy kernel. Only for the case of $n = -1$ is this operation uniquely defined. To see this consider the fact that $H^{0,1}(\mathbb{CP}^1, \mathcal{O}(n))$ is empty for $n \geq -1$, that is every $k \in \Omega^{0,1}(\mathbb{CP}^1, \mathcal{O}(n))$ is exact, and so $k = \overline{\partial}\omega$ for some $\omega \in \Omega^0(\mathbb{CP}^1, \mathcal{O}(n))$. Additionally, as $H^0(\mathbb{CP}^1, \mathcal{O}(n))$ is also empty for $n \leq -1$, there is no freedom in the definition of ω. Thus ω gives a unique definition of $\overline{\partial}^{-1}k$. We can give an explicit construction for this form in our case: suppose $\Xi \simeq \mathbb{CP}^1$ is parametrised by coordinates σ_A, we can define ω by

$$\overline{\partial}\Big|_{\Xi}^{-1} k(\sigma) = \frac{1}{2\pi i}\int_{\Xi} \frac{D\sigma'}{\langle\sigma\sigma'\rangle} \wedge k(\sigma') \ . \tag{7.80}$$

That this formula is correct can be seen by acting with $\overline{\partial}$ and using some facts about $\overline{\delta}$-functions.

We can define the delta-function on the complex plane as a $(0,1)$-form through

$$\overline{\delta}(az - b) = \frac{1}{2\pi i}\,\mathrm{d}\overline{z}\frac{\partial}{\partial\overline{z}}\left(\frac{1}{az-b}\right) \ . \tag{7.81}$$

In particular, this satisfies for a holomorphic function $f(z)$

$$\int_{\mathbb{C}} \mathrm{d}z \wedge \bar{\delta}(az-b) f(z) = \frac{1}{a} f\left(\frac{b}{a}\right) , \tag{7.82}$$

which can be rewritten as

$$\bar{\partial}\left(\frac{1}{\langle \sigma\sigma'\rangle}\right) = 2\pi i\, \bar{\delta}^{(2)}(\sigma-\sigma') D\bar{\sigma}' . \tag{7.83}$$

When we consider forms with $n \geq 0$ we can see from (7.80) that simply due to scaling in σ the formula must be modified and we need to include additional factors to give the correct weight under coordinate rescalings. This can be done by using a reference spinor: $\left(\frac{\langle \xi\sigma\rangle}{\langle \xi\sigma'\rangle}\right)^{n+1}$. The arbitrariness in the choice of ξ and thus the non-uniqueness of the resulting ω corresponds to the non-triviality of $H^0(\mathbb{CP}^1, \mathcal{O}(n))$ for $n > -1$. This gives rise to a gauge freedom which should drop out of any physical observable. We also chose to define the Cauchy kernel to have only one pole of highest degree at $\sigma' = \xi$ as opposed to introducing many different reference spinors ξ_i, in each of which there is only a simple pole. In the case $n < -1$ by modifying the formula (7.80) by adding additional factors to fix the scaling we find additional singularities which need to be specified. Using the specific homogeneity degrees, we find from the expression (7.52) for the $(0,1)$-form $\mathbf{f}$, that we have the following weights

$$\left(\bar{\partial}^{-1}\mathbf{f}\Big|_{\Xi}\right)(Z(\sigma)) = \frac{1}{2\pi i}\int_{\Xi} \frac{D\sigma'}{\langle \sigma\sigma'\rangle} \sum_{|J|,|K|,|L|=0}^{\infty} \left(\frac{\langle \xi\sigma\rangle}{\langle \xi\sigma'\rangle}\right)^{1+|J|+|K|-|L|} \mathbf{f}_{\delta_L}^{\gamma_K;\beta_J}(Z(\sigma'))\, \mathrm{T}^{\delta_L}_{\gamma_K;\beta_J} , \tag{7.84}$$

where

$$\mathbf{f}_{\delta_L}^{\gamma_K;\beta_J}(Z(\sigma'))\, \mathrm{T}^{\delta_L}_{\gamma_K;\beta_J} = \delta_{|J|,0}\, \mathbf{f}_{\delta_L}^{\gamma_K}(Z(\sigma'))\, \mathrm{T}_{\gamma_K}\, \mathrm{T}^{\delta_L} - \delta^{|L|,0}\, \mathbf{f}^{\gamma_K;\beta_J}(Z(\sigma'))\, \mathrm{T}_{;\beta_J}\, \mathrm{T}_{\gamma_K} .$$

The full action of the conformal higher spin theory is given as

$$S\left[f^{(\bullet)}, g^{(\bullet)}\right] = S_{\text{s.d.}}\left[f^{(\bullet)}, g^{(\bullet)}\right] - \varepsilon^2 S_{\text{a.s.d.}}\left[f^{(\bullet)}, g^{(\bullet)}\right] , \tag{7.85}$$

where ε is a dimensionless parameter.

To fully justify the choice for our proposed action (7.78), a proof of its gauge invariance is still outstanding. One piece of evidence that it indeed is gauge invariant is given by the fact that the three-point spin-s MHV amplitude is the parity conjugate of the $\overline{\text{MHV}}$ amplitude, both of which will be presented in the next chapter. We can however choose to expand around a self-dual graviton background, where the expansion is well-defined and has been understood [68]. This procedure should be equivalent to the proposed expansion above, at least for the spin-2 case that we will consider, but a full proof is still missing. Note however that the expansion (7.84) is completely general and allows for interactions of particles with arbitrary spin.

Chapter 8

Amplitudes

Amongst the simplest semi-classical observables in any field theory with an action description are tree-level scattering amplitudes. For $s > 2$ little is known about the tree-level S-matrix of CHS theory; in this section we use the twistor action to compute all 3-point tree amplitudes with two positive helicity external particles (*i.e.*, $\overline{\text{MHV}}$ amplitudes) and all n-point tree amplitudes for two helicity $-s$ states and $(n-2)$ helicity $+2$ states (*i.e.*, MHV amplitudes). The advantage of the twistor framework is that these amplitudes are directly generated by the SD and non-local parts of the twistor action, respectively. In the space-time formulation of CHS theory, these amplitudes would receive multiple Feynman diagram contributions involving complicated interaction vertices and space-time propagators (*c.f.*, [28, 29]).

In general, the definition of the S-matrix for massless particles with higher spin requires some subtlety, but can be formally defined through e.g. the (A)dS/CFT correspondence, e.g. [97]. However, as we are only considering tree-level the definition of scattering amplitudes needed for our purposes can be fairly naïve. The semi-classical S-matrix for any theory with a Lagrangian description is encoded by a generating functional defined purely in terms of the classical action – this is the basic content of the LSZ prescription. The external states of any scattering process are free fields which solve the linearised equations of motion on the scattering background.

A n-point tree-level scattering amplitude is given by the piece of the generating functional which is multi-linear in n such on-shell external states. This perspective applies just as well to CHS theory formulated on twistor space, with the classical action given by (7.85) and a 'flat' scattering background $\mathbb{PT}$.

In four space-time dimensions, the S-matrix of any field theory which admits a perturbative expansion around the SD sector will posses an important structure: scattering amplitudes can be classified by their 'MHV degree'. This means that the amplitudes are specified by on-shell four-momenta and a helicity label (rather than a polarization). Consistency and integrability of the SD sector imply that amplitudes for which fewer than two external states have negative helicity vanish; the 'maximal helicity violating' – or MHV – amplitudes are those with two negative helicity external states and the rest positive helicity. At three points (for complexified kinematics), there is also the possibility of an $\overline{\text{MHV}}$ amplitude, with one negative helicity and two positive helicity external legs. We denote the n-point tree-level amplitude with k negative helicity external states by $\mathcal{M}_{n,k-2}$.

An advantage of the twistor formulation of CHS theory is that the generating functionals for $\overline{\text{MHV}}$ and MHV amplitudes are provided by the classical action itself. The SD portion of the action generates all 3-point amplitudes with a single negative helicity and two positive helicity CHS external states (of arbitrary spin). These are precisely the $\overline{\text{MHV}}$ amplitudes, packaged in the generating functional

$$I^{\overline{\text{MHV}}} = \sum_{|I|=0}^{\infty} \int_{\mathbb{P}\mathscr{T}} D^3\mathcal{Z} \wedge g_{\alpha_I} \wedge \sum_{|J|=0}^{|I|} \sum_{|K|=0}^{\infty} \binom{|J|+|K|}{|J|} f^{\beta_K(\alpha_J} \wedge \partial_{\beta_K} f^{\alpha_{I-J})} \,, \tag{8.1}$$

where g_{α_I} and f^{α_I} obey the linearised equations of motion: $\overline{\partial} g_{\alpha_I} = 0 = \overline{\partial} f^{\alpha_I}$.

All amplitudes with $(n-2)$ positive helicity conformal gravitons and two helicity $-s$ CHS fields are generated by the non-local term (7.78) in the twistor action. These constitute all the tree-level MHV amplitudes of CHS theory on a conformal gravity

background. Instead of (7.78) we consider the generating functional

$$I^{\mathrm{MHV}} = \int\limits_{\mathbb{P}\mathscr{T}\times_{\mathbb{M}}\mathbb{P}\mathscr{T}} \Omega_1 \wedge \Omega_2 \wedge \sum_{|I|=0}^{\infty} \mathcal{Z}_1^{\alpha_I}\, \mathcal{Z}_2^{\beta_I}\, g_{\beta_I}(\mathcal{Z}_1) \wedge g_{\alpha_I}(\mathcal{Z}_2)\,. \tag{8.2}$$

This is equivalent to (7.78) because for the measure we have

$$\Omega_1 \wedge \Omega_2 = D^3\mathcal{Z}_1 \wedge D^3\mathcal{Z}_2 = \langle 12\rangle^2\, \mathrm{d}\mu \wedge D\sigma_1 \wedge D\sigma_2 \tag{8.3}$$

and the integrand obeys

$$\mathcal{Z}_i^{\alpha_I} g_{\alpha_I}(\mathcal{Z}_j) = \mathcal{X}^{\alpha_I A_I}\sigma_{iA_I}\, g_{\alpha_I}(\mathcal{Z}(\sigma_j)) = \sigma_{iA_I} g^{A_I}(\mathcal{Z}(\sigma_j))\,. \tag{8.4}$$

By assumption, the coordinates $\mathcal{Z}_i$ are homogeneous of degree one in the fibre coordinates σ_i, which means that we can always find a $\mathcal{X}^{\alpha_I A_I}$ such that this relation holds. The $\mathcal{X}^{\alpha_I A_I}$ act as a change of coordinates on $\mathbb{P}(\mathbb{S}^+\mathbb{M})$. The product $\mathbb{P}\mathscr{T}\times_{\mathbb{M}}\mathbb{P}\mathscr{T}$ is the fibre-wise product of the curved twistor space associated with the spin-two holomorphic structure $\overline{\partial}_f = \overline{\partial} + f^\alpha\partial_\alpha$. The fibres $\Xi \cong \mathbb{CP}^1$ are defined as the rational curves in $\mathbb{P}\mathscr{T}$ which are holomorphic with respect to this complex structure:

$$\overline{\partial}_f \mathcal{Z}^\alpha(x,\sigma) = 0 \qquad \Longrightarrow \qquad \overline{\partial}\mathcal{Z}^\alpha(x,\sigma) = f^\alpha(\mathcal{Z}(x,\sigma)) \tag{8.5}$$

for fixed $x \in \mathbb{M}$. The weight $+4$ $(3,0)$-form Ω is the top holomorphic form on this curved twistor space. The n-point amplitude is obtained from this functional by perturbatively expanding the integrand to order $(n-2)$ in $\mathcal{Z}$ from solving (8.5), thus generating interactions with f^α, which encodes the SD conformal gravity background, and evaluating the result on the flat fibre-wise product, $\mathbb{PT}\times_{\mathbb{M}}\mathbb{PT}$. Once more, on-shell external states obey $\overline{\partial} g_{\alpha_I} = 0 = \overline{\partial} f^{\alpha_I}$.

More generally, any MHV amplitude can be seen as the amplitude for a negative helicity incoming state to flip helicity after crossing a SD background [37, 36, 18]. In the case of (8.2), this is the amplitude for a negative helicity spin-s CHS field to flip helicity after crossing a SD conformal gravity background. The generating functional is quadratic in the negative helicity CHS states, defined on a non-linear, Bach-flat

(in particular, self-dual) spin-two background. It is therefore natural to conjecture that (8.2) is equivalent to the SD part of quadratic covariant action for CHS fields on a conformal gravity background, whose existence was argued in [98] (see also [99] and [90, 100]).

8.1 Spin Raising- and Lowering Operators

To evaluate the amplitudes produced by the generating functions (8.1) and (8.2), explicit representations for the external states are needed. How to construct these representatives is not immediately obvious since f^{α_I} and g_{α_I} have free twistor indices. A particularly useful framework is provided by the *helicity raising/lowering* formalism in twistor space [101].①

Consider the $s = 2$ example of conformal gravity; a negative helicity, on-shell conformal graviton is encoded by $g_\alpha \in H^{0,1}(\mathbb{PT}, \mathcal{O}(-5))$. This field has the same homogeneity as $\psi \in H^{0,1}(\mathbb{PT}, \mathcal{O}(-5))$, which describes a helicity $-\frac{3}{2}$ Rarita-Schwinger field by the Penrose transform:

$$\Psi_{ABC}(x) = \int_\Xi \lambda_A \lambda_B \lambda_C \, \psi\big|_\Xi \wedge D\lambda \,, \qquad \nabla^{AA'} \Psi_{ABC} = 0 \,, \tag{8.6}$$

where $D\lambda = \epsilon^{AB} \lambda_A \, \mathrm{d}\lambda_B$ is the weight $+2$ holomorphic measure on $\Xi \cong \mathbb{CP}^1$ in twistor space. The Penrose transform of g_α yields a space-time field with a twistor index

$$G_{\alpha BCD}(x) = \int_\Xi \lambda_B \lambda_C \lambda_D \, g_\alpha\big|_\Xi \wedge D\lambda \,, \qquad \nabla^{BB'} G_{\alpha BCD} = 0 \,, \tag{8.7}$$

which splits into primed and unprimed part $G_{\alpha BCD} = (G^A{}_{BCD}, \gamma_{A'BCD})$. The covariant derivative acting on a local twistor index gives on conformally flat background

$$\nabla^B{}_{B'} G^A{}_{BCD} = i\epsilon^{BA} \gamma_{B'BCD} \,, \qquad \nabla^B{}_{B'} \gamma_{A'BCD} = 0 \,, \tag{8.8}$$

① Our treatment here is valid on a conformally flat background; more generally, the equations will be modified by various correction terms involving the trace-free Ricci curvature.

subject to $G^A{}_{ABC} = 0$. These equations can be solved by choosing a fixed dual twistor $(\beta^A, \tilde{\beta}_{A'})$ obeying, in a conformally flat background,

$$\nabla^B{}_{B'}\beta^A = i\epsilon^{BA}\tilde{\beta}_{B'}\,, \qquad \nabla^B{}_{B'}\tilde{\beta}_{A'} = i\Lambda\,\epsilon_{B'A'}\beta^B\,. \tag{8.9}$$

Using these components, we define the space-time spinors

$$G^A{}_{BCD} = \beta^A\Psi_{BCD}\,, \qquad \gamma_{A'BCD} = \tilde{\beta}_{A'}\Psi_{BCD}\,. \tag{8.10}$$

By virtue of the dual twistor equation, these spinors satisfy (8.8) respectively. These equations (8.9) are solved in flat-space, $\Lambda = 0$, by taking $\beta^A = \beta_0^A + ix^{AA'}\tilde{\beta}_{0A'}$ for some constant spinors $\beta_0^A, \tilde{\beta}_{0A'}$. Solutions in arbitrary conformally flat geometries can be found by making the conformal transformation $g_{ab} \to \Omega^2 g_{ab}$. This transformation acts on the spinor components as

$$\beta^A \to \beta^A = \beta_0^A + ix^{AA'}\tilde{\beta}_{0A'}\,, \qquad \tilde{\beta}_{A'} \to \tilde{\beta}_{0A'} + i\Upsilon_{AA'}\beta^A\,, \tag{8.11}$$

where $\Upsilon_{AA'} = \Omega^{-1}\nabla_{AA'}\Omega$. Note that only the primed component transforms.

To obtain a general solution to (8.8), we also have to take into account the solution for the homogeneous case when $\tilde{\beta}_{A'} = 0$. Let $\xi^\alpha = (\xi_A, \tilde{\xi}^{A'})$ be a fixed twistor satisfying $\nabla_A{}^{A'}\tilde{\xi}^{B'} = -i\epsilon^{A'B'}\xi_A$, then the full solution is given by:

$$G^A{}_{BCD} = \tilde{\xi}^{B'}\nabla^A{}_{B'}\Psi_{BCD} - 4i\epsilon^{AE}\,\Psi_{(BCD}\xi_{E)} + \beta^A\,\Psi_{BCD}\,. \tag{8.12}$$

It is straightforward to see that the two equations (8.8) imply the Bach equation

$$\nabla_{AA'}\nabla^B{}_{B'}G^A{}_{BCD} = 0\,, \tag{8.13}$$

so the solution constructed in this way from the Rarita-Schwinger field encodes the ASD modes of conformal gravity. In order to write the solution (8.12) as $g_\alpha = B_\alpha\psi$ on twistor space for $\psi \in H^{0,1}(\mathbb{PT}, \mathcal{O}(-5))$ we note that

$$\int_\Xi \lambda_{(B}\lambda_C\lambda_D\xi_{A)}\,\psi\big|_\Xi \wedge D\lambda = -\frac{1}{4}\int_\Xi \lambda_B\lambda_C\lambda_D\lambda_A\xi_E\,\partial^E\psi\big|_\Xi \wedge D\lambda \tag{8.14}$$

since ξ_E is independent of λ, such that

$$B_\alpha = \left(\tilde{\beta}_{A'}\,,\ \ i\lambda^A \xi^\delta \partial_\delta + \beta^A\right)\,. \tag{8.15}$$

Since B_α is holomorphic and homogeneous, it follows that $g_\alpha \in H^{0,1}(\mathbb{PT}, T^*_{\mathbb{PT}}(-5))$, as desired. The degrees of freedom of conformal gravity are now manifest: in the flat space limit $(\xi_A, \tilde{\xi}^{A'})$ parametrizes the helicity -2 conformal graviton, $\tilde{\beta}_{A'}$ parametrizes the conformal ghost, and β^A parametrizes a conformal spin one state.

Similarly, $f^\alpha \in H^{0,1}(\mathbb{PT}, T_{\mathbb{PT}}(1))$ can be constructed from a helicity $+\frac{3}{2}$ representative $\tilde{\psi} \in H^{0,1}(\mathbb{PT}, \mathcal{O}(1))$ and a helicity raising operator A^α:

$$A^\alpha = \left(\zeta_\gamma Z^\gamma \frac{\partial}{\partial \mu_{A'}} + \tilde{\alpha}^{A'}\,,\ \ \alpha_A\right)\,, \tag{8.16}$$

where $\zeta_\alpha = (\zeta^A, \tilde{\zeta}_{A'})$ is a dual twistor parametrizing the helicity $+2$ conformal graviton, $\tilde{\alpha}^{A'}$ is a spinor parametrizing a conformal spin one state, and α_A parametrizes the conformal ghost in analogous fashion as above. Since this A^α is holomorphic and homogeneous, $f^\alpha = A^\alpha \tilde{\psi}$ encodes the appropriate on-shell information in twistor space.

8.2 Self-Dual Conformal Higher Spin Amplitudes

This procedure of helicity raising or lowering can be applied repeatedly to build on-shell representatives in twistor space for arbitrary spin CHS fields. For $|I| = s-1$, the twistor data g_{α_I}, f^{α_I} is constructed as

$$g_{\alpha_I} = B_{\alpha_I} \psi^{(-3-s)}\,, \qquad f^{\alpha_I} = A^{\alpha_I} \tilde{\psi}^{(s-1)}\,, \tag{8.17}$$

where the lowering/raising operators are products of copies of (8.15) or (8.16), and the fields $\psi^{(-3-s)} \in H^{0,1}(\mathbb{PT}, \mathcal{O}(-3-s))$, $\tilde{\psi}^{(s-1)} \in H^{0,1}(\mathbb{PT}, \mathcal{O}(s-1))$ are representatives for zero-rest-mass fields of helicity $\mp\frac{s+1}{2}$, respectively.

Of course, explicit representatives for $\psi^{(-3-s)}$ and $\tilde{\psi}^{(s-1)}$ are needed to obtain closed-form expressions for scattering amplitudes. A particularly convenient choice

is provided by *dual twistor* wavefunctions, which associate a fixed dual twistor $W_\alpha = (\tilde{\lambda}_{A'}, \tilde{\mu}^A)$ to each external field. The representatives take the form of plane-waves in twistor space, with this dual twistor serving as the 'momentum':

$$\psi^{(-3-s)}(Z;W) = \int \mathrm{d}t\, t^{s+2} e^{t\, W\cdot Z}\,, \qquad \tilde{\psi}^{(s-1)}(Z;W) = \int \frac{\mathrm{d}t}{t^s} e^{t\, W\cdot Z}\,. \tag{8.18}$$

The integrals over the scaling parameter t ensure that the expressions have the correct homogeneity on twistor space. A major advantage of working with dual twistor wavefunctions is that they render differential operators in twistor space as algebraic expressions in the dual twistors. Any amplitude expression in terms of dual twistors can then be transformed into an expression on momentum space by means of the half-Fourier transform. Note that since g_{α_I} and f^{α_I} are form-valued, the exponentials $e^{tW\cdot Z}$ are form-valued as well and yield signs in concrete computations when commuting them. To lighten the notation, we will treat any product $e^{t_i W_i\cdot Z}\cdot e^{t_j W_j\cdot Z}$ as an ordered product ($i<j$) and include the appropriate sign in the pre-factors accordingly. To summarize, arbitrary spin CHS states will be represented on twistor space by:

$$g_{\alpha_I} = B_{\alpha_I}\int \mathrm{d}t\, t^{|I|+3} e^{t\, W\cdot Z}\,, \qquad f^{\alpha_I} = A^{\alpha_I}\int \frac{\mathrm{d}t}{t^{|I|+1}} e^{t\, W\cdot Z}\,, \tag{8.19}$$

where $|I| = s-1$.

Consider the contribution to the generating function (8.1) with arbitrary but fixed external spins s_1, s_2, s_3. One state, say that with spin s_1, has negative helicity while the other two have positive helicity. The 3-point $\overline{\text{MHV}}$ amplitude is the symmetrisation over the positive helicity states:

$$\begin{aligned}\mathcal{M}_{3,-1}(-s_1,+s_2,+s_3) = \int D^3Z \wedge g_{1\,\alpha_I} \wedge \Bigg(& \frac{(|J|+|K|)!}{|K|!\,|J|!} f_2^{\beta_K(\alpha_J} \wedge \partial_{\beta_K} f_3^{\alpha_{I-J})} \\ & + \frac{(|I|-|J|)!}{|K|!\,(|I|-|J|-|K|)!} f_3^{\beta_K(\alpha_{I-J-K}} \wedge \partial_{\beta_K} f_2^{\alpha_{J+K})} \Bigg)\,,\end{aligned} \tag{8.20}$$

where $|I| = s_1 - 1$, $|J| + |K| = s_2 - 1$, and $|I| - |J| = s_3 - 1$. Evaluating the expression on the on-shell states (8.19), a bit of algebra shows that this expression is equal to

$$\begin{aligned}\mathcal{N}^{(s_1,s_2,s_3)} \int \frac{\mathrm{d}^4 Z}{\mathrm{vol}\,\mathbb{C}^*} \frac{\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3}{t_2^{|K|+|J|+1} t_3^{|I|-|J|+1}} t_1^{|I|+3}\, B_{1\,\alpha_I} \\ \times\Big((s_2-1)!\,(s_1-s_2)!\,t_3^{|K|}(A_2\cdot W_3)^{|K|} A_2^{\alpha_J} A_3^{\alpha_{I-J}} \\ + (-1)^{|K|+1}(s_3-1)!\,(s_1-s_3)!\,t_2^{|K|}(A_3\cdot W_2)^{|K|} A_2^{\alpha_{J+K}} A_3^{\alpha_{I-J-K}}\Big) e^{\sum_{i=1}^3 t_i W_i\cdot Z}\,,\end{aligned}$$

where we have re-written the projective measure D^3Z on $\mathbb{PT}$ as a non-projective measure on $\mathbb{C}^4$ modulo a quotient by $\mathbb{C}^*$-scalings. The overall normalisation constant is

$$\mathcal{N}^{(s_1,s_2,s_3)} := \frac{1}{\Gamma(-s_1+s_2+s_3)\Gamma(s_1-s_2+1)\Gamma(s_1-s_3+1)}\,,$$

in terms of the external spins.

Using the $\mathrm{vol}\,\mathbb{C}^*$ to fix $t_1 = 1$ and then performing the d^4Z integrals leaves

$$\begin{aligned}\mathcal{N}^{(s_1,s_2,s_3)} \int \frac{\mathrm{d}t_2\,\mathrm{d}t_3}{t_2^{s_2} t_3^{s_3}} \Big((s_2-1)!\,(s_1-s_2)!\,(B_1\cdot A_2)^{|J|}(B_1\cdot A_3)^{|I|-|J|}(A_2\cdot W_1)^{|K|} \\ + (-1)^{|K|+1}(s_3-1)!\,(s_1-s_3)!\,(B_1\cdot A_2)^{|J|+|K|}(B_1\cdot A_3)^{|I|-|J|-|K|}(A_3\cdot W_1)^{|K|}\Big) \\ \times \delta^4(W_1 + t_2 W_2 + t_3 W_3)\,, \quad (8.21)\end{aligned}$$

where the delta function and gauge conditions $A_i \cdot W_i = 0$ have been used to eliminate some powers of t_3 and t_2 from each term respectively. At this point, the delta functions can be rearranged as

$$\delta^4(W_1 + t_2 W_2 + t_3 W_3) = \frac{\delta^2(\tilde{\mu}_1 + t_2\tilde{\mu}_2 + t_3\tilde{\mu}_3)}{[23]}\,\delta\!\left(t_2 - \frac{[31]}{[23]}\right)\,\delta\!\left(t_3 - \frac{[12]}{[23]}\right)\,,$$

where $[ij] := \epsilon_{A'B'}\tilde{\lambda}_i^{A'}\tilde{\lambda}_j^{B'}$. Note that seemingly the right hand side is anti-symmetric in $2 \leftrightarrow 3$, where as the left hand side is not. However, $\delta^4(W_1 + t_2 W_2 + t_3 W_3)$ resulted from integrating $\exp\!\big(\sum_{i=1}^3 t_i W_i \cdot Z\big)$, which has an ordered sum as it arose from the ordering of the forms g_i and f_j, which itself is anti-symmetric in $2 \leftrightarrow 3$.

The $\mathrm{d}t_2$ and $\mathrm{d}t_3$ integrals are performed explicitly against these delta functions to give

$$\begin{aligned}\mathcal{M}_{3,-1}(-s_1,+s_2,+s_3)&\\ &= \mathcal{N}^{(s_1,s_2,s_3)}\Big((s_2-1)!\,(s_1-s_2)!\,(B_1\cdot A_3)^{s_3-1}(B_1\cdot A_2)^{s_1-s_3}(A_2\cdot W_1)^{s_{23|1}}\\ &\quad+(-1)^{s_{23|1}+1}(s_3-1)!\,(s_1-s_3)!\,(B_1\cdot A_2)^{s_2-1}(B_1\cdot A_3)^{s_1-s_2}(A_3\cdot W_1)^{s_{23|1}}\Big)\\ &\qquad\times\frac{[23]^{s_2+s_3+1}}{[12]^{s_3}[31]^{s_2}}\,\delta^2([23]\tilde{\mu}_1+[31]\tilde{\mu}_2+[12]\tilde{\mu}_3)\ ,\end{aligned}\tag{8.22}$$

using the shorthand

$$s_{ij|k}:=s_i+s_j-s_k-1\,.\tag{8.23}$$

In this expression, the helicity raising/lowering operators A_i, B_i should be thought of as differential operators acting on everything to the right. The dependence on the $\tilde{\mu}$ variables can be removed via the half-Fourier transform

$$\mathcal{M}_{3,-1}(\{\lambda_i,\tilde{\lambda}_i\}):=\int\mathcal{M}_{3,-1}(\{\tilde{\mu}_i,\tilde{\lambda}_i\})\prod_{i=1}^{3}D\tilde{\mu}_i\,e^{\langle\tilde{\mu}_i\lambda_i\rangle}\,,\tag{8.24}$$

if desired.

Although the formula (8.22) for a general $\overline{\mathrm{MHV}}$ amplitude may seem a bit unwieldy, it simplifies considerably in certain spin sectors. For instance, if all the external spins are equal ($s_1=s_2=s_3=s$) then the amplitude is

$$\begin{aligned}\mathcal{M}^{(s)}_{3,-1}(-,+,+)&=\Big((A_3\cdot B_1)^{s-1}(A_2\cdot W_1)^{s-1}+(-1)^s(A_2\cdot B_1)^{s-1}(A_3\cdot W_1)^{s-1}\Big)\\ &\qquad\times\frac{[23]^{2s+1}}{[12]^s[31]^s}\,\delta^2([23]\tilde{\mu}_1+[31]\tilde{\mu}_2+[12]\tilde{\mu}_3)\ .\end{aligned}\tag{8.25}$$

As we will see, even further simplification occurs when the A_i and B_i are chosen to encode the unitary subsector of the CHS theory.

8.3 MHV Amplitudes

8.3.1 n-Point MHV Amplitudes on a Conformal Gravity Background

To compute the n-point MHV amplitude with $(n-2)$ positive helicity spin-two states and two negative helicity spin-s states, we must perturbatively expand the generating functional (8.2). This expansion is operationalized using the same techniques as in [36]. On the SD background $\mathbb{M}$, composed of the helicity $+2$ states, the fibre-wise product $\mathbb{P}\mathscr{T} \times_{\mathbb{M}} \mathbb{P}\mathscr{T}$ is governed by (8.5), which can be re-written as an integral equation:

$$\mathcal{Z}^\alpha(x,\sigma) = X^{\alpha A}\sigma_A + \overline{\partial}^{-1}\Big|_\Xi f^\alpha(\mathcal{Z}(x,\sigma))\,. \tag{8.26}$$

Here, $X^{\alpha A}$ parametrize the homogenous solution and $\overline{\partial}^{-1}\Big|_\Xi$ is the inverse of the $\overline{\partial}$-operator restricted to the holomorphic curve labeled by $x \in \mathbb{M}$. Since f^α is homogeneous of weight $+1$ on twistor space, there is an ambiguity in the definition of $\overline{\partial}^{-1} f^\alpha$ which can be fixed by requiring $\overline{\partial}^{-1}\Big|_\Xi$ to have a second-order zero at some fixed point $\xi \in \Xi \simeq \mathbb{CP}^1$, in the same way as we did in section 7.3. Again, the choice of ξ is entirely arbitrary, constituting a 'gauge' in twistor space.

Having made this choice, (8.26) can be expanded perturbatively around the homogenous solution as

$$\mathcal{Z}^\alpha(x,\sigma) = X^{\alpha\,A}\,\sigma_A + \frac{1}{2\pi i}\int_\Xi \frac{D\sigma'}{\langle\sigma\sigma'\rangle}\frac{\langle\xi\sigma\rangle^2}{\langle\xi\sigma'\rangle^2} f^\alpha(X\cdot\sigma') + \cdots\,, \tag{8.27}$$

where the $+\cdots$ terms are higher-order terms in the expansion arising from expanding $f^\alpha(\mathcal{Z}(x,\sigma'))$ around $Z(x,\sigma') = X\cdot\sigma'$. The idea is to act with this expansion iteratively in the generating functional $(n-2)$ times. The first-order contribution is given by summing over all ways of shifting

$$\mathcal{Z}^\alpha(x,\sigma) \to \frac{1}{2\pi i}\int_\Xi \frac{D\sigma'}{\langle\sigma\sigma\rangle'}\frac{\langle\xi\sigma\rangle^2}{\langle\xi\sigma'\rangle^2} f^\alpha(Z(x,\sigma'))\,, \tag{8.28}$$

in (8.2). At second order, we sum over all the ways of similarly shifting the first-order contribution, and so on.

It will be useful to rewrite the generating functional (8.2) in a slightly different way:

$$I^{\mathrm{MHV}} = \sum_{|I|=0}^{\infty} \int_{\mathbb{M}\times\mathbb{CP}^1\times\mathbb{CP}^1} \frac{\mathrm{d}^8 X}{\mathrm{vol}\, GL(2,\mathbb{C})} \langle 12\rangle^2 D\sigma_1 \wedge D\sigma_2\, \mathcal{Z}^{\alpha_I}(x,\sigma_1)\, \mathcal{Z}^{\beta_I}(x,\sigma_2) \\ \wedge\, g_{1\beta_I}(\mathcal{Z}(x,\sigma_1)) \wedge g_{2\alpha_I}(\mathcal{Z}(x,\sigma_2))\,, \quad (8.29)$$

where the holomorphic measure on the fibre-wise product has been converted into a measure on $\mathbb{M}\times\mathbb{CP}^1\times\mathbb{CP}^1$, with $\mathrm{d}^8X/(\mathrm{vol}\, GL(2,\mathbb{C}))$ being the measure on the space of holomorphic curves in twistor space. The first iteration of the perturbative expansion can act either at $\mathcal{Z}(x,\sigma_1)$ or $\mathcal{Z}(x,\sigma_2)$, and in each case this action can be either in the explicit powers of $\mathcal{Z}^\alpha$ in (8.29) or in the wavefunctions $g_{1,2\alpha_I}$. In the latter case, the expansion takes the form of a derivation:

$$g_{i\alpha_I}(\mathcal{Z}(x,\sigma_i)) \to \frac{1}{2\pi i}\int_\Xi \frac{D\sigma'}{\langle i\sigma'\rangle}\frac{\langle \xi i\rangle^2}{\langle \xi\sigma'\rangle^2} f^\beta(Z(x,\sigma'))\, \frac{\partial}{\partial Z^\beta(x,\sigma_i)} g_{i\alpha_I}(Z(x,\sigma_i))\,. \quad (8.30)$$

All subsequent iterations act similarly: the perturbation is either at an explicit insertion of $\mathcal{Z}(x,\sigma_i)$ or at one of the wavefunctions, which now include f^α insertions from previous iterations. Note that this expansion differs from (7.84) which we introduced in the last chapter, though for a spin-2 background they are thought to be equivalent. The major difference can be seen in the computation of the three-point amplitudes later in this chapter. The expansion (8.30) has derivatives only acting on g, whereas (7.84) also has terms where derivatives are acting on f. For the three-point spin-s amplitudes both terms turn out to be proportional to each other (see below), up to a sign, amounting to a total pre-factor of $(1+(-1)^s)$. This means that odd-spin amplitudes, which are vanishing using (7.84), are present using (8.30).

After iterating $(n-2)$ times, the expansion is equivalent to summing over all n-point Feynman tree diagrams on the $\mathbb{CP}^1$ parametrized by $X^{\alpha A}$ and rooted at the

locations $\sigma_1, \sigma_2 \in \mathbb{CP}^1$. This means that the matrix tree theorem can be used to organize the expansion of the generating functional in terms of weighted determinants, in much the same way that it can be applied in the context of tree amplitudes of Einstein gravity [102, 43].

With dual twistor wavefunctions (8.19), the generating functional becomes

$$I^{\rm MHV} = \sum_{|I|=0}^{\infty} \int \frac{{\rm d}^8 X}{{\rm vol}\, GL(2,\mathbb{C})} \langle 12\rangle^2 \, D\sigma_1 \wedge D\sigma_2 \, (B_1 \cdot \mathcal{Z}(x,\sigma_2))^{|I|} (B_2 \cdot \mathcal{Z}(x,\sigma_1))^{|I|} \\ \wedge {\rm d}t_1 \, t_1^{|I|+3} \wedge {\rm d}t_2 \, t_2^{|I|+3} \wedge \exp(t_1 W_1 \cdot \mathcal{Z}(x,\sigma_1) + t_2 W_2 \cdot \mathcal{Z}(x,\sigma_2)) \ . \quad (8.31)$$

At this point, we take an arbitrary but fixed contribution to perform the expansion, with the two negative helicity states of spin s.

The basic object for applying the matrix tree theorem is a weighted Laplacian matrix, M, encoding the action of the perturbative expansion. This is a $n \times n$ matrix with entries

$$\mathsf{M}_{ij} = (-1)^{|i-j|} t_j \, A_i \cdot W_j \frac{\sqrt{D\sigma_i \wedge D\sigma_j}}{\langle ij\rangle} \,, \qquad \text{for } i \neq j \,, \quad (8.32)$$

$$\mathsf{M}_{ii} = -D\sigma_i \sum_{j\neq i} (-1)^{|i-j|} t_j \frac{A_i \cdot W_j}{\langle ij\rangle} \frac{\langle \xi j\rangle^2}{\langle \xi i\rangle^2} \,, \quad (8.33)$$

following from the structure of (8.30). The signs $(-1)^{|i-j|}$ arise from the ordering of the form-valued exponentials. The reduced determinant of this matrix, with rows and columns corresponding to $i = 1, 2$ removed, encodes the sum of all contributions arising from the expansion acting on wavefunctions. When the expansion acts at one of the explicit $\mathcal{Z}(x, \sigma_i)$ insertions, we must remove an additional row and column from the determinant, insert

$$\mathfrak{m}_j^i = (-1)^{|i-j|} \frac{D\sigma_j}{\langle ij\rangle} \frac{\langle \xi i\rangle^2}{\langle \xi j\rangle^2} \,, \quad (8.34)$$

and sum over all the possible ways of doing this.

Putting the pieces together gives the expression for the n-point MHV amplitude:

$$\mathcal{M}_{n,0} = \int \frac{\mathrm{d}^8 X}{\mathrm{vol}\, GL(2,\mathbb{C})} \langle 12\rangle^2 \, D\sigma_1 \wedge D\sigma_2 \wedge \Bigg((B_1{\cdot}X{\cdot}\sigma_2)^{s-1} (B_2{\cdot}X{\cdot}\sigma_1)^{s-1} \left|\mathsf{M}^{12}_{12}\right| + (s-1)\sum_{i=3}^{n} \Big(B_1{\cdot}A_i \, \mathfrak{m}_i^2 \, (B_1{\cdot}X{\cdot}\sigma_2)^{s-2} (B_2{\cdot}X{\cdot}\sigma_1)^{s-1} + (1 \leftrightarrow 2)\Big) \wedge \left|\mathsf{M}^{12i}_{12i}\right| + \cdots + ((s-1)!)^2 \sum_{\substack{i_1,\ldots,i_{s-1} \\ j_1,\ldots,j_{s-1}}} \left(\prod_{a=1}^{s-1} B_1{\cdot}A_{i_a} B_2{\cdot}A_{j_a} \mathfrak{m}^2_{i_a} \wedge \mathfrak{m}^1_{j_a} \right) \wedge \left|\mathsf{M}^{12 i_1 \cdots j_{s-1}}_{12 i_1 \cdots j_{s-1}}\right| \Bigg) \wedge e^{i\mathcal{P}\cdot X} (t_1 t_2)^{s+4} \prod_{k=1}^{n} \frac{\mathrm{d}t_k}{t_k^2} \,. \tag{8.35}$$

In the final line, the generalized 'momentum' $\mathcal{P}_{\alpha A}$ is defined to be

$$\mathcal{P}_{\alpha A} := -i \sum_{i=1}^{n} W_{i\,\alpha}\, \sigma_{i\,A} \,. \tag{8.36}$$

Though this formula may appear complicated at first glance, the structure of each contribution is quite simple. Schematically denoting the perturbative contributions of f as F, we can sketch the amplitude as

$$\mathcal{M}_{n,0} = \int \mathrm{d}\mu \, D\sigma_1 D\sigma_2 \Big(\sigma_1^{s-1}\sigma_2^{s-1} F^{n-2}[g_1 g_2] + \sigma_1^{s-1}\sigma_2^{s-2} F[\sigma_2] F^{n-3}[g_1 g_2] + (1 \leftrightarrow 2) + \cdots + F^{2s-2}[\sigma_1\sigma_2] F^{n-2s-4}[g_1 g_2] \Big) \tag{8.37}$$

The first line contains all contributions to the amplitude from the perturbative expansion acting on external wavefunctions; each of the subsequent contributions is a sum over the ways in which the expansion can also act on explicit coordinate insertions. This sum is exhausted by the contributions on the third line, where all explicit insertions of $\mathcal{Z}(x,\sigma)$ have been eaten by the perturbative expansion. Of course, for $n < 2s-4$ the expression for $\mathcal{M}_{n,0}$ will terminate sooner. When $s = 2$, (8.35) agrees with the MHV amplitude obtained from the twistor action of conformal gravity [36].

An initial concern about this expression for the amplitude is its gauge invariance: it seems far from clear that (8.35) is independent of the fixed point $\xi \in \mathbb{CP}^1$ used to

define the perturbative expansion. Since *any* choice of ξ suffices in (8.27), and this choice encodes no degrees of freedom relevant to the physical CHS theory on space-time, physical quantities (such as amplitudes) must be independent of ξ. This issue has arisen before in the context of gravitational amplitudes in Minkowski space [103] or AdS_4 [104], as well as conformal gravity [36, 18], and identical steps can be followed to check gauge invariance for (8.35). A lengthy but straightforward calculation, proceeding from the definitions of M, $\mathfrak{m}$, and the properties of determinants shows that

$$\frac{\partial \mathcal{M}_{n,0}}{\partial \xi^B} = \int \frac{\mathrm{d}^8 X}{\mathrm{vol}\, GL(2,\mathbb{C})} \frac{\partial \mathcal{V}_B^{\alpha A}}{\partial X^{\alpha A}} = 0\,. \tag{8.38}$$

for $\mathcal{V}_B^{\alpha A}$ a smooth function with respect to $X^{\alpha A}$. In other words, gauge invariance follows by Stokes' theorem on the moduli space of holomorphic curves in twistor space.

8.4 Unitary Truncation

So far, all amplitudes have been evaluated with totally general CHS external states. In this section, we restrict the external states to lie in the unitary sub-sector, defined on twistor space using the infinity twistor (6.51). On space-time, this restriction corresponds to choosing external states which solve the two-derivative equations of motion within the more general CHS framework. In this sub-sector, the external degrees of freedom are encoded in the twistor wavefunctions $h^{(s)}$ and $\tilde{h}^{(s)}$. This restriction can be understood as choosing particular helicity raising and lowering operators involving the infinity twistors $I^{\alpha\beta}$, $I_{\alpha\beta}$ describing a conformally flat background geometry as in (6.47). Schematically we choose

$$A^\alpha \to I^{\beta\alpha}\partial_\beta\,, \quad \text{and} \quad B_\alpha \to I_{\alpha\beta} Z^\beta \tag{8.39}$$

while the remaining factors provide the appropriate scaling of the wavefunctions.

The three-point $\overline{\text{MHV}}$ amplitudes can be easily extracted from (6.54) with an appropriate choice of wavefunctions. We choose plane wave momentum eigenfunctions following [105], see also [18], where we allow the space-time momenta to be complex. The unitary wavefunctions have dual twistor representatives

$$h^{(s)} = \int \frac{\mathrm{d}t}{t^{2s-1}} e^{t\,W\cdot Z}\,, \qquad \tilde{h}^{(s)} = \int \mathrm{d}t\, t^{2s+1} e^{t\,W\cdot Z}\,, \tag{8.40}$$

homogeneous of degree $2s-2$ and $-2s-2$, respectively. Since these wavefunctions solve the (two-derivative) zero-rest-mass equations on space-time, they are naturally based on plane waves with an on-shell (complex) four-momentum. Hence, it will be useful to compute amplitudes in terms of ordinary momentum eigenstates as well.

These have a straightforward representation on momentum space [106]:

$$h^{(s)} = \int \frac{\mathrm{d}t}{t^{2s-1}}\, \overline{\delta}^2(t\lambda - p)\, e^{t[\mu\tilde{p}]}\,, \quad \tilde{h}^{(s)} = \int \mathrm{d}t\, t^{2s+1}\, \overline{\delta}^2(t\lambda - p)\, e^{t[\mu\tilde{p}]}\,, \tag{8.41}$$

in terms of the on-shell four-momentum $k^\mu \leftrightarrow p^A\tilde{p}^{A'}$. Using these is equivalent to taking the half-Fourier transform of (8.25) for the truncated A and B.

It is worth noting that, in the unitary sub-sector, the helicity raising operator A^α is purely algebraic, while the helicity lowering operator B_α must still be treated as a differential operator. Indeed, at the level of components,

$$A^\alpha\big|_{\text{unitary}} = I^{\alpha\beta}W_\beta\,, \qquad B_\alpha\big|_{\text{unitary}} = I_{\alpha\beta}\frac{1}{t}\frac{\partial}{\partial W_\beta}\,, \tag{8.42}$$

when acting on dual twistor wavefunctions. For each insertion of A^α or B_α, the unitary sub-sector is given by a linear combination of the conformal graviton and ghost modes; the spin one conformal state is set to zero. In the Minkowski space limit, $\Lambda \to 0$, the ghost mode as well as the spin-1 mode is killed and the conformal graviton mode becomes the surviving Einstein mode.

8.4.1 Three-Point $\overline{\text{MHV}}$ Amplitudes

Going back to the self-dual action in the unitary sub-sector (6.54), we now perform all the twistor integrals to obtain an amplitude in momentum space by evaluating

the action $S[\tilde{h}, h]$ on the momentum eigenstates (8.41). For $\tilde{h}_1$ and h_2, h_3 of this form, the bracket acts as:

$$\{h_2, h_3\}_{(-s_1+s_2+s_3)} = t_2^{s_{23|1}} t_3^{s_{23|1}} \left([23] + \Lambda \left\langle \frac{\partial}{\partial p_2} \frac{\partial}{\partial p_3} \right\rangle \right)^{s_{23|1}} h_2 \wedge h_3 \,, \tag{8.43}$$

where again $s_{ij|k} = s_i + s_j - s_k - 1$, and all integrals can be performed analogously to the full CHS case to give the three-point amplitude:

$$\begin{aligned} \widetilde{\mathcal{M}}_{3,-1}(-s_1, +s_2, +s_3) = \Lambda^{s_1-1} \widetilde{\mathcal{N}}^{(s_1,s_2,s_3)} [23]^{2s_1+1} [12]^{-s_1+s_2-s_3} [31]^{s_3-s_1-s_2} \\ \times \left([23] + \Lambda \left\langle \frac{\partial}{\partial p_2} \frac{\partial}{\partial p_3} \right\rangle \right)^{s_{23|1}} \delta^4 \left(\sum_{i=1}^{3} |i\rangle [i| \right) \,, \end{aligned} \tag{8.44}$$

where

$$\widetilde{\mathcal{N}}^{(s_1,s_2,s_3)} = \frac{(-1)^{s_2+s_3}}{2} \frac{(1 + (-1)^{-s_1+s_2+s_3})}{\Gamma(-s_1+s_2+s_3)} \frac{\Gamma(s_1-s_2+s_3)\Gamma(s_1+s_2-s_3)}{\Gamma(s_1-s_2+1)\Gamma(s_1+1-s_3)} \,. \tag{8.45}$$

The differential operator appearing in the second line can be simplified by noting that when acting on the four-momentum conserving delta function, $\left\langle \frac{\partial}{\partial p_2} \frac{\partial}{\partial p_3} \right\rangle$ is equivalent to $[23]\Box_P$, where $\Box_P$ is the d'Alembertian with respect to the total 4-momentum:

$$\Box_P := \frac{\partial}{\partial P_{AA'}} \frac{\partial}{\partial P^{AA'}} \,, \qquad P^{AA'} := \sum_{i=1}^{n} p_i^A \tilde{p}_i^{A'} \,. \tag{8.46}$$

This reduces the $\overline{\text{MHV}}$ answer for the unitary sub-sector to:

$$\begin{aligned} \widetilde{\mathcal{M}}_{3,-1}(-s_1, +s_2, +s_3) = \Lambda^{s_1-1} \widetilde{\mathcal{N}}^{(s_1,s_2,s_3)} [23]^{s_2+s_3+s_1} [12]^{-s_1+s_2-s_3} [31]^{s_3-s_1-s_2} \\ \times (1 - \Lambda \Box_P)^{-s_1+s_2+s_3} \, \delta^4(P) \,. \end{aligned} \tag{8.47}$$

The presence of the differential operator acting on the momentum-conserving delta function can be viewed as a consequence of calculating the 'amplitude' on a conformally flat, rather than Minkowski, background [35].

In [28, 29], the definition of S-matrix for CHS theory was taken to have all external states in the unitary sub-sector, evaluated on a Minkowski background. Having already restricted to the unitary sub-sector in (8.47), the flat space limit is simply given by $\Lambda \to 0$:

$$\lim_{\Lambda \to 0} \widetilde{\mathcal{M}}_{3,-1}(-s_1, +s_2, +s_3) = 0 \,, \tag{8.48}$$

with the amplitude vanishing at $O(\Lambda^{s_1-1})$. So *all* three-point $\overline{\text{MHV}}$ amplitudes of CHS theory vanish when restricted to the unitary sub-sector on Minkowski space.

8.4.2 Three-Point Spin-s MHV Amplitudes

We can employ the expansion (7.84) in order to compute the three-point MHV amplitude for arbitrary, all equal spin. In particular we are interested in the case of even spin as the odd case is zero due to Bose statistics. To this end we start with the leading term in the expansion of the Green function (7.79) appearing the interaction action (7.78). Restricting to the case of self-interactions of spin-s fields, using $|I| = s-1$, we have

$$\begin{aligned}\mathcal{M}^{(s)}_{3,0}(-,-,+) = \frac{1}{2}\int \sigma_{1A_I}\sigma_{2B_I}\int D\sigma_3 \,\times\\ \left(\frac{1}{\langle 13\rangle}\left((-1)^s\sum_{|J|=0}^{|I|}\binom{|I|}{|J|}\frac{\langle\xi 1\rangle^{|J|+1}}{\langle\xi 3\rangle^{|J|+1}}\partial_{\beta_{I-J}}f_3^{\beta_I}\wedge\partial_{\beta_J}g_1^{\Omega B_I}+\frac{\langle\xi 1\rangle}{\langle\xi 3\rangle}\partial^{B_I}f_3^{\gamma_I}\wedge g^{\Omega}_{1\gamma_I}\right)\wedge g_2^{\Omega A_I}\right.\\ \left.+\, g_1^{\Omega B_I}\wedge\frac{1}{\langle 23\rangle}\left[(-1)^s\sum_{|J|=0}^{|I|}\binom{|I|}{|J|}\frac{\langle\xi 2\rangle^{|J|+1}}{\langle\xi 3\rangle^{|J|+1}}\partial_{\beta_{I-J}}f_3^{\beta_I}\wedge\partial_{\beta_J}g_2^{\Omega A_I}+\frac{\langle\xi 2\rangle}{\langle\xi 3\rangle}\partial^{A_I}f_3^{\gamma_I}\wedge g^{\Omega}_{2\gamma_I}\right)\right)\,.\end{aligned} \tag{8.49}$$

Now we further restrict to the diagonalisable sector by

$$\begin{aligned}f_i^{\alpha_1\ldots\alpha_n}(Z) &= I^{\beta_1\alpha_1}\cdots I^{\beta_n\alpha_n}\partial_{\beta_1}\cdots\partial_{\beta_n}h_i(Z)\,,\\ g_{i,\alpha_1\ldots\alpha_n}(Z) &= I_{\alpha_1\beta_1}\cdots I_{\alpha_n\beta_n}Z^{\beta_1}\cdots Z^{\beta_n}\tilde{h}_i(Z)\,.\end{aligned} \tag{8.50}$$

As $\partial_{\beta_i}f^{\cdots\beta_i\cdots}=0$ in this sector, this implies in the summations that only the terms where all the derivatives act on the g^{Ω} fields survive. It is also important to note that the fields in the Green function terms are restricted to the fibre; which is to say that on the fibre, for a conformally flat background where the incidence relations are $Z^\alpha = X^{\alpha A}\sigma_A$ with σ_A being homogeneous fibre coordinates, the parameters $X^{\alpha A}$ are taken to be constant. On the fibre, and in the diagonalisable sub-sector, we can thus take

$$\frac{\partial}{\partial Z_i^\alpha} = \frac{2I_{\alpha\beta}X^{\beta B}}{X^2}\frac{\partial}{\partial\sigma_i^B}\,, \tag{8.51}$$

where $X^2 = \epsilon_{AB} I_{\alpha\beta} X^{\alpha A} X^{\beta B}$. Furthermore, while in (8.50) the components of the fields are taken with respect to a holomorphic basis, in the interaction terms they are taken with respect to the non-holomorphic basis $\mathrm{d}\sigma_A$, hence we should use the substitution

$$g_i^{A_I \Omega} = X^{\alpha_I A_I} I_{\alpha_I \beta_I} Z_i^{\beta_I} \tilde{h}_i^{\Omega} = \left(\tfrac{X^2}{2}\right)^{s-1} \sigma_i^{A_I} \tilde{h}_i^{\Omega} \ . \tag{8.52}$$

After integrating by parts with respect to σ_3 and σ_1 or σ_2 as appropriate, we have for the three-point amplitude

$$\begin{aligned}
\widetilde{\mathcal{M}}_{3,0}^{(s)}(-,-,+) = \left(\frac{\Lambda}{2}\right)^{s-1} \int D\sigma_3 \wedge h_3 \wedge \tilde{h}_1^{\Omega} \wedge \tilde{h}_2^{\Omega} \langle 12\rangle^{s-1} \left(X^2\right)^{s-1} \\
\times \left(\langle \partial_3 \partial_1 \rangle^{s-1} \left(\frac{\langle 12\rangle^{s-1} \langle \xi 1\rangle^{s}}{\langle 13\rangle \langle \xi 3\rangle^{s}} \right) - \langle \partial_3 \partial_2 \rangle^{s-1} \left(\frac{\langle 21\rangle^{s-1} \langle \xi 2\rangle^{s}}{\langle 23\rangle \langle \xi 3\rangle^{s}} \right) \right. \\
\left. + \langle 2\partial_3\rangle^{s-1} \langle 1\partial_3\rangle^{s-1} \left(\frac{\langle \xi 1\rangle}{\langle 13\rangle \langle \xi 3\rangle} \right) - \langle 2\partial_3\rangle^{s-1} \langle 1\partial_3\rangle^{s-1} \left(\frac{\langle \xi 2\rangle}{\langle 23\rangle \langle \xi 3\rangle} \right) \right)
\end{aligned} \tag{8.53}$$

where we use the notation $\langle i\partial_j\rangle = \sigma_{iA}\partial_j^A$ and $\langle \partial_i \partial_j\rangle = \epsilon_{AB}\partial_i^A \partial_j^B$. This expression is in fact independent of ξ^A as can be seen by considering $\mathrm{d}_\xi \widetilde{\mathcal{M}} = \mathrm{d}\xi^A \frac{\partial}{\partial \xi^A} \widetilde{\mathcal{M}}$ and showing it vanishes. Alternatively it can be shown explicitly using

$$\begin{aligned}
\left(\frac{\langle 12\rangle^{s-1} \langle \xi 1\rangle^{s}}{\langle 13\rangle \langle \xi 3\rangle^{s}} \right) &= \frac{\langle 1\partial_3\rangle^{s-1}}{(-1)^{s-1}(s-1)!} \left(\frac{\langle 12\rangle^{s-1} \langle \xi 1\rangle}{\langle 13\rangle \langle \xi 3\rangle} \right) \\
\langle \partial_3 \partial_1 \rangle^{s-1} \left(\frac{\langle 12\rangle^{s-1} \langle \xi 1\rangle}{\langle 13\rangle \langle \xi 3\rangle} \right) &= (-1)^{s-1}(s-1)! \langle 2\partial_3\rangle^{s-1} \left(\frac{\langle \xi 1\rangle}{\langle 13\rangle \langle \xi 3\rangle} \right)
\end{aligned}$$

so that the first two terms in (8.53) are the same as the second two, which can be combined into

$$\langle 2\partial_3\rangle^{s-1} \langle 1\partial_3\rangle^{s-1} \left(\frac{\langle 12\rangle}{\langle 13\rangle \langle 23\rangle} \right) = ((s-1)!)^2 \frac{\langle 12\rangle^{2s-1}}{\langle 13\rangle^{s} \langle 23\rangle^{s}} \tag{8.54}$$

which is manifestly independent of ξ. The combination gives a factor or $(1 + (-1)^s)$, and hence we have

$$\begin{aligned}
&\widetilde{\mathcal{M}}_{3,0}^{(s)}(-,-,+) \\
&\quad = (1 + (-1)^s)\, \Gamma(s)^2 \left(\frac{\Lambda}{2}\right)^{s-1} \int D\sigma_3 \wedge h_3 \wedge \tilde{h}_1^{\Omega} \wedge \tilde{h}_2^{\Omega} (X^2)^{s-1} \frac{\langle 12\rangle^{3s-2}}{\langle 13\rangle^{s} \langle 23\rangle^{s}} \ .
\end{aligned}$$

From the definition of the holomorphic volume forms

$$\tilde{h}_1^\Omega \wedge \tilde{h}_2^\Omega = \langle 12\rangle^2\, D\sigma_1 \wedge D\sigma_2 \wedge \mathrm{d}^4x \wedge \tilde{h}_1 \wedge \tilde{h}_2 \tag{8.55}$$

and choosing plane wave momentum eigenfunctions describing particles with on-shell momenta given by the spinors $(p_i, \tilde{p}_i)$

$$\begin{aligned} h_i(Z) &= \int_{\mathbb{C}} \frac{\mathrm{d}u}{u} \frac{1}{u^{2s-2}} \wedge \overline{\delta}^2(u\lambda - p_i)\, e^{u[\mu\tilde{p}_i]}\ , \\ \tilde{h}_i(Z) &= \int_{\mathbb{C}} \frac{\mathrm{d}u}{u} \frac{1}{u^{-2s-2}} \wedge \overline{\delta}^2(u\lambda - p_i)\, e^{u[\mu\tilde{p}_i]}\ , \end{aligned}$$

we finally find

$$\widetilde{\mathcal{M}}^{(s)}_{3,0}(-,-,+) = \Lambda^{s-1} \frac{(1+(-1)^s)\,\Gamma(s)^2}{2^{s-1}} \left(\frac{\langle 12\rangle^3}{\langle 13\rangle\,\langle 23\rangle} \right)^s (1 - \Lambda\Box_P)^{s-1} \delta^4(P)\,. \tag{8.56}$$

We immediately see that this amplitude vanishes for odd spin, as expected. It also is the parity conjugate of the $\overline{\mathrm{MHV}}$ amplitude (8.47), and the mismatch in the normalisation constants can be adjusted for by a proper normalisation of the momentum eigenfunctions.

8.4.3 n-Point MHV Amplitudes on a Graviton Background

The generating functional (8.29) for MHV amplitudes on a conformal gravity background is easily restricted to the unitary sub-sector:

$$\begin{aligned} \widetilde{\Gamma}^{\mathrm{MHV}} = \sum_{s=0}^{\infty} \int \frac{\mathrm{d}^8X}{\mathrm{vol}\, GL(2,\mathbb{C})} \langle 12\rangle^2\ D\sigma_1 \wedge D\sigma_2\ \langle \mathcal{Z}(x,\sigma_1), \mathcal{Z}(x,\sigma_2)\rangle^{2(s-1)} \\ \wedge\, \tilde{h}_1^{(s)}(\mathcal{Z}(x,\sigma_1)) \wedge \tilde{h}_2^{(s)}(\mathcal{Z}(x,\sigma_2))\,, \end{aligned} \tag{8.57}$$

where $\langle \mathcal{Z}(x,\sigma_1), \mathcal{Z}(x,\sigma_2)\rangle = I_{\alpha\beta}\mathcal{Z}^\alpha(x,\sigma_1)\mathcal{Z}^\beta(x,\sigma_2)$. After choosing the negative helicity external states to have spin-two, the perturbative expansion is operationalized using

$$\mathcal{Z}^\alpha(x,\sigma) \to \frac{1}{2\pi i} \int_{\mathbb{CP}^1} \frac{D\sigma'}{\langle \sigma\sigma'\rangle} \frac{\langle \xi\sigma\rangle^2}{\langle \xi\sigma'\rangle^2}\, I^{\beta\alpha}\, \frac{\partial}{\partial Z^\beta(x,\sigma')} h^{(2)}(Z(x,\sigma'))\,. \tag{8.58}$$

Applying the matrix tree theorem with dual twistor wave functions results in

$$\widetilde{\mathcal{M}}_{n,0} = \int \frac{\mathrm{d}^8 X}{\mathrm{vol}\, GL(2,\mathbb{C})} \langle 12\rangle^2 \; D\sigma_1 \, D\sigma_2 \Bigg[\langle Z(x,\sigma_1), Z(x,\sigma_2)\rangle^{2s-2} \; \left|\mathbb{H}^{12}_{12}\right| + \; (2s-2)\Lambda \sum_{i=3}^{n} \left|\mathbb{H}^{12i}_{12i}\right| \langle Z(x,\sigma_1), Z(x,\sigma_2)\rangle^{2s-3} \; t_i W_i \cdot Z(x,\sigma_2)\, \mathfrak{m}^1_i + (1 \leftrightarrow 2) + \cdots + \Lambda^{2s-2}(2s-2)! \sum_{\substack{i_1,\ldots,i_{s-1} \\ j_1,\ldots,j_{s-1}}} \left|\mathbb{H}^{12i_1\cdots j_{s-1}}_{12i_1\cdots j_{s-1}}\right| \prod_{a=1}^{s-1} \mathfrak{m}^1_{i_a} \mathfrak{m}^2_{j_a} \, t_{i_a} t_{j_a} \, [W_{i_a}, W_{j_a}] \Bigg] \times e^{i\mathcal{P}\cdot X} \, (t_1 t_2)^{s+4} \prod_{k=1}^{n} \frac{\mathrm{d}t_k}{t_k^2} \,, \quad (8.59)$$

where $\mathbb{H}$ is the $n \times n$ matrix

$$\mathbb{H}_{ij} = (-1)^{|i-j|} t_i t_j \; [W_i, W_j] \; \frac{\sqrt{D\sigma_i D\sigma_j}}{\langle ij\rangle} \,, \quad \text{for } i \neq j \,, \quad (8.60)$$

$$\mathbb{H}_{ii} = -t_i \, D\sigma_i \sum_{j\neq i} (-1)^{|i-j|} t_j \frac{[W_i, W_j]}{\langle ij\rangle} \frac{\langle \xi j\rangle^2}{\langle \xi i\rangle^2} \,, \quad (8.61)$$

and $\mathfrak{m}^i_j$ is given by (8.34).

As it stands, little can be said about the flat space limit of (8.59) because the moduli integrals over $X^{\alpha A}$ remain to be performed, and there are no explicit overall powers of Λ. To proceed, we follow the method of [36] to partially evaluate the moduli integrals. First, the scaling parameters t_i can be absorbed into the homogeneous coordinates σ_i, at the expense of all $\mathbb{CP}^1$ integrals becoming $\mathbb{C}^2$ integrals: $\mathrm{d}t_i D\sigma_i \to \mathrm{d}^2\sigma_i$. We can also use the fact that $Z^\alpha(x,\sigma) = X^{\alpha A}\sigma_A$ to write

$$\langle Z(x,\sigma_1), Z(x,\sigma_2)\rangle = \langle 12\rangle \; I_{\alpha\beta} \, X^\alpha_A \, X^{\beta A} := \langle 12\rangle \; X^2 \,. \quad (8.62)$$

This allows us to rewrite the MHV amplitude as

$$\widetilde{\mathcal{M}}_{n,0} = \int \frac{\mathrm{d}^8 X}{\mathrm{vol}\, GL(2,\mathbb{C})} \langle 12\rangle^2 \; \mathrm{d}^2\sigma_1 \wedge \mathrm{d}^2\sigma_2 \Bigg[\langle 12\rangle^{2s-2} \, (X^2)^{2s-2} \; \left|\mathbb{H}^{12}_{12}\right| + \; (2s-2)\Lambda \sum_{i=3}^{n} \left|\mathbb{H}^{12i}_{12i}\right| \; \langle 12\rangle^{2s-3} \; (X^2)^{2s-3} \, (W_i \cdot X \cdot \sigma_2)\, \mathfrak{m}^1_i + (1 \leftrightarrow 2) + \cdots + \Lambda^{2s-2}(2s-2)! \sum_{\substack{i_1,\ldots,i_{s-1} \\ j_1,\ldots,j_{s-1}}} \left|\mathbb{H}^{12i_1\cdots j_{s-1}}_{12i_1\cdots j_{s-1}}\right| \prod_{a=1}^{s-1} \mathfrak{m}^1_{i_a} \mathfrak{m}^2_{j_a} \; [W_{i_a}, W_{j_a}] \Bigg] e^{i\mathcal{P}\cdot X} \,, \quad (8.63)$$

with generalized momentum $\mathcal{P} = \sum_i W_i \sigma_i$.

It is clear that the only potentially non-vanishing contribution in the flat space limit is the first term in (8.63), since every other term is of $O(\Lambda)$:

$$\lim_{\Lambda\to 0} \widetilde{\mathcal{M}}_{n,0} = \lim_{\Lambda\to 0} \int \frac{\mathrm{d}^8 X}{\mathrm{vol}\, GL(2,\mathbb{C})} \langle 12\rangle^{2s} \left|\mathbb{H}_{12}^{12}\right| \mathrm{d}^2\sigma_1 \wedge \mathrm{d}^2\sigma_2 \, (X^2)^{2s-2} \, e^{i\mathcal{P}\cdot X} \,. \tag{8.64}$$

Note that X^2 can be seen as a differential operator acting on $e^{i\mathcal{P}\cdot X}$ of the form

$$X^2 \longleftrightarrow \Box := \frac{1}{\langle 12\rangle} \left\langle \frac{\partial}{\partial W_1}, \frac{\partial}{\partial W_2} \right\rangle \,, \tag{8.65}$$

and (8.64) can then be written as

$$\lim_{\Lambda\to 0} \widetilde{\mathcal{M}}_{n,0} = \lim_{\Lambda\to 0} \int \frac{\mathrm{d}^8 X}{\mathrm{vol}\, GL(2,\mathbb{C})} \langle 12\rangle^{2s} \left|\mathbb{H}_{12}^{12}\right| \mathrm{d}^2\sigma_1 \wedge \mathrm{d}^2\sigma_2 \, \Box^{2s-2} \, e^{i\mathcal{P}\cdot X} \,. \tag{8.66}$$

The moduli integrals can now be performed to leave

$$= \lim_{\Lambda\to 0} \int \frac{\mathrm{d}^2\sigma_1 \wedge \mathrm{d}^2\sigma_2}{\mathrm{vol}\, GL(2,\mathbb{C})} \langle 12\rangle^{2s} \left|\mathbb{H}_{12}^{12}\right| \Box^{2s-2} \, \delta^8(\mathcal{P}) \,, \tag{8.67}$$

since the only X-dependence was in the exponential. Now, following [36], we integrate by parts twice to find

$$\begin{aligned} &\int \frac{\mathrm{d}^2\sigma_1 \wedge \mathrm{d}^2\sigma_2}{\mathrm{vol}\, GL(2,\mathbb{C})} \langle 12\rangle^{2s} \left|\mathbb{H}_{12}^{12}\right| \Box^{2s-2} \, \delta^8(\mathcal{P}) \\ &= \Lambda \int \frac{\mathrm{d}^2\sigma_1 \wedge \mathrm{d}^2\sigma_2}{\mathrm{vol}\, GL(2,\mathbb{C})} \langle 12\rangle^{2s} \left[\sum_i \frac{\langle 12\rangle^2}{\langle 1i\rangle^2 \langle 2i\rangle^2} \left|\mathbb{H}_{12i}^{12i}\right| \right. \\ &\qquad \left. + \sum_{i,j} \frac{\langle \xi 1\rangle^2 \langle i2\rangle \langle j2\rangle + \langle \xi 2\rangle^2 \langle i1\rangle \langle j1\rangle}{\langle 1i\rangle \langle 2i\rangle \langle 1j\rangle \langle 2j\rangle \langle \xi i\rangle \langle \xi j\rangle} \left|\mathbb{H}_{12j}^{12i}\right| \right] \Box^{2s-3} \delta^8(\mathcal{P}) \,. \end{aligned} \tag{8.68}$$

In particular, an overall power of Λ appears which was obscured in (8.64)-(8.67).

Therefore, all MHV amplitudes on a conformal gravity background vanish in the flat space limit upon restricting to the unitary sub-sector:

$$\lim_{\Lambda\to 0} \widetilde{\mathcal{M}}_{n,0} = 0 \,, \tag{8.69}$$

with the zero appearing as $O(\Lambda)$. Although slightly less obvious than the $\overline{\mathrm{MHV}}$ case, this provides an infinite set of amplitudes which confirm the conjecture of [28, 29] that the S-matrix of CHS theory restricted to the unitary sub-sector is trivial.

8.5 Discussion

It is useful to discuss the unitary sector theory in a more general fashion and to compare with some of the known results from the literature. In the gravitational case, as discussed in [13], one can relate, at the level of tree-diagrams, ordinary gravity in the presence of a cosmological constant to conformal gravity. In particular the dimensionless coupling of conformal gravity, c_W, is related to the dimensionless quantity appearing in gravity, c_E, here for the AdS case, by

$$c_W = \frac{1}{4} c_E = \frac{1}{8} \left(\frac{R^2}{8\pi G_2} \right) , \tag{8.70}$$

where $R \sim \Lambda^{-1/2}$ is the radius of the AdS space and G_2 is the usual gravitational constant.

For the CHS theory we might hope to find analogous behaviour, which is to say that upon truncation to the unitary sector we find a massless higher spin theory on an AdS (or dS) background where the dimensionful couplings are related to the dimensionless CHS parameters through powers of Λ. This corresponds to the powers of Λ appearing in the computation of the cubic amplitudes (8.47) and we see that the interactions involving a spin-s negative helicity excitation scale as Λ^{s-1}. Alternatively, given dimensionless couplings $c_{(s)}$ for the cubic interactions involving a negative helicity spin s field in the CHS theory, we expect

$$G_s \sim \frac{1}{c_{(s)}\, \Lambda^{s-1}} . \tag{8.71}$$

for G_s the analogous dimensionful coupling in massless higher spin theory.

Given this, it is unsurprising that the flat-space limit of the cubic couplings vanishes – it simply follows from the definition of G_s – however non-vanishing answers can be obtained when the amplitudes are normalized by certain powers of the cosmological constant *before* the flat space limit is taken. This of course can be done in the case of conformal gravity and results in Einstein's theory expanded around flat space; in particular it has been shown [35, 43, 36] to reproduce the MHV

amplitude formula of Hodges [44]. For the unitary sector CHS $\overline{\text{MHV}}$ amplitudes, this normalisation is easily read off from (8.47):

$$\lim_{\Lambda\to 0} \frac{\widetilde{\mathcal{M}}_{3,-1}(-s_1,+s_2,+s_3)}{\Lambda^{s_1-1}} = \widetilde{\mathcal{N}}^{(s_1,s_2,s_3)}\,\delta^4(P)\,\frac{[23]^{s_1+s_2+s_3}}{[12]^{s_1+s_3-s_2}\,[31]^{s_1+s_2-s_3}}\,. \tag{8.72}$$

This result is the unique combination of spinor invariants compatible with the $\overline{\text{MHV}}$ helicity configuration, four-momentum conservation (which emerges in the flat space limit), and Poincaré invariance [14].

For the MHV amplitudes with two negative helicity spin s particles and $n-2$ positive helicity spin two particles, the appropriate normalisation is to divide by a single power of Λ for any n. This reflects the fact that such amplitudes represent the coupling of the CHS fields to a conformal gravity background only. It follows that after normalisation, the resulting flat space amplitude can be given as an expression purely in terms of on-shell four-momenta:

$$\lim_{\Lambda\to 0} \frac{\widetilde{\mathcal{M}}_{n,0}}{\Lambda} \propto \delta^4(P)\,\frac{\langle 12\rangle^{2s+2}}{\langle 1i\rangle^2\,\langle 2i\rangle^2}\,\left|\Phi^{12i}_{12i}\right|\,, \tag{8.73}$$

where the $n\times n$ matrix Φ is the Hodges matrix [44]:

$$\Phi_{ij} = \frac{[ij]}{\langle ij\rangle}\,,\ \text{for } i\neq j\,,\quad \text{and}\quad \Phi_{ii} = -\sum_{j\neq i}\frac{[ij]}{\langle ij\rangle}\frac{\langle \xi j\rangle^2}{\langle \xi i\rangle^2}\,. \tag{8.74}$$

In reaching (8.73) one must invoke various properties of the Hodges matrix along with the twistor 'gauge' invariance of the amplitude itself.

Upon restricting the self-dual action to the unitary subsector one notes that it splits into an infinite sum of 'unitary' actions:

$$S_{\text{s.d.}}[h,\tilde{h}] = \sum_{s=1}^{\infty}\Lambda^{s-1}\,S^{(s)}_{\text{SD}}[h,\tilde{h}^{(s)}]\,, \tag{8.75}$$

where

$$\begin{aligned} S_{\text{s.d.}}\left[\tilde{h}^{(\bullet)},h^{(\bullet)}\right] = \sum_{s_1=1}^{\infty}\Lambda^{s_1-1}\int_{\mathbb{P}\mathscr{T}} D^3\mathcal{Z}\wedge\tilde{h}^{(s_1)} \\ \wedge\left(\frac{(2s_1-1)!}{2\,s_1!}\bar{\partial}h^{(s_1)} + \sum_{s_2=1}^{\infty}\sum_{s_3=1}^{s_1-1}\widetilde{\mathcal{N}}^{(s_1,s_2,s_3)}\left\{h^{(s_2)},h^{(s_3)}\right\}_{(-s_1+s_2+s_3)}\right). \end{aligned} \tag{8.76}$$

with the bracket $\{\cdot,\cdot\}_{(k)}$ defined as in (6.56).

So for each s, the action $S_{\rm SD}^{(s)}$ contains a single negative helicity state of spin s and an infinite tower of higher spin positive helicity states. These theories are purely self-dual, in the sense that their only tree-level scattering amplitudes are three-point $\overline{\rm MHV}$ amplitudes, and they possess two derivative massless equations of motion on space-time. Furthermore, Λ can be set to zero in (6.57) without any problems. While it is not guaranteed that this is a consistent truncation for the HS case, (6.57) appears to define an infinite family of non-conformal, chiral, flat space higher spin theories.

There are powerful no-go theorems, which are not applicable in the presence of a cosmological constant, that argue against the existence of such flat space HS theories, notably the Weinberg low-energy theorem [107] and the Coleman-Mandula theorem [4], and so the existence of a n-point amplitude such as (8.73) is puzzling. However, recently there has been renewed interest in the possibility of evading these theorems, in particular by the use of light-cone formulations, see [108] and [109], where it has been shown that there exists a non-trivial chiral theory with cubic interactions which, while it is non-parity invariant and non-unitary, is nonetheless consistent [110].

For the twistor action of (6.57), defined by the flat limit of the SD sector, one can read off the properties of cubic couplings, $C^{\lambda_1,\lambda_2,\lambda_3}$, involving helicities $\lambda_1,\lambda_2,\lambda_3$ that would appear in a corresponding light-cone Hamiltonian by examining the three-point amplitudes ②. The spinor-helicity formalism is in fact very closely related to the light-cone formalism and a direct translation is possible (*e.g.*, [112]). The $\overline{\rm MHV}$ amplitude would correspond to an interaction with coupling C^{-s_1,s_2,s_3} where there are $s_2+s_3-s_1$ powers of momentum in the light cone vertex and we have $s_1\geq s_2,s_3$ and $s_1\leq s_2+s_3$. Amongst these interactions are those involving helicities s, $-s$ and

② For a discussion of the match between the off-shell *Lagrangian* couplings and three-point amplitudes see [111].

$s' < s$ and which would correspond to s' derivatives in a covariant approach, however it is by no means clear that the twistor actions (6.57) are equivalent to manifestly covariant space-time actions. Such interactions are also present in the light-cone formalism; indeed all helicity configurations are allowed in this formalism.

In contradistinction, one immediate consequence of the structure of the twistor action is that it will not give rise to cubic interactions involving only positive helicities. Such interactions in Yang-Mills theory would correspond to adding F^3 terms in the space-time action; such vertices would require a dimensionful coupling parameter and so violate conformal invariance. For gravity this would correspond to adding an R^3 term to the action, but such terms cannot arise from the quadratic-in-curvature action of Weyl gravity. Similarly, in our CHS theory such 'Abelian' interactions are absent. These interactions are, however, present in the chiral higher-spin theory discussed in [110]. Specifically, by considering the kinematic consistency of the quartic couplings one finds constraints on the cubic couplings, a solution to which is given by the very simple formula

$$C^{\lambda_1,\lambda_2,\lambda_3} = \frac{(1+(-1)^{\lambda_1+\lambda_2+\lambda_3})\ell_P^{\lambda_1+\lambda_2+\lambda_3-1}}{2\Gamma(\lambda_1+\lambda_2+\lambda_3)}\,. \tag{8.77}$$

In addition to the presence of the all positive helicity coupling, the explicit expression for $\lambda_1 = -s_1$, $\lambda_2 = s_2$, $\lambda_3 = s_3$ differs from the normalisation $\widetilde{\mathcal{N}}^{(s_1,s_2,s_3)}$ in (6.55). The appearance of $\Gamma(-s_1+s_2+s_3)$ is in common but the remaining factorial terms are different and cannot be removed by rescaling the fields as they involve different spins. It is not clear that one can satisfy the light-cone kinematic constraints with only couplings of the form C^{-s_1,s_2,s_3} if spin greater than two is allowed, which would seem to rule out a potential match. However, it may be that the flat space limit requires better understanding, or the twistor description of SD CHS theory can be modified to include additional interactions.

While the self-dual theory appears to be self-consistent, we naturally wish to define the full parity invariant theory. As we have described, this is done by adding

anti-self-dual interaction terms and we have shown that, at least for the case of positive helicity spin-two fields, the resulting cubic interactions give MHV amplitudes which are the appropriate parity conjugates of the $\overline{\text{MHV}}$ amplitudes. As we saw in (8.73), all such MHV amplitudes have the same scaling in the flat space limit, so normalizing by a power of Λ^{-1} yields non-vanishing flat space expressions. Since these are candidates for n-point amplitudes, one must reckon with more recent on-shell arguments [14, 113] as well as the traditional no-go theorems. The light-cone theory potentially eludes the grasp of these on-shell arguments as it allows four-point amplitudes which are not BCFW constructible [114]. It would be interesting to perform the same analysis for the higher-point MHV amplitudes we found in (8.73).

Chapter 9

Conclusion and further Discussion

In this thesis we studied the twistor space description of higher spin theories. We applied the non-linear graviton construction to conformal higher spin theory and found a twistor space description for its self-dual sector, analogous to the spin-2 conformal gravity case. We identified a unitary truncation, which for the spin-2 case reduces to Einstein gravity, providing a higher spin generalization that should be related to Vasiliev theory. Introducing the star product and jet bundle formulation made contact with the usual higher spin literature and rigorously defined the holomorphic structure on twistor space. Using these results, we proposed an extension of the self-dual theory into the anti-self-dual sector by introducing an interaction term motivated from the Chalmers-Siegel-like space-time action. This anti-self-dual action is given as a perturbative expansion around the self-dual background, leading to an MHV amplitude expansion, from which we derived the explicit forms of the three-point spin-s amplitudes and general n-point amplitudes on a conformal gravity background. The unitary truncation of the external states reproduced the expected form of the three-point amplitudes, which are unique from general unitarity and Poincaré invariance arguments.

Continuing the discussion from the previous chapter in a more general setting, one immediate generalisation of the current work is to include supersymmetry. Self-dual

actions on super-twistor space have been previously considered for $\mathcal{N} = 4$ SYM [16], for $\mathcal{N} = 4$ CSG [34] and for $\mathcal{N} = 8$ Einstein gravity [40] while the twistor action for the full $\mathcal{N} = 4$ CSG was given in [36]. From a geometrical perspective $\mathcal{N} = 4$ supersymmetry is the most natural as it results in a Calabi-Yau super-twistor space, however it is by no means clear this theory is unique. At least in the spin two case, depending on the presence of certain additional global symmetries, the conformal supergravity theory has minimal and non-minimal versions and it is only the minimal version which contains Einstein supergravity as a truncation.

This raises the important question of what space-time theory our higher-spin twistor description actually corresponds to. At the level of the spectrum we have shown that the number of degrees of freedom matches with the number of on-shell states in the conformal higher spin theory described by Fradkin and Tseytlin [9]. Including the anti-self-dual interactions, a natural candidate is the four-dimensional case of Segal's conformal higher spin theory [11] which describes an infinite number of bosonic symmetric traceless tensor fields. In the unitary subsector we have seen that the spectrum of the linearised theory matches the spectrum of the Fronsdal theory [45]. As Vasiliev's theory [6, 8, 21] also reproduces the Fronsdal spectrum we may optimistically speculate that the full unitary subsector, given by h_i and $\tilde{h}_i$, is related to the non-linear massless higher spin theory on anti-de Sitter space, however the absence of any scalar field in the twistor theory means that an exact matching would require some modifications. Nonetheless there are certain similarities, for example the higher-spin symmetry underlying the space-time theories can be understood as acting on jet spaces of fields and the unfolded formulation, see [115] for the CHS case, involves a twistor space formulation with some resemblance to the twistor spaces considered here. Of course, to properly compare theories, we must better understand the structure of the interactions between the infinite tower of fields.

Our proposal (7.78) for the interaction terms of the anti-self-dual modes leads to an expansion about a given self-dual background which will generate an infinite series

of higher point vertices. Such expansions in $\mathcal{N} = 4$ super-Yang-Mills and conformal gravity have led to efficient formalisms for computing observables, see [106, 37, 116]. For example, fixing axial gauge for the the unitary truncation of the conformal gravity twistor action, Adamo and Mason [36] were able to re-sum the resulting Feynman diagrams by using the matrix-tree theorem as in [102, 43] to produce a formula for the de Sitter analogue of MHV amplitudes which reproduced Hodges' remarkable formula [44] in limit of vanishing cosmological constant. We achieved the same for a self-dual graviton background, as laid out in the last chapter, but understanding the full anti-self-dual interactions seems to be crucial for extending this work to the full higher-spin theory.

The space-time CHS kinetic operator is not known for general backgrounds, though there has been recent progress [91, 99, 117]. In the context of one-loop checks of the correspondence between massless higher spin theories on anti-de Sitter space and vector model CFTs, e.g. [27], this is an important object as it is needed for the calculation of the canonical partition function on curved boundary manifolds. The twistor space actions, which are in principle valid for arbitrary self-dual space-times, may provide an alternative method for such calculations.

Appendix A

Gauge Field Theory

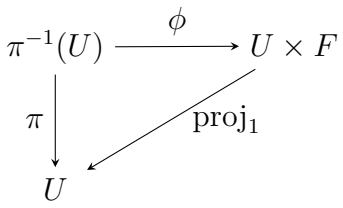
In this chapter, which is mainly based on [118], we will review the basics of gauge field theory. We introduce fibre bundles, with particular interest principal bundles, discuss connections on them, as well as their induced covariant derivative and curvature form, we define frames and the torsion form as well as the curvature endomorphism, also known as curvature tensor, and use all the ingredients to lay out the basic principles of Yang-Mills theory. At the end, we introduce the notion of self-duality in four dimensions, which is one of the essential pillars of this thesis.

A.1 Fibrations

Def Let B, E and F be topological spaces. If E is locally homeomorphic to $B \times F$ and $\pi : E \to B$ a continuous surjection, the collection (E, B, π, F) or the sequence $F \to E \xrightarrow{\pi} B$ is called **fibre bundle**. π is called projection, E is called total space, B is called base space, and for $b \in B$ the subspace $\{\pi^{-1}(b)\} =: E_b = F \subset E$ is called fibre[1]. More precisely, 'locally homeomorphic' means $\forall e \in E \; \exists U \subset B$ open with $\pi(e) \in U$ such that there is a homeomorphism $\phi : \pi^{-1}(U) \to U \times F$ such that the

[1] Since π is a surjection, it only admits right inverses, so that $\pi^{-1}(b)$ is not unique. $\{\pi^{-1}(b)\}$ denotes the space of images of all right inverses of π over b.

following diagram commutes:



The set of all $\{(U_i, \phi_i)\}$ is called a **local trivialization** of the bundle.

Example: the fibre bundle $F \hookrightarrow B \times F \overset{\mathrm{proj}_1}{\to} B$ is called a **trivial bundle**.

The concept of fibrations may become more intuitive if we consider B to be a smooth manifold (e.g. our space-time) and the fibers to be given by a Lie group G which acts on B. The group G is the group of

Def Let G be a topological group. A **G-bundle** is a fibre bundle $F \to E \overset{\pi}{\to} B$ together with a continuous left action $G \times E \to E$ that acts faithfully② on F. The group G can be thought of as a group of homeomorphisms of F. A **G-atlas** is a local trivialization such that for any two overlapping charts (U_i, ϕ_i) and (U_j, ϕ_j) the chart transition is given by

$$\phi_i \circ \phi_j^{-1} : (U_i \cap U_j) \times F \ni (b, x) \mapsto (b, t_{ij}(b)\, x) \in (U_i \cap U_j) \times F\,, \tag{A.1}$$

where $t_{ij} : U_i \cap U_j \to G$ are continuous maps called transition functions. Two G-atlases are equivalent if their union is also a G-atlas. A G-bundle is a fibre bundle with an equivalence class of G-atlases. The group G is called the **structure group** (or in physical terms **gauge group**) of the bundle.

The transition functions t_{ij} satisfy the following conditions

$$(1)\ \ t_{ii}(b) = \mathrm{id}\,, \tag{A.2}$$

$$(2)\ \ t_{ij}(b) = t_{ji}(b)^{-1}\,, \tag{A.3}$$

② $\forall g \neq \mathrm{id} \in G\, \exists\, x \in F : gx \neq x$

(3) $t_{ik}(b) = t_{ij}(b)\, t_{jk}(b)$ on a triple overlap $U_i \cap U_j \cap U_k$. (A.4)

A transition function with these properties completely determines the fibre. The third condition is also called the cocycle condition.

Def Let G be a topological group. A **principal G-bundle** is a fibre bundle $F \to E \xrightarrow{\pi} B$ together with a continuous right action $E \times G \to E$ that preserves the fibers of E and acts freely③ and transitively④ on them. In other words, a principal G-bundle is a G-bundle on which the structure group acts freely and tansitively. This implies that each fibre F is homeomorphic to G itself, and, for convenience, they are often identified with each other so that F inherits a right action of G (as well as a left action.)

Def A continuous right inverse s of the projection of a fibre bundle $F \to E \xrightarrow{\pi} B$ is called a **global section**, i.e. s is a continuous map $s : B \to E$ such that $(\pi \circ s)(x) = x \; \forall x \in B$.
A **local section** s is a continuous map $s : U \to E$ such that $\pi \circ s = \mathrm{id}_U$ (see figure A.1).

A section is the choice of a point $s(x)$ in each fibre. The space of local sections of E over U is denoted $\Gamma(U, E)$, while the space of global sections of E is denoted $\Gamma(B, E)$. The space of all local sections of E over open subsets of B is denoted $\Gamma(E)$.

Remark: let $G \to E \xrightarrow{\pi} B$ be a principal G-bundle, and let $s_i \in \Gamma(U_i, E)$ and $s_j \in \Gamma(U_j, E)$ be two local sections, where $U_i \cap U_j \neq \emptyset$. Then the transition function $t_{ij} : U_i \cap U_j \to G$ relates s_i and s_j by the formula

$$s_i(b) = t_{ij}(b)\, s_j(b) \quad \forall b \in U_i \cap U_j \,. \tag{A.5}$$

③ for $g \in G, \exists\, x \in F : xg = x \implies g = \mathrm{id}$
④ $\forall x, y \in F \; \exists\, g \in G : xg = y$

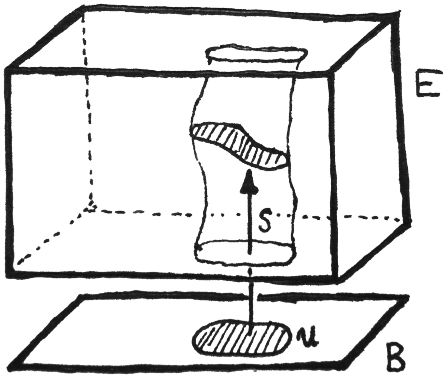


Figure A.1: A local section s of E over an open subset $U \subset B$.

Def Let $G \to E \xrightarrow{\pi} B$ be a principal bundle. In physical terms, a local section of E is called **gauge** and a condition that fixes a gauge is called **gauge condition**. A diffeomorphism $f : E \to E$ that preserves the fibers, $\pi \circ f = \pi$, and that is G-equivariant⑤, i.e. that is a transformation between two gauges, is a **gauge transformation**. The set of all gauge transformations forms a group under composition which is denoted $\mathcal{G}(E)$.

A.2 Connections on Principal Bundles

The differential of the projection π identifies a vertical component in the tangent space of the total space E of a fibre bundle in a natural way. To also identify a horizontal component through the condition of orthogonality to this vertical space, we would need the notion of a metric on E. A weaker, but still sufficient requirement is to introduce a mapping, called an (Ehresmann) connection, that assigns to a point in E the horizontal tangent space. What we usually understand as a *connection* is a connection form, that is a map that, which composed with the Ehresmann connection, vanishes. A connection form maps from the vertical tangent space. Both descriptions of a connection and connection form can be bijectively related, and we

⑤ $\forall e \in E \; \forall g \in G : f(eg) = f(e)g$

will, as usual, express everything in terms of the latter one. It also turns out that every principal bundle, which is already equipped with a (symmetry or gauge) group, has a connection.

Def Let $F \to E \xrightarrow{\pi} B$ be a smooth fiber bundle and $\mathbb{K}^n \to TE \xrightarrow{\tau} E$ the tangent bundle of E. Let

$$VE = \ker(\mathrm{d}\pi : TE \to \pi^* TB) \tag{A.6}$$

be the total space of the vertical bundle $TF \to VE \xrightarrow{\tau} E$ consisting of the vectors tangent to F, such that the fiber at $e \in E$ is $T_e E_{\pi(e)} \simeq T_e F$ (see figure A.2). An **Ehresmann connection** on E is a mapping $H : E \ni e \mapsto H_e E \subset T_e E$ that defines a smooth subbundle $TE \backslash TF \to HE \xrightarrow{\tau} E$ of the tangent bundle of E, called the horizontal bundle of the connection, which is complementary to VE in the sense that it defines a Whitney sum decomposition $TE = HE \oplus VE$.

Remark: let $G \to E \xrightarrow{\pi} B$ be a smooth principal G-bundle. An Ehresmann connection H on E is said to be a principal Ehresmann connection if it is invariant with respect to the G action on E in the sense that $\mathrm{d}R_g : H_e E \to H_{eg} E$ bijectively $\forall e \in E$ and $g \in G$.

A.2.1 Proposition

Let $\mathfrak{g}$ be the Lie algebra of G. The homeomorphism $\exp : [0,1] \times \mathfrak{g} \to G$ defines one-parameter family of subgroups $\exp(t\,\mathfrak{g}) =: G_t \subset G$. They act vertically on E by the map

$$\mathfrak{g} \ni v \mapsto \tilde{v}(e) := \frac{\mathrm{d}}{\mathrm{d}t}(e \exp(tv))\Big|_{t=0} \in V_e E\,, \tag{A.7}$$

which assigns to every element of $\mathfrak{g}$ its vector field on E. This map is a linear isomorphism.

A.2.2 Proposition

For $e \in E$, the differential of the projection $\mathrm{d}\pi_e : H_e E \to T_{\pi(e)} B$ is a linear isomorphism.

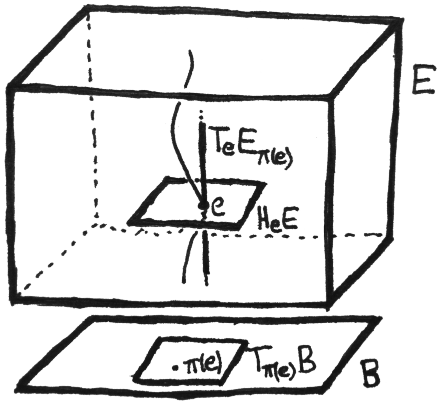


Figure A.2: Vertical and horizontal tangent space at $e \in E$.

Remark: these isomorphisms let us identify V_eE with $\mathfrak{g}$ and H_eE with $T_{\pi(e)}B$.

$$
\begin{array}{c}
G \to E \xrightarrow{\pi} B \\
HE \oplus VE = TE \xrightarrow{\mathrm{d}\pi} \pi^*TB \simeq HE \\
\wr \\
\mathfrak{g}
\end{array}
$$

Thus, in practice, $TE \simeq \pi^*TB \oplus \mathfrak{g}$. Note that we identified $VE \simeq \mathfrak{g}$, though this isomorphism actually holds only fiber-wise $V_eE \simeq \mathfrak{g}$.

Remark: an Ehresmann connection H defines the horizontal bundle with total space HE. Since the vertical bundle with total space VE is complementary to it, we would expect that there is also a complementary map to H

$$H : E \to HE \simeq \pi^*TB \qquad \Longleftrightarrow \qquad \tilde{H} : E \to VE \simeq \mathfrak{g}\,.$$

However, the structure of $\tilde{H}$ can be better captured in a map $A : TE \to \mathfrak{g}$ that satisfies $A \circ H \equiv 0$.

Def Let $G \to E \xrightarrow{\pi} B$ be a smooth principal G-bundle. A **connection form** on E is a $\mathfrak{g}$-valued 1-form $A \in \Omega^1(E, \mathfrak{g}) = \Gamma(T^*E \otimes \mathfrak{g})$ that is defined fibre-wise for all $e \in E$ by $A_e : T_eE \to \mathfrak{g}$, and which has the following properties

1) $\forall v \in \mathfrak{g} : A_e(\tilde{v}(e)) = v$, where $\tilde{v} \in \Gamma(VE)$ is the fundamental vector field associated to v; (A.8)

2) $\forall g \in G : R_g^* A_e = \mathrm{Ad}(g^{-1}) \circ A_e$.⑥ (A.9)

A.2.3 Lemma

There is a bijective relation between the Ehresmann connection and the connection form on a principal G-bundle $G \to E \xrightarrow{\pi} B$:

1. *Let $H : E \ni e \mapsto H_e E$ be an Ehresmann connection on E. Then a connection form is fibre-wise defined through*

$$A_e(h \oplus \tilde{v}(e)) := v \qquad \forall v \in \mathfrak{g}\,,\ e \in E\,,\ h \in H_e E\,. \tag{A.10}$$

2. *If $A \in \Omega^1(E, \mathfrak{g})$ is a connection form on E, then $H : E \ni e \mapsto H_e E := \ker A_e$ is a connection on E.*

Def Let $G \to E \xrightarrow{\pi} B$ be a smooth principal G-bundle. Let $A \in \Omega^1(E, \mathfrak{g})$ be a connection form on E and $\Gamma(E) \ni s : U \subset B \to E$ be a local section of E. The 1-form $A^s := s^* A \in \Omega^1(U, \mathfrak{g})$ which is point-wise defined by $A^s_b := (s^* A)(b) = A_{s(b)} \circ \mathrm{d}s$ is called the **local connection form** associated to s.

In a previous remark on transitions functions, we stated that two local sections of E are related through the action of a transition function on B in G. Likewise, we can define a transition function on TB with an action in $\mathfrak{g}$: consider the canonical 1-form (Maurer-Cartan form) $\theta \in \Omega^1(G, \mathfrak{g})$ of the group G, which for each $g \in G$ is defined as

$$\theta_g(y) := \mathrm{d}L_{g^{-1}}(y) \quad \forall y \in T_g G\,. \tag{A.11}$$

⑥ For any $g \in G$, let $\alpha_g = L_g \circ R_{g^{-1}} : G \to G$ be the action by conjugation. The adjoint action is defined as $\mathrm{GL}(\mathfrak{g}) \ni \mathrm{Ad}(g) = \mathrm{d}(\alpha_g)_e : T_e G \simeq \mathfrak{g} \to T_e G \simeq \mathfrak{g}$.

Consider further the pullback $t_{ij}^* \theta =: \theta_{ij} \in \Omega^1(U_i \cap U_j, \mathfrak{g})$, which for each $b \in U_i \cap U_j$ is given by

$$\theta_{ij}(u) = \mathrm{d}L_{t_{ij}(b)^{-1}}(\mathrm{d}t_{ij}(u)), \quad u \in T_b(U_i \cap U_j), \quad \mathrm{d}t_{ij}(u) \in T_{t_{ij}(b)}G. \tag{A.12}$$

Note that in the same way we identified $VE \simeq \mathfrak{g}$, we identify $\theta_{ij} \equiv (\theta_{t_{ij}(\cdot)})_{ij}$ fibre-wise. With the help of the pulled-back Maurer-Cartan form, which serves as transition function in $\mathfrak{g}$, we can extend gauge transformations, which relate two different gauges, also to the level of local connection forms.

A.2.4 Lemma

Let $G \to E \xrightarrow{\pi} B$ be a smooth principal G-bundle. Connections on E can be characterized through local connection forms in the following way:

1. *Let $A \in \Omega^1(E, \mathfrak{g})$ be a connection form on E and let $s_i \in \Gamma(U_i, E)$ and $s_j \in \Gamma(U_j, E)$ be two local sections, where $U_i \cap U_j \neq \emptyset$. Then, for $b \in U_i \cap U_j$ it holds*

$$A_b^{s_i}(u) = \mathrm{Ad}(t_{ij}(b)^{-1})(A_b^{s_j}(u)) + \theta_{ij}(u) \quad \forall u \in T_b(U_i \cap U_j). \tag{A.13}$$

2. *Conversely, if the images of $\{s_i \in \Gamma(U_i, E)\}_{i \in I}$ are a covering of E and $\{A_i \in \Omega^1(U_i, \mathfrak{g})\}_{i \in I}$ is a family of local 1-forms such that for $b \in U_i \cap U_j \neq \emptyset$ it holds that*

$$A_b^i(u) = \mathrm{Ad}(t_{ij}(b)^{-1})(A_b^j(u)) + \theta_{ij}(u) \quad \forall u \in T_b(U_i \cap U_j), \tag{A.14}$$

then there exists a connection form A on E that satisfies $A^{s_i} \equiv A^i$.

Example: let $G \subset \mathrm{GL}(r, \mathbb{K})$ be a matrix group. Because of the linearity of the action on E, we have $\mathrm{d}L_g(v) = g\,v$ and $\mathrm{Ad}(g)(v) = g\,v\,g^{-1}$ $\forall g \in G$ and $v \in \mathfrak{g}$. Thus, for $b \in U_i \cap U_j$ the transformation (A.14) takes the form

$$A_b^i(u) = t_{ij}(b)^{-1} A_b^j(u)\, t_{ij}(b) + t_{ij}(b)^{-1}\, \mathrm{d}t_{ij}(u) \quad \forall u \in T_b(U_i \cap U_j). \tag{A.15}$$

Remark: a gauge transformation is not a tensorial transformation because the Maurer-Cartan form depends on the differential of the transition functions.

A.2.5 Theorem

Every principal G-bundle has an Ehresmann connection.

Remark: we established that locally $TE = \pi^*TB \oplus \mathfrak{g}$. Given a section $s \in \Gamma(E)$, we have that $s_*TB \subset \pi^*TB$. Because of this, for $e \in E$, $h \in H_eE$, $\tilde{v}(e) \in V_eE$, where $\tilde{v} \in \Gamma(VE)$, the difference

$$\chi(h \oplus \tilde{v}(e)) = A(h \oplus \tilde{v}(e)) - s^*A(\pi_*(h \oplus \tilde{v}(e))) \tag{A.16}$$

is generically non-vanishing.

A.3 Differential Forms on Principal Bundles

Def Let $G \to E \xrightarrow{\pi} B$ be a principal G-bundle and $\rho : G \to GL(V)$ be a representation of G on V. Then for $\rho[E] := E \times_G V = {}^{E\times V}/_{G}$, the sequence $V \to \rho[E] \to E$ defines the **associated vector bundle**.

Def A k-form $\omega \in \Omega^k(E, V)$ is called

1) **horizontal** if $\omega_e(v_1, \ldots, v_k) = 0$ if any v_i is horizontal;

2) **of type ρ** if $R_g^*\omega = \rho(g^{-1}) \circ \omega \quad \forall g \in G$.

$\Omega^k_{\text{hor}}(E, V)^{(G,\rho)}$ denotes the space of all horizontal k-forms of type ρ.

Example: the connection form A is a horizontal 1-form of type Ad.

A.3.1 Proposition

For each $k \in \mathbb{N}$, $\Omega^k(B, \rho[E])$ is isomorphic to $\Omega^k_{\text{hor}}(E, V)^{(G,\rho)}$.

A.3.2 Corollary

The space of connections of a principal G-bundle $G \to E \xrightarrow{\pi} B$ is an affine space with vector space $\Omega^1(B, \text{Ad}[E])$.

So far, we only know how to construct an exterior derivative on the space of vector space-valued differential forms. Now, we would like to be able to define an exterior derivative on the vector space $\Omega^k(B, \rho[E])$. This space is isomorphic to the space of horizontal k-forms on E. However, the differential

$$\mathrm{d}\colon \Omega^k(E, V) \to \Omega^k(E, V)$$

does not necessarily map horizontal forms to horizontal ones again. Therefore, we have to project to the horizontal subspace.

Def The linear map $D_A : \Omega^k(E, V) \to \Omega^{k+1}(E, V)$ defined by

$$(D_A\omega)(v_1, \ldots, v_k) := \mathrm{d}\omega(\mathrm{proj}_h v_1, \ldots, \mathrm{proj}_h v_k) \tag{A.17}$$

is the **total derivative on E associated to A**.

A.3.3 Lemma

The total derivative on E related to A is a mapping

$$D_A : \Omega^k_{\mathrm{hor}}(E, V)^{(G,\rho)} \to \Omega^{k+1}_{\mathrm{hor}}(E, V)^{(G,\rho)}\,.$$

For any $\omega \in \Omega^k_{\mathrm{hor}}(E, V)^{(G,\rho)}$ we have that

$$D_A\omega = \mathrm{d}\omega + \rho_*(A) \wedge \omega\,, \tag{A.18}$$

where the second term is given by

$$(\rho_*(A) \wedge \omega)(v_1, \ldots, v_{k+1}) = \sum_{i=1}^{k} (-1)^i \rho_*(A(v_i))(\omega(v_1, \ldots, \widehat{v_i}, \ldots, v_{k+1})\,. \tag{A.19}$$

Remark: because of proposition A.3.1, the total derivative D_A induces a corresponding linear map on $\Omega^k(B, \rho[E])$.

Def Given $\sigma \in \Omega^k(B, \rho(E))$, let $\overline{\sigma} \in \Omega^k_{\mathrm{hor}}(E, V)^{(G,\rho)}$ denote the corresponding differential form to σ. The linear map $\mathrm{d}_A : \Omega^k(B, \rho[E]) \to \Omega^{k+1}(B, \rho[E])$ with $\overline{\mathrm{d}_A\omega} := D_A\overline{\omega}$ is called the **derivative induced by A**. It is locally given by

$$(\mathrm{d}_A\omega)_x(t_1, \ldots, t_{k+1}) = (p, \mathrm{d}\overline{\omega}_p(t_1^*, \ldots, t_{k+1}^*))\,, \tag{A.20}$$

$$(\mathrm{d}_A\omega)_x(t_1, \ldots, t_{k+1}) = (s(x), (D_A\overline{\omega})_{s(x)}(\mathrm{d}s(t_1), \ldots, \mathrm{d}s(t_{k+1})))\,, \tag{A.21}$$

where $p \in E_x$, $t_i \in T_xB$, $t_i^* \in T_pE$ their horizontal lifts, and $s \in \Gamma(U, E)$ is a local section over $x \in U \subset B$.

Remark: in contrast to d, which satisfies $\mathrm{d} \circ \mathrm{d} \equiv 0$, in general $\mathrm{d}_A \circ \mathrm{d}_A$ will be different from zero. However, d_A, being a differential, satisfies the same product rule:

A.3.4 Proposition

Let $\mathrm{d}_A : \Omega^*(B, \rho[E]) \to \Omega^{*+1}(B, \rho[E])$ *be the differential induced by* A. *Then it holds that*

$$\mathrm{d}_A(\omega \wedge \eta) = \mathrm{d}_A\omega \wedge \eta + (-1)^k \omega \wedge \mathrm{d}_A\eta \quad \forall \omega \in \Omega^k(B, \rho[E]), \eta \in \Omega^l(B, \rho[E]). \tag{A.22}$$

Def We call the operator

$$\nabla^A := \mathrm{d}_A\big|_{\Omega^0(B,\rho[E])} : \Gamma(\rho[E]) \to \Gamma(T^*B \otimes \rho[E]) \tag{A.23}$$

the **covariant derivative on $\rho[E]$ induced by A**.

A.3.5 Proposition

The covariant derivative on $\rho[E]$ *induced by the connection form* A *on* E *has locally the following form: the covariant derivative of a section* $r \in \Gamma(\rho[E])$ *in the direction of a vector field* $v \in \mathfrak{X}(B)$ *is given by*

$$(\nabla_v^A r)(x) = \Big(s(x), \mathrm{d}e_x(v_x) + \rho_*(A^s(v_x))(e(x))\Big) \quad \in \rho[E]_x, \tag{A.24}$$

where $s \in \Gamma(U, E)$ *is a local section over* $x \in U \subset B$, $e \in C^\infty(U, V)$ *is a smooth function with* $r\big|_U = (s, e)$, *and* $A^s = s^*A$ *is the local connection form.*

A.4 Curvature on Principal Bundles

Def The 2-form

$$F^A := D_A A \in \Omega^2(E, \mathfrak{g}) \tag{A.25}$$

is called **curvature form of A** or curvature form of the connection H. By definition, it is a horizontal 2-form of type Ad. Let $s \in \Gamma(U, E)$ be a local section of E. The 2-form $F^s := s^* F^A \in \Omega^2(U, \mathfrak{g})$ is called **local curvature form** related to s.

A.4.1 Proposition

Let $s_1 \in \Gamma(U_1, E)$ and $s_2 \in \Gamma(U_2, E)$ be two local sections of E and $t_{12} : U_1 \cap U_2 \to G$ their associated transition function (i.e. $s_1(x) = t_{12}(x)\, s_2(x)$.) The two local curvature forms related to s_1 and s_2 satisfy then for $b \in U_1 \cap U_2 \neq \emptyset$

$$F_b^{s_1} = \mathrm{Ad}(t_{12}(b)^{-1}) \circ F_b^{s_2} \,. \tag{A.26}$$

A.4.2 Lemma

Let $F^A \in \Omega^2(E, \mathfrak{g})$ be the curvature form of A, then

1. *given a k-form $\omega \in \Omega^k(E, V)^{(G,\rho)}$ of type ρ, then it is*

$$D_A D_A \omega = \rho_*(F^A) \wedge \omega \,; \tag{A.27}$$

2. *Bianchi identity: $D_A F^A = 0$.*

Remark: we now see that the curvature F^A is a measure for how much $D_A \circ D_A$ is different from zero.

A.5 Frames, Torsion and Curvature on the Tangent Bundle

So far, we just considered B to be some topological space. If we further endow it with a differentiable manifold structure, we can associate to its tangent bundle the principal bundle of general linear transformations of its frames, on which all the previously discussed structures of a connection form, a covariant derivative and a curvature form exist. Through the composition of their action with (co)frames, we can define those objects on the tangent bundle directly.

This section is tailored towards the discussion of gravitational theories in chapter 3, but more generally, the frame at a point x, that we will define below, can be replaced with a fiber isomorphism $f : V \to E_x$, with which we can define a covariant derivative, a torsion form, and a curvature endomorphism on any principal G-bundle in a similar fashion as layed out in this section. More details on this general set up can be found in [118], chapter 3.5.

Def Let $B = M$ be a d-dimensional smooth manifold M with signature (p, q). The tangent bundle over M is a vector bundle with an associated frame bundle

$$GL(p, q) \to F_{GL}(M) \xrightarrow{\pi} M \,. \tag{A.28}$$

A **frame at a point** $x \in M$ is a linear isomorphism $f : \mathbb{R}^{p,q} \to T_xM$ (or an ordered basis for T_xM), which has a natural right action of $GL(p, q)$. In a local trivialization, an element of $F_{GL}(M)$ is given by (x, f).

Given a frame field $\hat{e} : \mathbb{R}^{p,q} \to TM$ we can define a canonical 1-form (or solder form) field on the frame bundle as

$$\theta = \hat{e}^{-1} \circ \mathrm{d}\pi : TF_{GL}(M) \to \mathbb{R}^{p,q} \,. \tag{A.29}$$

For any local section $s : M \supset U \to F_{GL}(M)$, the pull-back of θ by s is $s^*\theta = \hat{e}^{-1} =: e : TM \to \mathbb{R}^{p,q}$. The map e is called **coframe field** and satisfies $e \circ \hat{e} = \mathrm{id}_{\mathbb{R}^{p,q}}$ and $\hat{e} \circ e = \mathrm{id}_{TM}$. Note that it is independent of which section s (i.e. which gauge) we choose.

Remark: We can understand e as an orientation-preserving (non-degenerate) $GL(p, q)$ transformation which maps a (generically) non-coordinate basis $(\boldsymbol{u}_\mu)$ of TM to a coordinate basis of $\mathbb{R}^{p,q}$. The mutual Lie derivatives of the basis vectors of a coordinate basis vanish, however this does not have to be the case for the non-coordinate basis:

$$\pounds_{\boldsymbol{u}_\mu} \boldsymbol{u}_\nu = [\boldsymbol{u}_\mu, \boldsymbol{u}_\nu] = \mathfrak{c}_{\mu\nu}{}^\tau \boldsymbol{u}_\tau \,. \tag{A.30}$$

We will consider non-vanishing structure constants $\mathfrak{c}$ only for completeness, as we will not make use of it throughout the thesis and set it to zero at the end of this section. It can thus be safely ignored.

Def The space $\mathbb{R}^{p,q}$ has a flat metric $\eta = \mathrm{diag}(+,\dots,+,-,\dots,-)$ which, under pull-back of e, induces a **semi-Riemannian metric** on M, i.e. $\Gamma(M, T^*M \otimes T^*M) \ni g = g_{\mu\nu}\sigma^\mu \otimes \sigma^\nu$, $g = e^*\eta$, with components

$$g_{\mu\nu} = e_\mu{}^a \eta_{ab} e_\nu{}^b \,. \tag{A.31}$$

Remark: The metric lets us relate e.g. $(0,2)$-tensors to $(1,1)$-tensors through

$$\begin{aligned} T(\zeta,\xi) = g(\zeta, T(\xi)) = g(T(\zeta),\xi) \qquad \forall \zeta,\xi \in \mathfrak{X}(M) \\ \zeta^\mu T_{\mu\nu}\xi^\nu = \zeta^\mu g_{\mu\lambda} T^\lambda{}_\nu \xi^\nu = \zeta^\mu T_\mu{}^\lambda g_{\lambda\nu}\xi^\nu \,, \end{aligned} \tag{A.32}$$

and more generally for (r,s)-tensors respectively.

We will now study the connection and all its induced structures on the frame and tangent bundles. Let $(\sigma^1,\dots,\sigma^d)$ be a basis of T^*M dual to $(\boldsymbol{u}_1,\dots,\boldsymbol{u}_d)$ (such that $\sigma^\nu(\boldsymbol{u}_\mu) = \delta^\nu_\mu$), and let furthermore $(\mathrm{L}^a{}_b)$ be a basis of the matrix representation of $\mathfrak{gl}(p,q)$ and (P_a) a basis of the vector representation of $\mathbb{R}^{p,q}$. In the following we will assume that we chose a gauge (i.e. a local section of the frame bundle), under which we pulled back the connection form to the base manifold M. Then the (so local) connection form of the $GL(p,q)$-principal bundle and the coframe field can be written in coordinate form as follows:

$$\Omega(M, \mathfrak{gl}(p,q)) \ni \omega = (\omega_\mu)^a{}_b \mathrm{L}^b{}_a \sigma^\mu \tag{A.33}$$

$$\Omega(M, \mathbb{R}^{p,q}) \ni e = e_\mu{}^a \mathrm{P}_a \sigma^\mu \,. \tag{A.34}$$

Let $\rho : GL(p,q) \to GL(V)$ be a representation of $GL(p,q)$ on $GL(V)$. The form ω defines a covariant derivative $D_\omega \alpha = \mathrm{d}\alpha + \rho_*(\omega) \wedge \alpha$ for $\alpha \in \Omega^k(M,V)^{(GL(p,q),\rho)}$. Using

the coframe field, this induces a covariant derivative $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ on the tangent bundle through

$$e(\nabla_\zeta \xi) := (D_\omega e(\xi))(\zeta) \qquad \forall \xi, \zeta \in \mathfrak{X}(M)\,. \tag{A.35}$$

A.5.1 Proposition

The coefficients of the covariant derivative

$$\nabla_{\boldsymbol{u}_\mu} \boldsymbol{u}_\nu = (\overline{\Gamma}_\mu)^\tau{}_\nu \boldsymbol{u}_\tau \tag{A.36}$$

are given by

$$(\overline{\Gamma}_\mu)^\lambda{}_\nu = e^\lambda{}_a (\omega_\mu)^a{}_b e_\nu{}^b + e^\lambda{}_a \partial_\mu e_\nu{}^a\,. \tag{A.37}$$

Note that $(\overline{\Gamma}_\mu)^\tau{}_\nu$ are not components of a tensor.

Def The covariant derivative associated to ω of the coframe field e defines the **torsion form**

$$\Omega^2(M, \mathbb{R}^{p,q}) \ni T = D_\omega e\,. \tag{A.38}$$

A.5.2 Proposition

*The action of T onto the tangent bundle is induced by e and defines the **torsion tensor** $\mathscr{T} = e^{-1} \circ T : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, which is given by*

$$\begin{aligned} \mathscr{T} &= e^{-1} \circ D_\omega e = \mathscr{T}^\tau{}_{\mu\nu}\, \boldsymbol{u}_\tau \otimes (\sigma^\mu \wedge \sigma^\nu) \\ &= \left((\overline{\Gamma}_\mu)^\tau{}_\nu - (\overline{\Gamma}_\nu)^\tau{}_\mu - \mathfrak{c}_{\mu\nu}{}^\tau \right) \boldsymbol{u}_\tau \otimes (\sigma^\mu \wedge \sigma^\nu)\,. \end{aligned} \tag{A.39}$$

A.5.3 Lemma

The subset of $GL(p,q)$-transformations that are isometries of the flat metric η is given by the set of $SO(p,q)$-transformations. Restricting ω to be a connection of this SO-frame bundle (its components being skew-symmetric) is equivalent to imposing $D_\omega \eta = 0$, which is called metric compatibility condition. Mapped to the tangent bundle, this conditions translates to $\nabla_\mu g_{\nu\lambda} = 0$.

More generally, we can find an explicit expression for the coefficients $(\overline{\Gamma}_\mu)^\lambda{}_\nu$ in terms of the metric, the structure constants and the torsion as

$$(\overline{\Gamma}_\mu)^\lambda{}_\nu = \frac{1}{2} g^{\lambda\tau} \Delta^{\alpha\beta\gamma}_{\mu\nu\tau} \left(\partial_\alpha g_{\beta\gamma} - \nabla_\alpha g_{\beta\gamma} + \mathfrak{c}_{\alpha\beta\gamma} + \mathscr{T}_{\alpha\beta\gamma} \right) , \tag{A.40}$$

where $\Delta^{\alpha\beta\gamma}_{\mu\nu\tau} = \delta^\alpha_\mu \delta^\beta_\nu \delta^\gamma_\tau + \delta^\alpha_\nu \delta^\beta_\tau \delta^\gamma_\mu - \delta^\alpha_\tau \delta^\beta_\mu \delta^\gamma_\nu$.

Def A basis of TM which satisfies $\mathfrak{c} \equiv 0$ is called holonomic basis. For such a basis, the coefficients corresponding to a connection that is metric compatible ($D_\omega \eta = 0$) and torsion-free ($D_\omega e = 0$) are known as Christoffel-symbols $\Gamma^\lambda_{(\mu\nu)}$, and the connection is known as Levi-Cevita connection. They are unique by the fundamental theorem of Riemannian geometry.

Def The **curvature form** of ω is defined as

$$\Omega^2(M, \mathfrak{gl}(p,q)) \ni R = D_\omega \omega \, . \tag{A.41}$$

A.5.4 Proposition

The action of R onto the tangent bundle is induced by a gauge transformation with transition function e and defines the ***Riemannian curvature tensor*** $\mathscr{R} = e^{-1} \circ R \circ e : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ *, which is given by*

$$\begin{aligned} \mathscr{R} &= e^{-1} \circ (D_\omega \omega) \circ e = \mathscr{R}^\tau{}_{\nu\alpha\mu} (\boldsymbol{u}_\tau \otimes_{GL} \sigma^\nu) \odot (\sigma^\alpha \wedge \sigma^\mu) \\ &= \left(\nabla_\alpha (\overline{\Gamma}_\mu)^\tau{}_\nu - \nabla_\mu (\overline{\Gamma}_\alpha)^\tau{}_\nu - \mathfrak{c}_{\alpha\mu}{}^\lambda (\overline{\Gamma}_\lambda)^\tau{}_\nu \right) (\boldsymbol{u}_\tau \otimes_{GL} \sigma^\nu) \odot (\sigma^\alpha \wedge \sigma^\mu) \, . \end{aligned} \tag{A.42}$$

A.5.5 Lemma

Restricting the structure group $GL \to SO$ and understanding $\mathscr{R}$ as a $(0,4)$-tensor through (A.32)*, the curvature tensor has the following symmetry properties*

$$\mathscr{R}_{\tau\nu\alpha\mu} = \mathscr{R}_{[\tau\nu][\alpha\mu]} = \mathscr{R}_{\alpha\mu\tau\nu} \, , \tag{A.43}$$

and satisfies the Bianchi identities

$$\mathscr{R}_{\tau[\nu\alpha\mu]} = 0 \qquad \textit{and} \qquad \nabla_{[\beta|} \mathscr{R}_{\tau\nu|\alpha\mu]} = 0 \, . \tag{A.44}$$

Proof: realizing that the first Bianchi identity is equivalent to $\mathrm{d}_\omega T = \mathrm{d}_\omega \mathrm{d}_\omega e = R \wedge e$, both Bianchi identities are a direct consequence of lemma A.4.2. The symmetry properties are obvious from the definitions. □

Remark: Considering $\mathscr{R}^\tau{}_{\nu\alpha\mu} = \sigma^\tau(\mathscr{R}(\boldsymbol{u}_\alpha, \boldsymbol{u}_\mu)(\boldsymbol{u}_\nu))$, the curvature tensor measures the non-commutativity of the parallel transport of $\boldsymbol{u}_\nu$ along directions $\boldsymbol{u}_\alpha$ and then $\boldsymbol{u}_\mu$ versus first along $\boldsymbol{u}_\mu$ and then $\boldsymbol{u}_\alpha$. Alternatively, it is a measure for the non-commutativity of the second covariant derivative $\mathscr{R}^\tau{}_{\nu\alpha\mu} = (\nabla_\alpha \nabla_\mu - \nabla_\mu \nabla_\alpha)^\tau{}_\nu$, and thus an integrability obstruction to the existence of an isometry between M and a flat space.

Def The **Ricci tensor** is defined as the trace of the mapping $\chi \mapsto \mathscr{R}(\chi, \xi)(\zeta)$,

$$\mathrm{Ric}(\xi, \zeta) = \mathrm{tr}(\chi \mapsto \mathscr{R}(\chi, \xi)(\zeta)) \qquad \forall \xi, \zeta \in \mathfrak{X}(M)$$
$$\mathrm{Ric} = \mathscr{R}^\tau{}_{\nu\tau\mu}\, \sigma^\nu \odot \sigma^\mu \,. \tag{A.45}$$

Taking the trace of the Ricci tensor, using the metric tensor, defines the **(Ricci) scalar curvature**

$$\mathcal{R} = \mathrm{tr}_g \,\mathrm{Ric} = \mathrm{Ric}^\mu{}_\mu \,. \tag{A.46}$$

Def The **Schouten tensor** is defined as

$$\mathscr{P} = \frac{1}{2}\left(\mathrm{Ric} - \frac{1}{6}\mathcal{R}\, g\right) , \tag{A.47}$$

A.5.6 Lemma

*The curvature $\mathscr{R}$ has a natural decomposition in terms of its irreducible representations for the action of the orthogonal group, called **Ricci decomposition**, which is given as*

$$\mathscr{R} = \mathscr{C} + \mathscr{P} \owedge g \,, \tag{A.48}$$

where $\mathscr{C}$ is the Weyl tensor and $\owedge$ is the Kulkarni–Nomizu product

$$(h \owedge k)(\boldsymbol{u}_\mu, \boldsymbol{u}_\nu, \boldsymbol{u}_\lambda, \boldsymbol{u}_\tau) = h_{\mu\lambda}k_{\nu\tau} + k_{\mu\lambda}h_{\nu\tau} - h_{\mu\tau}k_{\nu\lambda} + k_{\mu\tau}h_{\nu\lambda} \tag{A.49}$$

for two symmetric $(0,2)$-tensors h and k.

A.5.7 Proposition

An Einstein manifold is a manifold whose curvature takes the form

$$R = \Lambda\, e \wedge e \qquad \Longleftrightarrow \qquad \mathscr{R} = \Lambda\, g \owedge g\,. \tag{A.50}$$

The sectional curvature of an Einstein manifold is constant.

A.6 Yang-Mills Theory

Having defined the curvature on a principal bundle, it is now easy to define the Yang-Mills functional, which, by principle of variation, gives the equations of motion for physically propagating degrees of freedom, the Yang-Mills equations. Before we do so however, we need a few more definitions.

Def Let $(\sigma^1, \ldots, \sigma^d)$ be a basis of T^*M dual to $(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_d)$. The form

$$\Lambda^d(TM) \ni \mathrm{d}M = \sigma^1 \wedge \cdots \wedge \sigma^d \tag{A.51}$$

is called **volume form** on M. It is independent of the chosen basis and its length is defined to be

$$g(\mathrm{d}M, \mathrm{d}M) := (-1)^q\,, \tag{A.52}$$

where q is the sign of the metric.

Def The volume form $\mathrm{d}M$ is a basis element in $\Lambda^d(TM)$, and since $\Lambda^{d-k}(TM)$ has a scalar product, given a k-form $\omega \in \Lambda^k(TM)$, the relation

$$\omega \wedge \eta = g(*\,\omega, \eta)\,\mathrm{d}M \quad \forall \eta \in \Lambda^{n-k}(TM) \tag{A.53}$$

defines a $(d-k)$-form $*\,\omega \in \Lambda^{n-k}(TM)$. Thus,

$$* : \Lambda^k(TM) \to \Lambda^{n-k}(TM)$$

defines a linear map, called **Hodge operator**.

A.6.1 Proposition

The Hodge operator has the following properties:

1. $\forall \omega \in \Lambda^k(TM) : **\omega = (-1)^{k(d-k)+q}\omega$ (A.54)

2. $\forall \omega, \eta \in \Lambda^k(TM) : g(*\omega, *\eta) = (-1)^q g(\omega, \eta)$ *and* $\omega \wedge *\eta = (-1)^q g(\omega, \eta)\, \mathrm{d}M$. (A.55)

Def Let A be a connection form on the principal G-bundle $G \to E \to M$ and $\mathrm{d}_A : \Omega^k(M, \rho[E]) \to \Omega^{k+1}(M, \rho[E])$ the differential induced by A. The **codifferential** $\delta_A : \Omega^{k+1}(M, \rho[E]) \to \Omega^k(M, \rho[E])$ is defined by

$$\delta_A := (-1)^{dk+p+1} * \mathrm{d}_A * , \tag{A.56}$$

where $d = \dim(M)$ and p is the signature of (M, g).

A.6.2 Lemma

Let $\omega \in \Omega^k(M, \rho[E])$ *and* $\eta \in \Omega^{k+1}(M, \rho[E])$ *. Then*

$$\int_M \langle \mathrm{d}_A \omega, \eta \rangle \, \mathrm{d}M_g = \int_M \langle \omega, \delta_A \eta \rangle \, \mathrm{d}M_g \, , \tag{A.57}$$

where $\langle \cdot, \cdot \rangle$ *is a chosen bundle metric on* $\rho[E]$ *.*

Def A connection form A on E is called a **Yang-Mills connection** if its associated curvature form $F^A \in \Omega^2(M, \mathrm{Ad}[E])$ satisfies the Yang-Mills equations

$$\delta_A F^A = 0 \, . \tag{A.58}$$

Def The functional $L : \mathcal{C}(E) \to \mathbb{R}$ given by

$$S[A] := \int_M \left\langle F^A, F^A \right\rangle \mathrm{d}M_g \tag{A.59}$$

is called the **Yang-Mills functional**. The bundle metric is given by the Killing form $\langle \cdot, \cdot \rangle := -B_{\mathfrak{g}}$, for which $B_{\mathfrak{g}}(X, Y) := \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y))$ for $X, Y \in \mathfrak{g}$.

A.6.3 Lemma

The Yang-Mills functional is invariant under gauge transformations.

A.6.4 Lemma

The Yang-Mills equations are equivalent to $\mathrm{d}S[A] = 0$.

Remark: the Yang-Mills equations are equations of motion for a local connection form $s^*A \in \Omega(M, \mathfrak{g})$. In order to count physical degrees of freedom, we need to take three things into account:

1. s^*A has by itself $\dim(TM) \cdot \dim(\mathfrak{g})$ degrees of freedom, some of which are redundant and thus not physical in the following sense:

2. Looking at Lemma A.6.3, we see that the equations of motion are invariant under gauge transformations, i.e. the choice of s, which depends on $\dim(\mathfrak{g})$ parameters.

3. The equations of motion are constraining equations for another $\dim(\mathfrak{g})$ parameters.

This leaves us with $(d-2)\dim(\mathfrak{g})$ physical degrees of freedom. In $d = 4$ dimensions, each generator of $\mathfrak{g}$ has thus 2 degrees of freedom, which we can identify as massless particles.

Self-Duality in Four Dimensions

In even dimensions, $d = 2n$, the square of the Hodge operator is proportional to the identity, i.e. $** = (-1)^q \,\mathrm{id}_{\Lambda^n}$, and thus we can decompose the space of n-forms into those with eigenvalue $+1$ (or $+i$ for odd q) or -1 (or $-i$ for odd q) under this map.

We are mainly interested in four dimensions as this allows us to apply this decomposition to the curvature form.

Def An 2-form ω is called **self-dual** if $*\omega = \omega$ if q is even and $*\omega = i\,\omega$ if q is odd, and **anti-self-dual** if $*\omega = -\omega$ if q is even and $*\omega = -i\,\omega$ if q is odd. We

denote the bundles of self-dual and anti-self-dual 2-forms by $\Lambda^2_+(T^*M)$ and $\Lambda^2_-(T^*M)$ respectively. This gives a decomposition

$$\Lambda^2(T^*M) = \Lambda^2_+(T^*M) \oplus \Lambda^2_-(T^*M)\,. \tag{A.60}$$

Def A connection form A on a principal G-bundle is called self-dual or anti-self-dual if its curvature form $F^A \in \Omega^2(M, \mathrm{Ad}[E])$ is self-dual or anti-self-dual respectively.

A.6.5 Theorem

Every (anti-)self-dual connection is a Yang-Mills connection.

A.6.6 Lemma

Let $G \to E \xrightarrow{\pi} M$ and $G \to \widetilde{E} \xrightarrow{\widetilde{\pi}} \widetilde{M}$ two principal G-bundles over oriented, four-dimensional semi-Riemannian manifolds (M, g) and $(\widetilde{M}, \tilde{g})$. Let $\phi : E \to \widetilde{E}$ be a G-equivariant, smooth map which projects onto a conformal, orientation-preserving diffeomorphism $f : M \to \widetilde{M}$, i.e. the diagram

$$\begin{array}{ccc} E & \xrightarrow{\phi} & \widetilde{E} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\widetilde{\pi}} \\ M & \xrightarrow{f} & \widetilde{M} \end{array}$$

commutes. If $\widetilde{A}$ is an (anti-)self-dual connection form on $\widetilde{E}$, then the pull-back $\phi^*\widetilde{A}$ is an (anti-)self-dual connection form on E.

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