Direct Probability Integral Method-based System Reliability Sensitivity Analysis and Design Optimization

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ABSTRACT: System reliability-based design optimization (SRBDO) can take an optimal trade-off between economy and safety in structural design with multiple failure modes, in which the evaluation of system reliability and its sensitivity is a challenging problem. Due to the complexity introduced by multiple failure modes, a unified and efficient method is required. In this study, the direct probability integral method (DPIM) is proposed to evaluate system reliability and its sensitivity efficiently, and a decoupled approach is established to attack time-invariant and time-variant SRBDO problems. Firstly, the decoupled framework based on first-order and second-order sensitivity is established for SRBDO problems. Furthermore, the first-order and second-order system reliability sensitivity with respect to distribution parameters of random variables are derived from the probability density integral equation, which significantly improves the computational efficiency. Finally, numerical example verifies the high accuracy and efficiency of DPIM for system reliability and its sensitivity analysis. It is shown that the proposed decoupled approach can address the intractable SRBDO problems efficiently.

1. INTRODUCTION

Compared with the traditional reliability-based design optimization, the system reliability-based design optimization (SRBDO) provides an optimal design scheme for the system with the constraint of system reliability involving multiple complex failure modes (Zhang et al. 2020). As a consequence, SRBDO is extremely important to the design of practical engineering system with uncertainties.

The gradient-based optimization algorithm is most popular in SRBDO, which requires evaluating the system reliability and its sensitivity. Some researches were done, and the SRBDO problems can be solved by adopting different methods of system reliability and sensitivity analysis, such as approximate analytical methods (e.g. , the first-order reliability method (FORM), the second-order reliability method), stochastic sampling methods and surrogate model methods, etc. Approximate analytical methods-based SRBDO algorithms are carried out relatively early. The Taylor expansion of the performance function is carried out at the most probable point to avoid solving multidimensional integrals, in which the system reliability constraints are transformed into reliability index constraints. Ba-abbad et al. (2006) employed sequential optimization and reliability assessment method to solve SRBDO for the series system problem. Liang et al. (2007) proposed the single-loop approach in which a second-order Ditlevsen upper bound is adopted for estimating the system failure probability. The extended sequential compound method was invented to obtain the correlation coefficients between compound events and its internal events by Xing et al. (2021), and further FORM was utilized to attack SRBDO problem. The correlation coefficients between the failure modes are unnecessary for stochastic sampling methods, because the correlation is accounted in the governing equations for each sample. Wang et al. (2012) employed Monte Carlo simulation (MCS) to perform SRBDO for vehicle side impact problem. Weighted average simulation is developed to achieve SRBDO for truss structures by Okasha (2016). Yu and Zhang (2019) utilized weighted average simulation to establish a decoupling approach for time- and space-variant SRBDO problems. The surrogate model methods significantly reduce the computational effort of SRBDO problem. MCS and Kriging metamodel were combined to solve the SRBDO problems (Zhang et al. 2020). A unified and efficient approach is extremely necessary for SRBDO issues due to the limitations of existing methods in applicability and computational efficiency.

In fact, the SRBDO problem with strong nonlinear performance functions is difficult. The determination of correlation coefficients between all failure modes is also a tricky problem. Aiming at these challenges, in this study, direct probability integral method (DPIM) is proposed for system reliability and its sensitivity analyses, and the SRBDO problems including time-variant system are attacked in a unified and decoupled framework.

2. DECOUPLED FRAMEWORK FOR SRBDO

SRBDO seeks a design scheme that minimizes the weight, volume, or cost of a system while considering system reliability as the constraint. The unified SRBDO formulation for time-invariant and time-variant systems is expressed as

$$
\begin{aligned}
&\text{find} \quad \mathbf{d} \\
&\min_{\mathbf{d}} \quad C(\mathbf{d}) \\
&\text{s.t.} \quad h(\mathbf{d}) = P_{f,\text{sys}}(\mathbf{d}) - P_{f_0} \le 0 \\
&\qquad (\text{time-invariant system}), \\
&\qquad h(\mathbf{d}) = P_{f,\text{sys}}(\mathbf{d}, t) - P_{f_0} \le 0 \\
&\qquad (\text{time-variant system}), \\
&\qquad \mathbf{d}^{\mathbf{L}} \le \mathbf{d} \le \mathbf{d}^{\mathbf{U}}
\end{aligned}
\tag{1}
$$

where $\mathbf{d}$ stands for the design vector, which can include the distribution parameters of the random variables (e.g., the mean values). $\mathbf{d}^{\mathbf{U}}$ and $\mathbf{d}^{\mathbf{L}}$ are the upper bounds and lower bounds of the design variables, respectively. $C(\mathbf{d})$ refers to the cost, weight or volume of the system. $P_{f,\text{sys}}(\mathbf{d})$ and $P_{f,\text{sys}}(\mathbf{d}, t)$ indicate the failure probability of time-invariant system and time-variant system separately. P_{f_0} is the maximum allowable failure probability.

In practical applications, the objective function $C(\mathbf{d})$ is usually expressed explicitly and easy to solve. Consequently, the key of SRBDO problem is to calculate the constraint function (i.e., failure probability) and its sensitivity. In order to improve the solving efficiency of Eq. (1), researchers put forward the decoupled approach. For the decoupled approach, the constraint functions are approximated by explicit formulas so that the reliability and sensitivity analysis of system can be decoupled from the optimization problem. This approximation is usually local, and first-order Taylor expansion is a common approach. In this study, second-order Taylor expansion is employed to approximate the constraint function more accurately, which makes the decoupled approach more robust. Eq. (1) is rewritten by

$$
\begin{aligned}
&\text{find} \quad \mathbf{d} \\
&\min_{\mathbf{d},\,\boldsymbol{\mu}_{\mathbf{X}}} \quad C(\mathbf{d}) \\
&\text{s.t.} \quad P_{f,\text{sys}}^{\mathbf{d}_k}(\mathbf{d}_k) + \sum_{j=1}^{n} \frac{\partial P_{f,\text{sys}}^{\mathbf{d}_k}}{\partial d_j}\left(d_j - d_j^k\right) \\
&\qquad + \sum_{j=1}^{n} \frac{\partial^2 P_{f,\text{sys}}^{\mathbf{d}_k}}{\partial d_j^2}\left(d_j - d_j^k\right)^2 / 2 \le P_{f_0} \\
&\qquad P_{f,\text{sys}}^{\mathbf{d}_k}(\mathbf{d}_k, t) + \sum_{j=1}^{n} \frac{\partial P_{f,\text{sys}}^{\mathbf{d}_k}}{\partial d_j}\left(d_j - d_j^k, t\right) \\
&\qquad + \sum_{j=1}^{n} \frac{\partial^2 P_{f,\text{sys}}^{\mathbf{d}_k}}{\partial d_j^2}\left(d_j - d_j^k, t\right)^2 / 2 \le P_{f_0} \\
&\qquad \mathbf{d}^{\mathbf{L},\text{new}} \le \mathbf{d} \le \mathbf{d}^{\mathbf{U},\text{new}}
\end{aligned}
\tag{2}
$$

where $\mathbf{d}_k$ refers to the vector of design variables at k-th iteration. $\frac{\partial P_{f,\mathrm{sys}}^{\mathbf{d}_k}}{\partial d_j}$ and $\frac{\partial^2 P_{f,\mathrm{sys}}^{\mathbf{d}_k}}{\partial d_j^2}$ mean first-order and second-order sensitivity, respectively. $\mathbf{d}^{\mathrm{L,\,new}}$ and $\mathbf{d}^{\mathrm{U,new}}$ indicate the motion limit of design variables.

The motion limit of the design variable is the trust region, and selecting an appropriate trust region is crucial to solving Eq. (2). In this study, a moving trust region is introduced, namely

$$\begin{aligned}\mathbf{d}^{\mathbf{L,new}} &= \max\left(\mathbf{d}^{\mathbf{L}}, \mathbf{d}^{k} - \gamma_{(k)}\left(\mathbf{d}^{\mathbf{U}} - \mathbf{d}^{\mathbf{L}}\right)\right) \\ \mathbf{d}^{\mathbf{U,new}} &= \min\left(\mathbf{d}^{\mathbf{U}}, \mathbf{d}^{k} + \gamma_{(k)}\left(\mathbf{d}^{\mathbf{U}} - \mathbf{d}^{\mathbf{L}}\right)\right)\end{aligned} \tag{3}$$

in which

$$\gamma_{(k+1)} = \max\left(0.05, \xi\gamma_{(k)}\right) \tag{4}$$

where ξ and $\gamma_{(1)}$ are adjustable. In this research, $\xi = 0.8$, and $\gamma_{(1)} = 0.8$.

Eq. (2) is solved by the sequential quadratic programming. After solving Eq. (2) for several times, the optimal solution of the final Eq. (2) converges to the optimal solution of Eq. (1). Consequently, the system reliability and its sensitivity analysis at each iteration step is the core issue.

3. SYSTEM RELIABILITY AND SENSITIVITY ANALYSIS

Recently, Chen and Yang (2019) proposed direct probability integral method (DPIM) to obtain the probability density function (PDF) of the response and reliability of static and dynamic systems. The effectiveness of DPIM for reliability analysis with complex performance functions was demonstrated (Li et al. 2021). In this section, DPIM is introduced for efficient analysis of system reliability and its sensitivity.

3.1. System reliability analysis based on DPIM

For a time-invariant system, the performance function of each failure modes is expressed as

$$Z_i = g_i\left(\boldsymbol{\theta}\right), i = 1, 2, \ldots, m \tag{5}$$

where $\boldsymbol{\theta} = \{\theta_1, \theta_2, \cdots, \theta_n\}$ denotes the input random vector of system. $g_i(\boldsymbol{\theta})$ stands for the performance function of system, when $g_i > 0$, the failure mode is safe, and when $g_i \le 0$, the failure mode fails.

According to the previous work of our group, for a static system in Eq. (5), the joint PDF of multiple performance functions can be expressed by the probability density integral equation, i.e.,

$$\begin{aligned}p_{\mathbf{Z}}\left(Z_1, Z_2, \ldots, Z_m\right) = \int_{\Omega_{\boldsymbol{\Theta}}} p_{\boldsymbol{\Theta}}\left(\boldsymbol{\theta}\right)\delta\left[z_1 - g_1\left(\boldsymbol{\theta}\right)\right] \\ \delta\left[z_2 - g_2\left(\boldsymbol{\theta}\right)\right], \ldots, \delta\left[z_m - g_m\left(\boldsymbol{\theta}\right)\right]\mathrm{d}\boldsymbol{\theta}\end{aligned} \tag{6}$$

in which $p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$ means the joint PDF of input random variables, $\delta[\cdot]$ denotes Dirac delta function.

The time-invariant series system reliability based on DPIM can be expressed as

$$\begin{aligned}P_{\mathrm{r,sys}}^{\mathrm{s}} &= \Pr\left[\bigcap_{i=1}^{m} g_i\left(\boldsymbol{\theta}\right) > 0\right] \\ &= \int_0^{+\infty}\cdots\int_0^{+\infty} p_{\mathbf{Z}}\left(z_1, z_2, \cdots, z_m\right)\mathrm{d}z_1\mathrm{d}z_2\cdots\mathrm{d}z_m \\ &= \int_{\Omega_{\boldsymbol{\Theta}}} p_{\boldsymbol{\Theta}}\left(\boldsymbol{\theta}\right)\mathcal{H}\left[g_1\left(\boldsymbol{\theta}\right)\right]\mathcal{H}\left[g_2\left(\boldsymbol{\theta}\right)\right]\cdots \\ &\quad \mathcal{H}\left[g_m\left(\boldsymbol{\theta}\right)\right]\mathrm{d}\boldsymbol{\theta}\end{aligned} \tag{7}$$

where $\mathcal{H}[\cdot]$ is the Heaviside function.

For parallel and mixed system, reliability is written as (Chen et al. 2022)

$$\begin{aligned}P_{\mathrm{r,sys}}^{\mathrm{p}} &= \Pr\left[\bigcup_{i=1}^{m} g_i\left(\boldsymbol{\theta}\right) > 0\right] \\ &= 1 - \int_{\Omega_{\boldsymbol{\Theta}}} p_{\boldsymbol{\Theta}}\left(\boldsymbol{\theta}\right)\mathcal{H}\left[-g_1\left(\boldsymbol{\theta}\right)\right] \\ &\quad \mathcal{H}\left[-g_2\left(\boldsymbol{\theta}\right)\right]\cdots\mathcal{H}\left[-g_m\left(\boldsymbol{\theta}\right)\right]\mathrm{d}\boldsymbol{\theta}\end{aligned} \tag{8}$$

$$\begin{aligned}P_{\mathrm{r,sys}}^{\mathrm{m}} &= \Pr\left[\left(\bigcup_{i=1}^{m_1} g_i\left(\boldsymbol{\theta}\right) > 0\right)\cap\left(\bigcup_{i=m_1+1}^{m_1+m_2} g_i\left(\boldsymbol{\theta}\right) > 0\right)\right. \\ &\quad \left.\cap\cdots\cap\left(\bigcup_{i=m_1+\cdots+m_k+1}^{m_1+m_2+\cdots+m_k} g_i\left(\boldsymbol{\theta}\right) > 0\right)\right] \\ &= \prod_{i=1}^{k}\left(1 - \int_{\Omega_{\boldsymbol{\Theta}}} p_{\boldsymbol{\Theta}}\left(\boldsymbol{\theta}\right)\mathcal{H}\left[-g_1\left(\boldsymbol{\theta}\right)\right]\right. \\ &\quad \left.\mathcal{H}\left[-g_2\left(\boldsymbol{\theta}\right)\right]\cdots\mathcal{H}\left[-g_{m_i}\left(\boldsymbol{\theta}\right)\right]\mathrm{d}\boldsymbol{\theta}\right)\end{aligned} \tag{9}$$

For the time-variant system, the equivalent extreme-value mapping of its performance function is considered in the time interval (0, t],

$$g_{i,\min}(\boldsymbol{\theta},\mathbf{X},t) = \underset{0<\tau\leq t}{\text{ext}}\left\{g_i(\boldsymbol{\theta},\mathbf{X},\tau)\right\} = \min_{0<\tau\leq t}\left\{g_i(\boldsymbol{\theta},\mathbf{X},\tau)\right\} \tag{10}$$

Replacing $g_i(\boldsymbol{\theta})$ of Eqs. (7), (8) and (9) with $g_{i,\min}(\boldsymbol{\theta},\mathbf{X},t)$ is the time-invariant system reliability. Due to space limitation, time-variant system reliability will not be introduced in detail.

To solve the integral in Eqs. (7), (8) and (9), Chen and Yang adopted the representative point selection technique based on generalized *F*-discrepancy. This technique can generate a series of representative points $\theta_r, r=1,2,\cdots,N$ and their assigned probabilities $p_{\theta_r}, r=1,2,\cdots,N$. Then, Eqs. (7), (8) and (9) can be expressed by the representative points and the assigned probabilities, respectively,

$$P_{\mathrm{r,sys}}^{\mathrm{s}} = \sum_{r=1}^{N} p_{\theta_r}\mathcal{H}\left[g_1(\theta_r)\right]\mathcal{H}\left[g_2(\theta_r)\right]\cdots\mathcal{H}\left[g_m(\theta_r)\right] \tag{11}$$

$$P_{\mathrm{r,sys}}^{\mathrm{p}} = 1-\sum_{r=1}^{N} p_{\theta_r}\mathcal{H}\left[-g_1(\theta_r)\right]\mathcal{H}\left[-g_2(\theta_r)\right]\cdots\mathcal{H}\left[-g_m(\theta_r)\right] \tag{12}$$

$$P_{\mathrm{r,sys}}^{\mathrm{m}} = \prod_{i=1}^{k}\left\{1-\sum_{r=1}^{N} p_{\theta_r}\mathcal{H}\left[-g_1(\theta_r)\right]\mathcal{H}\left[-g_2(\theta_r)\right]\cdots\mathcal{H}\left[-g_{m_i}(\theta_r)\right]\right\} \tag{13}$$

Since the correlations of failure modes are coupled in the performance functions $Z_i = g_i(\boldsymbol{\theta})$, the correlation coefficients are not necessary for system reliability analysis with Eqs. (11), (12) and (13).

3.2. System reliability sensitivity analysis based on DPIM

Reliability sensitivity is the partial derivative of failure probability with respect to the distribution parameters of random variables (Bjerager and Krenk 1989; Song et al. 2009), consequently, for series system, first-order system reliability sensitivity is written as

$$\begin{aligned}\frac{\partial P_{f,\mathrm{sys}}^{\mathrm{s}}}{\partial \eta_{\theta_s}} &= -\frac{\partial P_{r,\mathrm{sys}}}{\partial \eta_{\theta_s}} \\ &= -\int_{\boldsymbol{\Omega}_{\boldsymbol{\Theta}}}\mathcal{H}\left[g_1(\boldsymbol{\theta})\right]\mathcal{H}\left[g_2(\boldsymbol{\theta})\right]\cdots \mathcal{H}\left[g_m(\boldsymbol{\theta})\right]\frac{\partial p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})}{\partial \eta_{\theta_s}} + p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})\frac{\partial \mathcal{H}\left[g_1(\boldsymbol{\theta})\right]\mathcal{H}\left[g_2(\boldsymbol{\theta})\right]\cdots\mathcal{H}\left[g_m(\boldsymbol{\theta})\right]}{\partial \eta_{\theta_s}}\mathrm{d}\boldsymbol{\theta}\end{aligned} \tag{14}$$

where η_{θ_s} is a distribution parameter of the *s*-th random variable θ_s. Note that the performance function depends on the realizations of random variables $\theta_r, r=1,2,\cdots,N$, but is independent to the corresponding probability density of random variables. Consequently, the partial derivatives of the performance function with respect to distribution parameters of random variables are zero.

In most cases, the joint PDF of the input random variables can be expressed explicitly by its distribution parameters. Eq. (14) is rewritten by

$$\begin{aligned}\frac{\partial P_{f,\mathrm{sys}}^{\mathrm{s}}}{\partial \eta_{\theta_s}} &= -\int_{\boldsymbol{\Omega}_{\boldsymbol{\Theta}}}\mathcal{H}\left[g_1(\boldsymbol{\theta})\right]\mathcal{H}\left[g_2(\boldsymbol{\theta})\right]\cdots\mathcal{H}\left[g_m(\boldsymbol{\theta})\right]\frac{\partial p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})}{\partial \eta_{\theta_s}}\mathrm{d}\boldsymbol{\theta} \\ &= -\int_{\Omega_{\Theta}}\frac{\partial p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})}{\partial \eta_{\theta_s}}\frac{1}{p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})}p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})\mathcal{H}\left[g_1(\boldsymbol{\theta})\right]\mathcal{H}\left[g_2(\boldsymbol{\theta})\right]\cdots\mathcal{H}\left[g_m(\boldsymbol{\theta})\right]\mathrm{d}\boldsymbol{\theta}\end{aligned} \tag{15}$$

Let the random variables be independent of each other, and the joint PDF of the random variables is then expressed as the multiplication of the PDFs of all variables. The derivative of the joint PDF to the distribution parameter of the *s*-th variable is

$$\frac{\partial p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})}{\partial \eta_{\theta_s}} = \frac{\partial p_{\theta_s}}{\partial \eta_{\theta_s}} \bullet \prod_{b=1,b\neq s}^{n} p_{\theta_b} \tag{16}$$

in p_{θ_b} which is the PDF of *b*-th random variable.

By substituting Eq. (16) into Eq. (15), Eq. (15) becomes

$$\frac{\partial P_{f,\mathrm{sys}}^{\mathrm{s}}}{\partial \eta_{\theta_s}} = -\int_{\Omega_{\boldsymbol{\Theta}}} \frac{\partial p_{\theta_s}}{\partial \eta_{\theta_s}} \frac{1}{p_{\theta_s}} p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) \mathcal{H}\left[g_1(\boldsymbol{\theta})\right] \mathcal{H}\left[g_2(\boldsymbol{\theta})\right] \cdots \mathcal{H}\left[g_m(\boldsymbol{\theta})\right] \mathrm{d}\boldsymbol{\theta} \tag{17}$$

Similarly, first-order reliability sensitivities of parallel and mixed system are deduced, respectively.

$$\frac{\partial P_{f,\mathrm{sys}}^{\mathrm{p}}}{\partial \eta_{\theta_s}} = \int_{\Omega_{\boldsymbol{\Theta}}} \frac{\partial p_{\theta_s}}{\partial \eta_{\theta_s}} \frac{1}{p_{\theta_s}} p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) \mathcal{H}\left[-g_1(\boldsymbol{\theta})\right] \mathcal{H}\left[-g_2(\boldsymbol{\theta})\right] \cdots \mathcal{H}\left[-g_m(\boldsymbol{\theta})\right] \mathrm{d}\boldsymbol{\theta} \tag{18}$$

$$\begin{aligned} \frac{\partial P_{f,\mathrm{sys}}^{\mathrm{m}}}{\partial \eta_{\theta_s}} = \sum_{i=1}^{k} \Bigg\{ \prod_{b=1,b\neq i}^{k} \Bigg(& \int_{\Omega_{\boldsymbol{\Theta}}} p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) \mathcal{H}\left[-g_1(\boldsymbol{\theta})\right] \mathcal{H}\left[-g_2(\boldsymbol{\theta})\right] \cdots \\ & \mathcal{H}\left[-g_{m_l}(\boldsymbol{\theta})\right] \mathrm{d}\boldsymbol{\theta} \Bigg) \cdot \int_{\Omega_{\boldsymbol{\Theta}}} \frac{\partial p_{\theta_s}}{\partial \eta_{\theta_s}} \frac{1}{p_{\theta_s}} p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) \\ & \mathcal{H}\left[-g_1(\boldsymbol{\theta})\right] \mathcal{H}\left[-g_2(\boldsymbol{\theta})\right] \cdots \mathcal{H}\left[-g_{m_i}(\boldsymbol{\theta})\right] \mathrm{d}\boldsymbol{\theta} \Bigg\} \end{aligned} \tag{19}$$

Eqs. (17), (18) and (19) are numerically solved by referring to the procedures of reliability solution based on DPIM.

$$\frac{\partial P_{f,\mathrm{sys}}^{\mathrm{s}}}{\partial \eta_{\theta_s}} = -\sum_{r=1}^{N} \left[\frac{\partial p_{\theta_s}}{\partial \eta_{\theta_s}} \frac{1}{p_{\theta_s}} \right]\Bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}_{s,r}} p_{\theta_r} \mathcal{H}\left[g_1(\theta_r)\right] \mathcal{H}\left[g_2(\theta_r)\right] \cdots \mathcal{H}\left[g_m(\theta_r)\right] \tag{20}$$

$$\frac{\partial P_{f,\mathrm{sys}}^{\mathrm{p}}}{\partial \eta_{\theta_s}} = \sum_{r=1}^{N} \mathcal{H}\left[-g_1(\theta_r)\right] \mathcal{H}\left[-g_2(\theta_r)\right] \cdots \mathcal{H}\left[-g_m(\theta_r)\right] \left[\frac{\partial p_{\theta_s}}{\partial \eta_{\theta_s}} \frac{1}{p_{\theta_s}} \right]\Bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}_{s,r}} p_{\theta_r} \tag{21}$$

$$\begin{aligned} \frac{\partial P_{f,\mathrm{sys}}^{\mathrm{m}}}{\partial \eta_{\theta_s}} = \sum_{i=1}^{k} \Bigg\{ \prod_{b=1,b\neq i}^{k} \Bigg(& \sum_{r=1}^{N} p_{\theta_r} \mathcal{H}\left[-g_1(\theta_r)\right] \mathcal{H}\left[-g_2(\theta_r)\right] \cdots \\ & \mathcal{H}\left[-g_{m_l}(\theta_r)\right] \Bigg) \cdot \sum_{r=1}^{N} \mathcal{H}\left[-g_1(\theta_r)\right] \\ & \mathcal{H}\left[-g_2(\theta_r)\right] \cdots \mathcal{H}\left[-g_{m_i}(\theta_r)\right] \left[\frac{\partial p_{\theta_s}}{\partial \eta_{\theta_s}} \frac{1}{p_{\theta_s}} \right]\Bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}_{s,r}} p_{\theta_r} \Bigg\} \end{aligned} \tag{22}$$

The second-order reliability sensitivity can be obtained by calculating the partial derivative of the first-order reliability sensitivity with respect to the distribution parameters of random variables. Consequently, the second-order reliability sensitivities of series, parallel and mixed system are derived.

$$\frac{\partial P_{f,\mathrm{sys}}^{\mathrm{s}}}{\partial \eta_{\theta_s}} = -\sum_{r=1}^{N} \left[\frac{\partial^2 p_{\theta_s}}{\partial \eta_{\theta_s}^2} \frac{1}{p_{\theta_s}} \right]\Bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}_{s,r}} p_{\theta_r} \mathcal{H}\left[g_1(\theta_r)\right] \mathcal{H}\left[g_2(\theta_r)\right] \cdots \mathcal{H}\left[g_m(\theta_r)\right] \tag{23}$$

$$\frac{\partial P_{f,\mathrm{sys}}^{\mathrm{p}}}{\partial \eta_{\theta_s}} = \sum_{r=1}^{N} \mathcal{H}\left[-g_1(\theta_r)\right] \mathcal{H}\left[-g_2(\theta_r)\right] \cdots \mathcal{H}\left[-g_m(\theta_r)\right] \left[\frac{\partial^2 p_{\theta_s}}{\partial \eta_{\theta_s}^2} \frac{1}{p_{\theta_s}} \right]\Bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}_{s,r}} p_{\theta_r} \tag{24}$$

$$\begin{aligned} \frac{\partial P_{f,\mathrm{sys}}^{\mathrm{m}}}{\partial \eta_{\theta_s}} = \sum_{i=1}^{k} \Bigg\{ \prod_{b=1,b\neq i}^{k} \Bigg(& \sum_{r=1}^{N} p_{\theta_r} \mathcal{H}\left[-g_1(\theta_r)\right] \mathcal{H}\left[-g_2(\theta_r)\right] \cdots \\ & \mathcal{H}\left[-g_{m_l}(\theta_r)\right] \Bigg) \cdot \sum_{r=1}^{N} \mathcal{H}\left[-g_1(\theta_r)\right] \\ & \mathcal{H}\left[-g_2(\theta_r)\right] \cdots \mathcal{H}\left[-g_{m_i}(\theta_r)\right] \left[\frac{\partial^2 p_{\theta_s}}{\partial \eta_{\theta_s}^2} \frac{1}{p_{\theta_s}} \right]\Bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}_{s,r}} p_{\theta_r} \Bigg\} \end{aligned} \tag{25}$$

Note that Eqs. (11), (12) and (13) or Eqs. (23), (24) and (25) employ the same representative points and assigned probabilities as

Eqs. (20), (21) and (22). Consequently, first-order and second-order system reliability sensitivity analyses in the context of DPIM are coproducts of reliability analysis. Accordingly, reliability sensitivity analysis is performed simultaneously with reliability analysis without any additional efforts, and the computational efficiency is extremely high.

4. COMPUTATIONAL PROCEDURES

In this section, the specific computational procedures are shown as follows.

Step 1: Initialize design variables **d**(*k*)=**d**(0).

Step 2: Select representative point set $\theta_r, r=1,2,\cdots,N$, and calculate the corresponding assigned probabilities $p_{\theta_r}, r=1,2,\cdots,N$ by combining the current design point **d**(*k*) and parameters of other random variables. Generate the corresponding stochastic processes if necessary.

Step 3: Calculate the performance function at the representative points, and then substitute the structural responses into Eq. (11), (12) or (13) to obtain the failure probability of system.

Step 4: Derive the explicit form of $\frac{\partial p_{\theta_s}}{\partial \eta_{\theta_s}} \frac{1}{p_{\theta_s}}$ and $\frac{\partial^2 p_{\theta_s}}{\partial \eta_{\theta_s}^2} \frac{1}{p_{\theta_s}}$. The first-order and second-order reliability sensitivities are solved by substituting the results of $\frac{\partial p_{\theta_s}}{\partial \eta_{\theta_s}} \frac{1}{p_{\theta_s}}$ and $\frac{\partial^2 p_{\theta_s}}{\partial \eta_{\theta_s}^2} \frac{1}{p_{\theta_s}}$ into Eq. (20), (21) or (22) and Eq. (23), (24) or (25), respectively.

Step 5: Update the trust region by Eq. (3).

Step 6: Solve Eq. (2) by the sequential quadratic programming. Set the optimal solution of Eq. (2) as **d** (*k*+1).

Step 7: Determine whether the change of design variables $\max\left(\left|\mathbf{d}_{(k+1)}-\mathbf{d}_{(k)}\right| \big/ \left|\mathbf{d}_{(k)}\right|\right)$ is less than *ε*. If the convergence criterion is satisfied, then stop the iteration and output the optimal design results; otherwise, go back to step 2.

5. NUMERICAL EXAMPLE

The liquid hydrogen fuel tank is an important transportation and storage tool (Rashki et al. 2014). In this example, a liquid hydrogen fuel tank with honeycomb sandwich design is considered. The tank is considered as a series system with three failure modes: von Mises strength, isotropic strength, or honeycomb buckling. The statistical properties of random variables are listed in Table 1. The formulation of RBDO problem is written as follows.

$$
\begin{aligned}
&\min\ t_p \\
&\text{s.t.}\quad h = P_{f,\text{sys}}^{s}(\mathbf{d}) \le P_{f_0} \\
&\qquad g_1 = 84000 t_p \big/ \left|N_y\right| - 1 \\
&\qquad g_2 = \frac{84000 t_p}{\sqrt{N_x^2 + N_y^2 - N_x N_y + 3N_{xy}^2}} - 1 \\
&\qquad g_3 = 0.847 + 0.96X_1 + 0.98X_2 - \\
&\qquad\quad 0.21X_3 + 0.077X_1^2 + 0.11X_2^2 \\
&\qquad\quad + 0.0007X_3^2 + 0.378X_1X_2 - \\
&\qquad\quad 0.106X_1X_3 - 0.11X_2X_3 \\
&\qquad \textit{where}\ \ X_1 = 4\left(t_p - 0.075\right) \\
&\qquad\qquad X_2 = 20\left(t_h - 1\right) \\
&\qquad\qquad X_3 = -6000\left(1/N_{xy} - 0.003\right) \\
&\qquad 0.01 \le t_p \le 1
\end{aligned}
\tag{26}
$$

Table 1: Random variables for a liquid hydrogen fuel tank.

Variables	Distribution type	Mean	Standard deviation
t_{plate}	Normal	Design variable	0.005
t_h	Normal	0.1	0.01
N_x	Normal	13	60
N_y	Normal	4751	48
N_{xy}	Normal	−684	11

Without loss of generality, the initial design variable is taken as the upper bound of the design variable. The results of the SRBDO problem with different threshold values are listed in Table 2, where P_f^{MCS} represent the failure probability corresponding to the optimal solution verified by MCS with 10^6 samples. G-evals means the number of evaluations for performance functions. DPIM refers to the combination of DPIM and the method of moving asymptote (MMA) (Svanberg 1987). Decoupled-DPIM indicates the proposed approach in this study.

According to the results, the optimal results of DPIM with MMA and decoupled-DPIM are consistent with those of the reference, which demonstrates the effectiveness of the proposed method for SRBDO problem. Additionally, for threshold value is 0.0007, the number of performance function evaluations required by decoupled-DPIM (i.e., 11000) is much lower than that of the reference (i.e., 50000) and DPIM with MMA (i.e., 20000). This indicates that DPIM is more efficient in the analysis of system reliability and sensitivity, and the decoupled approach proposed in this study can further improve the computational efficiency of SRBDO problem.

Figure 1 displays the iterative histories of design variable or objective function with different threshold values.

6. CONCLUSIONS

In this study, firstly, a decoupled SRBDO framework based on DPIM is established to address time-invariant and time-variant SRBDO problems. Further, the DPIM-based approach is proposed to solve first-order and second-order system reliability sensitivity efficiently and accurately. Finally, an example validates the high efficiency of the proposed method. The main conclusions are drawn as follows:

(1) The same representative points are adopted in the assessments of system reliability, first-order and second-order system reliability sensitivity, which makes reliability sensitivity analysis more efficient in the context of DPIM.

(2) A decoupled SRBDO framework based on DPIM is established to address time-invariant or time-variant RBDO problems. It is indicated from the example that the DPIM-based decoupled approach is more efficient and powerful than the reference in SRBDO problem.

Finally, the proposed method is suitable for most system reliability sensitivity analysis and SRBDO problems due to the wide applicability of DPIM. In addition, the computational efficiency of the proposed method can be further enhanced by using the surrogate model or other techniques.

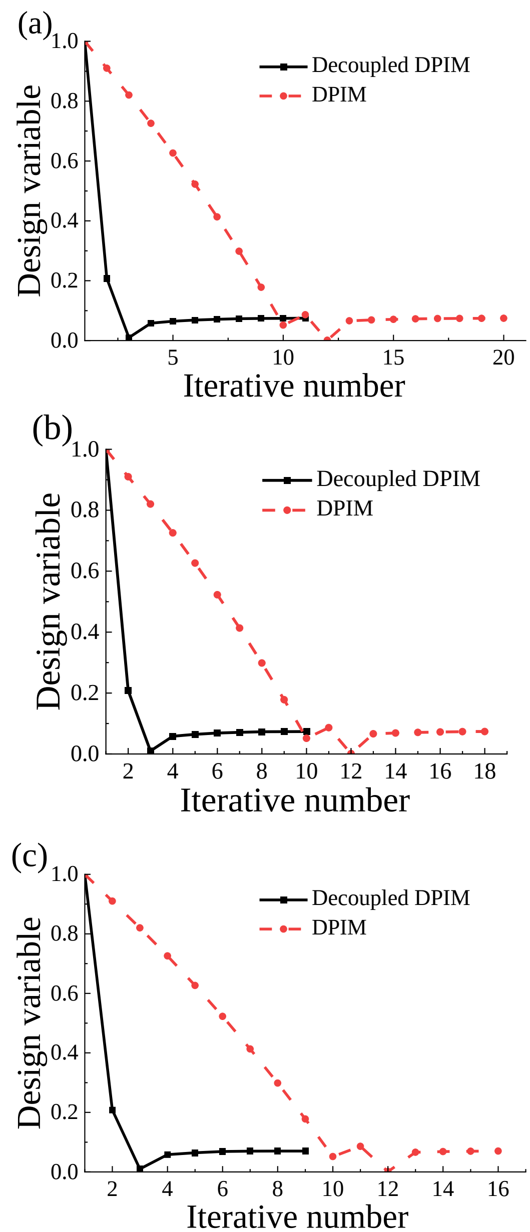

Figure 1: Iterative histories of design variable with (a) $P_{f_0}=0.0007$; (b) $P_{f_0}=0.001$; (c) $P_{f_0}=0.01$.

Table 2: Results of SRBDO problem.

Threshold value	Method	G-evals	Design variable	P_f^{MCS}
0.0007	M Rashki et al. 2014	50000	0.07433	6.964×10^{-4}
	DPIM	20000	0.07460	5.847×10^{-4}
	Decoupled-DPIM	11000	0.07470	5.509×10^{-4}
0.001	DPIM	18000	0.07345	1.278×10^{-3}
	Decoupled-DPIM	10000	0.07350	1.200×10^{-3}
0.01	DPIM	16000	0.07007	0.941×10^{-2}
	Decoupled-DPIM	9000	0.07006	0.945×10^{-2}

7. REFERENCES

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