Jordan systems, bounded symmetric domains and associated group orbits with holomorphic and CR extension theory.

by

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B.A. (Mod.)

A Thesis submitted to
The University of Dublin
for the degree of

M.Sc.

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Oct, 2005

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31 Oct, 2005

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Summary

The first chapter will deal with the one to one correspondence between the *positive hermitian Jordan triple systems* and the *bounded symmetric domains*. We start by defining the various Jordan systems. Then we continue by giving classification results leading to the classification of the Jordan pairs under finiteness conditions. This is followed by defining bounded symmetric domains, and some motivation on why they became a topic of interest. Up to this point we will be dealing with the Jordan systems over a general ring of scalars, we now restrict ourselves to the complex number field. We show bounded symmetric domains are connected to the positive hermitian Jordan triple systems. A look at *Peirce decompositions* of the Jordan systems will prove important. With this completed, we deal with the one to one correspondence mentioned. This will involve some results relating to Riemannian manifolds and Lie theory, we can then present the result as presented by *Ottmar Loos* in [Loos1]. A table of the classification is also included to conclude the chapter. We note here that throughout this thesis when we say we are dealing with bounded symmetric domains we mean *circled* bounded symmetric domains.

In the second chapter we look at some *holomorphic extension theory*. This is started by looking at and defining manifolds and varieties. We then look at *normal spaces* and *normalisations of spaces* with some extension theory. After that we look briefly at *Stein manifolds* and *domains of holomorphy*, explaining their importance in relation to (maximal) holomorphic and CR extension theory. We will also look at a CR extension theorem due to *Albert Boggess* and *John Polking*. To complete this section we will present a application to the extension theorem from the paper [KaupZait] by *Dimitri Zaitsev* and *Wilhelm Kaup*.

Our third chapter is based on [KaupZait] giving an overview of results contained within. We will look at orbits associated with bounded symmetric domains with respect to (the connected identity component of) the automorphism group of the domain. We look at various hulls and domains that are relevant to the extension theory results of the paper. Then we state the results, and sketch the extension theorem proof.

In the final chapter we will generalise these results. In [KaupZait] , the results are for *irreducible* bounded symmetric domains, we will look at what happens in the reducible case. We will introduce a generalised notation and generalise the results from the irreducible case when needed.

Acknowledgements

To my family, friends and Gill.
I would like to thank my supervisor, Dr. Richard Timoney, for his time and patience.

Chapter 1

Correspondence between bounded symmetric domains and positive hermitian Jordan triple systems

We first outline the various definitions of Jordan Structure, keeping in mind situations when our field may not be of characteristic equal to 2, (Jordan algebras, quadratic Jordan algebras, Jordan pairs and Jordan triple systems), and how they relate to one another. We continue to discuss some classification results without proofs, including non-degenerate *alternative* pairs with finiteness conditions and simple Jordan pairs with a maximal idempotent.

The remainder of the chapter will be in more detail. Restricting to the case where the base field (or module) is the complex numbers, we introduce positive hermitian Jordan triple systems. We define the bounded symmetric domains. We give a proof, using a method from [Loos1] of the one to one correspondence between bounded symmetric domains and the positive hermitian Jordan triple systems.

1.1 Jordan Systems

1.1.1 Jordan Algebras

Definition A *Jordan algebra* structure $\mathcal{A}$ over a vector space J in a field k of characteristic different from 2 is a binary operation $\circ$ on J such that for all $x, y \in J$ we have

1. It is commutative: $x \circ y = y \circ x$,

2. It is linear: $x \circ (y + z) = x \circ y + y \circ z, (\alpha x) \circ y = \alpha(x \circ y)$,

3. It satisfies the following (partial associativity) property: $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$.

We use the notation $\mathcal{A} := (J, \circ)$.

By defining a linear operator

$$L(x)y = x \circ y$$

we get a mapping from J to the space $\mathrm{End}_k(J)$ of J. Note: $\mathrm{End}_k(J)$ is the set of mappings from J to J which are linear with respect to the scalars k, i.e.

$$\mathrm{End}_k(J) = \{f : J \mapsto J : f(x+y) = f(x) + f(y) \text{ and } f(\alpha x) = \alpha f(x) \forall \alpha \in k, x, y \in J\}.$$

So we can restate the 3 conditions above as follows,

1. $L(x)y = L(y)x$,
2. $L(x)$ is linear in x,
3. $L(x)L(x^2) = L(x^2)L(x)$,

where $x \in J$.

The Quadratic Representation

In the literature, considerable use is made of a *quadratic* map defined as follows

$$U(x) = 2L(x)^2 - L(x^2), \text{ for } x \in J,$$

It can be shown that the algebra operation $\circ$ can be defined in terms of the quadratic operator and vice versa when an identity element $c \in J$ (with respect to $\mathcal{A}$) exists,

$$x \circ y = \frac{1}{2}(U(x+y)c - U(x)c - U(y)c)$$

when we have the following axioms,

1. $x \mapsto U(x)$ is a quadratic map (or equivalently a homogeneous polynomial of degree 2) which is a map $X \mapsto \mathrm{End}_k(X)$
2. $U(c) = \mathrm{Id}$
3. $U(c + x) - \mathrm{Id} - U(x) = 0$ implies $x = 0$
4. $U(U(x)y) = U(x)U(y)U(x)$ for every $x, y \in X$.

See, [Koech1].

We define a quadratic map in the following, and note that we are going to use a k as a ring of scalars, rather than just a field, to introduce more generality.

Definition A map $Q : V \mapsto W$ between k-modules (k a commutative, associative ring) is called *quadratic* if $Q(\lambda x) = \lambda^2 Q(x)$ for all $\lambda \in k$, and if

$$Q(x, y) := Q(x + y) - Q(x) - Q(y),$$

is a symmetric k-bilinear map from $V \times V$ into W.

Recall that when the ring k is a field a k-module is just a k-vector space.

Homotopes

A *homotope* (sometimes known as a mutation) of Jordan Algebra $\mathcal{A}$ w.r.t $z \in \mathcal{A}$ is an algebra with operation $\perp_z$ defined as follows:

$$x \perp_z y := x \circ (y \circ z) + y \circ (x \circ z) - (x \circ y) \circ z.$$

This is also a Jordan Algebra, and its associated quadratic and linear maps are

$$U_z(x) := U(x)U(z), L_z(x)y := x \perp_z y.$$

It follows from, for example [Koech1] p.78, that

$$L_z(x)Q(x) = Q(x)L_x(z)$$

and it can be shown that the following holds

$$Q_z(Q_z(x)y) = Q_z(x)Q_z(y)Q_z(x).$$

So there is not necessarily a unique Jordan algebra structure defined on the vector space J, we may have different Jordan algebra structures defined for different elements of $\mathcal{A}$.

With this in mind we continue by defining Jordan algebras with a unit element in a more general fashion. This more abstract, and general definition of a unital Jordan algebra is given in [Loos2]. This approach is useful in motivating the definition of the other Jordan systems. First we will give some preliminary definitions and results.

Definition By an *extension* of k we mean a commutative unital k-algebra. The symbol $\otimes$ stands for $\otimes_k$. By $\otimes_k$ we mean the tensor product (of k-modules) with respect to k as the underlying ring of scalars. If V is an k-module and R is an extension of k then the R-module

$$V_R = V \otimes R$$

is called the module obtained from V by extending scalars to R or simply a scalar extension of V.

If we have a map $T : V \mapsto W$, then we can define $T_R : T \otimes Id$, that is $T_R : V_R = V \otimes R \mapsto W \otimes R = W_R$. For a bilinear map $L : V \times V \mapsto W$ we define $L_R : V_R \times V_R \mapsto W_R$ by $L_R(v_1 \otimes r_1, v_2 \otimes r_2) = L(v_1, v_2) \otimes r_1 r_2$ for all $r_1, r_2 \in V$ and $v_1, v_2 \in R$ respectively. So for every extension R of k there is a unique quadratic map $Q_R : V_R \mapsto W_R$ such that $Q_R(x_R) = Q(x)_R$. We will write Q rather that Q_R if no confusion will occur.

It should be noted that we use the notation, $U_x = U(x)$, $Q_x = Q(x)$ and $P_x = P(x)$ which are standard in the literature. The quadratic map in a Jordan algebra is usually denoted by U while in Jordan pairs the letter Q is commonly used, we will use P for the triples.

We now introduce a more abstract definition of a Jordan algebra, this definition and notation will helps to motivate the introduction of *Jordan Pairs* and *Jordan Triples.*

Definition A *unital quadratic Jordan algebra* is a triple $(J, U, 1)$ consisting of a module J over some commutative unital associative ring k of scalars, a quadratic map U from J into the endomorphism ring $\mathrm{End}_k(J)$ of J, and an element $1 \in J$, called the unit element, such that the following identities hold in all scalar extensions.

$$U_1 = \mathrm{Id}, \tag{1.1}$$

$$\{x, y, U_x z\} = U_x\{xyz\}, \tag{1.2}$$

$$U(U_x y) = U_x U_y U_x. \tag{1.3}$$

Here $\{xyz\} = U_{x+z}y - U_x y - U_z y$ is the linearisations of $U_x y$.

Note the correspondence between this definition and our previous definition. A standard example is $U_x y = xyx$, where $x, y \in M_n(\mathbb{C})$.

1.1.2 Jordan Pairs

Before we define the other Jordan systems it may be useful to mention some reasons why these generalisations came about. As we mentioned earlier, for a Jordan Algebra $\mathcal{A}$ defined over a space E there is not necessarily a unique Jordan algebra structure defined on E. We can produce more Jordan algebras if we already have one. It turns out that it is useful to define the Jordan pairs (and Jordan triples) These systems give us the ability to study a Jordan algebra and all its homotopes as one object. More precisely, the automorphism group of a Jordan pair corresponds to the *structure group* of a corresponding Jordan algebra, see [Loos2] p.iv. The structure group contains Jordan algebra homomorphisms between different homotopes.

Let V^+ and V^- be k-modules, and let

$$Q_\sigma : V^\sigma \to \mathrm{Hom}(V^{-\sigma}, V^\sigma)$$

be quadratic maps (the index σ always takes the values + and -). We define trilinear maps $V^\sigma \times V^{-\sigma} \times V^\sigma \mapsto V^\sigma$, $(x, y, z) \to \{xyz\}$, and bilinear maps $D_\sigma : V^\sigma \times V^{-\sigma} \to \mathrm{End}(V^\sigma)$, by the formula

$$\{xyz\} := D_\sigma(x, y)z := Q_\sigma(x, z)y$$

where $Q_\sigma(x, z) = Q_\sigma(x + z) - Q_\sigma(x) - Q_\sigma(z)$. Obviously, we have $\{xyz\} = \{zyx\}$ and

$$\{xyx\} = 2Q_\sigma(x)y.$$

Definition A *Jordan pair* over k is a pair $V = (V^+, V^-)$ of k-modules together with a pair (Q_+, Q_-) of quadratic maps $Q_\sigma : V^\sigma \to \mathrm{Hom}(V^{-\sigma}, V^\sigma)$ such that, with the notations introduced above, the following identities hold in all scalar extensions V_R of V:

$$D_\sigma(x, y)Q_{-\sigma}(x) = Q_\sigma(x)D_{-\sigma}(y, x), \tag{1.4}$$

$$D_\sigma(Q_\sigma(x)y, y) = D_\sigma(x, Q_{-\sigma}(y)x), \tag{1.5}$$

$$Q_\sigma(Q_\sigma(x)y) = Q_\sigma(x)Q_{-\sigma}(y)Q_\sigma(x). \tag{1.6}$$

The validity of (1.4 - 1.6) in all scalar extensions is equivalent with the condition that all linearisations of (1.4 - 1.6) hold in V. This turns out to be quite simple. For example since the second identity is of at most degree two, its validity is immediate from the quadratic map extension we mentioned earlier. For more, see [Loos2].

It is not difficult to see that a Jordan algebra can be embedded in a Jordan pair. Let $V^+ = V^- = J$ and $Q_+ = Q_- = U$. If we look at the pair denoted (J, J), it satisfies (1.4) and (1.5) by definition and can be shown to satisfy (1.6), thus making is a Jordan pair.

A *homotope* of the Jordan pair $V = (V^+, V^-)$ with respect to some $y \in V^-$ is a (not necessarily unital) Jordan algebra over the module V^+ denoted by V_y^+, with quadratic operators $U_x = Q_+(x)Q_-(y)$ and squaring operation $x^{(2,y)} = Q_+(x)y$.

1.1.3 Jordan Triples

Definition A *Jordan triple system* is defined as a k-module T together with a quadratic map $P : T \mapsto \mathrm{End}_k(T)$ satisfying the following identities in all scalar extensions.

$$D(x, y)P(x) = P(x)D(y, x), \tag{1.7}$$

$$D(P(x)y, y) = D(x, P(y)x), \tag{1.8}$$

$$P(P(x)y) = P(x)P(y)P(x). \tag{1.9}$$

Now, if we compare these defining identities to those of a Jordan pair we see the following.

T gives rise to a Jordan pair (T, T) by setting $V^+ = V^- = T$ and $Q_+ = Q_- = P$. Not every Jordan pair is of this form since it is possible to construct counter examples where V^+ and V^- are not isomorphic as k-modules. So to obtain a Jordan triple system from a Jordan pair, we must have some way of identifying V^+ and V^-. We define an involution of a Jordan pair to be a module isomorphism $\alpha : V^+ \to V^-$ such that $Q_-(\alpha(x)) = \alpha Q_+(x)\alpha$ for all $x \in V^+$. Then a Jordan pair with involution gives rise to a Jordan triple system by setting $T = V_+$ and $P_x y = Q_+(x)\alpha(y)$. So we have established a one to one correspondence between Jordan triple systems and Jordan pairs with involution.

The reason the term *triple* is used is because we now have a triple product

$$T \times T \times T : \{xyz\} \mapsto T$$

which takes 3 elements of T and maps them to T. This product has many properties such as being symmetric in the outer variables, amonst many others which we will not look at at this time.

1.2 Classification of the non-degenerate simple Jordan algebras

The proof needed to give the full classification will not be presented here. For a complete proof of the classification classification, see [McCrimm]. We give some results,but first we give some definitions.

We say that the quadratic map U is non-degenerate if it has zero radical $\mathcal{R}ad(U) := \{z : U(z) = U(z,c) = 0\} = 0$, where U is a map from a space V onto itself. We say a Jordan algebra is non-degenerate if its associated quadratic map is non-degenerate.

A *Quadratic form* U is a quadratic map from a module to the ring of underlying scalars, $U : V \mapsto k$.

A *composition* algebra is a unital linear algebra C which carries a nondegenerate quadratic form U permitting composition in the sense that it is unital and satisfies the composition law, i.e.

$$U(1) = 1, U(xy) = U(x).U(y)$$

for all $x, y \in C$. We refer to U as the norm.

A *basepoint* for a quadratic form, U, is a point, c, such that $U(c) = 1$.

We refer to an algebra with an involution, $x \mapsto x^*$, as a $*$*-algebra*. In such an algebra we define a norm $n(x)$ and a trace $t(x)$ as follows,

$$n(x) := x \cdot x^*, t(x) := x + x^*.$$

In the case of a quadratic form, i.e. $U : C \to k$, with basepoint c is unital if $U(c) = 1$. A trace T is defined by $T(x) := U(x,c)$, and the standard trace involution is the linear map

$$\overline{x} := T(x)c - x.$$

If we have an associative algebra A with product (of the elements x and y) denoted by xy, we define A^+ with the same space as A but with a new, commutative product, $x \circ y := \frac{1}{2}(xy + yx)$, this being the Jordan product, thus giving A Jordan algebra structure.

An element e of a Jordan algebra is called an idempotent if $e^2 = e$. A *division* idempotent is one such that $U_e J$ forms a Jordan division subalgebra, i.e. forms a subalgebra where every non-zero element is invertible. Two idempotents e and f in J are called orthogonal if $e \circ f = 0$, this is denoted as $e \perp f$.

A Jordan algebra is said to have *capacity* n if it has a unit 1 which can be written as a finite sum of n orthogonal division idempotents. It has *finite* capacity if it has capacity n for some finite number n.

Let N be a cubic form on a k-module M over k with $N : M \to k$ being homogeneous of degree 3 and it extends to arbitrary scalar extensions K by

$$N(\sum_i w_i x_i) = \sum_i w_i^3 N(x_i) + \sum_{i \neq j} w_i^2 w_j N(x_i, x_j) + \sum_{i<j<k} w_i w_j w_k N(x_i, x_j, x_k),$$

where the $N(x_i, x_j)$ is quadratic in x_i and linear in x_j, while $N(x_i, x_j, x_k)$ is symmetric and trilinear and $w_i \in K$. If we write $N(x, y, z)$ as the linearisation of $N(x)$, then we define the trace, $T(x) := N(c, x)$ and the spur, $S(x) := N(x, c)$.

A cubic form is a Jordan cubic form if

1. it is nondegenerate at the base point in the sense that the trace bilinear form

$$T(x, y) := T(x)T(y) - S(x, y)$$

is a non degenerate bilinear form, where

$$S(x, y) := N(x, y, c),$$

and

2. the quadratic sharp map $\# : M \to M$, defined uniquely by

$$T(x^{\#}, y) = N(x, y).$$

Note that in the case of nondegeneracy the adjoint identity is all we need.

A ring is called *Artinian*, if it satisfies the condition that any descending sequence of ideals stabilises, i.e. becomes constant, after a finite number of steps. By descending we mean $I_i \supset I_{i+1}$, where I_i are the elements of the sequence. *Noetherian* is in a sense the opposite of Artinian because it involves replacing the condition of descending sequences with the same condition on ascending sequences, that is, ascending sequences stabilise after a finite number of elements. *acc* is used to denote the ascending chain condition and *dcc* to denote the descending chain condition.

A *division algebra* is an algebra where every non-zero element has a multiplicative inverse contained in that algebra.

We say an involution, $W \mapsto X^{sp}$ on $M_{2n}(K)$ is of *symplectic type* if $X^{sp} = SX^{tr}S^{-1}$, for S the symplectic $2n \times 2n$ matrix diag$\{s, s, \ldots, s\}$ with $s = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

Definition Here is a list of the types of algebras we need:

1. $\mathcal{J}ord(U, c)$: a degree-2 Jordan algebra of the quadratic form U with base point c and product
$$2x \circ y = T(x)y + T(y)x - U(x, y)c,$$
and $U : M \mapsto k$. (We don't divide by 2 in case our field is of characteristic 2.)

2. $\mathcal{J}ord(N, c)$: a Jordan algebra constructed from a Jordan cubic form, N, with basepoint c, (i.e. $N(c) = 1$) and product
$$x \circ y := \frac{1}{2}(x\#y + T(x)y + T(y)x - S(x, y)1).$$
with
$$x\#y := (x + y)^{\#} - x^{\#} - y^{\#}$$
where the sharp mapping is defined as
$$x^{\#} = x^2 - T(x)1 + S(x)1.$$
Also $N : M \mapsto K$.

3. $\mathcal{H}(A, *)$: algebra of hermitian elements (w.r.t. the involution $X \mapsto X^*$), i.e. $x = x^*$, of a linear algebra A with involution $X \mapsto X^*$.

4. $\mathcal{M}_n(k)$: algebra of $n \times n$ matrices with entries over a ring of scalars k.

5. $\mathcal{E}x(A) := (A \oplus A^{op}, ex)$, where $ex(a, b) := (b, a)$ is called the exchange involution. $\mathcal{E}x(A)$ also preserves the Jordan algebra structure on each component, i.e. $U(x_1 \oplus x_2) = U(x_1) \oplus Q(x_2)$. A^{op} is called the *opposite* algebra of A, if A has product $x \cdot y$, then we define $x \cdot_{op} y := y \cdot x$.

6. $\mathcal{H}_n(k, \Gamma))$: the twisted Jordan matrix algebra $\mathcal{H}(A, *_\Gamma)$ of hermitian elements of $A = \mathcal{M}_n(k)$ for the involution $*_\Gamma$, we define $X^{*\Gamma} = \Gamma \bar{X}^{tr} \Gamma^{-1}$ for the hermitian matrix $\Gamma = diag(\gamma_1, \ldots, \gamma_n)$ hermitian .So $X^{*\Gamma}$ is clearly an involution.

Theorem 1.2.1 *A Jordan algebra is nondegenerate of finite capacity if and only if it is the direct sum of a finite number of simple ideals of the following (Division, Spin, or matrix) types:*

Division type; a Jordan division algebra,

Quadratic type; a quadratic factor $\mathcal{J}\mathrm{ord}$*(U,c) determined by a nondegenerate quadratic form U with base point c over a field* Ω *(not* split*(see[McCrimm]) of dimension 2),*

Albert type; a cubic factor $\mathcal{J}\mathrm{ord(N,c)}$ *determined by a Jordan cubic form N with basepoint c (an exceptional Albert algebra) of dimension 27 over a field* Ω*,*

Hermitian type; $\mathcal{H}(A,*)$ *for a* $*$*-simple Artinian associative algebra A, where a* $*$*-simple algebra is an algebra with no* $*$*-simple ideals, i.e. no ideals invariant under that* $*$ *involution.*

More specifically, the algebras of Hermitian type (w.r.t. A for a $*$*-simple Artinian associative algebra A) are twisted Jordan Algebras of the following types:*

Exchange type; $\mathcal{M}_n(\Delta)^+$ *for an associative division ring* Δ *(A is a* $*$*-simple but not simple,* $A = \mathcal{E}x(B)$ *with exchange involution for a simple Artinian algebra* $B = \mathcal{M}_n(\Delta)$*),*

Orthogonal type; $\mathcal{H}_n(\Delta, \Gamma)$ *for an associative division ring* Δ *with involution (*$A = \mathcal{M}_n(\Delta)$ *simple Artinian with involution),*

Symplectic type; $\mathcal{H}_n(Q, \Gamma)$ *for a quaternion algebra Q over a field* Γ *with standard involution (*$A = \mathcal{M}_n(U)$ *simple Artinian with involution of symplectic type).*

For the proof, see [McCrimm] p.357.

1.3 Classification of the simple Jordan pairs with a maximal idempotent

We need the structure theorem from the last section and *alternative pairs* to be able to give the structure theorem we require in this section. [Loos2] gives details.

As in the last section, we will give a few definitions enabling understanding of the classification.

Definition Let $A = (A^+, A^-)$ be a pair of k-modules with trilinear maps $\langle,\rangle$: $A^\sigma \times A^{-\sigma} \times A^\sigma \to A^\sigma, \langle x, y, z\rangle \to \langle xyz\rangle$. A is called an *associative pair* if the identities

$$\langle uv\langle xyz\rangle\rangle = \langle u\langle yxv\rangle z\rangle = \langle\langle uvx\rangle yz\rangle \tag{1.10}$$

are satisfied. A is called *commutative* if $\langle xyz\rangle = \langle zyx\rangle$. A is an *alternative pair* if the following identities hold.

$$\langle uv\langle xyz\rangle\rangle + \langle xy\langle uvz\rangle\rangle = \langle\langle uvx\rangle yz\rangle + \langle x\langle vuy\rangle z\rangle, \tag{1.11}$$

$$\langle uv\langle xyx\rangle\rangle = \langle\langle uvx\rangle yx\rangle, \tag{1.12}$$

$$\langle xy\langle xyz\rangle\rangle = \langle\langle xyx\rangle yz\rangle. \tag{1.13}$$

Since these identities are of at most degree two in each variable they remain valid in every scalar extension, and hence, A_K is an alternative pair over K, for every extension K of k. Note associative pairs are alternative by definition.

We define left, middle and right multiplication as follows,

$$L_\sigma(x, y)z = M_\sigma(x, z)y = R_\sigma(z, y)x = \langle xyz\rangle.$$

We will only be concerned with $M_\sigma(x, z)y$ briefly in the following theorem.

Theorem 1.3.1 *Let $A = (A^+, A^-)$ be an alternative pair over k. Then A becomes a Jordan pair over k, denoted by A^J, with quadratic maps $Q : A^\sigma \mapsto Hom(A^{-\sigma}, A^\sigma)$ given by*

$$Q(x)y = \langle xyx\rangle = M(x, x)y.$$

([Loos2] p.68)

Definition Let $V = (V^+, V^-)$ be a Jordan pair over a ring k. A submodule $m \subset V^\sigma$ is called an *inner ideal* if $Q(m)V^{-\sigma} \subset m$.

Now, for each $x \in V^{-\sigma}$, $Q(x)V^\sigma$ is an inner ideal, we refer to it as the *principal inner ideal generated be x*. See [Loos2] p.103.

Definition A pair $(x, y) \in V$ is said to be *quasi invertible* if x is quasi invertible in the Jordan Algebra V_y^+. That is, we say x is quasi invertible in V_y^+ if $1 - x$ is invertible in the unital Jordan algebra $k.1 \oplus V_y^+$ obtained from V_y^+ by adjoining a unit element.

We say an element $x \in V^+$ is *properly* quasi invertible if (x, y) is quasi invertible for all $y \in Y^-$.

A Jordan pair V is said to be *semisimple* if it contains no properly quasi-invertible elements, i.e. its Jacobson radical is the null set, were the Jacobson radical consists of the properly quasi invertible elements.

Definition A pair $(x, y) \in V$, where V is a Jordan pair, is called an *idempotent* if $x = Q(x)y$ and $y = Q(y)x$. It is standard notation to denote an idempotent as $e = (e^+, e^-)$. This corresponds to the Jordan algebra definiton of an idempotent in the sense that e^σ is an idempotent in the jordan algebra $V^\sigma_{e^{-\sigma}}$. With this in mind we define

$$V_0^\sigma := Ker(Q(e^{-\sigma})) \cap Ker(D(e^\sigma, e^\sigma)),$$

$$V_1^\sigma := Ker(\mathrm{Id} - D(e^\sigma, e^{-\sigma})),$$

$$V_2^\sigma := Im(Q(e^\sigma)),$$

A *maximal idempotent* is an idempotent witch satisfies the property that its the Pierce component, $V_0(e)$ is 0.

Definition The associative pairs $A(M, R, \phi)$: Let R be an associative (not necessarily commutative) k-algebra, and let $M = (M^+, M^-)$ be a pair of k-modules such that M^+ is a left and M^- is a right module. Consider a k-bilinear map $\phi : M^+ \times M^- \to R$ which is R-bilinear in the sense that

$$\phi(ax, yb) = a\phi(x, y)b,$$

for all $a, b \in R, x \in M^+, y \in M^-$. We say that ϕ is non-degenerate if $\phi(x, M^-) = 0 \implies x = 0$ and $\phi(M^+, y) = 0 \implies y = 0$. In order to describe this situation in a more symmetrical way, let $R^+ = R$ and $R^- = R^{op}$ (R^{op} is called the *opposite* algebra, it has the same underlying k-module and multiplication $a \cdot b := ba$. The identity maps $R \mapsto R^{op}$ and $R^{op} \mapsto R$ are anti-isomorphisms, usually written $a \mapsto \overline{a}$.) Consider M^- as a left R^--module in the obvious way. Also let $a \to \bar{a}$ denotes the anti-isomorphism $R^\sigma \to R^{-\sigma}$ given by the identity map, and define $\phi_+ = \phi, \phi_- : M^- \times M^+ \to R^-$ by $\phi_-(y, x) = \overline{\phi(x, y)}$. Then $\phi_\sigma : M^{-\sigma} \times M^\sigma \to R^\sigma$ is *hermitian* in the sense that $\phi(ax, y) = a\phi_\sigma(x, y)$ and $\overline{\phi_\sigma(x, y)} = \phi_{-\sigma}(y, x)$.

The pair of modules (M^+, M^-) becomes an associative pair over k, denoted by $A = A(M, R, \phi)$, with

$$\langle xyz \rangle = \phi_\sigma(x, y)z,$$

for $x, z \in A^\sigma = M^\sigma, y \in A^{-\sigma} = M^{-\sigma}$. For more properties see [Loos2] p.59.

Definition The alternative pairs $A(X, K, \alpha)$: Let K be an extension of k, and let

X be a K-module. Suppose that there is given an alternating K-bilinear form $\alpha : X \times X \to K$. Let $A^+ = A^- = X$ as K-modules, and let $J_+ : A^- \to A^+$ be an arbitrary K-linear isomorphism. Define $J_- : A^+ \to A^-$ by $J_- = -(J_+)^{-1}$. Also set $\alpha = \alpha_+$ and define $\alpha_- : A^- \times A^- \to K$ by $\alpha_-(x, y) = \alpha_+(J_+(x), J_+(y))$, so that

$$J_\sigma J_{-\sigma} = -\mathrm{Id}_{A^\sigma}, \alpha_\sigma(J_\sigma(x), J_\sigma(y)) = \alpha_{-\sigma}(x, y).$$

Then $A = (A^+, A^-)$ becomes an alternative pair over k of K, denoted by $A = A(X, K, \alpha)$, with

$$\langle xyz \rangle = x\alpha_\sigma(z, J_\sigma(y)) + J_\sigma(y)\alpha_\sigma(x, z),$$

for $x, z \in A^\sigma, y \in A^{-\sigma}$. Again, for properties, see [Loos2] p.61.

Now we have enough to give a classification of the alternative pairs under certain conditions.

Theorem 1.3.2 ***Alternative pairs classification.*** *For a semisimple alternative pair over k with dcc on principal inner ideals the following are equivalent.*

1. *The acc on principal inner ideals*
2. *the chain condition on idempotents (i.e. every increasing sequence of idempotents must stabilise after a finite number of increases)*
3. *the existence of a maximal idempotent.*

The simple pairs with these properties are up to isomorphism the following.

(A) *$A(M, R, \phi)$ with R a simple Artinian algebra over k and ϕ non-degenerate.*

(A') *The reverse A' of a pair A of type (A).*

(B) *(C, C^{op}) with C a Cayley algebra (sometimes referred to as an octonion algebra) over an extension field K of k.*

(C) *$A(X, K, \alpha)$ with X a vector space over an extension field K of k and α non-degenerate.*

For the proof, see [Loos2] p.127.

We want to show more of a link between the alternative pairs and Jordan pairs, we do this by first giving the following definitions, then we give the standard embedding theorem to give a correspondence.

Definition Let A be an alternative pair, and let $E = E(A)$ be the associative algebra $\mathrm{End}(A^+) \times \mathrm{End}(A^-)^{op}$ with componentwise operations, that is

$$ab = (a_+ b_+, b_- a_-)$$

for $a = (a_+, a_-)$ and $b = (b_+, b_-)$ in E. Let E^J be the Jordan algebra associated with E so that $U_a b = aba$, $a \circ b = ab + ba$ in E^J.

The *Jordan structure algebra* is a quadratic Jordan subalgebra of E containing the unit element of E,

$$J = J(A) = \{a \in J^* : a^2 \in J^*\},$$

where $J^* = J^*(A)$ is the set of all $a \in E$ such that

$$a\langle xyz\rangle + \langle x, y, az\rangle = \langle ax, y, z\rangle + \langle x, az, y\rangle,$$

$$a\langle xyx\rangle = \langle ax, y, x\rangle,$$

for all $x, z \in A^\sigma, y \in a^{-\sigma}$. Note that these also imply that

$$\langle x, y, ax\rangle = \langle x, ay, x\rangle.$$

now we look at the *inner structure algebra*, first we define

$$l(x, y) := (L(x, y), L(y, x))$$

where we use $L(x, y)z = \langle x, y, z\rangle$. The k-submodule F of E spanned by all $l(x, y) \in A$ is an ideal of J. We call F the inner structure algebra. To show these are well defined, see [Loos2] p.81.

There exists a method of embedding an alternative pair into the corresponding Jordan pair, this is known as the *standard embedding*, and is described in the following theorem, which can be found in [Loos2] p.85.

Theorem 1.3.3 ***Standard embedding theorem***. *Let $A = (A^+, A^-)$ be an alternative pair over k. With the above notations, let $W^\sigma = J^\sigma \oplus A^\sigma$ (the direct sum of k-modules), and define quadratic maps $Q_\sigma : W^\sigma \mapsto Hom(W^{-\sigma}, W^\sigma)$ by*

$$Q_\sigma(a \oplus x)(b \oplus y) = (a\bar{b}a + l_\sigma(x, \bar{a}y)) \oplus (a(\bar{b}x) + \langle xyx\rangle),$$

where $a \in J^\sigma, b \in J^{-\sigma}, x \in A^\sigma, y \in A^{-\sigma}$. *Then* $W = (W^+, W^-)$ *is a Jordan pair, called the standard embedding of* A*, with the following properties.*

(a) $e = (e^+, e^-)$ *is an idempotent of* W*, and the Peirce spaces of* W *with respect to* e *are*

(1)

$$W_2^\sigma(e) = J^\sigma, W_1^\sigma(e) = A^\sigma, W_0^\sigma(e) = 0.$$

(2) *For all* $x, z \in A^\sigma, y \in A^{-\sigma}$ *we have*

$$\langle xyz \rangle = \{\{xye^\sigma\}e^{-\sigma}z\},$$

i.e. $A = W_1(e)$ *as an alternative pair as in [Loos2] p8.2.*

(b) *Conversely, let* $V = V_2 \oplus V_1$ *be a Jordan pair and* $A = V_1$ *the associated alternative pair as in [Loos2] p8.2. Define* $f_\sigma : V_2^\sigma \to J^\sigma$ *by* $f_+(a) = f(a)$ *as in [Loos2] p8.10, that is, for* $(x, y) \in A$ *we have* $f(\{xye^+\}) = l(x, y)$ *and hence* f *maps* $\{V_1^+, V_1^-, e^+\}$ *onto the inner algebra of* A. $f_-(a) = \overline{f(\overline{a})}$*, and extend* f_σ *to a map* $h_\sigma : V^\sigma \to W^\sigma$ *by* $h_\sigma(a \oplus x) = f_\sigma(a) \oplus x, (a \oplus x \in V_2 \oplus V_1)$. *Then* $h = (h_+, h_-)$ *is a homomorphism of Jordan pairs which is the identity on* V_1*. In general,* h *is neither injective nor surjective.*

(c) *If* A *is non-degenerate (resp. semisimple) then so are the standard embedding and the Jordan structure algebra of* A*.*

So now we can divide the Jordan pairs with an empty V_0 space into two mutually exclusive types.

Theorem 1.3.4 *The simple Jordan pairs* V *containing an idempotent* e *such that* $V(e)_0 = 0$ *are up to isomorphism either*

1. *Jordan pairs associated with simple unital Jordan algebras, or*

2. *standard embeddings of simple alternative pairs for which Jordan structure algebras and inner structure algebras coincide.*

So now with the classification of the Jordan algebras and the alternative pairs we can give the following theorem, breaking the classification of the Jordan pairs (with a maximal idempotent). For details of the proof, see [Loos2] p.133.

Theorem 1.3.5 *For a semisimple Jordan pair over* k *with dcc on principal inner ideals the following are equivalent.*

1. *The acc on principal inner ideals;*
2. *the chain condition on idempotents;*
3. *the existence of a maximal idempotent.*

The simple pair with these properties are precisely the following up to isomorphism.

(0) *Jordan division pairs over k*

(I) $A(M, R, \phi)^J$ *with R a simple Artinian k-algebra and ϕ non-degenerate*

(II) $(A_n(K), A_n(K))$ *with K an extension field of k, $n \geq 4$ and $A_n(K)$ are alternating matrices over K*

(III) $(H_n(D, D_0), H_n(D, D_0))$ *with D a division algebra with involution over k and D_0 an ample subspace, $n \geq 2$*

(IV) (I, I) *where I is an outer ideal containing 1 in a Jordan algebra of a non-degenerate quadratic form on a vector space over a an extension field K of k, (U is an outer ideal of V if it satisfies $Q_\sigma(V^\sigma)U^{-\sigma} + \{V^\sigma, V^{-\sigma}, U^\sigma\} \subset U^\sigma$, with this it can be shown that if $1/2$ is in the underlying scalars, then an outer ideal is an ideal.)*

(V) $(M_{1,2}(C), (M_{1,2}(C^{op}))$ *with C a Cayley algebra over an extension field K of k.*

(VI) $(H_3(C, K), H_3(C, K))$ *with C as above.*

We will not note the quadratic maps for this list, it can be found in [Loos2] p.138 along with the proof of the previous theorem, but we will do so for the next list. The quadratic maps will be noted in the following.

Now we look at the case of an algebraically closed *field* k. It is known that the only finite dimensional division pair over k is $(k, k)^J$ (see [Loos2] p.195). Also, up to isomorphism, there is only one Cayley algebra over k. It follows for the previous list that every simple finite-dimensional Jordan pair over k is contained in the following list.

$(I_{p,q})$ $(M_{p,q}(k), M_{p,q}(k))$, $p \times q$ matrices over k.

(II_n) $(A_n(K), A_n(K))$ alternating $n \times n$ matrices over k.

(III_n) $(H_n(k), H_n(k))$ symmetric $n \times n$ matrices over k.

In these first three cases, the Jordan pair structure is given by $Q(x)y = xy^tx$.

(IV_n) (k^n, k^n), with $Q(x)y = q(x,y)x - q(x)y$. Here q is the standard quadratic form on k_n, given by

$$q(x_1, \cdots, x_{2m}) = \sum_{i=1}^{m} x_i x_{m+i} \text{ if } n = 2m,$$

$$q(x_0, \cdots, x_{2m}) = x_0^2 + \sum_{i=1}^{m} x_i x_{m+i} \text{ if } n = 2m+1.$$

(Note that there are no other ideals containing 1 in a Jordan algebras of quadratic forms except the whole algebra since k is algebraically closed. This is not hard to see using the fact that Q_1 will generate the entire algebra.)

(V) $(M_{1,2}(C), (M_{1,2}(C^{op}))$ 1×2 matrices over the Cayley algebra C over k with $Q(x)y = x(y^*x)$.

(VI) $(H_3(C,K), H_3(C,K))$ the Jordan pair associated with the exceptional Jordan algebra of 3×3 hermitian matrices over C.

See [Loos2] p.195.

1.4 Bounded symmetric domains and their classification

1.4.1 What is a bounded symmetric domain and why did they arise?

Definition A bounded symmetric domain is an bounded open set which is symmetric in the following sence, for any point x in a bounded symmetric domain, there exists an biholomorphic involution, s_x whose only fixed point is x. (A holomorphic function which has a holomorphic inverse is called a biholomorphic function.) They arose when trying to generalise the *Riemann mapping theorem* to higher (complex) dimensions. See [Vági].

We now mention the Riemann mapping theorem, and outline its proof.

Theorem 1.4.1 The Riemann mapping Theorem *Let G be a simply connected region which is not the whole plane and let $a \in G$. Then there is a unique analytic function $f : G \to \mathbb{C}$ having the properties:*

1. *$f(a) = 0$ and $f'(a) > 0$, i.e. the mapping is real and positive;*

2. *f is one to one;*

3. *$f(G) = \{z : |z| < 1\}$*

Lemma 1.4.2 *Let $f : \mathcal{D} \to \mathcal{D}$ be a one to one analytic map of the unit disk $\mathcal{D}$ onto itself and suppose $f(a) = 0$. Then there is a complex number c with $|c| = 1$ such that $f = c\phi_\alpha$. Where*

$$\phi_\alpha(z) := \frac{z-a}{1-\bar{a}z},$$

i.e. a Mobius transformation. See [Conway] p.161

It can be shown that there is a one to one correspondence between analytic functions from $\mathcal{D} \to \mathcal{D}$. Mobius transformations can be shown to be a group under composition. So we have a our group of biholomorphic maps and with the Riemann mapping theorem we have transitivity and the remaining properties needed to make the unit disc into a bounded symmetric domain. (To visualise this we can take the map that rotates the unit disc by some angle, say 180^o, this is an involution that leaves the origin fixed, by composing this map with another biholomorphic map we can construct maps that will have any other point in the unit disc as the unique fixed point.)

Note also that the Riemann mapping theorem shows that each simply connected region in the complex plane can be biholomorphically mapped to the unit disc in the complex plane.

The unit disc is the simplest example of a bounded symmetric domain. In trying to generalise this theorem to that of holomorphic functions of several variables, it was discovered by Poincaré in 1907 that the theorem fails to extend to higher dimensions. He proved that the Cartesian product of two unit discs is not holomorphically equivalent to the unit ball $\mathcal{C}^2 = \{z \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\}$ in $\mathbb{C}^2$. But both of these objects are bounded symmetric domains, so the question of classifying these objects came about.

So a study developed of domains with the essential properties of the above, these became known as the bounded symmetric domains. Cartan eventually gave a complete classification of them. (Note: Cartan's method is different from the one we use here, we use the method used by *Ottmar Loos*. As noted later, his method of classification can be found in the book [Loos1].)

1.4.2 Bounded symmetric domains, positive hermitian jordan triple systems and the their 1-1 correspondence

Classification of the positive hermitian Jordan triple systems

Restricting the final classification result of the last section, that is the classification of simple finite-dimensional Jordan pairs, to the field $\mathbb{C}$ instead of the more generic ring k we obtain the following table. The table is then shown to correspond to the *positive hermitian Jordan triple systems*, which we show to correspond to the *jordan pairs with involution*, i.e. and involution $x \mapsto x^*$ and is positive in the sense that $Q(x)x^* \geq 0$ for all x in the Jordan pair.

($I_{p,q}$) $(M_{p,q}(\mathbb{C}), M_{p,q}(\mathbb{C}))$, $p \times q$ matrices over the complex numbers.

(II_n) $(A_n(\mathbb{C}), A_n(\mathbb{C}))$ alternating $n \times n$ matrices over the complex numbers.

(III_n) $(S_n(\mathbb{C}), S_n(\mathbb{C}))$ symmetric $n \times n$ matrices over the complex numbers.

In these 3 cases, the Jordan pair structure is given by

$$Q(x)y = xy^t x,$$

with standard matrix product. A positive involution is given by

$$\tau(x) = \bar{x},$$

the complex conjugate. Now, suppose $Q(x)\tau(x) = x\bar{x}^t x = \lambda x$ where $\lambda \in \mathbb{C}$. Then $(x\bar{x}^t)^2 = \lambda(x\bar{x}^t)$, and since $x\bar{x}^t$ is positive semidefinite hermitian, it follows (for instance by taking traces) that $\lambda > 0$ for $x \neq 0$.

(IV_n) $(\mathbb{C}^n, \mathbb{C}^n)$

$Q(x)y = q(x,y)x - q(x).y$ where q is a non-degenerate quadratic form on $\mathbb{C}^n$ and $q(x,y) = q(x+y) - q(x) - q(y)$. After a change of basis, we may assume that $q(x) = \langle x, x\rangle = \sum x_i^2$, and then $q(x,y) = 2\langle x, y\rangle$. A positive involution is given by $\tau(x) = \bar{x}$ (complex conjugation in each variable). Indeed, assume that $x \neq 0$ and $Q(x)\bar{x} = 2\langle x, \bar{x}\rangle x - \langle x, x\rangle \bar{x} = \lambda x$, $\lambda x \in \mathbb{C}$. Then if $\langle x, \bar{x}\rangle = 0$, we have $\lambda = 2\langle x, \bar{x}\rangle > 0$ and if $\langle x, x\rangle \neq 0$, it follows by taking the scalar product with x that $2\langle x, \bar{x}\rangle\langle x, x\rangle - \langle x, x\rangle\langle x, \bar{x}\rangle = \langle x, \bar{x}\rangle\langle x, x\rangle = \lambda\langle x, x\rangle$, hence $\lambda = \langle x, \bar{x}\rangle > 0$.

(V) $(M_{1,2}(\mathbb{O}_\mathbb{C}), (M_{1,2}(\mathbb{O}_\mathbb{C}))$ 1×2 matrices over the 8-dimensional Cayley algebra $\mathbb{O}_\mathbb{C}$.

$Q(x)y = x(\tilde{x}^t x)$. where $\tilde{}$ denotes the ($\mathbb{C}$-linear) canonical involution of $\mathbb{O}_\mathcal{C}$.

(VI) $(H_3(\mathbb{O}_\mathbb{C}), H_3(\mathbb{O}_\mathbb{C}))$ the Jordan pair associated with the exceptional Jordan algebra of 3×3 hermitian matrices over $\mathbb{O}_\mathbb{C}$. These are hermitian with respect the canonical involution, i.e. they satisfy $\tilde{x}^t = x$. The Jordan structure is the one induced from the Jordan algebra $H_3(\mathbb{O}_\mathbb{C})$, thus $Q(x)y = \frac{1}{2}(x \circ (x \circ y) - x^2 \circ y)$ where $x \circ y = xy + yx$. Note that type (V) may be embedded into type (VI) by

$$(x_1, x_2) \to \begin{pmatrix} 0 & x_1 & x_2 \\ \tilde{x}_1 & 0 & 0 \\ \tilde{x}_2 & 0 & 0 \end{pmatrix}$$

A positive involution for these two cases is give by $\tau(x) = \bar{x}$, complex conjugation relative to the real Cayley division algebra $\mathbb{O}$. Since the above embedding commutes with τ, it suffices to prove that for type (VI). Here the fixed point set of τ is the real Jordan algebra $A = H_3(\mathbb{O})$ which is known to be formally real, i.e. $x^2 + y^2 = 0 \implies x = y = 0$. Hence, $H_3(\mathbb{O}_\mathbb{C})$ is the hermitification of A in the following way. Let A be a real Jordan algebra, and let $z^* = x - iy$ denote the complex conjugate of $z = x + iy \in A_\mathbb{C} = A \oplus iA$, the complexification of A. Then the Jordan pair $(V^-, V^+) = (A_\mathbb{C}, A_\mathbb{C})$ carries a hermitian involution given by $\tau(z) = z^*$, in other words, we have

$$Q(z)\bar{w} = P(z)w^*$$

Therefore τ is positive for $z, w \in V = A_\mathbb{C}$. This follows easily from the fact that $z \to z^*$ is an antilinear automorphism of period 2 of the Jordan algebra $A_\mathbb{C}$. We call this the hermitification of A.

So as we see, all are positive Hermitian Jordan triple systems, hence we have all of the positive hermitian Jordan triple systems.

Peirce decompositions of positive hermitian jordan triple systems

If we have a positive hermitian Jordan triple system with vector space E, then a Peirce decomposition with respect to same idempotent can be defined on E as follows after a few preliminarys.

We will take an arbitrary sequence of mutually orthogonal idempotents, $e_1, e_2, \ldots, e_s$, and then let $\mathbf{e} = (e_1, e_2, \ldots, e_s)$.

Definition For all integers $0 \le j, k \le s$ we define

$$V_{j,k} = V_{j,k}(\mathbf{e}) = \{x \in V : \{e_l e_l x\} = \frac{1}{2}(\delta_{jl} + \delta_{lk})x \text{ for all } 1 \le l \le s\}$$

Now these are mutually complex orthogonal, so

$$V = \bigoplus_{0\le j,k\le s} V_{j,k}.$$

The Peirce spaces multiply according to the rules

$$\{V_{j,m}V_{m,n}V_{n,k}\} \subset V_{j,k}$$

and all products that cannot be brought to this form will vanish.
It follows that

$$L_a = \sum_{0\le j,k\le s} \frac{1}{2}\left(\lambda_j^2 + \lambda_y^2\right) P_{j,k}.$$

We now show a further decomposition

$$V^{j,k} := \{x \in V_{|j|,|k|} : \{exe\} = \mathrm{sign}(jk).x\}.$$

So clearly every $V^{j,k}$ is an $\mathbb{R}$-linear subspace of V with $V^{-j,k} = iV^{j,k} = V^{j,-k}$, and

$$V = \bigoplus_{|j|\le k\le s} V^{j,k}.$$

This is called the *refined* Peirce decomposition. Note that $V_{j,k} = V^{j,k} \oplus V^{-j,k}$.
So now we can define the following spectral resolutions

$$L_a = \sum_{|j|\le k\le s} \frac{1}{2}\left(\lambda_j^2 + \lambda_k^2\right) P^{j,k}, Q_a = \sum_{|j|\le k\le s} \lambda_j\lambda_k P^{j,k},$$

Definition

$$V_1(a) = \bigoplus_{1\le j\le s} V_{j,j}, V_{1/2}(a) := \bigoplus_{0\le j<k\le s} V_{j,k},$$

$$V_0(a) = V_{0,0} \text{ and } A(a) = \bigoplus_{1\le j\le s} V^{j,j}.$$

Then

$$V = V_1(a) \oplus V_{1/2}(a) \oplus V_0 \text{ and } V_1 = A(a) \oplus iA(a).$$

Definition If we take an element, a of a positive hermitian Jordan triple system, there exists a unique up to order set of idempotents $e_1, \cdots, e_n$ such that

$$a = \lambda_1 e_1 + \cdots + \lambda_1 e_1,$$

where the λ_i are positive definite real numbers. This decomposition is known as the *spectral decomposition*. We can also define a norm of some $x \in V$, this is known as the *spectral norm* as

$$|x| := \max_i(\lambda_i).$$

See [Loos2] p.3.8-3.11.

(We will see this norm and composition again in chapter 3.)

The one to one correspondence

Definition Let V be a finite-dimensional complex vector space, we define the space V^- by having the same underlying space as V but with the multiplication operation changed by complex conjugation, i.e. av denotes scalar multiplication in V and $a \cdot v$ in V^- as follows

$$a \cdot v := \overline{a}v$$

where $a \in \mathbb{C}$ and $v \in V^-$. Let $D \subset V$ be open, $f : D \to W$ a differentiable map, and $df(z) :_R V \to_R W$ the differential of f at $z \in D$. (Where $_RV$ denotes the underlying real vector space.) Then $df(z)$ decomposes

$$df(z) = \partial(z) + \overline{\partial} f(z)$$

where $\partial f(z)$ is $\mathbb{C}$-linear and $\overline{\partial} f(z)$ is $\mathbb{C}$-antilinear, i.e $\overline{\partial} f(z) : V_- \to W$ is $\mathbb{C}$-linear. For directional derivatives, we use the notation

$$d_v f(z) = df(z) \cdot v, \partial_v f(z) = \partial f(z) \cdot v, \overline{\partial}_{\overline{v}} f(z) = \overline{\partial} f(z) \cdot \overline{v}.$$

Looking at a *circled* bounded symmetric domain, we would like to define a Jordan structure on it, in particular, that of a positive hermitian Jordan triple system. We begin with this lemma, which can be found in [Loos2] p.2.2.

Definition We will use $\mathcal{D}$ to denote the unit ball with respect to the spectral norm of the vecor space V. Aut($\mathcal{D}$) is the automorphism group of $\mathcal{D}$, it is known to be a Lie group.

Lemma 1.4.3 *For every $v \in V$ let ξ_v be the unique vector field in $\mathcal{L}(Aut(\mathcal{D}))$, (the lie algebra of $Aut(\mathcal{D})$) which takes the value v at the origin. Then $v - \xi_v(z)$ is a homogeneous quadratic polynomial in z and is $\mathbb{C}$-antilinear in v.*

With this in mind, we shall write

$$v - \xi_v(z) = Q(z)\overline{v},$$

now we have $Q(z) : V^- \to V$ is a complex linear map and $Q : V \to \mathrm{Hom}(V^-, V)$ is a homogeneous quadratic polynomial. Hence

$$Q(x, z) = Q(x + z) - Q(x) - Q(z) : V^- \to V$$

is bilinear and symmetric in x and z. For $x, y, z \in V$ we define

$$\{x\overline{y}z\} = D(x, \overline{y}) \cdot z = Q(x, z) \cdot \overline{y}.$$

Finally, we define $\overline{Q} : V^- \mapsto \mathrm{Hom}(V, V^-), \overline{D} : V^- \times V \to \mathrm{End}(V^-)$ and $\{\overline{x}y\overline{z}\} \in V^-$ for $x, y, z \in V$ by

$$\overline{Q(z)} \cdot v = \overline{Q}(\overline{z}) \cdot v = \overline{Q(z) \cdot \overline{v}}$$

$$\{\overline{x}y\overline{z}\} = \overline{D}(\overline{x}, y) \cdot \overline{z} = \overline{\{x\overline{y}z\}}.$$

Now we have Jordan structure on $\mathcal{D}$. we have the required quadratic maps form the previous lemma and from above.

Theorem 1.4.4 *Let $\mathcal{D}$ be a circled bounded symmetric domain. Then $\mathcal{D}$ is the open unit ball of the spectral norm of the associated Jordan pair with positive hermitian involution. Conversely, given a Jordan pair with positive hermitian involution, the open unit ball of the spectral norm is a circled bounded symmetric domain whose associated Jordan pair is the one given.*

Before proving this we require some lemmas and definitions.

Definition Let M be a differentiable manifold. Let X be a vector field on M (i.e a smooth section of the tangent bundle TM). It is known (see [Jöst] p.42) that for each $p \in M$,there exists an open interval $I_p \subset \mathbb{R}$ with $0 \in I_p$ and a smooth curve

$$c_p : I_p \mapsto M$$

with

$$\dot{c}_p(t) = \frac{dc_p}{dt}(t) = X(c_p(t))$$

$$c_p(0) = p.$$

The curve c_p is called the *integral curve* of X through p.

A *Riemannian metric* on a differentiable manifold M is given by a scalar product on each tangent space T_pM which depends smoothly on the base point p. A *Riemannian manifold* is a differentiable manifold equipped with a Riemannian metric.

For a Riemannian manifold, a *geodesic* is a smooth curve, $\gamma = [a,b] \mapsto M$, such that $\frac{d}{dt}\langle \dot\gamma, \dot\gamma \rangle = 0$ (with respect the the Riemannian metric $\langle \cdot, \cdot \rangle$). (See [Jöst] for details of *parallel transport*).

A diffeomorphism $h : M \to N$ between Riemannian manifolds is an *isometry* if it preserves the Riemannian metric. That is for $p \in M, v, w \in T_pM$, and if $\langle \cdot, \cdot \rangle_M$ and $\langle \cdot, \cdot \rangle_N$ denotes the scalar products in T_pM and $T_{h(p)}N$ respectively, we have

$$\langle v, w \rangle_M = \langle dh(v), dh(w) \rangle_N.$$

A vector field on a Riemannian manifold is a *Killing* vector field if $L_X h = 0$, where L_X is the *Lie derivative* and h is the metric. L_X is the derivative of h with respect to change in X. We define $L_X Y = [X, Y]$, for more see [Jöst] p.47.

For more on the following lemma see [Jöst] sec 6.2 (p.259) for further detail.

Lemma 1.4.5 *Let ξ be a Killing vector field on a locally symmetric Riemannian manifold, and assume that $ds_p \cdot \xi \cdot s_p = -\xi$ for some point p. Then the integral curve of ξ through p is a geodesic.*

Proof Let ϕ_t be the local one-parameter group generated by ξ (see [Jöst] p. 49), and let $x(t) = \phi_t(p)$ be the integral curve of ξ through p. Since ξ is Killing, ϕ_t consists of local isometries and therefore $s_{x(t)} = s_{\phi_t(p)} = \phi_t \circ s_p \circ \phi_t^{-1}$. The assumption on ξ implies that $s_p(x(t)) = x(-t)$. It follows that

$$\begin{aligned} s_{x(t/2)}(x(t')) &= \phi_{t/2} \cdot s_p(x(t' - \frac{1}{2}t)) = \phi_{t/2} \cdot x(\frac{1}{2}t - t') \\ &= x(t - t'). \end{aligned}$$

But differentiating with respect to t' at $t' = 0$ we see that $x(t)$ satisfies the differential equation

$$\dot x = -ds_{x(t/2)} \cdot v$$

where $v = x(0) = \xi(p)$. On the other hand, let $y(t)$ be the geodesic through p with initial vector v. Then $\dot y(t/2)$ is obtained from v by parallel transport along $y(t)$, and since $s_{y(t/2)}$ is an isometry, $ds_{y(t/2)} \cdot v$ is obtained from $ds_{y(t/2)} \dot y(t/2) = -\dot y(t/2)$ by parallel transport along $y(t)$. This shows that

$$\dot y = -ds_{y(t/2)} \cdot v,$$

and hence $x(t)$ and $y(t)$ satisfy the same differential equation with the same initial condition $x(0) = y(0) = p$. And the result follows. ∎

Lemma 1.4.6 *Let (V, V^-) be a Jordan pair with positive hermitian involution, and for $v \in V$ define the vector field ξ_v on V by*

$$\xi_v(z) = v - Q(z) \cdot \bar{v}.$$

Then the integral curve of ξ_v through 0 is defined for all t and is given by

$$x(t) = \tanh(tv).$$

We define tanh *coordinate-wise as follows,*

$$\tanh(x) = \tanh(\lambda_1)e_1 + \cdots \tanh(\lambda_n)e_n,$$

for $x = \lambda_1 e_1 + \cdots + \lambda_n e_n$, the spectral decomposition.

Proof Let $v = \lambda_1 e_1 + \cdots + \lambda_n e_n$ be the spectral decomposition. Then we have $x(0) = 0$ and $\dot{x} = \sum \lambda_i \cosh^{-2}(\lambda_i t)e_i$. By orthogonality of the e_i,

$$\xi(x(t)) = v - Q(x(t))\bar{v} = \sum \lambda_i(1 - \tanh^2(\lambda_i t))e_i = \dot{x}.$$

∎

Definition

$$B(x, \bar{y}) := \mathrm{Id} - D(x, y) + Q(x)\overline{Q(y)},$$

Lemma 1.4.7 *Let $\mathcal{D}$ be the open unit ball of the spectral norm, and pick an Aut(V)-invariant scalar product $\langle , \rangle$ on V, Then*

$$h_z(u, \bar{v}) = \langle B(z, \bar{z})^{-1} \cdot u, \bar{v}\rangle \tag{1.14}$$

defines a hermitian metric on $\mathcal{D}$ with respect to which the vector fields ξ_v are Killing vector fields. The geodesic symmetry around 0 is $s_0 = -id$, it is a global isometry.

Proof From the definition of the spectral norm and [Loos2] 5.14, we can write

$$B(x, \bar{x})y_{ij} = (1 - |\lambda_i|^2)(1 - |\lambda_j|^2)y_{ij}$$

with $y_{ij} \in V_{ij}$. so it is clear that $B(z,\bar{z})$ is positive and self adjoint with respect to $<,>$. Thus 1.14 defines a hermitian metric on $\mathcal{D}$. An easy computation shows that a vector field ξ is Killing if and only if

$$d_{\xi(z)}B(z,\bar{z}) = d\xi(z)\cdot B(z,\bar{z}) + B(z,\bar{z})\cdot d\xi(z)^*, \tag{1.15}$$

where * denotes the adjoint relative to $\langle,\rangle$.

Indeed, by definition, for X to be Killing we require $L_X h = 0$. It can be shown (see [Jöst] p.127) that this means

$$X\cdot h(Y,Z) = h([X,Y],Z) + h(Y,[X,Z])$$

for all vector fields Y, Z where $[\cdot,\cdot]$ is the lie bracket. It is enough to have this as a basis (over the ring of differentiable functions) of the vector fields. In our situation, where the manifold is just an open set in a vector space, this implies that we may assume that $Y = v$ and $Z = w$ to be constant vector fields, and we choose to do so for simplicity. If $X = \xi(z)\cdot(\frac{\partial}{\partial z})$ in obvious notation, suppressing the indices which would appear when using a basis, then $[X,Y](z) = -(\xi'(z)\cdot v)(\frac{\partial}{\partial z})$ and similarly for $[X,Z]$.

$$(L_X h)(v,w) = 0$$

$$\Longleftrightarrow$$

$$\xi(z)(\frac{\partial}{\partial z})\langle B(z,\overline{z})^{-1}v,w\rangle - \langle B(z,\overline{z})^{-1}\xi'(z)v,w\rangle - \langle B(z,\overline{z})^{-1}v,\xi'(z)w\rangle = 0$$

$$\Longleftrightarrow$$

$$\xi(z)(\frac{\partial}{\partial z})B(z,\overline{z})^{-1} = -B(z,\overline{z})^{-1}\xi'(z) - \xi'(z)^*B(z,\overline{z})^{-1},$$

since this must hold for all v, w. Now use the standard formula for differentiating the inverse of a matrix or operator A, i.e. $dA^{-1} = -A^{-1}\cdot dA\cdot A^{-1}$, which here yields

$$\xi(z)\frac{\partial}{\partial z}B(z,\overline{z})^{-1} = -B(z,\overline{z})^{-1}\cdot d_{\xi(z)}B(z,\overline{z})\cdot B(z,\overline{z})^{-1},$$

and the rest follows easily.

For $\xi = \xi_v$ we have $d\xi(z) = -D(z,\bar{v})$ and hence $d\xi(z)^* = -D(v,\bar{z})$. The left hand side of 1.15 is $-D(\xi(z),\bar{z}) - D(z,\bar{\xi(z)}) + Q(z,\xi(z))\bar{Q}(\bar{z}) + Q(z)\bar{Q}(\bar{z},\bar{\xi(z)})$. Now 1.15 follows by a straight forward computation, by expanding both sides and using the identities

$$D(x,y)Q(x) = Q(x,Q(x)y) = Q(x)D(y,x), \tag{1.16}$$

$$D(x,y)D(z,y) = Q(x,z)Q(y) + D(x,Q(y)z), \tag{1.17}$$

$$D(x,y)D(x,z) = D(Q(x)y,z) + Q(z,\{xyz\}). \tag{1.18}$$

These are JP4, JP9 and JP13 as stated in [Loos2] respectively. From $B(z,\bar{z}) = B(-z,-\bar{z})$ it follows that $z \to \bar{z}$ is an isometry and since its only fixed point is 0, it is the geodesic symmetry. ∎

Now we can prove Theorem 1.4.4;

Proof Suppose $\mathcal{D}$ is a (circled) bounded symmetric domain in V, and let ξ_v be as defined earlier. Then ξ_v satisfies the hypothesis of 1.4.5 (with $p = 0$). Now $\xi_v(z) = v - Q(z)\bar{v}$ by definition of the Jordan structure associated with $\mathcal{D}$ and 1.4.6, the geodesic through 0 with initial vector v is given by $Exp_0(tv) = \tanh(tv)$, ([Jöst] p.18). Since the Bergmann metric is complete, $\mathcal{D} = Exp_0(V)$ ([Jöst] p.26). The function tanh is a real analytic diffeomorphism of R onto the open interval $(-1,1)$. It follows from 3.18 and 3.19 of [Loos1] that $x \to \tanh(x)$ is a real analytic diffeomorphism of V onto $\{x \in V : |x| < 1\}$, and hence $\mathcal{D}$ is the open unit ball of the spectral norm.

Conversely, let $\mathcal{D}$ be the open unit ball of the spectral norm of a Jordan pair with positive involution. Clearly $\mathcal{D}$ is bounded and circled. Form 1.4.6 and the properties of tanh, every point of $\mathcal{D}$ lies on an integral curve of some vector field ξ_v and the integral curves are defined for all $t \in R$. Since the ξ_v are Killing by 1.4.7, it follows that every point of $\mathcal{D}$ has a neighbourhood which is isometric to a neighbourhood of 0. (That is by the definition of a Killing vector, it will preserve the metric in the direction it points.) Again, by 1.4.7, $\mathcal{D}$ is locally symmetric. By 1.4.5, the geodesics through 0 are precisely the integral curves through 0 of the ξ_v and are therefore defined for all $t \in R$. This shows that $\mathcal{D}$ is complete as a Riemannian manifold. Since Killing vector fields on a complete Riemannian manifold are complete, the holomorphic vector fields ξ_v generate global one-parameter sub-groups of holomorphic automorphisms of $\mathcal{D}$, and therefore $\mathcal{D}$ is homogenous. Being symmetric around 0, it is symmetric around every point by homogeneity. Finally , it is clear that the vector fields, ξ_v, are precisely those used in sec2 of [Loos1] to define the Jordan structure associated with $\mathcal{D}$. Hence the Jordan pair associated with $\mathcal{D}$ is the given one. ∎

We say the unit ball of a Jordn pair V is an *irreducible* bounded symmetric domain if there does not exist non zero subspaces V_1 and V_2 such that $V = V_1 \oplus V_2$ and

$$L(V_1, V_2) = L(V_2, V_1) = 0.$$

Table 1.1 contains the complete classification of the *circled bounded symmetric domains* as presented in [KaupZait]. As noted earlier, every bounded symmetric domain is biholomorphic to a circled bounded symmetric domain. We refer to the first 4 countable categories of bounded symmetric domains as the *classical* ones, and the remaining two elements as the *exceptional* elements.

Up to linear bijections, every circled bounded symmetric domain is on in this list. The (reducible) circled bounded symmetric domains are the direct products of the irreducible domains, i.e. $\mathcal{D}_1 \times \mathcal{D}_2 \times \cdots \times \mathcal{D}_n$.

Bounded Symmetric Domains		dim	rank
$\mathbf{I}_{p,q}$	$D = \{z \in \mathbb{C}^{p\times q} : \mathbb{I}_p - zz^* > 0\}$	pq	r
$\mathbf{II}_p$	$D = \{z \in \mathbb{C}^{p\times p} : z' = -z \text{ and } \mathbb{I}_p - zz^* > 0\}$, where $p \geq 5$	$\binom{p}{2}$	$\left[\frac{p}{2}\right]$
$\mathbf{III}_p$	$D = \{z \in \mathbb{C}^{p\times p} : z' = z \text{ and } \mathbb{I}_p - zz^* > 0\}$, where $p \geq 2$	$\binom{p+1}{2}$	p
$\mathbf{IV}_p$	$D = \{z \in \mathbb{C}^{n} : (z\|z) + \sqrt{(z\|z)^2 - \|\langle z\|z\rangle\|^2} < 2\}$	n	2
$\mathbf{V}$	An exceptional domain in dimension $n = 16$	16	2
$\mathbf{VI}$	An exceptional domain in dimension $n = 27$	27	3

Table 1.1: The bounded symmetric domains

Chapter 2

Holomorphic extension theory

In this section we look at some *holomorphic extension* theory. We begin with some definitions and results related to complex manifolds and varieties. This is followed by a brief look at normal spaces. We then concentrate on extension theory. Moving to the study of *CR manifolds*, we look at various definitions and finally state an extension theorem for CR functions on CR manifolds.

2.1 Varieties, manifolds and vector fields

In the study of complex analysis (in several variables) *polydisks* are of significant importance. $\Delta(a, r) = \Delta(a_1, \ldots, a_n; r_1, \ldots, r_n) = \{z \in \mathbb{C}^n : |z_j - a_j| < r_j \text{ for } 1 \leq j \leq n\}$ where $a = (a_1, \ldots, a_n)$ and $r = (r_1 \ldots, r_n)$. We call this the *polydisk* with *polyradius* r centred at a.

We will define complex manifolds and varieties. First we define a holomorphic functions in $\mathbb{C}^n$, $n \in \mathbb{N}_0$.

Definition A complex valued function f defined on an open subset $D \subset \mathbb{C}^n$ is *holomorphic* in D if each point $a \in D$ has an open neighbourhood U such that the function f has a power series expansion

$$f(z) = \sum_{i_1, i_2, \ldots, i_n = 0}^{\infty} c_{i_1, i_2, \ldots, i_n} (z_1 - a_1)^{i_1} (z_2 - a_2)^{i_2} \cdots (z_n - a_n)^{i_n}$$

which converges for all $z \in U$. The set of all functions holomorphic in D is denoted by $\mathcal{O}_D$. (See [Gunn2] p.2.) By relatively closed we mean that the set D can be considered as a subset of a set G which is closed in $\mathbb{C}^n$ and $G \cap M = D$.

Definition A *complex submanifold* of complex dimension k of an open subset $D \in \mathbb{C}^n$

is a relatively closed subset $M \subset D$ such that for every point $a \in M$ there exists an open neighbourhood U_a of a and a biholomorphic mapping $F := U_a \to \Delta(0, R)$ onto an open polydisc in $\mathbb{C}^n$ such that $F(a) = 0$ and that

$$F(U_a \cap M) = \{z \in \Delta(0, R) : z_{k+1} = \ldots = z_n = 0\}$$

for some integer k, $0 \leq k \leq n$.

We will now define a *complex manifold*. A second countable Hausdorff topological space M is called a *topological manifold* of dimension d if each point $a \in M$ has an open neighbourhood homeomorphic to an open subset of $\mathbb{R}^d$. (We say a topology is second countable if its open sets are countable.) A *coordinate covering* $\{U_\alpha, F_\alpha\}$ of such a manifold is a covering of M by open sets U_α together with homeomorphisms $F_\alpha : U_\alpha \to V_\alpha$, U_α are called the *coordinate neighbourhoods* and F_α are the *coordinate mappings*.

If $d = 2n$, the space $\mathbb{R}^{2n}$ can be identified with $\mathbb{C}^n$, and the images $F_\alpha(U_\alpha) = V_\alpha$ of the coordinate neighbourhoods can be viewed as subsets of $\mathbb{C}^n$. The coordinate covering $\{U_\alpha, F_\alpha\}$ is called holomorphic if all of the coordinate transition maps are holomorphic maps. A coordinate transition map is a map of the form $F_\beta \circ F_\alpha^{-1} : F_\alpha(U_\alpha \cap U_\beta) \mapsto F_\beta(U_\alpha \cap U_\beta)$. Two holomorphic coordinate coverings are then called equivalent if their union is again a holomorphic coordinate covering. An equivalence class of holomorphic coordinate coverings is called a *holomorphic structure*, and a topological manifold together with a fixed holomorphic structure is called a *complex manifold of (complex) dimension n*. See [Gunn2] for further details.

Definition A *holomorphic subvariety* V of an open set $U \subset \mathbb{C}^n$ is a subset $V \subset U$ with the property that for each point $a \in U$ there exists an open neighbourhood U_a of a in U and finitely many holomorphic functions $f_{a_1}, \ldots$ in U_a such that

$$V \cap U_a = \{z \in U_a : f_{a_1}(z) = f_{a_2}(z) = \cdots = 0\}$$

Definition If V is a holomorphic subvariety of an open subset of $\mathbb{C}^n$, a *holomorphic function* on V is a complex valued function f defined on the point set V, with the property that for each point $a \in V$ there exists an open neighbourhood U_a of the point a in the ambient space $\mathbb{C}^n$ and a holomorphic function F_a in U_a such that $f|U_a \cap V = F_a|U_a \cap V$.

Definition A map between holomorphic subvarieties V, W of open subsets $U_V \subset \mathbb{C}^m, U_W \subset \mathbb{C}^n$ is called a *holomorphic mapping* if its coordinate functions are holomorphic functions on V. The mapping F is called a *biholomorphic* mapping if it is holomorphic and has a holomorphic inverse, i.e. it is *bijective*.

A *holomorphic variety* is a second-countable Hausdorff topological space V for which there exists a covering by open sets V_α and homeomorphisms $F_\alpha : V_\alpha \to W_\alpha$ between the subsets $V_\alpha \subseteq V$ and holomorphic subvarieties W_α of open sets $U_\alpha \subseteq \mathbb{C}^{n_\alpha}$, such that for each nonempty intersection $V_\alpha \cap V_\beta$ the composition

$$F_{\alpha\beta} := F_\alpha \circ F_\beta^{-1} : F_\beta(V_\alpha \cap V_\beta) \to F_\alpha(V_\alpha \cap V_\beta)$$

is a *biholomorphic mapping.* see [Gunn2], p.23-24.

As is clear, complex manifolds are a special case of varieties. For further results on varieties, see [Gunn2].

Before we continue, we define germs of holomorphic varieties. To motivate this we start with the standard definition of a germ of a (holomorphic) function on $\mathbb{C}^n$.

Definition Two functions f_U and g_V that are holomorphic on $\mathbb{C}^n$ are called equivalent at the point $a \in U \cap V$ if there is a neighbourhood W of the point a such that $W \subset U \cap V$ and $f_U|W = g_V|W$. This is an equivalence relation (in the usual sense), and an equivalence class is called a *germ of a complex valued function at the point a.*

Next for a holomorphic subvariety. Holomorphic subvarieties V_1, V_2 of open neighbourhoods U_1, U_2 of a point $a \in \mathbb{C}^n$ are called equivalent at the point a if there is an open neighbourhood W of the point a such that $W \subset U_1 \cap U_2$ and $V_1 \cap W = V_2 \cap W$. This is an equivalence relation, again, in the usual sense, and an equivalence class is called a *germ of a holomorphic subvariety at the point a.* This germ will be denoted by $\mathbf{V}$ or $\mathbf{V}_a$ for the holomorphic subvariety V at the point a. The definition of a germ of a holomorphic variety follows easily from this.

Definition We can define the structure of a ring with the germs of the holomorphic functions on $\mathbb{C}^n$ as follows. For any two germs $\mathbf{f}$ and $\mathbf{g}$ at a, select representatives f_U and g_V, respectively. If W is an open neighbourhood of a such that $W \subset U \cap V$, the germ of the function $f_U|W + g_V|W$ is defined to be the sum $\mathbf{f} + \mathbf{g}$, and the germ of the function $(f_U|W) \cdot (g_V|W)$ is defined to be the product $\mathbf{f} \cdot \mathbf{g}$. It follows easily by verification of the axioms of a ring that this is indeed a ring.

So we say that the ring of germs of holomorphic functions at a point a of a holomorphic subvariety V is called the *local ring* of the subvariety at the point a or of the germ V_a, and is denoted by ${}_V\mathcal{O}_a$ or ${}_{V_a}\mathcal{O}$.

In the case of the germs of holomorphic varieties (at a point a), we proceed as follows. Since the union and intersection of holomorphic subvarieties is a holomorphic subvariety we speak of the holomorphic subvarieties $V_1 \cup V_2$ and $V_1 \cap V_2$ for holomorphic subvarieties V_1 and V_2.

It is possible to define the germs of *continuous* functions at some point a in our holomorphic variety instead. We could construct a ring, similar to the method above, the ring is called the *ring of germs of continuous complex-valued functions* at the point a and is denoted by $\mathcal{C}_a$. We are concerned with holomorphic functions, so we define the *ring of germs of holomorphic functions* at a and denote it by $\mathcal{O}_a$. This is an integral domain, so it has a well defined quotient field, which is called the *field of germs of meromorphic functions* and is denoted $\mathcal{M}_a$. See [Gunn2] p.2.

The standard notation for the germ of holomorphic functions in $\mathbb{C}^n$ at a is ${}_n\mathcal{O}_a$. Since we can easily change to coordinates making a the origin, the notation $\mathcal{O} = \mathcal{O}_0 = {}_n\mathcal{O} = {}_n\mathcal{O}_0$ is commonly used.

Definition A germ V of a holomorphic subvariety is *reducible* if $V = V_1 \cup V_2$ where V_1 and V_2 are germs of holomorphic subvarieties and $V_j \neq V$. A germ that is not reducible is *irreducible.*

We will need to look briefly at some of the important relations between complex manifolds and holomorphic varieties. Again, [Gunn2] will be the main reference.

Definition A *thin set* in a holomorphic variety V is a subset $X \subset V$ with the property that for each point $a \in V$ there are an open neighbourhood U_a of the point a in V and a holomorphic function F_a in U_a such that the germ of f_a in ${}_V\mathcal{O}_a$ does not vanish on any irreducible component of the germ of the variety V at a and $X \cap U_a \subset \{z \in U_a : f_a(z) = 0\}$.

With this in mind we state the following.

Theorem 2.1.1 *A holomorphic variety is a complex manifold outside a thin set. [Gunn2] p.61.*

Definition A point in a holomorphic variety V at which V is a complex manifold is called a *regular* point of V. The set of all regular points comprises the *regular locus* $\mathcal{R}(V) \subset V$, while the set of all singular points makes up the *singular locus* $\mathcal{S}(V) = V - \mathcal{R}(V) \subset V$.

So sometimes a complex manifold is called a *regular holomorphic variety.*

2.2 Normal Spaces

We deal with the theory as it is dealt with in [Gunn2]. We will define normal spaces and normalisations, then we will give extension results, which we use in chapter 3 and 4.

Definition A function is said to be regular of degree v if $f(0, \cdots, 0, z)$ is a function of z alone and has a zero of order v at the origin.

Definition A *weakly holomorphic* function in an open set U of a holomorphic variety is a holomorphic function on the set of all points of U that are regular, i.e. $\mathcal{R}(V)$, and it is locally bounded in U. The set of all weakly holomorphic functions in U will be denoted by ${}_V\hat{\mathcal{O}}_U$.

As noted earlier, ${}_V\mathcal{O}_a$ is used to denote the germ of holomorphic functions at a point a of a holomorphic variety V, we use ${}_V\hat{\mathcal{O}}_a$ to denote the germ of weakly holomorphic functions Therefore we have that ${}_V\hat{\mathcal{O}}_a \supset {}_V\mathcal{O}_a$. See section Q of [Gunn2] for more results.

Definition A *holomorphic* variety V is said to be normal at a point $a \in V$ if ${}_V\hat{\mathcal{O}}_a = {}_V\mathcal{O}_a$. That variety is said to be normal if it is normal at each of its points.

2.2.1 Normalisations

Again, [Gunn2] will be the main reference.

Definition The *normalisation* of an irreducible germ V of a holomorphic variety is that germ $\hat{V}$ of a holomorphic variety for which ${}_{\hat{V}}\mathcal{O} = {}_V\hat{\mathcal{O}}$, The normalisation of a reducible germ of a holomorphic variety is the disjoint union of the normalisation of the irreducible components.

The next definition is important to show a correspondence between germs and their normalisations.

Definition A mapping $\pi : V \to W$ between two second-countable Hausdorff spaces is a *finite branched covering* if:

1. π is a continuous, finite, proper, surjective mapping.

2. There are dense open subsets $V_0 \subset V, W_0 \subset W$ such that $V_0 = \pi^{-1}(W_0)$ and the restriction $\pi|V_0 : V_0 \to W_0$ is a covering mapping.

$\pi|V_0 : V_0 \rightarrow W_0$ is called the regular part.

(Note: A mapping between two topological spaces is called *finite* if the inverse image of each point is a finite set of points, and is called *proper* if the inverse image of each compact set is compact.)

If $\pi : V \rightarrow W$ is a finite branched covering with regular part $\pi|V_0 : V_0 \rightarrow W_0$, then over each connected component of W_0 this regular part is a covering with a finite number of sheets (a sheet corresponds to a connected component of V, but not necessarily the same number of sheets over different connected components. The maximum number of sheets of coverings over all connected components of W_0 will be called the *order* of the branched covering $\pi : V \rightarrow W$ and will be denoted by o_π.

Definition A holomorphic mapping $\pi : V \rightarrow W$ between two holomorphic varieties is a *finite branched holomorphic covering* if:

1. π is a finite branched covering.
2. There is a regular part $\pi|V_0 : V_0 \rightarrow W_0$ of this branched covering for which $W - W_0$ is a holomorphic subvariety of W and $\pi|V_0$ is a locally biholomorphic mapping.

Definition The *normalisation* of a *holomorphic* variety V is that normal holomorphic variety $\hat{V}$ for which there exists a finite branched holomorphic covering $\pi : \hat{V} \rightarrow V$ of branching order one. For proof of the existence and uniqueness of $\hat{V}$ and to see how this related to the definiton of the normalisation of an irreducible germ of a holomorphic variety. see [Gunn2].

2.2.2 Extension results for normalised spaces

We now mention some extension theory relating to varieties and normal spaces. We will use the results in the following chapters.

Theorem 2.2.1 *Suppose that V is an irreducible normal holomorphic variety, that W is a proper holomorphic subvariety of V, and that f is a holomorphic function on $V - W$, then if dim $W \leq$ dim $V - 2$, then f extends to a holomorphic function on all of V. [Gunn2] p.196.*

Theorem 2.2.2 *If V is a normal holomorphic variety, W is a holomorphic subvariety of V, and f is a holomorphic function on $V - W$ that is locally bounded in V, then there is a holomorphic function $\tilde{f}$ on V such that $\tilde{f}|(V - W) = f$. If W does not contain any irreducible components of V, the extension $\tilde{f}$ is unique. [Gunn2] p.191.*

Specifically, we will use these results in the main extension theorem in chapter 3, and the generalisation in chapter 4.

2.3 Stein manifolds and domains of holomorphy

If you have a holomorphic function defined in some open set of $\mathbb{C}^n$ it may be possible to extend its domain of definition with some uniqueness properties (due to the identity theorem of holomorphic functions). In the classical study of complex analysis, certain domains known as *domains of holomorphy* were defined. They were defined to be maximal domains in the sense that it is not true that all holomorphic defined on the domain of holomorphy can be extended to a larger domain. We will begin this discussion with *domains of holomorphy*.

Definition A *domain of holomorphy* in $\mathbb{C}^n$ is an open subset $D \subset \mathbb{C}^n$ for which there exists at least one function $f \in \mathcal{O}_D$ that cannot be extended as a holomorphic function through the boundary. That is not all functions defined on the domain can be extended holomorphically to a larger open subset of $\mathbb{C}^n$.

Definition An open subset $D \subset \mathbb{C}^n$ is called *holomorphically convex* if for any compact subset $K \subset D$ the set

$$\hat{K}_D = \{a \in D : |F(a)| \leq ||f||_K \text{ for all } f \in \mathcal{O}_D\}$$

is also compact. The set $\hat{K}_D$ is the *holomorphically convex hull* of K in D.

Definition A mapping $u : D \to [-\infty, +\infty)$ defined in an open subset $D \subset \mathbb{C}$ is a *subharmonic function if*:

1. u is *upper semi continuous* in D,
2. each point in a has an open neighbourhood $U_a \subset D$ such that

$$u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta})d\theta$$

whenever $\overline{\Delta}(a,r) \subset U_a$.

(Recall that a mapping is *upper semicontinuous* if $\{z \in D : u(z) < r\}$ is an is an open subset of D for every real number r, or equivalently if

$$u(a) \geq \limsup_{z \to a} u(z) = \lim_{\epsilon \to 0} \left(\sup_{0<|z-a|<\epsilon} u(z) \right)$$

for every point $a \in D$.

Definition A mapping $u : D \to [-\infty, +\infty)$ defined in an open subset $D \subset \mathbb{C}^n$ is *plurisubharmonic* if:

1. u is upper semicontinuous in D and
2. the restriction of u to any complex line through any point of D is a subharmonic function on that line in D. (A complex line $l_{a,b}$ w.r.t. $a, b \in \mathbb{C}^n$ is defined as $l_{a,b} = \{a + bz : z \in \mathbb{C}, \text{ for fixed } a, b \in \mathbb{C}^n\}$), so we require $u(z) := u(a + bz)$ to be harmonic in the set $\{z \in \mathbb{C} : a + bz \in D\}$.

Definition An open subset $D \subset \mathbb{C}^n$ is *pseudoconvex* if for any compact subset $K \subset D$ the set $\hat{K}_D = \{a \in D : u(a) \leq \sup_{z \in K} u(z), u \in \mathcal{P}(D)\}$ is also compact. Where $\mathcal{P}(D)$ is the set of continuous plurisubharmonic functions on D. The set $\hat{K}_D$ is the *plurisubharmonically convex hull* of K in D.

Now, an extremely important theorem that shows how these definitions relate;

Theorem 2.3.1 *For an open subset $D \subset \mathbb{C}^n$ the following are equivalent:*

1. *D is a domain of holomorphy,*
2. *D is holomorphically convex,*
3. *D is pseudoconvex.*

See [Gunn1] p. 166.

With the study of complex manifolds and holomorphic varieties, we need a new way to define domains of extension. We now look at *Stein manifolds* which are more general.

Definition A complex analytic (i.e holomorphic) manifold Ω of dimension n which is countable at infinity (i.e. it is a countable union of compact subsets) is said to be *Stein* if

1. Ω is holomorph-convex, that is,

$$\hat{K} = \{z; z \in \Omega, |f(z)| \leq \sup_K |f| \text{ for all } f \in \mathcal{O}(\Omega)\}$$

2. If z_1 and z_2 are different points in Ω, then $f(z_1) \neq f(z_2)$ for some $f \in \mathcal{O}(\Omega)$.

3. For every $z \in \Omega$, we can find n functions $f_1, \cdots, f_n \in \mathcal{O}(\Omega)$ which form a coordinate system at z.

Where we are using $\mathcal{O}(\Omega)$ to denote the space of analytic functions in Ω. [Hörman] p.116. It is quite easy to see that any domain of holomorphy satisfies these conditions.

It is easy to see that domains of holomorphy are Stein spaces. We say that a manifold $\tilde{\Omega}$ is a *holomorphic extension* of another manifold Ω if

1. Ω is an open subset of $\tilde{\Omega}$,
2. The analytic structure of Ω is induced by that in $\tilde{\Omega}$,
3. For every $u \in \mathcal{O}(\Omega)$ we can find $\tilde{u} \in \mathcal{O}(\tilde{\Omega})$ so that $u = \tilde{u}$ in Ω. (The function $\tilde{u}$ is uniquely determined by u, for $\tilde{\Omega}$ is connected.)

Theorem 2.3.2 *If Ω is a Stein manifold, and $\tilde{\Omega}$ is a holomorphic extension of Ω, then $\Omega = \tilde{\Omega}$. That is Stein manifolds are maximal in this sense. (See [Hörman] p.138.)*

The following theorem will be used in the following chapter.

Theorem 2.3.3 *A complex manifold Ω is a Stein manifold if and only if there exists a strictly plurisubharmonic function $\psi \in C^\infty(\Omega)$ such that $\Omega_c = \{z : z \in \Omega, \psi(z) < c\}$ is relatively compact in Ω for every real number c. The set $\overline{\Omega}_c$ are then $\mathcal{O}(\Omega)$ convex. (See [Hörman] p.127)*

Note the similarities to results on domains of holomorphy.

2.4 A CR-extension Theorem and modification

First we start with an approximation theorem. We will learn about the *analytic disc technique*. After this we will see *Bishop's equation*. Finally we give an overview of a CR extension theorem by Baouendi and Treves.

2.4.1 An approximation theorem

This is due to *Baouendi and Treves*, its proof can be found in [Boggess] p.191.

Theorem 2.4.1 *Suppose p is a point in a generic CR submanifold M of $\mathbb{C}^n$ of class C^2 with $dim_{\mathbb{R}}M = 2n - d, 0 \leq d \leq n$. Given an open neighbourhood ω_1 of p in M, there exists an open set ω_2 in M with $p \in \omega_2 \subset \omega_1$ so that each CR function of class C^1 on ω_1 can be uniformly approximated on ω_2 by an infinite sequence of entire functions in $\mathbb{C}^n$.*

To clarify this statement we recall definitions..

Suppose X is a smooth N-dimensional manifold. A vector at a point $p \in X$ is any operator of the form

$$\sum_{j=1}^{N} a_j \left(\frac{\partial}{\partial x_j}\right)_p \text{ where each } a_j \in \mathbb{R}.$$

The set of all vectors on X at p is called the (real) tangent space of X at p and is denoted $T_p(X)$.

Suppose V is a real vector space. The *complexification* of V is the tensor product $V \otimes_{\mathbb{R}} \mathbb{C}$ (for short, $V \otimes \mathbb{C}$). As a vector over the reals $V \otimes \mathbb{C}$ is generated by $v \otimes 1$ and $v \otimes i$ for $v \in V$. $V \otimes \mathbb{C}$ can be made into a complex vector space by defining

$$\alpha(v \otimes \beta) = v \otimes \alpha\beta \text{ for } \alpha, \beta \in \mathbb{C} \text{ and } v \in V.$$

Suppose V is a real vector space. A linear map $J : V \to V$ is called a complex structure map if $J \circ J = -I$ where $I : V \to I$ is the identity map.

For $J : V \mapsto V$ a complex structure map we consider the complexification $J \otimes \mathrm{id} : V \otimes_{\mathbb{R}} \mathbb{C} \mapsto V \otimes_{\mathbb{R}} \mathbb{C}$ and we denote the $+i$ and $-i$ eigenspaces of the complexified map respectively by $V^{1,0}$ and $V^{0,1}$. So we have the decomposition

$$V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1},$$

see [Boggess] p.43.

For example if we look at $V = T_p(\mathbb{R}^{2n}) \cong T_p(\mathbb{C}^n)$. We give $\mathbb{R}^{2n}$ coordinates $(x_1, y_1, \cdots, x_n, y_n)$. The standard complex structure for $T_p(\mathbb{R}^{2n})$ is given as follows

$$J\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j} \text{ and}$$

$$J\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j} \text{ for } 1 \leq j \leq n.$$

So we have J acting on $\mathbb{R}^{2n}$ similar to how i acts on $\mathbb{C}^n$

For a point $p \in M$, M a real submanifold of $\mathbb{R}^{2n} \simeq \mathbb{C}^n$, the complex tangent space of M at p is the vector space

$$H_p(M) = T_p(M) \cap J\{T_p(M)\}.$$

The *complexified tangent bundle* $T^{\mathbb{C}}(M)$ of any real manifold M is defined as

$$T^{\mathbb{C}}(M) = \cup_{p \in M} T_p(M) \otimes \mathbb{C}.$$

We can define a complex conjugation on $T^{\mathbb{C}}$ by requiring that $\overline{v \otimes z} = v \otimes \bar{z}$ for $v \in T_p M$ and $z \in \mathbb{C}$.

A submanifold M of $\mathbb{C}^n$ is called an *embedded* CR manifold or a CR submanifold of $\mathbb{C}^n$ if $\dim_{\mathbb{R}} H_p(M)$ is independent of $p \in M$.

Definition Let M be a C^∞ manifold and suppose $\mathbb{L}$ is a sub bundle of $T^{\mathbb{C}}(M)$. The pair $(M, \mathbb{L})$ is called (an abstract) CR manifold or CR structure if

1. $\mathbb{L} \cap \overline{\mathbb{L}} = \{0\}$ for each $p \in M$.

2. $\mathbb{L}$ is involutive, that is, $[L_1, L_2]$ belongs to $\mathbb{L}$ whenever $L_1, L_2 \in \mathbb{L}$.

An example would be if M is a CR submanifold of $\mathbb{C}^n$, then the pair $(M, \mathbb{L})$ with $\mathbb{L} = H^{1,0}(M)$ is a CR structure.

Suppose $(M, \mathbb{L})$ is a CR structure. A function (or distribution) $f : M \to \mathbb{C}$ is called a *CR function* if $\bar{\partial} f = 0$ on M. So we need to define the operator ∂.

Definition The Cauchy-Riemann operator $\bar{\partial} : \mathcal{E}^{0,0}(M) \mapsto \mathcal{E}^{0,1}(M)$ and the operator $\partial : \mathcal{E}^{0,0}(M) \mapsto \mathcal{E}^{1,0}(M)$ are defined by

$$\bar{\partial} = \pi^{0,1} \circ d$$

$$\partial = \pi^{1,0} \circ d,$$

where we define

$$\pi^{1,0} : T^{*1,0}(M) \oplus T^{*0,1}(M) \mapsto T^{*1,0}(M) \text{ and}$$

$$\pi^{0,1} : T^{*1,0}(M) \oplus T^{*0,1}(M) \mapsto T^{*0,1}(M).$$

Where d is the exterior derivative operator (as defined, for example, in [Boggess] p.28).

A CR submainfold M is called *generic* if $\dim_{\mathbb{R}} H_p(M)$ is *minimal* (i.e. if the manifold is of dim $2n-d$ in $\mathbb{C}^n$ then $\dim_{\mathbb{R}}(H_pM) = 2n-2d$ for $p \in M$). For some equivalent conditions see [Boggess] p.102.

2.4.2 A CR extension Theorem

The extenstion theorem we reach at the end of this section is due to *Al Boggess and John C. Polking*. As stated throughout the section, it can be found in [Boggess].

The Levi form

Definition We define the natural projection π_p at $p \in M$ as

$$\pi_p : T_p(M) \otimes \mathbb{C} \to \{T_p(M) \otimes \mathbb{C}\}/(\mathbb{L}_p \oplus \overline{\mathbb{L}}_p).$$

For a CR manifold M with involutive tangent bundle $\mathbb{L}$ defined as above we define the following. The *Levi form* at a point $p \in M$ is the map $\mathcal{L}_p : \mathbb{L}_p \mapsto \{T_p(M) \otimes \mathbb{C}\}/(\mathbb{L}_p \oplus \overline{\mathbb{L}}_p)$ defined by

$$\mathcal{L}_p(L_p) = \frac{1}{2i}\pi_p\{[\overline{L}, L]_p\} \text{ for } L_p \in \mathbb{L}_p$$

where L is any vector field in $\mathbb{L}$ that equals L_p at p. This is independent of the choice of $L \in \mathbb{L}$. See [Boggess] p.156-7

The next definition is of the extrinsic Levi form, it is used for embedded CR-manifolds. It relies on use of the metric of the ambient space, this results in the orthogonal projection map

$$\tilde{\pi}_p : T_p(\mathbb{C}^n) \mapsto N_p(M)$$

where $N_p(M)$ is the orthogonal complement of $T_p(M)$ in $T_p(\mathbb{R}^{2n})$. We extend this map to the complexification of the underlying space as follows,

$$\tilde{\pi}_p := \tilde{\pi}_p \otimes \mathrm{id} : T_p(\mathbb{C}^n) \otimes \mathbb{C} \mapsto N_p(M) \otimes \mathbb{C}.$$

Definition The *extrinsic* Levi form of M at p is the map $\tilde{\mathcal{L}}_p : H_p^{1,0}(M) \to N_p(M)$ given by $\tilde{\pi}_p \circ J \circ \mathcal{L}_p$.

Since $H_p^{1,0}$ and $H_p^{0,1}$ are J-invariant, we have

$$\tilde{\mathcal{L}}_p(L_p) = \frac{1}{2i}\tilde{\pi}_p(J[L_p, \overline{L}_p])$$

where L is any $H^{1,0}(M)$-vector field extension of L_p.

Now we will define the *Levi cone.*

Definition For $p \in M$, let

$$\Gamma_p = \{\text{the convex hull of the image of } \tilde{\mathcal{L}}_p\} \subset N_p(M).$$

Γ_p is called the Levi cone of M at p. For two cones Γ_1 and Γ_2 in $N_p(M)$ we say that Γ_1 is *smaller than* Γ_2 if the closure of $\Gamma_1 \cap S_p$ is a compact subset of the interior of $\{\Gamma_2\} \cap S_p$, where S_p is the unit sphere in $N_p(M)$, and denote this by $\Gamma_1 < \Gamma_2$.

So we will state the extension theorem central to this section.

Theorem 2.4.2 *Suppose M is a generic, CR submanifold of $\mathbb{C}^n$ of class $C^k(4 \leq k \leq \infty)$ with $\dim_{\mathbb{R}} M = 2n - d, 1 \leq d \leq n-1$. Let p be a point in M so that Γ_p has nonempty interior (with respect to $N_p(M)$). Then for every neighbourhood ω of p in M, there is an open set ω' in M and an open set Ω in $\mathbb{C}^n$ such that*

1. $p \in \omega' \subset \overline{\Omega} \cap M \subset \omega$

2. *for each open cone $\Gamma_1 < \Gamma_p$, there is a connected neighbourhood ω_1 of p in M and an $\epsilon > 0$ so that*
$$\omega_1 + \{\Gamma_1 \cap B_\epsilon\} \subset \Omega$$

3. *for each CR function f of class C^1 on $\omega \subset M$, there is a unique holomorphic function F defined on Ω and continuous on $\Omega \cup \omega'$ with $F = f$ on ω', where B_ϵ is a open ball of radius ϵ about the origin.*

In what follows we outline the proof of this extension theorem. (The proof of this theorem can be found in [Boggess] sec. 14 and 15.)

The analytic disc technique

We are going to use the approximation theorem of Baouendi and Treves which we stated earlier, this showed that a CR function on a CR submanifold M can be approximated on an open set $\omega \subset M$ by a sequence of entire functions. We want to extend a given CR function to an open subset Ω of $\mathbb{C}^n$, when we do this we will want our sequences of approximating entire functions to be uniformly convergent on the compact subsets of Ω. This is were the use of *analytic discs* comes in. An *analytic disc* is a continuous map $A : \overline{D} \to \mathbb{C}^n$ which is holomorphic on D. The *boundary* of the analytic disc is the restriction of A to the unit circle $S^1 = \partial D$. Usually in the literature, the analytic disc and its boundary are identified with their images in $\mathbb{C}^n$. We want to show that each point in Ω is contained in (the image of) an analytic disc whose boundary image is contained in ω. Suppose that $\{F_j\}$ is a sequence of entire functions that is uniformly convergent to a given CR function f on the open set $\omega \subset \mathbb{C}^n$. From the maximal modulus principle for analytic functions, the sequence of entire functions $\{F_j\}$ must also converge uniformly on Ω. So the CR extension theorem now becomes a theorem about analytic discs, i.e. if we prove the following theorem then by the methods of this previous discussion the extension theorem will follow.

Theorem 2.4.3 *Suppose M is a generic CR submanifold of $\mathbb{C}^n$ of class $C^k, 4 \leq k \leq \infty$ with $dim_{\mathbb{R}} M = 2n - d, 1 \leq d \leq n-1$. Let p be a point in M such that the interior of Γ_p in $N_p(M)$ is nonempty. Then for each neighbourhood ω of p in M and for each cone $\Gamma < \Gamma_p$, there is a neighbourhood $\omega_\Gamma \subset \omega$ and a positive number ϵ_Γ such that each point in $\omega_\Gamma + \{\Gamma \cap B_{\epsilon_\Gamma}\}$ is contained in the image of an analytic disc whose boundary image is contained in ω.*

So we need to find suitable analytic discs. The proof of this requires *Bishop's equation* which we discuss in the next section.

Bishops equation

We will presume the coordinates are now in normal form (see [Boggess] p. 103), that is we will fix a point $p \in M$ and assume the coordinates are in normal form, so that $p = 0$ and

$$M = \{(z = x + iy, w) \in \mathbb{C}^d \times \mathbb{C}^{n-d}; y = h(x, w)\}$$

when h is defined and where $h : \mathbb{R}^d \times \mathbb{C}^{n-d} \to \mathbb{R}^d$ is smooth of class C^4 and $h(0) = 0, Dh(0) = 0$.

Given an analytic disc $W : \overline{D} \to \mathbb{C}^{n-d}$, we want to find an analytic disc $G : \overline{D} \to \mathbb{C}^d$ so that the boundary of the disc $A = (G, W) : \overline{D} \to \mathbb{C}^n$ is contained in M. This means we must satisfy

$$\mathrm{Im}G(\zeta) = h(\mathrm{Re}G(\zeta), W(\zeta)) \text{ for } |\zeta| = 1.$$

We want to relate $\mathrm{Im}G(\zeta)$ and $\mathrm{Re}G(\zeta)$, we go about this as follows. Let S^1 be the unit circle in $\mathbb{C}$. If $u : S^1 \to \mathbb{R}^d$ is a smooth function, then u extends to a unique harmonic function on the unit disc D, which we also call u. This harmonic function has a unique harmonic conjugate in $\overline{D}$, denoted by v, which vanishes at the origin. The *Hilbert transform* of u is denoted by $Tu : S^1 \to \mathbb{R}^d$ is defined to be $v|_{S^1}$.

If $G = u + iv : \overline{D} \to \mathbb{C}^d$ is analytic and continuous up to S^1, then $T(u|_{S^1}) = v|_{s^1} + C$ where $C = -v(\xi = 0)$. The function $-iG = v - iu$ is also analytic and so $T(v|_{s^1}) = -u + c$ where $c = u(\xi = 0)$. Conversely, if $u, v : S^1 \to \mathbb{R}^d$ are continuous functions with $u = -Tv + c$, then $u + iv : S^1 \to \mathbb{C}^d$ is the boundary values of a unique analytic disc $G : \overline{D} \to \mathbb{C}^d$ with $\mathrm{Re}G(\xi = 0) = c$.

Suppose $u + iv = G : \overline{D} \to \mathbb{C}^d$ is an analytic disc with $v(e^{i\phi}) = h(u(e^{i\phi}), W(e^{i\phi}))$ for $0 \leq \phi \leq 2\pi$. We apply $-T$ to both sides of this equation and obtain

$$u(e^{i\phi}) = -T(h(u, W))(e^{i\phi}, W(e^{i\phi})) + x, 0 \leq \phi \leq 2\phi$$

where $x \in \mathbb{R}^d$ is the value of u at $\xi = 0$. The above equation is referred to as *Bishops equation*. Conversely, suppose the analytic disc $W : \overline{D} \to \mathbb{C}^{n-d}$ and the vector $x \in \mathbb{R}^d$ are given, and suppose $u : S^1 \to \mathbb{R}^d$ is a solution to Bishop's equation. From the above discussion , the function

$$\phi \mapsto u(e^{i\phi}) + ih(u(e^{i\phi}), W(e^{i\phi}))$$

is the boundary values of a unique analytic disc $G : \overline{D} \to \mathbb{C}^d$. Since $\mathrm{Re}G(e^{i\phi}) = u(e^{i\phi})$, the boundary of the analytic disc $A = (G, W) : \overline{D} \to \mathbb{C}^n$ is contained in M. Furthermore, $ReG(\xi = 0) = x$. A theorem on the solutions of Bishop's equation can be found in [Boggess] p.220, we state it here, after we define Hölder continuous functions.

Definition Let L be ant normed linear space. A continuous function $f : S^1 \mapsto L$ is said to be Hölder continuous with exponent α $(0 < \alpha < 1)$ if there is a finite positive number M so that $|f(e^{i\phi_1}) - f(e^{i\phi_2})| \leq M|\phi_1 - \phi_2|^\alpha$ for all $0 \leq \phi_1, \phi_2 \leq 2\pi$. The set of all such functions is denoted by $C^\alpha(S^1, L)$.

Now we state a theorem on solutions to Bishop's equation, [Boggess] p.220.

Theorem 2.4.4 *Fix $0 < \alpha < 1$. Suppose $h : \mathbb{R}^d \times \mathbb{C}^{n-d} \mapsto \mathbb{R}^d$ is of class C^2 with $h(0) = 0, Dh(0) = 0$. There is a $\delta > 0$ such that if $|x| < \delta$ and $W \in C^\alpha(S^1, \mathbb{C}^{n-d})$ with $||W||_\alpha < \delta$, then there is a unique element $u = u(W, x) \in C^\alpha(S^1, \mathbb{C}^d)$ that solves Bishop's equation with $u(W = 0, x = 0) = 0$. In addition, if h is of class $C^k (k \geq 2)$, then there is a $\delta > 0$ such that $u : C^\alpha(S^1, \mathbb{R}^d)$ depends in a C^{k-1} fashion on $x \in \mathbb{R}$ and $W \in C^\alpha(S^1, \mathbb{C}^{n-d})$ with $|x| < \delta$ and $||W||_\alpha < \delta$.*

The Levi form $\tilde{\mathcal{L}} : \mathbb{C}^{n-d} \times \mathbb{C}^{n-d}$ is

$$\tilde{\mathcal{L}}(w, \hat{w}) = \sum_{j,k=1}^{n-d} \frac{\partial^2 h(0)}{\partial w_j \partial \bar{w}_k} w_j \hat{w}_k, w, \hat{w} \in \mathbb{C}^{n-d}.$$

This is a bilinear form. It can be shown that the the extrinsic Levi form $w \mapsto \tilde{\mathcal{L}}_0(w)$ can be identified with the map $w \mapsto \tilde{\mathcal{L}}(w, \bar{w}) \in \mathbb{R}^d$ for $w \in \mathbb{C}^{n-d}$. See p.160, p222 [Boggess] for further details.

Suppose $\Gamma < \Gamma_0$ is the cone given in the statement of the analytic disc theorem. We can find vectors $X_1, \ldots, X_N$ which lie in the image of $\tilde{\mathcal{L}}_0$ with

$$\Gamma < \Gamma_{X_1,\ldots,X_N} \subset \Gamma_0$$

where $\Gamma_{X_1,\ldots,X_N}$ is the convex cone generated by $X_1, \ldots, X_N$.

Since each X_j lies in the image of $\tilde{\mathcal{L}}_0$, there are vectors $\alpha_j \in \mathbb{C}^{n-d}$ with

$$X_j = \tilde{\mathcal{L}}(\alpha_j, \bar{\alpha}_j) \text{ where } 1 \leq j \leq N.$$

For $t \in \mathbb{R}^N$ and $w \in \mathbb{C}^{n-d}$, define the analytic disc

$$W(t, w)(\zeta) = w + \sum_{j=1}^{N} t_j \alpha_j \zeta_j.$$

W depends smoothly on the parameters $t \in \mathbb{R}^N$ and $w \in \mathbb{C}^{n-d}$ and we have $W(0, 0)(\zeta) = 0$.

Using a theorem on the solutions to Bishop's equation (2.4.4) we have solutions which form the analytic discs we require. Now we present the following lemmas, these complete the proof of the analytic disc theorem.

The first lemma gives us conditions for the existance of the analytic discs.

Lemma 2.4.5 *Given w an open neighbourhood of the origin in M, there is a $\delta > 0$ and a map*

$$A : \{(t, x, w) \in \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{C}^{n-d}; |t|, |x|, |w| < \delta\} \mapsto \mathcal{O}^\alpha(D, \mathbb{C}^n)$$

of class C^3 such that the boundary of each $A(t, x, w)$ is contained in $w \subset M$ for $|t|, |x|, |w| < \delta$. Furthermore, $A(t, x, w)(\zeta) = (G(, x, w)(\zeta), W(t, w)(\zeta))$ and $ReG(t, x, w)(\zeta = 0) = x$.

[Boggess] p.223.

Now we have a result on the error term.

Lemma 2.4.6 *Given $\eta > 0$ there are constants $\delta > 0, \epsilon' > 0$ such that for $|x|, |w| < \delta$ and $|t| < \epsilon'$*

$$A(t,x,w)(\zeta = 0) = (x, h(x,w), w) + \left(0, \sum_{j=1}^{N} t_j^2 X_j, 0\right) + (0, \xi(t,x,w), 0)$$

where $\xi : \mathbb{R}^N \times \mathbb{R}^d \times \mathbb{C}^{n-d} \mapsto \mathbb{R}^d$ is of class C^3 and

$$|\xi(t,x,w)| \leq \eta |t|^2,$$

where η is a constant.

This lemma shows that each point in the set

$$\Omega = \{A(x,w,t)(\zeta = 0); |t|, |x|, |w| < \delta\}$$

belongs to the image of an analytic disc whose boundary is contained in the given open set ω for the analytic disc theorem. The set

$$\omega_\Gamma = \{(x, h(x,w), w); |x|, |w| < \delta\}$$

is an open subset of M that contains the origin. So the proof of the analytic disc theorem is complete once we have shown that for fixed x, w with $|x|, |w| < \delta$, the map

$$t \mapsto f(x,w)(t) = \sum_{j=1}^{N} t_j^2 X_j + \xi(t,x,w) \text{ with } t \in \mathbb{R}^N, |t| < \epsilon'$$

parameterises a set that contains an open set of the origin in the cone Γ. We define

$$\hat{F}(t_1, \ldots, t_N) = \sum_{j=1}^{N} t_j X_j$$

and

$$\begin{aligned} F(x,w)(t_1, \ldots, t_N) &= f(x,w)(\sqrt{t_1}, \ldots, \sqrt{t_N}) \\ &= \hat{F}(t_1, \ldots, t_N) + e(t,x,w) \end{aligned}$$

where the error $e(t, x, w) = \eta(\sqrt{t_1}, \ldots, \sqrt{t_N})$ is continuous and satisfies the estimate

$$|e(t, x, w) \leq \eta |t|$$

under the correct conditions, see [Boggess] p.227.

Lemma 2.4.7 *Suppose $X_1, \ldots, X_d$ are linearly independent vectors in $\mathbb{R}^d$. Suppose $\hat{F}(t) = \sum_{j-1}^{d} t_j X_j$. Suppose $\Gamma < \Gamma_{X_1,\ldots,X_d}$ and $\epsilon' > 0$ are given. Then there exists $\eta, \epsilon > 0$ such that if $F : \mathbb{R}^d \mapsto \mathbb{R}^d$ is a continuous map with $|F(t) - \hat{F(t)}| \leq \eta |t|$ for $t = (t_1, \ldots, t_d)$ with $t_j \geq 0, 1 \leq j \leq d$, and $|t| \leq \epsilon'$, then the image of $\{t = (t_1, \ldots, t_d) : t_j \geq 0, |t| < \epsilon'\}$ under F contains $B_\epsilon \cap \Gamma$.*

And with this result we have the required proof of theorem 2.4.3, the analytic disc theorem. And with this in mind and the discussion before theorem 2.4.3, the proof of the CR extension theorem follows.

2.4.3 A modified version of the extension theorem

We give a modified version of the CR extension theorem 2.4.2 as used in [KaupZait], where it is refered to as a *deformation* version of the extension theorem. We will have a look at it now.

We will use $N_x \subset T_x X$ to denote the (real) orthogonal complement of $T_x M_x$ in $T_x X$. The Levi cone at x of M_x we denote by C_x. For every cone $B \subset N_x$ we write $B < C_x$ if the intersection of the closure $\overline{B}$ with the unit sphere in N_x is contained in the interior of the cone C_x. For every $\epsilon > 0$ denote by B_ϵ the intersection of B and the open ball with center $0 \in N_x$ and radius ϵ in N_x.

Lemma 2.4.8 *Let $X = \mathbb{C}^n$ with hermitian metric given by the standard scalar product and let $M \subset X$ be a smooth submanifold. Let $\varphi : M \to \mathbb{R}$ be a smooth submersion such that the submanifold $M_x := \varphi^{-1}(\varphi(x))$ is generic in X for every $x \in M$. Suppose furthermore that $B < C_y$ is an open cone in N_y for some $y \in M$, where $N_y \subset T_y X$ is the normal space and $C_y \subset N_y$ is the Levi cone of M_y at y. Then there exists an open neighbourhood U of y in M and an $\epsilon > 0$ such that for every $x \in U$*

1. *$W_x := (U \cap M_x) + B_\epsilon$ is open in X and*

2. *every continuous CR-function on M_x extends to a function in $\mathcal{C}(M_x \cup W_x) \cap \mathcal{O}(W_x)$*

(Note: A *submersion* is a smooth map $f : M \to N$ such that

$$\dim M \geq \dim N$$

and the differential, (or Jacobian), is surjective at every $x \in M$.)

We now look at a proposition as proved in [KaupZait] which will be used for the CR extension results in the next section.

Proposition 2.4.9 *Let X be a complex manifold, $M \subset X$ a smooth connected submanifold and $\varphi : M \to \mathbb{R}$ a smooth function such that $M_x := \varphi^{-1}(\varphi(x))$ is a connected compact generic submanifold of X for every $x \in M$. Assume that there exists on M a smooth vector field ξ with $d\varphi(\xi) > 0$ such that $\pi_x(\xi_x)$ is the canonical projection $T_xX \to T_xX/T_xM_x$. (This is the canonical projection with respect to the norm on $\mathbb{C}^n$ and the chart mappings between X and $\mathbb{C}^n$) Then for every $a \in M$ and $W := \{x \in M : \varphi(x) > \varphi(a)\}$ there exists an open neighbourhood U of W in X such that every continuous CR-function on M_a extends to a function in $\mathcal{C}(U \cup M_a) \cap \mathcal{O}(U)$.*

Chapter 3

Orbits associated with bounded symmetric domains

We start by defining orbits of bounded symmetric domains and state defining equations for them. The Pierce decomposition will then be looked at. We define important domains and show them to be (maximal) holomorphic and CR extensions spaces, and note that their closures are maximal for continuous extension. A look at the Levi cone will be then necessary to allow the use of the extension theory from the last section. This section is based on the paper [KaupZait].

3.1 Orbits of elements in bounded symmetric domains

Let E be a complex vector space of finite dimension with a Jordan structure on it. Let $\mathcal{D} \subset E$ be a bounded symmetric domain which we assume to be *circular*. We define $\mathrm{GL}(\mathcal{D}) = \{g \in \mathrm{GL}(E) : g(\mathcal{D}) = \mathcal{D}\}$ and $\mathbf{K}$ to be the connected component of the identity of $\mathrm{GL}(\mathcal{D})$. We will choose an arbitrary element $a \in \mathcal{D}$ and define the orbit $K := \mathbf{K}(a)$. Note also that $\mathrm{GL}(\mathcal{D}) = \mathrm{Aut}(E)$, where $\mathrm{Aut}(E)$ is the group of all *triple* automorphisms of E, i.e. $\{g \in \mathrm{GL}(E) : g\{xyz\} = \{gx, gy, gz\}$ for all $x, y, z \in E\}$.

Let $\mathbf{K} = \mathbf{ZS}$, where $\mathbf{Z}$ is the connected identity component of the center and $\mathbf{S}$ is the semisimple (see [Bourbaki] III.9.8) commutator subgroup of $\mathbf{K}$. With this in mind we define $S := \mathbf{S}(a)$. We will now list the group $\mathbf{S}$ corresponding to each bounded symmetric domain.

$\mathbf{I}_{p,q}$: $\mathbf{S}$ is the group of all transformations $z \mapsto uzv'$ with $u \in \mathrm{SU}(p)$ and $v \in \mathrm{SU}(q)$,

$\mathbf{II}_p$: $\mathbf{S}$ is the group of all transformations $z \mapsto uzu'$ with $u \in \mathrm{SU}(p)$,

III$_p$: S is the group of all transformations $z \mapsto uzu'$ with $u \in \mathrm{SU}(p)$,

IV$_n$: S $= \mathrm{SO}(n)$ acting in the standard way on $\mathbb{R}^n$ and $\mathbb{C}^n$,

V: S $= \mathrm{Spin}(10)$,

VI: S is a compact exceptional group of type E_6.. In each case z is an element of the bounded symmetric domain in question..

Singular value decompositions

A rectangular matrix $= (z_{jk}) \in E$, where E is the $n \times m$ space of complex values matrices, is called *diagonal* if $z_{jk} = 0$ holds for all $j \neq k$. Presume E has a Jordan structure defined on it, so it will contain the bounded symmetric domain $\mathbf{I}_{n,m}$. Then from the *singular value decomposition* there exists a unique diagonal matrix d with $z = g(d)$ where $g \in K$ and $d \in \Delta_p := \{x \in \mathbb{R}^p : x_1 \geq \cdots \geq x_p \geq 0\}$, where $x_1, \ldots, x_p$ are the diagonal entries of the diagonal matrix formed from our rectangular matrix and $p = \min\{x, y\}$. Here K can be identified with the action of $\mathrm{SU}(n) \otimes \mathrm{SU}(m)$ on $z \in D$, the open unit ball of the spectral norm in E. The diagonal entries $\sigma_j(z) := d_{jj}$ for $1 \leq j \leq p$ are called the *singular values* of the matrix z. We have $\sigma_1(z) \geq \sigma_2(z) \geq \cdots \sigma_p(z) \geq 0$. Note that $\sigma_j(z)$ is the jth biggest eigenvalue of the hermitian matrix $\left(\begin{smallmatrix} 0 & z \\ z^* & 0 \end{smallmatrix}\right)$ and $\sigma(z)^2$ is the jth biggest eigenvalue of the hermitian matrix zz^*. We see later that we can use this to define norms and hence the *spectral decomposition* from chapter 1.

Recall that an element of a positive hermitian Jordan triple system, E, is called a *tripotent* if $e = \{eee\}$. A tripotent is *minimal* if it cannot be written as the sum of two non-zero tripotents. Denote by $\mathcal{E}$ the set of all sequences $\mathbf{e} = (e_1, \ldots, e_s)$ of non-zero, mutually (triple) orthogonal tripotents (that is $L(e, f) = L(f, e) = 0$ for tripotents e and f means e and f are orthogonal) $e_j \in E$ and call $l(\mathbf{e}) := s$ the length of $\mathbf{e}$. Then necessarily $l(\mathbf{e}) \leq n = \dim E$ and $r := \max\{l(\mathbf{e}) : \mathbf{e} \in \mathcal{E}\}$ is called the *rank* of E. Every $\mathbf{e} \in \mathcal{E}$ with the maximal possible length $l(\mathbf{e}) = r$ is called a *frame* in E. Every tripotent in a frame is minimal. See [Loos1].

In chapter 1 we noted that every element $a \in E$ has a representation

$$a = \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_s e_s$$

for a suitable sequence $\mathbf{e} = (e_1, \ldots, e_s) \in \mathcal{E}$ and real coefficients λ_j. For convenience we put

$$\lambda_0 := 0 \text{ and } \lambda_{-j} := -\lambda_j \text{ for } 1 \leq j \leq s.$$

There always exists two extremal choices for the sequence $\mathbf{e}$ and the given element $a \in E$.
1. The maximal length choice: Here $\mathbf{e}$ is a frame, i.e. $s = r$, and we assume in addition that

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0$$

holds. Then the coefficient λ_j is uniquely determined by $a \in E$ and is called the *jth singular value* of a, denoted by $\sigma_j(a)$. In the matrix cases, this corresponds to the simple example given earlier.
2. The minimal length choice: Here $\mathbf{e}$ is not necessarily a frame, but we require

$$\lambda_1 > \lambda_2 > \cdots > \lambda_s > 0.$$

Under these assumptions the coefficients λ_j and the tripotents e_j are uniquely determined by the element a. The integer s is called the *reduced rank* of a.

The functions $\sigma_j : E \to \mathbb{R}$ can be shown to be K-invariant and continuous. Hence also $\sigma := (\sigma_1, \sigma_2, \ldots, \sigma_r) : E \to \mathbb{R}^r$ is K-invariant. For every $z \in E$, every $1 \leq p \leq \infty$ and every $k = 1, 2, \ldots, \infty$ put

$$||z||_p := \left(\sum_{j=1}^{r} \sigma_j(z)^p \right)^{1/p}, ||z||_\infty := \sigma_1(z) \text{ and } |||z|||_k := \sum_{j=1}^{k} \sigma_j(z).$$

As we have noted earlier, $||z||_\infty$ is sometimes known as the spectral norm.

Recall that the bounded symmetric domain $\mathcal{D} \subset E$ given by

$$\mathcal{D} := \{z \in E : \mathrm{id}_E - L(z, z) > 0\}$$

is the open unit ball with respect to the (spectral) norm $|| \, ||_\infty$. Furthermore, $|| \, ||_2$ is the unique K-invariant Hilbert norm on E such that all minimal tripotents have norm 1, so we have the unique $\mathrm{Aut}(E)$-invariant inner product $(x|y)$ on E with

$$(z|z) = ||z||_2^2 \text{ for all } z \in E.$$

We define $\mu_k(a) := \prod_{i=1}^{k} \sigma_i(a)$.

For an odd function $f : \mathbb{R} \mapsto \mathbb{C}$ and $a \in$ for some E we define

$$f(a) := f(\lambda_1)e_1 + f(\lambda_2)e_2 + \cdots + f(\lambda_s)e_s.$$

We define $\mathrm{sign}(t) : \mathbb{R} \mapsto \mathbb{C}$ where $\mathrm{sign}(t) = t/|t|$ for $t \neq 0$ and $\mathrm{sign}(0) = 0$, this can be generalised to $a \in E$ in the obvious manner, the result is a tripotent. So if we take $a \in E$ for a triple space E and then look at $e = \mathrm{sign}(a)$, we can look at the jordan algebra E^e. Later we will use $a^+ := \lambda_1^+ e_1 + \lambda_2^+ e_2 + \cdots + \lambda_s^+ e_s$ where we are using $x^+ = x$ if $x \geq 0$ and 0 otherwise for $x \in \mathbb{R}$. We also define $x^\dagger$ below,

Definition $x^\dagger := \lambda_1^\dagger e_1 + \lambda_2^\dagger e_2 + \cdots + \lambda_s^\dagger e_s$ where we are using $\lambda_i^\dagger = \lambda_i^{-1}$ if $a \neq 0$ and 0 otherwise, $\lambda_i \in \mathbb{R}$.

E^e is the Jordan algebra formed from the triple space E by defining the Jordan product $x \circ y := \{xey\}$.

It is known that a complex homogeneous polynomial $N : E \to \mathbb{C}$ of degree $r := \mathrm{rank}(E)$ called a *generic norm* on E which satisfies the following properties exists on E^e: (i) $N(e) = 1$ for some tripotent $e \in E$ and (ii) $z \in E$ is invertible if and only if $N(z) \neq 0$ for every $z \in E$. See [Loos2] sec. 16. which defines $N(z)$ (and $m_k(z,z)$ which we see in the following).

Defining equations for orbits

First, from [KaupZait], defining equations for complexifications of orbits are given.

It is known that when a is not an invertible element of the bounded symmetric domain in question then $\mathbf{K}^{\mathbb{C}}(a) = E_{[\rho]}$ where $\rho = \mathrm{rank}(a)$ and $E_{[\rho]} := \{a \in E : \mathrm{rank}(a) = \rho\}$. And in the case when a is an invertible element of the Jordan algbra E then $\mathbf{K}^{\mathbb{C}}(a) = \{z \in E : N(z) = N(a)\} \subset E_{[r]}$ where r is the rank of the space E.

Under appropriate restriction we get the following equations for the orbits $K(a)$ and $S(a)$:

$$\begin{aligned} K(a) &= \{z \in E : \sigma(z) = \sigma(a)\} \\ &= \{z \in E : ||z||_k = ||a||_k \text{ for } 1 \leq k \leq r\} \\ &= \{z \in E : \mu_k(z) = \mu_k(a) \text{ for } 1 \leq k \leq r\} \\ &= \{z \in E : m_k(z,z) = m_k(a,a) \text{ for } 1 \leq k \leq r\}, \end{aligned}$$

where $m_k(z,z) := \sum_{k_1<\cdots<k_j} \sigma_{k_1}^2(z)\sigma_{k_2}^2(z)\cdots\sigma_{k_j}^2(z)$. Or since $m_j(z,z) = 0$ for all $j > \mathrm{rank}(z)$ we can say

$$K(a) = \{z \in E_{[\rho]} : m_j(z,z) = m_j(a,a) \text{ for } 1 \leq k \leq \rho\} \text{ if } \rho = \mathrm{rank}(a),$$

where $E_{[\rho]} := \{a \in E : \mathrm{rank}(a) = \rho\}$. $\mathrm{S}(a) = \mathrm{K}(a)$ if E is not a Jordan algebra or if E is a Jordan algebra and $a \in E$ is not invertable. In the case of $S(a)$ we have

$$\begin{aligned} S(a) &= \{z \in K(a) : N(z) = N(a)\} \\ &= \{z \in E : N(z) = N(a) \text{ and } m_j(z,z) = m_j(a,a) \text{ for } 1 \leq j < r\} \end{aligned}$$

since $m_r(z,z) = |N(z)|^2$.

3.2 Tangent spaces of the orbits in terms of the Pierce decomposition

In this section we note the relationship between the tangent spaces and the Peirce decomposition.This correspondence helps us understand and work with them in a more structured and elegant way.

We noted earlier the following proposition, the proof can be found in [KaupZait] p.68.

Proposition 3.2.1 *For every $a \in E$ the tangent spaces to the orbits $S = S(a)$ and $K = K(a)$ at a satisfy $T_aS \subset T_aK = iA(a) \oplus E_{1/2}(a)$ and $H_aS = H_aK = E_{1/2}(a)$. (See chapter 1 for details on these decompositions.)*

Using the operators L_a and Q_a (defined in chapter 1), we can define some more useful operators:

$$\Phi_a := 2(L_a + Q_a), \Psi_a := 2(L_a - Q_a) \text{ and}$$

$$\Theta_a := \Phi_a \Psi_a = 4(L_a^2 - Q_a^2),$$

where the last operator is complex linear unlike the other two. Obviously,

$$\Phi_a = \sum_{|j| \leq k \leq s} (\lambda_j + \lambda_k)^2 P^{j,k}, \Psi_a = \sum_{|j| \leq k \leq s} (\lambda_j - \lambda_k)^2 P^{j,k}$$

$$\Theta_a = \sum_{0 \leq j \leq k \leq s} (\lambda_j^2 - \lambda_k^2)^2 P_{j,k},$$

where $P_{j,k}$ and $P^{j,k}$ are the orthogonal projections onto the subspaces $E_{j,k}$ and $E^{j,k}$ respectively.

So it follows easily that

Lemma 3.2.2 *The operators Φ_a, Ψ_a and Θ_a satisfy*

$$\Phi_a(E) = E_{1/2}(a) \oplus A(a) = i\Psi_a(E),$$

and

$$\Theta_a(E) = E_{1/2}(a) = \Phi_a(E) \cap \Psi_a(E)$$

is the maximal complex linear subspace of $\Phi_a(E)$ as well as of $\Psi_a(E)$.

See [KaupZait] section 7.

3.3 Domains of (maximal) extension of orbits

The domains $\mathcal{D}(a)$ and $\mathcal{B}(a)$

Let E be a factor and $a \in E$ be an arbitrary element.

$$\mathcal{D}(a) := \{z \in \overline{E}_{[p]} : \mu_k(z) < \mu_k(a) \text{ for } 1 \leq k \leq \rho\}, \rho := \operatorname{rank}(a),$$

is a bounded *balanced* domain in the complex analytic cone $\overline{E}_{[\rho]}$, that is, $t\mathcal{D}(a) \subset \mathcal{D}$ for every complex number t with $|t| \leq 1$. In the case $a \neq 0$ the domain $\mathcal{D}(a)$ is non-empty, and its closure is the compact set $\mathcal{Z}(a)$. Its boundary $\partial\mathcal{D}(a)$ consists of all $z \in \mathcal{Z}$ with $\mu_k(z) = \mu_k(a)$ for some $k \leq \rho$. The orbit $K = K(a)$ coincides with the subset $\{z \in \mathcal{Z}(a) : \mu_k(z) = \mu_k(a) \text{ for all } k \leq \rho\} \subset \partial\mathcal{D}(a)$.

The domain $\mathcal{D}(a)$ is shown to be Stein in [KaupZait], section 10.5. The proof starts with the case of a reduced (with respect to its singular values) a. The Levi form points inside the domain in all sections of the boundary, showing that different boundary sections are locally Stein and thus their intersection, $\partial\mathcal{D}(a)$, is locally Stein. The theorem referred to in [Hörman], Theorem 2.6.10, proves that if this is the case, then $\mathcal{D}(a)$ is Stein. Then a theorem of Behnke-Stein is used to prove that $\mathcal{D}(a)$ is also Stein for non-reduced $a \in E$.

For every relatively compact domain D in a normed complex space X, where $\dim(x) < \infty$, we have the concept of *minimal boundary*. This is a minimal subset $M \subset \overline{D}$ such that every $f \in \mathcal{C}(\overline{D}) \cap \mathcal{O}(D)$ attains its maximum modulus in M. Existence and uniqueness of the minimal boundary for D is well known, and its closure is called the *Shilov boundary* of D. Both boundarys are contained in the topological boundary ∂D of D by the maximum modulus principle. Now, for the domain $\mathcal{D}(a) \subset E_{[\rho]}, a \in E$ arbitrary of rank $\rho > 0$, both boundaries coincide with the orbit $K(a)$. This is verified in [KaupZait] p.85.

In the case a is invertible, the orbit $S(a)$ is a generic and minimal CR-submanifold of the complex manifold $S^{\mathbb{C}}(a) = \{z \in E : N(z) = N(a)\}$. For such a we put

$$\mathcal{B}(a) := \{z \in S^{\mathbb{C}}(a) : \mu_k(z) \leq \mu_k(a) \text{ for all } k < r\},$$

if $r := \text{rank}(a) = \text{rank}(E) > 1$, and $\mathcal{B}(a) := 0$ otherwise. We can see from a proof similar to that for $\mathcal{D}(a)$ that $\mathcal{B}(a)$ is also Stein. It can be shown, similar to [KaupZait] p.85, that $S(a)$ coincides with the Shilov boundary of $B(a)$ in $S^{\mathbb{C}}(a)$ if $\mathcal{B}(a) \neq 0$.

The domains $\mathcal{Z}(a)$, $\mathcal{Y}(a)$ and $\mathcal{X}(a)$

$$\mathcal{Z}(a) := \{z \in E : \mu_k(z) \leq \mu_k(a) \text{ for } 1 \leq k \leq r\} r := \text{rank}(E)$$

$$\mathcal{Y}(a) := \{z \in \mathcal{Z}(a) : \mu_\rho(z) = \mu_\rho(a)\}.\rho := \text{rank}(a)$$

$$\mathcal{X}(a) := \{z \in \mathcal{Z}(a) : N(z) = N(a)\}$$

see [KaupZait] p.81.

Note that $\mathcal{Z}(a) = \overline{\mathcal{D}(a)}$ and $\mathcal{X}(a) = \overline{\mathcal{B}(a)}$

The following definition and the lemma that follows show that these spaces are maximal extension sets (for continuous functions).

Definition Let M be Stein and let $K \subset M$ which is a compact subset strictly contained in M. Then we say that K has a *Stein neighborhood basis*, or is *holomorphically convex*, if for every open set U with $K \subset U$, there is a Stein domain V, such that $K \subset V \subset U$.

Now with this definition in mind, we state and prove the following short lemma from [KaupZait].

Lemma 3.3.1 *For every a with rank$(a) = \rho$, the compact subset $\mathcal{Z}(a) \subset \overline{E}_{[\rho]}$ has a Stein neighbourhood basis in $\overline{E}_{[\rho]}$. If a is invertible, then $\mathcal{X}(a) \subset \mathbf{S}^{\mathbb{C}}(a)$ also has a Stein neighbourhood basis in $\mathbf{S}^{\mathbb{C}}(a)$.*

Proof Write a in the standard form

$$a = \lambda_1 e_1 + \cdots + \lambda_\rho e_\rho$$

using the maximal length choice. For every $t > 1$ consider $a_t := t\lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_\rho e_\rho \in \overline{E_\rho}$ and $b_t := t\lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_{r-1} e_{r-1} + t^{-1}\lambda_r e_r \in \mathbf{S}^{\mathbb{C}}(a)$ in case

a is invertible. Then the sets $\mathcal{D}(a_t)$ and $\mathcal{B}(b_t)$ form the required basis. This follows by looking at the singular value decompositions of the sets. The domains $\mathcal{Z}(a)$ is bounded by the singular values of a, the domains $\mathcal{D}(a_t)$ are Stein and contain $\mathcal{Z}(a)$. This is similar when a is invertible.

This will be noted later to show that we have *maximal* domains of holomorphic extension.

3.4 The Levi cone

Definition The *Levi form* at a

$$\Lambda_a : H_aK \times H_aK \to \mathbb{C}T_aK/H_aK$$

is given by

$$\Lambda_a(x, y) \equiv ([\xi, \eta]_a + i[i\xi, \eta]_a) \mod H_aK,$$

where ξ and η are any real CR vector fields on K with $x = \xi_a$ and $y = \eta_a$. Also, see [Boggess] p.156 and section 2.4.2.

Definition In a real vector space V of finite dimension a nonempty subset $C \subset V$ is called a *cone* if $tC \subset C$ holds for all $t > 0$. With C' we denote the dual cone of C, that is the set of all linear forms τ on V with $\tau(C) \geq 0$. If we have a cone C in a space V equipped with an inner product $(\cdot|\cdot)$ we identify the dual space of V with V via $y(x) = (y|x)$ and then we have

$$C' := \{z \in V : (z|C) \geq 0\}.$$

space.

By proposition 3.2.2 we can identify $\mathbb{C}T_aK/H_aK$ with $iA(a)$ and hence $\Lambda_a(u, u) \in A(a)$ holds for all $u \in H_aK$. Denote $C_a \subset A(a)$ the convex hull of all such vectors and call is the *Levi cone* of K at a.

In the following we define two closed cones, their use will is in the proposition that follows. (We used Γ to denote the Levi cone in the last chapter.)

We define the following two cones to help understand the structure of the Levi cone of K at a. See [KaupZait] 9.5.

Definition

$$Z(a) := \{x \in A(a) : (x_j^+|a^\dagger) + \sum_{k<j}(x_k|a^\dagger) \le 0 \text{ for } j = 1, \dots, s\}$$

$$X(a) := \{x \in Z(a) : (x|a^\dagger) = 0\} \subset Z(a)$$

Definition $W(a) \in \overline{\Omega}(a)$ is the convex cone generated by all the vectors $w \in \overline{\Omega}(a)$ satisfying for some $1 \le j \le s$ the conditions

$$\lambda_j w_j = \begin{cases} e_k & k < j \\ 0 & k > j \end{cases} \quad \text{and } \lambda_k w_k \text{ is a minimal idempotent in } A(e_j).$$

So it can be shown that $Z(a)$ is the dual of the cone $-W(a)$ and $X(a)$ of the cone $\mathbb{R}a^\dagger - W(a)$ in $A(a)$, where $a^\dagger$ is the pseudo inverse as defined earlier (page 52).

We also note the following important propositions that can be found in [KaupZait] as Propositions 9.12 and 9.17.

Proposition 3.4.1 *The Levi cone C_a at a is $X(a)$ if a is invertible and is $Z(a)$ if a is not invertible. In particular, C_a is always a closed cone.*

Proposition 3.4.2 *The tangent cone at a to $\mathcal{Z}(a)$ satisfies $T_a\mathcal{Z}(a) = Z(a) \oplus T_aK$ for $K = \mathsf{K}(a)$.*

3.5 The polynomially convex hull

The *polynomial convex hull* of the orbit $K = \mathbf{K}(a)$ is defined as

$$\text{pch}(K) = \{z \in E : |f(z)| \le \sup_{z \in K} |f(z)| \text{ for all } f \in \mathbb{C}[z]\},$$

where $\mathbb{C}[z]$ are the complex polynomials in z. It has been shown in [KaupZait] that the polynomial convex hull pch(K) contains the set $\mathcal{Z}(a)$ and, in the classical case, they are equal. In the recent paper [Kaup] the result has been extended to the exceptional cases.

We know that any set to which we can uniquely extend each CR fruction defined on K must be contained in the polynomially convex hull.

3.6 Extensions theorems applied to bounded symmetric domains

Using the natation from [KaupZait] we write

$$Z := \overline{\mathbf{K}^{\mathbb{C}}(a)} \subset E \text{ and } X := \overline{\mathbf{S}^{\mathbb{C}}(a)} \subset E.$$

We write $\hat{Z}$ for the normalisation of Z that is homeomorphic to Z and biholomorphic outside 0. See [KaupZait] p.91.

The main result of this section is the following theorem, it is one of the main results of the paper [KaupZait].

Theorem 3.6.1 *If a is not invertible, every continuous CR-function on $K = S$ has a unique continuous extension to $\hat{\mathcal{Z}}(a)$ that is holomorphic in its interior with respect to $\hat{Z}$. If a is invertible, every continuous CR-function on K (resp. S) has a unique continuous extension to $\mathcal{Y}(a)$ (resp. $\mathcal{X}(a)$) that is holomorphic (resp. CR) in its interior with respect to Z (resp. X). Furthermore, the sets $\mathcal{Z}(a), \mathcal{Y}(a)$ and $\mathcal{X}(a)$ are maximal in the following sense. If $\mathcal{H}$ is any domain in $\hat{Z}$ (resp. Z or X) containing the interior of $\hat{\mathcal{Z}}(a)$ (resp. $\mathcal{Y}(a)$ or $\mathcal{X}(a)$) with the above extension property, then necessarily $\mathcal{H} \subset \hat{\mathcal{Z}}(a)$ (resp. $\mathcal{H} \subset \mathcal{Y}(a)$ or $\mathcal{H} \subset \mathcal{Z}(a)$).*

The proof follows from the next two propositions.

Proposition 3.6.2 *Suppose that E is a factor and $a \in E$ is not invertible. Then every continuous CR-function f on $K = \mathbf{K}(a)$ has a unique continuous extension to $\mathcal{Z}(a)$ that is holomorphic on $\mathcal{D}(a)\backslash\{0\}$.*

In proving the above proposition, we use a lemma mentioned in [KaupZait].

Lemma 3.6.3 *Let Y be a complex manifold and let $A \subset Y$ be a closed real-analytic submanifold of real codimension ≥ 2. Then every bounded holomorphic function on Y/A has a holomorphic extension to Y.*

The proof of this lemma is very similar to that of the extension theorem of normal spaces, theorem 2.2.1.

The following is the proof of proposition 3.6.2, see [KaupZait] sec 12 for further details.

Proof Fix a continuous CR-function f on $K = \mathbf{K}(a)$. Without loss of generality it is assumed that $a \neq 0$, i.e. $\rho := \mathrm{rank}(a) > 0$. We ignore the trivial case when

$a = 0$. For every $k \leq \rho$ denote by $\mathcal{D}_k \subset \mathcal{D}(a)$ the open subset of all elements $x \in \mathcal{D}(a)$ of rank ρ that have at least k pairwise different singular values $\neq 0$. Then $\mathcal{D}_\rho \subset \mathcal{D}_{\rho-1} \subset \cdots \subset \mathcal{D}_1 = \mathcal{D}(a) \cap E_{[\rho]}$ is the set of all reduced elements of rank ρ in $\mathcal{D}(a)$. Furthermore, for every $k \leq \rho$ the compliment $A_k := \mathcal{D}_{k-1} \backslash \mathcal{D}_k$ is a (not necessarily connected) real-analytic submanifold of $\mathcal{D}_{k-1}$. All K-orbits in $\mathcal{D}_\rho$ have the same dimension and all other K-orbits in $\mathcal{D}(a)$ have lower dimensions. In particular, A_k has codimension ≤ 2 in $\mathcal{D}_{k-1}$ for all $k \leq \rho$. We first prove that f extends holomorphically to $\mathcal{D}(a) \backslash \{0\}$.

We look at two cases, the case where a is reduced and the case where a is not reduced.

We start with the case that a is reduced, that is,

$$a = \sum_{j=1}^{\rho} \lambda_j e_j$$

for real coefficients $\lambda_1 > \ldots > \lambda_j > 0$.

Denote by $\mathcal{S}$ the set of all elements $x \in \sum_{j=1}^{\rho} x_j e_j \in \mathcal{D}(a)$ with $x_1 > \cdots > x_\rho > 0$. Then clearly $\mathcal{D}_\rho = \mathcal{K}(\mathcal{S})$.

For every $x \in \mathcal{S}$ define $\gamma : \mathbb{R} \to E$ for an open interval I containing $\{0, 1\}$, by

$$\gamma(t) := \sum_{j=1}^{\rho} \lambda_j^{1-t} x_j^t e_j.$$

Then $\gamma(0) = a$, $\gamma(1) = x$ and there is an open interval $I \subset \mathbb{R}$ with $0, 1 \in I$ and $\gamma(t) \in \mathcal{D}_\rho$ for every $t \in I$. In particular the orbits $\mathbf{K}(\gamma(t))$ all have the same dimension for $t \in I$ and there is a unique $\mathbf{K}$ invariant map $\phi : M \mapsto I$ with $\phi \circ \gamma|_I = \mathrm{id}_I$ for $M := \mathbf{K}(\phi(I))$. Also, there is a unique K-invariant smooth vector field ξ on the smooth submanifold $M \subset E_{[\rho]}$ with $\xi_{\gamma t} = \gamma'(t)$ for all $t \in I$. For $X := E_{[\rho]}$ it follows from the conditions of propositions 3.4.2 and 3.4.1 (page 57) are met and hence if we apply theorem 2.4.9 with $\xi_{\gamma(t)} = \gamma'(t)$ for all $t \in I$, it follows that $f \in \mathcal{C}(W \cup K) \cap \mathcal{O}(W)$, where W is an open set such that $\phi^{-1}(t) \subset W$ for all $0 < t \leq 1$. Since $x \in \mathcal{S}$ was arbitrary we get a holomorphic extension of f to $\mathcal{D}_\rho$, we also denoted it by f, and it is continuous up to K in the following *non-tangential* sense; the function is continuous in any wedge $W = (U \cap K) + B_\epsilon$, where U is any sufficently small neighbourhood of a point $b \in K, B < C_b$ where C_b is the Levi cone of K at b, and $\epsilon > 0$ also sufficently small. Since for every $c \in \mathbb{C} \backslash f(K)$ the CR-function $(f - c)^{-1}$ on K also has a holomorphic extension to $\mathcal{D}_\rho$, we have $f(\mathcal{D}_\rho) \subset f(K)$. In particular, f is bounded on $\mathcal{D}_\rho$.

Now suppose that for $k \leq \rho$ the function f has a holomorphic extension to $\mathcal{D}_k$. We know that $A_k := \mathcal{D}_{k-1}/\mathcal{D}_k$ is of dimension ≥ 2. Then applying lemma 3.6.3 (page 58), we see that f can be extended holomorphically to $\mathcal{D}_{k-1}$. Using induction down from $k = \rho$ we conclude that f extends holomorphically to $\mathcal{D}_1 = \mathcal{D}(a) \cap E_{[\rho]}$. Now since $\mathcal{D}_1$ has a complex analytic compliment in $\mathcal{D}(a)\backslash\{0\}$, we can apply theorem 2.2.2 (page 35) and we get f has a holomorphic extension to $\mathcal{D}(a)\backslash\{0\}$.

For the non-reduced case, we take a sequence of reduced elements who's limit is our chosen non-reduced element, we now show this.

Let a be non-reduced. As a consequence of Prop 3.4.2 (page 57) we can choose a cone $B < C_a$ and a sequence (a_n) of reduced points in $(a+B)\cap\mathcal{D}(a)$ with $\lim a_n = a$ and $\mathcal{D}(a_n) \subset \mathcal{D}(a_{n+1})$ for all n. Then it follows from lemma 2.4.8 (page 47) that there exists $n_0 \in \mathbb{N}$ and a K-invariant open subset V of $\mathcal{D}(a)$ with $a_n \in V$ for all $n \geq n_0$ such that f has a holomorphic extension to V. But then by case 1 the function extends holomorphically to $V \cup (\mathcal{D}(a_n)\backslash\{0\})$ for all $n \geq n_0$ and hence to $\mathcal{D}(a)\backslash\{0\} = \bigcup_{n\geq n_0} \mathcal{D}(a_n)\backslash\{0\}$. In any case, the normalisation of $\mathcal{D}(a)$ is homeomorphic to $\mathcal{D}(a)$ and as a consequence we get a holomorphic extension $f \in \mathcal{O}\left(\mathcal{D}(a)\backslash\{0\}\right)$ that is continuous up to K in the nontangential sense. For every $0 < t < 1$ the function f_t defined by $f_t(z) = f(tz)$ is holomorphic on a neighbourhood of $\mathcal{Z}(a)\backslash\{0\}$ in $\overline{E}_{[\rho]}$. Since $f = \lim_{t\nearrow 1} f_t$ is a uniform limit on K it is also uniform on $\mathcal{D}(a)$, that is, f extends from $\mathcal{D}(a) \cup K$ to a continuous function on $\mathcal{Z}(a) = \overline{\mathcal{D}(a)}$. ∎

Proposition 3.6.4 *Suppose that E is a factor and $a \in E$ is invertible. Then every continuous CR-function f on $S = \mathbf{S}(a)$ (resp. $K = \mathbf{K}(a)$) has a unique continuous extension to $\mathcal{X}(a)$ (resp. to $\mathcal{Y}(a) = \mathbb{T}(\mathcal{X}(a))$) that is holomorphic on $\mathcal{B}(a)$ (resp. CR on $\mathbb{T}(\mathcal{B}(a))$).*

This is similar to the last proposition for the holomorphic extension part, the proof of the continuous extension is more complicated. Proving the extension to $\overline{\mathcal{B}(a)}$ will require the use of *analytic annuli*, as the existence of a suitable family of holomorphic functions on $\mathcal{B}(a)$ converging uniformly to f is unclear.

An *analytic annulus* in $\mathbb{C}^n$ is a complex submanifold $R \subset \mathbb{C}^n$ such that there is an analytic annulus $A := \{\zeta \in \mathbb{C} : s < |\zeta| < t\}$ for $0 < s < t$ and a biholomorphic mapping $A \to R$ that extends to a homeomorphism $\overline{A} \to \overline{R}$ of the closures. It should be noted that unlike discs, different analytic annuli may not be biholomorphically equivalent. So we need to note the following lemma, whose proof can be found in [KaupZait], Lemma 12.10.

Lemma 3.6.5 *Let (R_n) be a sequence of analytic annuli in E converging to an analytic annulus $R \subset E$ in the following sense: There is a sequence (ϕ_n) of homeomorphisms $\phi_n : \overline{R} \to \overline{R}_n$ converging uniformly to the identity transformation on $\overline{R}$. Suppose that (f_n) is a sequence of functions $f_n \in \mathcal{C}(\overline{R_n}) \cap \mathcal{O}(R_n)$ such that the sequence $(f_n \circ \phi_n)$ converges uniformly on the boundary $\partial R := \overline{R} \backslash R$. Then $(f_n \circ \phi_n)$ converges uniformly on $\overline{R}$ to a function $f \in \mathcal{C}(\overline{R}) \cap \mathcal{O}(R)$.*

Now we will give the proof of proposition 3.6.4 which can also be found in [KaupZait].

Proof Without loss of generality, assume $\mathcal{B}(a) \neq 0$ which happens precisely when the orbit S is not totally real ($H_a S \neq 0$). For the extension to $\mathcal{B}(a)$, the main steps of the proof are similar to those of prop 3.6.2. Fix a continuous CR-function on f on S and put $r := \mathrm{rank}(a) = \mathrm{rank}(E)$ as well as $X := S^{\mathbb{C}}(a)$. For every $k \leq r$ denote by $\mathcal{B}_k \subset \mathcal{B}(a)$ the open subset of all elements $x \in \mathcal{B}(a)$ that have at least k pairwise different singular values. Again, in a first case suppose that a is reduced and define $\mathcal{S}$ with $\mathcal{B}_r = S(\mathcal{S})$ as in the proof of prop 3.6.2. Also, for every $x \in \mathcal{S}$ define $\gamma : \mathbb{R} \to E$ the same formula as prop 3.6.4. Then $\gamma(t) \in \mathcal{S}(t)$ for $0 \leq t < 1$ and as in the proof of prop 3.6.2 we conclude that f has a holomorphic extension to $\mathcal{B}(a)$ that is continuous up to S in the nontangential sense (as in prop 3.6.2). The same extension property follows in case a is not reduced (by using a suitable sequence (a_n) of reduced points in $\mathcal{B}(a)$ converging to a, as in the proof of prop 3.6.2.

The proof that the extension of f to $\mathcal{B}(a) \cup S$ can be extended continuously to the closure of $\overline{\mathcal{B}(a)} = \mathcal{X}(a)$ requires some more care. Here the final step of the proof of 3.6.2 cannot be carried out since the existance of a suitable family (f_t) of holomorphic functions on $\mathcal{B}(a)$ converging uniformly to f is not clear. The proof given here uses the fine stratification structure of the set $\mathcal{X}(a)$ and a construction of analytic annuli connecting different strata.

Put $N := \{1, 2, \ldots, r-1\}$. We define the sets

$$\mathcal{X}_I := \mathcal{X}_I(a) := \{z \in \mathcal{X}(a) : \mu_k(z) = \mu_k(a) \Leftrightarrow k \in I\},$$

where I runs over all subsets $I \subset N$ with $\mathcal{X}_i \neq 0$. For every integer $j \geq 1$ denote by $\mathcal{K}_j$ the union of all $\mathcal{X}_I$ with cardinality $\geq j$. Clearly $\mathcal{K}_{r-1} = S$ and $\mathcal{K}_1 = \partial\mathcal{B}(a)$. Denote by m the smallest integer such that the holomorphic extension $f \in \mathcal{O}(\mathcal{B}(a))$ is continuous up to $\mathcal{K}_m$ in the following non-tangential sence: For every $b \in \mathcal{K}_m$ the limit $\lim_n f(z_n)$ exists for every sequence (z_n) in $\mathcal{B}(a)$ converging to b and satisfying for some $\epsilon > 0$ and all n the inequality $\mathrm{dist}(z_n, \partial\mathcal{B}(a)) \geq \epsilon \mathrm{dist}(z_n, \mathcal{K}_m)$, where *dist* stands for the distance from a point to a subset with respect to a fixed norm on the vector space E. From the first part of the proof we have non-tangential extension of f to $S = \mathcal{K}_{r-1}$ where $m \leq r-1$.

We wish to show that $m = 1$. Assume the contrary that $m > 1$ holds and fix a point $b \in \mathcal{K}_{m-1} \subset \partial\mathcal{B}(a)$ in the following. Fix $q \in N$ in such a way that $\sigma_q(a) > \sigma_{q_1}(a)$ in case $b \in \mathcal{X}_I$. For every $z \in \mathcal{B}(a)$ we construct an analytic annulus $R(z) \subset \mathcal{B}(a)$ with $z \in R(z)$ in the following way: Write z as a linear combination $z = \sum_j \sigma_j(z)e_j$ for some frame $\mathbf{e} = (e_1, \ldots, e_r)$ and define $R(z)$ to be the choice of all complex linear combinations $\sum_j e_j$ in $\mathcal{B}(a)$ converging to b and satisfying $z_j = \sigma_j(z)$ for all $j \neq q, q+1$. Clearly, $R(z)$ depends on the choice of the frame for z. Consider a sequence (z_n) in $\mathcal{B}(a)$ converging to b and satisfying $\text{dist}(z_n, \partial\mathcal{B}(a)) \geq \epsilon \text{dist}(z_n, \mathcal{K}_{m-1})$ for some $\epsilon > 0$ and all n.

For every n there is a decomposition $z_n = \sum_j \sigma_j(z_n)e_j^n$ for some frame $\mathbf{e}^n = (e_1^n, \ldots, e_r^n)$ in E. Define with respect to this frame the analytic annulus $R(z_n) \subset \mathcal{B}(a)$ as above. The space of all frames in E is compact, therefore the sequence $((e)^n$ has a frame $\mathbf{e}$ as a point of accumulation. Let us assume for a while that $\mathbf{e}$ actually is a limit. Then the sequence $(R(z_n))$ converges to an analytic annulus R in the sence of 3.6.5. For this annulus $b \in \overline{R}$ as well as $\partial R \subset \mathcal{K}_m$ holds by the choice of the index q. By choosing smaller annuli $R_n \subset R(z_n)$ with $z_n \in R_n$ we can achieve that the sequence (R_n) also converges to the annulus R and hence on the frame $\mathbf{e}$.

Suppose that $(\tilde{z}_n)$ is another sequence converging to b as above such that the corresponding sequence of frames $\tilde{\mathbf{e}}^{\mathbf{n}}$ converges to b as above such that the corresponding sequence of frames $\tilde{\mathbf{e}}^{\mathbf{n}}$ converges to a frame $\tilde{\mathbf{e}}$ and hence gives a limit to the sequence $(f(\tilde{z}_n))$. We clain that the two limits coincide. Indeed, choose a sequence (w_n) in $\mathcal{B}(a)$ converging nontangnetially to b with respect to $\mathcal{K}_{m-1}$, where every w_n has the form $w_n = \sum_j w_{n,j}e_j$ for suitable coefficients $w_{n,j}$ satisfying $\sigma_j(b) = \lim_n w_{n,j}$ and $w_{n,j} = w_{n,k}$ if $\sigma_j(b) = \sigma_k(b)$. Then $\lim f(z_n) = \lim f(w_n)$ is clear since both limit frame agree. Now $b = \sum_j \sigma_j(b)e_j = \sum_j \sigma_j(b)\tilde{e}_j$ implies that every w_n also has the representation $w + n = \sum_j w_{n,j}\tilde{e}_j$, which implies $\lim f(\tilde{z}_n) = \lim f(w_n)$. As a consequence, $f(b) := \lim f(z_n)$ does not depend on the sequence (z_n). since $b \in \mathcal{K}_{m-1}$ was arbitrarily chosen, f has a continuous extension to $\mathcal{K}_{m-1}$ in the nontangential sense and therefore m is not minimal with respect to the property used for its definition, that is, $m = 1$, or equivalently, for every $b \in \partial\mathcal{B}(a)$ and every sequence (z_n) in $\mathcal{B}(a)$ converging to b the sequence $(f(z_n))$ converges. Since every convergent sequence in $\partial\mathcal{B}(a)$ can be approximated by a sequence in $\mathcal{B}(a)$ we derive that f has a continuous extension to $\overline{\mathcal{B}(a)} = \mathcal{X}(a)$, completing the proof fot the extension to $\mathcal{X}(a)$ of CR-functions on the orbit S.

Now we must prove the extension result for a continuous CR function on $K = \mathbf{K}(a) = \mathbb{T}(S)$. Since the orbit K is foliated by S-ofbits, every continuous CR-function on K extends to a function on $\mathcal{Y}(a)$ that is continuous on each subset $t(\mathcal{X}(a)) \subset \mathcal{Y}(a)$ with $t \in \mathbb{T}$ and holomorphic on its interior. Since the norm of the extension equals the norm of the function itself and $\mathbb{T}$ acts transitively on the set of all S-orbits in $\mathcal{Y}(a)$. Moreover, since the holomorphic tangent spaces of $\mathbb{T}(\mathcal{B}(a))$ coincide with those of $t(\mathcal{B}(a)), t \in \mathbb{T}$, the extension is also CR on $\mathbb{T}(\mathcal{B}(a))$. This completes the proof. ∎

To complete thm 3.6.1, we just note that the domains $\mathcal{Z}(a), \mathcal{Y}(a)$ and $\mathcal{X}(a)$ are in fact maximal in the sense mentioned earlier, we just note lemma 3.3.1.

Chapter 4

From the irreducible case to the reducible

In this section we will generalise from the irreducible to the reducible case, i.e. we will be dealing with bounded symmetric domains that may be the non-trivial direct product of bounded symmetric domains.

Our main concern will be in showing how the extension results of the last chapter generalise to the reducible case, i.e. when we have a (non trivial) product of factors.

4.1 The new product automorphism group

We now consider general (circled, but not necessarily irreducible) bounded symmetric domains. $\mathcal{D}$ will be the unit ball of the spectral norm of a hermitian Jordan triple system E, as before.

$$\mathrm{Aut}(E) := \{g \in \mathbf{GL}(E) : g\{xyz\} = \{(gx)(gy)(gz)\} \text{ for all } x, y, z \in E\}$$

We denote by $\mathbf{K}$ the connected identity component of $\mathrm{Aut}(E)$, as we did for an irreducible factor.

We know that E is a finite direct product of simple hermitian Jordan triple systems,

$$E = E^1 \oplus E^2 \oplus \ldots \oplus E^n,$$

where E^i, $i = 1, \ldots, n$ are irreducible factors. We will write $x \in E$ as follows

$$x = x_1 + x_2 + \ldots + x_n$$

where $x_i \in E^i$.

Theorem 4.1.1 *For E a hermitian Jordan triple system and write $E = E^1 \oplus E^2 \oplus \ldots \oplus E^n$ the decomposition of E as a sum of simple hermitian Jordan triple systems $E_i, 1 \leq i \leq n$. Replacing E by an isomorphic copy of itself we assume that E_i is isomorphic to E_j iff $E_i = E_j, (1 \leq i, j \leq n)$.*

Then $g \in Aut(E)$ if and only if there exists $g_i \in Aut(E^i), (1 \leq i, j \leq n)$ and a permutation σ of $\{1, 2, \cdots, n\}$ satisfying

1. *$E^{\sigma(i)} = E^i$ for all $i \in \{1, 2, \cdots, n\}$*
2. *$g(x_1 + x_2 + \cdots + x_n) = g_1(x_{\sigma(1)}) + g_2(x_{\sigma(2)}) + \cdots + g_n(x_{\sigma(n)})$*

We can summarise this as $Aut(E) = \left(\prod_{i=1}^{n} Aut(E^i)\right)\Gamma(E)$ where $\Gamma(E)$ consists of permutations of the factors (those $g \in Aut(E)$ with $g_i = id$ for $1 \leq i \leq n$).

Proof Choose $x \in E^i, y \in E^j, z \in E$. Presume $E^i \neq E^j$. We now choose x, y such that $\{xyz\} \neq 0$. Let $g_i(x) \in E^j, g_j(y) = y$ and $g(z) = z$, then $\{g(x), g(y), g(z)\} = 0$ for all $z \in E$. Hence $g\{xyz\} = 0$ for all $z \in E$ and hence $\{xyz\} = 0$ for all $z \in E$ leading to a contradiction.

The identity component of $\mathbf{K}$ of $\mathrm{Aut}(E)$ will not contain any permutations (apart from the identity) and so

$$\mathbf{K} = \mathbf{K}_1 \times \mathbf{K}_2 \times \cdots \times \mathbf{K}_n,$$

where $\mathbf{K}_i$ is the connected component of the identity in $\mathrm{Aut}(E_i)$. Then

$$[\mathbf{K}, \mathbf{K}] = [\mathbf{K}_1, \mathbf{K}_1] \times [\mathbf{K}_2, \mathbf{K}_2] \times \cdots \times [\mathbf{K}_n, \mathbf{K}_n]$$

and since $\mathbf{S} = [\mathbf{K}, \mathbf{K}]$, $\mathbf{S}_i = [\mathbf{K}_i, \mathbf{K}_i]$ we have

$$\mathbf{S} = \mathbf{S}_1 \times \mathbf{S}_2 \times \cdots \times \mathbf{S}_n.$$

We also consider the complexification $\mathbf{S}^{\mathbb{C}} = \mathbf{S}_1^{\mathbb{C}} \times \mathbf{S}_2^{\mathbb{C}} \times \cdots \times \mathbf{S}_n^{\mathbb{C}}$. We will continue to use this notation throughout the chapter.

4.2 The new Orbits

We define $\sigma_i^k(z) := \sigma_i(z_k)$ for $1 \leq k \leq r_i$ as the singular values of E, $z = z_1 + z_2 + \cdots + z_n$, $r_i = \mathrm{rank} E^i$. This is in analogy with the singular values as used in section

3.1 for irreducible components E_i of rank r_i. Similarity we define $\mu_i^k(z) := \mu_i(z_k)$ for $1 \leq k \leq r_i$. We know from the last chapter that

$$\mathbf{K}_i(a_i) = \{z \in E_i : \mu_k^i(z) = \mu_k^i(a) \text{ for } 1 \leq k \leq r_i\}$$

and

$$\mathbf{S}_i(a_i) = \{z \in \mathbf{K}_i(a_i) : N_i(z) = N_i(a)\} \text{ for}$$

where N_i is the a generic norm associated with the component E_i. We also use

$$\mathbf{K}(a_i) = \mathbf{K}_i(a_i)$$

and

$$\mathbf{S}(a_i) = \mathbf{S}_i(a_i).$$

4.3 Tangent spaces

We are concerned with the following proposition:

Proposition 4.3.1 *For every $a \in E$ the tangent spaces to the orbits $S = \mathbf{S}(a)$ and $K = \mathbf{K}(a)$ at a satisfy $T_aS \subset T_aK = iA(a) \oplus E_{1/2}$ and $H_aS = H_aK = E_{1/2}(a)$.*

This is similar to proposition 3.2.1, the irreducible version. The result follows from proposition 3.2.1 without difficulty which will be explained in the following. For $a \in E = E^1 \oplus \cdots \oplus E^n$, we have $E_\alpha(a) = E_\alpha^1(a_1) \oplus \cdots \oplus E_\alpha^n(a_n)$ for $\alpha = 0, 1/2, 1$, and $A(a) = A(a_1) \oplus \cdots \oplus A(a_n)$ given the decomposition $a = a_1 + \cdots + a_n$ with respect to the above decomposition of E.

The operators $\Phi_a = 2(L_a + Q_a), \Psi_a = 2(L_a + Q_a)$ and $\Theta_a = \Phi_a\Psi_a$ all become direct sums, $\Phi_a = \bigoplus_{i=1}^n \Phi_{a_i}$, $\Psi_a = \bigoplus_{i=1}^n \Psi_{a_i}$ and $\Theta_a = \bigoplus_{i=1}^n \Theta_{a_i}$. This can be verified by orthogonality of the factors $E_1, E_2, \cdots, E_n$ under Jordan multiplication. We have that $T_aK = T_{a_1}K_1 \times \cdots \times T_{a_n}K_n$, this also holds for T_aS and H_aK. So it follows from this that we have the above proposition for the *reducible* case.

4.4 The domains $\mathcal{Z}(a)$, $\mathcal{X}(a)$ and $\mathcal{Y}(a)$

Definition Let $i = 1, \ldots, n$ and $k = 1, \ldots, r_i$. Let $a \in E, a = a_1 + a_2 + \cdots + a_n$ with $a_i \in E_i$. Let

$$\mathcal{Z}(a) := \prod_{i=1}^n \mathcal{Z}(a_i).$$

Similarly we define

$$\mathcal{Y}(a) := \prod_{i=1}^{n} \mathcal{Y}(a_i).$$

and

$$\mathcal{X}(a) := \prod_{i=1}^{n} \mathcal{X}(a_i)$$

where $\mathcal{Z}(a_i), \mathcal{Y}(a_i)$ and $\mathcal{X}(a_i)$ are as defined in section 3.3.

Now for example we can write

$$\mathcal{Z}_i(a) = \mathcal{Z}(a_i) := \{z \in E_i : \mu_k^i(z) \leq \mu_k^i(a_i) \text{ for } 1 \leq k \leq r_i\}, r_i := \text{ rank}(E_i),$$

where $i = 1, \ldots, n$ and $k = 1, \ldots, r_i$. Let $a \in E, a = a_1 + a_2 + \cdots + a_n$ with $a_i \in E_i$.

Definition We also need to define the cone $Z(a)$,

$$Z(a) := \prod_{i=1}^{n} Z_i(a_i)$$

where $Z(a_i)$ as defined in section 3.4, we have

$$T_a\mathcal{Z}(a) = \prod_{i=1}^{n} T_{a_i}\mathcal{Z}_i(a_i) \text{ and } T_aK = \prod_{i=1}^{n} T_{a_i}K.$$

The next two propositions follow easily and are the generalisations of proposition 3.4.1 and proposition 3.4.2. The above remarks help to clarify them.

Proposition 4.4.1 *The Levi cone C_a at a is $X(a)$ if a is invertible and is $Z(a)$ if a is not invertible. In particular, C_a is always a closed cone.*

Proposition 4.4.2 *The tangent cone $T_a\mathcal{Z}(a) = Z(a) \oplus T_aK$ for $K = \mathbf{K}(a)$.*

4.5 The domains $\mathcal{D}(a)$ and $\mathcal{B}(a)$

We now look at $\mathcal{D}(a_i)$ and $\mathcal{B}(a_i)$ which where defined in 3.3.

Definition We have from the definition and the last chapter that $a \in E, a = a_1 + a_2 + \cdots + a_n$ with $a_i \in E_i, (1 \leq i \leq n)$ and put $\rho_i = \text{rank}(a_i)$. Let

$$\mathcal{D}(a_i) := \{z \in \overline{E}_{i[\rho_i]} : \mu_k^i(z) < \mu_k^i(a_i) \text{ for } 1 \leq k \leq \rho_i\}, \rho_i := \text{ rank } (a_i).$$

Then we define

$$\mathcal{D}(a) := \prod_{i=1}^{n} \mathcal{D}(a_i).$$

$$\mathcal{B}(a_i) := \{z \in \mathbf{S}_i^{\mathbb{C}}(a) : \mu_k^i(z) < \mu_k^i(a_i) \text{ for all } 1 \le k < r_i\}.$$

We define

$$\mathcal{B}(a) := \prod_{i=1}^{n} \mathcal{B}(a_i).$$

Theorem 4.5.1 *$\mathcal{D}(a)$ is Stein*

Proof It is known that the direct product of Stein spaces/manifolds is also Stein, so the result follows.

4.6 Hulls of orbits

Definition The *polynomial convex hull* of a compact set $K \subset \mathbb{C}^n$ is defined as follows,

$$\mathrm{pch}(K) = \{z \in E : |f(z)| \le \sup |f(K)| \text{ for all } f \in \mathbb{C}[z]\}.$$

We are concerned with showing the following,

Theorem 4.6.1 *Let $K_i \subset \mathbb{C}^{n_i}, 1 \le i \le m$ be nonempty compact subsets and $K = K_1 \times \cdots \times K_n \subset \mathbb{C}^n$. Then*

$$pch(K) = pch(K_1) \times \cdots \times pch(K_m),$$

Proof We will do this first for the case of $K = K_1 \times K_2$, i.e. when it is the product of two components. We will prove the inclusion in both directions, thus giving equality. First we prove that

$$\mathrm{pch}(K) \subset \mathrm{pch}(K_1) \times \cdots \times \mathrm{pch}(K_m),$$

Begin with $(a_1, a_2) \in \mathrm{pch}(K)$. We will show that $a_1 \in \mathrm{pch}(K)$. For this we start with $f(z_1)$ which is a polynomial in z_1. Now, put $p(z_1, z_2) = f(z_1)$. Now,

$$\sup_{z_1 \in K_1} |f(z_1)| = \sup_{(z_1, z_2) \in K} |p(z_1, z_2)|,$$

This is true since we are assuming $K_2 \neq \emptyset$. So $|f(a_1)| = |p(a_1, a_2)| \le \sup_{(z_1,z_2)\in K} |p(z_1, z_2)| = \sup_{z_1 \in K_1} |f(z_1)|$. This is true for all f in z_1 so $a_1 \in \mathrm{pch}(K_1)$. The proof is similar for $a_2 \in \mathrm{pch}(K_2)$, so the result follows.

Now the converse. We start with $(a_1, a_2) \in \mathrm{pch}(K_1) \times \mathrm{pch}(K_2)$. we want to show $(a_1, a_2) \in \mathrm{pch}(K)$. So we look at an arbitrary $p(z_1, z_2)$. We claim

$$|p(a_1, a_2)| \leq \sup_{(z_1,z_2)\in K} |p(z_1, z_2)|,$$

Fix $b_2 \in K_2$. Look at $f(z_1) = p(z_1, b_2)$, a polynomial in z_1. Then

$$\begin{aligned} \sup_{z_1\in K_1} |f(z_1)| &= \sup_{z_1\in K_1} |p(z_1, b_2)| \\ &\leq \sup_{(z_1,z_2)\in K} |p(z_1, z_2)| \end{aligned}$$

Therefore, since $a_1 \in \mathrm{pch}(K_1)$, we have

$$|p(a_1, b_2)| = |f(a_1)| \leq \sup_{(z_1,z_2)\in K} |p(z_1, z_2)|$$

and this is true for any $b_2 \in K_2$.

Now put $g(z_2) = p(a_1, z_2)$, this is a polynomial in z_2. so

$$|g(a_2)| \leq \sup_{z_2\in K_2} |g(z_2)| = \sup_{z_2\in K_2} |p(a_1, z_2)| \leq \sup_{z_1,z_2\in K} |p(z_1, z_2)|,$$

and we conclude

$$|p(a_1, a_2)| = |g(a_2)| \leq \sup_{(z_1,z_2)\in K} |p(z_1, z_2)|.$$

∎

Theorem 4.6.2 *The polynomial convex hull of K, pch(K), contains the set $\mathcal{Z}(a)$.*

Proof This is as consequence of the previous theorem, Lemma 11.3 of [KaupZait] and the fact $\mathcal{Z}(a) = \prod_{i=1}^{n} \mathcal{Z}_i(a)$.

From [KaupZait] it follows that for the classical cases we have $pch(\mathrm{K}) = \mathcal{Z}(a)$. This also follows for the exceptional cases (and therefore a mixture of both) from a recent paper published by *Wilhelm Kaup*, [Kaup].

4.7 Extension theorems

We need to generalise some of the notation for section 3.6. Using the natation from [KaupZait] we write

$$Z := \overline{\mathbf{K}^{\mathbb{C}}(a)} \subset E \text{ and } X := \overline{\mathbf{S}^{\mathbb{C}}(a)} \subset E.$$

We consider a (possibly) reducible product

$$E = E^1 \oplus \cdots \oplus E^n,$$

where each E^i is irreducible. For $a = a_1 + \cdots + a_n \in E$ we say a is strongly noninvertible if $a_i \in E^i$ is noninvertible for each $1 \leq i \leq n$. It is invertible if each a_i is invertible in its respective factor.

We write $\hat{Z}$ for the normalisation of Z. It is straight forward to show that $\hat{Z} = \prod_{i=1}^n \hat{Z}_i$. Where $Z_i = \overline{\mathbf{K}^{\mathbb{C}}(a_i)} \subset E$ and $X_i := \overline{\mathbf{S}^{\mathbb{C}}(a_i)} \subset E$. We restate theorem 3.6.1 from page 58, but keep in mind we may be dealing with reducible factors, theorem 3.6.1 referred only to the irreducible case.

Theorem 4.7.1 *If a is strongly noninvertible, every continuous CR-function on $K = S$ has a unique continuous extension to $\hat{\mathcal{Z}}(a)$ that is holomorphic in its interior with respect to $\hat{Z}$. If a is invertible, every continuous CR-function on K (resp. S) has a unique continuous extension to $\mathcal{Y}(a)$ (resp. $\mathcal{X}(a)$) that is holomorphic (resp. CR) in its interior with respect to Z (resp. X). Furthermore, the sets $\mathcal{Z}(a), \mathcal{Y}(a)$ and $\mathcal{X}(a)$ are maximal in the following sense. If $\mathcal{H}$ is any domain in $\hat{Z}$ (resp. Z or X) containing the interior of $\hat{\mathcal{Z}}(a)$ (resp. $\mathcal{Y}(a)$ or $\mathcal{X}(a)$) with the above extension property, then necessarily $\mathcal{H} \subset \hat{\mathcal{Z}}(a)$ (resp. $\mathcal{H} \subset \mathcal{Y}(a)$ or $\mathcal{H} \subset \mathcal{Z}(a)$).*

As with the reducible case, we break the theorem down into smaller parts, the majority being contained the the following two theorems.

Proposition 4.7.2 *Suppose that E is a product of factors and $a \in E$ is strongly noninvertible. Then every continuous CR-function f on $K = \mathbf{K}(a)$ has a unique continuous extension to $\mathcal{Z}(a)$ that is holomorphic on $\mathcal{O} := \prod_{i=1}^n (\mathcal{D}(a_i) \backslash \{0\})$.*

Proposition 4.7.3 *Suppose that E is a product of factors and $a \in E$ is invertible. Then every continuous CR-function f on $S = \mathbf{S}(a)$ (resp. $K = \mathbf{K}(a)$) has a unique continuous extension to $\mathcal{X}(a)$ (resp. to $\mathcal{Y}(a) = \mathbb{T}(\mathcal{X}(a))$) that is holomorphic on $\mathcal{B}(a)$ (resp. CR on $\mathbb{T}(\mathcal{B}(a))$).*

We assume we are looking at domains that may be reducible. We note that lemma 2.4.8 and lemma 2.4.9 (lemmas 12.6 and 12.7 of [KaupZait], the deformation type CR-extension theorem and the result that follows it, do not need to be generalised for the reducible case.

Now, the majority of Prop. 4.7.2 remains similar to the irreducible version of the proposition, proposition 3.6.2 on page 58. We write out the main steps as a list to show clearer what must be done.

- a is reduced with respect to the spectral decomposition (i.e. each a_i is reduced with respect to its spectral decomposition).

- For E_i we will use $\mathcal{D}_{ij} \subset \mathcal{D}_i(a)$ the open subset of all $x \in \mathcal{D}_i(a)$ of rank ρ_i that has at least j pairwise different singular values that are not equal to 0. So $\mathcal{D}_{ip} \subset \mathcal{D}_{iq}$ when $p \leq q$.

- We define a vector $\rho = (\rho_1, \rho_2, \cdots, \rho_n)$ and hence define $E_{[\rho]} = \prod_{i=1}^n E_{[\rho_i]}$. We also write $\mathcal{D}_\rho = \prod_{i=1}^n \mathcal{D}_{i\rho_i}$ where $\mathcal{D}_{i\rho_i} \subset \mathcal{D}(a_i)$ is the set of all $x \in \mathcal{D}(a_i)$ of rank ρ that have at least k pairwise different singular values not equal to 0. So $\mathcal{D}_p \subset \mathcal{D}_q$ when $p \leq q$. (by $p \leq q$ we mean $p_i \leq q_i$ for $1 \leq i \leq n$.)

- $\mathcal{S}_i$ is defined to be the set of elements

$$x_i = \sum_{j=1}^{\rho_i} x_{ij} e_{ij} \in \mathcal{D}_i(a_i)$$

  with $x_{i1} > \cdots > x_{i\rho_i} > 0$. Then $\mathcal{D}_i(\rho_i) = \mathbf{K}(\mathcal{S}_i)$ holds. So we define

$$\mathcal{S} := \mathcal{S}_1 \times \cdots \times \mathcal{S}_n.$$

- Similarly, for each $x \in \mathcal{S}$ we define $\gamma : \mathbb{R} \to E$ as

$$\gamma(t) := \gamma_1(t) + \cdots + \gamma_n(t),$$

  where $\gamma_i(t) : \mathbb{R} \to E$ is defined for the irreducible case as follows

$$\gamma_i(t) := \sum_{j=1}^{\rho_i} \lambda_{ij}^{1-t} x_{ij}^t e_{ij}.$$

  Which obviously has $\gamma_i(0) = a_i, \gamma_i(1) = x_i$.

- Since the conditions of propositions 4.4.1 and 4.4.2 are satisfied - the generalised versions of 3.4.1 and 3.4.2 (page 57) - we can apply the extension result presented in proposition 2.4.9 (page 48) since the conditions for this theorem to be applied are therefore satisfied and, similar to the proof of the irreducible version (prop 3.6.2), we get holomorphic extension up to $\mathcal{D}_\rho$ in the same non-tangential sense as the irreducible version of the theorem. We now show this.

  - We start with $\mathcal{D}_\rho = \prod_{i=1}^n \mathcal{D}_{i\rho_i}$ and we extend to $\prod_{i=1} \mathcal{D}_{i1}$ using an inductive argument involving lemma 3.6.3. Lemma 3.6.3 allows us to extend from $\mathcal{D}_\rho$ to $\prod_{i=1} \mathcal{D}_{i1}$ when used with an obvious inductive argument that is very similar to the irreducible case.

- We know $\mathcal{D}(a)_1 = \mathcal{D}(a) \cap E_{[\rho]}$ and that $\prod_{i=1}^{n}(\mathcal{D}_i(a)\backslash\{0\})$ is normal since $\mathcal{D}_i(a)\backslash\{0\}$ for $1 \leq i \leq n$. Since $\mathcal{D}_1$ (where the vector $1 = (1, \cdots, 1)$) has a complex-analytic complement in $\prod_{i=1}^{n}(\mathcal{D}_i(a)\backslash\{0\})$ we get that f has a holomorphic extension to $\prod_{i=1}^{n}(\mathcal{D}_i(a)\backslash\{0\})$. This follows from theorem 2.2.2 which deals with holomorphic extensions from compliments of varieties to varieties under conditions.
- The proof for a, when a is not necessarily reduced with respect to its spectral norm follows by the same method as the irreducible case.
- The continuous extension to the boundary follows, also, just like the irreducible case.

We now want to look at prop 4.7.3 which is the generalisation of prop 3.6.4 from page 60. We start by looking at a continuous CR function defined on $\mathbf{S}(a)$, when this is complete we look at $\mathbf{K}(a)$. The holomorphic extension part of the proof is very similar to the previous proposition, so we leave out the details and look at the continuous extension part, first for $\mathbf{S}(a)$.

- We define
$$N := \{N_1, \cdots, N_n\},$$
where $N_i := \{1, 2, \ldots, r_i - 1\}$. We will use
$$\mathcal{X}_I := \prod_{i=1}^{n} \mathcal{X}_{I_i}$$
where $I := (I_1, \ldots, I_n)$ and where $\mathcal{X}_{I_i}$ corresponds to E_i and is defined as
$$\mathcal{X}_{I_i} = \mathcal{X}_{I_i}(a) = \{z \in \mathcal{X}(a_i) : \mu_k(z) = \mu_k(a) \iff k \in I_i\}.$$
- We will use the vector m as follows, $m = (m_1, \cdots, m_n)$ and we will define $m < w$ to mean $m_i < w_i$ for $i = 1, \ldots, n$. Similarly we define $>$.
- So with this in mind we can write $\mathcal{K}_m = \mathcal{K}_{m_1} \times \cdots \times \mathcal{K}_{m_n}$ where $\mathcal{K}_i$ is the union of all $\mathcal{X}_{I_i}$ with I_i of cardinality $\geq i$.
- What remains is the proof of the continuous extension up to $\overline{\mathcal{B}(a)}$.

- We trivially have an extension up to $S = \mathcal{K}_{r-1}$ where $r = (r_1, r_2, \cdots, r_n)$ and $r - 1 = (r_1 - 1, r_2 - 1, \cdots, r_n - 1)$.
- We want to show that we can extend our function to $\overline{\mathcal{B}(a)} = \mathcal{K}_1$. We will presume we have continuous extension to $\mathcal{K}_m > 1$ but not $\mathcal{K}_{m-1}$ and then show that this leads to contradiction, implying $m = 1$.
  - Lemma 3.6.5 (page 60) concerning analytic discs is used to get the extension to $\mathcal{K}_{I_{m-1}}$. $\overline{\partial\mathcal{B}(a)} = \mathcal{K}_1$ so we have the continuous extension to $\overline{\mathcal{B}(a)}$ as we require. We go about this as follows.
    * We need $b \in \mathcal{K}_{m-1}$ as follows.
    * Fix $q \in N$ satisfying $\sigma_q(a) > \sigma_{q+1}(a)$ in case $b \in \mathcal{K}_m$ and $q \notin I$ in case $b \notin \mathcal{K}_m$.
    * Now we can apply Lemma 3.6.5 since we have a suitable family of analytic discs, this does not depend on choice of frame as shown in [KaupZait].
    * This gives us extension to $\mathcal{K}_{m-1}$ and we have a contradiction, implying that we have continuous extension to $\partial\mathcal{B}(a) = \mathcal{K}_{r-1}$ as we required, and it follows by induction hypothesis that we have extension up to $\mathcal{K}_{r-1} = \partial\mathcal{B}(a)$.
- We also need to look at the case of a CR function defined on the orbit $K = \mathbb{T}S$, where our function will extend continuously to $\mathcal{Y}(a) = \mathbb{T}\mathcal{X}(a)$ and extends holomorphically to $\mathbb{T}\mathcal{B}(a)$. The proof for this part follows form the method of the irreducible case.

4.7.1 Another method

In this section we will attempt a more intuitive and general method. The last method did not manage to generalise the results to the case of arbitrary products of bounded symmetric domains. We have only worked on $a = a_1 + \cdots + a_n$ that where invertible (i.e. each a_i is invertible in its respective E_i or products of non-invertible elements) We wish to prove the result for a combination. We will do this by using the results from the end of chapter 3, prop 3.6.2 (page 58) and prop 3.6.4 (page 60).

Before we start the main proposition it will prove useful to define the following domains, we choose a fixed a,

$$\mathcal{U}(a) = \prod_{i=0}^{n} \mathcal{U}_i(a) \text{ where } \mathcal{U}_i(a) = \mathcal{Z}_i(a) \text{ if } a_i \text{ is not invertible and}$$

$$\mathcal{U}_i(a) = \mathcal{Y}_i(a) \text{ if } a_i \text{ is invertible, also}$$

$$\mathcal{T}(a) = \prod_{i=0}^{n} \mathcal{T}_i(a) \text{ where } \mathcal{T}_i(a) = \mathcal{Z}_i(a) \text{ if } a_i \text{ is not invertible and}$$

$$\mathcal{T}_i(a) = \mathcal{X}_i(a) \text{ if } a_i \text{ is invertible.}$$

Note that $\mathcal{U}_i(a) = \mathcal{T}_i(a)$ in the case that a_i is not invertible. The domains $\mathcal{U}(a)$ and $\mathcal{T}(a)$ will be useful for the study of the CR extensions relating to the orbits K and S respectively. We note that their normalisations are

$$\hat{\mathcal{U}}_i(a) = \hat{\mathcal{Z}}_i(a) \text{ if } a_i \text{ is invertible,}$$

$$\hat{\mathcal{U}}_i(a) = \hat{\mathcal{Y}}_i(a) = \mathcal{Y}_i(a) \text{ if } a_i \text{ is not invertible,}$$

$$\hat{\mathcal{T}}_i(a) = \hat{\mathcal{Z}}_i(a) \text{ if } a_i \text{ is invertible,}$$

$$\hat{\mathcal{T}}_i(a) = \hat{\mathcal{X}}_i(a) = \mathcal{X}_i(a), \text{ in the case that } a_i \text{ is not invertible.}$$

It should also be noted that a function that is CR and continuous on an open set of $\mathbb{C}^n$ can also be said to be holomorphic on that set. With this notation in mind we now state our main proposition.

Theorem 4.7.4 *For a point $a \in E$, we take domains $\mathcal{D}_i(a)$ and $\mathcal{B}_i(a)$. We look at the following boundary sets, which we denote by $\tilde{D}_i$, with $\tilde{D}_i = K_i(a)$ ($\tilde{D}_i = S_i(a)$).*

Any CR function defined on $\tilde{D} = \prod_{i=1}^{n} \tilde{D}_i$ can be extended to a CR function on the domain $\hat{D} = \prod_{i=1}^{n} \hat{D}_i$. In the case $\tilde{D} = K(a)$, $\hat{D}_i = \hat{\mathcal{D}}_i(a)$ if a_i is noninvertible and $\hat{D}_i = \mathbb{T}\hat{\mathcal{B}}(a)$ if a_i is invertible. In the other case where $\tilde{D} = S(a)$, $\hat{D}_i = \hat{\mathcal{D}}_i(a)$ when a_i is noninvertible and $\hat{D}_i = \hat{\mathcal{B}}_i(a)$ when a_i is invertible.

Proof We break the proof into several parts.

1. We will prove the theorem for the case of the product of 2 domains, the result for an arbitrary product of n domains then follows by induction.

   We are going to deal with S orbits first. We will generalise to K orbits in a later step by using the fact that the K orbits can be foliated by S orbits.

   For the non-invertible a_i we will let $F_i = F_i(a) = \mathcal{D}_i(a)$, $\tilde{F}_i = \tilde{F}_i(a) = S(a) = K(a)$ and for invertible a_i we let $F_i(a) = F_i = \mathcal{B}_i(a)$ and $\tilde{F}_i(a) = \tilde{F}_i = S(a)$. If f is a function defined on the $S(a)$ orbit then we just continue, otherwise if the function is defined on the $K(a)$, then we will restrict it (to a function on the $S(a)$ orbit) as follows. We take a function f defined on the boundary $\tilde{D}_1 \times \tilde{D}_2$.

We first restrict this function to $\tilde{F}_1 \times \tilde{F}_2$. Now we are dealing with the case of $S(a)$ orbits.

We restrict this function to a point b_2 on F_2 giving us a function on $\tilde{F}_1$, which we call f_{b_2}. We will choose our points b_1 and b_2 to be reduced with respect to their individual factors.

2. We apply extension theory to our restricted function f_{b_2}, with $b_2 \in \tilde{F}_2$ to get an extension of f_{b_2} to $\hat{F}_1$. For each $b_2 \in F_2$ we now have an extension of f_{b_2} in $\hat{F}_1 \times \tilde{F}_2$ that is holomorphic on $\hat{F}$.

3. For f defined on $\hat{F}_1 \times \tilde{F}_2$, we show that for any fixed $b_1 \in \mathcal{D}_{\rho_1}$,we have the function f_{b_1} is

   (a) continuous and

   (b) CR

   on the domain $\tilde{F}_2$. This leaves us with a function that is CR and continuous on $\mathcal{D}_{\rho_i} \times \tilde{F}_2$ with respect to $\tilde{F}_2$. We will define $\mathcal{D}_\rho$ again later. We have used it in the proof of Prop. 3.6.2 and Prop. 4.7.2.

4. Because of the last step, for a fixed $b_1 \in \mathcal{D}_{\rho_1}$, we apply extension theory to the function f_{b_1} defined on $\tilde{F}_2$ to give us an extension to $\mathcal{D}_{\rho_2}$.

5. We show that we now have a function f that is defined on $\mathcal{D}_{\rho_1} \times \mathcal{D}_{\rho_2}$. We must show that thos function is holomorphic

6. We note how to extend this result by induction to the case of an arbitrary product of n irreducible factors.

7. Now that we have holomorphic extension to the set $\prod_{i-1}^n \mathcal{D}_{\rho_i}$, we will show how we can further extend to a function which is holomorphic on $\prod_{i=1}^n \hat{F}_i$

8. We have so far chosen b_i to be reduced points. Now we look at the case when they might not be reduced.

   The proof follows in a similar manner to Thm. 3.6.2

9. We now extend the result to K orbits

10. We show the uniqueness of the extension and the maximality of the domain we have extended to.

Part 1

First we take a CR-function f which is defined on $\tilde{D}_1 \times \tilde{D}_2$. We restrict this function to one defined on $\tilde{F}_1 \times \tilde{F}_2$. We will choose an arbitrary point $b = b_1 + b_2$ on the boundary set $\tilde{F}_1 \times \tilde{F}_2$. We define two CR functions as follows, $f_{b_2}(x) = f(x, b_2), x \in \tilde{F}_1$ and $f_{b_1}(y) = f(b_1, y), y \in \tilde{F}_2$. So these functions are the restrictions of f to $\tilde{F}_1$ and $\tilde{F}_2$ respectively relative to the point $b \in \tilde{F}$.

Part 2

We can extend f_{b_2} to a larger domain in the following fashion, f_{b_2} extends as a function that is holomorphic on $\hat{F}_1$. We still denote this function by f_{b_2} since the extension is unique. The element $b_2 \in \tilde{F}_2$ was chosen arbitrarily so it follows that we have an extension of the function f to $\hat{F}_1 \times \tilde{F}_2$ that is holomorphic on $\hat{F}_1$.

Part 3

We wish to show that f_{b_1} is CR and continuous on $\tilde{F}_2$. We already know that f_{b_2} is holomorphic on $\hat{F}_1$ with respect to $\hat{Z}_1$ (and continuous up to F_1 in each slice $F_1 \times b_2$ for all $b \in \hat{F}_1 \times \tilde{F}_2$).

Part 3 (a)

We will now show that f is continuous on $\mathcal{D}_{\rho_1} \times \tilde{F}_2$. Recall the definition of $\mathcal{D}_{\rho_i}$ from p.59 (and p.70 for the reducible case). For $b_2 \in \tilde{F}_2$, we know that the function f_{b_2} is continuous on F_1 up to the boundary $\mathcal{U}_1$. So all that remains to be shown is that the function is continuous with respect to $b_2 \in \tilde{F}_2$.

We will use a proof by contradiction. We assume that for a fixed $b_1 \in \mathcal{D}_{\rho_1}$ that the function f_{b_1} is not continuous for some $b_2 \in \tilde{F}_2$. We show below that this will imply that the function is not continuous at some point in the boundary $\tilde{F}_1$. This implies a contradiction since we have assumed the function to be continuous on $\tilde{F}_1$. This will lead to the conclusion that the function is continuous at $b_2 \in \tilde{F}_2$.

For a fixed $b_2 \in \tilde{F}_2$, we know that for any $b_1 \in \mathcal{D}_{\rho_1}$, that the value of the function $f_{b_2}(b_1)$ can be found by integrating around a closed path on the boundary set $\tilde{F}_1$. This is true since we used analytic disks to prove the extension to these values in our CR extension theorem for manifolds Thm. 2.4.2. Thm. 2.4.8 relys on Thm. 2.4.2 in a manner that we can see the existence of the analytic disks also. So does Thm. 2.4.9 since it used the Thm. 2.4.8 in a manner where we are still relying on the analytic disks.

We now look at analytic disks. (We have already seen analytic disks in our section on Bishops equation.) If we have a continuous function $u : S^1 \to \mathbb{R}^d$ defined on the boundary of the unit complex disk, we can extend this to a function $w = u + iv$ which is holomorphic on the interior of the unit complex disk, v is the harmonic conjugate of u. The function can be written as follows using Poisson integrals

$$w(z_0) = \frac{1}{2\pi} \int_0^{2\pi} w(e^{i\theta}) \frac{1 - |z_0|^2}{|z_0 - e^{i\theta}|^2} d\theta.$$

We have analytic disks defined on $\mathcal{D}_{\rho_i} \subset F_1$ as follows: For a specific $b_1 \in F_1$, A_{b_1} is a holomorphic mapping from the unit complex disk to F_1 in a manner such that the boundary of the complex disk is mapped into the boundary $\tilde{F}_1$ and we have $A_{b_1}(0)$ maps to b_1.

We continue keeping $b_1 \in \mathcal{D}_{\rho_1}(a)$ fixed. So the function will take the following form

$$f_{b_1}(b_2) = f(b_1, b_2) = f(A_{b_1}(z_0), b_2) = \frac{1}{2\pi} \int_0^{2\pi} f_{b_2}(A_{b_1}(e^{i\theta})) \frac{1 - |z_0|^2}{|z_0 - e^{i\theta}|^2} d\theta$$

for any $b_2 \in \mathcal{D}_{\rho_i}$. We have chosen to integrate around a closed path on the boundary of $\tilde{F}_1$.

The path of integration is continuous. The function f_{b_2} is continuous along this path. The analytic disk mapping A_{b_2} is continuous. Finally, the remaining function in the integral is continuous on the range of integration. If the function was discontinuous at the value $f_{b_1}(b_2)$, the it would imply the integral must have a discontinuity, but as we have just discussed, this cannot be the case.

We will later prove continuity of the function on the remaining values in the set F_1. We now know that the function is continuous on each slice $F_1 \times a_2$ and that it is continuous on the set $\mathcal{D}_{\rho_1}$.

Part 3 (b)

First we will prove that f_{b_1} is CR at a fixed $b_2 \in \tilde{F}_2$ for some $b_1 \in F_1$. If we show that this is CR on $\tilde{F}_2$ (with respect to $\tilde{F}_2$ for fixed $b_2 \in \tilde{F}_2$) then the result follows since b_2 was chosen as a arbitrary element in $\tilde{F}_2$.

Take the function $g_{b_2} = \overline{\partial}_2 f_{b_2}$. as stated earlier, b_2 is fixed, so we don't have to worry about $\overline{\partial}_2$ working on different tangent tangent spaces, it specifically works on the tangent space at the point b_2 restricted to $\tilde{F}_2$. We know this function is defined on the set $\tilde{F}_1$. We also know that $g_{b_2} = 0$ identically on the boundary set $\tilde{F}_1$.

We note that the $\overline{\partial}_2$ operator and the integration operator applied to f_{b_2} commute since they are independent. This can be seen more clearly since b_2, and the integral is being evaluated over values in $\tilde{F}_1$, and the operator $\overline{\partial}$ is independent from $\tilde{F}_1$.

$$\begin{aligned}
\partial_2 f(A_{b_1}(z_0), b_2) &= \partial_2 \frac{1}{2\pi}\int_0^{2\pi}\left(f(A_{b_1}(e^{i\theta}), b_2)\frac{1-|z_0|^2}{|z_0-e^{i\theta}|^2}\right)d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\partial_2\left(f(A_{b_1}(e^{i\theta}), b_2)\frac{1-|z_0|^2}{|z_0-e^{i\theta}|^2}\right)d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} 0 d\theta \\
&= 0
\end{aligned}$$

We can move ∂_2 inside the integral in the above equation since the variables we are applying the ∂_2 to are independent from the values the integral is being evaluated over.

$\tilde{F}_1$ is the Shilov boundary of F_1, so by the maximum modulus theorem and the conditions of our theorem that all CR functions on $\tilde{F}_1$ extend to functions on F_1, we have g_{b_2} extends to F_1 as the trivial function $g_{b_2} = 0$. So now we have $g_{b_2} = \overline{\partial}_2 f_{b_2} = 0$ on F_1. This implies that the function f_{b_2} is CR (with respect to $\tilde{F}_2$) at the fixed point b_2 for any choice of $b_1 \in \mathcal{D}_{\rho_1}$ and hence this means the function f_{b_2} is CR on the set $\tilde{F}_2$ at the point b_2. Finally, since our choice of b_2 was arbitrary we now have that f (defined on $D_1 \times \tilde{F}_2$) is CR with respect to $\tilde{F}_2$.

Part 4

Now we return to look at the function f which is defined on the domain $\mathcal{D}_{\rho_1}(a) \times F_2$ as described earlier. We choose a restriction to some point b_1 in $\mathcal{D}_{\rho_1}$ and call this function f_{b_1}. From part 3, we know this function is CR and continuous on $\tilde{F}_2$, so it satisfies the conditions of our extension theorem, Thm. 3.6.1 and hence we can extend it to a function which is holomorphic on F_2 and continuous up to the boundary of F_2, i.e. $\mathcal{D}_{\rho_2}(a)$, we continue to notate it by f_{b_1} since we such an extension to be unique.

Part 5

We now have a function f that is defined on $\mathcal{D}_{\rho_1} \times \mathcal{D}_{\rho_2}$. We look at the function f defined on $\mathcal{D}_{\rho_1} \times \mathcal{D}_{\rho_2}$. We must show for some $b_2 \in \mathcal{D}_{\rho_2}$ that this function is holomorphic on the domain $\mathcal{D}_{\rho_1}$. This follows by similar methods to Part 3(b).

$$\begin{aligned}
\partial_{b_1} f(b_1, A_{b_2}(z_0)) &= \partial_{b_1} \frac{1}{2\pi} \int_0^{2\pi} \left(f(b_1, A_{b_2}(e^{i\theta})) \frac{1-|z_0|^2}{|z_0 - e^{i\theta}|^2} \right) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \partial_{b_1} \left(f(b_1, A_{b_2}(e^{i\theta})) \frac{1-|z_0|^2}{|z_0 - e^{i\theta}|^2} \right) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} 0 d\theta \\
&= 0
\end{aligned}$$

We note that in the above equation, the operator ∂_{b_1} is the usual complex derivative operator acting on a subset of $\mathbb{C}^m$ that is biholomorphic to neighbourhood of b_1 in the set $\mathcal{D}_{\rho_1}$. We explain more below how the equation follows.

Use an operator that maps the (locally a point back to $\mathbb{C}^m$) and then apply ∂_{b_1} as it is defined on $\mathbb{C}^m$. We call the operator Δ. So in our case ∂_{b_1} is a function on a subset of $\mathbb{C}^m$.

$$\partial_{b_1} f(b_1, b_2) := \partial_{b_1} f(\Delta(x_1), b_2) := \partial_{b_1} f_{b_2}(\Delta(x_1))$$

where $\Delta(x_1) = b_1$.

We know that the function is holomorphic with respect with respect to $\mathcal{D}_{\rho_1}$ on the boundary set $\tilde{D}_2$ we are integrating with respect to. Also we know that this range of integration is independent from the variables that ∂_{b_1} is diffentiating with respect to.

We now know that this function is holomorphic on $\mathcal{D}_{\rho_1} \times \mathcal{D}_{\rho_2}$. This follows since we know the restrictions f_{b_1} to $b_1 \times \mathcal{D}_{\rho_2}$ for $b_1 \in \mathcal{D}_{\rho_1}$ and f_{b_2} to $\mathcal{D}_{\rho_1} \times b_2$ for $b_2 \in \mathcal{D}_{\rho_2}$ are holomorphic.

Part 6

We have only worked so for with the product to two factors. To continue, we must now generalise the result for the product of n factors. To do this we will use the $\mathcal{D}_\rho$, for the vector ρ. We note that all the results we have proven so far will continue to be valid with its usage.

Part 7

Here we will use similar result to last section to show extensions from $\mathcal{D}_\rho$ to $\prod_{i=1}^n \mathcal{D}_{i1}$. Then we note that $\prod_{i=1}^n \mathcal{D}_{i1}$ has an analytic compliment in F. With this we can apply a previous theorem.

We will use a method from Prop. 4.7.2 (page. 70). We look at the vector $\rho = (\rho_1, \cdots, \rho_2)$ and hence define $E_{[\rho]} = \prod_{i-1}^{n} E_{[\rho_i]}$. We also write $\mathcal{D}_{1\rho} = \prod_{i=1}^{n} \mathcal{D}_{1\rho_i}$ where $\mathcal{D}_{i\rho_i} \subset \mathcal{D}(a_i)$ is the set of all $x \in \mathcal{D}(a_i)$ of rank ρ that have at least k pairwise different singular values not equal to 0. So $\mathcal{D}_p \subset \mathcal{D}_q$ when $p \leq q$. (by $p \leq q$ we mean $p_i \leq q_i$ for $1 \leq i \leq n$.)

Since the conditions of propositions 4.4.1 and 4.4.2 are satisfied - the generalised versions of 3.4.1 and 3.4.2 (page 57) - we can apply the extension result presented in proposition 2.4.9 (page 48) since the conditions for this theorem to be applied are therefore satisfied and, similar to the proof of the irreducible version (prop 3.6.2), we get holomorphic extension up to $\mathcal{D}_\rho$ in the same non-tangential sense as the irreducible version of the theorem.

We start with $\mathcal{D}_\rho = \prod_{i=1}^{n} \mathcal{D}_{1\rho_i}$ and we extend to $\prod_{i=1}^{n} \mathcal{D}_{i1}$ using an inductive argument based on (the Hartogs type result) lemma 3.6.3. Lemma 3.6.3 allows us to extend from $\mathcal{D}_\rho$ to $\prod_{i=1}^{n} \mathcal{D}_{i1}$ when used with an inductive argument that is very similar to the irreducible case. We know $\mathcal{D}(a)_1 = \mathcal{D}(a) \cap E_{[\rho]}$. Since $\mathcal{D}_1$ (where the vector $1 = (1, \cdots, 1)$) has a complex-analytic complement in $\prod_{i=1}^{n} \hat{F}_i$ we get that f has a holomorphic extension to $\prod_{i=1}^{n}(F \backslash \{0\})$. This follows from theorem 2.2.2 which deals with holomorphic extensions from compliments of varieties to varieties under conditions.

Part 8

We have so far chosen b_i for $i = 1, \cdots, n$ to be reduced points (with respect to their spectral norm). Now we look at the case when they might not be reduced.

The proof follows in a similar manner to Thm. 3.6.2. Take a point a and presume at least one of our $a_i, i = 1, \cdots, n$ is not reduced (otherwise we have nothing to prove). We know we can choose a closed cone $B < C_a$ by Prop. 4.4.1 and Prop. 3.4.1. We can therefore choose a sequence of reduced points (a^i) in $(a + B) \cap \hat{F}(a)$ with $\lim a^i = a$ and $\hat{F}(a^i) \subset \hat{F}(a^{i+1})$ for all i. It then follows from Thm. 2.4.8 that there exists $i_0 \in \mathbb{N}$ and an S invariant open subset V of $\hat{F}(b)$ with $b^i \in V$ for all $i \geq i_0$ such that f has a holomorphic extension to V. But then from the last parts of the proof, since a^i is not reducible, the function f extends holomorphically to $V \cup \left(\hat{F}(a^i) \backslash \{0\}\right)$ for all $i \geq i_0$ and hence to $\hat{F} \backslash \{0\} = \bigcup_{i \geq i_0} \hat{F}(a^i) \backslash \{0\}$ that is continuous up to $S(a)$ in the nontangential sense discussed in Thm. 3.6.2.

We know want to show we have continuity up to the boundary for these elements. fix b_2. We look at Prop. 4.7.2 and Prop. 4.7.3.

Part 9

We have shown the result for S orbits and K_i orbits for invertible a_i elements (since $K_i(a) = S_i(a)$ in these cases). We now need to show that our generalisation also works for any K orbit. We note the final part of Prop. 3.6.4 to show that this is the case as our result follows similarly. This will give us the extension to the set $\mathcal{Y}_i(a) = \mathbb{T}\mathcal{X}_i(a)$ as required. This goes as follows.

For an invertible a_i, since the orbit $K_i(a) = \mathbb{T}(S_i(a))$ is foliated by S orbits, every continuous CR-function on $K_i(a)$ extends to a function on $\mathcal{Y}_i(a)$ that is continuous on each subset $t(\mathcal{X}_i(a)) \subset \mathcal{Y}_i(a)$ with $t \in \mathbb{T}$ and holomorphic on its interior. Since the norm of the extension equals the norm of the function itself and $\mathbb{T}$ acts transitively on the set of all S_i orbits in $\mathcal{Y}_i(a)$, the extension is continuous on $\mathcal{Y}_i(a)$. We are more interested in the following, since the holomorphic tangent spaces of $\mathbb{T}(\mathcal{B}_i(a)))$ coincide with those of $t(\mathcal{B}_i(a)), t \in \mathbb{T}$, the extension is also CR on $\mathbb{T}(\mathcal{B}_i(a))$.

If we are looking at our function f, the above result applies to a restriction of this function to the orbit relating ot the *ith* factor. For the entire function will become CR on the required set because of the same reasoning mentioned above, the holomorphic tangent spaces will of $\mathbb{T}_i(\mathcal{B}(a))$ coincide with those of $t_i(\mathcal{B}(a))$ for $t_i \in \mathbb{T}_i$. We are using $\mathbb{T}_i$ as the restriction of the of $\mathbb{T}$ to the *ith* factor.

Part 10

To complete the theorem we must show that the extension is unique and the set we have extended to is maximal.

The maximality of the extension follows from the maximality of the extension to the restriction to either D_1 or D_2. If the function was able to extend to a larger domain, then one of the restrictions would be able to extend to a larger domain, and since we have that the set we extended to already, this would lead to a contradiction. So we have that our extension is maximal. ∎

We have not shown if the function f is continuous up to the boundary. By observation of the previous theorem, we can get continuous extension up to sections of the boundary, but at present I cannot find an extension to the complete boundary.

One method that might work is to use an argument similar to the method used in the sketched proof of Thm. 4.7.3. This method can be modified to include $K_i(a)$ orbits too.

4.8 Conclusions

We have extended results from irreducible bounded symmetric domains to reducible bounded symmetric domains. This involved generalising notation and some observations on generalising results from the irreducible case.

We described one method of extension, which only gives us a partial extension. In the final section we completed a theorem that allows for the generalisation of the holomorphic extension results for a product of (irreducible) bounded symmetric domains.

The author now believes that a combination of the first sketched proof and the second method we have shown would be another method of proving the extension of the proof. Using this method we could also show the continuity part of the proof that we mentiond at the end of the last section. Perhaps this approach would be more natural than the second approach.

The original aim of the second method was to complete a theorem that would use the results of the irreducible case to generalise. This was not how the result worked out. Similar in both methods, we had to use methods from the proofs of the results for the irreducible case.

It might be a better approach to use combination of the two methods as mentioned above.

Further work could be done on explicitly stating what the equation is for the extended function f. The author believes the equation will be an integral equation with its range of integration being over the boundary values of our bounded symmetric domain.

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