

Nonperturbative results for the mass dependence of the QED fermion determinant

M. P. Fry

School of Mathematics, University of Dublin, Trinity College, Dublin 2, Ireland

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Abstract

The fermion determinant in four-dimensional quantum electrodynamics in the presence of $O(2) \times O(3)$ symmetric background gauge fields with a nonvanishing global chiral anomaly is considered. It is shown that the leading mass singularity of the determinant's nonperturbative part is fixed by the anomaly. It is also shown that for a large class of such fields there is at least one value of the fermion mass at which the determinant's nonperturbative part reduces to its noninteracting value.

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Every physical process calculable within the standard model ultimately depends on the model's fermion determinants. These are part of the effective functional measure for the gauge fields when the fermion fields are integrated. Without them, charge and color screening, quark fragmentation into hadrons and unitarity would be lost. Accordingly, they are fundamental, and the nonperturbative structure of the standard model requires corresponding information about its determinants.

Taking Schwinger's 1951 paper [1] on vacuum polarization as the beginning of the modern era of fermion determinants, progress on elucidating their nonperturbative properties has been extremely slow. In this paper we focus on the fermion mass dependence of the determinant in four-dimensional QED. We note here the recent advance in determining the quark mass dependence of the QCD₄ instanton determinant [2].

We begin by summarizing the main analytic results for QED₄'s fermion determinant on non-compact, Euclidean space-time. Formally, a fermion field integration produces the ratio of determinants $\det(\not{p} - e\not{A} + m)/\det(\not{p} + m) = \det(1 - eS\not{A})$, where S is the free fermion propagator. Since the operator $S\not{A}$ is not trace class, $\det(1 - eS\not{A})$ is undefined no matter how well-behaved the gauge field A_μ is. Nevertheless, sense is made of it based on the following results.

(a) The operator $S\not{A}$ is a non-Hermitian compact operator in the trace ideal $\mathcal{J}_p$ for $p > 4$ and fermion mass $m \neq 0$ provided $A_\mu \in L^p(\mathbb{R}^4)$ [3, 4]. This includes the instanton-like case of A_μ having a $1/r$ falloff. The theorem means that the traces $\text{Tr}(S\not{A})^n$, $n \geq 5$, are absolutely convergent and really do correspond to sums of eigenvalues of $S\not{A}$.

(b) A renormalized determinant can be defined:

$$\det_{ren} = \exp(\Pi_2 + \Pi_3 + \Pi_4) \det_5(1 - eS\not{A}), \quad (1)$$

where

$$\ln \det_5 = \text{Tr} \left[\ln(1 - eS\not{A}) + \sum_{n=1}^4 \frac{(eS\not{A})^n}{n} \right], \quad (2)$$

and $\Pi_{2,3,4}$ are the second, third and fourth-order contributions to the one-loop effective action, $\ln \det_{ren}$, defined by some consistent regularization procedure [5]. The graph Π_2 contains a charge renormalization subtraction. The regularization should result in $\Pi_3 = 0$ by C -invariance and give the unique gauge invariant result for Π_4 .

(c) As corollaries of (a) and (b), $\det_5$ is an entire function of the coupling e and can be represented in terms of the discrete complex eigenvalues $1/e_n$ of $S\mathcal{A}$:

$$\det_5 = \prod_n \left[\left(1 - \frac{e}{e_n} \right) \exp \left(\sum_{k=1}^4 \frac{(e/e_n)^k}{k} \right) \right]. \quad (3)$$

By C -invariance and the reality of $\det_5$ for real e the eigenvalues can appear in quartets $\pm e_n, \pm \bar{e}_n$ or as imaginary pairs [6].

(d) $\det_{ren}$ has no zeros for real e when $m \neq 0$ [7], and since $\det_{ren}(e=0) = 1$, $\det_{ren} > 0$ for real e .

(e) $\det_{ren}$ is an entire function of e of order 4 since $S\mathcal{A} \in \mathcal{J}_{4+\epsilon}$ when $A_\mu \in L^{4+\epsilon}(\mathbb{R}^4)$, $\epsilon > 0$ [8]. This conclusion was first reached for a restricted class of gauge fields by Adler [9] and later by Balian et al. [10], for another restricted class of fields. The growth of $\det_{ren}$ for real values of e is unknown.

(f) $\det_5$ is an analytic function of m throughout the complex m -plane cut along the negative real axis [11].

This comprises the general analytic knowledge of $\det_{ren}$ obtained since 1951. The present sparse knowledge of such a central part of the standard model is noteworthy. There is a large body of results for background gauge fields which do not fall off sufficiently rapidly in all directions in $\mathbb{R}^4$ to satisfy the theorem in (a) [12]. These require the introduction of an *ad hoc* volume cutoff, and none of the results (b) – (f) necessarily hold for such fields.

We report here on an extension of results (a) – (f) for a large class of $O(2) \times O(3)$ symmetric background gauge fields of the form $A_\mu(x) = M_{\mu\nu} x_\nu a(r^2)$, where the profile function $a(r^2)$ is at least three times differentiable, regular at the origin, and $a(r^2) = \nu/r^2$ for $r > R$, R being a range parameter, $r^2 = x_\mu x_\mu$. The constant ν is assumed positive without loss of generality. For $r < R$, $a(r^2)$ may have multiple zeros. The constant antisymmetric matrix M has nonvanishing entries $M_{12} = M_{30} = 1$. Letting $*F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}$ and noting that $*F F = \partial_\alpha(\epsilon_{\alpha\beta\mu\nu}A_\beta F_{\mu\nu})$, it is evident that A_μ must have a $1/r$ falloff for the Dirac operator $\mathcal{D} = \not{P} - e\mathcal{A}$ to have a nonvanishing global chiral anomaly $\mathcal{A} = -\int d^4x *F_{\mu\nu}F_{\mu\nu}/16\pi^2$. In our case $\mathcal{A} = \nu^2/2$. From here on we set $e = 1$ to reduce notation.

In [11] we proved a vanishing theorem for this choice of A_μ . Consequently all the square-integrable zero modes of $\mathcal{D}$ have positive chirality, and such modes first appear when $\nu > 2$. The zero modes can be shifted to the negative chirality sector by replacing M with the

antisymmetric matrix N with entries $N_{03} = N_{12} = 1$. Since $F_{\mu\nu}$ is not self-dual it extends the vanishing theorem of [13] to such $U(1)$ fields. Because the calculation here is in non-compact Euclidean space-time the index theorem has to be modified to account for the continuum part of the spectrum of $\not{D}^2$ extending down to zero energy, thereby contributing an additional part to the index of $\not{D}$ [14, 15]. These low energy states play an essential role in the analysis discussed below.

We can now add the following results to the list (a) – (f) above:

(g) For the class of gauge fields defined above the leading mass singularity of $\ln \det_5$ is governed by the chiral anomaly, that is,

$$\ln \det_5 \underset{m \rightarrow 0}{\sim} \frac{\nu^2}{4} \ln m^2 + \text{less singular}, \quad (4)$$

so that $\ln \det_5$ becomes negative as $m \rightarrow 0$. Inconclusive evidence for (4) was first reported in [11].

The presence of a zero mode in the spectrum of the Dirac operator and its control of the leading quark mass dependence of QCD_4 instanton determinant has been known for many years [16]. Establishing (4) relies on the above vanishing theorem and showing that the zero-mode-free negative chirality sector contributes terms to (4) less singular than $\ln m^2$. With more effort it should be possible to prove (4) under the more general assumption that $a(r^2) \sim \nu/r^2$, for $r \gg R$.

(h) If

$$\int_0^{R^2} dr^2 [2r^{14} a'^6 + 12r^{12} a a'^5 + 23r^{10} a^2 a'^4 + 12r^8 a^3 a'^3 - 19r^6 a^4 a'^2] < \frac{9\nu^6}{2R^8}, \quad (5)$$

then $\ln \det_5$ becomes positive before dropping off to zero for $m \rightarrow \infty$ [11]. This and (4) imply that there is at least one value of m for which $\ln \det_5 = 0$. That is, $\ln \det_5$ has a mass zero, or possibly an odd number of such zeros, at which $\det_5 = 1$.

Subject to the conditions on a stated above, (5) can be satisfied in general for an $a(r^2)$ that can oscillate between positive and negative values before curving downward to join smoothly with its long-range form ν/r^2 at $r = R$.

As ν is varied the eigenvalues e_n in (3) will shift, presumably shifting the position of the mass zero as well. That a particular value of m can cause $\det_5$ to assume its noninteracting

value indicates that mass has a profound effect on the distribution of the eigenvalues e_n in the complex plane.

To establish (4) we begin by defining $\det_{ren}$ in (1) by

$$\begin{aligned} \ln \det_{ren} = & \frac{1}{2} \lim_{\delta \rightarrow 0} \int_{\delta}^{\infty} \frac{dt}{t} \left[\text{Tr} \left(e^{-tP^2} - e^{-t[(P-A)^2 + \frac{1}{2}\sigma F]} \right) \right. \\ & \left. \times e^{-tm^2} + \frac{1}{24\pi^2} \int d^4x F^2(x) e^{-t\mu^2} \right], \end{aligned} \quad (6)$$

where μ is the renormalization scale. Although $\det_{ren}$ is finite with on-shell renormalization of Π_2 when $F_{\mu\nu} \sim 1/r^2$, this complicates the small mass analysis of $\det_5$, and so we prefer to deal with the off-shell case. One can always go back on-shell once $\det_5$ is understood.

In the representation where γ_5 is diagonal with entries $\pm \mathbb{1}_2$, $(P-A)^2 + \frac{1}{2}\sigma F$ is diagonal with corresponding positive and negative chirality entries $H_{\pm} = (P-A)^2 - \sigma \cdot (\mathbf{B} \pm \mathbf{E})$. Differentiating (6) with respect to m^2 yields the renormalization independent result

$$\begin{aligned} m^2 \frac{\partial}{\partial m^2} \ln \det_5 = & \frac{1}{2} m^2 \text{Tr} [(H_+ + m^2)^{-1} - (H_- + m^2)^{-1}] \\ & + m^2 \int_0^{\infty} dt e^{-tm^2} \int d^4x \text{tr} \langle x | e^{-tH_-} - e^{-tP^2} | x \rangle \\ & - m^2 \partial \Pi_2 / \partial m^2 - m^2 \partial \Pi_4 / \partial m^2, \end{aligned} \quad (7)$$

where we have taken δ to zero and set $\Pi_3 = 0$. The first and last terms in (7) are well-defined for the background fields considered here, but the second and third terms are not. Specifically, the perturbative expansion of (6) gives $\partial \Pi_2 / \partial m^2 \sim \int_0^{\infty} dk/k$ which must be cancelled.

The strategy is this: There must be a corresponding infrared divergence in the second term in (7) that cancels that in $\partial \Pi_2 / \partial m^2$, as the left-hand side of (7) is well-defined for $m^2 > 0$. The second term in (7) will be calculated by summing over the exact eigenstates of H_- . As already noted, these are scattering states only. An infrared regulator is introduced by cutting off the low energy spectra of H_- and P^2 at λ^2 . Then the infrared divergent part is isolated; it must be second order. Identify it as the divergent part of $\partial \Pi_2 / \partial m^2$ to effect the cancellation of infrared divergences. That is, the second term in (7) defines the divergent part of $\partial \Pi_2 / \partial m^2$ according to (9) and (13) below, consistent with our way of calculating $\det_{ren}$. Then set $\lambda = 0$ and finally study the $m \rightarrow 0$ limit of (7) to find the small mass dependence of the well-defined quantity $\ln \det_5$. It is essential to proceed in this way. Pulling out the contribution to $\partial \Pi_2 / \partial m^2$ from the second term in (7) by a straightforward

perturbation expansion results in a gauge invariant remainder that is a sum of separately non-gauge invariant terms, leading to a computational impasse.

For the fields under consideration we find Π_4 is well-defined for $m \neq 0$ and is less singular than $\ln m^2$ as $m \rightarrow 0$ and so gives a vanishing contribution to the right-hand side of (7) as $m \rightarrow 0$. This result relies in part on the finiteness at $m = 0$ of the photon-photon scattering subgraph in Π_4 [17].

We now turn to the calculation of the rest of the right-hand side of (7). Here we use the definition of the chiral anomaly on non-compact manifolds due to Musto et al. [15], $\mathcal{A} = \lim_{m \rightarrow 0} m^2 \text{Tr} [(H_+ + m^2)^{-1} - (H_- + m^2)^{-1}]$. This combined with $\mathcal{A} = \nu^2/2$, Eq.(7), and an integration with respect to m^2 gives result (4), provided that the remainder in (7) contributes terms to (4) less singular than $\ln m^2$ for $m \rightarrow 0$.

Denote the second term on the right-hand side of (7) by I and obtain for $m \rightarrow 0$

$$I(m^2) = m^2 \lim_{\lambda \rightarrow 0} \lim_{L \rightarrow \infty} \int_{\lambda^2}^{\Lambda^2} \frac{dk^2}{k^2 + m^2} \int_0^L dr r^3 \int d\Omega_4 \quad (8)$$

$$\times \sum_{jMmm'} \left(|\Psi_{EjMmm'}^-(x)|^2 - |\Psi_{EjMmm'}^0(x)|^2 \right),$$

where $\Psi_{EjMmm'}^-$ are the eigenstates H_- derived in [11], and $\Psi_{EjMmm'}^0$ are the associated free-particle states. Here $E = k^2$, and $j = 0, \frac{1}{2}, 1, \dots$; $M = -j - \frac{1}{2}, \dots, j + \frac{1}{2}$; $m = -j, \dots, j$; $m' = \pm \frac{1}{2}$ are the quantum numbers associated with the $O(2) \times O(3)$ symmetry of the background fields; λ is the infrared cutoff introduced above, and Λ , with $\Lambda R < 1$, limits the range of k needed to study the small-mass dependence of I .

Divide I into $I_>(I_<)$, the exterior (interior) parts of I from $r > R$ ($r < R$), and consider first the most singular part in m^2 , $I_>$. The radial wave functions associated with Ψ^- for $r > R$ are calculated from the outgoing wave combination of Bessel functions $\sqrt{r}J_\sigma(kr) \cos \Delta_\alpha(k) - \sqrt{r}Y_\sigma(kr) \sin \Delta_\alpha(k)$, where α denotes jMm' , $\sigma = \sqrt{(2j+1)^2 + 4\nu M + \nu^2}$, and Δ_α is the energy-dependent part of the low-energy phase shifts δ_α , $\Delta_\alpha(k) = \pi(\sigma - 2j - 1)/2 + \delta_\alpha(k)$, mod π . Since $\Delta_\alpha(0) = 0$, we can expand in powers of Δ_α . For $|M| \neq j + \frac{1}{2}$, $\tan \Delta_\alpha = C_\alpha(\sigma)(kR/2)^{2\sigma}(\sigma\Gamma^2(\sigma))^{-1} [1 + O((kR)^2, (kR)^{2\sigma})]$, where $C_\alpha(\sigma)$ is a bounded function of σ [11]. The rapid falloff of Δ_α with j and energy allows one to terminate the expansion after Δ_α^2 . Terms in $I_>$ containing Δ_α and Δ_α^2 are uniformly convergent and can be integrated term-by-term and the limit $L = \infty$ taken. There are some oscillating k -integrals containing $\cos(2kL)$ and $\sin(2kL)$. These are set equal to zero by the

Riemann-Lebesgue lemma following the sequence of limits in (8). The result is contributions to $I_>$ less singular than $m^2 \ln m^2$ and $O(1)$ contributions to $\ln \det_5$. Terms from $M = j + \frac{1}{2}$ contribute $O(m^2)$ terms to $I_>$. Terms of $O(\Delta_\alpha^0)$ will be considered below.

The zero modes of H_+ appear in the sector $M = -j - \frac{1}{2}$ for values of j satisfying $\nu > 2j + 2$, $j = 0, \frac{1}{2}, \dots$. The most singular contribution to $I_>$ occurs at the zero mode thresholds $M = -j - \frac{1}{2}$, $\nu = 2j + 2$ at which Δ_α 's energy dependence drops to $\tan \Delta_\alpha = \frac{\pi}{2}(1 + O(kR)^2)[\ln(kR) + C + O(kR)^2 \ln(kR)]^{-1}$ where C is a negative k -independent constant [11]. This results in a contribution to $I_>$ of $O(1/\ln(mR))$ and a $\ln |\ln(mR)|$ contribution to $\ln \det_5$ in (4). This covers all terms in (8) from Δ_α and Δ_α^2 .

The zero mode thresholds also dominate the region $r < R$. Specifically, they are responsible for the radial wave function contributing to (8) with the slowest k falloff, whose form is $(\ln kR + C)^{-1}\psi(k^2, r)$, $\psi(0, r) \neq 0$, and C as above. Here ψ is analytic in k^2 and is a smooth function of r behaving near $r = 0$ as $r^{2j+3/2}$. This results in contributions of $O(m^2)$ to $I_<$ and $O(1)$ to $\ln \det_5$. Other cases in the $M = -j - \frac{1}{2}$ sector have a faster small k falloff. The study of the $M \neq -j - \frac{1}{2}$ sectors is facilitated by the $1/(2j+1)!$ falloff of the radial wave functions (also true for $M = -j - \frac{1}{2}$), their small k falloff of at least $(kR)^\sigma$, and their $r^{2j+1/2}$ behavior near $r = 0$ [11]. These results allow the $m \rightarrow 0$ limit of $I_<$ to be taken term-by-term, giving a final $O(1)$ contribution to $\ln \det_5$.

Now consider the terms in $I_>$ of $O(\Delta_\alpha^0)$, here denoted by $I_>^0$. For fixed L the integral and sum over j in (8) can be interchanged since $|J_\sigma(z)| \leq |z/2|^\sigma / \Gamma(\sigma + 1)$, z real, and because only J_σ is present in $I_>^0$. The result for the L -dependent terms is

$$I_>^0 = \frac{m^2}{4} \lim_{\lambda \rightarrow 0} \lim_{L \rightarrow \infty} \int_{\lambda^2}^{\Lambda^2} \frac{dk^2}{k^2 + m^2} L^2 (S_1(kL) + S_2(kL)), \quad (9)$$

where

$$S_1 = \sum_{j=0, \frac{1}{2}, \dots} (2j+1) \left[J_{|2j+2-\nu|}^2 - J_{|2j+1-\nu|} J_{|2j+3-\nu|} + (\nu \rightarrow -\nu) - (\nu = 0) \right], \quad (10)$$

$$S_2 = \sum_{j=\frac{1}{2}, 1, \dots} (2j+1) \sum_{M=-j+\frac{1}{2}}^{j-\frac{1}{2}} \left[J_{\sigma+1}^2 - J_{\sigma+2} J_\sigma + J_{\sigma-1}^2 - J_{\sigma-2} J_\sigma - (\nu = 0) \right]. \quad (11)$$

The Bessel functions are evaluated at kL . These series are not uniformly convergent in L

and must be summed before taking $L \rightarrow \infty$. S_1 can be summed to give for $kL \gg 1$

$$S_1 = \frac{2\nu^2}{\pi kL} + \frac{1}{\pi(kL)^2} \cos(2kL) \sin^2\left(\frac{\pi\nu}{2}\right) + O(kL)^{-3}. \quad (12)$$

The leading term in (12) must be cancelled by S_2 to make $I^0_>$ finite. Because of the absolute value signs in (10), Eq.(12) is valid for $\nu \leq 1$. For $\nu > 1$ we show in Appendix A that the leading term in (12) is unchanged.

We have not been able to sum S_2 without resorting to its expansion in ν . This is a well-behaved expansion as it occurs in the Bessel function's order, and J_σ is an entire function of σ . To $O(\nu^2)$ we find for $kL \gg 1$

$$S_2 = -\frac{2\nu^2}{\pi kL} + \left(\frac{\nu}{kL}\right)^2 \left[C + \frac{\pi}{12} \cos(2kL) + \left(\frac{7}{30} + \frac{2}{3} \ln 2\right) \sin(2kL) \right] + O(kL)^{-3}. \quad (13)$$

As expected, the leading term in (13) cancels that in (12). Referring to (9), the second term in (13) results in the expected infrared divergent term discussed above. The constant C is given by a complicated, but absolutely convergent, series of Bessel functions. Its value is irrelevant to our analysis as it will be cancelled by the counterterm $\partial\Pi_2/\partial m^2$. The remaining oscillating terms in (12) and (13) give vanishing contribution to $I^0_>$ by the Riemann-Lebesgue lemma.

The second-order calculation of S_2 may be extended to all orders in ν . Structures generated in second order appear again in higher orders differentiated with respect to Bessel function order. No further infrared divergences appear, only $\cos(2kL)$ and $\sin(2kL)$ terms as in (12) and (13). Appendices A and B show that (13) and higher order results for S_2 hold for $\nu \leq 2$ and that they can be continued to $\nu > 2$. There is no loss of information in the expansion of S_2 and its continuation. The R -dependent terms from the lower bound of integration of $I^0_>$ are uniformly convergent, and no expansion is necessary. They result in $O(1)$ contributions to $\ln \det_5$. This establishes Eq.(4).

The conclusion that $\det_5$ can be reduced to its noninteracting value by varying its mass for a class of background gauge fields points to an unexpected nonperturbative role of mass in QED₄'s effective action. It would be surprising if result (4) – the chiral anomaly's control of $\ln \det_5$'s leading mass singularity – is limited to our background fields. Presumably it is generally true and, if so, mass zero(s) in $\ln \det_5$ are also present more generally.

The communication of Christian Schubert that Π_4 in scalar QED_4 with a r^{-2} falloff profile function has no $\ln m^2$ singularity is gratefully acknowledged.

APPENDIX A

To calculate $S_1(\nu)$ first assume $\nu \leq 1$ so that the absolute value signs can be dropped. Combine products of Bessel functions using the identity [18]

$$J_\mu(z)J_\nu(z) = \frac{2}{\pi} \int_0^{\pi/2} d\theta J_{\mu+\nu}(2z \cos \theta) \cos[(\mu - \nu)\theta], \quad (\text{A1})$$

for $\text{Re}(\mu + \nu) > -1$. Then calculate sums using

$$\sum_{n=0}^{\infty} (2n + \rho) J_{2n+\rho}(z) = \frac{z}{2} J_{\rho-1}(z) + \frac{z}{2} \int_0^z dt J_\rho(t), \quad (\text{A2})$$

for $\text{Re} \rho > -1$ from [19], entry 5.7.2.6 and

$$\sum_{n=0}^{\infty} J_{2n+\rho}(z) = \frac{1}{2} \int_0^z dt J_{\rho-1}(t), \quad (\text{A3})$$

from entry 5.7.1.6. Calculation of standard integrals followed by the limit $z = kL \gg 1$ gives Eq.(12).

When $\nu > 1$ care has to be taken in summing over j and the sign of the absolute values in Eq.(10). Let $\nu = N + \epsilon$, $0 < \epsilon \leq 1$, $N = 1, 2, \dots$. Then the absolute value signs can be omitted for $j \geq N/2$. The sum in (10) is the same as in (12) with the addition of the extra term

$$(-1)^N \left[\frac{N+1}{2} \right] \frac{\cos(2kL) \cos(\pi\nu)}{2\pi(kL)^2} + O(kL)^{-3},$$

where $[x]$ is the integral part of x .

Now consider S_2 in Eq.(11). From (A1) we get

$$\begin{aligned} & J_{\sigma+1}^2 - J_{\sigma+2}J_\sigma + J_{\sigma-1}^2 - J_{\sigma-2}J_\sigma \\ &= \frac{4}{\pi} \int_0^{\pi/2} d\theta \sin^2 \theta [J_{2\sigma+2}(2z \cos \theta) + J_{2\sigma-2}(2z \cos \theta)]. \end{aligned}$$

Expand σ in powers of ν , initially assuming $\nu \leq 2$ so that the expansion of σ converges for $j \geq 1/2$. Keeping all terms through order ν^2 we get after summing over M ,

$$\begin{aligned} S_2 &= \frac{8\nu^2}{3\pi} \frac{\partial}{\partial \rho} \int_0^1 dx \sqrt{1-x^2} \sum_{j=1/2, 1, \dots} \left(2j+1 - \frac{1}{2j+1} \right) J_{4j+4+\rho}(2xz) \Big|_{\rho=0} \\ &+ \frac{16\nu^2}{3\pi} \frac{\partial^2}{\partial \rho^2} \int_0^1 dx \sqrt{1-x^2} \sum_{j=1/2, 1, \dots} (2j^2 - 1) J_{4j+4+\rho}(2xz) \Big|_{\rho=0}. \end{aligned} \quad (\text{A4})$$

In addition to (A2) and (A3) use is made of the following results from [19]:

$$\sum_{n=0}^{\infty} (2n + \rho)^2 J_{2n+\rho}(z) = \frac{\rho z}{2} J_{\rho-1}(z) + \frac{z^2}{2} J_{\rho}(z) + \frac{z^2}{2} \int_0^z dt J_{\rho+1}(t), \quad (\text{A5})$$

from entry 5.7.2.24 and

$$\sum_{n=0}^{\infty} \frac{1}{2n+1} J_{2n+\rho}(z) = \frac{\pi(\rho-1)}{4} \int_0^z dt \frac{1}{t} J_{\rho-1}(t) \mathbf{H}_0(z-t), \quad (\text{A6})$$

for $\text{Re } \rho > 1$ from entry 5.7.3.5, where $\mathbf{H}_0$ is the Struve function of order zero. Then (A4) together with (A2), (A3), (A5), (A6) becomes

$$\begin{aligned} S_2 = & \frac{8\nu^2}{3\pi} \frac{\partial}{\partial \rho} \int_0^1 dx \sqrt{1-x^2} \left[\left\{ \int_0^{2xz} dt \left[\frac{1}{2} xz J_{4+\rho}(t) - \left(\frac{1}{2} + \frac{\rho}{4} \right) J_{3+\rho}(t) \right] \right. \right. \\ & + \frac{1}{2} xz J_{3+\rho}(2xz) + \sum_{n=1}^{\infty} \frac{J_{2n+4+\rho}(2xz)}{(2n+1)(n+1)} \left. \left. \right\} + \{\rho \rightarrow \rho-4\} \right]_{\rho=0} \\ & + \frac{8\nu^2}{3\pi} \frac{\partial}{\partial \rho} \int_0^1 dx \sqrt{1-x^2} \left[-\pi(\rho+3) \int_0^{2xz} dt \frac{1}{t} J_{3+\rho}(t) \mathbf{H}_0(2xz-t) + J_{4+\rho}(2xz) \right. \\ & - J_{\rho}(2xz) - \frac{2}{3} J_{\rho+2}(2xz) + 8 \sum_{n=0}^{\infty} \frac{J_{2n+4+\rho}(2xz)}{(2n+1)(2n+5)} \left. \right]_{\rho=0} + \frac{16\nu^2}{3\pi} \frac{\partial^2}{\partial \rho^2} \int_0^1 dx \sqrt{1-x^2} \\ & \times \left[\left\{ \int_0^{2xz} dt \left[\left(\frac{1}{4} (xz)^2 + \frac{3}{2} + \frac{5\rho}{8} + \frac{\rho^2}{16} \right) J_{3+\rho}(t) - \frac{1}{4} (5+\rho) xz J_{4+\rho}(t) \right] \right. \right. \\ & - \frac{1}{4} (xz)^2 J_{4+\rho}(2xz) - \left(\frac{3}{4} + \frac{1}{8} \rho \right) xz J_{3+\rho}(xz) \left. \left. \right\} + \{\rho \rightarrow \rho-4\} \right]_{\rho=0} + O(\nu^4). \end{aligned} \quad (\text{A7})$$

The only terms in (A7) needing further discussion are the infinite series (see below) and

$$I(z) \equiv \int_0^1 dx \sqrt{1-x^2} \int_0^{2xz} dt \frac{1}{t} J_{3+\rho}(t) \mathbf{H}_0(2xz-t) = I_1 + I_2, \quad (\text{A8})$$

where

$$\begin{aligned} I_1 = & \int_0^1 dx \sqrt{1-x^2} \int_0^{\infty} dt \frac{1}{t} J_{3+\rho}(t) \mathbf{H}_0(2xz-t) \\ = & \int_0^1 dx \sqrt{1-x^2} \int_0^N dt \frac{1}{t} J_{3+\rho}(t) \mathbf{H}_0(2xz-t) (\equiv I_{11}) \\ & + \int_0^1 dx \sqrt{1-x^2} \int_N^{\infty} dt \frac{1}{t} J_{3+\rho}(t) \mathbf{H}_0(2xz-t) (\equiv I_{12}). \end{aligned} \quad (\text{A9})$$

Since for $z \gg 1$ [18]

$$\mathbf{H}_0(z) \sim Y_0(z) + \frac{2}{\pi} \left(\frac{1}{z} - \frac{1}{z^3} + \dots \right), \quad (\text{A10})$$

then $\lim_{z \rightarrow \infty} I_{11} = 0$. Since $|\mathbf{H}_0(z)| < 1$ for all $z \geq 0$,

$$|I_{12}| \leq \int_0^1 dx \sqrt{1-x^2} \int_N^\infty dt \frac{1}{t} |J_{3+\rho}(t)| < \infty, \quad (\text{A11})$$

Hence, I_{12} converges uniformly for all $z > 0$. Therefore, the limit $z \rightarrow \infty$ can be interchanged with the integrals defining I_{12} to give $\lim_{z \rightarrow \infty} I_{12} = 0$, and so $\lim_{z \rightarrow \infty} I_1 = 0$. By inspection

$$I_2 = - \int_0^1 dx \sqrt{1-x^2} \int_{2xz}^\infty dt \frac{1}{t} J_{3+\rho}(t) \mathbf{H}_0(2xz-t), \quad (\text{A12})$$

vanishes as $z \rightarrow \infty$. Hence, $\lim_{z \rightarrow \infty} I(z) = 0$.

From [18] $\mathbf{H}_0 = 2/\pi - \mathbf{H}_1$ so that

$$\frac{dI}{dz} = 2 \int_0^1 dx \sqrt{1-x^2} \int_0^{2xz} dt \frac{1}{t} J_{3+\rho}(t) \left[\frac{2}{\pi} - \mathbf{H}_1(2xz-t) \right], \quad (\text{A13})$$

since $\mathbf{H}_0(0) = 0$. Substitute [20]

$$\mathbf{H}_1(z) = \frac{2}{\pi} - \frac{2}{\pi} J_0(z) + \frac{4}{\pi} \sum_{n=1}^\infty \frac{J_{2n}(z)}{4n^2 - 1}, \quad (\text{A14})$$

in (A13) and use [18] for $\text{Re}(\mu) > 0$, $\text{Re}(\nu) > -1$,

$$\int_0^z dt \frac{1}{t} J_\mu(t) J_\nu(z-t) = \frac{J_{\mu+\nu}(z)}{\mu}, \quad (\text{A15})$$

to get

$$\frac{dI}{dz} = \frac{4/\pi}{3+\rho} \int_0^1 dx x \sqrt{1-x^2} J_{3+\rho}(2xz) + I_3, \quad (\text{A16})$$

where

$$I_3 = -\frac{8}{\pi} \int_0^1 dx x \sqrt{1-x^2} \int_0^{2xz} dt \frac{1}{t} J_{3+\rho}(t) \sum_{n=1}^\infty \frac{J_{2n}(2xz-t)}{4n^2 - 1}. \quad (\text{A17})$$

The series in (A17) for $0 \leq 2xz-t \leq 2z$ is uniformly convergent by inspection using $|J_\mu(z)| \leq 1$, $\mu \geq 0$. Therefore, the series defines a continuous function of $2xz-t$, and

$J_{3+\rho}(t)/t$ is continuous for $0 \leq t \leq 2xz$. Hence the series can be integrated term-by-term with respect to t . Using (A15),

$$I_3 = -\frac{8/\pi}{3+\rho} \int_0^1 dx x \sqrt{1-x^2} \sum_{n=1}^{\infty} \frac{J_{2n+3+\rho}(2xz)}{4n^2-1}. \quad (\text{A18})$$

By the same reasoning as above the integral and sum in (A18) can be interchanged to give, with (A1) employed in both (A16) and (A18),

$$\begin{aligned} \frac{dI}{dz} &= \frac{1/2}{3+\rho} \left[J_{2+\rho/2}(z) J_{1+\rho/2}(z) - J_{3+\rho/2}(z) J_{\rho/2}(z) \right] \\ &\quad - \frac{1}{3+\rho} \sum_{n=1}^{\infty} \frac{J_{n+2+\rho/2}(z) J_{n+1+\rho/2}(z) - J_{n+3+\rho/2}(z) J_{n+\rho/2}(z)}{4n^2-1}. \end{aligned} \quad (\text{A19})$$

From (A7) we need $\partial[(\rho+3)I]/\partial\rho \Big|_{\rho=0}$ for large $z (=kL)$. By inspection the series in (A19) is uniformly convergent in z , allowing the interchange of limit and sum. Keeping terms of order z^{-3} we get

$$\frac{\partial}{\partial\rho}(\rho+3) \frac{dI}{dz} \Big|_{\rho=0} = -\frac{\pi \cos(2z)}{4z^2} + O\left(\frac{\cos(2z)}{z^3}, \frac{\sin(2z)}{z^3}\right), \quad (\text{A20})$$

where we used

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1} = \frac{1}{2} - \frac{\pi}{4}. \quad (\text{A21})$$

The remainder in (A20) has been verified to be finite. From our previous result $\lim_{z \rightarrow \infty} I = 0$ we get

$$\frac{\partial}{\partial\rho}(\rho+3) \int_0^1 dx \sqrt{1-x^2} \int_0^{2xz} dt \frac{1}{t} J_{3+\rho}(t) \mathbf{H}_0(2xz-t) \Big|_{\rho=0} = -\frac{\pi \sin(2z)}{8z^2} + O\left(\frac{\cos(2z)}{z^3}, \frac{\sin(2z)}{z^3}\right), \quad (\text{A22})$$

as a contribution to (A7).

Regarding the constant C in (13), refer to (A7) and let

$$f(z, \rho) = \int_0^1 dx \sqrt{1-x^2} \left[\sum_{n=1}^{\infty} \frac{J_{2n+4+\rho}(2xz) + (\rho \rightarrow \rho-4)}{(2n+1)(n+1)} + 8 \sum_{n=0}^{\infty} \frac{J_{2n+4+\rho}(2xz)}{(2n+1)(2n+5)} \right]. \quad (\text{A23})$$

These series can be integrated term-by-term due to their uniform convergence in x . From (A1),

$$f(z, \rho) = \frac{\pi}{4} \sum_{n=1}^{\infty} \frac{J_{n+2+\rho/2}^2(z) - J_{n+3+\rho/2}(z)J_{n+1+\rho/2}(z) + (\rho \rightarrow \rho - 4)}{(2n+1)(n+1)} + 2\pi \sum_{n=0}^{\infty} \frac{J_{n+2+\rho/2}^2(z) - J_{n+3+\rho/2}(z)J_{n+1+\rho/2}(z)}{(2n+1)(2n+5)}. \quad (\text{A24})$$

Since the series in (A24) are uniformly convergent in z the limit $z \gg 1$ can be interchanged with the sums. The leading term is $O(z^{-1})$, independent of ρ . Therefore it makes no contribution to (A7). The subleading terms are $O(z^{-2})$ and ρ -dependent, but the remainder, formally $O(z^{-3})$, is logarithmically divergent. Hence, the series in (A24) has to be summed before taking the $z \gg 1$ limit in order to get the required subleading terms that give C in Eq.(13). The constant C can only originate from (A24) through $\partial f(z, \rho)/\partial \rho \Big|_{\rho=0}$. We now give a demonstration of how C arises.

Replace the Bessel functions in (A24) with integrals using

$$\int_0^z dx x J_\mu^2(x) = \frac{1}{2} z^2 [J_\mu^2(z) - J_{\mu+1}(z)J_{\mu-1}(z)]. \quad (\text{A25})$$

The integrals and sums can again be interchanged for all finite z . Then (A24) becomes

$$\partial [z^2 f(z, \rho)] / \partial z = \frac{\pi}{2} z \sum_{n=1}^{\infty} \frac{J_{n+2+\rho/2}^2(z) + J_{n+\rho/2}^2(z)}{(2n+1)(n+1)} + 4\pi \sum_{n=0}^{\infty} \frac{J_{n+2+\rho/2}^2(z)}{(2n+1)(2n+5)}. \quad (\text{A26})$$

As required by (A7) the series can be differentiated term-by-term with respect to ρ as $J_\mu(z)$ is an entire function of μ . The limit $z \gg 1$ can be interchanged with the sums, giving

$$\frac{\partial}{\partial z} \left[z^2 \partial f(z, \rho) / \partial \rho \Big|_{\rho=0} \right] = \left(\frac{\pi}{2} + \frac{\pi}{2} \ln 2 - \frac{\pi^2}{4} \right) \cos(2z) - \frac{\pi}{30} \cos(2z) + O \left(\frac{\cos(2z)}{z}, \frac{\sin(2z)}{z} \right), \quad (\text{A27})$$

where first (second) term in (A27) comes from the first two (last) series in (A26); the indicated remainder has been verified to be finite. Hence, for $z \gg 1$

$$\partial f(z, \rho) / \partial \rho \Big|_{\rho=0} = \left(\frac{\pi}{4} \ln 2 + \frac{7\pi}{30} - \frac{\pi^2}{8} \right) \frac{\sin(2z)}{z^2} + \frac{D}{z^2} + O \left(\frac{\cos(2z)}{z^3}, \frac{\sin(2z)}{z^3} \right), \quad (\text{A28})$$

where D is the constant of integration of (A27). In Eq.(13), $C = 8D/3$. The reason D is hard to calculate from (A24) is because it arises in the transitional region where $J_\mu(z)$ has

$\mu \sim z \gg 1$. Of course we cannot *a priori* rule out that C is zero in our way of calculating the infrared divergent part of $\partial\Pi_2/\partial m^2$.

The remaining terms in (A7) are tedious but straightforward to calculate. The result for $z = kL \gg 1$ is given by Eq.(13).

Finally, the terms $O(\nu^4)$ and higher in S_2 can be calculated by continuing the expansion of σ in powers of ν , which we noted converges when $\nu \leq 2$ and $j \geq 1/2$. This will involve higher derivatives in ρ of the second-order result together with new terms that are easily estimated. Each higher order only contains terms $O\left(\frac{\cos(2kL)}{(kL)^2}, \frac{\sin(2kL)}{(kL)^2}\right)$; there are no terms of the form $C/(kL)^2$. The case $\nu > 2$ is dealt with in Appendix B.

APPENDIX B

The purpose of this appendix is to continue the results of Appendix A to values of $\nu > 2$. Recall that the first zero mode in the positive chirality sector occurs when $\nu > 2$.

The estimates below require the following results: From

$$\begin{aligned} \sigma &= [(2j+1)^2 + 4M\nu + \nu^2]^{1/2}, \quad -j+1/2 \leq M \leq j-1/2, \\ \sigma &\geq \sqrt{8j}, \quad j = \frac{1}{2}, 1, \dots, \end{aligned} \tag{B1}$$

for all real values of ν . Also

$$\frac{4M\nu + \nu^2}{(2j+1)^2} \leq 1, \quad j \geq 1/2, \quad \nu \leq 2; \quad j \geq j_*, \quad \nu > 2, \tag{B2}$$

where

$$j_* = \left\{ \frac{\nu-1}{2} + \sqrt{\frac{1}{2}\nu^2 - \nu} \right\}, \tag{B3}$$

and $\{x\}$ is the greatest integer/half-integer value of x exceeding or equal to x . Therefore the expansion of σ in powers of ν converges for $j \geq j_*$ for $\nu > 2$ and for $j \geq 1/2$ for $\nu \leq 2$.

Consider the first term in S_2 given by Eq.(11). For all $\nu > 0$ it can be rewritten as

$$S_{21}(\nu, z) = \sum_{j=\frac{1}{2}}^{j_*-\frac{1}{2}} (2j+1) \sum_{M=-j+\frac{1}{2}}^{j-\frac{1}{2}} J_{\sigma+1}^2(z) + \sum_{j=j_*}^{\infty} (2j+1) \sum_{M=-j+\frac{1}{2}}^{j-\frac{1}{2}} J_{\sigma+1}^2(z), \tag{B4}$$

with $z = kL$. Note that for real z

$$|J_\sigma(z)| \leq \frac{|\frac{1}{2}z|^\sigma}{\Gamma(\sigma+1)}, \quad \sigma \geq -\frac{1}{2}. \quad (\text{B5})$$

From (B1) and $\sigma \leq \sqrt{2}(2j+1)$ from (B2), it follows from (B5) with $z = kL > 2$ that there is a ν - independent upper bound

$$|J_\sigma(z)| \leq \frac{(\frac{1}{2}z)^{\sqrt{2}(2j+1)}}{\Gamma(\sqrt{8j}+1)}, \quad j \geq \frac{1}{2}, \quad \nu \leq 2; \quad j \geq j_*, \quad \nu > 2, \quad (\text{B6})$$

on the terms in the second series in (B4). The sum of the bounded terms is absolutely convergent, and so the second series in (B4) is uniformly convergent for all $\nu > 0$.

Now $J_{\sigma+1}$ is an entire function of σ , and hence so is $J_{\sigma+1}^2$. Therefore the expansion of $J_{\sigma+1}^2$ about the point $\sigma = 2j+1$ has an infinite radius of convergence for any ν :

$$J_{\sigma+1}^2(z) = \sum_{n=0}^{\infty} J_{2j+2}^{2(n)}(z) \frac{(\sigma - 2j - 1)^n}{n!}, \quad (\text{B7})$$

where $J_{2j+2}^{2(n)} = d^n J_\mu^2 / d\mu^n \Big|_{\mu=2j+2}$. Because of the uniform convergence of the second series in (B4) the sums over n and j can be interchanged when (B7) is inserted in (B4):

$$S_{21}(\nu, z) = \sum_{j=\frac{1}{2}}^{j_*-\frac{1}{2}} (2j+1) \sum_M J_{\sigma+1}^2(z) + \sum_{n=1}^{\infty} S_n(\nu, z)/n!, \quad (\text{B8})$$

where

$$S_n(\nu, z) = \sum_{j=j_*}^{\infty} (2j+1) J_{2j+2}^{2(n)}(z) \sum_M (\sigma - 2j - 1)^n. \quad (\text{B9})$$

Note that

$$(\sigma - 2j - 1)^n = \left[\frac{1}{2} \frac{4M\nu + \nu^2}{2j+1} - \frac{1}{8} \frac{(4M\nu + \nu^2)^2}{(2j+1)^3} + \dots \right]^n = 2^{-n} \left(\frac{4M\nu + \nu^2}{2j+1} \right)^n (1 + R)^n, \quad (\text{B10})$$

where, because the series is convergent and alternating,

$$0 < |R| < \frac{1}{4} \frac{4M\nu + \nu^2}{(2j+1)^2} \leq \frac{1}{4}, \quad j \geq j_*. \quad (\text{B11})$$

Hence,

$$\sum_{n=1}^{\infty} S_n(\nu, z)/n! = \sum_{n=1}^{\infty} \frac{1}{2^n n!} \sum_{j=j_*}^{\infty} (2j+1) J_{2j+2}^{2(n)}(z) \sum_M \left(\frac{4M\nu + \nu^2}{2j+1} \right)^n (1+R)^n. \quad (\text{B12})$$

For $j \gg j_*$ in (B12) the leading term is

$$\sum = 2 \sum_{n=1}^{\infty} \frac{\nu^n}{n!} \sum_{j \gg j_*}^{\infty} j^{1-n} J_{2j+2}^{2(n)}(z) \sum_M M^n, \quad (\text{B13})$$

where $R = O(\nu/j)$ has been neglected. From the estimate

$$\sum_{M=-j+\frac{1}{2}}^{j-\frac{1}{2}} M^n = \frac{2}{n+1} \left(j - \frac{1}{2} \right)^{n+1}, \quad n = 2, 4, \dots, \quad (\text{B14})$$

and letting $n \rightarrow 2n$ one gets

$$\sum = 4 \sum_{n=1}^{\infty} \frac{\nu^{2n}}{(2n+1)(2n)!} \sum_{j \gg j_*}^{\infty} j^2 J_{2j+2}^{2(2n)}(z). \quad (\text{B15})$$

The convergence of the series in (B15) is investigated by estimating the sum over j by the integral

$$\int_{Nj_*}^{\infty} dj j^2 J_{2j+2}^{2(2n)}(z),$$

for $N, n \gg 1$ and employing the method of steepest descents for fixed z using

$$J_{\mu}(z) \underset{\mu \gg 1}{\sim} \frac{1}{\sqrt{2\pi\mu}} \left(\frac{ez}{2\mu} \right)^{\mu}. \quad (\text{B16})$$

This gives

$$\sum_{j \gg j_*} j^2 J_{2j+2}^{2(2n)}(z) \underset{n \gg 1}{\sim} \frac{1}{8} \left(\frac{n}{\pi(\ln n)^3} \right)^{1/2} e^{2n \ln \ln n + O(n \ln \ln n / \ln n)}. \quad (\text{B17})$$

In view of (B17) it is evident that the power series in ν in (B15) converges for all $\nu > 0$ satisfying (B2), (B3) and for all $z > 0$, and hence so does the series S_{21} in (B8).

Repeat the above analysis for the remaining terms in S_2 in Eq.(11). With obvious notation,

$$\begin{aligned}
S_2(\nu, z) = & \sum_{j=\frac{1}{2}}^{j_*-\frac{1}{2}} (2j+1) \sum_M \left[J_{\sigma+1}^2 - J_{\sigma+2} J_\sigma + J_{\sigma-1}^2 - J_{\sigma-2} J_\sigma - (\nu = 0) \right] \\
& + \sum_{n=1}^{\infty} \sum_{j=j_*}^{\infty} (2j+1) \sum_M [\bullet]_{\sigma=2j+1}^{(n)} (\sigma - 2j - 1)^n / n!.
\end{aligned} \tag{B18}$$

Denote the first sum in (B18) by $\sum_1$. Setting $z = kL$ then

$$\sum_1 = \sum_{kL \gg 1} \sum_{j=\frac{1}{2}}^{j_*-\frac{1}{2}} (2j+1) \sum_{M=-j+\frac{1}{2}}^{j-\frac{1}{2}} \left(\frac{2 \cos(2kL - \pi\sigma)}{\pi(kL)^2} + O(kL)^{-3} \right). \tag{B19}$$

We know from our analysis of S_{21} that the second series in (B18) is absolutely convergent, and so it can be rearranged in powers of ν . Due to the symmetric sum over M only even powers of ν survive. Denoting the second series by $\sum_2$, it takes the form

$$\sum_2 = \sum_{n=1}^{\infty} \frac{\nu^{2n}}{(2n)!} \sum_{j=j_*}^{\infty} (2j+1) \sum_{M=-j+\frac{1}{2}}^{j-\frac{1}{2}} a_{2n}(j, kL). \tag{B20}$$

In second order we found for $kL \gg 1$, $\nu \leq 2$,

$$\frac{\nu^2}{2!} \sum_{j=\frac{1}{2}}^{\infty} (2j+1) \sum_M a_2(j, kL) = -\frac{2\nu^2}{\pi kL} + \frac{C\nu^2}{(kL)^2} + O\left(\frac{\nu^2 \cos(2kL)}{(kL)^2}, \frac{\nu^2 \sin(2kL)}{(kL)^2}\right). \tag{B21}$$

In fourth and higher orders following Appendix A only terms of $O\left(\frac{\cos(2kL)}{(kL)^2}, \frac{\sin(2kL)}{(kL)^2}\right)$ are present. Therefore set for $n \geq 2$, $kL \gg 1$

$$\sum_{j=\frac{1}{2}}^{\infty} (2j+1) \sum_{M=-j+\frac{1}{2}}^{j-\frac{1}{2}} a_{2n}(j, kL) = b_{2n} \frac{\sin(2kL)}{(kL)^2} + c_{2n} \frac{\cos(2kL)}{(kL)^2} + O((kL)^{-3}). \tag{B22}$$

Then for $kL \gg 1$

$$\sum_2 = -\frac{2\nu^2}{\pi kL} + \frac{C\nu^2}{(kL)^2} + \sum_{n=1}^{\infty} \frac{\nu^{2n}}{(2n)!} b_{2n} \frac{\sin(2kL)}{(kL)^2} + \sum_{n=1}^{\infty} \frac{\nu^{2n}}{(2n)!} c_{2n} \frac{\cos(2kL)}{(kL)^2} + O((kL)^{-3}). \tag{B23}$$

Since $\sum_2$ is convergent for $\nu \leq 2$ with $j_* = 1/2$ and sine and cosine are linearly independent, both series in (B23) are convergent as well. For $\nu > 2$ the sums in (B20)-(B22) begin at $j = j_* \geq 1$ given by (B3), and following the above analysis they are convergent. Terms with $j = 1/2, \dots, j_* - 1/2$ result in the oscillating terms in (B19) whose power series expansion in ν is not guaranteed to converge for $\nu > 2$. Therefore, the results (B21), (B22), as well as (B23) and Eq.(13) continue to hold for $\nu > 2$, except that the coefficients of $\cos(2kL)/(kL)^2$

and $\sin(kl)/(kL)^2$ will change. This result is consistent with what we found for the fully summed series for S_1 in Eq.(12).

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