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The Herrero conditions on norm limits of hypercyclic operators

James Boland

School of Mathematics, Trinity College, Dublin 2
E-mail address: jaboland@tcd.ie

ABSTRACT. For a separable infinite dimensional Banach space X it is known that X supports many bounded linear operators T which are hypercyclic, that is such that there is a vector $x \in X$ with dense orbit $\{x, Tx, T^2x, \ldots\}$. Here we investigate the operator norm closure of the class of hypercyclic operator on X, and of various subclasses that have been considered extensively in recent literature. In the case of a Hilbert space the norm closures we consider are all known to be characterised by conditions of Herrero.

Our first positive results are that operators $T = \lambda I + S$ are norm-approximable by mixing operators on X if $|\lambda| = 1$ and S has a property we call generalised nilpotent. Mixing operators are a subclass of hypercyclic operators.

We then discuss extensions of classical reults of Salas and Rolewicz concerning operators related to backward shifts.

Finally we consider $T = \lambda I + S$ where $|\lambda| = 1$ and S is quasinilpotent but with restrictions on the space X. In particular we find that the Herrero conditions characterise the norm closure of the hypercyclic operators for some recently constructed pathological Banach spaces X.

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CHAPTER 1

Introduction

Hypercyclic operators $T\colon X \to X$ on a Banach space X (those bounded linear operators for which there is a vector $v \in X$ with dense orbit $\{v, Tv, T^2v, \ldots\}$) have been the subject of intense study in recent decades and we refer to [**6**][**18**] for a recent overview of the topic. Many problems that were posed [**15**][**16**] about hypercyclicity have now been solved. In particular, it is well known that every separable infinite dimensional Banach space X supports hypercyclic operators and in fact many such operators. We use $\mathcal{B}(X)$ for the bounded linear operators on X and $\mathcal{HC}(X)$ for the subclass of hypercyclic operators. An elegant result of [**19**] allows one to conclude that, for $T \in \mathcal{HC}(X)$, the similarity class $\{S^{-1}TS : S \in \mathcal{B}(X) \text{ invertible}\}$ is always dense in the strong operator topology of $\mathcal{B}(X)$ (and consists of hypercyclic operators).

In the operator norm on $\mathcal{B}(X)$, there are some obvious restrictions such as $\|T\| > 1$ for $T \in \mathcal{HC}(X)$. In [**21**] Herrero proved the following:

THEOREM 1.1.1. *For H a separable infinite dimensional Hilbert space over $\mathbb{C}$, $\overline{\mathcal{HC}(H)}$ is the class of all those operators T in $\mathcal{B}(H)$ satisfying the conditions*

(1) $\sigma_W(T) \cup \mathbb{T}$ is connected;
(2) $\sigma_0(T) = \varnothing$; and
(3) $\operatorname{ind}(\lambda - T) \geq 0$ for all $\lambda \in \rho_{s-F}(T)$.

where

- $\mathbb{T}$ is the unit circle,
- $\operatorname{ind}(T) = \dim \ker T - \operatorname{codim} TH$ for any semi-Fredholm operator $T \in \mathcal{B}(H)$,
- $\sigma_W(T) = \{\lambda \in \mathbb{C} : \lambda - T$ is not a semi-Fredholm operator with $\operatorname{ind}(\lambda - T) = 0\}$ is the Weyl spectrum of T.
- $\sigma_0(T)$ is the set of normal eigenvalues, i.e. the set of isolated points λ in the spectrum $\sigma(T)$ for which the spectral projection associated with $\{\lambda\}$ is of finite rank.
- $\rho_{s-F}(T) = \{\lambda \in \mathbb{C} : \lambda - T \text{ is semi-Fredholm}\}$.

We will discuss the terminology and notation in more detail in Chapter 2. We refer to conditions (1), (2), (3) as *Herrero's conditions*. Herrero [**21**, remark 2.3] notes that these necessary and sufficient conditions for a Hilbert space operator $T \in \mathcal{B}(H)$ to belong in $\overline{\mathcal{HC}(H)}$ are also necessary for the Banach space case (over $\mathbb{C}$). We provide a proof of this in §2.7.

In Chapter 2 we will cover all of the background information required for the later chapters. This will include, among other things, quick introductions to hypercyclicity and spectral theory.

In Chapter 3 we will consider generalised nilpotent operators. These are operators for which the span of the generalised eigenvectors associated with 0 is dense in the whole space. Weighted backward shifts are examples of these kinds of operators and we will look at some of their spectral properties in §3.2. The main result in this chapter are from §3.5 where we show that operators of the form $T = \lambda I + S$ where S is a generalised nilpotent and $\lambda \in \mathbb{K}$ is such that $|\lambda| = 1$, are norm limits of mixing operators and are hence norm limits of hypercyclic operators.

In Chapter 4 we will discuss operators such that $[N^\infty(T - \lambda) : \lambda \in \rho^r_{s-F}(T)] = X$. In §4.2 we will see that, for operators of this kind, if each bounded component of $\rho_{\text{s}-\text{F}}(T)$ intersects the unit circle, $\mathbb{T}$, and if we assume that T has a dense set of eigenvectors corresponding to eigenvalues in $\rho^r_{\text{s}-\text{F}}(T)$, then

$$T \in \mathcal{FHC}(X) \cap \mathcal{M}ix(X) \subset \mathcal{HC}(X)$$

(Theorem 4.2.7). We will also see some applications of this including extensions of well-known results of Salas [**29**] and Rolewicz [**28**].

In Chapter 5 we discuss operators of the form $T = \lambda I + S \in \mathcal{B}(X)$ where $|\lambda| = 1$, S is a quasinilpotent operator on spaces containing a complemented copy of ℓ^p or c_0. We will start the chapter by going over statements and proofs of results found in [**11**], which will be used to show that, on spaces having FDpBD, quasinilpotents are norm limits of generalised nilpotents. We will also look at applications of these results to the norm closure of the hypercyclic operators.

CHAPTER 2

Preliminaries

In this chapter, we recall some definitions and results that will be required later.

2.1. Hypercyclic operators

As usual, if X is a normed space we refer to the bounded linear operators $T\colon X \to X$ as the operators on X and then use $\mathcal{B}(X)$ for the collection of all such operators. For two Banach (or normed) spaces X and Y we use $\mathcal{B}(X,Y)$ for the bounded linear operators $T\colon X \to Y$. We use the operator norm $\|T\|$ on $\mathcal{B}(X)$ and on $\mathcal{B}(X,Y)$.

For $T \in \mathcal{B}(X)$, we use T^n ($n \in \{0\} \cup \mathbb{N}$) for the powers of T, with $T^0 = \mathrm{id}_X$ the identity operator, $T^1 = T$ and $T^{n+1} = TT^n$ for $n \in \mathbb{N}$.

DEFINITION 2.1.1. If X is a Banach space, then $T \in \mathcal{B}(X)$ is called a *hypercyclic operator* if there exists $x \in X$ such that the *orbit of x under T*

$$\mathrm{Orb}_T(x) = \{x, Tx, T^2x, \ldots\} = \{T^n x : n \in \{0\} \cup \mathbb{N}\}$$

is dense in X.

Such an element $x \in X$ is called a *hypercyclic vector for T* and we write $\mathcal{HC}(T)$ for the set of all hypercyclic vectors for T.

We write $\mathcal{HC}(X)$ for the set of all hypercyclic operators on X.

There is extensive literature on hypercyclic operators for instance the monograph [**6**]. It is usually hard to show that an operator is hypercyclic directly so the Hypercyclicity Criterion is often used:

THEOREM 2.1.2. [**6**, Theorem 1.6] *Let $T \in \mathcal{B}(X)$ where X is a separable Banach space. Suppose there exists an increasing sequence of integers (n_k), two dense sets $D_1, D_2 \subset X$ and a sequence of maps $S_{n_k} : D_2 \to X$ such that:*

(1) $T^{n_k}x \to 0$ for any $x \in D_1$;
(2) $S_{n_k}y \to 0$ for any $y \in D_2$;
(3) $T^{n_k}S_{n_k}y \to y$ for each $y \in D_2$.

Then T is hypercyclic.

The following is a stronger form of hypercyclicity:

DEFINITION 2.1.3. $T \in \mathcal{B}(X)$ is called *weakly mixing* if $T \oplus T$ is hypercyclic on $X \oplus X$.

We will use $\mathcal{WM}\text{ix}(X)$ to denote the set of weakly mixing operators on X.

In [**7**], Bès and Peris show that an operator is weakly mixing iff it satisfies the Hypercyclicity Criterion. This also tells us that weakly mixing operators are hypercyclic. It can also be shown that there are hypercyclic operators that are not weakly mixing and therefore do not satisfy the Hypercyclicy Criterion [**6**, Theorem 4.14].

In addition, we have three other forms of hypercyclicity that we will be concerned with:

DEFINITION 2.1.4. If X is a Banach space, then $T \in \mathcal{B}(X)$ is called a *mixing operator* if for any non-empty open sets $U, V \subset X$ there exists $N \in \mathbb{N}$ such that

$$T^n U \cap V \neq \varnothing$$

for all $n > N$. $\mathcal{M}\text{ix}(X)$ will denote the set of all mixing operators on X.

This will be the main form of hypercyclicity that we will be looking at. It can be shown that mixing operators are always weakly mixing and hence hypercyclic [**18**, Observation 2.42].

Mixing operators also have the following useful property:

PROPOSITION 2.1.5. [**18**, Proposition 2.40] *Suppose* $X = \sum_{i=1}^{N} X_i$, *where each* X_i *is a separable infinite dimensional Banach space and let* $T = (\bigoplus_{i=1}^{N} T_i) \in B(X)$. *Then* T *is mixing on* X *iff each* T_i *is mixing on* X_i.

DEFINITION 2.1.6. $T \in \mathcal{B}(X)$ is called *frequently hypercyclic* if there is $x \in X$ such that for every non-empty open set $V \subset X$:

$$\liminf_{N\to\infty} \frac{\#\{n \in \{1, 2, \ldots, N\} : T^n x \in V\}}{N} > 0.$$

$\mathcal{FHC}(X)$ will denote the set of frequently hypercyclic operators on X.

Again, we have that every frequently hypercyclic operator is weakly mixing [**17**, Theorem 2.2].

DEFINITION 2.1.7. $T \in \mathcal{B}(X)$ is called *chaotic* if

(1) T is hypercyclic,
(2) T has a dense set of periodic points
(3) T has sensitive dependence on initial conditions: $\exists \delta > 0$ such that $\forall x \in X$ and every neighbourhood U of x, one can find $y \in U$ and $\mathbb{N} \ni n \geq 0$ such that $\|T^n x - T^n y\| \geq \delta$.

$\mathcal{CHC}(X)$ will denote the set of chaotic operators on X.

However we have that hypercyclicity implies sensitive dependence on initial conditions ([**6**, Proposition 6.6]) and so we have the following:

PROPOSITION 2.1.8. *$T \in \mathcal{B}(X)$ is chaotic iff it is hypercyclic and has a dense set of periodic points.*

Clearly chaotic operators are hypercyclic but they are also weakly mixing [**7**, Proposition 2.14].

The diagram shows the relationship between the various kinds of hypercyclicity mentioned above:

$$\begin{array}{ccccc} \mathcal{CHC}(X) & & & & \\ & \subset & & & \\ \mathcal{M}\mathrm{ix}(X) & \subset & \mathcal{WM}\mathrm{ix}(X) & \subset & \mathcal{HC}(X) \\ & \subset & & & \\ \mathcal{FHC}(X) & & & & \end{array}$$

In [**1**], Ansari showed that every separable infinite dimensional Banach space admits a hypercyclic operator. In fact we have that every Banach space admits a mixing operator, since the operator constructed is of the form $I + T$ where T is a generalised backward shift (see 3.3). However, there are Banach spaces admitting no chaotic or frequently hypercyclic operators [**6**, Theorem 6.36]. Therefore, there are mixing operators that are neither chaotic nor frequently hypercyclic. There are also chaotic or frequently hypercyclic operators that are not mixing [**6**, Corollary 6.46] and operators that are frequently hypercyclic but not chaotic [**6**, Theorem 6.41]. It is not known if $\mathcal{M}\mathrm{ix}(X) = \mathcal{WM}\mathrm{ix}(X)$ or if $\mathcal{CHC}(X) \subset \mathcal{FHC}(X)$.

2.2. Banach space properties

NOTATION 2.2.1. If X is a Banach space (over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) we denote its dual space by X^*. Given two Banach spaces, X and Y, and an operator $T \in B(X, Y)$ we denote its adjoint by $T^* \in \mathcal{B}(Y^*, X^*)$.

If X is a normed space, and $E \subseteq X$ is a subset, we use $\mathrm{span}(E)$ for the linear span of E, the set of all finite linear combinations of elements of E (taken to be $\{0\}$ if $E = \emptyset$).

We use $[E]$ to denote the closure of the linear span, that is $[E] = \overline{\mathrm{span}(E)}$. If $E = \{x_n : 1 \leq n < N\}$ is a finite or infinite countable set $(0 \leq N \leq \infty)$ we write $[x_n]_{1 \leq n < N}$ (or just $[x_n]$ if the range of indices n is clear from the context) for $[E]$.

We use $\approx$ to denote isomorphism.

DEFINITION 2.2.2. If X is a Banach space a pair of sequences $(x_n)_{n=1}^{\infty}$ in X and $(x_n^*)_{n=1}^{\infty}$ in X^* is called a *biorthogonal system* if $x_i^*(x_j) = \delta_{i,j}$.

A sequence $(x_n)_{n=1}^{\infty}$ in X is called *minimal* if there exists a sequence $(x_n^*)_{n=1}^{\infty}$ such that $((x_n)_{n=1}^{\infty}, (x_n^*)_{n=1}^{\infty})$ is a biorthogonal system. In this case we say that x_n^* is the biorthogonal functional associated with x_n.

A minimal sequence $(x_n)_{n=1}^{\infty}$ in X is called *fundamental* if $[x_n] = X$.

A minimal sequence $(x_n^*)_{n=1}^\infty$ in X^* is called *total* if: $x_n^*(x) = 0$ for all $n \Rightarrow x = 0$.

A sequence in X is called an *M-basis* (for X) if it is minimal fundamental and has a total biorthogonal sequence.

$\{x_n\}_{n=1}^\infty$ is called *normalised* if $\|x_n\| = 1$ for all n.

It is known that every separable Banach space has an M-basis [**24**, Theorem 1.f.4].

DEFINITION 2.2.3. If X is a finite dimensional Banach space, a Schauder basis for X is a linearly independent sequence $\{x_1, x_2, \ldots, x_n\}$ which spans X.

If X is an infinite dimensional Banach space, a sequence $(x_n)_{n=1}^\infty$ in X is called a *Schauder basis for* X if for every $x \in X$ there is a unique sequence of scalars $(a_n)_{n=1}^\infty$ so that $x = \sum_{n=1}^\infty a_n x_n$.

By a *basic sequence* we mean any sequence which is a basis for its closed linear span.

In particular, we have that every Schauder basis is an M-basis [**24**, Definition 1.f.2].

Suppose X is a Banach space with a Schauder basis $(x_n)_{n=1}^\infty$. Let P_n be the functions defined by

$$P_n\left(\sum_{i=1}^\infty a_i x_i\right) = \sum_{i=1}^n a_i x_i.$$

These operators are bounded and $\sup_n \|P_n\| < \infty$ by [**24**, Proposition 1.a.2]. The number $K = \sup_n \|P_n\|$ is called the *basis constant* for $(x_n)_{n=1}^\infty$.

DEFINITION 2.2.4. A basis $(x_n)_{n=1}^\infty$ for a Banach space X is said to be *unconditional* if the convergence of $\sum_{i=1}^\infty a_i x_i$ implies the convergence of $\sum_{i=1}^\infty \theta_i a_i x_i$ for every choice of signs $\theta_i \in \{+1, -1\}$.

Given an unconditional basis $(x_n)_{n=1}^\infty$, we have every sequence $\theta = (\theta_n)_{n=1}^\infty$ with $\theta_n \in \{+1 - 1\}$ defines an operator

$$M_\theta\left(\sum_{n=1}^\infty a_n x_n\right) = \sum_{n=1}^\infty \theta_n a_n x_n$$

and we also have that $\sup_\theta \|M_\theta\| < \infty$.

We let $K := \sup_\theta \|M_\theta\|$ be the *unconditional basis constant* for $(x_n)_{n=1}^\infty$.

We have the following:

PROPOSITION 2.2.5. [**24**, Proposition 1.c.7] *Suppose $(x_n)_{n=1}^\infty$ is an unconditional basic sequence with an unconditional constant K. Then, for every choice of scalars $(a_n)_{n=1}^\infty$ such that $\sum_{n=1}^\infty a_n x_n$ converges and*

every choice of bounded scalars $(\lambda_n)_{n=1}^{\infty}$ *we have*

$$\left\| \sum_{n=1}^{\infty} \lambda_n a_n x_n \right\| \leq 2K \sup_n |\lambda_n| \left\| \sum_{n=1}^{\infty} a_n x_n \right\|$$

(in the real case we can take K *instead of* $2K$*).*

PROPOSITION 2.2.6. [**24**, Proposition 1.c.6] *A basic sequence* $(x_n)_{n=1}^{\infty}$ *is unconditional iff any of the following conditions holds:*

(1) For every permutation π *of the integers the sequence* $(x_{\pi(n)})_{n=1}^{\infty}$ *is a basic sequence.*

(2) For every subset σ *of the integers the convergence of* $\sum_{n=1}^{\infty} a_n x_n$ *imples the convergence of* $\sum_{n\in\sigma} a_n x_n$*.*

(3) The convergence of $\sum_{n=1}^{\infty} a_n x_n$ *implies the convergence of* $\sum_{n=1}^{\infty} b_n x_n$ *whenever* $|b_n| \leq |a_n|$ *for all* n*.*

DEFINITION 2.2.7. Two bases, $(x_n)_{n=1}^{\infty}$ of X and $(y_n)_{n=1}^{\infty}$ of Y are called *equivalent* provided a series $\sum_{i=1}^{\infty} a_i x_i$ converges iff $\sum_{i=1}^{\infty} a_i y_i$ converges.

DEFINITION 2.2.8. A basis $(x_n)_{n=1}^{\infty}$ of a Banach space X is said to be *symmetric* if, for any permutation π of the integers, $(x_{\pi(n)})_{n=1}^{\infty}$ is equivalent to $(x_n)_{n=1}^{\infty}$.

Clearly, by Proposition 2.2.6, every symmetric basis is unconditional.

Given a symmetric basis $(x_n)_{n=1}^{\infty}$, every permutation π defines an operator

$$V_\pi \left(\sum_{n=1}^{\infty} a_n x_n \right) = \sum_{n=1}^{\infty} a_n x_{\pi(n)}.$$

We have that $\sup_{\pi,\theta} \|M_\theta V_\pi\| < \infty$, where π runs over all permutations of $\mathbb{N}$ and θ over all sequences $\theta = (\theta_n)_{n=1}^{\infty}$ of signs ($\theta_n \in \mathbb{K}$, $|\theta_n| = 1$ for all $n \in \mathbb{N}$).

We call $K := \sup_{\pi,\theta} \|M_\theta V_\pi\|$ the *symmetric constant* of $(x_n)_{n=1}^{\infty}$. Clearly we have that the symmetric constant of a symmetric basis is at least as big as the unconditional constant.

DEFINITION 2.2.9. We say that a Banach Space has the *Approximation Property* (or AP) if every compact operator is a norm limit of finite rank operators.

The following theorem about the AP will be used later on:

THEOREM 2.2.10. [**24**, Theorem 1.e.7 (a)] *Let* X *be a Banach space. If* X^* *has the AP then* X *has the AP. In particular, if* X *is reflexive then* X *has the AP iff* X^* *has the AP.*

We will use the following consequence of the Principle of Local Reflexivity [**22**, Section 3]:

THEOREM 2.2.11. [**22**, Corollary 3.2] *Let X be a Banach space and let $U \subset X^*$ be a finite dimensional subspace. Let F be a finite rank operator on X^* and let $\epsilon > 0$. Then there is a finite rank operator $K \in \mathcal{B}(X^*)$ such that:*

(1) $K(X^*) = F(X^*)$
(2) $K|_U = F|_U$,
(3) $\|K\| \leq \|F\|(1+\epsilon)$,
(4) $K^*y^* = F^*y^*$ whenever F^*y^* belongs to X,
(5) K is weak-continuous.*

We will show the following well known fact:

LEMMA 2.2.12. *Suppose $\phi \in X^*$ and $x_0 \in X$ such that $\phi(x_0) = 1$. Then*

$$\|\phi\| = \frac{1}{\operatorname{dist}(x_0, \ker \phi)}$$

PROOF.

$$\begin{aligned} \|\phi\| &= \inf\{M : \|\phi(x)\| \leq M\|x\|\} \\ &= \inf\{M : |\phi(cx_0 + k)| \leq M\|cx_0 + k\|, k \in \ker\phi, c \in \mathbb{K}\} \\ &= \inf\{M : |c| \leq M\|cx_0 + k\|, k \in \ker\phi, c \in \mathbb{K}\} \end{aligned}$$

We can assume $c \neq 0$ since $c = 0$ doesn't give us any restriction on M so

$$\begin{aligned} \|\phi\| &= \inf\left\{M : \frac{|c|}{\|cx_0 + k\|} \leq M, k \in \ker\phi, c \in \mathbb{K}\right\} \\ &= \inf\left\{M : \frac{1}{\|x_0 + k\|} \leq M, k \in \ker\phi\right\} \\ &= \sup\left\{\frac{1}{\|x_0 + k\|}; k \in \ker\phi\right\} \\ &= \frac{1}{\inf\{\|x_0 + k\| : k \in \ker\phi\}} \\ &= \frac{1}{\operatorname{dist}(x_0, \ker\phi)} \end{aligned}$$

□

2.3. Finite dimensional decompositions

NOTATION 2.3.1. If Y and Z are normed spaces and $1 \leq p < \infty$ we write $Y \oplus_p Z$ for the direct sum vector space $Y \oplus Z$ with the norm $\|y \oplus z\|_p = (\|x\|^p + \|y\|^p)^{1/p}$. We also write $Y \oplus_\infty Z$ for $Y \oplus Z$ with the norm $\|y \oplus z\|_\infty = \max(\|x\|, \|y\|)$.

DEFINITION 2.3.2. If X is a normed space, and $Y, Z \subseteq X$ are closed subspaces, we write $X = Y \oplus Z$ if each $x \in X$ can be written uniquely

as $x = y + z$ with $y \in Y$, $z \in Z$ and if there is a constant $M > 1$ such that $\|y\| + \|z\| \leq M\|y + z\|$ holds for all $y \in Y$, $z \in Z$.

The *projection of X onto Y with kernel Z* means the operator $P_Y \in \mathcal{B}(X)$ given by

$$P_Y(y + z) = y. \tag{2.3.1}$$

We say that a closed subspace Y of a normed space X is 1-*complemented* in X if there exists a closed subspace Z of X such that $X = Y \oplus Z$ and $\|y\| \leq \|y+z\|$ holds for all $y \in Y$, $z \in Z$. This means that $\|P_Y\| \leq 1$ and is equivalent to the existence of $P \in \mathcal{B}(X)$ with $P(X) = Y$, $\|P\| \leq 1$ and $P^2 = P$.

DEFINITION 2.3.3 ([**24**, §1.g]). We say that a Banach space X has a *finite dimensional block decomposition* (or *FDBD*), also known as a finite dimensional Schauder decomposition (abbreviated FDD), if there are finite dimensional subspaces $X_1, X_2, \ldots$ of X such that each $x \in X$ has a unique representation in the form $x = \sum_{n=1}^{\infty} x_n$ with $x_n \in X_n$ for each n.

Note that if there is n with $X_n = \{0\}$ we can omit all such X_n from the list of subspaces provided X is not itself finite dimensional. We will restrict our attention to infinite dimensional X and assume $\dim X_n > 0$ for all n.

DEFINITION 2.3.4 ([**11**]). We say that a Banach space X has a *finite dimensional p-block decomposition (FDpBD)*, where $1 \leq p \leq \infty$, if there are finite dimensional subspaces $X_1, X_2, \ldots$ of X such that each $x \in X$ has a unique representation in the form $x = \sum_{n=1}^{\infty} x_n$ with $x_n \in X_n$ for each n (that is an FDBD) such that

(i) $\|x\| = \left(\sum_{n=1}^{\infty} \|x_n\|^p\right)^{1/p}$ in case $1 \leq p < \infty$, or

(ii) $\|x\| = \sup_n \|x_n\|$ if $p = \infty$.

We write $X = \bigoplus_{(p)} X_n$ in this situation.

Again we will use this for $\dim X = \infty$ and $\dim X_n > 0 (\forall n)$. Associated with such an FDpBD $X = \bigoplus_{(p)} X_n$ we have natural projections $P_n \colon X \to X_n$ given by

$$P_n\left(\sum_{j=1}^{\infty} x_j\right) = x_n$$

and $R_n \colon X \to X_1 \oplus X_2 \oplus \cdots \oplus X_n$ given by

$$R_n = \sum_{j=1}^{n} P_j.$$

The projections P_n and R_n are all bicontractive, that is satisfy $\|P_n\| = \|1 - P_n\| = 1$ and $\|R_n\| = \|1 - R_n\| = 1$. Moreover we note for future reference that $x \in X \Rightarrow \lim_{n\to\infty} \|P_n x\| = 0$.

LEMMA 2.3.5. *If an infinite dimensional Banach space X is an FDpBD ($1 \leq p \leq \infty$) then it contains a 1-complemented isometric copy of ℓ^p (if $1 \leq p < \infty$) or of c_0 if $p = \infty$.*

PROOF. Suppose $X = \bigoplus_{(p)} X_n$ is an FDpBD of X and $\dim X_n > 0 (\forall n)$. Choose unit vectors $y_n \in X_n$ for each n and take $Y = [y_n]$ and choose linear functionals $\phi_n \in X_n^*$ of ;norm one such that

$$1 = \|y_n\| = \phi_n(y_n) = \|\phi_n\|.$$

We can naturally extend ϕ_n to $\phi_n \circ P_n \in X^*$ and note that $\|\phi_n \circ P_n\| = 1$. We will (treating P_n as the canonical projection onto X_n) use the same notation ϕ_n for the extension $\phi_n \circ P_n$ to X. Then we have $\phi_n(y_m) = 0$ when $n \neq m$, or

$$\delta_{m,n} = \phi_n(y_m), \quad 1 = \|y_n\| = \phi_n(y_n) = \|\phi_n\|.$$

The closed linear span $Y_0 = [y_n : n \in \mathbb{N}]$ is clearly isometric to ℓ^p if $1 \leq p < \infty$ or to c_0 if $p = \infty$, that is

$$\left\| \sum_{n=1}^{\infty} \lambda_n y_n \right\| = \|(\lambda_n)_{n=1}^{\infty}\|_p$$

and there is a natural norm one projection

$$P_0 \colon X \to Y_0, \quad P_0(x) = \sum_{n=1}^{\infty} \phi_n(x) y_n.$$

To see that $\|P_0\| \leq 1$ (and that P_0 is well-defined) note that $x \in X \Rightarrow \|x\| = \|(\|P_n x\|)_{n=1}^{\infty}\|_p$ and that $\phi_n = \phi_n \circ P_n \Rightarrow |\phi_n(x)| \leq \|P_n x\|$. □

2.4. Spectral theory

In this section, we summarise properties of the spectrum of an operator on a Banach space and how the spectrum changes under compact perturbations of the operator. Most of the following comes from [**23**] (with different notation used) or [**25**].

Given a Banach space X over $\mathbb{C}$ and $T \in \mathcal{B}(X)$, we denote by $\sigma(T)$ the *spectrum of* T:

$$\sigma(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}.$$

The spectrum of any (bounded) operator on X is a compact set. We denote by $r(T)$ the *spectral radius of* T:

$$r(T) := \max\{|\lambda| : \lambda \in \sigma(T)\} = \lim_{n\to\infty} \|T^n\|^{1/n}.$$

We denote by $\rho(T) := \mathbb{C} \setminus \sigma(T)$ the *resolvent set of* T,

The *point spectrum of* T, $\sigma_p(T)$, is the set of all eigenvalues for T and the *approximate point spectrum of* T, $\sigma_{ap}(T)$, is the set of all approximate eigenvalues, i.e. the set of all $\lambda \in \mathbb{C}$ for which there exists a sequence $x_n \in X$ with $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and $\|(T - \lambda)x_n\| \to 0$. Clearly we have that $\sigma_p(T) \subset \sigma_{ap}(T) \subset \sigma(T)$.

DEFINITION 2.4.1. We say that $T \in \mathcal{B}(X)$ is a *Fredholm operator* if it has closed range and both $\dim \ker T$ and $\operatorname{codim} TX$ are finite.

We say that T is *semi-Fredholm operator* if it has closed range and either $\dim \ker T$ or $\operatorname{codim} TX$ is finite.

Given a semi-Fredholm (or Fredholm) operator $T \in \mathcal{B}(X)$, we define the index of T by $\operatorname{ind}(T) := \dim \ker T - \operatorname{codim} TX$, so that $\operatorname{ind}(T) \in \mathbb{Z} \cup \{\pm\infty\}$.

It can be shown ([**23**, Chapter IV, Theorem 5.26]) that the property of being semi-Fredholm and the index of a semi-Fredholm operator are unaffected by compact perturbation.

The *essential spectrum*, $\sigma_{ess}(T)$ is the set of all $\lambda \in \mathbb{C}$ such that $T - \lambda$ is not Fredholm. The *Calkin algebra over* X is the algebra $C(X) = \mathcal{B}(X)/\mathcal{K}(X)$, where $\mathcal{K}(X)$ is the set of all compact operators over K. Let $\hat{T}$ denote the element of the Calkin algebra induced by $T \in \mathcal{B}(X)$. Then it is known that $\sigma(\hat{T}) = \sigma_{ess}(T)$.

Suppose A is a Banach algebra with identity I. For each $a \in A$ we define the corresponding left multiplication operator $L_a : A \to A$ by $L_a x = ax$ for all $x \in A$ and the corresponding right multiplication operator $R_a : A \to A$ by $R_a x = xa$. The following is also well-known:

PROPOSITION 2.4.2. *Suppose A is a Banach algebra with identity. Then for any $a \in A$, $\sigma(A) = \sigma(L_a) \cup \sigma(R_a)$.*

PROOF. This is the same as showing that $\rho(A) = \rho(L_a) \cap \rho(R_a)$. Note that $L_{a-\lambda I} = (L_a - \lambda)$ and $R_{a-\lambda I} = (R_a - \lambda)$. Suppose $a - \lambda I$ has inverse b, i.e. $\lambda \in \rho(a)$. Then $L_{a-\lambda I} L_b x = L_b L_{a-\lambda I} x = x$ so $L_a - \lambda$ and therefore $\lambda \in \rho(L_a)$. Similarly $\lambda \in \rho(R_a)$.

Suppose $\lambda \in \rho(L_a) \cap \rho(R_a)$. Then $L_a - \lambda$ has inverse B and $R_a - \lambda$ has inverse C. Then $L_{a-\lambda I} b = I$ has a solution $b = BI$ and so $(a - \lambda_I)b = I$. Similarly, $R_{a-\lambda I} c = I$ has a solution $c = CI$ and so $c(a - \lambda_I) = I$.

So $b = (c(a - \lambda I))b = c((a - \lambda I)b) = c$ and $(a - \lambda_I)b = b(a - \lambda_I) = I$. Therefore $\lambda \in \rho(a)$. □

Combining these facts gives us:

COROLLARY 2.4.3. $\sigma_{ess}(T) = \sigma(L_{\hat{T}}) \cup \sigma(R_{\hat{T}})$.

The *semi-Fredholm spectrum of* T, $\sigma_{s-F}(T)$, is the set of all $\lambda \in \mathbb{C}$ such that $T - \lambda$ is not semi-Fredholm and the *semi-Fredholm domain* $\rho_{s-F}(T) := \mathbb{C} \setminus \sigma_{s-F}(T)$.

We have that

$$\{\lambda \in \mathbb{C} : |\lambda| > \|T\|\} \subseteq \rho(T) \subseteq \rho_{\mathrm{s-F}}(T).$$

The semi-Fredholm domain is an open set that consists of at most countably many connected components (and all but one of them is bounded). On each component $\operatorname{ind}(T-\lambda)$ is constant and both $\dim\ker(T-\lambda)$ and $\operatorname{codim}(T-\lambda)X$ are constant except for at most countably many points. We call these exceptional points singular points of the semi-Fredholm domain and we denote them by $\rho^s_{s-F}(T)$. The non-exceptional points we call regular points of the semi-Fredholm domain and denote them by $\rho^r_{s-F}(T)$. $\rho^s_{s-F}(T)$ only has accumulation points in $\sigma_{s-F}(T)$. For each $i \in \mathbb{Z}$ we denote by $\rho^i_{s-F}(T) := \{\lambda \in \rho_{s-F}(T) : \operatorname{ind}(T-\lambda) = i\}$ (or equivalently the union of all components of $\rho_{s-F}(T)$ for which the index is equal to i) and $\rho^{r,i}_{s-f}(T) := \rho^i_{s-F}(T) \cap \rho^r_{s-F}(T)$.

Suppose ω is a connected component of $\rho_{s-F}(T)$. As stated before, $\dim\ker(T-\lambda)$ and $\operatorname{codim}(T-\lambda)X$ are constant on $\omega \cap \rho^r_{s-F}(T)$. Let $k = \dim\ker(T-\lambda)$ and $r = \operatorname{codim}(T-\lambda)X$ for any $\lambda \in \omega \cap \rho^r_{s-F}(T)$. For every $\mu \in \omega \cap \rho^s_{s-F}(T)$ there is a $d \in \mathbb{N}$ such that $\dim\ker(T-\mu) = k+d$ and $\operatorname{codim}(T-\mu)X = r+d$.

We call the singular points of a component of the semi-Fredholm domain with index 0 *normal eigenvalues* and denote the set of normal eigenvalues by $\sigma_0(T)$.

The *Weyl spectrum of* T, $\sigma_W(T)$, is defined by

$$\sigma_W(T) := \bigcap_{K\in\mathcal{B}(X),K \text{ compact}} \sigma(T+K),$$

i.e. the part of the spectrum that is unchanged by compact perturbation. It can be shown ([**25**, Theorem 19.7]) that this can also be written as

$$\sigma_W(T) = \{\lambda \in \mathbb{C} : T-\lambda \text{ is not semi-Fredholm or } \operatorname{ind}(T-\lambda) \neq 0\}.$$

We also have that

$$\partial\sigma(T) \subset \sigma_{ap}(T)$$

and

$$\begin{aligned}\partial\sigma(T) \setminus \sigma_0(T) \quad \subset \quad & \partial\sigma_W(T) \subset \partial\sigma_{ess}(T) \subset \partial\sigma_{s-F}(T) \subset \sigma_{s-F}(T) \\ & \subset \sigma_{ess}(T) \subset \sigma_W(T) \subset \sigma(T) \setminus \sigma_0(T). \qquad (2.4.1)\end{aligned}$$

We say that an operator, $T \in \mathcal{B}(X)$, is a *Riesz operator* if $\sigma_{ess}(T) = \{0\}$. Note that (by Equation 2.4.1) this is true iff $\sigma_{s-F}(T) = \{0\}$ iff $\sigma_W(T) = \{0\}$. If this is the case then $\sigma(T) = \sigma_0(T) \cup \sigma_{ess}(T)$. Every compact operator, K, on a Banach space is a Riesz operator. Since the essential spectrum is unaffected by compact perturbation, we have that $\sigma_{ess}(T+K) = \{0\}$ for any compact operator $K \in \mathcal{B}(X)$ and Riesz operator T. Therefore, compact perturbations of Riesz operators are also Riesz operators.

PROPOSITION 2.4.4. [**10**, Proposition 4.11] *Suppose $T \in \mathcal{B}(X)$ and $\sigma(T) = F_1 \cup F_2$, where F_1 and F_2 are disjoint nonempty closed sets. Then there is a nontrivial projection $P \in \mathcal{B}(X)$ such that:*

(1) If $S \in \mathcal{B}(X)$ has $ST = TS$, then $SP = PS$;
(2) if $T_1 = TP$ and $T_2 = T(I - P)$, then $T = T_1 + T_2$ and $T_1T_2 = T_2T_1 = 0$;
(3) $\sigma(T_1) = F_1 \cup \{0\}$ and $\sigma(T_2) = F_2 \cup \{0\}$.

The projection, P, above is called the *Riesz spectral projection* corresponding to F_1. It is also true that $\sigma(T_1|_{PX}) = F_1$.

2.5. Triangular operators

Let X be a Banach space and let $(Y_n)_{n=1}^{\infty}$ be a sequence of subspaces. Then we define

$$\lim_{n\in\mathbb{N}} Y_n := \{x \in X : \lim_{n\to\infty} \operatorname{dist}(x, Y_n) = 0\}.$$

DEFINITION 2.5.1. $T \in B(X)$ is *triangular* if there exists a sequence of subspaces Y_n such that (for all $n \in \mathbb{N}$)

$$Y_n \subset Y_{n+1}, \dim Y_n = n, X = \lim_{n\in\mathbb{N}} Y_n, TY_n \subset Y_n$$

Note that since $Y_n = \bigcup_{i=1}^{n} Y_i$, $X = \lim_{n\in\mathbb{N}} Y_n$ is the same as $\overline{\bigcup_{i=1}^{\infty} Y_i} = X$.

Let $N^{\infty}(T) = \bigcup_{i=1}^{\infty} \ker T^i$, which is known as the generalised kernel of T (see §3.1).

LEMMA 2.5.2. *Let X be a separable infinite dimensional Banach space over $\mathbb{C}$. Then $T \in B(X)$ is a triangular operator iff there exists an infinite sequence $\lambda_i \in \mathbb{C}$ (where the λ_i are not necessarily distinct) such that $[N^{\infty}(T - \lambda_i) : i \in \mathbb{N}] = X$.*

PROOF. Suppose T is triangular and let Y_n be a sequence of subspaces as above. Let $y_1 \in Y_1$. Then y_1 is an eigenvector for T with eigenvalue λ_1 (say). Suppose we have $y_1, y_2, \dots, y_k$ that are generalised eigenvectors for T that form a basis for Y_k. $T|_{Y_{k+1}}$ is a finite dimensional operator so Y_{k+1} has a basis consisting of generalised eigenvectors for T. In fact, since $y_1, \dots, y_k$ are already generalised eigenvectors, we can extend the basis for Y_k by some generalised eigenvector y_{k+1} (with eigenvalue λ_{k+1}, say) to get a basis for Y_{k+1}. Continuing in this way we get a sequence, $\{y_i\}_{i=1}^{\infty}$, of generalised eigenvectors, i.e. vectors in $N^{\infty}(T - \lambda)$ for some $\lambda \in \sigma_p(T)$. Therefore $[N^{\infty}(T - \lambda_i) : i \in \mathbb{N}] = [y_k] = \lim_{n\in\mathbb{N}} Y_n = X$.

Suppose now we have a sequence $(\lambda_i)_{i=1}^{\infty}$ in $\mathbb{C}$ with $[N^{\infty}(T - \lambda_i) : i \in \mathbb{N}] = X$. Eliminating repetitions, we assume now that the λ_i are all distinct and that there are N of them, $1 \le N \le \infty$. Then $N^{\infty}(T - \lambda_i) \cap N^{\infty}(T - \lambda_j) = \{0\}$ if $i \neq j$.

Let $d_i = \dim N^{\infty}(T - \lambda_i)$ if the dimension is finite and $d_i = \infty$ otherwise. We can choose $x_j^i \in N^{\infty}(T - \lambda_i)$ for $1 \le j < d_i + 1$ such that

$$[x_j^i]_{1\le j<d_{i+1}} = \overline{N^{\infty}(T - \lambda_i)}$$

and $(T-\lambda_i)(x_k^i) \in \operatorname{span}\{x_j^i : 1 \le j \le k-1\}$ for each $k < d_i + 1$. If there exists some $\alpha \in \mathbb{N}$ such that $\ker(T-\lambda_i)^\alpha = \ker(T-\lambda)^{\alpha+k}$ for all $k \in \mathbb{N}$ denote it by α_i. If no such α exists let $\alpha_i = \infty$.

(1) Suppose $\dim\ker(T-\lambda_i) < \infty$. Let $c_j := \dim\ker(T-\lambda_i)^j < \infty$ for $1 \le j < \alpha_i + 1$. Let $\{x_j^i : 1 \le j \le c_1\}$ be a basis for $\ker(T-\lambda_i)$ and inductively find $\{x_j^i : c_{k-1}+1 \le j \le c_k\}$ such that $\{x_j^i : 1 \le j \le c_k\}$ is a basis for $\ker(T-\lambda_i)^k$ for $1 \le k < \alpha_i + 1$.

(2) If $\dim\ker(T-\lambda_i) = \infty$ let y_i be a sequence of linearly independent vectors in $N^\infty(T-\lambda_i)$ whose span is dense in $\overline{N^\infty(T-\lambda_i)}$. We will inductively construct a sequence x_j^i satisfying the above criteria and satisfying the following: There exists a sequence r_j such that $x_{r_j}^i = y_j$.

Since $y_1 \in N^\infty(T-\lambda_i)$ there exists k_1 such that $(T-\lambda_i)^{k_1} y_1 = 0$ but $(T-\lambda_i)^{k_1-1} y_1 \neq 0$. Let $x_j^i = (T-\lambda_i)^{k_1-j} y_1$ for $1 \le j \le k_1$ and let $r_1 = k_1$.

Suppose we have found $k_1, k_2, \ldots k_m$, $r_1, r_2, \ldots, r_m$ and x_j^i for $1 \le j \le k_m$ for some $m < \alpha$. Then there exists k_{m+1} such that $(T-\lambda_i)^{k_{m+1}} y_{m+1} \in \operatorname{span}\{x_j^i : 1 \le j \le r_m\}$ but $(T-\lambda_i)^{k_{m+1}-1} y_{m+1} \notin \operatorname{span}\{x_j^i : 1 \le j \le r_m\}$. Let $x_{r_m+j}^i = (T-\lambda_i)^{k_1-j} y_1$ for $1 \le j \le k_{m+1}$ and let $r_{m+1} = r_m + k_{m+1}$. Then x_j^i is linearly indendent, has dense linear span in $\overline{N^\infty(T-\lambda_i)}$ (since $y_j \in \{x_j^i : j \in \mathbb{N}\}$), and $(T-\lambda_i)x_{k+1}^i \in \operatorname{span}\{x_j^i : 1 \le j \le k\}$.

Let λ_{n_α} denote the (empty, finite or infinite) sequence of λ_i where $d_i < \infty$ and λ_{m_β} those with $d_i = \infty$. For x_j^i with $d_i < \infty$ (if any) we can enumerate them $u_1, u_2, \ldots$ by starting with $\{x_j^i : 1 \le j \le d_i\}$ for $i = n_1$, then $i = n_2$, etc. Then we have $\operatorname{span}\{u_1, \ldots, u_r\}$ a T-invariant space for each r. For the x_j^i with $d_i = \infty$ we can enumerate them $v_1, v_2, \ldots$ in such a way that $\operatorname{span}\{v_1, \ldots, v_s\}$ is T-invariant for each s. For example begin with $v_1 = x_1^{m_1}$, $v_2 = x_1^{m_2}$ (provided there are at least two values of β), $v_3 = x_2^{m_1}$, $v_4 = x_2^{m_2}$ etc. We can then create a single sequence $(w_i)_{i=1}^\infty$ with $\operatorname{span}\{w_1, \ldots, w_t\}$ T-invariant for each T and $[w_i]_{i\in\mathbb{N}} = X$ by interspersing the u's and v's. Then T is triangular. □

2.6. The gamma function, $\gamma(T)$, and the gap functions

For a nonzero operator T to a Banach space X, we can consider $\tilde{T} : X/\ker T \to X$ given by $\tilde{T}(x + \ker T) = Tx$. Then

$$\begin{aligned} \gamma(T) &= \inf\{\|\tilde{T}(x+\ker T)\| : \|x+\ker T\| = 1\} \\ &= \inf\{\|Tx\| : x \in X, \operatorname{dist}(x, \ker T) = 1\} \end{aligned}$$

By convention $\gamma(0) = \infty$. [**25**, section 10]. This function is called the *reduced minimum modulus* of T.

Note that if T has closed range then $\tilde{T}$ induces an isomorphism of $T/\ker T$ onto $T(X)$ and so $\gamma(T) > 0$. In fact:

LEMMA 2.6.1. [**25**, Theorem 10.2] *Let $T : X \to Y$ be an operator. Then $T(X)$ is closed if and only if $\gamma(T) > 0$.*

PROPOSITION 2.6.2. [**25**, Theorem 10.3] *If $T : X \to Y$ be an operator. Then $\gamma(T) = \gamma(T^*)$.*

Suppose M,N are closed subspaces of X, then the *gap* between M and N is defined as:

$$\delta(M, N) := \sup_{x \in M, \|x\|=1} \inf_{y \in N} \|x - y\|$$

and the *symmetrised gap* is defined as:

$$\hat{\delta}(M, N) := \max\{\delta(M, N), \delta(N, M)\}$$

LEMMA 2.6.3. [**25**, p 96] *If E,F are two subspaces of X with $\dim E < \dim F$ then $\hat{\delta}(E, F) = 1$.*

In [**23**], Kato defined the gap and symmetrised gap between operators: For an operator $T \in \mathcal{B}(X)$ let $G(T)$ denote the graph of T, a subspace of $X \oplus X$. Then we define the gap and symmetrised gap between two operators, $T, S \in \mathcal{B}(X)$, by $\delta(S, T) := \delta(G(S), G(T))$ and $\hat{\delta}(S, T) := \hat{\delta}(G(S), G(T))$.

THEOREM 2.6.4. [**23**, Chapter IV, Theorem 2.14] *If $T, S \in \mathcal{B}(X)$ then*

$$\hat{\delta}(S, T) \leq \|S - T\|,$$

Kato also proved the following:

THEOREM 2.6.5. [**23**, Chapter IV, Theorem 3.1] *Let $T \in \mathcal{B}(X)$ and let Γ be a compact subset of the resolvent set $\rho(T)$. Then there is a $\delta > 0$ such that $\Gamma \subset \rho(S)$ for any $S \in \mathcal{B}(X)$ with $\hat{\delta}(S, T) < \delta$.*

THEOREM 2.6.6. [**23**, Chapter IV, Remark 3.3,Upper semi-continuity of the spectrum] *For any $T \in \mathcal{B}(X)$ and $\epsilon > 0$ there exists a $\delta > 0$ such that $\rho(\sigma(S), \sigma(T)) = \sup_{\lambda \in \sigma(S)} \operatorname{dist}(\lambda, \sigma(T)) < \epsilon$ if $\|S - T\| < \delta$.*

LEMMA 2.6.7. [**23**, Chapter IV, Theorem 3.16] *Let $T \in \mathcal{B}(X)$ and let $\sigma(T)$ be separated into two parts, $\sigma_1(T)$, $\sigma_2(T)$ by a closed curve Γ.*

Let $X = M_1(T) \oplus M_2(T)$ be the associated decomposition of X.

Then there exists a $\delta > 0$, depending on T and Γ, with the following properties.

If $S \in \mathcal{B}(X)$ has $\hat{\delta}(S, T) < \delta$ then the spectrum $\sigma(S)$ is separated by Γ into two parts $\sigma_1(S)$, $\sigma_2(S)$ (and $\Gamma \subset \rho(S)$). In the associated deomposition $X = M_1(S) \oplus M_2(S)$, $M_1(S)$ and $M_2(S)$ are respectively isomorphic with $M_1(T)$ and $M_2(T)$. In particular $\dim M_1(S) =$

$\dim M_1(T)$, $\dim M_2(S) = \dim M_2(T)$ *and both* $\sigma_1(S)$ *and* $\sigma_2(S)$ *are nonempty if this is true for* T. *The decomposition* $X = M_1(S) \oplus M_2(S)$ *is continuous in* S *in the sense that the projection* $P[S]$ *of* X *onto* $M_1(S)$ *along* $M_2(S)$ *tends to* $P[T]$ *in norm as* $\hat{\delta}(S,T) \to 0$.

This last statement tells us that spectral projections are continuous.

THEOREM 2.6.8. [**23**, Chapter IV, Theorem 5.17] *Let* $T, S \in \mathcal{B}(X)$ *and let* T *be Fredholm/semi-Fredholm. If* $\hat{\delta}(S,T) < \gamma(1+\gamma^2)^{-1/2}$ *where* $\gamma = \gamma(T)$, *then* S *is Fredholm/semi-Fredholm and* $\dim \ker S \leq \dim \ker T$, $\operatorname{codim} SX \leq \operatorname{codim} TX$.

Furthermore, there is a $\delta > 0$ *such that* $\hat{\delta}(S,T) < \delta$ *implies* $\operatorname{ind} S = \operatorname{ind} T$.

In particular, combining Theorems 2.6.4 and 2.6.8, we get:

COROLLARY 2.6.9. *Let* $S, T \in \mathcal{B}(X)$ *and let* T *be a semi-Fredholm operator. Then there exists* $\delta > 0$ *such that* $\|S - T\| < \delta$ *implies that* S *is semi-Fredholm with* $\operatorname{ind} S = \operatorname{ind} T$.

2.7. The Herrero conditions on Banach spaces

As mentioned above, [**21**, remark 2.3] states that if $T \in \mathcal{HC}(X)$ then T must satisfy Herrero's conditions. We provide a proof of this for completeness. Firstly, we need the following:

THEOREM 2.7.1. [**25**, Chapter II, Section 1, Theorem 25] *Let* X *be a Banach space and let* $T \in \mathcal{B}(X)$. *Then:*

$$\operatorname{dist}(0, \sigma_{ap}(T)) = \lim_{n\to\infty} j(T^n)^{1/n} \text{ where } j(T) = \inf_{\|x\|=1} \|Tx\|$$

LEMMA 2.7.2. *Let* X *be a Banach space and let* $T \in \mathcal{B}(X)$. *If* $\sigma(T) \subset \{\lambda \in \mathbb{C} : |\lambda| > 1\}$ *then for each non-zero* $x \in X$ *there exists* $C > 0$ *such that* $\|T^n x\| > C$ *for all* $n \in \mathbb{N}$.

PROOF. Since $\sigma(T) \subset \{\lambda \in \mathbb{C} : |\lambda| > 1\}$ and $\partial\sigma(T) \subset \sigma_{ap}(T)$ we have that $\operatorname{dist}(0, \sigma_{ap}(T)) = 1 + \delta$. By Theorem 2.7.1:

$$\operatorname{dist}(0, \sigma_{ap}(T)) = 1 + \delta = \lim_{n\to\infty} j(T^n)^{1/n}.$$

Therefore for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$(1+\delta)^n - \epsilon < j(T^n) < (1+\delta)^n + \epsilon$$

for all $n > N$. Choose $\epsilon < \delta$. In general $\|T^n y\| \geq (\inf_{\|x\|=1} \|T^n x\|)\|y\| = j(T^n)\|y\|$, so

$$(1 + \delta - \epsilon)\|x\| < ((1+\delta)^n - \epsilon)\|x\| < j(T^n)\|x\|$$

for all $n > N$.

Since $0 \notin \sigma(T)$, T is invertible so $T^n x \neq 0$ for all non-zero $x \in X$. Therefore for any x, there exists C' so that $C' \leq \min\{\|T^i x\| : 1 \leq i \leq N\}$. Letting $C := \min\{C', 1 + \delta - \epsilon\}$ gives us our result. □

PROPOSITION 2.7.3. [**21**, Proposition 2.2, Remark 2.3] *Let X be a separable infinite dimensional Banach space and suppose $T \in \mathcal{B}(X)$ is hypercyclic. Then:*

(i) For each invariant subspace M of T (with $M \neq X$), the induced operator $\hat{T} \in \mathcal{B}(X/M)$ is hypercyclic,
(ii) $\sigma_p(T^*) = \varnothing$,
(iii) $\operatorname{ind}(\lambda - T) \geq 0$ for all $\lambda \in \rho_{s-F}(T)$,
(iv) $\sigma_W(T) = \sigma(T)$,
(v) $\sigma_W(T) \cup \mathbb{T}$ is a connected set.

If instead $T \in \overline{\mathcal{HC}(X)}$ then T satisfies Herrero's conditions.

PROOF. (i) Let $x \in X$ be a hypercyclic vector for T. Clearly $x \notin M$ (as otherwise $T^n x \in M$ for all $n \in \mathbb{N}$). $\operatorname{Orb}_T(x)$ is dense in X so $\operatorname{Orb}_{\hat{T}}(x + M)$ is dense in X/M and therefore $x + M$ is hypercyclic for $\hat{T}$.

(ii) Suppose $\lambda \in \sigma_p(T^*)$ and $\phi \in X^*$ is such that $T^*\phi = \lambda\phi$. Then we have that $\phi(Tx) = \lambda\phi(x)$ and $\phi(T^n x) = \lambda^n \phi(x)$. If x is a hypercyclic vector for T then $\operatorname{Orb}_T(x)$ is dense in X and $\phi(\operatorname{Orb}_T(x)) = \{\lambda^n \phi(x) : n \in \{0\} \cup \mathbb{N}\}$ is dense in $\mathbb{K}$. So we have a contradiction.

(iii) $\operatorname{ind}(T - \lambda) = \dim\ker(T - \lambda) - \operatorname{codim}(T - \lambda)X$. $\operatorname{codim}(T - \lambda) = 0$ for all $\lambda \in \rho_{s-F}(T)$ since $\sigma_p(T^*) = 0$ so $\operatorname{ind}(T - \lambda) = \dim\ker(T - \lambda) \geq 0$.

(iv)

$$\sigma_W(T) = \sigma(T) \setminus \{\lambda \in \mathbb{C} : \lambda \in \rho_{s-F}(T), \operatorname{ind}(T - \lambda) = 0\}$$

Suppose $\lambda \notin \sigma_W(T)$. Then $\lambda \in \rho_{s-F}(T)$ and $\operatorname{ind}(T - \lambda) = 0$. But if T is hypercyclic then $\operatorname{ind}(T - \lambda) = \dim\ker(T - \lambda)$ so $\dim\ker(T - \lambda) = \operatorname{codim}(T - \lambda) = 0$ and hence $T - \lambda$ is invertible. So $\sigma_W(T) = \sigma(T)$.

(v) Since $\sigma(T) = \sigma_W(T)$, this is the same as showing that $\sigma(T) \cup \mathbb{T}$ is connected. If σ is a connected component of $\sigma(T)$ and $\sigma \subset B(0, 1)$ then (using Proposition 2.4.4) there is a Riesz decomposition $X = M_1 \oplus M_2$ of X into T-invariant subspaces such that $\sigma(T|_{M_1}) = \sigma$. The operator $\hat{T} : X/M_2 \to X/M_2$ is equivalent to $T|_{M_1}$ and is hypercyclic by (1). Since it has spectral radius strictly smaller than 1, we have $\|(T|_{M_1})^n\| < 1$ for all large n and this contradicts hypercyclicity of $T|_{M_1}$.

If $\sigma \subset \{\lambda \in \mathbb{C} : |\lambda| > 1\}$ we again look at the Riesz spectral invariant subspace M associated with $\sigma(T) \setminus \sigma$ and the operator $\hat{T} : X/M \to X/M$ induced by T on X/M (which is hypercyclic by (i)). Let $x + M$ be hypercyclic for $\hat{T}$. Since $\sigma(\hat{T}) \subset \{\lambda \in \mathbb{C} : |\lambda| > 1\}$ we have that there exists $C > 0$ so that $\|\hat{T}^n(x + M)\| > C$ (by Lemma 2.7.2) so $\operatorname{Orb}_{\hat{T}}(x)$ is not dense in X/M, which contradicts the hypercyclicity of $x + M$.

Therefore $\sigma(T) \cup \mathbb{T} = \sigma_W(T) \cup \mathbb{T}$ is connected

Now suppose $T \in \overline{\mathcal{HC}(X)}$ with $T_n \in \mathcal{HC}(X)$ and $T_n \to T$. We show that T must satisfy the Herrero conditions.

(1) Suppose $\sigma_W(T) \cup \mathbb{T}$ is not connected. Then there is some connected component, σ, of $\sigma_W(T)$ that does not intersect the circle.

Either σ is a connected component of $\sigma(T) \cup \mathbb{T}$ or it is surrounded by some component of $\rho_{s-F}(T)$ with index 0.

If σ is a connected component of $\sigma(T) \cup \mathbb{T}$ then let Γ be a closed curve in $\mathbb{C}$ that separates $(\sigma_W(T) \cup \mathbb{T}) \setminus \sigma$ and $\sigma(T)$. Then there is a decomposition $X = E_1 \oplus E_2$ with $\sigma(T|_{E_1}) = \sigma$ and $\sigma(T|_{E_2}) = \sigma(T) \setminus \sigma$. By Lemma 2.6.7 there exists $\delta > 0$ so that if $\hat{\delta}(S, T) < \delta$ then there is a similar decomposition $X = F_1 \oplus F_2$, where F_1 and F_2 are S-invariant subspaces of X with $F_1 \sim E_1$ and $F_2 \sim E_2$. But we can find T_n with $\hat{\delta}(T_n, T) < \|T_n - T\| < \delta$ (by Theorem 2.6.4), so γ separates $\sigma(T_n) = \sigma_W(T_n)$ by (iv) into two components and one of them does not intersect the circle, contradicting (v) above.

If σ is surrounded by some component ω of $\rho_{s-F}(T)$ with index 0 then we can find a closed path $\Gamma \subset \omega \cap \rho^r_{s-F}(T)$ that surrounds σ and separates σ from $\mathbb{T}$. Since $T - \lambda$ is a Kato operator for each $\lambda \in \Gamma$, and $\gamma(T - z) > 0$ and is continuous at every point z where $T - z$ is Kato there must be some $C > 0$ such that $\gamma(T - \lambda) > C$ for all $\lambda \in \Gamma$. Choose n such that $\|T_n - T\| < C/\sqrt{1 + C^2}$. Then by Theorem 2.6.8 $T_n - \lambda$ is Fredholm for all $\lambda \in \Gamma$. Since Γ is connected, it must be contained in some connected component of $\rho_{s-F}(T_n)$ and hence the index must be constant on Γ.

$\partial\sigma \subset \partial\sigma_W(T) \subset \sigma_{ess}(T)$ and $\sigma_{ess}(T) = \sigma(L_{\hat{T}}) \cup \sigma(R_{\hat{T}})$ by Corollary 2.4.3 so there is some component of either $\sigma(L_{\hat{T}})$ or $\sigma(R_{\hat{T}})$ separated from $\mathbb{T}$ by Γ, Without loss of generality, we may assume that there is a component of $\sigma(L_{\hat{T}})$ separated from $\mathbb{T}$ by Γ, Using Lemma 2.6.7 with Γ as above and $L_{\hat{T}}$ we know there is some $\delta > 0$ so that the projection associated with the inside of Γ and $\hat{S}$ with $\delta(\hat{T}, \hat{S}) < \delta$ is non-zero. Choosing m such that $\|T - T_m\| < \delta$ gives us $\delta(\hat{T}, \hat{T_m}) \leq \|\hat{T} - \hat{T_m}\| \leq \|T - T_m\| < \delta$, so there is some connected component of $\sigma_{ess}(T_m)$, σ', from $\mathbb{T}$. $\sigma_{ess}(T) \subset \sigma_W(T)$. Let $k = \max\{n, m\}$, then $\mathrm{ind}(T - \lambda) = 0$ for all $\lambda \in \Gamma$ so σ' is contained in some connected component of $\sigma_W(T_k)$ that is separated from $\mathbb{T}$ by Γ. So T_k is hypercyclic and $\sigma_W(T_k) \cup \mathbb{T}$ is not connected, contradicting (v) above.

(2) Suppose $\lambda \in \sigma_0(T)$. Let Γ be a closed curve in $\mathbb{C}$ that separates λ and $\sigma(T) \setminus \{\lambda\}$. Then there is a decomposition $X = E_1 \oplus E_2$ with $\sigma(T|_{E_1}) = \{\lambda\}$ and $\sigma(T|_{E_2}) = \sigma(T) \setminus \{\lambda\}$. By Lemma

2.6.7 there exists $\delta > 0$ so that if $\hat{\delta}(S,T) < \delta$ then there is a similar decomposition $X = F_1 \oplus F_2$, where F_1 and F_2 are S-invariant subspaces of X with $F_1 \sim E_1$ and $F_2 \sim E_2$. Choose n large enough so that $\|T - T_n\| < \delta$. By Theorem 2.6.4 we then have $\hat{\delta}(T_n, T) < \|T - T_n\| < \delta$. Since E_1 is finite dimensional, so is F_1. Let $d := \dim E_1 = \dim F_1$. Then $\dim X/F_2 = \dim F_1 = d < \infty$ and $\hat{T_n} : X/F_2 \to X/F_2$ is a hypercyclic operator on a finite dimensional space, which is impossible.

(3) Suppose $\operatorname{ind}(T - \lambda) < 0$ for some $\lambda \in \rho_{s-F}(T)$. By Corollary 2.6.9 there exists $\delta > 0$ such that $\|(T - \lambda) - (T_n - \lambda)\| = \|T - T_n\| < \delta$ implies $\operatorname{ind}(T_n - \lambda) = \operatorname{ind}(T - \lambda) < 0$, which contradicts (iii) above.

□

2.8. Block representations

If $Y \subseteq X$ is a closed subspace and $T \in \mathcal{B}(X)$ has $T(X) \subseteq Y$, we can naturally regard T as an operator $T\colon X \to Y$ (with the same values as T but formally a different operator since it has different codomain Y instead of X). On the other hand if $X = Y \oplus Z$ is a topological direct sum of (closed) subspaces $Y, Z \subseteq X$, then we can let P_Y denote the projection on X with range Y and kernel Z (as in (2.3.1)). As just mentioned, we allow an ambiguity of notation and also consider $P_Y\colon X \to Y$.

As is common, when we have $X = Y \oplus Z$ and $T \in \mathcal{B}(X)$ a general operator, we may consider T as represented by a block matrix

$$T = \begin{pmatrix} T_{1,1} & T_{1,2} \\ T_{2,1} & T_{2,2} \end{pmatrix} \tag{2.8.1}$$

where

$$\begin{aligned} T_{1,1} &= P_Y \circ T|_Y \in \mathcal{B}(Y), \\ T_{2,2} &= (I - P_Y) \circ T|_Z \in \mathcal{B}(Z), \\ T_{2,1} &= (I - P_Y) \circ T|_Y \in \mathcal{B}(Y, Z), \text{ and} \\ T_{1,2} &= P_Y \circ T|_Z \in \mathcal{B}(Z, Y). \end{aligned}$$

LEMMA 2.8.1. *In this convention, powers T^n of T may be computed using standard matrix multiplication rules for block matrices and we have the inequalities*

$$\begin{aligned} \|T\| &\leq \sum_{i=1}^{2} (\|T_{i,1}\|\|P_Y\| + \|T_{i,2}\|\|I - P_Y\|) \\ \|T_{1,j}\| &\leq \|T\|\|P_Y\| \\ \|T_{2,j}\| &\leq \|T\|\|I - P_Y\| \quad (j = 1, 2) \end{aligned} \tag{2.8.2}$$

PROOF. $T = T_{1,1}P_Y + T_{2,1}P_Y + T_{1,2}(I - P_Y) + T_{2,2}(I - P_Y)$, the matrix rules follows and also

$$\|T\| \leq \sum_{i=1}^{2}(\|T_{i,1}\|\|P_Y\| + \|T_{i,2}\|\|I - P_Y\|).$$

We have that $T_{1,j}$ is a restriction of $P_Y \circ T$ so $\|T_{1,j}\| \leq \|T\|\|P_Y\|$. Similarly we have that $\|T_{2,j}\| \leq \|T\|\|I - P_Y\|$ □

With the aid of P_Y we can consider operators $S \in \mathcal{B}(Y)$ as inducing operators $T = S \circ P_Y$ on X. We can say then that T *acts on* Y (provided that the complementary subspace Z for Y in X is clearly understood from the context).

In other words an operator $S \in \mathcal{B}(Y)$ gives rise to

$$T = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{B}(X)$$

(acting on Y with $T_{1,1} = S$). However $\|T\|$ and $\|S\|$ will usually differ and hence we need to take care of the distinction between S and T.

2.9. Complexifications of Banach spaces

There are different natural ways to define a complex norm on the complexification of a real Banach space X, however all of these are complex-isomorphic to each other ([**27**]).

Let $\tilde{X}$ be the complexification of X. Then $\tilde{X} = X \oplus X$ as an $\mathbb{R}$-vector space and multiplication by $a+bi \in \mathbb{C}$ is given by $(a+bi)(x, y) = (ax - by, ay + bx)$. Let $T \in \mathcal{B}(\tilde{X})$, that is suppose T is a $\mathbb{C}$-linear operator on $\tilde{X}$. Then $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$ as an $\mathbb{R}$-linear operator with respect to the decomposition $\tilde{X} = X \oplus X$.

$$\begin{aligned}
(a + bi)T(x, y) &= (a + bi)\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\
&= (a + bi)\begin{pmatrix} T_{11}x + T_{12}y \\ T_{21}x + T_{22}y \end{pmatrix} \\
&= \begin{pmatrix} a(T_{11}x + T_{12}y) - b(T_{21}x + T_{22}y) \\ a(T_{21}x + T_{22}y) + b(T_{11}x + T_{12}y) \end{pmatrix} \quad (2.9.1)
\end{aligned}$$

$$\begin{aligned}
T(a + bi)(x, y) &= T(ax - by, ay + bx) \\
&= \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}\begin{pmatrix} ax - by \\ ay + bx \end{pmatrix} \\
&= \begin{pmatrix} T_{11}(ax - by) + T_{12}(ay + bx) \\ T_{21}(ax - by) + T_{22}(ay + bx) \end{pmatrix} \\
&= \begin{pmatrix} a(T_{11}x + T_{12}y) + b(T_{12}x - T_{11}y) \\ a(T_{21}x + T_{22}y) - b(T_{21}y - T_{22}x) \end{pmatrix} \quad (2.9.2)
\end{aligned}$$

Hence complex linearity, $(a + bi)T(x, y) = T[(a + bi)(x, y)]$, implies (2.9.1) must agree with (2.9.2) so

$$\begin{pmatrix} -T_{21}x - T_{22}y \\ T_{11}x + T_{12}y \end{pmatrix} = \begin{pmatrix} T_{12}x - T_{11}y \\ -T_{21}y + T_{22}x \end{pmatrix}$$

This is true for all $x, y \in X$, so setting $y = 0$ gives us that $T_{11}x = T_{22}x$ and $T_{12}x = -T_{21}x$ for all $x \in X$, i.e. $T_{11} = T_{22}$ and $T_{12} = -T_{21}$.

Conversely, given any two $\mathbb{R}$-linear operators $T_1, T_2 \in \mathcal{B}(X)$ the operator $T = \begin{pmatrix} T_1 & T_2 \\ -T_2 & T_1 \end{pmatrix}$ is a bounded $\mathbb{C}$-linear operator on $\mathcal{B}(\tilde{X})$. Therefore we have that

$$\mathcal{B}(\tilde{X}) = \left\{ \begin{pmatrix} T_1 & T_2 \\ -T_2 & T_1 \end{pmatrix} : T_1, T_2 \in \mathcal{B}(X) \right\}. \tag{2.9.3}$$

There are many ways to define a norm on a complexification and different ones suit different purposes. However, all such complexification will be complex-isomorphic to each other (see [**27**] for details). We also have that if a complexification is reasonable (in the sense of [**27**]) then the complexification of the dual space and the dual of the complexification are isomorphic [**27**, Proposition 7].

DEFINITION 2.9.1. Given a Banach space X over $\mathbb{R}$ and $T_1 \in \mathcal{B}(X)$ we define the spectrum of T_1 to be $\sigma \begin{pmatrix} T_1 & 0 \\ 0 & T_1 \end{pmatrix}$.

2.10. Banach spaces with few operators

In [**13**] Gowers and Maurey constructed a space that is hereditarily indecomposable such that all operators on this space are of the form $\lambda + S$ where $\lambda \in \mathbb{C}$ and S is strictly singular. In this sense they had constructed a space with few operators. They also show that every space that is hereditarily indecomposable has few operators.

They also asked whether it was possible to construct a space where all of the operators were of the form $\lambda + K$ where K was a compact operator. In [**5**], Argyros and Haydon constructed such a space over the reals. In addition, this space was hereditarily indecomposable and the dual of this space is isomorphic to ℓ^1.

This was later followed by constructions by Tarbard ([**30**]). The Tarbard constructions are also hereditarily indecomposable, have dual isomorphic to ℓ^1 and the operators on this space all have the form $\sum_{i=0}^{k-1} \lambda_i N^i + K$, where N is a nilpotent of order k and K is a compact operator.

The complexifications of both the Argyros-Haydon space and the Tarbard spaces satisfy the same properties by 2.9.3 and since the dual of a complexification is isomorphic to the complexification of the dual. We will show in Chapter 5 that the Herrero conditions characterise the norm closure of $\mathcal{HC}(X)$ for any Banach space, X, over the complex

numbers that is hereditarily indecomposable and has dual isomorphic to ℓ^1,

CHAPTER 3

Generalised nilpotent operators

In this chapter we discuss operators (always on an infinite dmensional separable Banach space X over $\mathbb{C}$) of the form $T = \lambda I + S$, where $|\lambda| = 1$ and S is a generalised nilpotent operator. We say that S is a generalised nilpotent operator if it has dense generalised kernel $N^\infty(S)$ (see §3.1). Examples of such operators S include backward shifts with respect to a basis for X and "generalised shifts".

In particular, in §3.2 we discuss circularity properties of parts of the spectrum of a weighted shift when the basis is unconditional. We also introduce a notion of Fredholm radius for a backward shift (which will be applied in §4.2). Further we note that quasinilpotent weighted backward shifts are norm limits of nilpotents (Proposition 3.2.16).

In §3.3 we consider a notion of generalised shift (which does not require a basis) and show that such operators are both generalised nilpotents and have dense hyper-range. In Theorem 3.3.3, we show that if $S \in \mathcal{B}(X)$ has both of these properties, then $T = \lambda I + S \in \mathcal{M}\mathrm{ix}(X) \subset \mathcal{HC}(X)$ for $|\lambda| = 1$.

The main results in this chapter are in §3.5 where we show that operators of the form $T = \lambda I + S$ are norm limits of mixing operators (hence norm limits of hypercyclic operators) when $|\lambda| = 1$ and $S \in \mathcal{B}(X)$ a generalised nilpotent operator. These results have appeared in [**8**].

Finally in §3.6 we consider applications to other kinds of operators like forward shifts and bilateral shifts S, again concluding that

$$T = \lambda I + S \subset \overline{\mathcal{M}\mathrm{ix}(X)} \subset \overline{\mathcal{HC}(X)}$$

for $|\lambda| = 1$ and various assumptions on $S \in \mathcal{B}(X)$.

3.1. The generalised kernel

DEFINITION 3.1.1. Given an operator $T \in \mathcal{B}(X)$ we denote by $N^\infty(T) := \bigcup_{i=1}^\infty \ker T^i$ the generalised kernel of T.

REMARK 3.1.2. Note that any nilpotent operator N has $N^k = 0$ for some $k \in \mathbb{N}$ so $N^\infty(N) = X$. Operators with dense generalised kernel may be viewed as *generalised nilpotent operators.*

Note also that the converse is true: if $N^\infty(N) = X$ then $N^k = 0$ for some k. For suppose otherwise. Then for all k there exists a vector x_k such that $N^k x_k \neq 0$. Then $\ker N^k$ is nowhere dense for all $k \in \mathbb{N}$

(since it is a proper closed subspace) and $X = N^\infty(T) = \bigcup_{i=1}^\infty \ker N^i$ which contradicts the Baire Category Theorem.

LEMMA 3.1.3. *Suppose σ_1 is a connected component of $\sigma(T)$ and σ_2 its complement in $\sigma(T)$ and let P be the spectral projection associated with σ_1. If σ_1 has any eigenvalues then all eigenvectors and generalised eigevectors associated with those eigenvalues are contained in PX.*

PROOF. Let $x \in \ker(T-\lambda)^n$. Then $x = (x_1, x_2)$ where $x_1 \in PX$ and $x_2 \in (I-P)X$. Also, $T = T_1 \oplus T_2$ where $T_1 = T|_{PX} \in \mathcal{B}(PX)$, $T_2 = T|_{(I-P)X} \in \mathcal{B}((I-P)X)$. Then

$$0 = (T-\lambda)^n x = ((T_2-\lambda)^n x_1, (T_2-\lambda)^n x_2) \Rightarrow x_2 = 0$$

since $\lambda \notin \sigma(T_2) = \sigma_2$. So $x \in PX$. □

COROLLARY 3.1.4. *If $N^\infty(T)$ is dense in X then $\sigma(T)$ has only one connected component. Therefore, T has no isolated eigenvalues and $\sigma_0(T) = \varnothing$. Further, $\sigma_0(\lambda I + T) = \varnothing$ for $\lambda \in \mathbb{C}$.*

We note this because we will be considering the Herrero conditions for operators $\lambda I + T$ with $|\lambda| = 1$ and this corollary says that the Herrero condition (2) is satisfied for such operators.

COROLLARY 3.1.5. *If T is a triangular quasinilpotent operator then T is a generalised nilpotent.*

PROOF. If Y_n is a sequence of finite dimensional T invariant subspaces with $\lim_{n\in\mathbb{N}} Y_n = X$. Then $T|_{Y_n}$ is a finite dimensional operator on Y_n whose eigenvalues are contained in $\sigma(T) = \{0\}$. So then $T|_{Y_n}$ is a nilpotent and $Y_n \subset N^\infty(T)$. So $\cup_{n=1}^\infty Y_n \subset N^\infty(T)$ and so $X = \overline{\cup_{n=1}^\infty Y_n} \subset \overline{N^\infty(T)}$. □

COROLLARY 3.1.6. *If T has dense generalised kernel then T is triangular.*

PROOF. Apply Lemma 2.5.2 to T. □

3.2. Backward Shifts

In this section we give the (standard) notion of a weighted backward shift T with respect to a Schauder basis for X (and we always take X to be a separable infinite dimensional Banach space). While this notion makes sense for a real Banach space (with a basis), we need to pass to the complexification to make good sense of the spectrum (see §2.9) and hence we are justified in considering only the complex case for our purposes.

In the case when the basis is unconditional, we establish that the spectrum $\sigma(T)$ must be a disc about the origin, together with circular symmetry of parts of the spectrum of T. The point spectrum $\sigma_p(T)$ is either an open disc $\{\lambda \in \mathbb{C} : |\lambda| < r_p(T)\}$ or the closed disc about

the origin with a radius we also denote by $r_p(T)$, and this allows us to deduce circularity properties for the semi-Fredholm part $\sigma(T) \cap \rho_{s-F}(T)$ of $\sigma(T)$. With this we motivate a definition for what we call the Fredholm radius $r_F(T)$. We have $r_F(T) \leq r_p(T) \leq r(T)$ and in general both inequalities can be strict (as we shall show in Example 3.2.14).

In the case of a symmetric basis, simple norm estimates on T can be given in terms of the weights and this allows us to show (Proposition 3.2.16) that if T is quasinilpotent, then it is a norm limit of nilpotent operators. While this is not needed in our later results on the norm closure of $\mathcal{HC}(X)$ because we can allow more general types of operators in §3.6, it relates to the question of when quasinilpotents are norm limits of nilpotents which we discuss later in §5.4.

DEFINITION 3.2.1. We say that an operator $T \in \mathcal{B}(X)$ is a *weighted backward shift* if there exists a basis $\{e_n\}_{n=1}^{\infty}$ for X such that $Te_1 = 0$ and $Te_{n+1} = w_n e_n$ for some sequence of non-zero scalars $\{w_n\}_{n=1}^{\infty}$.

For the case of an orthonormal basis on a Hilbert space results similar to the following can be found in [**20**].

LEMMA 3.2.2. *Suppose $T \in B(X)$ is a weighted backward shift on an unconditional basis e_n. Then*

$$\sigma_{ap}(T) = \sigma(T) = \overline{r\mathbb{D}}$$

for $r = r(T)$ the spectral radius.

PROOF. Let $\lambda \in \sigma_{ap}(T)$, and let $x_n = \sum_{i=1}^{\infty} a_i^n e_i$ be such that $\|x_n\| = 1$ and $\|Tx_n - \lambda x_n\| \to 0$. We have $Te_{n+1} = w_{n+1}e_n$. So

$$\|Tx_n - \lambda x_n\| = \left\| \sum_{i=1}^{\infty} (w_i a_{i+1}^n - \lambda a_i^n) e_i \right\| \to 0$$

Let $\mu \in \mathbb{C}$ such that $|\mu| \leq 1$. Since e_n is unconditional basis we have that $y_n := \sum_{i=1}^{\infty} \mu^i a_i^n e_i$ converges, has a non-zero limit and

$$\begin{aligned}
\|(T - \mu\lambda)y_n\| &= \left\| (T - \mu\lambda) \sum_{i=1}^{\infty} \mu^i a_i^n e_i \right\| \\
&= \left\| \sum_{i=1}^{\infty} (w_i \mu^{i+1} a_{i+1}^n - \lambda\mu\mu^i a_i^n) e_i \right\| \\
&= \left\| \sum_{i=1}^{\infty} \mu^{i+1} (w_i a_{i+1}^n - \lambda a_i^n) e_i \right\| \\
&\leq 2|\mu| K \left\| \sum_{i=1}^{\infty} (w_i a_{i+1}^n - \lambda a_i^n) e_i \right\| \to 0
\end{aligned}$$

where K is the unconditional constant for e_i. Therefore $\mu\lambda \in \sigma_{ap}(T)$ for all $|\mu| \leq 1$. We also have that $\partial\sigma(T) \subset \sigma_{ap}(T)$ so $\sigma(T) = \sigma_{ap}(T) = \overline{r\mathbb{D}}$ for some $r \geq 0$. □

LEMMA 3.2.3. *Let X be a separable infinite dimensional Banach space with an unconditional basis, e_n, and let $T \in \mathcal{B}(X)$ be a backward weighted shift on e_n with some non-zero eigenvalue λ. Then $\mu\lambda$ is also an eigenvalue for $|\mu| \leq 1$ so $|\lambda|\mathbb{D} \subset \sigma_p(T)$.*

Moreover there exists $r = r_p(T) > 0$ such that $\sigma_p(T)$ is either $r\mathbb{D}$ or $\overline{r\mathbb{D}}$.

PROOF. Suppose $Te_{n+1} = w_n e_n$. λ is an eigenvalue so there exists an $x = \sum_{n=1}^{\infty} x_n e_n \in X$ such that $Tx = \lambda x$. Then

$$\sum_{n=1}^{\infty} \lambda x_n e_n = T\left(\sum_{n=1}^{\infty} x_n e_n\right) = \sum_{n=1}^{\infty} w_n x_{n+1} e_n$$

$$\Rightarrow \lambda x_n = w_n x_{n+1} \Rightarrow x_{n+1} = \frac{\lambda x_n}{w_n} \Rightarrow x_{n+1} = \frac{\lambda^n x_1}{\Pi_{i=1}^{n} w_i}$$

So every eigenvector associated with λ is some scalar multiple of

$$v_\lambda := \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{\Pi_{i=1}^{n-1} w_i} e_n. \tag{3.2.1}$$

Since e_n is an unconditional basis we have that $v_{\mu\lambda}$ converges and

$$T(v_{\mu\lambda}) = \mu\lambda v_{\mu\lambda}.$$

Let $r := \sup\{|\lambda| : \lambda \in \sigma_p(T)\}$. Then by above we have that $r\mathbb{D} \subset \sigma_p(T) \subset \overline{r\mathbb{D}}$. However $r \in \sigma_p(T)$ iff $\partial(r\mathbb{D}) \subset \sigma_p(T)$ by above so $\sigma_p(T)$ must be either $r\mathbb{D}$ or $\overline{r\mathbb{D}}$. □

NOTATION 3.2.4. For T a weighted backward shift on an unconditional basis of X, we put $r_p(T) = r(T) = 0$ if T is quasinilpotent and take $r_p(T)$ as in Lemma 3.2.3 otherwise. We call $r_p(T)$ the (inner) point spectral radius of T.

LEMMA 3.2.5. *Suppose $T \in \mathcal{B}(X)$ is a weighted backward shift on a basis e_n. Then $T - \lambda$ has dense range for all $\lambda \in \mathbb{C}$.*

PROOF. Suppose $Te_{n+1} = w_n e_n$ for all $n \in \mathbb{N}$. We have that, for any $\lambda \in \mathbb{C}\backslash\{0\}$, $(T-\lambda)\lambda^{-1}e_1 = e_1$ and $e_k = (T-\lambda)(\sum_{i=1}^{k} \lambda^{-i}(\Pi_{j=k-i+1}^{k-1} w_j)e_{k-i+1})$ we have that $(T-\lambda)X$ is dense in X. When $\lambda = 0$ $e_k = T(w_k^{-1}e_{k+1})$ so TX is dense in X. □

COROLLARY 3.2.6. *If $T \in \mathcal{B}(X)$ is a weighted backward shift on a Schauder basis of X, then*

$$\sigma(T) \cap \rho_{s-F}(T) \subset \sigma_p(T).$$

PROOF. If $\lambda \in \rho_{s-F}(T)$, then $T-\lambda$ has closed range and so Lemma 3.2.5 implies that $(T-\lambda)(X) = X$. Thus if also $\lambda \in \sigma(T)$, we must have $\ker(T-\lambda) \neq \{0\}$. □

LEMMA 3.2.7. *Suppose $T \in \mathcal{B}(X)$ is a weighted backward shift on an unconditional basis e_n with strictly positive real weights w_n and there exists $0 < \lambda \in \mathbb{R} \cap \sigma_p(T)$ such that $T-\lambda$ has closed range. Then $T-\mu$ has closed range for any $|\mu| \leq \lambda$.*

PROOF. Take $y = \sum_{i=1}^{\infty} y_i e_i \in X$ with $0 \leq y_i \in \mathbb{R}$ and assume $0 < \mu \in \mathbb{R}$. We show first that $y \in (T-\mu)X$ in this case. Since $T-\lambda$ has closed range and has dense range we must have that $(T-\lambda)X = X$. Therefore y is in the range of $(T-\lambda)$ and there exists $x \in X$ such that $(T-\lambda)x = y$. Since $\lambda \in \sigma_p(T)$, we have that $x + cv_\lambda \in X$ (with v_λ as in 3.2.1), so we can assume that $x = \sum_{i=1}^{\infty} x_i e_i$ is such that $x_1 = 0$.

$$\begin{aligned} & (T-\lambda)x = \sum_{i=1}^{\infty} (w_i x_{i+1} - \lambda x_i) e_i = y \\ \Rightarrow \quad & y_i = w_i x_{i+1} - \lambda x_i \\ \Rightarrow \quad & x_{i+1} = \frac{y_i + \lambda x_i}{w_i}. \end{aligned}$$

Since $|\mu| \leq |\lambda|$ we must have that $\mu \in \sigma_p(T)$ also. Therefore if we have $z \in X$ such that $(T-\mu)z = y$ then we also have that $(T-\mu)(z+cv_\mu) = y$ for all $c \in \mathbb{C}$ and so we can assume that $z = \sum_{i=1}^{\infty} z_i e_i$ has $z_1 = 0$ as well and just as above we have

$$z_{i+1} = \frac{y_i + \mu z_i}{w_i}.$$

CLAIM 1. $\sum_{i=1}^{\infty} z_i e_i$ converges.

PROOF. We have that $z_1 = x_1$. We show, by induction that $z_{k+1} \leq x_{k+1}$. Assuming $z_k \leq x_k$ we have

$$\begin{aligned} z_k \leq x_k \quad & \Rightarrow \quad \mu z_k \leq \lambda x_k \Rightarrow y_i + \mu z_k \leq \lambda y_i + x_k \\ & \Rightarrow \quad z_{i+1} = \frac{y_i + \mu z_i}{w_i} \leq \frac{y_i + \lambda x_i}{w_i} \\ & \Rightarrow \quad z_{k+1} \leq x_{k+1} \end{aligned}$$

Since e_n is an unconditional basis we must then have that $\sum_{i=1}^{\infty} z_i e_i$ converges. □

Now we look at arbitrary $y = \sum_{i=1}^{\infty} y_i e_i$ and arbitrary $|\mu| \leq \lambda$. We complete the proof by showing that $y = (T-\mu)z$ for some $z \in X$. There exists a sequence of $\theta_i \in \mathbb{T}$ such that $0 \leq \theta_i y_i \in \mathbb{R}$ and $\pi \in \mathbb{T}$ such that $0 \leq \pi\mu \in \mathbb{R}$. We can still assume that $z_1 = 0$ and so we must have

$$z_{i+1} = \frac{y_i + \mu z_i}{w_i}.$$

Let $y'_i := \theta_i y_i$. Since e_n is an unconditional basis we have that $\sum_{i=1}^{\infty} y'_i e_i$ converges and so, by above, there exists $z' = \sum_{i=1}^{\infty} z'_i e_i$ with $(T - \pi\mu)z' = y$ and $z'_1 = 0$.

CLAIM 2. $|z_i| \leq |z'_i|$.

PROOF. We have that $z_1 = z'_1 = 0$. For all other terms we have

$$z_{k+1} = \sum_{i=1}^{k} \frac{\mu^{k-i} y_i}{\Pi_{j=1}^{k-i+1} w_{k-j+1}}.$$

and

$$z'_{k+1} = \sum_{i=1}^{k} \frac{(\pi\mu)^{k-i} \theta_i y_i}{\Pi_{j=1}^{k-i+1} w_{k-j+1}}.$$

Hence

$$\begin{aligned} |z_{k+1}| &= \left| \sum_{i=1}^{k} \frac{\mu^{k-i} y_i}{\Pi_{j=1}^{k-i+1} w_{k-j+1}} \right| \leq \sum_{i=1}^{k} \frac{|\mu|^{k-i} |y_i|}{\Pi_{j=1}^{k-i+1} w_{k-j+1}} \\ &= \sum_{i=1}^{k} \frac{(\pi\mu)^{k-i} \theta_i y_i}{\Pi_{j=1}^{k-i+1} w_{k-j+1}} = z'_{k+1} = |z'_{k+1}| \end{aligned}$$

Since $z' = \sum_{i=1}^{\infty} z'_i e_i$ converges, $|z_i| \leq |z'_i|$ and e_n is an unconditional basis, we must have that $\sum_{i=1}^{\infty} z_i e_i$ converges. □

Note that $(T - \mu)z = y$. Therefore $(T - \mu)X = X$ for all μ with $|\mu| \leq \lambda$. □

COROLLARY 3.2.8. *Suppose $T \in \mathcal{B}(X)$ is a weighted backward shift on an unconditional basis e_n and there exists $0 < \lambda \in \mathbb{R} \cap \sigma_p(T)$ such that $T - \lambda$ has closed range. Then $T - \mu$ has closed range for any $\mu \in \mathbb{C}$ with $|\mu| \leq \lambda$.*

PROOF. For each weight w_n there exists $\theta_n \in \mathbb{T}$ such that $\theta_n w_n \in \mathbb{R}^+$. If we look at T on the unconditional basis $(f_n)_{n=1}^{\infty}$ where $f_1 = e_1$ and $f_{n+1} := \Pi_{i=1}^{n} \theta_i^{-1} e_{n+1}$ for all $n \geq 1$, then the weights of T with respect to this new basis are $w'_n = \theta_n w_n$, and so we have that $w'_n \in \mathbb{R}^+$. If we apply the previous lemma with respect to this new basis we get the required result. □

LEMMA 3.2.9. *Suppose $T \in \mathcal{B}(X)$ is a weighted backward shift on an unconditional basis e_n and there exists $\lambda \in \mathbb{C} \cap \sigma_p(T)$ such that $T - \lambda$ has closed range. Then $T - |\lambda|$ has closed range.*

PROOF. We have that $Te_{n+1} = w_n e_n$. Suppose θ is such that $\lambda = \theta|\lambda|$. We have that $T - |\lambda|$ has closed range iff $\theta T - \lambda$ has closed range. Let $f_n := \theta^{1-n} e_n$. Note that f_n and e_n are equivalent bases and that $\theta T f_{n+1} = w_n f_n$. Choose $\sum_{i=1}^{\infty} y_i f_i \in X$. Since f_i and e_i are equivalent,

we must also have that $\sum_{i=1}^{\infty} y_i e_i$ converges. Since $(T-\lambda)X = X$, there exists $\sum_{i=1}^{\infty} x_i e_i \in X$ such that $(T-\lambda)\left(\sum_{i=1}^{\infty} x_i e_i\right) = \sum_{i=1}^{\infty} y_i e_i$. Again, since e_n and f_n are equivalent, we must have that $\sum_{i=1}^{\infty} x_i f_i$ converges and $(\theta T - \lambda)\left(\sum_{i=1}^{\infty} x_i f_i\right) = \sum_{i=1}^{\infty} y_i f_i$. Therefore $(\theta T - \lambda)X = X$ and so $\theta T - \lambda$ has closed range and therefore $T - |\lambda|$ has closed range, as required. □

COROLLARY 3.2.10. *Suppose $T \in \mathcal{B}(X)$ is a weighted backward shift on an unconditional basis and there exists $\lambda \in \mathbb{C} \cap \sigma_p(T)$ such that $T - \lambda$ has closed range. Then $T - \mu\lambda$ has closed range for all $\mu \in \mathbb{C}$ with $|\mu| \leq 1$.*

Therefore given any weighted backward shift T on an unconditional basis of X, there is at most one bounded component, ω of $\rho_{s-F}(T)$ and $\omega = B(0,r)$ for some $r = r_F(T) \geq 0$.

PROOF. The first statement follows from Lemma 3.2.3, Corollary 3.2.8 and Lemma 3.2.9.

Then applying Corollary 3.2.6, we see that the (open set) $\sigma(T) \cap \rho_{s-F}(T)$ must be a disc $B(0,r)$. □

DEFINITION 3.2.11. Given a weighted backward shift on an unconditional basis, we define the *Fredholm radius* of T to be the radius of the bounded component of $\rho_{s-F}(T)$, and we denote it by $r_F(T)$. When there is no bounded component of $\rho_{s-F}(T)$, we let $r_F(T) = 0$.

For a general backward shift (on a basis that is not necessarily unconditional), we define the *Fredholm radius* of T to be the radius of the largest disk centered at the origin that is contained in $\rho_{s-F}(T)$. Again, if no such disk is contained in $\rho_{s-F}(T)$ we set $r_F(T)$ to 0.

REMARK 3.2.12. For a general weighted backward shift T with respect to an unconditional basis of X, we have three radii, $r(T)$, $r_p(T)$ and $r_F(T)$ which satisfy

$$r_F(T) \leq r_p(T) \leq r(T)$$

(by Corollary 3.2.10). An example in [**20**, Solution 94] on $X = \ell^2$ and using the standard basis shows that $0 < r_F(T) < r(T)$ can occur. We give the details now.

EXAMPLE 3.2.13. [**20**, Solution 94] For $0 < r < R < \infty$, there is a weighted backward shift B on $X = \ell^2$ (with respect to the standard basis) such that $0 < r_F(B) \leq r$ and $r(B) = R$.

For the construction of the weights , we choose a sequence $(\lambda_n)_n$ which is dense in the annulus $\{\lambda : r \leq \lambda \leq R\}$, and then construct $\alpha = (\alpha_n)_n$ such that for each n, and each $k \geq 1$, there are k consecutive terms starting at α_m, where $m = m(k,n)$, such that

$$\alpha_m = \alpha_{m+1} = \ldots = \alpha_{m+k-1} = \lambda_n. \tag{3.2.2}$$

For instance α could start with λ_1, then λ_2 twice, λ_1 twice, then λ_3 3 times, λ_2 3 times, etc.

Let e_n denote the standard basis of ℓ^2, and $F \in \mathcal{B}(\ell^2)$ the forward shift given by $F(e_n) = \alpha_n e_{n+1}$, $B = F^*$ the dual backward shift given by $B(e_1) = 0$, $B(e_{n+1}) = \alpha_n e_n$. We claim that $F - \lambda$ does not have closed range if $\lambda = \lambda_n$ for some n. To see this, fix λ_n and $\epsilon > 0$. choose m as in (3.2.2) where k is such that $k > 2R^2/\epsilon^2$ and consider $x = (1/\sqrt{k}) \sum_{i=m}^{m+k-1} e_i$. Then

$$\begin{aligned} \|(F - \lambda_n)x\| &= \left\| \frac{1}{\sqrt{k}} \sum_{i=m}^{m+k-1} \lambda_n e_{i+1} - \frac{1}{\sqrt{k}} \sum_{i=m}^{m+k-1} \lambda_n e_i \right\| \\ &= \left\| \frac{\lambda_n}{\sqrt{k}} (e_m - e_{m+k}) \right\| \\ &= \frac{|\lambda_n|}{\sqrt{k}} \sqrt{2} \leq \frac{\sqrt{2}R}{\sqrt{k}} < \epsilon \end{aligned}$$

As $\ker(F - \lambda_n) = \{0\}$, it follows that $\gamma(F - \lambda_n) = 0$ (see §2.6). By Proposition 2.6.2, the adjoint of $F - \lambda_n$, which can be identified with $B - \lambda_n$ under the complex linear identification of $(\ell^2)^*$ with ℓ^2, has $\gamma(B - \lambda_n) = 0$ and $\lambda_n \in \sigma_{s\text{-}F}(B)$. Clearly then $|\lambda_n| \leq r(B)$, and by Corollary 3.2.10, $r_F(B) \leq |\lambda_n|$. As this holds for each n, we have $r(B) \geq R$ and $r_F(B) \leq r$.

In fact $r(B) = \|B\| = R$, $r_p(T) \geq r$ and $0 < r_F(B) \leq R - \sqrt{R^2 - r^2} < r$. To see that $r_F(B) \leq R - \sqrt{R^2 - r^2}$, consider the forward shift F again and let $a = \sum_{m=1}^{\infty} a_n e_n \in \ell^2$ have $\|a\| = 1$, that is $\sum_{n=1}^{\infty} |a_n|^2 = 1$. Then

$$\begin{aligned} \|(F - \lambda)a\|^2 &= \left\| \sum_{i=1}^{\infty} \alpha_i a_i e_{i+1} - \lambda a_i e_i \right\|^2 \\ &= \left\| \lambda a_1 e_1 + \sum_{i=2}^{\infty} (\alpha_{i-1} a_{i-1} - \lambda a_i) e_i \right\|^2 \\ &\geq |\lambda|^2 |a_1|^2 + \sum_{i=2}^{\infty} |\alpha_{i-1}|^2 |a_{i-1}|^2 + |\lambda|^2 |a_i|^2 - 2|\lambda||\alpha_{i-1}||a_i||a_{i-1}| \\ &\geq |\lambda|^2 \|a\|^2 + r^2 \|a\|^2 - 2|\lambda| R \sum_{i=2}^{\infty} |a_i||a_{i-1}| \\ &= |\lambda|^2 + r^2 - 2|\lambda| R \sum_{i=2}^{\infty} |a_i||a_{i-1}| \end{aligned}$$

CLAIM.

$$\sum_{i=2}^{\infty} |a_i||a_{i-1}| \leq 1$$

PROOF. Let T be the standard forward shift on e_n and let $a' = \sum_{i=1}^{\infty} |a_i| e_i$. Then

$$\sum_{i=2}^{\infty} |a_i||a_{i-1}| = \sum_{i=1}^{\infty} |a_i||a_{i+1}| = \langle Ta', a' \rangle \leq \|T\| \langle a', a' \rangle = 1$$

□

Therefore:

$$\begin{aligned} \|(F-\lambda)a\|^2 &\geq |\lambda|^2 + r^2 - 2|\lambda| R \sum_{i=2}^{\infty} |a_i||a_{i-1}| \\ &\geq |\lambda|^2 + r^2 - 2|\lambda| R \\ &= |\lambda|^2 + r^2 - 2|\lambda| R \\ &= (|\lambda| - R)^2 + r^2 - R^2 \\ &= (R - |\lambda|)^2 - (R^2 - r^2). \end{aligned}$$

But

$$(R-|\lambda|)^2 - (R^2 - r^2) > 0 \Leftrightarrow R - |\lambda| > \sqrt{R^2 - r^2} \Leftrightarrow |\lambda| < R - \sqrt{R^2 - r^2}.$$

In other words $\|(F-\lambda)a\| > 0$ for all λ with $|\lambda| < R - \sqrt{R^2 - r^2}$. Clearly $R - \sqrt{R^2 - r^2} = 0$ iff $r = 0$ so we must have that there exists $\rho > 0$ such that $B - \lambda$ has closed range whenever $|\lambda| < \rho$. Since $r < R$ we also have:

$$\begin{aligned} r - R < 0 &\Leftrightarrow 2r^2 - 2Rr < 0 \Leftrightarrow R^2 + r^2 - 2Rr < R^2 - r^2 \\ &\Leftrightarrow (R-r)^2 < R^2 - r^2 \\ &\Leftrightarrow R - \sqrt{R^2 - r^2} < r \end{aligned}$$

Therefore $\gamma(F-\lambda) = \inf_{\|x\|=1} \|(F-\lambda)x\| > 0$ and so, by Propostion 2.6.2,$\gamma(B-\lambda) > 0$. Hence $B - \lambda$ has closed range for $|\lambda| < R - \sqrt{R^2 - r^2} < r$.

By Lemma 3.2.2, $\overline{R\mathbb{D}} \subset \sigma_{ap}(B) = \sigma(B)$. Let $\sum_{i=1}^{\infty} a_i e_i \in \ker(B-\lambda)$

$$(B-\lambda)\left(\sum_{i=1}^{\infty} a_i e_i\right) = \sum_{i=1}^{\infty} (\alpha_{i+1} a_{i+1} - \lambda a_i) e_i = 0$$

$$\Leftrightarrow \alpha_{i+1} a_{i+1} - \lambda a_i = 0 \Leftrightarrow a_{i+1} = \frac{\lambda a_i}{\alpha_{i+1}} = \frac{\lambda^i}{\Pi_{j=1}^{i} \alpha_j} a_1.$$

Since $|\lambda| < r \leq |\alpha_i| < R$ for all i we have

$$\frac{|\lambda|^i}{\Pi_{j=1}^{i} |\alpha_j|} < \left(\frac{\lambda}{r}\right)^i \tag{3.2.3}$$

so $\sum_{i=1}^{\infty} a_i e_i$ converges and $\lambda \in \sigma_p(B)$ whenever $|\lambda| < r$, i.e. $r_p(B) \geq r$. Combining this fact with the fact that $\gamma(B-\lambda) > 0$ for $|\lambda| < R - \sqrt{R^2 - r^2} < r$ we get that $r_F(B) \leq r$.

In fact, by modifying the previous example, we can ensure that all three radii are distinct:

EXAMPLE 3.2.14. If $0 \leq r < a_1 < a_2 < R < \infty$, there is a weighted backward shift B on ℓ^2 (with respect to the standard basis) such that

$$0 < r_F(B) < r < a_1 < r_p(B) < a_2 < r(T) = R.$$

The construction is a modification of that in the previous example. Fix $C > 0$ such that $r < Ca_1$. The idea is to add extra weights between the runs of constant terms among the α_j's to get a new sequence β that still satisfies the consecutive term condition (3.2.2) but also has some extra properties. Let n_k denote the first term of the kth run of constant terms amongst the α_j and $\ell(k)$ the number of terms among them. So $n_1 = 1$, $n_{k+1} = n_k + \ell(k)$, α_j is constant for $n_k \leq j < n_k + \ell(k)$, and for each λ_n we can find k with $\ell(k)$ arbitrarily large and $\alpha_j = \lambda_n$ for $m_k \leq j < m_k + \ell(k)$.

We construct $\beta = (\beta_k)_k$ together with m_k such that

(1) $\beta_{m_k+i-1} = \alpha_{n_k+i-1}$ for $1 \leq i \leq \ell(k)$
(2) $C^i \Pi_{j=1}^i |\beta_j| > a_1^i$ for all $i \in \mathbb{N}$ with $m_k + \ell(k) \leq i \leq m_{k+1} + \ell(k+1) - 1$
(3) for each $k \in \mathbb{N}$ there is some $i \in \mathbb{N}$ with $m_k + \ell(k) \leq i \leq m_{k+1} + \ell(+1k) - 1$ such that $\Pi_{j=1}^i |\beta_j| < a_2^i$

If $a_2 < |\alpha_1| = |\lambda_1|$ we let $w_1 \in (Ca_1, a_2)$, $\beta_1 = w_1$, $\beta_2 = \alpha_1$ and $m_1 = 2$. Since both $|\alpha_1|, w_1 > Ca_1$ so (2) is satisfied for $k = 1$ and since $w_1 < a_2$ (3) is also satisfied for $k = 1$.

If $Ca_1 < |\alpha_1| = |\lambda_1| < a_2$ we let $m_1 = n_1 = 1$ and $\beta_1 = \alpha_1$ so (2) and (3) are satisfied for $k = 1$.

If $|\alpha_1| = |\lambda_1| < Ca_1$, we let $w_1 \in (Ca_1, a_2)$ we let p_1 be such that $w_1^{p_1} |\alpha_1| > Ca_1^{p_1+1}$. Let $m_1 = p_1 + 1$, $\beta_i = w_1$ for $1 \leq i \leq p_1$ and $\beta_{m_1} = \alpha_1$. Then $(Ca_1)^i < \Pi_{j=1}^i |\beta_j|$ for all $i \leq p_1$ since $Ca_1 < w_1$ and $(Ca_1)^{m_1} < \Pi_{j=1}^{m_1} |\beta_j|$ by construction so (2) is satisfied for $k = 1$. Since $w_1 = \beta_1 < a_2$ (3) is satisfied for $k = 1$.

Suppose we have $m_1, \ldots, m_k$ and $\beta_1, \ldots, \beta_{m_k} + \ell(k) - 1$ for some $k \in \mathbb{N}$ satisfying (1), (2) and (3) above.

Suppose

$$\left(\Pi_{j=1}^{m_k+\ell(k)-1} |\beta_j|\right) \Pi_{j=1}^i |\alpha_{n_{k+1}+j-1}| < (Ca_1)^{m_k+\ell(k)+i-1}$$

for some $1 \leq i \leq \ell(k+1)$. $\alpha_{n_{k+1}+j-1} = \lambda_q$ for some $q \in \mathbb{N}$. Clearly $|\lambda_q| < Ca_1$. There exists p_{k+1} such that

$$\left(\Pi_{j=1}^{m_k+\ell(k)-1} |\beta_j|\right) \left(\Pi_{j=1}^i |\alpha_{n_{k+1}+j-1}|\right) a_2^{p_{k+1}} > (Ca_1)^{m_k+\ell(k)+i-1+p_{k+1}}$$

for each $i \leq \ell(k+1)$.Therefore there exists $w_{k+1} \in (Ca_1, a_2]$ such that

$$\left(\Pi_{j=1}^{m_k+\ell(k)-1} |\beta_j|\right) \left(\Pi_{j=1}^i |\alpha_{n_{k+1}+j-1}|\right) w_{k+1}^{p_{k+1}} > (Ca_1)^{m_k+\ell(k)+i-1+p_{k+1}}$$

for $1 \leq i \leq \ell(k+1)$ and there exists some $i \leq \ell(k+1)$ such that

$$\left(\Pi_{j=1}^{m_k+\ell(k)-1}|\beta_j|\right)\left(\Pi_{j=1}^{i}|\alpha_{n_{k+1}+j-1}|\right) w_{k+1}^{p_{k+1}} < a_2^{m_k+\ell(k)+i-1+p_{k+1}}.$$

Setting $\beta_{m_k+\ell(k)+i-1} = w_{k+1}$ for $1 \leq i \leq p_{k+1}$, $m_{k+1} = m_k + \ell(k) + p_{k+1}$ and $\beta_{m_{k+1}+i-1} = \alpha_{n_{k+1}+i-1}$ for $1 \leq i \leq \ell(k+1)$. Then β_i and m_j satisfy (2) and (3) for $j \leq k+1$.

Suppose

$$\left(\Pi_{j=1}^{m_k+\ell(k)-1}|\beta_j|\right)\Pi_{j=1}^{i}|\alpha_{n_{k+1}+j-1}| > a_2^{m_k+\ell(k)+i-1}$$

for $1 \leq i \leq \ell(k+1)$. Again, $\alpha_{n_{k+1}+j-1} = \lambda_q$ for some $q \in \mathbb{N}$. We can find p_{k+1} such that

$$\left(\Pi_{j=1}^{m_k+\ell(k)-1}|\beta_j|\right)\left(\Pi_{j=1}^{i}|\alpha_{n_{k+1}+j-1}|\right)(Ca_1)^{p_{k+1}} < a_2^{m_k+\ell(k)+i-1+p_{k+1}}$$

for each $i \leq \ell(k+1)$.Therefore there exists $w_{k+1} \in (a_1, a_2)$ such that

$$\left(\Pi_{j=1}^{m_k+\ell(k)-1}|\beta_j|\right)\left(\Pi_{j=1}^{i}|\alpha_{n_{k+1}+j-1}|\right) w_{k+1}^{p_{k+1}} > (Ca_1)^{m_k+\ell(k)+i-1+p_{k+1}}$$

for $1 \leq i \leq \ell(k+1)$ and there exists some $i \leq \ell(k+1)$ such that

$$\left(\Pi_{j=1}^{m_k+\ell(k)-1}|\beta_j|\right)\left(\Pi_{j=1}^{i}|\alpha_{n_{k+1}+j-1}|\right) w_{k+1}^{p_{k+1}} < a_2^{m_k+\ell(k)+i-1+p_{k+1}}.$$

Setting $\beta_{m_k+\ell(k)+i-1} = w_{k+1}$ for $1 \leq i \leq p_{k+1}$, $m_{k+1} = m_k + \ell(k) + p_{k+1}$ and $\beta_{m_{k+1}+i-1} = \alpha_{n_{k+1}+i-1}$ for $1 \leq i \leq \ell(k+1)$. Then β_i and m_j satisfy (2) and (3) for $j \leq k+1$.

Otherwise we set $m_{k+1} := m_k + \ell(k)$ and $\beta_{m_k+\ell(k)+i-1} = \alpha_{n_k+i-1} = \alpha_{n_k}$ for $1 \leq i \leq \ell(k+1)$.

Continuing in this way we get β_i satisfying (1), (2) and (3) above.

Since β_i contains all of the intervals of α_i and $r < |\beta_i| < R$ we still get that $\gamma(B-\lambda) = 0$ for $R > |\lambda| > r$ and $\gamma(B-\lambda) > 0$ for $|\lambda| < R - \sqrt{R^2 - r^2} < r$.

However we now have that $\frac{a_1^i}{\Pi_{j=1}^{i}|\beta_j|} < C^i$ (where $C < 1$) for all $i \in \mathbb{N}$. Therefore the radius of the point spectrum, r_p, is greater that a_1 (by (3.2.3)). Also, there exists a sequence $(b_i)_{i=1}^{\infty} \subset \mathbb{N}$ such that $\frac{a_2^{b_i}}{\Pi_{j=1}^{b_i}|\beta_j|} > 1$ and therefore $a_2 \notin \sigma_p(T)$ (again by (3.2.3)). So $0 < r_F(T) \leq r < a_1 < r_p(T) < a_2 < R = r(T)$ and the three radii are distinct.

LEMMA 3.2.15. *Let X be a separable Banach space with a symmetric basis. Suppose C is the symmetric basis constant. Then every weighted backward shift $B \in \mathcal{B}(X)$ on this basis having bounded weights is bounded. In particular $\|B\| \leq 2C^2 \sup_i |w_i|$ where w_i are the weights of B.*

PROOF. Let e_n be the symmetric basis and let $\epsilon > 0$ be arbitrary. For sufficiently large N we have:

$$\begin{aligned}
\left\| B\left(\sum_{i=1}^{\infty} a_i e_i\right)\right\| &= \left\|\sum_{i=1}^{\infty} w_i a_{i+1} e_i\right\| < \left\|\sum_{i=1}^{N} w_i a_{i+1} e_i\right\| + \epsilon \\
&\leq C \sup|w_i| \left\|\sum_{i=2}^{N+1} a_i e_i\right\| + \epsilon \\
&\leq C \sup_i |w_i| \left\|(P_{n+1} - P_1)\left(\sum_{i=1}^{\infty} a_i e_i\right)\right\| + \epsilon \\
&\leq C \sup_i |w_i| \|(P_{n+1} - P_1)\| \left\|\sum_{i=1}^{\infty} a_i e_i\right\| + \epsilon \\
&= 2C^2 \sup_i |w_i| \left\|\sum_{i=1}^{\infty} a_i e_i\right\| + \epsilon
\end{aligned}$$

Since ϵ is arbitrary, we have that

$$\|B\| \leq 2C^2 \sup_i |w_i|.$$

□

PROPOSITION 3.2.16. *Every quasinilpotent weighted backward shift on a symmetric basis is a norm limit of nilpotents.*

PROOF. Let e_n be the normalised unconditional basis and let C be the symmetric basis constant for e_n. Let $B \in \mathcal{B}(X)$ be a quasinilpotent backward shift with weights w_n. Let $\epsilon > 0$ be given. Since B is quasinilpotent we have that $\|B^n x\| < \left(\frac{\epsilon}{2C^2}\right)^n \|x\|$ for all $n \geq N$. Therefore we have that for all $n \geq N$

$$\begin{aligned}
\Pi_{i=1}^{N} |w_{n-i}| &= \Pi_{i=1}^{N} |w_{n-i}| \|e_{n-N}\| = \|B^N e_n\| < \left(\frac{\epsilon}{2C^2}\right)^N \|e_n\| \\
&= \left(\frac{\epsilon}{2C^2}\right)^N
\end{aligned}$$

So we have that in every interval in $\mathbb{N}$ of length N there exists at least one n such that $w_n < \frac{\epsilon}{2C^2}$. Choose $n_i \in [(i-1)N + 1, iN] \cap \mathbb{N}$ such that $w_{n_i} < \frac{\epsilon}{2C^2}$ Let $A := \sum_{i=1}^{\infty} w_{n_i} e^*_{n_i+1} \otimes e_{n_i}$. Then

$$\|A\| = 2C^2 \sup_i |w_{n_i}| < 2C^2 \left(\frac{\epsilon}{2C^2}\right) = \epsilon.$$

We also have that $(B - A)e_{n_i+1} = 0$ and $n_{i+1} - n_i \leq 2N$ and therefore $B - A$ is a nilpotent having order at most $2N$. □

3.3. Generalised shifts

In this section we introduce a notion of generalised (backward) shift where we do not need a basis of the space. Instead we just need a (doubly indexed) linearly independent collection of vectors with dense span.

For $T \in \mathcal{B}(X)$ we use the notation $R^\infty(T) = \bigcap_{n=1}^\infty T^n(X)$ for the hyperrange. We use $\mathbb{K}$ for $\mathbb{R}$ or $\mathbb{C}$.

It will be convenient for us to introduce the (lexicographic) ordering $<$ on $\mathbb{N} \times \mathbb{N}$ given by $(p,q) < (\ell,k)$ if either $p < l$ or $p = \ell$ and $q < k$. Given $(\ell,k) \in \mathbb{N} \times \mathbb{N}$ with $k > 1$ we refer to $(\ell, k-1)$ as the *immediate predecessor* of (ℓ,k). This helps to explain the notions of generalised (backward) shift that we will use. Notice that we base these notions of shifts on sets of vectors which span a dense linear subspace, but need not be Schauder bases for X.

DEFINITION 3.3.1. Let X be a separable infinite dimensional Banach space. Given a set of linearly independent vectors $\{x_{\ell,k} : (\ell,k) \in \{1,\dots,n\} \times \mathbb{N}\}$ whose span is dense in X, we call $S \in \mathcal{B}(X)$ a *generalised backward n-shift adapted to the set* if for each $(\ell,k) \in \{1,\dots,n\} \times \mathbb{N}$, $T(x_{\ell,k})$ is a finite linear combination of $x_{p,q}$ with $(p,q) < (\ell,k)$ and is such that the coefficient of the immediate predecessor, $x_{\ell,k-1}$, is non-zero when $k > 1$. In more formal terms, we insist that

$$Sx_{\ell,k} = \sum_{i=1}^{k-1} a_i^{\ell,k} x_{\ell,i} + \sum_{i=1}^{\ell-1} \sum_{j=1}^{m_j^{\ell,k}} b_{i,j}^{\ell,k} x_{i,j} \qquad a_i^{\ell,k}, b_{i,j}^{\ell,k} \in \mathbb{K}, a_{k-1}^{\ell,k} \neq 0 \text{ if } k > 1, m_j^{\ell,k} \in \mathbb{N}. \tag{3.3.1}$$

Similarly, given a set of linearly independent vectors $\{x_{\ell,k} : (\ell,k) \in \mathbb{N} \times \mathbb{N}\}$ whose span is dense in X, we call $S \in \mathcal{B}(X)$ a *generalised backward ∞-shift (adapted to the set)* if for each $(\ell,k) \in \mathbb{N} \times \mathbb{N}$, $T(x_{\ell,k})$ is a finite linear combination of the form (2.1).

We think of the sequence $(x_{\ell,k})_{k=1}^\infty$ as the ℓ^{th} chain of the set.

LEMMA 3.3.2. *If $S \in \mathcal{B}(X)$ is a generalised n-shift or ∞-shift adapted to a set $\{x_{\ell,k}\}$, then $x_{\ell,k} \in R^\infty(S) \cap N^\infty(S)$ (for each $x_{\ell,k}$ in the set).*

PROOF. The proof is a simple induction on ℓ.

For $\ell = 1$, $Sx_{1,k} = \sum_{i=1}^{k-1} a_i^{1,k} x_{1,i} \in \operatorname{span}\{x_{1,i} : 1 \leq i \leq k-1\}$, $x_{1,1} \in \ker S$. Clearly $x_{1,k} \in \ker S^k \subseteq N^\infty(S)$ for each k. Also,

$$S^r \left(\frac{x_{1,\ell+1}}{\prod_{i=1}^r a_i^{1,i+1}} \right) = x_{1,1}$$

and so $x_{1,1} \in S^r(X)$ for all r. Hence $\mathbb{K}x_{1,1} \subseteq R^\infty(S)$. Using a subinduction on k, assume $x_{1,i} \in R^\infty(S)$ for $1 \leq i \leq k$. Then

$$S^r\left(\frac{x_{1,r+k+1}}{\prod_{i=1}^r a_{i+k}^{1,i+k+1}}\right) = x_{1,k+1} + u$$

with $u \in \operatorname{span}\{x_{1,i} : 1 \leq i \leq k\} \subseteq R^\infty(S)$ and $u \in S^r(X) \Rightarrow x_{1,k+1} \in S^r(X)$ for $r \geq 1$. Hence $x_{1,k+1} \in R^\infty(S)$. By induction we get $x_{1,k} \in R^\infty(S)$ for all k.

Similar arguments (using finiteness of the sums (2.1)) allow us to show the induction step on ℓ. □

The following is a simplified version of [**26**, theorem 1] which is a simplification of [**6**, theorem 2.2].

THEOREM 3.3.3. *Let X be a separable Banach space, $S \in \mathcal{B}(X)$ with $N^\infty(S) \cap R^\infty(S)$ dense in X and let $\lambda \in \mathbb{K}$ with $|\lambda| = 1$. Then $\lambda + S \in \mathcal{M}\mathrm{ix}(X)$.*

PROOF. Although the statement in [**26**, theorem 1] is for the case of complex numbers, the proof in [**6**, theorem 2.2] works over both $\mathbb{R}$ and $\mathbb{C}$. □

COROLLARY 3.3.4. *If X is a Banach space, $S \in \mathcal{B}(X)$ a generalised n-shift or ∞-shift (adapted to some set $\{x_{\ell,k}\}$) and if $\lambda \in \mathbb{K}$ with $|\lambda| = 1$, then $\lambda + S \in \mathcal{M}\mathrm{ix}(X)$.*

PROOF. This follows from Theorem 3.3.3 and Lemma 3.3.2 □

Note that backward weighted shifts are generalised shifts so these reults apply to backward shifts as well.

3.4. Herrero Conditions

In [**21**], Herrero proved (see Theorem 1.1.1) that the Herrero conditions describe the norm closure of the hypercyclic operators on a separable Hilbert space. In [**21**, Remark 2.3], Herrero also notes that these conditions are necessary for an operator $T \in \mathcal{B}(X)$ to belong to $\overline{\mathcal{HC}(X)}$ for any Banach space over $\mathbb{C}$.

Later we will find that the Herrero conditions do in fact describe the norm closure of $\mathcal{HC}(X)$ on a number of other Banach spaces. In this chapter, however, we restrict our focus to operators of the form $\lambda + S$ where S is a generalised nilpotent. As mentioned in Corollary 3.1.4, any such operator has no normal eigenvalues and so it satisfies Herrero's second condtion. They will also satisfy Herrero's third condition: Any such operator is triangular (see Section 2.5). This fact together with the fact that every triangular operator is pseudotrinagular and [**2**, Theorem 6.11] gives us that $\lambda + S$ satisfies Herrero's third condition. Herrero's third condition is in fact equivalent to quasitriangularity (in the sense of [**2**]) on ℓ^p for $1 \leq p < \infty$ or c_0 by [**4**].

Also by Corollary 3.1.4, such an operator will necessarily satisfy Herrero's first condition whenever $|\lambda| = 1$ and the rest of this chapter will be focused mainly on proving that $\lambda + S \in \overline{\mathcal{HC}(X)}$ whenever $|\lambda| = 1$ and S a generalised nilpotent for any Banach space (over $\mathbb{R}$ or $\mathbb{C}$). This will be used in Chapter 5 and Chapter 6 to prove some further results about quasinilpotent operators on certain Banach spaces.

3.5. A class of operators approximable by hypercyclics

Our aim now is to show that if an operator $S \in \mathcal{B}(X)$ (with X as usual a separable infinite dimensional Banach space) is a generalised nilpotent then it can be approximated in norm by generalised shifts (as defined in Definition 3.3.1). Our approach is to first find linearly independent vectors

$$\{y_{\ell,j} : 1 \leq \ell < \infty, 1 \leq j \leq d_\ell\}$$

with dense linear span in X which are well adapted to S in a sense somewhat similar to a forward shift version of (2.1). There will however be some important differences between what we can achieve for general S and what is required in (2.1). In particular, the vectors $y_{\ell,j}$ will not necessarily belong to $R^\infty(S)$ and our approach will be to adjust the operator S (with an adjustment of at most a prescribed $\varepsilon > 0$ in norm) so as to be able to apply Corollary 3.3.4. Indeed, since $R^\infty(S)$ need not be dense in X we must modify S.

For any sequence or set of vectors $\{x_n\}$ we will use square brackets $[x_n]$ for the closure of their linear span.

We describe our construction as an algorithm, starting from a fixed sequence $\{x_n : n \in \mathbb{N}\} \subset N^\infty(S)$ so that $[x_n : n \in \mathbb{N}] = X$ (which certainly exists since X is assumed to be separable). We suppose also that $\{x_n : n \in \mathbb{N}\}$ is algebraically linearly independent (or at least that $x_1 \neq 0$).

We begin by setting

$$y_{1,1} = x_1. \tag{3.5.1}$$

Then there is a smallest $r \geq 1$ with

$$S^r y_{1,1} = 0.$$

and we choose $d_1 = r$. Then the space $E_1 = \operatorname{span}\{y_{1,1}, Sy_{1,1}, \ldots, S^{r-1}y_{1,1}\}$ is S-invariant and we let

$$y_{1,j} = S^{j-1}y_{1,1} \quad (2 \leq j \leq d_1) \tag{3.5.2}$$

so that $E_1 = \operatorname{span}\{y_{1,1}, y_{1,2}, \ldots, y_{1,d_1}\}$.

Assuming that we have already chosen $d_1, \ldots, d_\ell \in \mathbb{N}$ and vectors $\{y_{i,j} : 1 \leq i \leq \ell, 1 \leq j < d_i + 1\}$, we will use the notation

$$E_m = \operatorname{span}\{y_{i,j} : 1 \leq i \leq m, 1 \leq j \leq d_i\} \tag{3.5.3}$$

(even for $m = 0$ when $E_m = \{0\}$) and insist during the algorithm that $S(E_m) \subseteq E_m$ holds for $m = 1, 2, \ldots, \ell$ as well as

$$Sy_{m,j} = y_{m,j+1} \qquad (1 \leq m \leq \ell, 1 \leq j < d_m), \tag{3.5.4}$$

$$Sy_{m,d_m} \in E_{m-1} \qquad (1 \leq m \leq \ell), \tag{3.5.5}$$

and

$$\operatorname{span}\{y_{m,j} : 1 \leq j \leq d_m\} \cap E_{m-1} = \{0\} \qquad (1 \leq m \leq \ell). \tag{3.5.6}$$

We will have in addition that

$$\{x_1, \ldots, x_m\} \subset E_m \quad (1 \leq m \leq \ell), \tag{3.5.7}$$

$$y_{m,1} \in \{x_1, x_2, \ldots\} \quad (1 \leq m \leq \ell) \tag{3.5.8}$$

and

$$\{y_{m,j} : 1 \leq m \leq \ell, 1 \leq j \leq d_m\} \text{ linearly independent.} \tag{3.5.9}$$

Notice that the construction so far has ensured that these conditions hold initially for $\ell = 1$.

The next step is to choose the smallest n such that $x_n \notin E_\ell$ and to put $y_{\ell+1,1} = x_n$ (which will ensure that (3.5.7) and (3.5.8) hold for $m = \ell + 1$) and then there is a smallest $r \geq 1$ such that

$$S^r y_{\ell+1,1} \in \operatorname{span}\{y_{\ell+1,1}, Sy_{\ell+1,1}, \ldots, S^{r-1} y_{\ell+1,1}\} + E_\ell \tag{3.5.10}$$

We put $d_{\ell+1} = r$. Then we take

$$y_{\ell+1,j+1} = S^j y_{\ell+1,1} \qquad (1 \leq j < d_{\ell+1}).$$

It is clear then that (3.5.4) and (3.5.6) hold for $m = \ell + 1$. To show (3.5.5), we can consider the operator induced by S on the quotient X/E_ℓ, which we temporarily denote $\tilde{S}$. Writing $\tilde{x}$ for the coset $x + E_\ell$ we then have $\tilde{S}(\tilde{x}) = \widetilde{Sx}$ and $\tilde{S}$ is a nilpotent operator with an invariant subspace $\operatorname{span}\{\widetilde{y_{l+1,j}} : 1 \leq j \leq r\}$ of dimension r. Hence $\tilde{S}^r = 0$ on this subspace and (3.5.5) holds for $m = \ell + 1$, so that also $S(E_{\ell+1}) \subseteq E_{\ell+1}$. Finally, (3.5.9) is easy to check for $\ell + 1$ in place of ℓ.

THEOREM 3.5.1. *Let X be a separable infinite dimensional Banach space (over $\mathbb{K}$) and let $S \in \mathcal{B}(X)$ be such that $N^\infty(S)$ is dense in X. Then S is in the norm closure of the generalised backward 1-shift operators on X and $T = \lambda + S \in \overline{\mathcal{M}\mathrm{ix}(X)}$ (the norm closure of $\mathcal{M}\mathrm{ix}(X)$) for each $\lambda \in \mathbb{K}$ with $|\lambda| = 1$.*

PROOF. Let $\{x_n\}_{n=1}^\infty \subset N^\infty(S)$ be a linearly independent set whose span is dense in X. We now apply the above construction to produce vectors $\{y_{\ell,j} : 1 \leq \ell < \infty, 1 \leq j \leq d_\ell\}$ which satisfy the conditions (3.5.4) through (3.5.9) for all $\ell < \infty$.

For each (ℓ, j) with $1 \leq j \leq d_\ell$, we can use the Hahn-Banach theorem and choose $y^*_{\ell,j} \in X^*$ (the dual space of X) so that

$$y^*_{\ell,j}(y_{m,k}) = \delta_{\ell,m}\delta_{j,k} \text{ if } m \leq \ell. \tag{3.5.11}$$

Given any $\varepsilon > 0$, we then choose $\varepsilon_\ell > 0$ such that

$$\sum_{\ell=2}^{\infty} \varepsilon_\ell \|y^*_{\ell,d_\ell}\| \|y_{\ell-1,1}\| < \varepsilon.$$

We now define $S_\ell \colon X \to X$ for $\ell \geq 2$ by $S_\ell = 0$ if $y^*_{\ell-1,1}(Sy_{\ell,d_\ell}) \neq 0$ and $S_\ell(x) = \varepsilon_\ell y^*_{\ell,d_\ell}(x) y_{\ell-1,1}$ otherwise. Then $\|S_\ell\| \leq \varepsilon_\ell \|y^*_{\ell,d_\ell}\| \|y_{\ell-1,1}\|$ and so $\sum_{\ell=2}^{\infty} \|S_\ell\| < \varepsilon$. Put $S' = S + \sum_{\ell=2}^{\infty} S_\ell$.

We claim that S' is then a generalised 1-shift in the sense of Definition 3.3.1 with respect to the set obtained by linearly ordering $\{y_{\ell,j} : \ell \in \mathbb{N}, 1 \leq j \leq d_\ell\}$ by taking those vectors in the order

$$y_{1,d_1}, \ldots, y_{1,1}, y_{2,d_2}, \ldots, y_{2,1}, y_{3,d_3}, \ldots.$$

In order for this to be true, we need to verify two properties:

(a)
$$S'y_{j,k} \in E_{j-1} + \operatorname{span}\{y_{j,\ell} : k < \ell \leq d_\ell\},$$
i.e. $S'y_{j,k}$ is a finite linear combination of previous $y_{n,m}$ in the new order.

(b)
$$y^*_{j,k+1}(S'y_{j,k}) \neq 0 \text{ for } k < d_j \text{ and } y^*_{j-1,1}(S'y_{j,d_j}) \neq 0 \text{ for } j > 1,$$
i.e. the coefficient of the immediate predecessor of $y_{j,k}$ is non-zero in $S'(y_{j,k})$ except for $(j,k) = (1,d_1)$.

Note that $S_\ell(y_{j,k}) = 0$ if $\ell > j$ and so $S'(y_{j,k}) = (S + \sum_{\ell=2}^{j} S_\ell)(y_{j,k})$. In particular we have

$$S'y_{1,k} = Sy_{1,k} = \begin{cases} y_{1,k+1} & \text{for } k < d_1 \\ 0 & \text{if } k = d_1 \end{cases}$$

For $j > 1$, we note that S_ℓ is an operator of rank at most one with range in $\mathbb{C}y_{\ell-1,1}$ and so

$$S_\ell(y_{j,k}) \in E_{j-1} \text{ if } \ell \leq j. \tag{3.5.12}$$

Hence $S'(y_{j,k}) - S(y_{j,k}) \in E_{j-1}$.

For $1 \leq k < d_j$ this gives

$$S'(y_{j,k}) - y_{j,k+1} \in E_{j-1}. \tag{3.5.13}$$

For $k = d_j$ it gives

$$S'(y_{j,d_j}) \in E_{j-1} \tag{3.5.14}$$

since $S(y_{j,d_j}) \in E_{j-1}$ by (3.5.5). By (3.5.13) and (3.5.14) we have verified (a).

Also

$$S'(y_{j,d_j}) - (S(y_{j,d_j}) + S_j(y_{j,d_j})) \in E_{j-2} \text{ (for } j \geq 2)$$

implies

$$y^*_{j-1,1}(S'(y_{j,d_j})) = y^*_{j-1,1}(S(y_{j,d_j}) + S_j(y_{j,d_j})) \tag{3.5.15}$$

and S_j is chosen to make this non-zero. Now (b) follows from (3.5.13) and (3.5.15) with (3.5.11).

Since, for $|\lambda| = 1$, we have $\lambda + S' \in \mathcal{M}\mathrm{ix}(X)$ by Corollary 3.3.4, and $\|(\lambda + S) - (\lambda + S')\| < \varepsilon$, we conclude that $\lambda + S \in \overline{\mathcal{M}\mathrm{ix}(X)}$ as required. □

COROLLARY 3.5.2. *Let X be a separable infinite dimensional Banach space and let $N \in \mathcal{B}(X)$ be a nilpotent operator. Then $\lambda + N \in \overline{\mathcal{M}\mathrm{ix}(X)} \setminus \mathcal{HC}(X)$ for any $\lambda \in \mathbb{K}$ with $|\lambda| = 1$.*

PROOF. From Theorem 3.5.1 and $N^\infty(N) = X$ we deduce immediately that $\lambda + N \in \overline{\mathcal{M}\mathrm{ix}(X)}$. The Binomial Theorem implies that the orbit $\{(\lambda + N)^n(x)\}$ of any $x \in X$ is in a finite dimensional subspace and so $\lambda + N$ cannot be hypercyclic. □

3.6. Applications to operators that are not generalised nilpotents

By a *forward shift* with respect to a basis $\{x_n\}_{n=1}^\infty$ we mean an operator $S \in \mathcal{B}(X)$ satisfying $Sx_n = w_n x_{n+1}$ for $n \geq 1$, where $w_n \in \mathbb{K}$ are weights. We will also consider bilateral shifts when the basis is $\{x_n\}_{n=-\infty}^\infty$.

In the next Proposition we consider a more general type of operator akin to a mixture of shifts but we do not insist on a Schauder basis.

PROPOSITION 3.6.1. *Let X be a separable infinite dimensional Banach space. Suppose $S \in B(X)$ is an operator such that*

$$Sx_{i,j} = a_{i,j}x_{i,j+1} \text{ and } Sy_{i,j} = b_{i,j}y_{i,j+1}$$

where $a_{i,j}, b_{i,j} \in \mathbb{K}$ and

$$\{x_{i,j} : 1 \leq i < I_1+1, 1 \leq j < \infty\} \cup \{y_{i,j} : 1 \leq i < I_2+1, -\infty < j < \infty\}$$

is a normalised minimal fundamental set for X with biorthogonal functionals

$$\{x^*_{i,j} : 1 \leq i < I_1+1, 1 \leq j < \infty\} \cup \{y^*_{i,j} : 1 \leq i < I_2+1, -\infty < j < \infty\}$$

where $0 \leq I_1, I_2 \leq \infty$ (but we can't have $I_1 = I_2 = 0$).

If for each i with $1 \leq i < I_1 + 1$ there is an increasing sequence $n_{i,j}$ with $|a_{i,n_{i,j}}|\|x^*_{i,n_{i,j}}\| \to 0$ and for each i with $1 \leq i < I_2 + 1$ there is an increasing sequence $m_{i,j}$ with $|b_{i,m_{i,j}}|\|y^*_{i,m_{i,j}}\| \to 0$ then $\lambda + S$ is in the closure of the mixing operators for any $\lambda \in \mathbb{K}$ with $|\lambda| = 1$.

PROOF. Let $\epsilon > 0$ be given and let $\epsilon_{i,j} > 0, \delta_{i,j} > 0$ be such that

$$\sum_{i=1}^{I_1}\sum_{j=1}^{\infty}\epsilon_{i,j} + \sum_{i=1}^{I_2}\sum_{j=1}^{\infty}\delta_{i,j} < \epsilon/2.$$

We have that $|a_{i,n_{i,j}}|\|x^*_{i,n_{i,j}}\| \to 0$ so for each i choose a subsequence n_{i,k_j} such that $|a_{i,n_{i,k_j}}|\|x^*_{i,n_{i,k_j}}\| < \epsilon_{i,j}$ and similarly choose a subsequence m_{i,k_j} such that $|b_{i,m_{i,k_j}}|\|y^*_{i,m_{i,k_j}}\| < \delta_{i,j}$. Let

$$K = \sum_{i=1}^{I_1}\sum_{j=1}^{\infty} a_{i,n_{i,k_j}} x^*_{i,n_{i,k_j}} \otimes x_{i,n_{i,k_j}+1} + \sum_{i=1}^{I_2}\sum_{j=1}^{\infty} b_{i,m_{i,k_j}} y^*_{i,m_{i,k_j}} \otimes y_{i,m_{i,k_j}+1}.$$

Then

$$\begin{aligned}\|K\| &\leq \sum_{i=1}^{I_1}\sum_{j=1}^{\infty} |a_{i,n_{i,k_j}}|\|x^*_{i,n_{i,k_j}}\| + \sum_{i=1}^{I_2}\sum_{j=1}^{\infty} |b_{i,m_{i,k_j}}|\|y^*_{i,m_{i,k_j}}\| \\ &< \sum_{i=1}^{I_1}\sum_{j=1}^{\infty} \epsilon_{i,j} + \sum_{i=1}^{I_2}\sum_{j=1}^{\infty} \delta_{i,j} < \epsilon/2\end{aligned}$$

CLAIM. $N^\infty(S-K)$ is dense in X.

PROOF. First, we will show that $x_{i,j} \in \mathcal{N}^\infty(S-K)$ for each i,j. Note that $(S-K)x_{i,j} = Sx_{i,j} = a_{i,j}x_{i,j+1}$ for $(i,j) \neq (i,n_{i,k_m})$. We have $(S-K)x_{i,n_{i,k_j}} = a_{i,n_{i,k_j}}x_{i,n_{i,k_j}+1} - a_{i,n_{i,k_j}}x_{i,n_{i,k_j}+1} = 0$. Suppose $n_{i,k_j} < j \leq m_{i,k}$, say $j = m_{i,k} - \ell$, then $(S-K)^{\ell+1}x_{i,j} = (a_{i,j}a_{i,j+1}\dots a_{i,m_{i,k}-1})(S-K)x_{i,j+\ell}) = 0$. So $x_{i,j} \in \mathcal{N}^\infty(S-K)$.

Similarly $y_{i,j} \in \mathcal{N}^\infty(S-K)$ for $j \geq m_{i,1}$. But we also have that $(S-K)y_{i,j} = Sy_{i,j}$ for $j < m_{i,1}$. Let $j = m_{i,1} - \ell$ for some $\ell \in \mathbb{N}$. Then $(S-K)^{\ell+1}y_{i,j} = (b_{i,j}b_{i,j+1}\dots b_{i,m_{i,1}-1})(S-K)y_{i,j+l} = 0$ so $y_{i,j} \in \mathcal{N}^\infty(S-K)$.

Then $\operatorname{span}\{x_{i,j}\} \cup \{y_{i,j}\} \subset \mathcal{N}^\infty(S-K)$ and therefore $\mathcal{N}^\infty(S-K)$ is dense in X. □

Therefore by Theorem 3.5.1 we have that there is a 1-shift B such that $\|S-K-B\| < \epsilon/2$. So $\|S-B\| \leq \|S-(S-K)\| + \|(S-K)-B\| < \epsilon$ and therefore (via Corollary 3.3.4) $\lambda + S$ is in the closure of the mixing operators. □

COROLLARY 3.6.2. *Let X be a separable infinite dimensional Banach space with a Schauder basis. If S is a weighted forward or bilateral shift on a normalised basis for X with weights w_j such that there exists an increasing sequence n_j with $|w_{n_j}| \to 0$ then $\lambda + S$ is in the closure of the mixing operators for any $\lambda \in \mathbb{K}$ with $|\lambda| = 1$.*

In particular if S is a quasinilpotent forward or bilateral shift on a basis then $\lambda + S$ is in the closure of the mixing operators (for $|\lambda| = 1$).

PROOF. Suppose e_n is our normalised basis. By a remark at the beginning of [**24**, 1.b] we have that $\sup_n \|e^*_n\| = K < \infty$. Then

$|w_{n_j}|\|e_n^*\| \leq K|w_{n_j}| \to 0$ and the rest follows from Proposition 3.6.1 by setting $I_1 = 1$, $I_2 = 0$ and $x_{1,j} = e_j$ for the forward shift case and $I_1 = 0$, $I_2 = 1$ and $y_{1,j} = e_j$ for the bilateral shift case.

For the second part we need only show that if the shift is quasinilpotent then there must be a subsequence of the weights going to 0. Wlog we can take the basis to be normalised. Assume for the sake of contradiction that this is not true. Then there is $j \in \mathbb{N}$ such that $C = \inf_{n \geq j}\{|w_n|\} \neq 0$. Choose $0 < \epsilon < C$. Then there is $N \in \mathbb{N}$ such that $\|S^n x\| < \epsilon^n \|x\|$ for all $n > N$ and all $x \in X$.

$$C^n \leq \|w_j w_{j+1} \dots w_{j+n-1} e_{j+n}\| = \|S^n e_j\| < \epsilon^n < C^n$$

for $n > N$.

□

CHAPTER 4

The semi-Fredholm domain and hypercyclicity

In this chapter we consider the case of operators $T \in B(X)$ such that $[N^\infty(T-\lambda) : \lambda \in \rho^r_{s-F}(T)] = X$. For such T, $\sigma_0(T) = \emptyset$ and so the Herrero conditions for $T \in \overline{\mathcal{HC}(X)}$ amount to (1) $\sigma_W(T) \cup \mathbb{T}$ connected and (3) $\operatorname{ind}(\lambda - T) \geq 0$ for each $\lambda \in \rho_{\text{s-F}}(T)$.

For operators of this kind, if each bounded component of $\rho_{\text{s-F}}(T)$ intersects the unit circle, $\mathbb{T}$, and if we assume that T has a dense set of eigenvectors corresponding to eigenvalues in $\rho^r_{\text{s-F}}(T)$, then

$$T \in \mathcal{FHC}(X) \cap \mathcal{M}ix(X) \subset \mathcal{HC}(X)$$

(Theorem 4.2.7). Applications of this include $T = \lambda + B$ (Corollary 4.2.11) and $T = \lambda B$ (Corollary 4.2.13) with B a weighted backward shift on X and restrictions on λ expressed in terms of the Fredholm radius of B. These Corollaries extend well-known results of Salas [**29**] and Rolewicz [**28**].

To establish these results we use Theorem 4.1.1, a result of [**11**], which shows that components Ω of $\rho_{\text{s-F}}(T)$ give rise to invariant subspaces M for T by taking the closed linear span of eigenvectors of T corresponding to eigenvalues $\lambda \in \Omega \cap \rho^r_{\text{s-F}}(T)$. In fact the same M can be obtained by taking eigenvectors for a sequence $\lambda_n \in \Omega \cap \rho^r_{\text{s-F}}(T)$ converging to some $\lambda_0 \in \Omega \cap \rho^r_{\text{s-F}}(T)$, or by taking $N^\infty(T-\lambda)$ for a single $\lambda \in \Omega \cap \rho^r_{\text{s-F}}(T)$. We provide our own proof of Theorem 4.1.1. We also rely for the result $T \in \mathcal{M}ix(X)$ on a criterion due to Grivaux and Shkarin which can be found in [**6**] and requires that T has plentiful generalised eigenvectors (see Theorem 4.2.1 for the precise statement). To establish $T \in \mathcal{FHC}(X)$ we rely on a result of [**14**] (stated below as Theorem 4.2.4), which requires that T has a plentiful supply of eigenvectors for unimodular eigenvalues.

4.1. The span of eigenvectors associated with eigenvalues in $\rho_{s-F}(T)$

In this section we present proofs for results stated [**11**] with complete proofs that differ from those indicated by [**11**]. The proofs from [**11**] seem to depend on Hilbert space techniques. Our proofs rely on the theory of Kato operators and the Kato decomposition of operators with closed range. We include statements of the relevant results from this theory with references to [**25**]. We show (Lemma 4.1.7) that $T - \lambda$ is a Kato operator if and only if λ is a regular point in $\rho_{s-F}(T)$ (for

$T \in \mathcal{B}(X)$, $\lambda \in \mathbb{C}$). This allows us to esblish different ways to characterise invariant subspaces for T arising from the connected components of $\rho^r_{s-F}(T)$ that intersect the point spectrum of T.

Our goal in this section is to establish the following theorem, which is nontrivial only when Ω is a bounded component of $\rho_{s-F}(T)$:

THEOREM 4.1.1. [**11**, Lemma 4.4] *Let X be an infinite dimensional Banach space, $T \in B(X)$ and Ω a component of $\rho_{s-F}(T)$. Define M as the closed linear span*

$$M = [\ker(T - \lambda) : \lambda \in \Omega \cap \rho^r_{s-F}(T)].$$

Then M is invariant under T.

Moreover if $\lambda_1, \lambda_2, \ldots$ is a sequence of distinct points in $\Omega \cap \rho^r_{s-F}(T)$ which has a limit point, λ_0 in $\Omega \cap \rho^r_{s-F}(T)$, then

$$M = [\ker(T - \lambda_n) : n \geq 1]$$

and if $\lambda \in \Omega \cap \rho^r_{s-F}(T)$ then

$$M = [\ker(T - \lambda)^n : n \geq 1] = \overline{N^\infty(T - \lambda)}$$

The proof of this will be split into two cases: the case where $\dim \ker(T - \lambda_0) < \infty$ and the case where $\dim \ker(T - \lambda_0) = \infty$. This split starts at Lemma 4.1.14.

REMARK 4.1.2. Note that since $\rho^s_{s-F}(T)$ consists of isolated points in (the open set) $\rho_{s-F}(T)$ (see Section 2.4), if Ω is a connected component of $\rho_{s-F}(T)$, then $\Omega \cap \rho^r_{s-F}(T)$ is a connected component in $\rho^r_{s-F}(T)$.

Moreover, $\Omega \cap \rho^r_{s-F}(T)$ is open in $\mathbb{C}$ so for any $\lambda \in \Omega \cap \rho^r_{s-F}(T)$ we can find a sequence $\lambda_n \in \Omega \cap \rho^r_{s-F}(T)$ converging to λ. Hence it suffices to prove the last statement for $\lambda = \lambda_0$.

Given any set $S \subset \Omega \cap \rho^r_{s-F}(T)$ we define

$$Y_S = [\ker(T - \lambda) : \lambda \in S]. \tag{4.1.1}$$

Given a point $\lambda \in \Omega \cap \rho^r_{s-F}(T)$ we define

$$Z_\lambda := \overline{N^\infty(T - \lambda)}.$$

If $\lambda_0 \in \rho^r_{s-F}(T)$ then we denote the component of $\rho^r_{s-f}(T)$ containing λ_0 by Ω_{λ_0}.

LEMMA 4.1.3. [**25**, Chapter 2 section 12 Theorem 2] *For X a Banach space and $T \in B(X)$ with closed range, the following are equivalent conditions on T (where $z \in \mathbb{C}$):*

(1) the function $z \mapsto \gamma(T - z)$ is continuous at 0;
(2) $\limsup_{z \to 0} \gamma(T - z) > 0$
(3) $\lim_{z \to 0} \hat{\delta}(\ker T, \ker(T - z)) = 0$
(4) $\lim_{z \to 0} \hat{\delta}(\text{Ran } (T - z), \text{Ran } T) = 0$
(5) $\ker T \subset R^\infty(T)$

(6) $N^\infty(T) \subset \operatorname{Ran} T$
(7) $N^\infty(T) \subset R^\infty(T)$
(8) $\ker T \subset \overline{R^\infty(T)}$

Moreover an operator satisfying these conditions has $T^k(X)$ *closed* $\forall k \in \mathbb{N}$.

DEFINITION 4.1.4. A *Kato operator* is an operator $T \in B(X)$ on a Banach space X such that T has closed range and T satisfies the conditions (1)-(8) of Lemma 4.1.3.

REMARK 4.1.5. If $T \in B(X)$ is a Kato operator and $\ker T$ is nonzero, then $N^\infty(T)$ is infinite dimensional.

PROOF. By condition (6) $0 \neq \ker T \subset N^\infty(T) \subset T(X) \Rightarrow \ker T^2$ properly contains $\ker T$. Similarly $\ker T^k$ is strictly increasing with k. □

THEOREM 4.1.6 (Kato decomposition). [**25**, Theorem 10.21] *Let X be a Banach space and $T \in B(X)$ semi-Fredholm. Then there are closed subspaces $X_1, X_2 \subset X$ with $X = X_1 \oplus X_2$, $T(X_j) \subset X_j$ $(j = 1, 2)$, $\dim X_1 < \infty$, $T|_{X_1}$ nilpotent and $T|_{X_2}$ Kato.*

LEMMA 4.1.7. *Let X be a Banach space and $T \in B(X)$, $\lambda \in \rho_{s-F}(T)$. Then $T - \lambda$ is a Kato operator if and only if $\lambda \in \rho^r_{s-F}(T)$.*

PROOF. Suppose first that $(T-\lambda)$ is a Kato operator. If $\dim\ker(T-\lambda) < \infty$, $\lambda \in \rho_{s-F}(T)$ and by (3) of Lemma 4.1.3

$$\lim_{\mu\to\lambda} \hat{\delta}(\ker(T-\mu), \ker(T-\lambda)) = 0.$$

Then by Lemma 2.6.3 $\dim\ker(T-\mu)$ is locally constant near $\mu = \lambda$ and so $\lambda \in \rho^r_{s-F}(T)$.

If $\dim\ker(T-\lambda) = \infty$ then we must have $\dim\ker(T-\lambda)^* < \infty$ and $(T-\lambda)^*$ semi-Fredholm (since $\lambda \in \rho_{s-F}(T)$). Since $\gamma(T^*) = \gamma(T)$ we have $(T-\lambda)^*$ a Kato operator and then re-use the earlier argument to show $\dim\ker(T-\mu)^* = \operatorname{codim}(T-\mu)(X)$ is locally constant near $\mu = \lambda$. Hence $\lambda \in \rho^r_{s-F}(T)$ holds in this case also. We've proved the only if part of the statement.

For the converse, suppose $\lambda \in \rho^r_{s-F}(T)$ and $(T-\lambda)$ is not Kato. If $\dim\ker(T-\lambda) < \infty$ we have by the definition of $\rho^r_{s-F}(T)$ that $\mu \longmapsto \dim\ker(T-\mu)$ is constant near λ. Since $(T-\lambda)$ is semi-Fredholm but not Kato, we can use Theorem 4.1.6 to get $(T-\lambda) = N \oplus (T'-\lambda)$, where $(T'-\lambda)$ is Kato and N is a finite dimensional nilpotent operator. So $\mu \longmapsto \dim\ker(T'-\mu)$ is also constant near λ. Then $\mu \longmapsto \dim\ker(N-(\mu-\lambda)) + \dim\ker(T'-\mu)$ is not constant near λ. But $\dim\ker(N-(\mu-\lambda)) + \dim\ker(T'-\mu) = \dim\ker(T-\mu)$ is constant near λ, a contradiction. If $\ker(T-\lambda)$ is infinite dimensional, we can argue with $(T-\lambda)^*$ instead (since $\rho^r_{s-F}(T) = \rho^r_{s-F}(T^*)$), using $\dim\ker(T-\mu)^* = \operatorname{codim}(T-\mu)(X)$. We again reach a contradiction. □

COROLLARY 4.1.8. *Let X be a Banach space and $T \in B(X)$.*

(1) If $\lambda \in \rho^r_{s-F}(T)$ with $\dim\ker(T-\lambda) > 0$ (i.e.if $\lambda \in \rho^r_{s-F}(T) \cap \sigma_p(T)$) then

$$\overline{N^\infty(T-\lambda)} = [\ker(T-\lambda)^n : n \in \mathbb{N}]$$

is infinite dimensional.

(2) If $\lambda, \mu \in \rho^r_{s-F}(T)$ then

$$(T-\mu)N^\infty(T-\lambda) = N^\infty(T-\lambda)$$

and

$$(T-\mu)\overline{N^\infty(T-\lambda)} = \overline{N^\infty(T-\lambda)}.$$

(3) If S is an infinite subset contained in some connected component of $\rho^r_{s-F}(T) \cap \sigma_p(T)$ then

$$\dim[\ker(T-\lambda) : \lambda \in S] = \infty$$

and for $\mu \in \rho^r_{s-F}(T) \setminus S$ we have

$$(T-\mu)[\ker(T-\lambda) : \lambda \in S] = [\ker(T-\lambda) : \lambda \in S]$$

PROOF. (1) If $\lambda \in \rho^r_{s-F}(T)$, then $(T-\lambda)$ is a Kato operator (by Lemma 4.1.7) and $N^\infty(T-\lambda)$ must be infinite dimensional by Remark 4.1.5.

(2) If $\mu = \lambda$ then

$$\begin{aligned}(T-\mu)(\ker(T-\lambda)^{k+1}) &= (T-\lambda)(\ker(T-\lambda)^{k+1}) \\ &= \ker(T-\lambda)^k\end{aligned}$$

(since $N^\infty(T-\lambda) \subset (T-\lambda)(X)$ by Lemma 4.1.3 (6) so $\ker(T-\lambda)^k \subset (T-\lambda)(X)$)

$$\Rightarrow (T-\lambda)(N^\infty(T-\lambda)) = N^\infty(T-\lambda).$$

If $\mu \neq \lambda$ then

$$\begin{aligned}(T-\mu)(\ker(T-\lambda)^{k+1}) &= ((T-\lambda) \\ &\quad +(\lambda-\mu))(\ker(T-\lambda)^{k+1}) \\ &= \ker(T-\lambda)^{k+1}\end{aligned}$$

$$\Rightarrow (T-\lambda)(N^\infty(T-\lambda)) = N^\infty(T-\lambda).$$

Also note that, given any T-invariant subspace, Y, of X, we have $\gamma(T|_Y) \geq \gamma(T)$. Therefore, since $T-\mu$ has closed range, $(T-\mu)|_Y$ has closed range. Letting $Y = \overline{N^\infty(T-\lambda)}$ then we get that $(T-\mu)\overline{N^\infty(T-\lambda)} = \overline{N^\infty(T-\lambda)}$.

(3) Let $(\lambda_n)_{n=1}^\infty$ be any sequence of distinct elements in S. Then $\dim\operatorname{span}\{\ker(T-\lambda_i) : 1 \leq i \leq n+1\} > \dim\operatorname{span}\{\ker(T-\lambda_i) : 1 \leq i \leq n\}$ since eigenvectors associated with different eigenvalues are linearly independent. Clearly $\infty = \dim\operatorname{span}\{\ker(T-\lambda_i) : i \in \mathbb{N}\}$ and $\operatorname{span}\{\ker(T-\lambda_i) : i \in \mathbb{N}\} \subset Y_S$ (with Y_S as in equation 4.1.1) and hence $\dim Y_S = \dim[\ker(T-\lambda) : \lambda \in S] =$

∞. Suppose $\mu \in \rho^r_{s-F}(T) \setminus S$. Then $(T-\mu)(\ker(T-\lambda)) = \ker(T-\lambda)$ for all $\lambda \neq \mu$ and so $(T-\mu)(\operatorname{span}\{\ker(T-\lambda) : \lambda \in S\}) = \operatorname{span}\{\ker(T-\lambda) : \lambda \in S\}$. Since $\mu \in \rho^r_{s-F}(T)$, $T-\mu$ has closed range and so $(T-\mu)|_{Y_S}$ has closed range. Therefore

$$\begin{aligned}(T-\mu)Y_S &= (T-\mu)\overline{(\operatorname{span}\{\ker(T-\lambda) : \lambda \in S\})} \\ &= \overline{\operatorname{span}\{\ker(T-\lambda) : \lambda \in S\}} \\ &= Y_S.\end{aligned}$$

□

REMARK 4.1.9. If $\lambda \in \rho^s_{s-F}(T)$ then, using the Kato Decomposition (Theorem 4.1.6), we can split $(T-\lambda)$ into two parts: $(T-\lambda) = N \oplus (T'-\lambda)$ with respect to the decomposition $U \oplus V$ for some subspaces U and V, where N is nilpotent and $T'-\lambda$ is a Kato operator. If $\dim\ker(T'-\lambda) > 0$ and $\lambda \neq \mu$ in (2) then Corollary 4.1.8 is still true .

However, in (2), if $\mu = \lambda$ then U will not be contained in the range of $(T-\mu) = (T-\lambda)$.

Let Ω_{λ_0} be the connected component of $\rho^r_{s-F}(T)$ containing λ_0.

LEMMA 4.1.10. *Suppose $\lambda_0 \in \rho^r_{s-F}(T)$ and $(\lambda_n)_{n=0}^\infty \subset \Omega_{\lambda_0}$ with $\lambda_n \to \lambda_0$ and $\lambda_n \neq \lambda_0$ for any n. Then*

$$Z_{\lambda_0} \subset Y_{(\lambda_n)_{n=1}^\infty} \subset Y_{\Omega_{\lambda_0}}.$$

PROOF. If $\lambda_0 \notin \sigma_p(T)$, then $\dim\ker(T-\lambda_0) = \dim\ker(T-\lambda)$ for each $\lambda \in \Omega_{\lambda_0}$ (see §2.4) and then $\{0\} = Z_{\lambda_0} = Y_{(\lambda_n)_{n=1}^\infty} = Y_{\Omega_{\lambda_0}}$. Thus we can assume $\lambda_0 \in \sigma_p(T)$ (and $\Omega_{\lambda_0} \subset \sigma_p(T)$).

Since $T-\lambda_0$ is Kato we get that $\lim_{z\to\lambda_0} \hat{\delta}(\ker(T-z), \ker(T-\lambda_0)) = 0$. Therefore $\ker(T-\lambda_0) \subset Y_{(\lambda_n)_{n=1}^\infty}$. Without loss of generality, we may assume that $\lambda_0 = 0$ so that $\lambda_n \neq 0$ for all $n \in \mathbb{N}$ and we have that $\ker T \subset Y_{(\lambda_n)_{n=1}^\infty}$.

Using Corollary 4.1.8 (3) we have that $TY_{(\lambda_n)_{n=1}^\infty} = Y_{(\lambda_n)_{n=1}^\infty}$. If $x \in \ker T^2$, $Tx \in \ker T \Rightarrow Tx = Tz$ for some $z \in T^k Y_{(\lambda_n)_{n=1}^\infty}$, so that $x - z \in \ker T \Rightarrow x \in T^k Y_{(\lambda_n)_{n=1}^\infty}$. Inductively we establish that

$$\ker T^m \subset T^k Y_{(\lambda_n)_{n=1}^\infty}$$

for all $m, k \in \mathbb{N}$, and hence

$$Z_{\lambda_0} \subset T^k Y_{(\lambda_n)_{n=1}^\infty}.$$

Obviously $Y_{(\lambda_n)_{n=1}^\infty} \subset Y_{\Omega_{\lambda_0}}$ so

$$Z_{\lambda_0} \subset Y_{(\lambda_n)_{n=1}^\infty} \subset Y_{\Omega_{\lambda_0}}$$

as required. □

In the next definition, we discuss X-valued analytic functions $f(\lambda)$ (of a complex variable λ). One way to define them is as functions admitting a convergent power series expansion $f(\lambda) = \sum_{n=1}^\infty a_n(\lambda-\lambda_0)^n$

(with $a_n \in X$) valid in a disk of positive radius about each λ_0 in the domain of f.

DEFINITION 4.1.11. Let $T : X \to X$ be a closed linear operator mapping a Banach space X into itself, and let λ_0 be a complex number. The operator T has the *single valued extension property at* λ_0 if $f = 0$ is the only solution to $(\lambda - T)f(\lambda) = 0$ that is analytic in a disk $\{\lambda : |\lambda - \lambda_0| < r\}$ about λ_0 (with $r > 0$). Also, T has the *single valued extension property* if it has this property at every point λ_0 in the complex plane.

THEOREM 4.1.12. [**12**, Theorem 2] *Let T be a closed linear operator mapping the Banach space X into itself. If T is onto but not one-one, then T does not have the single valued extension property at $\lambda = 0$.*

REMARK 4.1.13. The proof in [**12**] actually gives us a non-zero analytic function: $f(\lambda) = \sum_{i=0}^{\infty} x_i(\lambda - \lambda_0)^i$, where $x_0 \in \ker(T - \lambda_0)$ is arbitrary and x_i is chosen such that $(T - \lambda_0)x_i = x_{i-1}$ and $\|x_i\| \leq k\|x_{i-1}\|$ for some constant $k > 0$ (using the open mapping theorem). So then the series defining f converges for $|\lambda - \lambda_0| < k^{-1}$.

Moreover, since $(T-\lambda)f(\lambda) = 0$, we must have that $f(\lambda) \in \ker(T - -\lambda)$ and $f(\lambda) \in Z_{\lambda_0}$ for all $|\lambda - \lambda_0| < k^{-1}$ since $x_i \in Z_{\lambda_0}$ for all i.

LEMMA 4.1.14. *Suppose $\lambda_0 \in \rho^r_{s-F}(T)$. If $\dim\ker(T - \lambda_0) < \infty$ then there exists $r > 0$ such that $|\mu - \lambda_0| < r \Rightarrow \ker(T - \mu) \subset Z_{\lambda_0}$.*

PROOF. If $\dim\ker(T - \lambda_0) = 0$ then $\dim\ker(T - \mu) = 0$ for all μ close enough to λ (since $\lambda \in \rho^r_{s-F}(T)$) so $\ker(T - \mu) = \{0\} \subset Z_{\lambda_0}$.

Let $x_{1,0}, x_{2,0}, \ldots, x_{d,0}$ be a basis for $\ker(T - \lambda_0)$. For each $1 \leq i \leq d$ choose $x_{i,n+1}$ and $f_i(z)$ as in Remark 4.1.13 such that $f_i(\lambda_0) = x_{i,0}$, then $\{f_1(0), f_2(0), \ldots, f_d(0)\}$ is linearly independent. Then there exists $r \leq k^{-1}$ such that $\{f_1(\lambda), f_2(\lambda), \ldots, f_d(\lambda)\}$ is linearly independent for $|\lambda - \lambda_0| < r$ (by the continuity of determinants and the continuity of f_i). But $f_i(\lambda) \in \ker(T - \lambda))$ (since $(T - \lambda)f(\lambda) = 0$ by Remark 4.1.13) so $\{f_1(\lambda), f_2(\lambda), \ldots, f_d(\lambda)\}$ is a basis for $\ker(T - \lambda))$. Therefore $\ker(T - \lambda) \subset Z_{\lambda_0}$ for $|\lambda - \lambda_0| < r$, i.e. $|\mu - \lambda_0| < r \Rightarrow \ker(T - \mu) \subset Z_{\lambda_0}$. □

REMARK 4.1.15. Note that if $\mu \in \rho^s_{s-F}(T)$ (a singular point) then, using the Kato decomposition we get $(T - \mu) = N \oplus (T' - \mu)$ on some decomposition $X = X_1 \oplus X_2$ (with $\dim X_1 < \infty$, N nilpotent and $T' - \mu$ Kato on X_2) and $(T - \lambda_0) = (N + (\mu - \lambda_0)) \oplus (T' - \lambda_0)$. So $Z_{\lambda_0} \subset X_2$ and $\ker(T - \mu) \cap X_1 \neq \{0\}$ and therefore $\ker(T - \mu)$ is not contained in Z_{λ_0}.

LEMMA 4.1.16. *Let $\lambda_0 \in \rho^r_{s-F}(T) \cap \sigma_p(T)$. If $\dim\ker(T - \lambda_0) < \infty$ then the set $\{\mu \in \Omega_{\lambda_0} : \ker(T - \mu) \subset Z_{\lambda_0}\} = \Omega_{\lambda_0}$ so*

$$Y_{\Omega_{\lambda_0}} \subset Z_{\lambda_0}$$

PROOF. Let $\mu \in S = \{\mu \in \Omega_{\lambda_0} : \ker(T-\mu) \subset Y_{\lambda_0}\}$. Then since $(T-\mu)$ is Kato (by Lemma 4.1.7), we can argue as in the previous lemma that, for any η close enough to μ, $\ker(T-\eta) = \operatorname{span}\{f_i(\lambda)\}$ for some λ. Therefore, if $x \in \ker(T-\eta)$ then $x = \sum_{i=1}^{d}\sum_{j=1}^{\infty} a_{i,j}x_{i,j}$, where $x_{i,j} \in Z_\mu$. But $\ker(T-\mu) \subset Z_{\lambda_0}$ and $(T-\mu)Z_{\lambda_0} = Z_{\lambda_0}$ by Corollary 4.1.8 (2). Thus, for $x \in \ker(T-\mu)^2$ there is $z \in Z_{\lambda_0}$ with $(T-\mu)x = (T-\mu)z$, hence $x - z \in \ker(T-\mu) \subset Z_{\lambda_0} \Rightarrow x \in Z_{\lambda_0}$. Inductively $ker(T-\mu)^n \subset Z_{\lambda_0}$ for all $n \in \mathbb{N}$.

Therefore, for η close enough to μ:

$$\ker(T-\eta) \subset Z_\mu \subset Z_{\lambda_0} \Rightarrow \eta \in S.$$

So S is an open set.

Now suppose $\mu \in \rho^r_{s-F}(T)$ is a limit point of S. Then there is a sequence $(\mu_i)_{i=1}^{\infty} \subset S$ with $\mu_i \to \mu$. We have

$$\mu_i \in S \Rightarrow \ker(T-\mu_i) \subset Z_{\lambda_0} \Rightarrow Y_{(\mu_i)_{i=1}^{\infty}} \subset Z_{\lambda_0}.$$

But by Lemma 4.1.10 $\ker(T-\mu) \subset Z_\mu \subset Y_{(\mu_i)_{i=1}^{\infty}}$. Therefore $\mu \in S$.

So, S is a clopen subset of the connected set Ω_{λ_0} and therefore $S = \Omega_{\lambda_0}$. □

REMARK 4.1.17. A similar proof using the Kato decomposition works when λ_0 is a singular point so we also have that:

If ω is a connected component of $\rho_{s-F}(T)$ with $\dim\ker(T-\lambda) > 0$ for all $\lambda \in \omega$. Let $\Omega = (\omega \cap \rho^r_{s-F}(T)) \cup \{\lambda_0\}$ where $\lambda_0 \in \omega$. Then $Y_\Omega \subset Z_{\lambda_0}$.

PROOF OF THEOREM 4.1.1 WHEN $\dim\ker(T-\lambda_0) < \infty$. Putting together Lemma 4.1.10 and 4.1.16 gives

$$Z_{\lambda_0} \subset Y_{(\lambda_n)_{n=1}^{\infty}} \subset Y_{\Omega_{\lambda_0}} \subset Z_{\lambda_0}$$
$$\Rightarrow Z_{\lambda_0} = Y_{(\lambda_n)_{n=1}^{\infty}} = Y_{\Omega_{\lambda_0}}$$

as required. □

LEMMA 4.1.18. *If $\lambda_0 \in \rho^r_{s-F}(T) \cap \sigma_p(T)$ and $\dim\ker(T-\lambda_0) = \infty$ then there exists $r > 0$ such that $|\mu - \lambda_0| < r \Rightarrow \ker(T-\mu) \subset Z_{\lambda_0}$.*

PROOF. If $\dim\ker(T-\lambda_0) = \infty$ then $\operatorname{codim}((T-\lambda_0)X) = d < \infty$. Let $x_1, x_2, \ldots, x_d$ be chosen so that $(T-\lambda_0)X + \operatorname{span}\{x_1, x_2, \ldots, x_d\} = X$. By Corollary 4.1.8, Z_{λ_0} is $(T-\lambda)$-invariant for each $\lambda \in \Omega_{\lambda_0}$ and we now consider the operators $\widehat{(T-\lambda)}$ induced by $T-\lambda$ on X/Z_{λ_0}. Since $X = (T-\lambda_0)X + \operatorname{span}\{x_1, x_2, \ldots, x_d\}$ we have that $X/Z_{\lambda_0} = \widehat{(T-\lambda_0)}(X/Z_{\lambda_0}) + \operatorname{span}\{x_1 + Z_{\lambda_0}, x_2 + Z_{\lambda_0}, \ldots, x_d + Z_{\lambda_0}\}$ and therefore $\widehat{T-\lambda_0}$ has closed range (by [**25**, Chapter III, Lemma 16.2]). We also have that $\{0\} = \ker\widehat{(T-\lambda_0)} \subset R^\infty(\widehat{T-\lambda_0})$ so $\widehat{T-\lambda_0}$ is a Kato operator. Therefore $\lambda_0 \in \rho^r_{s-F}(\hat{T})$ which is an open set so there is some $r > 0$ so that $|\lambda - \lambda_0| < r \Rightarrow \lambda \in \rho^r_{s-F}(\hat{T})$. In fact

all such λ are in the same connected component of $\rho^r_{s-F}(\hat{T})$ as λ_0, so $\dim\ker\widehat{(T-\lambda)} = \dim\ker\widehat{(T-\lambda)} = 0$. Therefore $\ker(T-\lambda) \subset Z_{\lambda_0}$ for $|\lambda - \lambda_0| < r$ as required.

□

LEMMA 4.1.19. *If $\lambda_0 \in \rho^r_{s-F}(T) \cap \sigma_p(T)$ and $\dim\ker(T-\lambda_0) = \infty$ then the set $\{\mu \in \Omega_{\lambda_0} : \ker(T-\mu) \subset Z_{\lambda_0}\} = \Omega_{\lambda_0}$ so*

$$Y_{\Omega_{\lambda_0}} \subset Z_{\lambda_0}$$

PROOF. The fact that $S := \{\mu : \ker(T-\mu) \subset Z_{\lambda_0}\}$ is closed follows by the same argument as in Lemma 4.1.16.

The fact that S is an open set follows from the previous lemma.

So, S is a clopen subset of the connected set Ω_{λ_0} and therefore $S = \Omega_{\lambda_0}$. □

PROOF OF THEOREM 4.1.1 WHEN $\dim\ker(T-\lambda_0) = \infty$. Putting together Lemma 4.1.10 and 4.1.19 gives

$$Z_{\lambda_0} \subset Y_{(\lambda_n)_{n=1}^\infty} \subset Y_{\Omega_{\lambda_0}} \subset Z_{\lambda_0}$$

$$\Rightarrow Z_{\lambda_0} = Y_{(\lambda_n)_{n=1}^\infty} = Y_{\Omega_{\lambda_0}}$$

as required. □

COROLLARY 4.1.20. [**11**, Lemma 4.5] *Let σ be a compact subset of $\rho^r_{s-F}(T) \cap \sigma_p(T)$ and let $\phi \in X^*$; then $\{\lambda \in \sigma : \phi|\ker(T-\lambda) = 0\}$ is either finite or all of σ.*

PROOF. Suppose $S := \{\lambda \in \sigma : \phi|\ker(T-\lambda) = 0\}$ is not finite. Then S contains a sequence $(\lambda_n)_{n=1}^\infty$. Since $(\lambda_n)_{n=1}^\infty$ is contained in the compact set σ, it has a limit point, λ_0, in $\sigma \subset \rho^r_{s-F}(T)$. So $[\ker(T-\lambda_n) : n \geq 1] = Y_{(\lambda_n)_{n=1}^\infty} = Y_{\Omega_{\lambda_0}}$. Therefore, if $\phi \in X^*$ such that $\phi(\ker(T-\lambda_n)) = 0$ for all $n \geq 1$ then $\phi(\ker(T-\lambda)) = 0$ for all $\lambda \in \Omega_{\lambda_0} \supset \sigma$. □

4.2. Some applications to hypercyclicity

Let $T \in \mathcal{B}(X)$ and for this section let Ω_n $(1 \leq n < N+1)$ be distinct bounded components of $\rho^r_{s-F}(T) \cap \sigma(T)$ where $N \in \mathbb{N} \cup \{\infty\}$ and let M_n be the span of the eigenvectors associated with the eigenvalues in Ω_n. We don't insist that every bounded component of $\rho^r_{s-F}(T) \cap \sigma(T)$ is listed among Ω_n $(1 \leq n < N+1)$.

In this section we will show (Theorem 4.2.7) that $T \in \mathcal{M}\mathrm{ix}(X)$ if we can choose Ω_n's so that $[M_n; 1 \leq n < N+1] = X$ and $\Omega_n \cap \mathbb{T} \neq \varnothing$ for all n. If $N = 1$, $[M_1] = X$ and $\overline{\Omega_1} \cap \mathbb{T} \neq \varnothing$ then $T \in \overline{\mathcal{M}\mathrm{ix}(X)}$ (Corollary 4.2.8). We apply these results to various special cases of operators T, including examples arising from shifts.

THEOREM 4.2.1. [**6**, Theorem 2.2] *Let X be a topological vector space, and let $T \in B(X)$. Set*

$$\Lambda_1(T) := \operatorname{span}\left(\bigcup_{|\lambda|=1, N\in\mathbb{N}} \ker(T-\lambda)^N \cap (T-\lambda)^N X\right),$$

$$\Lambda^+(T) := \operatorname{span}\left(\Lambda_1(T) \cup \bigcup_{|\lambda|>1, N\in\mathbb{N}} \ker(T-\lambda)^N\right),$$

$$\Lambda^-(T) := \operatorname{span}\left(\Lambda_1(T) \cup \bigcup_{|\lambda|<1, N\in\mathbb{N}} \ker(T-\lambda)^N\right),$$

If $\Lambda^+(T)$ and $\Lambda^-(T)$ are both dense then T is mixing.

PROPOSITION 4.2.2. *Let $T \in \mathcal{B}(X)$ and Ω_n $(1 \leq n < N+1)$ be such that $[M_n : 1 \leq n < N+1] = X$ and $\Omega_n \cap \mathbb{T} \neq \varnothing$ for $1 \leq n < N+1$. Then T is mixing.*

PROOF. Since each Ω_n is open and $\Omega_n \cap \mathbb{T} \neq \varnothing$ we can find $\lambda_n^+, \lambda_n^- \in \Omega_n$ such that $|\lambda_n^-| < 1 < |\lambda_n^+|$. Then by Theorem 4.1.1

$$M_n = \left[\bigcup_{i\in\mathbb{N}} \ker(T-\lambda_n^+)^i\right] = \left[\bigcup_{i\in\mathbb{N}} \ker(T-\lambda_n^-)^i\right]$$

so $X = [M_n : 1 \leq n < N+1] \subset \overline{\Lambda^+(T)}, \overline{\Lambda^-(T)}$ so $\Lambda^+(T)$ and $\Lambda^-(T)$ are dense in X and therefore, by Theorem 4.2.1, T is mixing. □

DEFINITION 4.2.3. We say that a bounded operator T on X has a *perfectly spanning set of eigenvectors associated to unimodular eigenvalues* if there exists a continuous probability measure σ on the unit circle $\mathbb{T}$ such that for every σ-measurable subset A of $\mathbb{T}$ which is of σ-measure 1, $[\ker(T-\lambda) : \lambda \in A]$ is dense in X.

THEOREM 4.2.4. [**14**] *Let T be a bounded operator acting on a complex Banach space X. If the eigenvectors of T associated to eigenvalues of modulus 1 are perfectly spanning, then T is frequently hypercyclic.*

LEMMA 4.2.5. *If σ is a continuous measure on $\mathbb{T}$, then any subset of $\mathbb{T}$ with positive σ-measure must have limit points.*

PROOF. If $A \subset \mathbb{T}$ has positive measure, then it must have infinitely many points since finite subsets have σ-measure 0. Then $A \subset \mathbb{T}$ contains a sequence of points and, since $\mathbb{T}$ is compact, this sequence must have a convergent subsequence, i.e. the subsequence has a limit point. Therefore A has a limit point. □

PROPOSITION 4.2.6. *Let $T \in \mathcal{B}(X)$ and Ω_n $(1 \leq n < N+1)$ be such that $\overline{\sum_{1\leq n<N+1} M_n} = X$ and $\Omega_n \cap \mathbb{T} \neq \varnothing$ for $1 \leq n < N+1$. Then T is frequently hypercyclic.*

PROOF. Let σ be normalised Borel length measure on $\mathbb{T}$ and let $A \subset \mathbb{T}$ be a set such that $\sigma(A) = 1$. Then $\sigma(A \cap \Omega_n) > 0$ since otherwise

$$\sigma(A) = \sigma(A \cap \Omega_n^c) + \sigma(A \cap \Omega_n) = \sigma(A \cap \Omega_n^c) \leq \sigma(\Omega_n^c \cap \mathbb{T}) < 1.$$

Since $A \cap \Omega_n$ has positive measure we can find a sequence $\lambda_{m,n} \in A \cap \Omega_n$ converging to $\lambda_n \in \Omega_n \cap \mathbb{T}$ by Lemma 4.2.5. Since $\lambda_n \in \Omega_n$ is the limit of $\lambda_{m,n}$ as $m \to \infty$ we have that $M_n = [\ker(T - \lambda_{m,n}) : m \in \mathbb{N}]$ and

$$X = \overline{\sum_{1 \leq n < N+1} M_n} = [\ker(T - \lambda_{m,n}) : m \in \mathbb{N}, 1 \leq n < N + 1].$$

Since we can find such $\lambda_{m,n} \in A$ for any A with $\sigma(A) = 1$, T has a perfectly spanning set of eigenvectors associated to unimodular eigenvalues and therefore T is frequently hypercyclic by Theorem 4.2.4. □

Our main result in this section is the following:

THEOREM 4.2.7. *Let $T \in \mathcal{B}(X)$ and Ω_n $(1 \leq n < N + 1)$ be such that $[M_n : 1 \leq n < N + 1] = X$ and $\Omega_n \cap \mathbb{T} \neq \varnothing$ for $1 \leq n < N + 1$. Then T is mixing and frequently hypercyclic.*

PROOF. This follows from Propositions 4.2.2 and 4.2.6. □

Note that for any connected open set $A \subset \mathbb{C}$, $A \cup \mathbb{T}$ is connected iff $\overline{A} \cap \mathbb{T} \neq \varnothing$.

Also note that by Remark 4.1.2 $\Omega \cap \rho_{s-F}^r(T)$ is a connected component of $\rho_{s-F}^r(T)$.

COROLLARY 4.2.8. *Suppose $T \in \mathcal{B}(X)$ and Ω is a (bounded) connected component of $\rho_{s-F}(T)$. If $\overline{\Omega} \cap \mathbb{T} \neq \varnothing$ and there exists $\mu \in \Omega \cap \rho_{s-F}^r(T)$ such that $N^\infty(T - \mu)$ is dense, then $T \in \overline{\mathcal{M}\mathrm{ix}(X) \cap \mathcal{FHC}(X)}$.*

PROOF. If $\Omega \cap \mathbb{T} \neq \varnothing$ we can apply Theorem 4.2.7.

If $\Omega \cap \mathbb{T} = \varnothing$ then we must have that $\partial\Omega \cap \mathbb{T} \neq \varnothing$ and there exists a sequence $(\lambda_n)_n$ in $\mathbb{C}$ such that $(a_n + \Omega) \cap \mathbb{T} \neq \varnothing$ and $a_n \to 0$. Then, by Theorem 4.2.7, $a_n + T$ is both mixing and frequently hypercyclic and $a_n + T \to T$ as $n \to \infty$ so T is in the closure of the intersection of those sets of operators. □

COROLLARY 4.2.9. *Suppose $T \in B(X)$ and Ω is a (bounded) connected component of $\rho_{s-F}(T)$. If there exists $\mu \in \Omega \cap \rho_{s-F}^r(T)$ such that $N^\infty(T - \mu)$ is dense in X, then $\lambda + T \in \mathcal{M}\mathrm{ix}(X) \cap \mathcal{FHC}(X)$ whenever $(\lambda + \Omega) \cap \mathbb{T} \neq \varnothing$.*

PROOF. This follows by applying Theorem 4.2.7 to $\lambda + T$. □

PROPOSITION 4.2.10. *If T is a backward shift then $\rho_{s-F}(T)$ has no singular points, i.e. $\rho_{s-F}^r(T) = \rho_{s-F}(T)$. In particular, if $0 \in \rho_{s-F}(T)$ then $0 \in \rho_{s-F}^r(T)$.*

PROOF. By Lemma 3.2.5 we have that $T - \lambda$ has dense range. Therefore, if $\lambda \in \rho_{s-F}(T)$ then $(T-\lambda)X = X$ and $\operatorname{codim}(T-\lambda)X = 0$.

Suppose $\mu \in \rho^s_{s-F}(T)$. Then (see Section 2.4)

$$\operatorname{codim}(T-\mu)X - \operatorname{codim}(T-(\mu+\epsilon))X = k > 0$$

for all $\epsilon \neq 0$ small enough. But $\operatorname{codim}(T-\mu)X = \operatorname{codim}(T-(\mu+\epsilon))X = 0$ so $k = 0$ which is a contradiction. □

In [**29**], Salas proved that $I + B$ is hypercyclic for any weighted backward shift B on any ℓ^p space. The following is a generalisation using the concept of Fredholm radius from Section 3.2:

COROLLARY 4.2.11. *If T is a weighted backward shift with Fredholm radius r_F then $\lambda + B$ is mixing and frequently hypercyclic whenever $1 - r_F < |\lambda| < 1 + r_F$.*

PROOF. If T is a weighted shift then $N^\infty(T)$ is dense in X. Suppose T is a Fredholm operator and Ω is the connected component of $\rho_{s-F}(T)$ containing 0. Note that by Proposition 4.2.10, $\rho_{s-F}(T) = \rho^r_{s-F}(T)$. We then have that $(\lambda + \Omega) \cap \mathbb{T} \neq \varnothing$ when $1 - r < |\lambda| < 1 + r$, and the result follows by applying Corollary 4.2.9. □

Note that by Corollary 3.3.4, if T is a weighted backward shift then $\lambda + T$ will be mixing when $|\lambda| = 1$ even when $r_F(T) = 0$.

COROLLARY 4.2.12. *Suppose $T \in B(X)$ and Ω is a connected component of $\rho^r_{s-F}(T)$. Suppose $N^\infty(T-\mu)$ is dense for some $\mu \in \Omega$. Then λT is mixing and frequently hypercyclic whenever $\lambda\Omega \cap \mathbb{T} \neq \varnothing$.*

PROOF. This follows by applying Theorem 4.2.7 to λT. □

One of the first examples of a hypercyclic operator was constructed by Rolewicz in [**28**]. He showed that λB was hypercyclic for $|\lambda| > 1$, where B is the standard backward shift on ℓ^p. The following is a generalisation of this result:

COROLLARY 4.2.13. *If T is a weighted backward shift with Fredholm radius $r_F > 0$ then λB is mixing and frequently hypercyclic whenever $|\lambda| > 1/r_F$.*

PROOF. Note that $\lambda(T-\mu) = \lambda T - \lambda\mu$ has closed range iff $(T-\mu)$ has, $\dim\ker\lambda(T-\mu) = \dim\ker(T-\mu)$ and $\operatorname{codim}\lambda(T-\mu) = \operatorname{codim}(T-\mu)$ so $\rho_{s-F}(\lambda T) = \lambda\rho_{s-F}(T)$.

Let ω the connected component of $\rho_{s-F}(T) = \rho^r_{s-F}(T)$ containing 0. Since $r_F(T) > 0$, we must have that ω contains an open disk of radius $r_F(T)$ centered at 0. If $|\lambda| > 1/r_F$ then the connected component of $\rho_{s-F}(\lambda T)$ containing 0 is $\lambda\omega$, which contains a disk with radius greater than 1. Therefore $\lambda\omega \cap \mathbb{T} \neq \varnothing$ and the result follows by applying Corollary 4.2.12. □

CHAPTER 5

Special Banach Spaces

In this chapter we discuss operators of the form $T = \lambda I + S \in \mathcal{B}(X)$ where $|\lambda| = 1$, S is a quasinilpotent operator.

The first section will go over the statements and proofs of some results found in [**11**]. These will be necessary for section 5.2 where we will prove that if a space, X, contains a complemented copy of ℓ^p $(1 \leq p < \infty)$ or c_0 then for every Riesz operator $T \in B(X)$ containing a complemented copy of ℓ^p $(1 \leq p < \infty)$ or c_0 in its kernel there is a compact operator K such that $T + K$ is quasinilpotent. We also show that if, instead, the dual space X^* and the dual operator T^* satisfy these hypotheses, then the same conclusion holds for T. (A result that a Riesz operator is the sum of a quasinilpotent operator and a compact operator is known as a West decomposition [**31**][**11**].) In fact we will give bounds on $\|K\|$. In Section 5.3 will give some applications of the results in Section 5.2, including a characterisation of the norm-closure of the hypercyclic operators on the complexification of the Argyros-Haydon space (Corollary 5.3.6). In Section 5.4 we will show that if either X or its dual has FDpBD then every quasinilpotent operator is a norm limit of generalised nilpotents and therefore if $T \in \mathcal{B}(X)$ is as above then T is in the closure of the mixing operators.

5.1. Nilpotent matrices with large diagonal

In [**11**, Lemma 2.2], Davidson and Herrero outline a construction of operators on finite dimensional spaces $X = \ell_m^p$ which are nilpotent but also are almost block diagonal with respect to a decomposition of the space into a 1-dimensional subspace and a hyperplane. The operator has norm approximately 1 and the 1×1 block is 1, while the off-diagonal blocks have small norm.

In this section we elaborate the statement of [**11**, Lemma 2.2] and give a full proof because we rely on these details in future sections.

We use the standard notation ℓ_m^p for $\mathbb{C}^m$ with the p-norm $(1 \leq p \leq \infty)$.

LEMMA 5.1.1 ([**11**, Lemma 2.2]). *Let $1 \leq p \leq \infty$ and $\varepsilon > 0$ be given. Then for $m = n^2$ with $n > \max(24, 24/\varepsilon)$, we can find*

(a) a nilpotent operator $J' \in \mathcal{B}(\ell_m^p)$,
(b) a unit vector $x \in \ell_m^p$,
(c) a unit norm functional $f \in (\ell_m^p)^* = \ell_m^q$ $(1/p + 1/q = 1)$,

(d) an operator $Q \in \mathcal{B}(\ker f)$,
(e) $\phi \in (\ell^p_m)^*$, $u \in \ker f$, *and rank one operators* $x \otimes \phi, u \otimes f \in \mathcal{B}(\ell^p_m)$ *given (for* $y \in X$*) by*

$$(x \otimes \phi)(y) = \phi(y)x \in \mathbb{C}x, \quad (u \otimes f)(y) = f(y)u \in \ker f,$$

(f) an operator $R \in \mathcal{B}(\ell^p_m)$,
such that

(i) $\|J'\| < 1 + \varepsilon$ *and* J' *has the block representation*

$$J' = \begin{pmatrix} 1 & x \otimes \phi \\ u \otimes f & Q \end{pmatrix} \tag{5.1.1}$$

with respect to $X = \mathbb{C}x \oplus \ker f$,
(ii) $\phi(x) = 0$,

$$|f(x) - 1| < \frac{\varepsilon}{2}, \|u\| < \frac{\varepsilon}{2}, \|\phi\| < \frac{\varepsilon}{2}, \|Q\| < 1 + \varepsilon,$$

(iii)

$$R = \begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix}$$

(with respect to $X = \mathbb{C}x \oplus \ker f$*) satisfies*

$$R|_{\ker f} = Q, \quad \|R\| \le 2 + \varepsilon, \quad R(\ell^p_m) \subseteq \ker f, \quad R(x) = 0$$

and

$$J' = \frac{1}{f(x)} x \otimes f + x \otimes \phi + u \otimes f + R, \tag{5.1.2}$$

where the first summand is the projection $P_{x,f}$ *of* X *onto* $\mathbb{C}x$ *with kernel* $\ker f$, *that is* $P_{x,f} = (1/f(x))x \otimes f$.

PROOF. We will sometimes use the notation $X = \ell^p_m$ for brevity. Let $e_1, \dots, e_m$ denote the standard unit vector basis of ℓ^p_m and $\delta_1, \dots, \delta_m$ the dual basis of $(\ell^p_m)^*$, that is the standard basis of ℓ^q_m in the usual identification of $(\ell^p_m)^*$ with ℓ^q_m, or the functionals with $\delta_k(e_j) = \delta_{k,j}$. We assume $m = n^2$ with $n > \max(24, 24/\varepsilon)$.

Define J to be the nilpotent backward shift operator on X given by $Je_j = e_{j-1}$ $(2 \le j \le m)$ and $Je_1 = 0$. So $J^m = 0$, $\|J\| = 1$ and the matrix of J in the standard basis is a nilpotent Jordan block. By J^* we mean the dual operator, the forward shift given by $J^*\delta_j = \delta_{j+1}$ $(1 \le j < m)$ and $J\delta_m = 0$.

Let

$$x_0 = \sum_{i=1}^{n-1} \left(\frac{i}{n} e_i + \frac{i}{n} e_{m-i} \right) + \sum_{i=n}^{m-n} e_i$$

and

$$f_0 = \sum_{i=1}^{n-1} \left(\frac{i}{n} \delta_i + \frac{i}{n} \delta_{m-i} \right) + \sum_{i=n}^{m-n} \delta_i.$$

Then

$$Jx_0 = \sum_{i=2}^{n-1} \frac{i}{n} e_{i-1} + \sum_{i=1}^{n-1} \frac{i}{n} e_{m-i-1} + \sum_{i=n}^{m-n} e_{i-1},$$

$$Jx_0 - x_0 = \frac{1}{n} \sum_{i=1}^{n} e_i - \frac{1}{n} \sum_{i=1}^{n} e_{m-i}.$$

We take $x = x_0/\|x_0\|$ (the unit vector for (b)). For $p = \infty$, $x = x_0$ and $\|Jx - x\|_\infty = 1/n$. For $1 \le p < \infty$ (using $m = n^2$)

$$\|x_0\| = \left(\frac{2}{n^p} \sum_{i=1}^{n-1} i^p + n^2 - 2(n-1) \right)^{1/p} \ge (n^2 - 2(n-1))^{1/p} \ge n^{1/p},$$

$\|Jx_0 - x_0\| = (1/n)(2n)^{1/p}$ and

$$\|Jx - x\| = \frac{1}{\|x_0\|} \|Jx_0 - x_0\| \le \frac{2}{n}. \tag{5.1.3}$$

Similarly

$$J^* f_0 = \sum_{i=1}^{n-1} \frac{i}{n} \delta_{i+1} + \sum_{i=1}^{n-1} \frac{i}{n} \delta_{m+1-i} + \sum_{i=n}^{m-n} \delta_{i+1},$$

$$J^* f_0 - f_0 = -\frac{1}{n} \sum_{i=1}^{n} \delta_i + \frac{1}{n} \sum_{i=1}^{n} \delta_{m+1-i}.$$

We take $f = f_0/\|f_0\|$ (the unit norm functional for (c)). For $p = 1$ ($q = \infty$), $f = f_0$ and $\|J^* f - f\|_\infty = 1/n$. For $1 < p \le \infty$ (using $m = n^2$)

$$\|f_0\| = \left(\frac{2}{n^q} \sum_{i=1}^{n-1} i^q + n^2 - 2(n-1) \right)^{1/q} \ge (n^2 - 2(n-1))^{1/q} \ge n^{1/q},$$

$\|J^* f_0 - f_0\| = (1/n)(2n)^{1/q}$ and

$$\|J^* f - f\| = (1/\|f_0\|)\|J^* f_0 - f_0\| \le 2/n. \tag{5.1.4}$$

Note that $\|f_0\|\|x_0\| \le (n^2)^{1/q}(n^2)^{1/p} = n^2$ and $f_0(x_0) \ge n^2 - 2(n-1)$, which implies

$$|f(x) - 1| = 1 - f(x) = 1 - \frac{f_0(x_0)}{\|f_0\|\|x_0\|} \le 1 - \frac{n^2 - 2(n-1)}{n^2} < \frac{2}{n}. \tag{5.1.5}$$

We must then have (using the previous estimates (5.1.3), (5.1.4) and (5.1.5))

$$\begin{aligned} |f(Jx) - f(x)| &= |f(Jx - x)| \le \|Jx - x\| \\ &\le 2/n, \\ |f(Jx) - 1| &\le |f(Jx) - f(x)| + |f(x) - 1| \\ &< 4/n, \\ \left|\frac{f(x)}{f(Jx)} - 1\right| &< \frac{2/n}{1 - 4/n} \\ &< 3/n \qquad (n > 24) \end{aligned} \tag{5.1.6}$$

Since $P_{x,f}(Jx) = (f(Jx)/f(x))x$, we let

$$J' = \frac{f(x)}{f(Jx)} J$$

(the nilpotent operator for (a)). Note that

$$\|J'\| \le (1 + 3/n)\|J\| = 1 + 3/n, \tag{5.1.7}$$

so that $\|J'\| < 1+\varepsilon$ as claimed in (i). We have ensured that $P_{x,f}(J'x) = 1$, so that when we consider the block form of J' relative to $X = \mathbb{C}x \oplus \ker f$ we must get the form

$$J' = \begin{pmatrix} 1 & x \otimes \phi \\ u \otimes f & Q \end{pmatrix}$$

where $\phi\colon \ker f \to \mathbb{C}$ is a linear functional (which we extend to X below), $u = (1/f(x))(x - J'x)$, and we have $Q = (1 - P_{x,f})J'(1 - P_{x,f}) \in \mathcal{B}(\ker f)$. (Now Q will be the operator for (d) and $u \in \ker f$ for (e) but below we will replace ϕ by a suitable extension to $\phi \in X^*$. Modulo this J' has the block form (5.1.1).)

Now we have

$$\begin{aligned} \|x - J'x\| &\le \|x - Jx\| + \left|1 - \frac{f(x)}{f(Jx)}\right| \|Jx\| \\ &< \frac{2}{n} + \frac{3}{n} = \frac{5}{n}, \\ \|u\| &< \frac{1}{1 - 2/n}\frac{5}{n} < \frac{9}{n} < \frac{\varepsilon}{2} \end{aligned} \tag{5.1.8}$$

using $n > 24/\varepsilon$.

For $y \in \ker f$ we have

$$\phi(y)x = (x \otimes \phi)(y) = P_{x,f}(J'y) = \frac{f(J'y)}{f(x)}x$$

implying

$$\phi(y) = \frac{1}{f(x)}J'^*(f)(y) = \frac{1}{f(x)}\left(J'^*(f) - \frac{1}{f(Jx)}f\right)(y).$$

We can then define (an extension) $\tilde{\phi} \in X^*$ by

$$\tilde{\phi} = \frac{1}{f(x)}\left(J'^*(f) - \frac{1}{f(Jx)}f\right) = \frac{1}{f(x)f(Jx)}\left(J^*(f) - f\right)$$

and then we have (by (5.1.6), (5.1.5) and (5.1.4)) the bound

$$\|\tilde{\phi}\| \leq \frac{1}{f(x)|f(Jx)|}\|J^*(f) - f\| \leq \frac{1}{(1-2/n)(1-4/n)}\frac{2}{n} < \frac{3}{n}$$

(for $n > 24$). Finally we define the required extension of ϕ from $\ker f$ to X^* to be $\phi = \tilde{\phi} \circ (I - P_{x,f})$ which is indeed an extension and satisfies $\|\phi\| \leq 6/n$, $\phi(x) = 0$.

In view of (5.1.7), if we take

$$\tilde{Q} = J' - x \otimes \phi \in \mathcal{B}(X),$$

then $\tilde{Q}(y) = Q(y)$ for $y \in \ker f$ and hence

$$\|Q\| \leq \|\tilde{Q}\| \leq \|J'\| + \|\phi\| \leq 1 + \frac{3}{n} + \frac{6}{n} = 1 + \frac{9}{n}$$

(for $n > 24$). Using this, $\|\phi\| < 6/n$, (5.1.8), (5.1.5) and $\phi(x) = 0$, we have now established (e), (ii) and (5.1.1).

We define $R \in \mathcal{B}(X)$ by $R = Q \circ (I - P_{x,f})$ (so that (f) holds). Then we have $\|R\| \leq 2\|Q\| < 2 + 18/n \leq 2 + \varepsilon$, $R|_{\ker f} = Q$ and $R(X) \subseteq \ker f$. For $z \in X$ we write $z = P_{x,f}z + y$ with $y \in \ker f$ and then (using the block matrix for J')

$$\begin{aligned} J'z &= P_{x,f}z + (x \otimes \phi)y + (u \otimes f)(P_{x,f}z) + Qy \\ &= P_{x,f}z + (x \otimes \phi)z + (u \otimes f)z + Rz. \end{aligned}$$

This established (ii). □

COROLLARY 5.1.2. *Let $1 \leq p \leq \infty$ and let X be a Banach space that admits a norm one projection $P \in \mathcal{B}(X)$ with range $Y = PX$ that is isometric to ℓ^p if $1 \leq p < \infty$ or to c_0 if $p = \infty$. For $m > 1$, let Y_m denote the span of the first m standard basis vectors of Y, π_m the coordinate projection of Y onto Y_m and $P_m = \pi_m P$.*

Let $\varepsilon > 0$ be given and $\lambda \in \mathbb{C} \setminus \{0\}$.

Then, there is $m = m(\lambda, \varepsilon) < \infty$ such that we can find

(a) a nilpotent operator $J \in \mathcal{B}(X)$ with $J = P_m J P_m$,
(b) $x \in P_m X = Y_m$,
(c) $f \in X^*$ with $f = f \circ P_m$, $f(x) < \min(1, \epsilon)/2$,
(d) $\phi \in X^*$ with $\phi = \phi \circ P_m$,
(e) $u \in (\ker f) \cap Y_m$,
(f) and $R \in \mathcal{B}(X)$ with $R = P_m R P_m$

such that

(i)

$$J = \frac{\lambda}{f(x)}x \otimes f + x \otimes \phi + u \otimes f + R, \tag{5.1.9}$$

where $(x \otimes f)(z) = f(z)x$, $(x \otimes \phi)(z) = \phi(z)x$, $(u \otimes f)(z) = f(z)u$ *(for* $z \in X$*),*

(ii) $\phi(x) = 0$, $R(X) \subseteq (\ker f) \cap Y_m$, $R(x) = 0$,

(iii)

$$\|x\| = 1 = \|f\|, \|\phi\| + \|u\| < \varepsilon, \|R\| < 2\lambda + \varepsilon,$$

(iv) and $\|J\| < |\lambda| + \varepsilon$.

Further, J *has finite rank (at most* m*) and* R *has finite rank.*

PROOF. Choose $\delta = \min(\varepsilon/|\lambda|, \varepsilon, 1)$ and m according to Lemma 5.1.1 with δ in place of ε (that is $m > n^2$ with $n > \max(24, 24/\delta)$). Choose $x \in Y_m$, $f_1 \in Y_m^*$, $u_1 \in \ker f_1$, $R_1 \in \mathcal{B}(Y_m)$ and $\phi_1 \in Y_m^*$ according to Lemma 5.1.1 so that

$$\|x\| = 1 = \|f_1\|, \|\phi_1\| + \|u_1\| < \delta, \|R_1\| < 2 + \delta,$$

and $J' \in \mathcal{B}(Y_m)$ is nilpotent and satisfies

$$J' = \frac{1}{f_1(x)} x \otimes f_1 + x \otimes \phi_1 + u_1 \otimes f_1 + R_1.$$

Also $\|J'\| < 1 + \delta$.

To conclude, let $J = \lambda J' P_m$, $f = f_1 \circ P_m$, $u = \lambda u_1$, $\phi = \lambda \phi_1 \circ P_m$ and $R = \lambda R_1 P_m$. □

REMARK 5.1.3. As a consequence of (5.1.9), we can say that J has the block form

$$J = \begin{pmatrix} \lambda & r \\ c & Q \end{pmatrix}$$

with respect to $X = \mathbb{C}x \oplus \ker f$, where Q is the restriction of R to $\ker f$, r is (the restriction of) $x \otimes \phi$, c is $u \otimes f$.

Note that if X is a Banach space (infinite dimensional) with FDpBD, then X contains a 1-complemented copy of ℓ^p (if $1 \leq p < \infty$) or c_0 (if $p = \infty$) and so Corollary 5.1.2 can be applied to X.

5.2. Removal of normal eigenvalues

Here we develop results based on [**11**, §3] and a result that allows us to remove (via a compact perturbation) infinitely many normal eigenvalues from certain operators on spaces containing an isomorphic copy of an ℓ^p space.

We show first (Proposition 5.2.9) that if the space X contains a 1-complemented copy of ℓ^p $(1 \leq p < \infty)$ and if $T \in \mathcal{B}(X)$ has a suitably large kernel and a normal eigenvalue λ, then there is a finite rank operator $K \in \mathcal{B}(X)$ with controlled norm $\|K\|$ such that $T + K$ has spectrum $\sigma(T) \setminus \{\lambda\}$. The conclusion of Proposition 5.2.9 contains further details about the construction of K because we will use these later to deal with sequences of normal eigenvalues of Riesz operators T. In Corollary 5.2.12 we elaborate Proposition 5.2.9 a little further to allow us to choose K in such a way that $\ker(T + K)$ contains a specified

part of $\ker T$, again with an eye to future uses of the conclusions. In Proposition 5.2.14 we establish that a finite rank perturbation $T + K$ of T can have spectrum $\sigma(T) \setminus \{\lambda_1, \ldots, \lambda_J\}$ if T is as in Proposition 5.2.9 and $\lambda_1, \ldots, \lambda_J$ are normal eigenvalues of T. In Corollary 5.2.18 we give a version for bounded projection with range isomorphic to ℓ^p (or to c_0). Corollary 5.2.18 is used in §5.3. If X is a dual space and T is a dual operator, we can also ensure that K is a dual operator. In Theorem 5.2.19, we extend our attention to allow infinitely many normal eigenvalues $(\lambda_j)_{j=1}^{\infty}$ but now we must assume in addition that T is a Riesz operator with spectrum $\sigma(T) = \{0\} \cup \{\lambda_j : j \in \mathbb{N}\}$. Theorem 5.2.19 is used in §5.4.

REMARK 5.2.1. If $T \in \mathcal{B}(X)$ is an operator and $E \in \mathcal{B}(X)$ is a projection, then $ETE = TE$ is the equivalent to $T(EX) \subseteq EX$ (or EX being invariant for T). So $ETE = TE$ means that the block form of T with respect to $X = EX \oplus (\ker E)$ has the form

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

On the other hand $TE = 0$ is equivalent to $EX \subseteq \ker T$ and means that the block form of T with respect to $X = EX \oplus (\ker E)$ has the form

$$\begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix}$$

LEMMA 5.2.2. *Let $1 \leq p \leq \infty$ and let X be a Banach space that admits a norm one projection $P \in \mathcal{B}(X)$ with range $Y = PX$ that is isometric to ℓ^p if $1 \leq p < \infty$ or to c_0 if $p = \infty$.*

Suppose also that $T \in \mathcal{B}(X)$ has $TP = 0$ and suppose $F \in \mathcal{B}(X)$ is a finite rank operator.

Then there exists a projection $P' = PP'P \in \mathcal{B}(X)$ with $\|P'\| = 1$ and range $Y' = P'X \subseteq Y$ isometrically isomorphic to Y such that $(T + F)P' = 0$.

Moreover, if $FX \cap Y \subset \operatorname{span}\{e_n : 1 \leq n \leq n_0\}$ for some $n_0 \in \mathbb{N}$ (where $(e_n)_n$ denotes the standard unit vector Schauder basis for Y) then we can choose P' such that $FX \cap Y' = \{0\}$ and

$$\|y_0 + y'\| = \|(\|y_0\|, \|y'\|)\|_p \tag{5.2.1}$$

holds for $y_0 \in \operatorname{span}\{e_n : 1 \leq n \leq n_0\}$, $y' \in Y'$.

PROOF. Since F is a finite rank operator, $\ker F$ has finite codimension. We denote by P_m the coordinate projection of Y onto $\operatorname{span}_{1 \leq n \leq m}\{e_n\}$.

Let $m_0 = 0$ and let m_1 be the smallest integer such that $((P_{m_1} - P_{m_0})Y) \cap \ker F \neq \{0\}$. Choose $y_1 \in ((P_{m_1} - P_{m_0})Y) \cap \ker F$ such that $\|y_1\| = 1$. We can inductively find m_k and y_k such that $\|y_{k+1}\| = 1$ and $y_{k+1} \in (P_{m_{k+1}} - P_{m_k})Y) \cap \ker F$.

Clearly $Y' = [y_k : k \in \mathbb{N}] \subset \ker T \cap \ker F_n \subset \ker(T + F_n)$ and Y' is isometrically isomorphic to ℓ^p (or c_0 if $p = \infty$). Let

$$y_k^* \in ((P_{m_{k+1}} - P_{m_k})(Y))^*$$

such that $y_k^*(y_k) = \|y_k^*\| = 1$ and extend each y_k^* to X by composing with P and the projection $P_{m_{k+1}} - P_{m_k}$ of Y. Then

$$P' = \sum_{i=1}^{\infty} y_k^* \otimes y_k$$

has $P' = PP'P$, $\|P'\| = 1$ (since the y_k are disjointly supported) and P' is a projection from X onto Y', as required.

For the last part, if $FX \cap Y \subset \operatorname{span}\{e_n : 1 \leq n \leq n_0\}$, we can proceed as before but with $m_0 = n_0$ instead of $m_0 = 0$. □

REMARK 5.2.3. Summarising the above proof, once we choose finitely and disjointly supported unit vectors $y_k \in Y$, they are equivalent to the standard basis for Y and their (closed) span is 1-complemented. This same idea will recur in several proofs below.

COROLLARY 5.2.4. *Let $1 \leq p \leq \infty$ and let X be a Banach space that admits a norm one projection $P \in \mathcal{B}(X)$ with range $Y = PX$ that is isometric to ℓ^p if $1 \leq p < \infty$ or to c_0 if $p = \infty$.*

Suppose $F \in \mathcal{B}(X)$ is a finite rank operator.

Then there exists a a projection $P' = PP'P \in \mathcal{B}(X)$ with $\|P'\| = 1$ and range $Y' = P'X \subseteq Y$ isometrically isomorphic to Y such that $FP' = 0$.

PROOF. Apply Lemma 5.2.2 to $T = 0$. □

PROPOSITION 5.2.5. *Let $1 \leq p \leq \infty$ and let X be a Banach space that admits a norm one projection $P \in \mathcal{B}(X)$ with range $Y = PX$ that is isometric to ℓ^p if $1 \leq p < \infty$ or to c_0 if $p = \infty$.*

Suppose $F \in \mathcal{B}(X)$ is a finite rank operator.

Then there exists a projection $P' = PP'P \in \mathcal{B}(X)$ with $\|P'\| = 1$ and range $Y' = P'X \subseteq Y$ isometrically isomorphic to Y such that $FP' = 0 = P'F$.

PROOF. By Corollary 5.2.4, we can suppose that P already satisfies $FP = 0$.

Let $(e_n)_n$ denote the standard unit vector Schauder basis for Y. For $r \geq 0$, let $Y_r = \operatorname{span}\{e_1, \ldots, e_r\}$, π_r the coordinate projection of Y onto Y_r and $P_r = \pi_r P$. Let $m = 1 + \dim(FX)$,

$$Y_{k,m} = \operatorname{span}\{e_{(k-1)m+n} : 1 \leq n \leq m\},$$

for $k \in \mathbb{N}$. Let $P_{k,m} = P_{k,m}P = P_{km} - P_{(k-1)m} \in \mathcal{B}(X)$ be the (norm one) projection of X onto $Y_{k,m}$ obtained by composing P with the appropriate coordinate projections of Y ($k \in \mathbb{N}$).

By Lemma 2.6.3, since $\dim P_{k,m}FX < m = \dim Y_{k,m}$, there is $y_k \in Y_{k,m}$ with $\|y_k\| = 1 = \operatorname{dist}(y_k, P_{k,m}FX)$. Then by Lemma 2.2.12, there is $y_k^* \in Y_{k,m}^*$ such that $y_k^*(y_k) = 1 = \|y_k^*\|$ and $P_{k,m}FX \subset \ker y_k^*$. Extend each y_k^* to $z_k^* = y_k^* \circ P_{k,m} \in X^*$ (so that $FX \subset \ker z_k^*$).

Then $Y' = [y_k : k \in \mathbb{N}]$ is isometrically isomorphic to Y (since $(y_k)_k$ is equivalent to the unit vector basis of Y). Moreover

$$P' = \sum_{i=1}^{\infty} z_k^* \otimes y_k$$

has $P' = PP'P$, $\|P'\| = 1$ (since the y_k are disjointly supported) and P' is a projection from X onto $Y' = P'Y = P'X$ with $P'F = 0$. □

LEMMA 5.2.6. *Let X be a Banach space and $T, K \in \mathcal{B}(X)$ with K compact. Suppose $\sigma(T+K) = \sigma(T) \setminus S$ for some finite (possibly empty) set $S \in \sigma_0(T)$. Then $\sigma_0(T+K) = \sigma_0(T) \setminus S$.*

PROOF. We have $\sigma_0(T+K) \subseteq \sigma(T+K) = \sigma(T) \setminus S$. Thus, if $\mu \in \sigma_0(T+K) \setminus S$, then $\mu \in \sigma_0(T)$ or $\mu \in \rho^{r,0}(T)$ since K is compact (see §2.4). Since $\rho^{r,0}(T)$ is open and μ is isolated in $\sigma(T+K)$, hence isolated in $\sigma(T)$, we must have $\mu \in \sigma_0(T)$. Thus $\sigma_0(T+K) \subseteq \sigma_0(T) \setminus S$.

For the reverse containment, if $\mu \in \sigma_0(T) \setminus S$, the same reasoning rules out $\mu \in \rho^{r,0}(T+K)$ and so $\mu \in \sigma_0(T+K)$. □

LEMMA 5.2.7. *Suppose $T \in \mathcal{B}(X)$ is a Riesz operator. Then $\lambda + T$ is left invertible iff it is right invertible.*

PROOF. If $-\lambda \notin \sigma(T)$ then $\lambda + T$ is invertible and is therefore both left and right invertible.

If $\lambda = 0$, then since T is not semi-Fredholm we have that either TX is not closed or $\dim\ker T = \operatorname{codim} TX = \infty$. If $\dim\ker T = \operatorname{codim} TX = \infty$ then T is neither left nor right invertible. If TX is not closed, T is clearly not right invertible. If $\ker T \neq \{0\}$ then T is not left invertible either. If $\ker T = \{0\}$ then $\gamma(T) = \inf_{\|x\|=1} \|Tx\|$ and (by Lemma 2.6.1) $\gamma(T) = 0$ so again T is not left invertible. Therefore if $\lambda = 0$ then $\lambda + T$ is neither left nor right invertible.

If $-\lambda \in \sigma_0(T)$, then $0 \neq \dim\ker(\lambda + T) = \operatorname{codim}(\lambda + T)X$ and again $\lambda + T$ is neither left nor right invertible.

Putting these three cases together we get that $\lambda + T$ is left invertible iff it is right invertible. □

COROLLARY 5.2.8. *Let X be a Banach space with $X = Y \oplus Z$ and let $P(y+z) = y$ denote the projection with kernel Z and range Y. Assume $T \in \mathcal{B}(X)$ has $T(Y) \subseteq Y$ and the block form*

$$T = \begin{pmatrix} T_{1,1} & T_{1,2} \\ 0 & T_{2,2} \end{pmatrix}$$

with respect to $X = Y \oplus Z$.

If $T_{1,1}$ is of the form $\mu + F$ where $\mu \in \mathbb{K}$ and F is a Riesz operator then $\sigma(T) = \sigma(T_{1,1}) \cup \sigma(T_{2,2})$.

If $\dim Y < \infty$ then $\sigma_0(T) = (\sigma_0(T_{1,1}) \setminus \sigma(T_{2,2})) \cup \sigma_0(T_{2,2})$

PROOF. If $\lambda \notin \sigma(T)$, we can write

$$(T-\lambda)^{-1}(T-\lambda) = \begin{pmatrix} S_{1,1} & S_{1,2} \\ S_{2,1} & S_{2,2} \end{pmatrix} \begin{pmatrix} T_{1,1}-\lambda & T_{1,2} \\ 0 & T_{2,2}-\lambda \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

to conclude that $S_{1,1} = (T_{1,1} - \lambda)^{-1}$ (since $F + (\mu - \lambda)$ is left invertible iff it is right invertible by Lemma 5.2.7) and $S_{2,1} = 0$. Then $S_{2,2} = (T_{2,2} - \lambda)^{-1}$ follows and we have $\lambda \notin \sigma(T_{1,1}) \cup \sigma(T_{2,2})$. Thus $\sigma(T_{1,1}) \cup \sigma(T_{2,2}) \subset \sigma(T)$. For the reverse, if $\lambda \notin \sigma(T_{1,1}) \cup \sigma(T_{2,2})$ then

$$(T-\lambda)^{-1} = \begin{pmatrix} (T_{1,1}-\lambda)^{-1} & -(T_{1,1}-\lambda)^{-1}T_{1,2}(T_{2,2}-\lambda)^{-1} \\ 0 & (T_{2,2}-\lambda)^{-1} \end{pmatrix}$$

and we conclude $\lambda \notin \sigma(T)$. Hence the first assertion is established.

If $\dim Y < \infty$, then the first assertion and Lemma 5.2.6 shows that

$$\sigma_0(T) = \sigma_0 \begin{pmatrix} T_{1,1} & 0 \\ 0 & T_{2,2} \end{pmatrix}$$

(since $T_{1,2}$ is compact). Then the second assertion now follows since the generalised λ-eigenspace of the latter block diagonal operator are direct sums of (finite dimensional) generalised eigenspaces for $T_{1,1}$ and generalised eigenspaces for $T_{2,2}$. □

PROPOSITION 5.2.9. *Let $1 \leq p \leq \infty$ and let X be a Banach space that admits a norm one projection $P \in \mathcal{B}(X)$ with range $Y = PX$ isometric to ℓ^p if $1 \leq p < \infty$ or to c_0 if $p = \infty$.*

Fix an isometry $\tau\colon Y \to \ell^p$ (if $1 \leq p < \infty$ or $\tau\colon Y \to c_0$ if $p = \infty$). (We will suppress mention of τ from now on, however.) Let Y_r denote the span of the first r canonical basis vectors for Y ($r \in \mathbb{N}$).

Suppose also that $T \in \mathcal{B}(X)$ has $TP = 0$, or equivalently the block form

$$T = \begin{pmatrix} 0 & B \\ 0 & A \end{pmatrix}$$

with respect to $X = Y \oplus \ker P$.

Let $\lambda \in \sigma_0(T) \setminus \{0\}$ and let $E = E_{T,\lambda} \in \mathcal{B}(X)$ denote the (finite rank) spectral projection onto the generalized λ-eigenspace of T. Let $\ell = \dim E(X)$.

Let $\varepsilon > 0$ be given.

Then, for a suitably chosen $m = m(\lambda, \varepsilon/(4\ell))$ there is

(a) a norm one projection $P' \in \mathcal{B}(X)$ with $P' = PP'P$ and range $Y' = P'(X)$ isometric to Y, such that

$$TP' = 0 \text{ and } E = P'E = EP';$$

We can ensure that $Y'_r \subseteq Y_{r(\ell+1)}$ holds for $r \in \mathbb{N}$, where Y'_r is defined similarly to Y_r with respect to the (canonical) basis for Y'.

(b) *a finite rank operator* $K \in \mathcal{B}(X)$ *with* $K = \pi K \pi$, *where* π *is a finite rank projection of* X *onto* $Y'_{m\ell} \oplus E(X)$.
(c) $\tilde{x}_i \in E(X)$ $(1 \leq i \leq \ell)$.
(d) $u_i \in Y'_{m\ell}$ $(1 \leq i \leq \ell)$.
(e) $\tilde{f}_i, \phi_i \in X^*$ *with* $\phi_i = \phi_i \circ P'_{m,i}$ *and* $\tilde{f}_i = \tilde{f}_i \circ (I - P')$ $(1 \leq i \leq \ell)$. *(Here* $P'_{m,i} = P'_{mi} - P'_{m(i-1)}$ *where* $P'_{m\ell} = P'_{m\ell} P'$ *is the coordinate projection of* Y' *onto* $Y'_{m\ell}$ *extended to* X*).*
(f) $R \in \mathcal{B}(X)$ *with* $R = P'_{m\ell} R P'_{m\ell}$.

such that

(i)

$$K = R + \sum_{i=1}^{\ell} (\tilde{x}_i \otimes \phi_i + u_i \otimes \tilde{f}_i),$$

(ii) $\|R\| < 2|\lambda| + \varepsilon/2$,
(iii) $\|\tilde{x}_i\|, \|\tilde{f}_i\| < 2$ $(1 \leq i \leq \ell)$,
(iv) $\|u_i\| + \|\phi_i\| < \varepsilon/(4\ell)$ $(1 \leq i \leq \ell)$,
(v) $\|K\| < 2|\lambda| + \varepsilon$,
(vi)

$$\sigma(T + K) = \sigma(T) \setminus \{\lambda\},$$

(vii)

$$\sigma_0(T + K) = \sigma_0(T) \setminus \{\lambda\}.$$

PROOF. Choose $m = m(\lambda, \varepsilon/(4\ell))$ as in Corollary 5.1.2.

Let $e_1, e_2, \ldots$ denote the standard basis of Y (or, more formally, be so that $\tau(e_1), \tau(e_2), \ldots$ are the standard basis vectors of ℓ^p (or c_0 if $p = \infty$). Write

$$Y_{i,r} = \operatorname{span}\{e_{(i-1)r+n} : 1 \leq n \leq r\}$$

(for $i, r \in \mathbb{N}$). Let $P_{i,r} = P_{i,r} P = P_{ir} - P_{(i-1)r} \in \mathcal{B}(X)$ be the norm one projection of X onto $Y_{i,r}$ obtained by composing P with the appropriate coordinate projections of Y $(i, r \in \mathbb{N})$.

By Lemma 2.6.3, there exists $e'_i \in Y_{i,\ell+1}$ such that

$$1 = \|e'_i\| = \operatorname{dist}(e'_i, P_{i,\ell+1}(E(X))) \qquad (i \in \mathbb{N}).$$

Hence (by Lemma 2.2.12) there is a linear functional

$$\zeta'_i \in (P_{i,\ell+1}(E(X)) + \mathbb{C}e'_i)^*$$

with

$$\zeta'_i(P_{i,\ell+1}(E(X))) = 0 \text{ and } \zeta'_i(e'_i) = 1 = \|\zeta'_i\|$$

(for each $i \in \mathbb{N}$). Via Hahn-Banach, we can extend each ζ'_i to

$$\zeta'_i \in (Y_{i,\ell+1})^*,$$

keeping the same properties, and then further extend to

$$\zeta_i = \zeta'_i \circ P_{i,\ell_1} \in X^* \qquad (i \in \mathbb{N}).$$

In summary, we have $\zeta_i(E(X)) = 0$, $1 = \zeta_i(e'_i) = \|\zeta'_i\| = \|e'_i\|$ and $e'_i \in Y_{i,\ell+1}$.

We define Y' to be the closed span of $\{e'_i : i \in \mathbb{N}\}$ and note that it is isometric to Y (in such a way that $e'_1, e'_2, \ldots$ becomes the canonical basis of Y') because of the disjoint supports of the vectors e'_i in the basis of Y. We have a norm one projection

$$P' = \sum_{i=1}^{\infty} e'_i \otimes \zeta_i = P'P$$

of X onto Y'. We have $TP' = TPP' = 0$.

Since $\zeta_i(E(X)) = 0$, we have $E(X) \subseteq \ker P'$, or $P'E = 0$, and E must have the block form

$$E = \begin{pmatrix} 0 & 0 \\ * & E' \end{pmatrix}$$

with respect to $X = Y' \oplus \ker P'$. By Proposition 2.4.4 (since E is a Riesz projection for T), $\ker T \subseteq \ker E$ and so $Y' = P'(X) \subseteq \ker T \subseteq \ker E$. Hence E has the block form

$$E = \begin{pmatrix} 0 & 0 \\ 0 & E' \end{pmatrix} \tag{5.2.2}$$

with respect to $X = Y' \oplus \ker P'$, or $E = (I - P')E(I - P')$.

We can now introduce the notations $Y'_r = \operatorname{span}\{e'_1, \ldots, e'_r\}$, P'_r, $Y'_{i,r}$ and $P'_{i,r}$ (for $i, r \in \mathbb{N}$) by following the pattern established before.

We have now established (a).

Using Corollary 5.1.2, there are (for $1 \leq i \leq \ell$) nilpotent operators

$$J_i \in \mathcal{B}(X)$$

with $J_i = P'_{i,m} J_i P'_{i,m}$,

$$x_i \in Y'_{i,m}, f_i, \phi_i \in X^*, u_i \in (\ker f_i) \cap Y'_{i,m}$$

and

$$R_i \in \mathcal{B}(X)$$

with

$$J_i = R_i + \lambda x_i \otimes \left(\frac{f_i}{f_i(x_i)} \right) + x_i \otimes \phi_i + u_i \otimes f_i. \tag{5.2.3}$$

Moreover we have the properties

$$R_i = P'_{i,m} R_i P'_{i,m}, \quad \|R_i\| < 2|\lambda| + \frac{\varepsilon}{4\ell}, \quad R_i(X) \subseteq Y'_{i,m} \cap \ker f_i, \quad R_i(x_i) = 0,$$

$$\|x_i\| = 1 = \|f_i\|, \quad |f_i(x_i) - 1| < \frac{1}{2}, \quad \|\phi_i\| + \|u_i\| < \frac{\varepsilon}{2\ell}, \quad \phi_i = \phi_i \circ P'_{i,m}$$

and $\phi_i(x_i) = 0$. Note that u_i satisfies (d), ϕ has the property in (e) and (iv) holds.

Reviewing the proof of Corollary 5.1.2, we can also ensure that the coordinates of x_i and u_i in the canonical basis $\{e'_{(i-1)m+n} : 1 \leq n \leq m\}$ of $Y'_{i,m}$ are independent of i, and that the coefficients of f_i and ϕ_i in the

cananical dual basis $\{\zeta_{(i-1}m+n : 1 \le n \le m\}$ also do not depend on i. We can fix a basis of $(\ker f_i) \cap Y'_{i,m}$ in such a way that the coordinates of the basis vectors with respect to $\{e'_{(i-1}m + n : 1 \le n \le m\}$ are independent of i. We can then treat the restriction of R_i to $(\ker f_i) \cap Y'_{i,m}$ as a constant $(m-1) \times (m-1)$ matrix Q (independent of i) in the chosen basis for $(\ker f_i) \cap Y'_{i,m}$. In short then we have a block matrix form

$$J_i|_{Y'_{i,m}} = \begin{pmatrix} \lambda & x_i \otimes \phi_i \\ u_i \otimes f_i & Q \end{pmatrix} = \begin{pmatrix} \lambda & r \\ c & Q \end{pmatrix}$$

where the block structure is with respect to $Y'_{i,m} = \mathbb{C}x_i \oplus ((\ker f_i) \cap Y'_{i,m})$ $(1 \le i \le \ell)$.

Next we show that $\left\|\sum_{i=1}^{\ell} R_i\right\| = \|R_1\|$. If $x \in X$, we write $x = \sum_{i=1}^{\ell} y_i + z$ with $y_i = P'_{i,m}x \in Y'_{i,m}$ and $z \in \ker\left(\sum_{i=1}^{\ell} P'_{i,m}\right) = \ker P'_{\ell m}$. In the case $p < \infty$, we can estimate

$$\begin{aligned} \left\|\sum_{i=1}^{\ell} R_i x\right\| &= \left\|\sum_{i=1}^{\ell} R_i y_i\right\| \\ &= \left(\sum_{i=1}^{\ell} \|R_i y_i\|^p\right) \\ &\le \max_i \|R_i\| \left(\sum_{i=1}^{\ell} \|y_i\|^p\right) \\ &= \|R_1\|\|P'_{\ell m}x\| \le \|R_1\|\|x\|, \end{aligned}$$

and a similar estimate holds in the case $p = \infty$. So $\left\|\sum_{i=1}^{\ell} R_i\right\| \le \|R_1\|$ and the reverse inequality follows easily from consideration of the restriction of the sum to $Y'_{1,m}$.

Let $R = \sum_{i=1}^{\ell} R_i$ so that R satisfies (f) and (ii).

Now choose (using Auerbach's Lemma [**24**, Proposition1.c.3]) a biorthogonal system $x'_i \in E(X)$, $f'_i \in (E(X))^*$ $(1 \le i \le \ell)$ such that $\|x'_i\| = 1 = \|f'_i\|$. We consider each f'_i as extended by Hahn-Banach (without change of norm) to $f'_i \in (\ker P')^*$ and let $\tilde{f}_i = f'_i \circ (I - P') \in X^*$. Then $\|\tilde{f}_i\| \le 2$ and the system consisting of $x'_i \in E(X)$ and $\tilde{f}_i \in X^*$ $(1 \le i \le \ell)$ is biorthogonal. Let $\tilde{x}_i = f_i(x_i)x'_i$. Note that (since $|f_i(x_i)| < 3/2$) $\tilde{x}_i$ and $\tilde{f}_i$ satisfy (c) and (iii), and (e) holds.

We now take (as in (i))

$$K = R + \sum_{i=1}^{\ell} (\tilde{x}_i \otimes \phi_i + u_i \otimes \tilde{f}_i),$$

a finite rank operator with range in

$$X_1 \stackrel{\text{def}}{=} Y'_{m\ell} \oplus E(X).$$

Since $R = R \circ P'_{m\ell}$ and $\phi_i = \phi_i \circ P'_{m\ell}$, $K(E(X)) = \operatorname{span}\{u_1, \ldots, u_\ell\}$ and so

$$K(X_1) = \left(R + \sum_{i=1}^{\ell} \tilde{x}_i \otimes \phi_i \right) (Y'_{m\ell}) + \operatorname{span}\{u_1, \ldots, u_\ell\} = K(X) \subseteq X_1.$$

Thus $X_1 + \ker K = X$ and there is a (closed) subspace $X_2 \subseteq \ker K$ with $X = X_1 \oplus X_2$. If π is the projection of X onto X_1 with kernel X_2, then $K = \pi K = K\pi$ and in addition to (i), K satisfies (b).

A simple estimate on $\|K\|$ (using (ii), (iii) and (iv)) gives (v).

Since X_1 is T-invariant and contains $E(X)$, T has a block form

$$T = \begin{pmatrix} T_1 & T_3 \\ 0 & T_2 \end{pmatrix}$$

with respect to $X = X_1 \oplus X_2$ and $\sigma(T_2) = \sigma(T) \setminus \{\lambda\}$, $\sigma(T_1) = \{\lambda, 0\}$. Since $K(X) \subset X_1$, $T + K$ has a similar block form

$$T + K = \begin{pmatrix} T'_1 & T'_3 \\ 0 & T_2 \end{pmatrix}$$

and it remains to show that the spectrum of the finite dimensional operator $T'_1 = (T_1 + K)|_{X_1}$ is $\{0\}$ so that (vi) will then follow from Corollary 5.2.8 and (vii) will follow from Lemma 5.2.6.

On X_1 define an invertible operator S by $S(y) = y$ for $y \in \bigoplus_{i=1}^{\ell} (Y'_{i,m} \cap \ker f_i)$,

$$S(x_i) = f_i(x_i)x'_i = \tilde{x}_i, \quad S(x'_i) = (1/f_i(x_i))x_i \quad (1 \leq i \leq \ell).$$

So $S(\tilde{x}_i) = x_i$. Since $\tilde{f}_i(y) = 0 = f_i(y)$ for $y \in \bigoplus_{i=1}^{\ell} (Y'_{i,m} \cap \ker f_i)$, we have $S^* \tilde{f}_i(y) = f_i(y)$ for those y. Also $f_i(x_j) = 0 = \tilde{f}_i(\tilde{x}_j)$ when $i \neq j$ so that $S^* \tilde{f}_i(x_j) = 0 = f_i(x_j)$ and $S^* f_i(x_j) = 0 = \tilde{f}_i(x_j)$ when $i \neq j$. For $1 \leq i \leq \ell$,

$$S^* \tilde{f}_i(x_i) = \tilde{f}_i(f_i(x_i)x'_i) = f_i(x_i)$$

and

$$S^* \tilde{f}_i(\tilde{x}_i) = f_i(x_i) = 0 = f_i(\tilde{x}_i).$$

Thus $\tilde{f}_i \circ S = S^* \tilde{f}_i = f_i$ on X_1. A more simple calculation gives $\phi_i \circ S = \phi_i$ and we have $Su_i = u_i$ for $1 \leq i \leq \ell$. We also have $R_i S = R_i = S R_i$ on X_1.

The projection $P_{x_i, f_i}(z) = (1/f_i(x_i))(x_i \otimes f_i)(z)$ has $S P_{x_i,f_i} S^{-1} = P_{\tilde{x}_i, \tilde{f}_i} = \tilde{x}_i \otimes \tilde{f}_i$ on X_1. We will use this a little later.

We can see that

$$S^{-1} K S = \sum_{i=1}^{\ell} (R_i + x_i \otimes \phi_i + u_i \otimes f_i) = \sum_{i=1}^{\ell} \left(J_i - \lambda x_i \otimes \frac{f_i}{f_i(x_i)} \right).$$

Omitting blocks of zero matrices for the summand $E(X)$ of X_1, and considering $S^{-1}KS$, $\sum_{i=1}^{\ell} J_i$ and $\sum_{i=1}^{\ell} \lambda P_{x_i,f_i}$ as operators on $Y'_{m\ell}$ we have an equality of $m\ell \times m\ell$ matrices. We can consider the right

hand side above as a direct sum of ℓ matrices of size $m \times m$, with the i^{th} summand an operator on $Y'_{m,i}$. Recalling the choice of a basis for $Y'_{m,i}$ respecting the decomposition $Y'_{m,i} = \mathbb{C}x_i \oplus (Y'_{m,i} \cap \ker f_i)$ with coefficients independent of i, we have (on $Y'_{m,i}$)

$$J_i = \begin{pmatrix} \lambda & r \\ c & Q \end{pmatrix}$$

and (on $Y'_{m\ell}$)

$$\sum_{i=1}^{\ell} J_i = \begin{pmatrix} \lambda & r \\ c & Q \end{pmatrix} \otimes I_\ell, \quad S^{-1}KS = \begin{pmatrix} 0 & r \\ c & Q \end{pmatrix} \otimes I_\ell.$$

Notice that $S^{-1}T_1S$ has a block diagonal form

$$\begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix}$$

with respect to $X_1 = Y'_{m\ell} \oplus E(X)$. Also T_1 has generalized λ-eigenspace $\operatorname{span}\{\tilde{x}_1, \ldots, \tilde{x}_\ell\}$ and $S^{-1}T_1S$ has generalized λ-eigenspace $\operatorname{span}\{x_1, \ldots, x_\ell\} \subset Y'_{m\ell}$. Hence, on $\operatorname{span}\{x_1, \ldots, x_\ell\}$, $S^{-1}T_1S$ agrees with $\lambda + N$ for some nilpotent N, and on $Y'_{m\ell}$ it can be expressed as

$$\begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} \otimes I_\ell + \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} \otimes N.$$

It follows that we have a representation for $S^{-1}(T_1 + K)S$ on $Y'_{m\ell}$ as

$$\begin{pmatrix} \lambda & r \\ c & Q \end{pmatrix} \otimes I_\ell + \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} \otimes N = J \otimes I_\ell + \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} \otimes N.$$

Expanding this sum to the power ℓm, we get sums of non-commuting products of $J \otimes I_\ell$ and $\begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} \otimes N$. All terms with ℓ or more powers of the second summand vanish (by tensor algebra) and there remain terms with a total of at most $\ell - 1$ occurrences of the second summand and at least $\ell m - \ell + 1$ occurrences of $J \otimes I_\ell$. But all of these remaining terms must have at least one string of m consecutive occurrences of $J \otimes I_\ell$ and so must vanish.

Thus $0 = (S^{-1}(T_1 + K)S|_{Y'_{m\ell}})^{m\ell} = S^{-1}((T_1 + K)|_{Y'_{m\ell}})^{m\ell}S$ and so we may conclude that $(T_1 + K)|_{X_1}$ is nilpotent.

As noted above, this is sufficient to show (vi) (by Corollary 5.2.8) and (vii) follows from (vi) and Lemma 5.2.6.

□

REMARK 5.2.10. Note that in the above proof we could have defined $Y' \oplus \ker P'$ so that $\ker P'$ contained any finite dimensional T-invariant subspace containing $E(X)$. Suppose we had replaced $E(X)$ in the proof by $E(X) \oplus S$ (where S is any finite dimensional T-invariant space in X, then $E(X) \oplus S \subset \ker P'$ exactly as before. Let $E' \in \mathcal{B}(\ker P')$ be a projection onto $E(X)$ with S in its kernel. If we were to keep

$\|\phi_i\| + \|u_i\| < \epsilon/(4\ell\|E'\|)$ and define $\tilde{f}_i = f_i' \circ E' \circ (I - P')$, we would have $\|\tilde{f}_i\| < 2\|E'\|$. Then $S \subset \ker \tilde{f}_i \cap \ker R_i \cap \ker \phi_i$ for each i so $S \subset \ker K$ and

$$\begin{aligned}
\|K\| &\leq \|R\| + \sum_{i=1}^{\ell} (\|\tilde{x}_i \otimes \phi_i\| + \|u_i \otimes \tilde{f}_i\|) \\
&< 2|\lambda| + \frac{\epsilon}{2} + \sum_{i=1}^{\ell} 2\|\phi_1\| + 2\|E'\|\|u_i\| \\
&\leq 2|\lambda| + \frac{\epsilon}{2} + \sum_{i=1}^{\ell} 2\|E'\|(\|\phi_1\| + \|u_i\|) \\
&< 2|\lambda| + \frac{\epsilon}{2} + \sum_{i=1}^{\ell} \frac{2\|E'\|\epsilon}{4\ell\|E'\|} \\
&\leq 2|\lambda| + \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= 2|\lambda| + \epsilon
\end{aligned}$$

In other words we can do the previous Proposition while also keeping any finite dimensional T-invariant subspace S with $S \cap E(X) = \{0\}$ in $\ker K$ without changing norm. In other words we have the following:

COROLLARY 5.2.11. *Let $1 \leq p \leq \infty$ and let X be a Banach space that admits a norm one projection $P \in \mathcal{B}(X)$ with range $Y = PX$ isometric to ℓ^p if $1 \leq p < \infty$ or to c_0 if $p = \infty$.*

Fix an isometry $\tau\colon Y \to \ell^p$ (if $1 \leq p < \infty$ or $\tau\colon Y \to c_0$ if $p = \infty$). (We will suppress mention of τ from now on, however.) Let Y_r denote the span of the first r canonical basis vectors for Y ($r \in \mathbb{N}$).

Suppose also that $T \in \mathcal{B}(X)$ has $TP = 0$, or equivalently the block form

$$T = \begin{pmatrix} 0 & B \\ 0 & A \end{pmatrix}$$

with respect to $X = Y \oplus \ker P$.

Let $\lambda \in \sigma_0(T) \setminus \{0\}$ and let $E = E_{T,\lambda} \in \mathcal{B}(X)$ denote the (finite rank) spectral projection onto the generalized λ-eigenspace of T. Let $\ell = \dim E(X)$.

Suppose S is a finite dimensional T-invariant space with $S \cap E(X) = \{0\}$. Let $E' \in \mathcal{B}(\ker P')$ be a projection onto $E(X)$ with S in its kernel.

Let $\varepsilon > 0$ be given.

Then, for a suitably chosen $m = m(\lambda, \varepsilon/(4\ell))$ there is

(a) a norm one projection $P' \in \mathcal{B}(X)$ with $P' = PP'P$ and range $Y' = P'(X)$ isometric to Y, such that

$$TP' = 0 \text{ and } E = P'E = EP';$$

We can ensure that $Y'_r \subseteq Y_{r(\ell+1)}$ *holds for* $r \in \mathbb{N}$, *where* Y'_r *is defined similarly to* Y_r *with respect to the (canonical) basis for* Y'.

(b) *a finite rank operator* $K \in \mathcal{B}(X)$ *with* $K = \pi K \pi$, *where* π *is a finite rank projection of* X *onto* $Y'_{m\ell} \oplus E(X)$.

(c) $\tilde{x}_i \in E(X)$ $(1 \leq i \leq \ell)$.

(d) $u_i \in Y'_{m\ell}$ $(1 \leq i \leq \ell)$.

(e) $\tilde{f}_i, \phi_i \in X^*$ *with* $\phi_i = \phi_i \circ P'_{m,i}$ *and* $\tilde{f}_i = \tilde{f}_i \circ E' \circ (I - P')$ $(1 \leq i \leq \ell)$. *(Here* $P'_{m,i} = P'_{mi} - P'_{m(i-1)}$ *where* $P'_{m\ell} = P'_{m\ell} P'$ *is the coordinate projection of* Y' *onto* $Y'_{m\ell}$ *extended to* X*).*

(f) $R \in \mathcal{B}(X)$ *with* $R = P'_{m\ell} R P'_{m\ell}$.

such that

(i)

$$K = R + \sum_{i=1}^{\ell} (\tilde{x}_i \otimes \phi_i + u_i \otimes \tilde{f}_i),$$

(ii) $\|R\| < 2|\lambda| + \varepsilon/2$,

(iii) $\|\tilde{x}_i\|, \|\tilde{f}_i\| < 2\|E'\|$ $(1 \leq i \leq \ell)$,

(iv) $\|u_i\| + \|\phi_i\| < \varepsilon/(4\ell\|E'\|)$ $(1 \leq i \leq \ell)$,

(v) $\|K\| < 2|\lambda| + \varepsilon$,

(vi)

$$\sigma(T + K) = \sigma(T) \setminus \{\lambda\},$$

(vii)

$$\sigma_0(T + K) = \sigma_0(T) \setminus \{\lambda\}.$$

(viii) $S \subset \ker K$.

In the following Lemma, we add an extra conclusion to Proposition 5.2.9 so as to ensure that the compact K has a pre-determined space in its kernel.

COROLLARY 5.2.12. *Let* $1 \leq p \leq \infty$ *and let* X *be a Banach space that admits a norm one projection* $P \in \mathcal{B}(X)$ *with range* $Y = PX$ *isometric to* ℓ^p *if* $1 \leq p < \infty$ *or to* c_0 *if* $p = \infty$.

Fix an isometry $\tau : Y \to \ell^p$ *(if* $1 \leq p < \infty$ *or* $\tau : Y \to c_0$ *if* $p = \infty$*). (We will suppress mention of* τ *from now on, however.) Let* Y_r *denote the span of the first* r *canonical basis vectors for* Y $(r \in \mathbb{N})$.

Let $Y'' \subseteq Y$ *be the span of finitely many basis vectors of* Y *and* Y''' *the closed span of the complementary basis vectors so that* $Y = Y'' \oplus_p Y'''$ *and there are norm one projections* $P'', P''' \in \mathcal{B}(X)$ *with ranges* Y'' *and* Y''' *(respectively) obtained by composing the coordinate projections on* Y *with* P. *Note that* Y''' *is isometric to* Y *via renumbering the basis vectors.*

Suppose also that $T \in \mathcal{B}(X)$ *has* $TP = 0$ *and the the block form*

$$T = \begin{pmatrix} 0 & 0 & B'' \\ 0 & 0 & B''' \\ 0 & 0 & A \end{pmatrix}$$

with respect to $X = Y'' \oplus Y''' \oplus \ker P$ *and* N *is a nilpotent.*

Let $\lambda \in \sigma_0(T) \setminus \{0\}$.

Then $\lambda \in \sigma_0((I - P'')T)$.

Let $E = E_{(I-P'')T,\lambda} \in \mathcal{B}(X)$ *denote the (finite rank) spectral projection onto the generalized* λ*-eigenspace of* $(I - P'')T$. *Let* $\ell = \dim E(X)$.

Let $\varepsilon > 0$ *be given.*

Then, for a suitably chosen $m = m(\lambda, \varepsilon/(4\ell))$ *there is*

(a) a norm one projection $P' \in \mathcal{B}(X)$ *with* $P' = P'''P'P'''$ *and range* $Y' = P'(X)$ *isometric to* Y, *such that*

$$(I - P'')TP' = 0 \text{ and } E = P'E = EP';$$

We can ensure that $Y'_r \subseteq Y'''_{r(\ell+1)}$ *holds for* $r \in \mathbb{N}$, *where* Y'_r *is defined similarly to* Y_r *with respect to the (canonical) basis for* Y' *(and* Y'''_r *is also similar).*

(b) a finite rank operator $K \in \mathcal{B}(X)$ *with* $K = \pi K \pi$, *where* π *is a finite rank projection of* X *onto* $Y'_{m\ell} \oplus E(X)$.

(c) $\tilde{x}_i \in E(X)$ $(1 \le i \le \ell)$.

(d) $u_i \in Y'_{m\ell}$ $(1 \le i \le \ell)$.

(e) $\tilde{f}_i, \phi_i \in X^*$ *with* $\phi_i = \phi_i \circ P'_{m,i}$ *and* $\tilde{f}_i = \tilde{f}_i \circ (I - P')$ $(1 \le i \le \ell)$. *(Here* $P'_{m,i} = P'_{mi} - P'_{m(i-1)}$, *where* $P'_{mi} = P'_{mil}P'$ *is the coordinate projection of* Y' *onto* Y'_{mi} *extended to* X*).*

(f) $R \in \mathcal{B}(X)$ *with* $R = P'_{m\ell}RP'_{m\ell}$

such that

(i)

$$K = R + \sum_{i=1}^{\ell} (\tilde{x}_i \otimes \phi_i + u_i \otimes \tilde{f}_i),$$

(ii) $\|R\| < 2|\lambda| + \varepsilon/2$,

(iii) $\|\tilde{x}_i\|, \|\tilde{f}_i\| < 2$ $(1 \le i \le \ell)$,

(iv) $\|u_i\| + \|\phi_i\| < \varepsilon/(4\ell)$ $(1 \le i \le \ell)$,

(v) $\|K\| < 2|\lambda| + \varepsilon$,

(vi)

$$\sigma(T + K) = \sigma(T) \setminus \{\lambda\},$$

(vii)

$$\sigma_0(T + K) = \sigma_0(T) \setminus \{\lambda\},$$

(viii) $Y'' \subset \ker K$.

PROOF. We first address the claim that $\lambda \in \sigma_0((I - P'')T)$.

$$(I - P'')T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & B''' \\ 0 & 0 & A \end{pmatrix} = 0 \oplus \begin{pmatrix} 0 & B''' \\ 0 & A \end{pmatrix}$$

and using Corollary 5.2.8, $\sigma((I - P'')T) = \sigma(T)$ (0 is an element of both) and by Lemma 5.2.6, $\sigma_0((I - P'')T) = \sigma_0(T)$.

We can now apply Proposition 5.2.9 to the operator $(I - P'')T$ on $Y''' \oplus \ker P$, with Y''' in place of Y, and from it deduce the existence of $Y' \subset Y'''$, K, R, etc. There are two operator norms to be considered for each operator or functional we construct, the norm on the subspace $Y''' \oplus \ker P$ and the norm on X. If we invoke Proposition 5.2.9 and later extend all the operators to X by composition with $I - P''$, we would lose a factor 2 in the estimates. But if we follow that same proof as used Proposition 5.2.9, and extend the operators to X as we go along (by composing with $I - (P' + P'')$ where we composed with $I - P'$ in the previous argument) we can save that factor 2.

We get the spectral conditions (vi) and (vii) for $(I - P'')T + K$ in place of $T + K$ but we can deduce (vi) and (vii) by invoking Corollary 5.2.8 and the proof of Lemma 5.2.6.

Property (viii) follows from the above.

□

REMARK 5.2.13. Notice that as a consequence of the above proof (of Corollary 5.2.12), there is a finite dimensional space

$$U = Y'' \oplus Y'_{m\ell} \oplus E(X)$$

which contains $K(X)$. It is also invariant under T because $E(X)$ is invariant under $(I - P'')T$, and that implies $T(E(X)) \subset Y'' \oplus E(X)$. Moreover $Y'' \oplus Y'_{m\ell} \subset Y \subset \ker T$.

Now if $\tilde{K} \in \mathcal{B}(X)$ is an operator with the properties

(1) $\tilde{K}(X) \subseteq U$ (so that $\tilde{K}$ is finite rank)
(2) $\tilde{K}|_U = K|_U$

then $\tilde{K}$ will still satisfy the spectral properties (vi) and (vii) and also satisfy (viii). That is $\tilde{K}$ will satisfy

(vi) $\sigma(T + \tilde{K}) = \sigma(T) \setminus \{\lambda\}$
(vii) $\sigma_0(T + \tilde{K}) = \sigma_0(T) \setminus \{\lambda\}$
(viii) $Y'' \subset \ker \tilde{K}$.

PROOF. Choose a closed complementary space V for U in X. With respect to the decomposition $X = U \oplus V$, T, K and $\tilde{K}$ all have the upper triangular block forms

$$T = \begin{pmatrix} T_{1,1} & T_{1,2} \\ 0 & T_{2,2} \end{pmatrix}, \quad K = \begin{pmatrix} K_{1,1} & K_{1,2} \\ 0 & 0 \end{pmatrix} \text{ and } \tilde{K} = \begin{pmatrix} K_{1,1} & \tilde{K}_{1,2} \\ 0 & 0 \end{pmatrix}.$$

Now (vi) follows from Corollary 5.2.8, (vii) from Lemma 5.2.6 and (viii) since $\tilde{K}|_U = K_U$ □

PROPOSITION 5.2.14. *Let $1 \leq p \leq \infty$ and let X be a Banach space that admits a norm one projection $P \in \mathcal{B}(X)$ with range $Y = PX$ isometric to ℓ^p if $1 \leq p < \infty$ or to c_0 if $p = \infty$.*

Fix an isometry $\tau\colon Y \to \ell^p$ (if $1 \leq p < \infty$ or $\tau\colon Y \to c_0$ if $p = \infty$). (We will suppress mention of τ from now on, however.) Let Y_r denote the span of the first r canonical basis vectors for Y ($r \in \mathbb{N}$).

Suppose that $T \in \mathcal{B}(X)$ has $TP = 0$, let $\lambda_1, \ldots, \lambda_J \in \sigma_0(T) \setminus \{0\}$ be $J < \infty$ distinct elements and let $\varepsilon > 0$ be given.

Let $Y'' \subseteq Y$ be the span of finitely many basis vectors of Y and Y''' the closed span of the complementary basis vectors so that $Y = Y'' \oplus_p Y'''$ and there are norm one projections $P'', P''' \in \mathcal{B}(X)$ with ranges Y'' and Y''' (respectively) obtained by composing the coordinate projections on Y with P. Note that Y''' is isometric to Y via renumbering the basis vectors.

Then there is a finite rank operator $K \in \mathcal{B}(X)$, a subspace $Y' \subseteq Y'''$ isometric to Y such that Y' is 1-complemented in X, and a finite dimensional space $U \subset X$ satisfying

(a) $\|K\| < 2\max_{1\leq j\leq J} |\lambda_j| + \varepsilon$,
(b) $T(U) \subseteq U$,
(c) $K(X) \subseteq U$,
(d) $\sigma(T + K) = \sigma(T) \setminus \{\lambda_1, \ldots, \lambda_J\}$,
(e) $\sigma_0(T + K) = \sigma_0(T) \setminus \{\lambda_1, \ldots, \lambda_J\}$,
(f) $Y'' \subset \ker K$ and $U \cap Y' \subseteq Y'_r$ for some finite r.

Moreover, if $\tilde{K} \in \mathcal{B}(X)$ has $\tilde{K}(X) \subset U$ and $\tilde{K}|_U = K|_U$, then $\tilde{K}$ will also satisfy $\sigma(T + K) = \sigma(T) \setminus \{\lambda_1, \ldots, \lambda_J\}$, $\sigma_0(T + K) = \sigma_0(T) \setminus \{\lambda_1, \ldots, \lambda_J\}$ and $Y'' \subset \ker \tilde{K}$.

PROOF. The idea is to apply Corollary 5.2.12 J times. First we apply it to T and λ_1 to produce K_1 satisfying the conclusions for K and λ of Corollary 5.2.12. Then we apply it to $T + K_1$ and λ_2, with the role of Y''' now taken by the Y' from the first application, to get K_2, etc. However, we need to take care of the details to get the bound $\|K_1 + K_2 + \cdots + K_J\| < 2\max_{1\leq j\leq J} |\lambda_j| + \varepsilon$. From Remark 5.2.13, we have no trouble coming up with U.

The details involve replacing ε by $\varepsilon/2^j$ and estimating the sum of the R terms using the disjointness of their supports.

Let us write Y_0 for Y''' and P_0 for P'''.

Let ℓ_1 denote the dimension of the generalised λ_1 eigenspace of T and $E_1 = E_{(I-P'')T,\lambda_1} \in \mathcal{B}(X)$ the spectral projection for $(I - P'')T$ corresponding to λ_1.

Applying Corollary 5.2.12, we can find m_1 and

(1-a) a norm one projection $P'_1 \in \mathcal{B}(X)$ with $P'_1 = P_0 P'_1 P_0$ and range $Y'_1 = P'_1(X)$ isometric to Y, such that

$$(I - P'')TP'_1 = 0 \text{ and } E_1 = P'_1 E_1 = E_1 P'_1;$$

We can ensure that $Y'_{1,r} \subseteq Y_{0,r(\ell_1+1)}$ holds for $r \in \mathbb{N}$, where $Y'_{1,r}$ is defined similarly to Y_r with respect to the (canonical) basis for Y'_1 (and $Y_{0,r}$ is also similar).

(1-b) a finite rank operator $K_1 \in \mathcal{B}(X)$ with $K_1 = \pi_1 K_1 \pi_1$, where π_1 is a finite rank projection of X onto $Y'_{1,m_1\ell_1} \oplus E_1(X)$.

(1-c) $\tilde{x}_{1,i} \in E_1(X)$ $(1 \le i \le \ell_1)$.

(1-d) $u_{1,i} \in Y'_{1,m_1\ell_1}$ $(1 \le i \le \ell_1)$.

(1-e) $\tilde{f}_{1,i}, \phi_{1,i} \in X^*$ with $\phi_{1,i} = \phi_{1,i} \circ P'_{1,m_1,i}$ and $\tilde{f}_{1,i} = \tilde{f}_{1,i} \circ (I - P'_1)$ $(1 \le i \le \ell_1)$. (Here $P'_{1,m_1,i} = P'_{1,m_1 i} - P'_{1,m_1(i-1)}$, where $P'_{1,r} = P'_{1,r} P'_1$ is the coordinate projection of Y'_1 onto $Y'_{1,r}$ extended to X).

(1-f) $R_1 \in \mathcal{B}(X)$ with $R_1 = P'_{1,m_1\ell_1} R P'_{1,m_1\ell_1}$,

such that

(1-i)

$$K_1 = R_1 + \sum_{i=1}^{\ell_1} (\tilde{x}_{1,i} \otimes \phi_{1,i} + u_{1,i} \otimes \tilde{f}_{1,i}),$$

(1-ii) $\|R_1\| < 2|\lambda_1| + \varepsilon/2^2$,

(1-iii) $\|\tilde{x}_{1,i}\|, \|\tilde{f}_{1,i}\| < 2$ $(1 \le i \le \ell)$,

(1-iv) $\|u_{1,i}\| + \|\phi_{1,i}\| < \varepsilon/(2^3 \ell_1)$ $(1 \le i \le \ell_1)$,

(1-v) $\|K_1\| < 2|\lambda_1| + \varepsilon/2$,

(1-vi)

$$\sigma(T + K_1) = \sigma(T) \setminus \{\lambda_1\},$$

(1-vii)

$$\sigma_0(T + K_1) = \sigma_0(T) \setminus \{\lambda_1\},$$

(1-viii) $Y'' \subset \ker K_1$.

Let $Y_1 = (P_1 - P_{1,m_1\ell_1})Y'_1$ so that $Y'' \oplus Y_1 = Y'' \oplus_p Y_1$ and is in the kernel of $T + K_1$. (To see the latter, observe that $\tilde{f}_{1,i}$ vanishes on $Y_1 = P_1 X$ and R_1 and $\phi_{1,i}$ vanish on $(P_1 - P_{1,m_1\ell_1})Y_1$ (as well as Y'').) Also $Y'' \oplus Y'_{1,m_1\ell_1} \oplus Y_1 = Y'' \oplus_p Y'_{1,m_1\ell_1} \oplus_p Y_1$. We are ready to apply Corollary 5.2.12 again, this time to $T + K_1$, λ_2 and $Y'' \oplus_p Y_1$.

Indeed we can repeat the initial step $J - 1$ times to get $K_2, \ldots, K_J$ such that (for each j with $2 \le j \le J$) we have

(j-a) a norm one projection $P'_j \in \mathcal{B}(X)$ with $P'_j = P_{j-1} P'_j P_{j-1}$ and range $Y'_j = P'_j(X)$ isometric to Y, such that

$$(I - P'')TP'_j = 0 \text{ and } E_j = P'_j E_j = E_j P'_j,$$

where E_j is the spectral projection for λ_j associated with the operator

$$(I - P'')(T + K_1 + \cdots + K_{j-1}).$$

We can ensure that $Y'_{j,r} \subseteq Y_{j-1,r(\ell_j+1)}$ holds for $r \in \mathbb{N}$, where $Y'_{j,r}$ is defined similarly to Y_r with respect to the (canonical) basis for Y'_j and $\ell_j = \dim E_j(X)$.

(j-b) a finite rank operator $K_j \in \mathcal{B}(X)$ with $K_j = \pi_j K_j \pi_j$, where π_j is a finite rank projection of X onto $Y'_{j,m_j\ell_j} \oplus E_j(X)$.

(j-c) $\tilde{x}_{j,i} \in E_j(X)$ $(1 \leq i \leq \ell_j)$.

(j-d) $u_{j,i} \in Y'_{j,m_j\ell_j}$ $(1 \leq i \leq \ell_j)$.

(j-e) $\tilde{f}_{j,i}, \phi_{j,i} \in X^*$ with $\phi_{j,i} = \phi_{j,i} \circ P'_{j,m_j,i}$ and $\tilde{f}_{j,i} = \tilde{f}_{j,i} \circ (I - P'_j)$ $(1 \leq i \leq \ell_j)$. (Here $P'_{j,m_j,i} = P'_{j,m_j i} - P'_{j,m_j(i-1)}$, where $P'_{j,r} = P'_{j,r} P'_j$ is the coordinate projection of Y'_j onto $Y'_{j,r}$ extended to X).

(j-f) $R_j \in \mathcal{B}(X)$ with $R_j = P'_{j,m_j\ell_j} R P'_{j,m_j\ell_j}$,

such that

(j-i)
$$K_j = R_j + \sum_{i=1}^{\ell_j} (\tilde{x}_{j,i} \otimes \phi_{j,i} + u_{j,i} \otimes \tilde{f}_{j,i}),$$

(j-ii) $\|R_j\| < 2|\lambda_j| + \varepsilon/2^{j+1}$,

(j-iii) $\|\tilde{x}_{j,i}\|, \|\tilde{f}_{j,i}\| < 2$ $(1 \leq i \leq \ell_j)$,

(j-iv) $\|u_{j,i}\| + \|\phi_{j,i}\| < \varepsilon/(2^{j+2}\ell_j)$ $(1 \leq i \leq \ell_j)$,

(j-v) $\|K_j\| < 2|\lambda_j| + \varepsilon/2^j$,

(j-vi)
$$\sigma(T + K_1 + \cdots + K_j) = \sigma(T) \setminus \{\lambda_1, \ldots, \lambda_j\},$$

(j-vii)
$$\sigma_0(T + K_1 + \cdots + K_j) = \sigma_0(T_1) \setminus \{\lambda_1 \ldots, \lambda_j\},$$

(j-viii) $Y'' \subset \ker K_j$.

As for $j = 1$, we define $Y_j = (P_j - P_{j,m_j\ell_j})Y'_j$ to be ready for the next iteration. We will have $Y'' \oplus Y'_{1,m_1\ell_1} \oplus \ldots \oplus Y'_{j,m_j\ell_j} \oplus Y_j = Y'' \oplus_p Y'_{1,m_1\ell_1} \oplus_p \ldots \oplus_p Y'_{j,m_j\ell_j} \oplus_p Y_j$.

To conclude we will take $K = K_1 + \cdots + K_J$, but we need to estimate $\|K\|$. The tensor terms can be estimated by the triangle inequality (using (j-iii) and (j-iv)) but the contribution from $\sum_{j=1}^{J} R_j$ can be estimated as

$$\left\| \sum_{j=1}^{J} R_j \right\| \leq \max_{1 \leq j \leq J} \|R_j\|$$

in the same way as the estimate in the proof of Proposition 5.2.9.

In this way, we have constructed K and we have the estimate (a) and the spectral properties (d) and (e). Also $Y'' \subset \ker K$, as required in (f).

To construct U, we argue as in Remark 5.2.13. Let

$$U_1 = Y'' \oplus Y'_{1,m_1\ell_1} \oplus E_1(X).$$

For $2 \leq j \leq J$,

$$U_j = U_{j-1} + Y'_{j,m_j\ell_j} + E_j(X).$$

By Remark 5.2.13, we know that $T(U_1) \subseteq U_1$ and $K_1(X) \subseteq U_1$, and hence $(T + K_1)(U_1) \subseteq U_1$. By a similar argument, for $2 \leq j \leq J$,

$$(T + K_1 + \cdots + K_{j-1})(U_j) \subset U_j$$

(and $K_j(X) \subseteq U_j$). As U_j contains the ranges of $K_1, \ldots, K_j$, we deduce that $T(U_j) \subseteq U_j$.

Taking $U = U_J$ and

$$Y' = Y'_{1,m_1\ell_1} \oplus_p \cdots \oplus_p Y'_{J-1,m_{J-1}\ell_{J-1}} \oplus_p Y'_J,$$

we can now verify (b), (c) and the remaining assertion in (f).

The final part follows as in Remark 5.2.13. □

REMARK 5.2.15. Let $S_0 \subset N^\infty(T)$ be some finite dimensional T-invariant subspace of X such that $S_0 \cap \bigoplus_{i=1}^J E_{T,\lambda_i}(X) = \{0\}$. Suppose we let $S_1 = S_0 \oplus \bigoplus_{i=2}^J E_{T,\lambda_i}(X) \oplus Y''$. Then we can apply Corollary 5.2.11 instead of Corollary 5.2.12 to remove λ_1. and $S_1 \subset \ker K_1$ and S_1 is still $T + K_1$ invariant. We will also have that $E_{T,\lambda_i} \subset E_{T+K_1,\lambda_i}$ for each $i > 1$.

Next, let

$$S_2 = S_0 \oplus E_{T,\lambda_i} \oplus Y'_{1,m_1\ell_1} + \bigoplus_{i=3}^J E_{T,\lambda_i}(X) \oplus Y'' = S_0 + U_1 + \bigoplus_{i=3}^J E_{T,\lambda_i}(X).$$

and repeat the above process. Then S_2 is $T + K_1 + K_2$ invariant and it contains E_{T,λ_1} and E_{T,λ_2} and S_0. We will also have that $E_{T,\lambda_i} \subset E_{T+K_1+K_2,\lambda_i}$ for each $i > 2$ and $S_0 \subset \ker K_1 \cap \ker K_2$.

Continuing in this way with

$$S_j = S_0 \oplus U_{j-1} \oplus \bigoplus_{i=3}^J E_{T+\sum_{i=1}^{j-1},\lambda_i}(X)'$$

we get rid of all of the eigenvalues but we can also ensure that there exists a $T + K$-invariant subspace S with

$$S_0 \oplus \bigoplus_{i=1}^J E_{T,\lambda_i}(X) \oplus Y'' \subset S$$

$S_0 \subset \bigcap_{i=1}^J \ker K_i \subset \ker K$ and $T + K$ is nilpotent on this space.

We then change the definition of U to be $U = S_J + Y'_{J,m_J\ell_J} + E_j(X)$.

In other words we have

PROPOSITION 5.2.16. *Let $1 \leq p \leq \infty$ and let X be a Banach space that admits a norm one projection $P \in \mathcal{B}(X)$ with range $Y = PX$ isometric to ℓ^p if $1 \leq p < \infty$ or to c_0 if $p = \infty$.*

Fix an isometry $\tau\colon Y \to \ell^p$ (if $1 \le p < \infty$ or $\tau\colon Y \to c_0$ if $p = \infty$). (We will suppress mention of τ from now on, however.) Let Y_r denote the span of the first r canonical basis vectors for Y ($r \in \mathbb{N}$).

Suppose that $T \in \mathcal{B}(X)$ has $TP = 0$, let $\lambda_1, \dots, \lambda_J \in \sigma_0(T) \setminus \{0\}$ be $J < \infty$ distinct elements and let $\varepsilon > 0$ be given.

Let $Y'' \subseteq Y$ be the span of finitely many basis vectors of Y and Y''' the closed span of the complementary basis vectors so that $Y = Y'' \oplus_p Y'''$ and there are norm one projections $P'', P''' \in \mathcal{B}(X)$ with ranges Y'' and Y''' (respectively) obtained by composing the coordinate projections on Y with P. Note that Y''' is isometric to Y via renumbering the basis vectors.

Suppose $S_0 \subset N^\infty(T)$ is some finite dimensional T-invariant subspace of X such that $S_0 \cap \bigoplus_{i=1}^J E_{T,\lambda_i}(X) = \{0\}$.

Then there is a finite rank operator $K \in \mathcal{B}(X)$, a subspace $Y' \subseteq Y'''$ isometric to Y such that Y' is 1-complemented in X, and a finite dimensional space $U \subset X$ with $S_0 \subset U$ satisfying

(a) $\|K\| < 2\max_{1\le j\le J} |\lambda_j| + \varepsilon$,
(b) $T(U) \subseteq U$,
(c) $K(X) \subseteq U$,
(d) $\sigma(T+K) = \sigma(T) \setminus \{\lambda_1, \dots, \lambda_J\}$,
(e) $\sigma_0(T+K) = \sigma_0(T) \setminus \{\lambda_1, \dots, \lambda_J\}$,
(f) $S_0 + Y'' \subset \ker K$ and $U \cap Y' \subseteq Y'_r$ for some finite r,
(g) $U \subset N^\infty(T+K)$ and $\operatorname{span}_{1\le i\le J}\{E_{T,\lambda_i}(X)\} \subset U$.

Moreover, if $\tilde{K} \in \mathcal{B}(X)$ has $\tilde{K}(X) \subset U$ and $\tilde{K}|_U = K|_U$, then $\tilde{K}$ will also satisfy $\sigma(T+K) = \sigma(T) \setminus \{\lambda_1, \dots, \lambda_J\}$, $\sigma_0(T+K) = \sigma_0(T) \setminus \{\lambda_1, \dots, \lambda_J\}$, $S_0 + Y'' \subset \ker \tilde{K}$, $U \subset N^\infty(T+K)$ and $\operatorname{span}\{E_{T,\lambda_i}(X)\} \subset U$.

LEMMA 5.2.17. *If X is a Banach space, $1 \le p \le \infty$ and $P \in \mathcal{B}(X)$ is a projection with range $Y = P(X)$ isomorphic to ℓ^p if $1 \le p < \infty$ or to c_0 in the case $p = \infty$, then there is an equivalent norm $|\!|\!| \cdot |\!|\!|$ on X and a constant $C \ge 1$, such that*

(a) $(Y, |\!|\!| \cdot |\!|\!|)$, that is Y with the norm $|\!|\!| \cdot |\!|\!|$, is isometric to ℓ^p (if $1 \le p < \infty$) or c_0 (if $p = \infty$);
(b) $|\!|\!|P|\!|\!| = 1$, where we use the same notation $|\!|\!| \cdot |\!|\!|$ for the operator norm on $\mathcal{B}(X, |\!|\!| \cdot |\!|\!|)$;
(c) $|\!|\!|T|\!|\!| \le C\|T\|$ and $\|T\| \le C|\!|\!|T|\!|\!|$ for $T \in \mathcal{B}(X)$;
(d) $|\!|\!|x|\!|\!| \le C\|x\|$ and $\|x\| \le C|\!|\!|x|\!|\!|$ for $x \in X$, and
(e) $|\!|\!|x^*|\!|\!| \le C\|x^*\|$ and $\|x^*\| \le C|\!|\!|x^*|\!|\!|$ for $x^* \in X^*$ (where again we use the same notation $|\!|\!| \cdot |\!|\!|$ for the dual norm on $(X, |\!|\!| \cdot |\!|\!|)^*$).

PROOF. If we fix an isomorphism $\tau\colon Y \to \ell^p$ (or $\tau\colon Y \to c_0$ in the case $p = \infty$), we can define

$$|\!|\!|x|\!|\!| = \|\tau P x\|_p + \|(I - P)x\|$$

and establish that C exists by elementary estimates. □

COROLLARY 5.2.18. *Let $1 \leq p \leq \infty$ and let X be a Banach space that admits a bounded projection $P \in \mathcal{B}(X)$ with range $Y = PX$ isomorphic to ℓ^p if $1 \leq p < \infty$ or to c_0 if $p = \infty$.*

Fix an isomorphism $\tau\colon Y \to \ell^p$ (if $1 \leq p < \infty$ or $\tau\colon Y \to c_0$ if $p = \infty$) and an equivalent norm $|||\cdot|||$ on X and a constant C that satisfy the conclusions of Lemma 5.2.17 and $\|\tau y\|_p = |||y|||$ for $y \in Y$. Let Y_r denote the span of the first r canonical basis vectors for Y ($r \in \mathbb{N}$).

Suppose that $T \in \mathcal{B}(X)$ has $TP = 0$, let $\lambda_1, \ldots, \lambda_J \in \sigma_0(T) \setminus \{0\}$ be $J < \infty$ distinct elements and let $\varepsilon > 0$ be given.

Let $Y'' \subseteq Y$ be the span of finitely many basis vectors of Y (interpreted using τ and the standard basis for ℓ^p, or c_0 in the case $p = \infty$) and Y''' the closed span of the complementary basis vectors so that

$$(Y, |||\cdot|||) = (Y'', |||\cdot|||) \oplus_p (Y''', |||\cdot|||).$$

Let $P'', P''' \in \mathcal{B}(X)$ be the projections with ranges Y'' and Y''' (respectively) obtained by composing the coordinate projections on Y with P. Note that $(Y''', |||\cdot|||)$ is isometric to ℓ^p (or c_0 in the case $p = \infty$) via renumbering the basis vectors.

Suppose $S_0 \subset N^\infty(T)$ is some finite dimensional T-invariant subspace of X such that $S_0 \cap \bigoplus_{i=1}^J E_{T,\lambda_i}(X) = \{0\}$.

Then there is a finite rank operator $K \in \mathcal{B}(X)$, a subspace $Y' \subseteq Y'''$ with $(Y', |||\cdot|||)$ isometric to $(Y, |||\cdot|||)$ such that $(Y', |||\cdot|||)$ is 1-complemented in $(X, |||\cdot|||)$, and a finite dimensional space $U \subset X$ with $S_0 \subset U$ satisfying

(a) $|||K||| < 2C \max_{1\leq j\leq J} |\lambda_j| + \varepsilon$,
(b) $T(U) \subseteq U$,
(c) $K(X) \subseteq U$,
(d) $\sigma(T + K) = \sigma(T) \setminus \{\lambda_1, \ldots, \lambda_J\}$,
(e) $\sigma_0(T + K) = \sigma_0(T) \setminus \{\lambda_1, \ldots, \lambda_J\}$,
(f) $S_0 + Y'' \subset \ker K$ *and* $U \cap Y' \subseteq Y'_r$ *for some finite r, (where Y'_r denotes the span of the first r canonical basis vectors of Y'),*
(g) $U \subset N^\infty(T + K)$ *and* $\operatorname{span}_{1\leq i\leq J}\{E_{T,\lambda_i}(X)\} \subset U$.

Moreover, if $\tilde{K} \in \mathcal{B}(X)$ has $\tilde{K}(X) \subset U$ and $\tilde{K}|_U = K_U$, then $\tilde{K}$ will also satisfy $\sigma(T + K) = \sigma(T) \setminus \{\lambda_1, \ldots, \lambda_J\}$, $\sigma_0(T + K) = \sigma_0(T) \setminus \{\lambda_1, \ldots, \lambda_J\}$ and $S_0 + Y'' \subset \ker \tilde{K}$, $U \subset N^\infty(T + K)$ and $\operatorname{span}\{E_{T,\lambda_i}(X)\} \subset U$.

If X is a dual space, we can ensure that K is weak-continuous.*

PROOF. This follows easily from Proposition 5.2.16 (applied in the equivalent norm $|||\cdot|||$) by replacing ε by ε/C. Thus the estimate

$$|||K||| < 2 \max_{1\leq j\leq J} |\lambda_j| + \varepsilon/C,$$

gives (a) via Lemma 5.2.17.

If X is a dual space, we can choose $\eta > 0$ small enough such that

$$\|K\|(1+\eta) < 2C \max_{1\le j\le J} |\lambda_j| + \varepsilon$$

and apply the principle of local reflexivity in the form Theorem 2.2.11 to show that there is a weak*-continuous $\tilde{K} \in \mathcal{B}(X)$ such that $\tilde{K}(X) = K(X) \subseteq U$, $\tilde{K}|_U = K|_U$ and $\|\tilde{K}\|$ satisfies the bound from (a). □

THEOREM 5.2.19. *Let $1 \le p \le \infty$ and let X be a Banach space that admits a bounded projection $P \in \mathcal{B}(X)$ with range $Y = PX$ isomorphic to ℓ^p if $1 \le p < \infty$ or to c_0 if $p = \infty$.*

Fix an isomorphism $\tau\colon Y \to \ell^p$ (if $1 \le p < \infty$ or $\tau\colon Y \to c_0$ if $p = \infty$) and an equivalent norm $|||\cdot|||$ on X and a constant C that satisfy the conclusions of Lemma 5.2.17 and $\|\tau y\|_p = |||y|||$ for $y \in Y$.

Suppose that $T \in \mathcal{B}(X)$ is a Riesz operator that has $TP = 0$, and suppose

$$\sigma_0(T) = \{\lambda_j : 1 \le j < J+1\},$$

where $J \le \infty$, and let $\varepsilon > 0$ be given.

Then there is a compact operator $K \in \mathcal{B}(X)$, a subspace $Y'' \subseteq Y$ with $(Y'', |||\cdot|||)$ isometric to $(Y, |||\cdot|||)$ such that $Y'' = P''(X)$ for a projection $P'' \in \mathcal{B}(X)$ with $|||P''||| = 1$, and the following hold:

(a) $\|K\| < 2C \max_{1\le j<J+1} |\lambda_j| + \varepsilon$,
(b) $\sigma(T+K) = \{0\}$, and
(c) $(T+K)P'' = 0$.

If X is a dual space, we can ensure that K is weak-continuous.*

PROOF. Note that the Riesz property is that $\sigma(T) = \{0\} \cup \sigma_0(T)$ (since X is not finite dimensional).

We consider first the case where $J < \infty$. In this case, Corollary 5.2.18 provides an operator K (of finite rank) with all the desired properties, except (c). To get the projection P'' we apply Lemma 5.2.2 on the space $(X, |||\cdot|||)$.

It remains to consider the case where $J = \infty$. We may assume that the λ_j are ordered so that $|\lambda_j| \ge |\lambda_{j+1}|$ for each $j \ge 1$. Choose a strictly decreasing sequence $(\delta_k)_{k=1}^\infty$ of positive scalars with $\sum_{k=1}^\infty \delta_k < \varepsilon/(8C)$ and a strictly increasing sequence $(n_k)_{k=1}^\infty$ of natural numbers such that

$$|\lambda_n| \le \delta_k \text{ for } n \ge n_k.$$

Let y_1 be the first basis vectors for Y. We first apply Corollary 5.2.18 to T and $\{\lambda_1, \dots, \lambda_{n_1}\}$ with Y'' in Corollary 5.2.18 taken as $Y_1'' = \mathbb{C}y_1$, Y''' in Corollary 5.2.18 taken as the closed span of the remaining basis vectors of Y, which we denote Y_1'''. We find a finite rank $K_1 \in \mathcal{B}(X)$ (weak*-continuous in the case X is a dual space) a subspace $Y_1' \subseteq Y_1'''$ with $(Y_1', |||\cdot|||)$ isometric to $(Y, |||\cdot|||)$ such that $(Y_1', |||\cdot|||)$ is 1-complemented in $(X, |||\cdot|||)$, satisfying

(a) $\|K_1\| < 2C|\lambda_1| + \delta_1$,

(b) $\sigma(T+K_1)=\sigma(T)\setminus\{\lambda_1,\dots,\lambda_{n_1}\}$,
(c) $\sigma_0(T+K_1)=\sigma_0(T)\setminus\{\lambda_1,\dots,\lambda_{n_1}\}$,
(d) $Y_1''\subset\ker K_1$ (and $Y_1''\subset\ker(T+K_1)$).

Note that $(Y_1'''\oplus Y_1',|\!|\!|\cdot|\!|\!|)=(Y_1''',|\!|\!|\cdot|\!|\!|)\oplus_p(Y_1',|\!|\!|\cdot|\!|\!|)$.

If P_1' is a projection in $\mathcal{B}(X)$ with $|\!|\!|P_1'|\!|\!|=1$ and range Y_1', we can replace it with $P_1'P_1'''$ (where P_1''' denotes P followed by the coordinate projection onto Y_1''') to guarantee $P_1'=P_1'P_1'''=P_1'P$.

We now apply Lemma 5.2.2 to find a projection $\tilde{P}_2\in\mathcal{B}(X)$ with $|\!|\!|\tilde{P}_2|\!|\!|=1$, $\tilde{P}_2=P_1'\tilde{P}_2P_1'$, range $\tilde{Y}_2=\tilde{P}_2(X)$, such that $(\tilde{Y}_2,|\!|\!|\cdot|\!|\!|)$ is isometric to $(Y,|\!|\!|\cdot|\!|\!|)$ and such that $(T+K_1)\tilde{P}_2=0$.

Note that $Y_2=Y_1''\oplus\tilde{Y}_2$ is isometric in $|\!|\!|\cdot|\!|\!|$ to $(Y,|\!|\!|\cdot|\!|\!|)$ and $(Y_2,|\!|\!|\cdot|\!|\!|)=(Y_1'''\oplus\tilde{Y}_2,|\!|\!|\cdot|\!|\!|)=(Y_1''',|\!|\!|\cdot|\!|\!|)\oplus_p(\tilde{Y}_2,|\!|\!|\cdot|\!|\!|)$. If P_1'' denotes the projection P onto Y followed by the coordinate projection onto Y_1'', then $P_2=P_1''+\tilde{P}_2$ has $|\!|\!|P_2|\!|\!|=1$ and $P_2(X)=Y_2$. If $\tilde{P}_2''$ is $\tilde{P}_2$ followed by the coordinate projection onto the first basis vector of Y_2, then $P_2''=P_1''+\tilde{P}_2''$ is a projection which has $|\!|\!|P_2''|\!|\!|=1$, $P_2''=P_2P_2''P_2$ and $P_2''(X)=Y_2''$.

Let y_2 denote the first basis vector of Y_2 and $Y_2''=\operatorname{span}\{y_1,y_2\}$, let Y_2''' be the closed span of the basis vectors of $\tilde{Y}_2$ apart from y_2. Note that $(Y_2'',|\!|\!|\cdot|\!|\!|)$ is a copy of a 2- dimensional ℓ^p space. We apply Corollary 5.2.18 again, this time to $T+K_1$ and $\{\lambda_{n_1+1},\dots,\lambda_{n_2}\}$ with Y'' in Corollary 5.2.18 taken as Y_2'' and Y''' in Corollary 5.2.18 taken as Y_2'''. We find a finite rank $K_2\in\mathcal{B}(X)$ (weak*-continuous in the case X is a dual space), a subspace $Y_2'\subseteq Y_1'''$ with $(Y_2',|\!|\!|\cdot|\!|\!|)$ isometric to $(Y,|\!|\!|\cdot|\!|\!|)$ such that $(Y_2',|\!|\!|\cdot|\!|\!|)$ is 1-complemented in $(X,|\!|\!|\cdot|\!|\!|)$, satisfying

(a) $\|K_2\|<2C\delta_1+\delta_2$,
(b) $\sigma(T+K_1+K_2)=\sigma(T)\setminus\{\lambda_1,\dots,\lambda_{n_2}\}$,
(c) $\sigma_0(T+K_1+K_2)=\sigma_0(T)\setminus\{\lambda_1,\dots,\lambda_{n_2}\}$,
(d) $Y_2''\subset\ker K_2$ (and $Y_2''\subset\ker(T+K_1+K_2)$)

Note that $(Y_2'''\oplus Y_2',|\!|\!|\cdot|\!|\!|)=(Y_2''',|\!|\!|\cdot|\!|\!|)\oplus_p(Y_2',|\!|\!|\cdot|\!|\!|)$.

If P_2' is a projection in $\mathcal{B}(X)$ with $|\!|\!|P_2'|\!|\!|=1$ and range Y_2', we can replace it with $P_2'P_2'''$ (where P_2''' denotes P followed by the coordinate projection onto Y_1''') to guarantee $P_2'=P_2'P_2'''=P_2'P$.

We now apply Lemma 5.2.2 to find a projection $\tilde{P}_3\in\mathcal{B}(X)$ with $|\!|\!|\tilde{P}_3|\!|\!|=1$, $\tilde{P}_3=P_2'\tilde{P}_3P_2'$, range $\tilde{Y}_3=\tilde{P}_3(X)$, such that $(\tilde{Y}_3,|\!|\!|\cdot|\!|\!|)$ is isometric to $(Y,|\!|\!|\cdot|\!|\!|)$ and such that $(T+K_1+K_2)\tilde{P}_3=0$.

Note that $Y_3=Y_2''\oplus\tilde{Y}_3$ is isometric in $|\!|\!|\cdot|\!|\!|$ to $(Y,|\!|\!|\cdot|\!|\!|)$ and $(Y_3,|\!|\!|\cdot|\!|\!|)=(Y_2'''\oplus\tilde{Y}_3,|\!|\!|\cdot|\!|\!|)=(Y_2''',|\!|\!|\cdot|\!|\!|)\oplus_p(\tilde{Y}_3,|\!|\!|\cdot|\!|\!|)$. If P_2'' denotes the projection P_2 onto Y_2 followed by the coordinate projection onto Y_1'', then $P_3=P_2''+\tilde{P}_3$ has $|\!|\!|P_3|\!|\!|=1$ and $P_3(X)=Y_3$. If $\tilde{P}_3''$ is $\tilde{P}_3$ followed by the coordinate projection onto the first basis vector of Y_3, then $P_3''=P_2''+\tilde{P}_3''$ is a projection which has $|\!|\!|P_3''|\!|\!|=1$, $P_3''=P_3P_3''P_3$ and $P_3''(X)=Y_3''$.

Continuing in this way, we can find $K_j \in \mathcal{B}(X)$ (for $j = 1, 2, \ldots$), we can suppose that each K_j is weak*-continuous if X is a dual space, we can find $y_1, y_2, \ldots \in Y$ such that $([y_1, y_2, \ldots], |\!|\!| \cdot |\!|\!|)$ is isometric to $(Y, |\!|\!| \cdot |\!|\!|)$ (since these y_i are finitely and disjointly supported in Y and hence also have a projection P'' with $|\!|\!|P''|\!|\!| = 1$),

(a) $\|K_1\| < 2C|\lambda_1| + \delta_1$, $\|K_{j+1}\| < 2C\delta_j + \delta_{j+1}$ (for $j \geq 1$),
(b) $\sigma(T + K_1 + \cdots + K_j) = \sigma(T) \setminus \{\lambda_1, \ldots, \lambda_{n_j}\}$,
(c) $\sigma_0(T + K_1 + \cdots + K_j) = \sigma_0(T) \setminus \{\lambda_1, \ldots, \lambda_{n_j}\}$,
(d) $\operatorname{span}\{y_1, \ldots, y_j\} \subset \ker K_j \cap \ker(T + K_1 + \cdots + K_j)$.

Since $\sum_{j=1}^{\infty} \delta_j < \infty$, $\sum_{j=1}^{\infty} K_j$ is convergent. We let $K = \sum_{j=1}^{\infty} K_j$ (and it will be weak*-continuous in the dual space case). We can estimate

$$\|K\| \leq \sum_{j=1}^{\infty} \|K_j\| < 2C|\lambda_1| + \sum_{j=1}^{\infty} (2C + 1)\delta_j < 2C|\lambda_1| + \varepsilon.$$

We have $y_k \in \ker(T + K_1 + \cdots + K_j)$ for $k \leq j$ and hence $y_k \in \ker(T + K)$ for all k.

Since $\sigma_{ess}(T+K) = \sigma_{ess}(T) = \{0\}$ we have $\partial\sigma(T+K) \setminus \sigma_0(T+K) = \{0\}$ and hence $\sigma(T + K) = \{0\} \cup \sigma_0(T + K)$

To show that $\sigma_0(T + K) = \varnothing$ we use Lemma 2.6.7. If $\lambda \in \sigma_0(T + K)$, choose a simple closed curve Γ enclosing λ and no other point of $\sigma(T + K)$. In addition:

(1) If $\lambda \in \sigma_0(T)$, say $\lambda = \lambda_k$, we ensure that Γ also encloses no other points, λ_j, of $\sigma(T)$ $(j \neq k)$.
(2) If $\lambda \notin \sigma_0(T)$, choose Γ so that no $\lambda_j \in \sigma_0(T)$ is enclosed by Γ. Let $k = 0$ in this case.

This is possible since λ cannot be an accumulation point of $\sigma(T + K)$ or $\sigma(T)$. Using Lemma 2.6.7 we can find $\delta > 0$ so that $\hat{\delta}(T, S) < \delta \Rightarrow M_1(S) \eqsim M_1(T)$. There exists N so that for all $n > N$, $\|K - F_n\| < \delta$, so that

$$\hat{\delta}(T, S) < \delta \Rightarrow M_1(S) \eqsim M_1(T).$$

But there is N so that for $n > N$

$$\|K - (K_1 + K_2 + \ldots + K_n)\| < \delta$$

and so

$$\hat{\delta}(T + K, T + K_1 + K_2 + \ldots + K_n) \leq \|K - (K_1 + K_2 + \ldots + K_n)\| < \delta$$

for all $n > N$ (by Theorem 2.6.4). Let $M := \max\{N, k\}$. Then, for all $n > M$, $\dim M_1(T + F_n) = \dim M_1(T + K)$. However, since $n > k$ we have that $0 = \dim M_1(T + F_n) = \dim M_1(T + K)$, contradictiong the fact that λ is an eigenvalue. Therefore $\sigma_0(T+K) = \varnothing$ and $\sigma(T) = \{0\}$, as required. □

5.3. Applications to compact operators

In this section we show that if either X or X^* has the approximation property and contains a complemented copy of ℓ^p and $T \in \mathcal{B}(X)$ has the form $T = \lambda I + K$ where K is compact then $T \in \overline{\mathcal{HC}(X)}$ iff $|\lambda| = 1$ and K is quasinilpotent.

In addition we show (Corollary 5.3.6) that the set of operators satisfying the Herrero conditions coincide with the norm closure of the hypercyclic operators in the case where the space X is the complexification of the Argyros-Haydon space (see [**5**]).

COROLLARY 5.3.1. *Let X be a space containing a complemented subspace Y that is isomorphic to either ℓ^p $(1 \leq p < \infty)$ or c_0 $(p = \infty)$. Then there exists $C > 0$ (as in Corollary 5.2.18, depending only on a projection onto Y and the isomorphism chosen between Y and ℓ^p or c_0) such that the following holds: For every finite rank $F \in B(X)$ and every $\epsilon > 0$ there exists a finite rank operator K such that $\|K\| < 2Cr(F) + \epsilon$ and $F + K$ is a nilpotent.*

PROOF. We know that $\ker F$ is a space of finite codimension. Then $Y \cap \ker F$ is a space of finite codimension in Y. Applying Corollary 5.2.4 in an equvalent norm $|||\cdot|||$ (for which Y is isometric to ℓ^p and 1-complemented in X as in the proof of Corollary 5.2.18) we conclude that $Y \cap \ker F$ contains an isomorphic copy Y' of Y that is also complemented in X (in fact Y' is 1-complemented in $(X, |||\cdot|||)$.

So we have that $Y' \subset \ker F$ is a complemented subspace of X that is isomorphic to Y. Suppose $X = Y' \oplus L$. Then $F = \begin{pmatrix} 0 & B \\ 0 & A \end{pmatrix}$ and so by the Corollary 5.2.18 we can find a finite rank operator K so that $|||K||| < 2r(F)+\epsilon/C$ (so $\|K\| < 2Cr(F)+\epsilon$ and $F+K$ is a nilpotent. □

COROLLARY 5.3.2. *Let X be a space such that its dual, X^* contains a complemented subspace Y that is isomorphic to either ℓ^p $(1 \leq p < \infty)$ or c_0 $(p = \infty)$. Then there exists $C > 0$ (as in Corollary 5.2.18, depending only on a projection onto Y and the isomorphism chosen between Y and ℓ^p or c_0) such that the following holds: For every finite rank $F \in B(X)$ and every $\epsilon > 0$ there exists a finite rank operator K such that $\|K\| < 2Cr(F) + \epsilon$ and $F + K$ is a nilpotent.*

PROOF. We proceed as in Corollary 5.3.1 except we work on F^* and, when we apply Corollary 5.2.18 we ensure that K is weak* continuous. Then there is a finite rank operator K' on X such that $(K')^* = K$ and K' is the required finite rank operator. □

PROPOSITION 5.3.3. *Suppose either X or X^* be a space with the approximation property containing a complemented subspace Y that is isomorphic to either ℓ^p $(1 \leq p < \infty)$ or c_0 (if $p = \infty$). Let $K \in B(X)$ be a compact quasinilpotent. Then K is a norm limit of finite rank nilpotents.*

PROOF. Let $\epsilon > 0$ be given. Let C satisfy Corollary 5.3.1 or 5.3.2. Since X has the approximation property (Theorem 2.2.10), we can find a finite rank operator F with $\|K - F\| < \epsilon/3$ and $r(F) < \epsilon/(6C)$ (by the upper semi-continuity of the spectrum (Theorem 2.6.6)). By either Corollary 5.3.1 or Corollary 5.3.2 we can find a finite rank nilpotent operator F' with $\|F - F'\| < 2Cr(F) + \epsilon/3 < 2\epsilon/3$. Then $\|K - F'\| \leq \|K - F\| + \|F - F'\| < \epsilon/3 + 2\epsilon/3 = \epsilon$. □

COROLLARY 5.3.4. *Let X be as in Proposition 5.3.3. In addition assume X is separable. Then $\lambda + K \in \overline{\mathcal{M}\mathrm{ix}(X)}$ for any $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and compact quasinilpotent K.*

PROOF. By Proposition 5.3.3, K is a norm limit of nilpotent operators, N_k. By Theorem 3.5.1, $\lambda + N_k \in \overline{\mathcal{M}\mathrm{ix}(X)}$ and therefore $\lambda + K \in \overline{\mathcal{M}\mathrm{ix}(X)}$. □

COROLLARY 5.3.5. *Every Banach space X with dual isomorphic to ℓ^1 has the property that every compact quasinilpotent is a norm limit of finite rank nilpotents.*

PROOF. This follows from Proposition 5.3.3. □

COROLLARY 5.3.6. *Suppose X is the complexification of the Argyros-Haydon space. Then:*

$$\overline{\mathcal{M}\mathrm{ix}(X)} = \overline{\mathcal{W}\mathcal{M}\mathrm{ix}(X)} = \overline{\mathcal{HC}(X)} = \{\lambda I + K : |\lambda| = 1, \sigma(K) = \{0\}\}$$

Moreover, this is exactly the set of operators given by Herreros conditions on the complexification of the Argyros-Haydon space.

PROOF. Using Proposition 2.7.3 $\overline{\mathcal{HC}(X)}$ is contained in the set of operators satisfying Herrero's conditions (Theorem 1.1.1). Every operator on X is of the form $\lambda + K$ where K is compact and $\lambda \in \mathbb{C}$. Applying Herrero's conditions gives us:

(1) $\sigma_W(\lambda + K) \cup \mathbb{T}$ is connected. $\sigma_W(\lambda + K) = \{\lambda\}$ so we must have $|\lambda| = 1$.
(2) $\sigma_0(\lambda + K) = \varnothing$ so $\sigma_0(K) = \varnothing$ and $\sigma(K) = \{0\}$.
(3) $\rho_{s-F}(\lambda + K) = \mathbb{C} \setminus \{\lambda\}$ and $\mathrm{ind}(\lambda + K - \mu) = 0$ for any $\mu \neq \lambda$.

Therefore $\overline{\mathcal{HC}(X)} \subset \{\lambda I + K : |\lambda| = 1, \sigma(K) = \{0\}$. We also have in general that $\mathcal{M}\mathrm{ix}(X) \subset \mathcal{W}\mathcal{M}\mathrm{ix}(X) \subset \mathcal{HC}(X)$.

We know that the dual space of the Argyros-Haydon space is isomorphic to ℓ^1. Using Corollary 5.3.5 together with Theorem 3.5.1 we get that $\{\lambda I + K : |\lambda| = 1, \sigma(K) = \{0\}\} \subset \overline{\mathcal{M}\mathrm{ix}(X)}$. Therefore

$$\overline{\mathcal{M}\mathrm{ix}(X)} = \overline{\mathcal{W}\mathcal{M}\mathrm{ix}(X)} = \overline{\mathcal{HC}(X)} = \{\lambda I + K : |\lambda| = 1, \sigma(K) = \{0\}\}$$

□

DEFINITION 5.3.7. A *perfect set* is a set that does not have any isolated points.

PROPOSITION 5.3.8 ([**6**], Proposition 6.37). *Let X be a Banach space, and let $T \in \mathcal{B}(X)$ be chaotic or frequently hypercyclic. Then $\sigma(T)$ is a perfect set.*

Let $\mathcal{CHC}(X)$ denote the set of chaotic operators on X and $\mathcal{FHC}(X)$ the frequently hypercyclic operators.

In [**26**, Theorem 11]. Müller proves that given a separable (infinite dimensional) Hilbert space H: $\overline{\mathcal{HC}(H)} = \overline{\mathcal{WM}\text{ix}(H)} = \overline{\mathcal{M}\text{ix}(H)} = \overline{\mathcal{CHC}(H)} = \overline{\mathcal{FHC}(H)} = \overline{\mathcal{M}\text{ix}(H) \cap \mathcal{CHC}(H) \cap \mathcal{FHC}(H)}$. The following is the analogous result for the complexification of the Argyros-Haydon space.

PROPOSITION 5.3.9. *Suppose X is the complexification of the Argyros-Haydon space. Then:*

(1) $\overline{\mathcal{M}\text{ix}(X)} = \overline{\mathcal{WM}\text{ix}(X)} = \overline{\mathcal{HC}(X)}$

(2) $\overline{\mathcal{CHC}(X)} = \overline{\mathcal{FHC}(X)} = \overline{\mathcal{M}\text{ix}(X) \cap \mathcal{CHC}(X) \cap \mathcal{FHC}(X)} = \varnothing$

PROOF. The first statement is the result of Corollary 5.3.6. The second statements follows from Proposition 5.3.8 and the fact that every compact operator has at least one isolated point in its spectrum. Since each $\mathcal{B}(X) = \{\lambda + K : K \in \mathcal{K}(X)\}$, for each $T \in \mathcal{B}(X)$ there is an isolated point in $\sigma(T)$. □

5.4. Approximation of quasinilpotents by generalised nilpotents

If $T \in \mathcal{B}(X)$ has $\sigma_{ess}(T) = \{\lambda\}$ then T satisfies Herrero conditions iff $|\lambda| = 1$ and $\sigma(T) = \{\lambda\}$ (which is equivalent to $T - \lambda$ being quasinilpotent). To show that T can be approximated by hypercyclic operators, because of Theorem 3.5.1, we are led to the question of approximating quasinilpotents by nilpotents (or quasinilpotent generalised nilpotents).

In the Hilbert space case, it is known (see [**3**]) that quasinilpotents are all norm limits of nilpotents. But in that case Herrero's theorem characterises $\overline{\mathcal{HC}(X)}$. In the case of more general X, there seems to be no known examples of quasinilpotents that fail to be norm limits of nilpotent operators, but a general positive result seems out of reach. We need extra conditions on X or T to prove positive results. If X is separable and there is a complemented copy of ℓ^p in $\ker T$, then we show (Lemma 5.4.8) that T is a limit of triangular quasinilpotents (which are the same as quasinilpotent generalised nilpotents).

We also show (Lemma 5.4.5) that if X or X* belongs to the special class of FDpBD spaces, we can perturb T via a small compact perturbation to ensure it has a complemented copy of ℓ^p (or c_0 if $p = \infty$) in its kernel and deduce that every quasinilpotent $T \in \mathcal{B}(X)$ is a norm limit of triangular quasinilpotents (Theorem 5.4.10).

LEMMA 5.4.1. *Given $T \in B(X)$ a quasinilpotent operator, $\epsilon > 0$, $x \in X$ with $\|x\| = 1$ and $E \subset X$ a closed subspace such that $T^j x \notin E$ for any j then we can find k such that*

$$\frac{\operatorname{dist}(T^{k+1}x, \operatorname{span}_{0\leq i\leq k}\{T^i x\} + E)}{\operatorname{dist}(T^k x, \operatorname{span}_{0\leq i\leq k-1}\{T^i x\} + E)} < \epsilon.$$

PROOF. Given any $\epsilon > 0$ we have that $\operatorname{dist}(T^{n+1}x, \operatorname{span}_{0\leq i\leq n}\{T^i x\} + E) \leq \|T^{n+1}x\| < \epsilon^n \|x\| = \epsilon^n$ for all n large enough. Suppose $\delta < \min\{\epsilon, \operatorname{dist}(x, E)\}$. Then

$$\operatorname{dist}\{x, E\}\frac{\operatorname{dist}\{Tx, \mathbb{K}x + E\}}{\operatorname{dist}\{x, E\}} \cdots \frac{\operatorname{dist}(T^{n+1}x, \operatorname{span}_{0\leq i\leq n}\{T^i x\} + E)}{\operatorname{dist}(T^k x, \operatorname{span}_{0\leq i\leq k-1}\{T^i x\} + E)} < \delta^n$$

So we must have that there is some $k \leq n$ such that

$$\frac{\operatorname{dist}(T^{k+1}x, \operatorname{span}_{0\leq i\leq k}\{T^i x\} + E)}{\operatorname{dist}(T^k x, \operatorname{span}_{0\leq i\leq k-1}\{T^i x\} + E)} < \delta < \epsilon.$$

□

LEMMA 5.4.2. [**9**, Lemma 4.4.5] *Suppose $T \in \mathcal{B}(X)$ and suppose T does not have closed range. Then for each $\epsilon > 0$ there exists an infinite dimensional closed subspace Z of X such that $\|T|_Z\| < \epsilon$.*

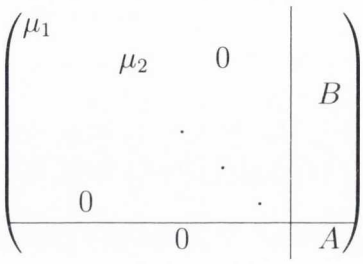
LEMMA 5.4.3. *Suppose $T \in \mathcal{B}(X)$ is a Riesz operator. Then for every $\epsilon > 0$ there exists an infinite dimensional closed subspace M_ϵ such that $\|T|_{M_\epsilon}\| < \epsilon$.*

PROOF. The proof of Lemma 5.4.2 actually shows that if an operator is not bounded below on any subspace of X with finite codimension then we get such a space M_ϵ for each ϵ. We will show that Riesz operators are not bounded below on any subspace of X with finite codimension.

$0 \in \sigma_{s-F}(T)$ since T is a Riesz operator. Therefore we either have that the range of T is not closed (and $\gamma(T) = 0$ by Lemma 2.6.1 hence T not bounded below) and we can apply the previous lemma (Lemma 5.4.2) or $\dim \ker T = \operatorname{codim} TX = \infty$. But then, for any subspace Y of X with finite codimension, we have that $Y \cap \ker T \neq \{0\}$ (in fact $\dim Y \cap \ker T = \infty$) and T is not bounded below. So T is not bounded below on any space with finite codimension and we can follow the proof of Lemma 5.4.2 ([**9**, Lemma 4.4.5]) to get the required result. □

LEMMA 5.4.4. [**11**, Lemma 2.1] *Let $X = \oplus_p X_n$ be a FDpBD space. Suppose T belongs to $\mathcal{B}(X)$ and $\mu_1, \mu_2, \ldots$ is a sequence of (not necessarily distinct points of $\sigma_{\ell e}(T)$, and let $\epsilon > 0$ be given. There is a compact operator K with $\|K\| < \epsilon$ and a complemented subspace Y isometrically isomorphic to ℓ^p $(1 \leq p < \infty)$ or to c_0 $(p = \infty)$, and*

projection constant 1 so that $T - K$ has the form



with respect to $X = Y \oplus Z$, $Z \neq X$ being a complementary subspace for Y.

LEMMA 5.4.5. *If X is isomorphic to an FDpBD, $T \in B(X)$ is a quasinilpotent operator and $\epsilon > 0$ then there is a compact operator K with $\|K\| < \epsilon$, a subspace Y of X which is complemented and isomorphic to ℓ^p (or c_0 if $p = \infty$) such that $Y \subset \ker(T + K)$ and $T + K$ is quasinilpotent.*

If X is a dual space then we can also ensure that K is weak continuous.*

PROOF. Fix an isomorphism $\tau\colon X \to X'$ where X' is an FDpBD, let $C = \|\tau\|\|\tau^{-1}\|$ and introduce the equivalent norm $|||\cdot|||$ on X given by $|||x||| = \|\tau x\|$. Then $(X, |||\cdot|||)$ is an FDpBD space and we have $\|x\| \leq C|||x|||$, $|||x||| \leq C\|x\|$ for $x \in X$. In addition, the dual norm of $(X, |||\cdot|||)$, which we also denote $|||\cdot|||$, satisfies $\|x^*\| \leq C|||x^*|||$, $|||x^*||| \leq C\|x^*\|$ for $x^* \in X^*$. We can also suppose C satisfies $|||S||| \leq C\|S\|$ and $\|S\| \leq C|||S|||$ holds for $S \in \mathcal{B}(X)$.

If X is not a dual space, we can work throughout with $|||\cdot|||$ and $\epsilon' > 0$ chosen so that $\epsilon' < \epsilon/C$, and finally switch back to the original norm $\|\cdot\|$ at the end. But for the dual space case we will need to switch to the original norm $\|\cdot\|$ at each step.

If we are not in the dual space case, we can apply [**11**, Lemma 2.1] (Lemma 5.4.4 above) to get a compact operator K_1 with $|||K_1||| < \epsilon'/3$, a projection $P \in \mathcal{B}(X)$ with $|||P||| = 1$ and range $Y = PX$ isometric in $|||\cdot|||$ to ℓ^p (or to c_0 if $p = \infty$) such that $(T - K_1)P = 0$. Indeed, since $\sigma_{ess}(T) = \{0\}$, we can take $0 = \mu_1 = \mu_2 = \ldots$ in Lemma 5.4.4 to establish the existence of K_1 and P. We can then apply Theorem 5.2.19 to find a compact K_2 with $|||K_2||| < 2\epsilon'/3$ and $T - K_1 + K_2$ has no normal eigenvalues (and is then still quasinilpotent), but still contains a 1-complemented copy of Y in its kernel.

For the case when X is a dual space, we need to adapt the argument in [**11**, Lemma 2.1] to get a weak*-continuous K_1. Here are the details.

Let R_n denote the projection onto the first n blocks of the FDpBD space $(X, |||\cdot|||)$. We will find a sequence x_k of unit vectors supported on disjoint blocks, that is such that there exists an increasing sequence

$0 = n_0 < n_1 < n_2 < \cdots$ with

$$(R_{n_k} - R_{n_{k-1}})x_k = x_k$$

(where $R_0 = 0$), where $|||x_k||| = 1$ but $|||Tx_k|||$ is small. We will then construct K_1 compact so that $x_k \in \ker(T - K_1)$ for all k. Being disjointly supported, in $|||\cdot|||$, the vectors x_k arre equivalent to the standard basis for ℓ^p (or c_0 if $p = \infty$) and their closed span is 1-complmented in $(X, |||\cdot|||)$.

First, by upper semi-continuity of the spectrum we can find $\delta > 0$ such that

$$S \in \mathcal{B}(X), |||S||| < \delta \Rightarrow \sigma(T + S) \subset (\sigma(T))_{\epsilon/(4C)}$$

(and hence implies $r(T + S) < \epsilon/(4C)$). We choose $\epsilon' > 0$ in a way to be exactly specified later but so that $\epsilon' < \min(\delta, \epsilon, 1/2)/C$.

To begin, we find $x \in X$ with $|||x||| = 1$ and $|||Tx||| < \epsilon'/32$. Then there exists $n = n_1$ large enough so that

$$|||(I - R_n)x||| < \frac{\epsilon'}{32(|||T||| + 1)}.$$

We then have

$$|||TR_n x||| \leq |||Tx||| + |||T(I - R_n)x||| < \frac{\epsilon'}{32} + \frac{\epsilon'}{32} = \frac{\epsilon'}{16}.$$

Let $x_1 = R_{n_1}x/|||R_{n_1}x|||$ so that $|||x_1||| = 1$ and

$$|||Tx_1||| = \frac{|||TR_{n_1}x|||}{|||R_{n_1}x|||} < \frac{\epsilon'}{8}$$

(since $|||R_{n_1}x||| > 1/2$).

Let $x_1^* \in X^*$ have $1 = |||x_1^*||| = x_1^*(x_1)$. Then $F_1 = x_1^* \otimes Tx_1$ is a rank one operator with $|||F_1||| = |||Tx_1|||$ small and $x_1 \in \ker(T - F_1)$. When X is a dual space, we need to replace F_1 by a weak*-continuous operator. We have $\|F_1\| \leq C|||F_1||| < C\epsilon'/8$ and we apply the principle of local reflexivity in the form Theorem 2.2.11 to find a weak*-continuous $G_1 \in \mathcal{B}(X)$ such that G_1 agrees with F_1 on $\mathbb{C}x_1$, has the same range $\mathbb{C}Tx_1$ as F_1 and has $\|G_1\| < C\epsilon'/8$. This operator G_1 will be of the form

$$G_1 = \phi_1 \otimes Tx_1$$

for a weak*-continuous $\phi_1 \in X^*$.

Suppose we have found $x_1, \ldots x_{k-1}$ and $n_1, \ldots n_{k-1}$ and $F_1, \ldots F_{k-1}$ (and $G_1, \ldots, G_{k-1}$ in the dual space case).

Let $T_{k-1} = T - \sum_{i=1}^{k-1} F_i$ (or $T_{k-1} = T - \sum_{i=1}^{k-1} G_i$ in the dual space case) and let $T_0 = T$. T_{k-1} is a Riesz operator since the essential spectrum is unaffected by finite rank perturbation (see §2.4). Therefore by Lemma 5.4.3 there exists an infinite dimensional subspace V_k such that $|||T_{k-1}|_{V_k}||| < \epsilon'/(2^{k+4})$. Since V_k is infinite dimensional and $(I - R_{n_{k-1}})X$ has finite codimension, there exists $x \in V_k \cap (I - R_{n_{k-1}})X$

such that $|||x||| = 1$ and $|||T_{k-1}x||| < \epsilon'/(2^{k+4})$. Then there exists n_k such that $|||(I - R_{n_k})x||| < \frac{\epsilon'}{(2^{k+4}(|||T_{k-1}|||+1))}$, so that

$$\begin{aligned} |||T_{k-1}(R_{n_k} - R_{n_{k-1}})x||| &= |||T_{k-1}(R_{n_k}x)||| \\ &\leq |||T_{k-1}x||| + |||T_{k-1}(I - R_{n_k})x||| \\ &< \epsilon'/(2^{k+4}) + \epsilon'/(2^{k+4}) \\ &< \epsilon'/(2^{k+3}). \end{aligned}$$

Let $x_k = \frac{R_{n_k}x}{\|R_{n_k}x\|}$. Note that $(R_{n_k} - R_{n_{k-1}})x_k = x_k$. We then have that

$$|||R_{n_k}x||| = |||x - (I - R_{n_k})x||| \geq ||||x||| - |||(I - R_{n_k})x|||| > 1/2$$

since $|||(I - R_{n_k})x||| < 1/2$ and hence

$$|||T_{k-1}x_k||| < \frac{\epsilon'/(2^{k+4})}{1/2} = \epsilon'/(2^{k+2}).$$

Now $x_k \in (R_{n_k} - R_{n_{k-1}})X$ and $x_i \in R_{n_{k-1}}X$ so we can find linear functionals $x_k^* \in X^*$ such that $|||x_k^*||| = 1$ and $x_k^*(x_j) = \delta_{i,k}$ for all $1 \leq i \leq k$. Then $F_k = x_k^* \otimes T_{k-1}x_k$ is a rank 1 operator with $|||F_k||| = |||T_{k-1}x_k|||$ and $\operatorname{span}\{x_1, x_2, \ldots, x_k\} \subset \ker(T_{k-1} - F_k)$.

However in the dual space case we need to replace F_k by a weak*-continuous operator. We have $\|F_k\| \leq C|||F_k||| < C\epsilon'/2^{k+2}$ and we apply the principle of local reflexivity in the form Theorem 2.2.11 to find a weak*-continuous $G_k \in \mathcal{B}(X)$ such that G_k agrees with F_k on $\operatorname{span}\{x_1, x_2, \ldots, x_k\}$, has the same range $\mathbb{C}Tx_k$ as F_k and has $\|G_k\| < C\epsilon'/2^{k+2}$. This operator G_k will be of the form

$$G_k = \phi_k \otimes Tx_k$$

for a weak*-continuous $\phi_k \in X^*$.

In the non dual space case $K_1 = \sum_{i=1}^{\infty} F_i$ and

$$\|K_1\| \leq \sum_{i=1}^{\infty} \|x_i^* \otimes Tx_i\| = \sum_{i=1}^{\infty} \|Tx_i\| < \sum_{i=1}^{\infty} \frac{C\epsilon'}{2^{i+2}} < \frac{C\epsilon'}{4} < \frac{\epsilon}{4}$$

and in the dual space case we let $K_1 = \sum_{i=1}^{\infty} G_i$ and

$$\|K_1\| \leq \sum_{i=1}^{\infty} \|G_i\| < \sum_{i=1}^{\infty} \frac{C\epsilon'}{2^{i+2}} < \frac{C\epsilon'}{4} < \frac{\epsilon}{4}.$$

So in either case we have that $\|K_1\| < \epsilon/4$. In the dual space case we also have that K_1 is a limit of weak* continuous operators and therefore is itself a weak* continuous operator.

Also, $(T - K_1)x_i = 0$ so $Y_1 \subset \ker(T - K_1)$ where $Y_1 = [x_n]_{n\in\mathbb{N}}$. Since these x_n are finitely and disjointly supported unit vectors, Y_1 is isometric to ℓ^p (or to c_0 if $p = \infty$) in the $|||\cdot|||$ norm (see Remark 5.2.3).

Since T is quasinilpotent we must have that $T - K_1$ is a Riesz operator (as it is a compact perturbation of a Riesz operator). It also has spectral radius less than $\epsilon/4C$ since $\|K_1\| < C\epsilon'/4 < \delta/4 < \delta$. By

Theorem 5.2.19 we have that there exists a compact operator K_2 with norm $\|K_2\| < 2Cr(T - K_1) + \epsilon/4 < 3\epsilon/4$ such that $T - K_1 + K_2$ is quasinilpotent, K_2 is weak* continuous if X is a dual space and a space Y isometric to Y_1 in $|||\cdot|||$ which is 1-complemented in $(X, |||\cdot|||)$. Then $\|K_2 - K_1\| < 3\epsilon/4 + \epsilon/4 < \epsilon$. □

REMARK 5.4.6. Davidson and Herrero showed that the West Decomposition can be extended to spaces with FDpDB ([**11**, Corollary 3.3]). In the following we prove that it can also be extended to preduals of spaces with FDpBD and put bounds on the norm of the compact operator K.

LEMMA 5.4.7. *If X or X^* is isomorphic to an FDpBD then there exists C such that the following holds: If $T \in B(X)$ is a Riesz operator with spectral radius $r(T) = R$ and $\epsilon > 0$, then there is a compact operator K with $\|K\| < 2CR + \epsilon$ and a quasinilpotent operator $T' \in \mathcal{B}(X)$ such that $T = T' + K$.*

PROOF. The proof is almost the same as the proof of Lemma 5.4.5 except $\|K_2\| < 2R + \epsilon/2$. □

LEMMA 5.4.8. *Suppose $1 \leq p \leq \infty$ and X is a separable Banach space such that $X \approx L \oplus X'$ where L is isometrically isomorphic to ℓ^p (or c_0 if $p = \infty$), $T \in B(X)$ is a quasinilpotent operator such that $L \subset \ker T$ and $\epsilon > 0$. Then there exists a compact operator, K, with $\|K\| < \epsilon$ so that $T + K$ is triangular and quasinilpotent.*

If X is a dual space then we can ensure that K is weak continuous.*

PROOF. We take an equivalent norm $|||\cdot|||$ and C as in Lemma 5.2.17, and work with $(X, |||\cdot|||)$. We then have $|||P||| = 1$ where P is the projection onto L with $\ker P = X'$.

Let $\{x_n : n \in \mathbb{N}\}$ be an M-basis for X such that $|||x_n||| = 1$. We are going to construct a sequence of quasinilpotent operators, T_q, and a sequence of finite dimensional T_q-invariant subspaces $U_q \subset N^\infty(T_q)$ such that $x_1, \ldots x_q \in U_q$. The first step in the construction of each T_q is to ensure that either $x_q \in N^\infty(T_{q-1})$ (in which case we let $T_q := T_{q-1}$) or we find an operator M_q with small norm such that x_q is in a $(T_{q-1} - M_q)$-invariant finite dimensional subspace, (and x_q is a linear combination of generalised eigenvectors for $T_{q-1} - M_q$). Then we remove all eigenvalues of $(T_{q-1} - M_q)$ using Theorem 5.2.19, leaving us with a quasinilpotent operator T_q such that $x_q \in N^\infty(T_q)$ and allowing us to construct U_q with the properties mentioned above. We also ensure that there is some L_q isometric to L (in $(X, |||\cdot|||)$, $L = L_q \oplus Y_q$ with a projection P_q onto L_q with $\ker P_q = Y_q \oplus X'$, $|||P_q||| = 1$ and $L_q \subset \ker T_q$.

Suppose we have already done this for $0 \leq i \leq q-1$. By the upper semi-continuity of the spectrum we get that there exists $\delta_q > 0$ such that $|||S||| < \delta_q \Rightarrow \sigma(T_{q-1} + S) \subset (\sigma(T))_{\epsilon/(C2^{q+2})}$ (where $T_0 := T$). Let

$\epsilon'_q = \min(\delta_q/C^2, \epsilon/(C2^{q+2}))$. (Note that we will then have $\sum_{q=1}^{\infty} \epsilon'_q < \epsilon/(4C)$.)

If $x_q \in N^\infty(T_{q-1})$ we let $T_q := T_{q-1}$ and $L_q := L_{q-1}$. Otherwise, since $|||x_q||| = 1$, (by Lemma 5.4.1) we can find k_q such that

$$\frac{\operatorname{dist}(T_{q-1}^{k_q+1}x_q, \operatorname{span}_{0\le i\le k_q}\{T_{q-1}^i x_q\} + S_{q-1})}{\operatorname{dist}(T_{q-1}^{k_q}x_q, \operatorname{span}_{0\le i\le k_q-1}\{T_{q-1}^i x\} + U_{q-1})} < \epsilon'_q/2.$$

So there exists $f_q \in X^*$ such that $f_q(U_{q-1}) = 0$, $f_q(T_{q-1}^i x_q) = \delta_{i,k_q}$ for $0 \le i \le k_q$ and $u_q \in \operatorname{span}_{0\le i\le k_q}\{T_{q-1}^i x_q\} + S_{q-1}$ such that $|||T_{q-1}^{k_q+1}x_q - u_q||| = \operatorname{dist}(T_{q-1}^{k_q+1}x_q, \operatorname{span}_{0\le i\le k_q}\{T_{q-1}^i x_q\} + U_{q-1})$. Note that by Lemma 2.2.12 we can ensure that

$$|||f_q||| = \frac{1}{\operatorname{dist}(T_{q-1}^{k_q}x_q, \operatorname{span}_{0\le i\le k_q-1}\{T_{q-1}^i x_q\} + U_{q-1})}.$$

Then, letting $M_q = f_q \otimes (T_{q-1}^{k_q+1}x_q - u_q)$,

$$\begin{aligned} |||T_{q-1} - (T_{q-1} - M_q))||| &= |||f_q \otimes (T_{q-1}^{k_q+1}x_q - u_q))||| \\ &= \frac{\operatorname{dist}(T_{q-1}^{k_q+1}x_q, \operatorname{span}_{0\le i\le k_q}\{T_{p-1}^i x_q\} + U_{q-1})}{\operatorname{dist}(T_{q-1}^{k_q}x_q, \operatorname{span}_{0\le i\le k_q-1}\{T_{q-1}^i x_q\} + U_{q-1})} \\ &< \epsilon'_q/2. \end{aligned}$$

Then $\|M_q\| \le C|||M_q||| < C\epsilon'_q$. In the dual space case we apply Theorem 2.2.11 to M_q with $U = S_q := U_{q-1} + \operatorname{span}\{T_{q-1}^i x_q : 0 \le i \le k_q\}$. We will also call this new operator M_q and it has norm $\|M_q\| < C\epsilon'_q$ and so $|||M_q||| < C^2\epsilon'_q \le \delta_q$. Thus $r(T - M_q) < \epsilon/(C2^{q+2})$.

Now S_q is $T_{q-1}-M_q$-invariant. We know $\sigma_{ess}(T_{q-1}-M_q) = \sigma_{ess}(T_{q-1}) = \{0\}$ and hence $\sigma(T_{q-1} - M_q) = \{0\} \cup \sigma_0(T_{q-1} - M_q)$. We claim that in fact $\sigma_0(T_q)$ is finite.

Let W_q be a complement for S_q in X with $W_q \subset \ker M_q$. Let the block form for $T_{q-1} - M_q$ with respect to $X = S_q \oplus W_q$ be

$$T_{q-1} - M_q = \begin{pmatrix} A_q & B_q \\ 0 & C_q \end{pmatrix}.$$

From Corollary 5.2.8, $\sigma(T_q) = \sigma(A_q) \cup \sigma(C_q)$ and $\sigma_0(T_q) = (\sigma_0(A_q) \setminus \sigma(C_q)) \cup \sigma_0(C_q)$. But if $\lambda \in \sigma_0(C_q)$, there is an eigenvector $w_q \in W_q$ for C_q and the eigenvalue $\lambda \ne 0$. That is $C_q w_q = \lambda w_q \ne 0$. But then $T_{q-1}w_q = \lambda w_q$ and $\lambda \in \sigma(T_{q-1}) = \{0\}$, contradicting $\lambda \ne 0$. Therefore the normal eigenvalues of $T_{q-1} - M_q$ are the (at most finitely many) normal eigenvalues of the finite dimensional $T_{q-1} - M_q$-invariant space, S_q.

If there are no normal eigenvalues for $T_{q-1} - M_q$ we let $T_q := T_{q-1} - M_q$, $U_q := S_q$ and $L_q := L_{q-1}$ except when X is a dual space, where L_q is gotten using Lemma 5.2.2 and is isometric to L and is 1-complemented in $(X, |||\cdot|||)$. Note that we then have $x_i \in N^\infty(T_q)$ for $1 \le i \le q$ (since

x_i is a generalised eigenvector and the only eigenvalue of $T_{q-1} - M_q$ is 0).

If $T_{q-1} - M_q$ has non-zero eigenvalues, λ_i we use Corollary 5.2.18 with $S_0 = U_{q-1}$ to get a compact operator K_q with $\|K_q\| < 3C\epsilon'_q$ that will remove all of the normal eigenvalues such that $T_q = T_{q-1} - M_q + K_q$ is quasinilpotent and will have a finite-dimensional T_q-invariant subspace $U_q \supset S_q = U_{q-1} + \operatorname{span}\{T_{q-1}^i x_q : 0 \leq i < k_q\}$ (since $\operatorname{span}\{T_{q-1}^i x_q : 0 \leq i \leq k_q\}$ is in the span of the generalised eigenvectors for $T_{q-1} - M_q$) and we let L_q be gotten using Lemma 5.2.2 (since K_q is finite rank). In the dual space case K_q will be weak* continuous.

This sequence T_q converges to some operator $T' \in B(X)$. Since $x_j \in U_j \subset N^\infty(T_q)$ for all $q \geq j$ and $U_j \subset \ker K_q \cap \ker M_q$ for all $q > j$ we must have $x_j \in N^\infty(T')$. But $\operatorname{span}\{x_i\}$ is dense in X so we have that $N^\infty(T')$ is dense in X. Therefore $\sigma(T')$ is connected (by Corollary 3.1.4) and contains 0. But $T' = T + K$ is a Riesz operator (since T is a Riesz operator) so $\sigma(T') = \{0\} \cup \sigma_0(T')$. Since $\sigma(T')$ is connected we must have $\sigma_0(T') = \varnothing$, in other words T' is a quasinilpotent operator with dense generalised kernel.

Also, letting $T_0 := T$,

$$\|T - T'\| \leq \sum_{i=1}^{\infty} \|T_{i-1} - T_i\| < \sum_{i=1}^{\infty} 4C\epsilon'_i \leq \epsilon$$

and $T - T' = \sum_{i=1}^{\infty}(M_i - K_i)$ is compact as required. In the dual space case $K = T' - T$ is weak* continuous since M_i and K_i are weak* continuous.

□

COROLLARY 5.4.9. *Suppose X is a separable Banach space such that $X \approx L \oplus X'$ where L is isomorphic to ℓ^p (or c_0 if $p = \infty$), $T \in B(X)$ is a quasinilpotent operator of the form $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ with respect to $L \oplus X'$ and $\epsilon > 0$. Then there exists a compact operator, K, with $\|K\| < \epsilon$ so that $T + K$ is triangular and quasinilpotent.*

If X is a dual space we can ensure that K is also weak continuous.*

PROOF. Let $|||\cdot|||$ and C be as in Theorem 5.2.19. Suppose P is the projection onto L with $\ker P = X'$. First apply Lemma 5.4.5 to T to produce a compact operator $K_1 \in \mathcal{B}(X')$, such that $|||K_1||| < \epsilon/2C$ and such that $T_1 + K_1$ is quasinilpotent with a 1-complemented (in $(X, |||\cdot|||)$) copy of ℓ^p (or c_0) in its kernel. Then $T' = \begin{pmatrix} T_1 + K_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ is quasinilpotent since $\sigma(T') = \sigma(T_1 + K_1) \cup \sigma(T_3) = \sigma(T_1) \cup \sigma(T_3) = \{0\}$ (by Corollary 5.2.8).

Then apply Lemma 5.4.8 to T' to produce a compact operator $K_2 \in \mathcal{B}(X)$ with $|||K_2||| < \epsilon/2C$ and such that $T'' := T' + K_2$ is a triangular quasinilpotent. Then $T'' = T + K_1P + K_2$ and $K := K_1P + K_2$ is

compact with:

$$\|K\| \leq C|||T + K_1P + K_2 - T||| \leq C(|||K_1|||||P||| + |||K_2|||) < \epsilon.$$

□

THEOREM 5.4.10. *Suppose X is isomorphic to an FDpBD space or X^* is isomorphic to an FDpBD and $T \in B(X)$ is a quasinilpotent operator. Then T is a norm limit of triangular quasinilpotents, i.e. quasinilpotents with dense generalised kernel.*

PROOF. Choose $\epsilon > 0$ First apply Lemma 5.4.5 to produce a compact operator $K_1 \in \mathcal{B}(X)$ such that $\|K\| < \epsilon/2$ and such that $T_1 = T + K_1$ has a complemented copy of ℓ^p in its kernel. Then apply Lemma 5.4.8 to T_1 to get another compact operator $K_2 \in \mathcal{B}(X)$ such that $\|K_2\| < \epsilon/2$ and $T_2 = T_1 + K_2$ is a triangular quasinilpotent. Then $\|T_2 - T\| = \|K_1 + K_2\| \leq \|K_1\| + \|K_2\| < \epsilon$. □

REMARK 5.4.11. So we have that, on ℓ^p, c_0, or any of their preduals the following is true:

Quasinilpotents are norm limits of nilpotents iff triangular quasinilpotents are norm limits of nilpotents.

COROLLARY 5.4.12. *If either X or its dual is isomorphic to an FDpBD space, $T \in B(X)$ is a quasinilpotent and $|\lambda| = 1$ then $\lambda + T \in \overline{\mathcal{M}\text{ix}(X)}$.*

PROOF. Apply Theorems 3.5.1 and 5.4.10 □

COROLLARY 5.4.13. *Suppose X is isomorphic to an FDpBD space or X^* is isomorphic to an FDpBD and $T \in \mathcal{B}(X)$ is such that $\sigma_{ess}(T) = \{\mu\}$. Then $T \in \overline{\mathcal{HC}(X)}$ iff $T \in \overline{\mathcal{M}\text{ix}(X)}$ iff $\sigma(T) = \{\mu\}$ and $|\mu| = 1$ iff*

(1) $\sigma_W(T) \cup \mathbb{T}$ is connected,
(2) $\sigma_0(T) = \varnothing$
(3) $ind(\lambda - T) \geq 0$ for all $\lambda \in \rho_{s-F}(T)$

Hence in the case where $\sigma_{ess}(T)$ is a singleton Herrero's conditions fully characterise the closure of the hypercyclic operators.

PROOF. By Equation 2.4.1

$$\partial\sigma(T) \setminus \sigma_0(T) \subset \sigma_{ess}(T) = \{\mu\}$$

and so $\sigma(T) \setminus \sigma)_0(T) = \{\mu\}$. Also

$$\partial\sigma(T) \setminus \sigma_0(T) = \sigma_W(T) = \sigma_{s-F}(T) = \sigma(T) \setminus \sigma_0(T),$$
$$\Rightarrow \sigma(T) = \sigma_0(T) \cup \sigma_W(T) = \sigma_0(T) \cup \sigma_{s-F}(T).$$

Since $\sigma(T) = \sigma_0(T) \cup \sigma_{s-F}(T)$, we have that $\text{ind}(\lambda - T) = 0$ for all $\lambda \in \rho_{s-F}(T)$. So Herrero's third condition is redundant when $\sigma_{ess}(T)$ is a singleton.

If $\sigma_0(T) = \varnothing$ then $\sigma(T) = \sigma_W(T)$. Therefore, $\sigma_W(T) \cup \mathbb{T}$ is connected iff $\sigma(T) = \sigma_W(T) \subset \mathbb{T}$.

Putting this all together we get:

$$\left\{ T \in \mathcal{B}(X) : \begin{array}{l} \sigma_{ess}(T) = \{\mu\}, \\ \sigma_W(T) \cup \partial\mathbb{D} \text{ is connected}, \\ \sigma_0(T) = \varnothing, \\ \operatorname{ind}(\lambda - T) \geq 0 \text{ for all } \lambda \in \rho_{s-F}(T) \end{array} \right\}$$

$$=$$

$$\left\{ T \in \mathcal{B}(X) : \begin{array}{l} \sigma(T) = \{\mu\}, \\ |\mu| = 1 \end{array} \right\}$$

Applying Corollary 5.4.12 we get that T can be approximated by mixing operators and hence:

$$\left\{ T \in \mathcal{B}(X) : \begin{array}{l} \sigma(T) = \{\mu\}, \\ |\mu| = 1 \end{array} \right\}$$

$$\subset$$

$$\overline{\mathcal{M}\text{ix}(X)} \cap \{T \in \mathcal{B}(X) : \sigma(T) \text{ is a singleton}\}$$

$$\subset$$

$$\overline{\mathcal{HC}(X)} \cap \{T \in \mathcal{B}(X) : \sigma(T) \text{ is a singleton}\}$$

By Proposition 2.7.3 we get:

$$\overline{\mathcal{HC}(X)} \cap \{T \in \mathcal{B}(X) : \sigma(T) \text{ is a singleton}\}$$

$$\subset$$

$$\left\{ T \in \mathcal{B}(X) : \begin{array}{l} \sigma_{ess}(T) \text{ is finite}, \\ \sigma_W(T) \cup \partial\mathbb{D} \text{ is connected}, \\ \sigma_0(T) = \varnothing, \\ \operatorname{ind}(\lambda - T) \geq 0 \text{ for all } \lambda \in \rho_{s-F}(T) \end{array} \right\}$$

So

$$\overline{HC(X)} \cup \{T \in \mathcal{B}(X) : \sigma(T) \text{ is a singleton}\}$$

$$=$$

$$\left\{ T \in \mathcal{B}(X) : \begin{array}{l} \sigma_{ess}(T) \text{ is a singleton}, \\ \sigma_W(T) \cup \partial\mathbb{D} \text{ is connected}, \\ \sigma_0(T) = \varnothing, \\ \operatorname{ind}(\lambda - T) \geq 0 \text{ for all } \lambda \in \rho_{s-F}(T) \end{array} \right\}$$

□

QUESTION 1. We are not certain whether Corollary 5.4.13 can be extended to the case where $\sigma_{ess}(T)$ is a finite set. If $\sigma_{ess}(T)$ is finite we can still use Equation 2.4.1 to show

$$\sigma(T) \setminus \sigma_0(T) = \sigma_W(T) - \sigma_{s-F}(T) = \sigma_{ess}(T).$$

COROLLARY 5.4.14. *Suppose X is a hereditarily indecomposable isomorphic predual of ℓ^1. Then:*

$$\overline{\mathcal{M}\mathrm{ix}(X)} = \overline{\mathcal{WM}\mathrm{ix}(X)} = \overline{\mathcal{HC}(X)} = \{\lambda I + S : |\lambda| = 1, \sigma(S) = \{0\}\}$$

Moreover this is exactly the set given by Herrero's conditions on such spaces.

We also have that

$$\overline{\mathcal{CHC}(X)} = \overline{\mathcal{FHC}(X)} = \overline{\mathcal{M}\mathrm{ix}(X) \cap \mathcal{CHC}(X) \cap \mathcal{FHC}(X)} = \varnothing$$

PROOF. By [**13**, Theorem 18], if X is hereditarily indecomposable then $\mathcal{B}(X) = \{\lambda + S : \lambda \in \mathbb{K}, \sigma_{ess}(S) = \{0\}\}$. Therefore $\sigma_{ess}(T)$ is finite for every $T \in \mathcal{B}(X)$ and we can apply Corollary 5.4.13 to get the first part.

By Proposition 5.3.8, the spectrum of any chaotic or frequently hypercyclic operator must be a perfect set. Since every operator, T, on a hereditarily indecomposable space is of the form $T = \lambda + S$ where $\sigma_{ess} = \{0\}$, i.e. S is a Riesz operator, there must be some isolated point in its spectrum (either λ if T has only finitely many normal eigenvalues or any one of its normal eigenvalues otherwise). Therefore, no $T \in \mathcal{B}(X)$ has a spectrum that is a perfect set and the second part follows by Proposition 5.3.8. □

REMARK 5.4.15. Corollary 5.4.14 gives us a version of both [**21**, Theorem 2.1] (Theorem 1.1.1) and [**26**, Theorem 11] for hereditarily indecomposable preduals of ℓ^1.

The Tarbard spaces, X_k, (see [**30**]) are hereditarily indecomposable spaces each admitting a non-compact nilpotent operator of order k. In addition

$$\mathcal{B}(X_k) = \{\lambda_0 I + \sum_{j=1}^{k-1} \lambda_j N^j + K : \lambda \in \mathbb{K}, K \text{ compact}\}.$$

Therefore given any $T \in \mathcal{B}(X_k)$, $T - \lambda_0$ is a Riesz operator, i.e. if $T \in \mathcal{B}(X_k)$ then $\sigma_{ess}(T) = \{\lambda_0\}$ is finite.

COROLLARY 5.4.16. *Suppose X_k is one of the Tarbard spaces. Then:*

(1)

$$\overline{\mathcal{M}\mathrm{ix}(X_k)} = \overline{\mathcal{WM}\mathrm{ix}(X_k)} = \overline{\mathcal{HC}(X_k)} = \{\lambda I + K : |\lambda| = 1, \sigma(K) = 0\}$$

(2)

$$\overline{\mathcal{CHC}(X_k)} = \overline{\mathcal{FHC}(X_k)} = \overline{\mathcal{M}\mathrm{ix}(X_k) \cap \mathcal{CHC}(X_k) \cap \mathcal{FHC}(X_k)} = \varnothing$$

PROOF. This follows from Corollary 5.4.14. □

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