

Charmed Tetraquarks from Lattice QCD

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Declaration

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Summary

We investigate the ground state effective masses of charmed tetraquark operators using lattice QCD. Algorithms and simulation data from the Hadron Spectrum Collaboration are used. The effective masses of a range of D and D^* mesons are first calculated. We then investigate the effective masses of three s-wave charmed tetraquarks close to the $D^*\bar{D}$ threshold with quantum numbers $I^G J^{PC}=1^-0^{+-}$, 1^+1^{++} and 1^-1^{+-} . The results for these tetraquarks are compared to those for combinations of $D\bar{D}$ mesons to determine if potential binding occurs in the tetraquark. The effective masses are also compared to the experimental results for the X, Y, Z tetraquark candidate particles.

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1 Introduction

Lattice QCD is a non-perturbative approach to calculating observable quantities with Quantum Chromodynamics, which, due to confinement, is not well suited to perturbative techniques at low energies. It works by discretizing space-time with a set of lattice sites, which essentially regularizes the quantum field theory. Observable quantities such as correlation functions can then be calculated using numerical techniques such as Monte Carlo integration.¹

In lattice QCD, fermions $\psi(x)$ are defined at lattice sites, $x = (\vec{x}, t) = (a\vec{n}, a_t n_t)$, where a and a_t are the lattice spacings in space and time respectively and $n_t = 0, 1, \dots, M-1$, $n_i = 0, 1, \dots, N-1$ for $i = 1, 2, 3$. Gauge bosons are link variables acting between lattice sites, with $U_\mu(n)$ acting between the site at n and the site in the unit direction $\hat{\mu}$. The Euclidean correlator is defined as

$$\langle \hat{\mathcal{O}}_2(t) \hat{\mathcal{O}}_1^\dagger(0) \rangle_T = \frac{1}{Z_T} \text{Tr} \left[e^{-(T-t)\hat{H}} O_2 e^{-t\hat{H}} O_1^\dagger \right], \quad (1.1)$$

with a normalisation factor Z_T , for some large time T , which can eventually be taken to infinity. As the quantum mechanical time transporter is $e^{-i\tau\hat{H}}$, the above definition involves a Wick rotation from real time τ to imaginary time $t = i\tau$. This leads to the removal of the relative minus-sign between time and space components in 4-vectors, hence the moniker *Euclidean*.

The path integral in lattice QCD can be written in stylized form as

$$\langle f \rangle = \frac{1}{Z} \int \mathcal{D}[\psi, \bar{\psi}] \mathcal{D}[U] e^{-S[U, \psi, \bar{\psi}]} f[U, \psi, \bar{\psi}] \quad (1.2)$$

where S is the Euclidean action and f is a generic observable. We can split S into fermionic and gauge actions and the fermionic part of the path integral can usually be performed using Gaussian integration. This reduces the path integral to

$$\langle f \rangle = \frac{1}{Z} \int \mathcal{D}[U] e^{-S[U]} \det M[U] \tilde{f}[U] \quad (1.3)$$

where $\det M[U]$ is the matrix (functional) determinant resulting from the Gaussian integration².

Because we have defined the path integral using the Euclidean, rather than

¹This overview and its notation is adapted from [1].

² $\det M[U]$ essentially represents the fermionic vacuum and provides contributions from the creation and annihilation of virtual pairs of sea quarks and anti-quarks. It is included in dynamical lattice QCD studies and is omitted in the quenched approximation.

Minkowski, metric, the term $\frac{1}{2}e^{-S[U]} \det M[U]$ in 1.3 is normalised, real and decays quickly in configuration space away from the minimum of the action. We can therefore treat this term as a probability measure and solve 1.3 using Monte Carlo simulation. This can be carried out by creating random samples of gauge field configurations and computing the functional $\tilde{f}$ for each. Each contribution is weighted with the probability measure and the results are summed up to approximate the integral.

Lattice QCD has been widely used for the accurate computation of ground and excited state energies of mesons and baryons - see, for example, [2][3] for state of the art results for the charmonium spectrum. More recently, lattice studies of tetraquark and meson-meson interactions have been carried out [4]. This analysis aims to extend this work by investigating the ground state energies of three charmed tetraquark interpolators with quantum numbers $I^G J^{PC} = 1^- 0^{+-}$, $1^+ 1^{++}$ and $1^- 1^{+-}$.

The quark content of mesons and baryons is given by the $SU(3)$ group which describes the gauge symmetry of the three color charges. However, this structure allows for exotic multiquark states such as tetraquarks ($q\bar{q}q\bar{q}$) and pentaquarks ($qqqq\bar{q}$).

Exotic particles are identified experimentally by examining their quantum numbers or charge ³. For example, according to the quark model, a meson with spin, S , and orbital angular momentum, L , has parity, $P = -1^{L+1}$ and charge conjugation, $C = -1^{L+S}$. Therefore quantum numbers such as 0^{--} , 0^{+-} and 1^{-+} are not allowed. In addition, a boson with charge $+2$ cannot be formed from less than 4 quarks and charmonium-like states cannot have non-zero charge. However, exotic candidates may often have the same charge and quantum numbers as conventional states and, in these cases, the presence of extra, supernumerary states at certain mass thresholds is often the main way of identifying exotic particles.

This was the case with the particle $X(3872)$ which was first observed in 2003 by the Belle collaboration while studying the weak decay of the B meson to the Kaon and, eventually, to $\pi^+\pi^-J/\psi$ [7]. It had an unusually narrow width (or long life) with a mass close to the $D^{0*}\bar{D}^0$ threshold. Moreover, while its quantum numbers were not yet determined, its decay to two pions suggested it contained some combination of light quarks with isospin 0 or 1. Eventually, it was confirmed that the $X(3872)$ had isospin 1 and quantum numbers $J^{PC} =$

³The overview of exotic particles and experimental results in this section are taken from the reviews [5] and [6]

1^{++} [8] and that it can decay to either a charmonium-light quark meson pair or $D^{*0}\overline{D}^0$.

The unexpected discovery of $X(3872)$ prompted searches involving other production mechanisms, which led to the identification of a whole range of exotic hadrons. Typically, exotic particles that can be produced through weak B -meson decay are called X , while those produced from Initial State Radiation, $e^+e^- \rightarrow \gamma_{ISR}Y$, are termed Y . Charged quarkonium-like states, or such states in a multiplet with at least one charged member, are usually referred to as Z .

Various theoretical pictures of these exotic particles have been explored. For example, it is posited that the $X(3872)$, being at the $D^{*0}\overline{D}^0$ mass threshold, is a bound state of two D mesons. Alternative explanations for various exotic particles include the hadro-quarkonium picture of a quarkonium core surrounded by a light meson cloud, as well as hybrid structures involving gluons. A more tightly bound model consisting of a diquark (qq) and an anti-diquark ($\overline{q}\overline{q}$) has also been applied to charmed tetraquark candidates [9] and it is this set-up that we aim to investigate using lattice QCD.

The rest of this manuscript is organised as follows: we begin in Section 2 by examining the color $SU(3)$ and spin $SU(2)$ irreducible representations of tetraquarks and implement an algorithm to produce the correct spin indices for a tetraquark interpolator; in Section 3, we review the construction of the meson two-point correlation function and extend the analysis to a charmed tetraquark correlation function; Section 4 reports the results of the lattice QCD calculations of a range of D mesons and three tetraquark operators and we investigate their relationship for possible binding in the tetraquarks; in this section we also compare our lattice QCD output with the established experimental results; Section 5 concludes.

2 Tetraquark Irreducible Representations

2.1 Color SU(3) Irreducible Representations

We wish to find the color singlet representations of the Color SU(3) product space of two anti-quarks and two quarks. We begin by using Young Tableaux to decompose the product space into irreducible representations and then use tensor methods to find the explicit form of the color singlet operators.

The tensor product of two anti-quarks and two quarks can be written as a sum of products of pairs.

$$\begin{aligned}\bar{\mathbf{3}} \otimes \bar{\mathbf{3}} \otimes \mathbf{3} \otimes \mathbf{3} &= [\bar{\mathbf{3}} \oplus \bar{\mathbf{6}}] \otimes [\bar{\mathbf{3}} \oplus \mathbf{6}] \\ &= [\bar{\mathbf{3}} \otimes \bar{\mathbf{3}}] \oplus [\bar{\mathbf{6}} \otimes \bar{\mathbf{3}}] \oplus [\bar{\mathbf{6}} \otimes \mathbf{3}] \oplus [\bar{\mathbf{6}} \otimes \mathbf{6}]\end{aligned}\tag{2.1}$$

Each of the tensor products in this sum can be decomposed using Young Tableaux as follows:

$$\bar{\mathbf{3}} \otimes \mathbf{3} = \mathbf{8} \oplus \mathbf{1}:$$

$$\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \otimes \begin{array}{|c|} \hline \square \\ \hline \end{array} = \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}\tag{2.2}$$

$$\mathbf{6} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8}:$$

$$\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \otimes \begin{array}{|c|} \hline \square \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}\tag{2.3}$$

$$\bar{\mathbf{6}} \otimes \bar{\mathbf{3}} = \bar{\mathbf{10}} \oplus \bar{\mathbf{8}}:$$

$$\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \otimes \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array}\tag{2.4}$$

$$\bar{\mathbf{6}} \otimes \mathbf{6} = \mathbf{27} \oplus \mathbf{8} \oplus \mathbf{1}:$$

$$\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}\tag{2.5}$$

This illustrates the origin of two color singlets in the decomposition of the original tensor product. In order to derive the precise form of the color singlet

operators, we must track the color indices of the quarks and anti-quarks fields through the decomposition of their tensor product.

Define anti-quarks $\bar{a}$ and $\bar{b}$ and quarks c and d , together with the tensor products $\bar{\mathbf{3}} \otimes \bar{\mathbf{3}}$ and $\mathbf{3} \otimes \mathbf{3}$ as $v_{ij} := \bar{a}_i \bar{b}_j$ and $u^{kl} := c^k d^l$, respectively. These tensors can be decomposed by enforcing the following two properties of irreducible representations:

1. Tensors are separately symmetric in upper indices and in lower indices.
2. Tensors are traceless.

Thus u can be decomposed as

$$\begin{aligned}
u^{kl} &= u^{(kl)} + u^{[kl]} \\
&= u^{(kl)} + \frac{1}{2}(u^{kl} - u^{lk}) \\
&= u^{(kl)} + \frac{1}{2}(\delta_a^k \delta_b^l - \delta_b^k \delta_a^l) u^{ab} \\
&= u^{(kl)} + \frac{1}{2} \epsilon^{klm} \epsilon_{mab} u^{ab}
\end{aligned} \tag{2.6}$$

where $u^{(kl)}$ is a symmetric rank 2 tensor corresponding to the $\mathbf{6}$, while $\epsilon_{mab} u^{ab}$ is a rank 1 tensor with a lower index corresponding to the $\bar{\mathbf{3}}$. Using the corresponding decomposition for v_{ij} , the $\bar{\mathbf{3}} \otimes \mathbf{3}$ product can be written as

$$w_m^n := \epsilon^{nij} v_{ij} \epsilon_{mkl} u^{kl}. \tag{2.7}$$

The irreducible representations of this can be found by extracting the trace:

$$\underbrace{w_m^n - \frac{1}{3} \delta_m^n w_a^a}_{\mathbf{8}} + \underbrace{\frac{1}{3} \delta_m^n w_a^a}_{\mathbf{1}}. \tag{2.8}$$

Thus, we can see that the color singlet observed in the diagram 2.2 above can be written as

$$\epsilon_{akl} \epsilon^{aij} v_{ij} u^{kl} \tag{2.9}$$

or, alternatively,

$$[\delta_k^i \delta_l^j - \delta_l^i \delta_k^j] \bar{a}_i \bar{b}_j c^k d^l. \tag{2.10}$$

Normalization of this sum can be carried out by re-writing it as

$$\bar{a}_i \bar{b}_j c^i d^j - \bar{a}_i \bar{b}_j c^j d^i \tag{2.11}$$

and noting that each of these terms is a sum of 9 terms, with each containing the 3 terms $\overline{a_i} \overline{b_i} c^i d^i$, which cancel due to the minus sign. We are therefore left with $9 - 3 + 9 - 3 = 12$ terms and the sum of the square of their co-efficients must equal 1. Thus the normalization factor for the above Color singlet is $\frac{1}{\sqrt{12}}$.

To obtain the form of the other color singlet, we must decompose the tensor product of the symmetric parts of v and u , corresponding to the $\overline{\mathbf{6}} \otimes \mathbf{6}$:

$$t_{ij}^{kl} := v_{(ij)} u^{(kl)}. \quad (2.12)$$

This tensor is already symmetric in the upper and in the lower indices, so we just need to extract the trace to render it irreducible.

$$t_{ij}^{kl} - \frac{1}{3} \delta_i^k t_{nj}^{nl} + \frac{1}{3} \delta_i^l t_{nj}^{nk}. \quad (2.13)$$

t , being the product of two symmetric rank 2 tensors has dimension 36, while its trace is of dimension 9. Therefore, the first two terms taken together have dimension 27.

The trace above can be contracted further to yield the irreducible representations, as follows.

$$\underbrace{t_{ij}^{kl} - \frac{1}{3} \delta_i^k t_{nj}^{nl}}_{27} + \underbrace{\frac{1}{3} \delta_i^k (t_{nj}^{nl} - \frac{1}{3} \delta_j^l t_{nm}^{nm})}_{8} + \underbrace{\frac{1}{9} \delta_j^l \delta_i^k t_{nm}^{nm}}_1. \quad (2.14)$$

Using the above decomposition and the symmetry of t , we can write the second color singlet that appears in the diagram 2.5 as

$$v_{nm} u^{nm} + v_{mn} u^{nm} = (\delta_k^i \delta_l^j + \delta_k^j \delta_l^i) \overline{a_i} \overline{b_j} c^k d^l. \quad (2.15)$$

Normalization of this operator follows in the same manner as the normalization of the Color singlet in 2.11. However, in this case, the $\overline{a_i} \overline{b_i} c^i d^i$ terms add rather than cancel and therefore the sum of the square of the coefficients is $2^2 + 2^2 + 2^2 + 12 = 24$. Thus the normalization factor is $\frac{1}{\sqrt{24}}$.

The above calculations can be checked by performing the tensor product in a different order:

$$\overline{\mathbf{3}} \otimes \mathbf{3} \otimes \overline{\mathbf{3}} \otimes \mathbf{3} = [\mathbf{1} \oplus \mathbf{8}] \otimes [\mathbf{1} \oplus \mathbf{8}]. \quad (2.16)$$

Each $\mathbf{1} \oplus \mathbf{8}$ has the form 2.8 with w_m^n equal to $\overline{a_m} c^n$ or $\overline{b_m} d^n$. By multiplying these singlet representations together and using the equivalent combination of

$\bar{a}$ with d and $\bar{b}$ with c , we get the symmetric color singlet representation given in 2.8.

In order to get the other singlet representation, we must decompose the $\mathbf{8} \otimes \mathbf{8}$. This is done using Young Tableaux in [10] Chapter 12, where we find irreducible representations that give us the same results as diagrams 2.2 to 2.5. In this decomposition, the tensor product

$$t_{ij}^{kl} := \bar{a}_i c^k \bar{b}_j d^l \quad (2.17)$$

must be symmetrized, which gives rise to

$$t_{[ij]}^{[kl]} := \epsilon^{nij} \epsilon_{mkl} \bar{a}_i c^k \bar{b}_j d^l. \quad (2.18)$$

By removing the trace from this tensor, we are left with the anti-symmetric color singlet given in 2.9.

In summary, the color singlet representations of anti-quarks $\bar{a}$ and $\bar{b}$ and quarks c and d can be written as

$$\Delta_{kl}^{ij} \bar{a}_i \bar{b}_j c^k d^l \quad (2.19)$$

with

$$\Delta_{kl}^{ij} = \frac{1}{\sqrt{12}} [\delta_k^i \delta_l^j - \delta_k^j \delta_l^i] \quad (2.20)$$

and

$$\Delta_{kl}^{ij} = \frac{1}{\sqrt{24}} [\delta_k^i \delta_l^j + \delta_k^j \delta_l^i]. \quad (2.21)$$

2.2 Spin SU(2) Irreducible Representations

2.2.1 SU(2) tensor product decomposition

Next, we determine the form of irreducible representations of the tensor product of the 2-component spinors of the 2 quarks and 2 anti-quarks, which can be written as

$$\begin{aligned} \mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} &= [\mathbf{3} \oplus \mathbf{1}] \otimes [\mathbf{3} \oplus \mathbf{1}] \\ &= [\mathbf{3} \otimes \mathbf{3}] \oplus [\mathbf{3} \otimes \mathbf{1}] \oplus [\mathbf{3} \otimes \mathbf{1}] \oplus [\mathbf{1} \otimes \mathbf{1}] \\ &= \mathbf{5} \oplus \mathbf{3} \oplus \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{1}. \end{aligned} \quad (2.22)$$

To determine the specific form of these irreducible representations, we assign

spinor indices $\alpha, \beta, \gamma, \delta = 1, 2$ to each of the quarks and antiquarks. We note that irreducible representations are described by symmetric tensors as, otherwise, a non-zero tensor would result from a contraction with the two dimensional Levi-Civita tensor, $\epsilon_{\alpha\beta}$, resulting in a reduced representation. A symmetric tensor with n indices represents an $(\mathbf{n} + \mathbf{1})$, as the number of indices assuming a value of 1, say, can vary between 0 and n .

Starting with the tensor product of the two anti-quarks and defining $v^{\alpha\beta} := \overline{a^\alpha b^\beta}$ (ignoring color indices), we find

$$v^{\alpha\beta} = v^{(\alpha\beta)} + \frac{1}{2}\epsilon^{\alpha\beta}\epsilon_{\mu\nu}v^{\mu\nu}. \quad (2.23)$$

Here, the symmetric part of the tensor product corresponds to the $\mathbf{3}$, while the fully contracted antisymmetric part corresponds to the $\mathbf{1}$.

We can similarly decompose the tensor product of the two quarks, $u^{\gamma\delta} := c^\gamma d^\delta$, and taking the tensor product with the two anti-quarks results in representations $\mathbf{1}$, $\mathbf{3}$ and $\mathbf{3}$, together with $v^{(\alpha\beta)}u^{(\gamma\delta)}$. This last term can be further decomposed to yield the completely symmetric $v^{(\alpha\beta)}u^{\gamma\delta}$ (a $\mathbf{5}$), the completely anti-symmetric $\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}v^{\alpha\beta}u^{\gamma\delta}$ (a $\mathbf{1}$) and the final $\mathbf{3}$ given by various linear combinations of the form $\epsilon_{\mu\nu}v^{(\mu\rho}u^{\nu\sigma)}$. Thus, we can determine the tensor form of the irreducible representations from the basic properties of two dimensional tensor products.

2.2.2 Weyl spinor decomposition

When considering color $SU(3)$ irreducible representations, we used generic tensor methods and ignored the properties of the spinors fields themselves. In this section, we specifically consider 2 component Weyl spinors and follow the notation and techniques outlined in [11], [12] Section 2 and Appendices B and G, as well as [13], Chapters 33-36.

Weyl spinors are defined by using the fact that the Lie algebra of the Lorentz group, $SO(3,1)$, decomposes into two $SU(2)$ Lie algebras that are related by complex conjugation. The representations of the Lorentz group then take the form $(2n+1, 2m+1)$, with n, m taking half-integer values. A two-component spinor field in the $(2,1)$ representation is referred to as a left-handed Weyl spinor and is denoted ψ_α , whereas a right-handed spinor fields in the $(1,2)$ representation and is given a dotted index: $\psi_\alpha^\dagger := (\psi_\alpha)^\dagger$. The fundamental or metric spinor, $\epsilon^{\alpha\beta}$, can be used to raise and lower spinor indices and is defined

such that $1 = \epsilon^{12} = -\epsilon^{21} = -\epsilon_{12} = \epsilon_{21}$.

To construct anti-fermions in this formalism, we define fields that are related by $U(1)$ symmetry, as follows:

$$\begin{aligned}\xi &= \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2) \\ \eta &= \frac{1}{\sqrt{2}}(\psi_1 - i\psi_2),\end{aligned}\tag{2.24}$$

where ψ_1 and ψ_2 are left-handed spinor fields. We then define the unitary charge conjugation operator C such that

$$\begin{aligned}C^{-1}\xi C &= \eta \\ C^{-1}\eta C &= \xi.\end{aligned}\tag{2.25}$$

Thus, η is the anti-particle of ξ and we choose ξ to represent a heavy quark and η a heavy anti-quark of the same flavor. We also define a light quark, χ , and a light anti-quark, ζ , in the same manner.

Next, we note that the tensor product of a fermion and anti-fermion can take the form

$$v_{\alpha\dot{\alpha}} = \frac{1}{2}v^\mu\sigma_{\mu\alpha\dot{\alpha}},\tag{2.26}$$

where $\sigma^\mu = (I, \vec{\sigma})$ and v^μ is a Lorentz 4-vector that can be written as

$$\xi^\alpha\sigma_{\alpha\dot{\alpha}}^\mu\eta^{\dagger\dot{\alpha}}.\tag{2.27}$$

The relationship 2.26 is posited in [11] as a connection between Hermitian rank 2 spinors, which are determined by 4 real numbers, and space-time vectors, where the quantities with both space-time and spinor indices, $\sigma_{\mu\alpha\dot{\alpha}}$, are arbitrary. By equating the space-time invariant form, $v^\mu v_\mu$, with the spinor invariant form, $v^{\alpha\dot{\alpha}}v_{\alpha\dot{\alpha}}$, and using 2.26, we can write

$$g_{\mu\nu} = \sigma^{\mu\alpha\dot{\alpha}}\sigma_{\alpha\dot{\alpha}}^\nu,\tag{2.28}$$

which allows us to solve for v^μ . It is only later shown that the anti-symmetry of the fundamental spinor, $\epsilon^{\mu\nu}$, results in $\sigma_{\mu\alpha\dot{\alpha}}$ taking the form of the Pauli matrices.

Here we see that that the tensor product decomposes into a singlet consisting of $\xi^\alpha\delta_{\alpha\dot{\alpha}}\eta^{\dagger\dot{\alpha}}$, for $\mu = 0$, corresponding to the **1** representation of the previous

section, and $\xi^\alpha \sigma_{\alpha\dot{\alpha}}^i \eta^{\dagger\dot{\alpha}}$, for $\mu = i = 1, 2, 3$, corresponding to the **3**.

The full tensor product for the two quarks and two anti-quarks can then be obtained from

$$\xi^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \eta^{\dagger\dot{\alpha}} \chi^\beta \sigma_{\beta\dot{\beta}}^\nu \zeta^{\dagger\dot{\beta}} \quad (2.29)$$

using the following Fierz identity (with $g^{\mu\nu} = \text{diag}\{-1, 1, 1, 1\}$ and $\sigma_\mu = (-I, \vec{\sigma})$)

$$\sigma_{\alpha\dot{\alpha}}^\mu \sigma_{\beta\dot{\beta}}^\nu = \frac{1}{2} \left[\underbrace{\sigma_{\alpha\dot{\beta}}^\mu \sigma_{\beta\dot{\alpha}}^\nu + \sigma_{\alpha\dot{\beta}}^\nu \sigma_{\beta\dot{\alpha}}^\mu}_{(a)} - \underbrace{g^{\mu\nu} \sigma_{\alpha\dot{\beta}}^\lambda \sigma_{\lambda\beta\dot{\alpha}}}_{(b)} + \underbrace{i\epsilon^{\mu\nu\rho\kappa} \sigma_{\rho\alpha\dot{\beta}} \sigma_{\kappa\beta\dot{\alpha}}}_{(c)} \right] \quad (2.30)$$

to yield

$$\begin{aligned} \xi^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \eta^{\dagger\dot{\alpha}} \chi^\beta \sigma_{\beta\dot{\beta}}^\nu \zeta^{\dagger\dot{\beta}} &= \frac{1}{2} [\xi^\alpha \sigma_{\alpha\dot{\beta}}^\mu \zeta^{\dagger\dot{\beta}} \chi^\beta \sigma_{\beta\dot{\alpha}}^\nu \eta^{\dagger\dot{\alpha}} \\ &\quad + \xi^\alpha \sigma_{\alpha\dot{\beta}}^\nu \zeta^{\dagger\dot{\beta}} \chi^\beta \sigma_{\beta\dot{\alpha}}^\mu \eta^{\dagger\dot{\alpha}} \\ &\quad - g^{\mu\nu} \xi^\alpha \sigma_{\alpha\dot{\beta}}^\lambda \zeta^{\dagger\dot{\beta}} \chi^\beta \sigma_{\lambda\beta\dot{\alpha}} \eta^{\dagger\dot{\alpha}} \\ &\quad + i\epsilon^{\mu\nu\rho\kappa} \xi^\alpha \sigma_{\rho\alpha\dot{\beta}} \zeta^{\dagger\dot{\beta}} \chi^\beta \sigma_{\kappa\beta\dot{\alpha}} \eta^{\dagger\dot{\alpha}}] \end{aligned} \quad (2.31)$$

From this expression we can determine equivalent forms of all irreducible spin representations derived in the previous section, as follows:

- $\mu = \nu = 0$: (a) and (b) combined gives the fully contracted $-\sigma_{\alpha\dot{\beta}}^\lambda \bar{\sigma}_{\lambda\beta\dot{\alpha}}$ (with $\bar{\sigma}^\mu = (I, -\vec{\sigma})$) corresponding to a **1**, while (c) is 0;
- $\mu = 0, \nu = i$: (b) is 0 and (a) and (c) combine to give

$$\delta_{\alpha\dot{\beta}} \sigma_{\beta\dot{\alpha}}^i + \sigma_{\alpha\dot{\beta}}^i \delta_{\beta\dot{\alpha}} + i\epsilon^{0i\rho\kappa} \sigma_{\rho\alpha\dot{\beta}} \sigma_{\kappa\beta\dot{\alpha}}, \quad (2.32)$$

which, with a single index $i = 1, 2, 3$, corresponds to a **3**;

- $\mu = j, \nu = 0$:

$$\delta_{\alpha\dot{\beta}} \sigma_{\beta\dot{\alpha}}^i + \sigma_{\alpha\dot{\beta}}^i \delta_{\beta\dot{\alpha}} - i\epsilon^{0i\rho\kappa} \sigma_{\rho\alpha\dot{\beta}} \sigma_{\kappa\beta\dot{\alpha}} \quad (2.33)$$

corresponds to another **3**;

- $\mu = i$ and $\nu = j$ with $i, j = 1, 2, 3$: (c) takes the form

$$i\epsilon^{ij\rho\kappa} \sigma_{\rho\alpha\dot{\beta}} \sigma_{\kappa\beta\dot{\alpha}} \quad (2.34)$$

which is a 3 dimensional, rank 2, anti-symmetric tensor corresponding to

a **3**; (a) and (b) together are

$$\sigma_{\alpha\dot{\beta}}^i \sigma_{\beta\dot{\alpha}}^j + \sigma_{\alpha\dot{\beta}}^j \sigma_{\beta\dot{\alpha}}^i - \sigma_{\alpha\dot{\beta}}^k \sigma_{k\beta\dot{\alpha}}, \quad (2.35)$$

$k = 1, 2, 3$, which forms a 3 dimensional, rank 2, traceless, symmetric tensor, corresponding to a **5**; the remaining term from (b), $\delta_{\alpha\dot{\beta}} \delta_{\beta\dot{\alpha}}$, corresponds to the final **1** representation.

The forms of the operators corresponding to each spin representation are summarized in the table at the end of the section (where, for convenience, the spinor indices have been dropped).

Generalization to 4 component spinors

The results of the previous section can be generalized to 4 component Dirac spinors and Dirac gamma matrices. This is conveniently done in the chiral or Weyl basis where

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma_{\alpha\dot{\beta}}^\mu \\ \bar{\sigma}_{\dot{\alpha}\beta}^\mu & 0 \end{pmatrix} \text{ and } \gamma_5 = i\gamma^1\gamma^2\gamma^3\gamma^4 = \begin{pmatrix} -\delta_\alpha^\beta & 0 \\ 0 & \delta_{\dot{\alpha}}^{\dot{\beta}} \end{pmatrix}.$$

A Dirac spinor field is made up of two 2 component spinor fields related by U(1) symmetry as before:

$$\Psi = \begin{pmatrix} \xi_\alpha \\ \eta^{\dagger\dot{\alpha}} \end{pmatrix}. \quad (2.36)$$

By defining the projection operator, $P_R := \frac{1}{2}(1 + \gamma_5)$, we can write a 4 component right-handed spinor as

$$\Psi_R = P_R \Psi = \begin{pmatrix} 0 \\ \eta^{\dagger\dot{\alpha}} \end{pmatrix}. \quad (2.37)$$

This allows us to generalize a typical 2 component bilinear form as

$$\chi \sigma^\mu \eta^\dagger = \bar{\Psi} \gamma^\mu P_R \Psi. \quad (2.38)$$

Note that in the term on the right hand side above, the chiral nature of γ^μ means that the lower two components of $\gamma^\mu P_R \Psi$ are 0. This in turn means that only the upper two components of $\bar{\Psi}$ are used in the bilinear, obviating the need for a left handed projection operator.

By repeatedly using projection operators in this way we could generalize the bilinears of the previous section to 4 components, with either the upper two or lower two components set to 0. However, for relativistic fermions with mass, all 4 components of the Dirac spinor are required. In this case, we use 4 component Fierz identities to obtain the full decomposition of 4 component Dirac spinors, in the following way.

We begin by defining Dirac fermions, Ψ_c and Ψ_d , and following the rationale of the 2 component case, write their tensor product as the diquark operator $\Psi_c^T \gamma^\nu \Psi_d$, there the superscript T denotes the transpose of the spinor indices (without complex conjugation). The full tetraquark operator can then be written as the product of the diquark operator with an anti-diquark operator:

$$[\bar{\Psi}_a^T \gamma^\mu \bar{\Psi}_b] [\Psi_c^T \gamma^\nu \Psi_d]. \quad (2.39)$$

This tensor product can be decomposed as in the 2 component case using a Fierz identity, as given in [14]:

$$\begin{aligned} 4[\gamma^\mu \otimes \gamma^\nu] &= \underbrace{\gamma^\mu \cdot \gamma^\nu + \gamma^\nu \cdot \gamma^\mu - g^{\mu\nu} \gamma^\lambda \cdot \gamma_\lambda}_{(a)} \\ &+ \underbrace{\gamma^\mu \gamma_5 \cdot \gamma^\nu \gamma_5 + \gamma^\nu \gamma_5 \cdot \gamma^\mu \gamma_5 - g^{\mu\nu} \gamma^\lambda \gamma_5 \cdot \gamma_\lambda \gamma_5}_{(a)} \\ &+ \underbrace{[(iI \cdot \sigma^{\mu\nu} + \gamma_5 \cdot \tilde{\sigma}^{\mu\nu} + i\epsilon^{\mu\nu\lambda\rho} \gamma_\lambda \cdot \gamma_\rho \gamma_5) - \leftrightarrow]}_{(b)} \\ &+ \underbrace{\frac{1}{2} g^{\mu\nu} \sigma^{\lambda\rho} \cdot \sigma_{\lambda\rho} - \sigma^{\mu\lambda} \cdot \sigma^\nu{}_\lambda - \sigma^{\nu\lambda} \cdot \sigma^\mu{}_\lambda}_{(c)} \\ &+ \underbrace{g^{\mu\nu} [I \cdot I - \gamma_5 \cdot \gamma_5]}_{(d)} \end{aligned} \quad (2.40)$$

Here, $\tilde{\sigma}^{\mu\nu} = -i\sigma^{\mu\nu}\gamma_5$ and the we work with the bases

$$\Gamma^A = 1; \gamma_5; \gamma^\mu; \gamma^\mu \gamma_5; \sigma^{\mu\nu} \quad (2.41)$$

$$\Gamma_A = 1; \gamma_5; \gamma_\mu; \gamma_5 \gamma_\mu; \sigma_{\mu\nu} \quad (2.42)$$

(with $\mu < \nu$), together with the orthonormality condition $tr(\Gamma^A \Gamma_B) = 4\delta_B^A$. The SU(2) irreducible combinations can then be determined as before.

- $\mu = \nu = 0$: (b) and (c) are identically 0 and (a) and (d) reduce to the

fully contracted $I \cdot I - \gamma_5 \cdot \gamma_5 + \gamma^\lambda \cdot \gamma^\lambda + \gamma^\lambda \gamma_5 \cdot \gamma^\lambda \gamma_5$ (summation implied);

- $\mu = 0, \nu = i$: (d) is 0 and the rest combine to give

$$\begin{aligned} & \gamma^0 \cdot \gamma^i + \gamma^i \cdot \gamma^0 + \gamma^0 \gamma_5 \cdot \gamma^i \gamma_5 + \gamma^i \gamma_5 \cdot \gamma^0 \gamma_5 \\ & + [i(I \cdot \sigma^{0i} + \gamma_5 \cdot \sigma^{0i} \gamma_5 + \epsilon^{0i\lambda\rho} \gamma_\lambda \cdot \gamma_\rho \gamma_5) - \leftrightarrow] \\ & - \sigma^{0\lambda} \cdot \sigma^i{}_\lambda - \sigma^{i\lambda} \cdot \sigma^0{}_\lambda \end{aligned} \quad (2.43)$$

which, with a single index $i = 1, 2, 3$, corresponds to a **3**;

- $\mu = i, \nu = 0$: corresponds to another **3**;
- $\mu = i$ and $\nu = j$ with $i, j = 1, 2, 3$: (a) takes the form

$$\begin{aligned} & \gamma^i \cdot \gamma^j + \gamma^j \cdot \gamma^i - g^{ij} \gamma^k \cdot \gamma_k + \\ & \gamma^i \gamma_5 \cdot \gamma^j \gamma_5 + \gamma^j \gamma_5 \cdot \gamma^i \gamma_5 - g^{ij} \gamma^k \gamma_5 \cdot \gamma_k \gamma_5 \end{aligned} \quad (2.44)$$

($k = 1, 2, 3$), which forms a 3 dimensional, rank 2, traceless, symmetric tensor, corresponding to a **5**;

$$\begin{aligned} & [i(I \cdot \sigma^{ij} + \gamma_5 \cdot \sigma^{ij} \gamma_5 + \epsilon^{ij\lambda\rho} \gamma_\lambda \cdot \gamma_\rho \gamma_5) - \leftrightarrow] \\ & \frac{1}{2} g^{ij} \sigma^{\lambda\rho} \cdot \sigma_{\lambda\rho} - \sigma^{i\lambda} \cdot \sigma^j{}_\lambda - \sigma^{j\lambda} \cdot \sigma^i{}_\lambda \end{aligned} \quad (2.45)$$

corresponds to a **3**, while the remaining terms form the final **1** representation:

$$I \cdot I - \gamma_5 \cdot \gamma_5 - \gamma^0 \cdot \gamma^0 - \gamma^0 \gamma_5 \cdot \gamma^0 \gamma_5 \quad (2.46)$$

As in the 2 component case, we summarize the forms of the operators corresponding to each spin representation in the table below.

J	Weyl Spinors	Dirac Spinors
1	$[\xi I \zeta^\dagger][\chi I \eta^\dagger]$	$ \begin{aligned} & [\bar{\Psi}_a^T I \Psi_c][\bar{\Psi}_b^T I \Psi_d] - [\bar{\Psi}_a^T \gamma_5 \Psi_c][\bar{\Psi}_b^T \gamma_5 \Psi_d] - \\ & [\bar{\Psi}_a^T \gamma^0 \Psi_c][\bar{\Psi}_b^T \gamma^0 \Psi_d] - \\ & [\bar{\Psi}_a^T \gamma^0 \gamma_5 \Psi_c][\bar{\Psi}_b^T \gamma^0 \gamma_5 \Psi_d] \end{aligned} $
1	$[\xi \sigma^\lambda \zeta^\dagger][\chi \bar{\sigma}_\lambda \eta^\dagger]$	$ \begin{aligned} & [\bar{\Psi}_a^T \gamma^\lambda \Psi_c][\bar{\Psi}_b^T \gamma_\lambda \Psi_d] + \\ & [\bar{\Psi}_a^T \gamma^\lambda \gamma_5 \Psi_c][\bar{\Psi}_b^T \gamma_\lambda \gamma_5 \Psi_d] \end{aligned} $
3	$[\xi I \zeta^\dagger][\chi \sigma^i \eta^\dagger] + [\xi \sigma^i \zeta^\dagger][\chi I \eta^\dagger] +$ $i \epsilon^{0i\rho\kappa} [\xi \sigma_\rho \zeta^\dagger][\chi \sigma_\kappa \eta^\dagger]$	$ \begin{aligned} & [\bar{\Psi}_a^T I \Psi_c][\bar{\Psi}_b^T \sigma^{0i} \Psi_d] + \\ & [\bar{\Psi}_a^T \gamma_5 \Psi_c][\bar{\Psi}_b^T \sigma^{0i} \gamma_5 \Psi_d] + \dots \end{aligned} $
3	$[\xi I \zeta^\dagger][\chi \sigma^i \eta^\dagger] + [\xi \sigma^i \zeta^\dagger][\chi I \eta^\dagger] -$ $i \epsilon^{0i\rho\kappa} [\xi \sigma_\rho \zeta^\dagger][\chi \sigma_\kappa \eta^\dagger]$	$ \begin{aligned} & [\bar{\Psi}_a^T I \Psi_c][\bar{\Psi}_b^T \sigma^{i0} \Psi_d] + \\ & [\bar{\Psi}_a^T \gamma_5 \Psi_c][\bar{\Psi}_b^T \sigma^{i0} \gamma_5 \Psi_d] + \dots \end{aligned} $
3	$i \epsilon^{ij\rho\kappa} [\xi \sigma_\rho \zeta^\dagger][\chi \sigma_\kappa \eta^\dagger]$	$ \begin{aligned} & [\bar{\Psi}_a^T I \Psi_c][\bar{\Psi}_b^T \sigma^{ij} \Psi_d] + \\ & [\bar{\Psi}_a^T \gamma_5 \Psi_c][\bar{\Psi}_b^T \sigma^{ij} \gamma_5 \Psi_d] + \dots \end{aligned} $
5	$[\xi \sigma^\mu \zeta^\dagger][\chi \sigma^\nu \eta^\dagger] + [\xi \sigma^\nu \zeta^\dagger][\chi \sigma^\mu \eta^\dagger] -$ $[\xi \sigma^k \zeta^\dagger][\chi \sigma_k \eta^\dagger]$	$ \begin{aligned} & [\bar{\Psi}_a^T \gamma^i \Psi_c][\bar{\Psi}_b^T \gamma^j \Psi_d] + [\bar{\Psi}_a^T \gamma^j \Psi_c][\bar{\Psi}_b^T \gamma^i \Psi_d] + \\ & [\bar{\Psi}_a^T \gamma^i \gamma_5 \Psi_c][\bar{\Psi}_b^T \gamma^j \gamma_5 \Psi_d] + \\ & [\bar{\Psi}_a^T \gamma^j \gamma_5 \Psi_c][\bar{\Psi}_b^T \gamma^i \gamma_5 \Psi_d] - \\ & g^{ij} [\bar{\Psi}_a^T \gamma^k \Psi_c][\bar{\Psi}_b^T \gamma_k \Psi_d] - \\ & g^{ij} [\bar{\Psi}_a^T \gamma^k \gamma_5 \Psi_c][\bar{\Psi}_b^T \gamma_k \gamma_5 \Psi_d] \end{aligned} $

2.3 Calculation of irreducible representation spin indices

In order to utilise the spin irreducible representations of tetraquark operators defined in the previous section, it is necessary to determine the non-zero values of various combinations of spinor linear operators $\Gamma = \{I, \gamma_5, \gamma^\mu \dots\}$. An algorithm was implemented in the statistical software package, R^4 , which takes two 4-dimensional matrices and determines which combinations of elementwise products are non-zero. These non-zero products and the contributing matrix indices are then outputted in table format.

The algorithm allows the use of either the Euclidean or Minkowski metric, with either the Chiral or Dirac basis.⁵ Results can be obtained for a single tetraquark quadrilinear as well as a combination of tetraquark creation and annihilation operators. Sample results are given in Appendix A.1 and the code

⁴www.r-project.org

⁵The USQCD convention is to use the Euclidean metric with the Chiral basis.

used is given in A.2.

Given the large number of matrix element products (a tetraquark creation and annihilation operator involves $16^4 = 65,536$ linear combinations), it is useful to check the results using symmetry transformations of the spinor operators Γ .⁶ This was carried out by calculating, in the vector and spinor representations, the Lorentz group elements associated with an arbitrary 3-dimensional rotation:

$$\Lambda_V = \exp(i\theta_{ij}V^{ij}), \Lambda_S = \exp(i\theta_{ij}S^{ij}), \quad (2.47)$$

where θ_{ij} , $i \neq j$ represents the three arbitrary angles and V^{ij} and S^{ij} are the generators of the rotations in the vector and spinor representations, respectively. The matrix exponentials in the above definitions were implemented using a Taylor series expansion and it was found that at least 15 terms were required in order to obtain sufficient convergence for the tests to complete accurately.

The spinor operator products were then checked using the relationship

$$(\Lambda_S^{-1})_{\alpha\gamma}\Gamma_{\gamma\delta}^{\mu}(\Lambda_S)_{\delta\beta} = (\Lambda_V)^{\mu\nu}\Gamma_{\alpha\beta}^{\nu}, \quad (2.48)$$

which relates Lorentz transformations in the spinor representation (acting on spinor indices, $\alpha, \beta \dots$) to the Lorentz transformation in the vector representation. This test was carried out by rotating Γ in the spinor representation using the left hand side of equation 2.48, performing the inverse vector transformation and then checking that the result is equal to the original matrix. This was carried out with success for $\Gamma = \gamma^{\mu}, \gamma^{\mu}\gamma_5$ and various product combinations of γ^{μ} matrices.

We can also check the spinor operator products with parity transformations using the relationship that is equivalent to 2.48 for parity:

$$P_{\alpha\gamma}^{-1}\Gamma_{\gamma\delta}^{\mu}P_{\delta\beta} = \mathcal{P}^{\mu\nu}\Gamma_{\alpha\beta}^{\nu}. \quad (2.49)$$

Here, $P^{-1} = P = \gamma_0$ operates on spinor indices and $\mathcal{P} : (t, \vec{x}) \rightarrow (t, -\vec{x})$ operates on vector indices. As before, the two sides of this equation can be checked for equivalence with various operator combinations.

⁶In this sub-section, the notation and equations of [15] are used.

3 Correlation functions on the lattice

3.1 Meson two-point correlation functions

In this section, we review the calculation of meson two-point correlation functions in Lattice QCD. This will be then be generalised to the tetraquark case in the next section. Here, we mostly following the notation given in [1].

A two-point correlation function is an expectation of the form $\langle \hat{\mathcal{O}}(t) \hat{\mathcal{O}}^\dagger(0) \rangle$, where quantum field operators $\hat{\mathcal{O}}^\dagger(0)$ and $\hat{\mathcal{O}}(t)$ represent particle creation at time 0 and annihilation at time t , respectively. By inserting a complete set of energy eigenstates, this expectation can be expanded to yield

$$C(t) = \langle \hat{\mathcal{O}}(t) \hat{\mathcal{O}}^\dagger(0) \rangle = \sum_n \langle 0 | \hat{\mathcal{O}} | n \rangle \langle n | \hat{\mathcal{O}}^\dagger | 0 \rangle e^{-E_n t}. \quad (3.1)$$

Where energy differences between the ground state and first excited state are large, only the leading term in the summation dominates and the expected asymptotic behaviour of the correlation function is

$$C(t) \sim A^{(0)} e^{-E_0 t} \quad (3.2)$$

for $t \rightarrow \infty$. Therefore, by computing the path integral $C(t)$ we can determine the ground state energy of the particle represented by the operator $\hat{\mathcal{O}}$.

In the lattice field path integral, a meson operator is represented by an interpolator, or a Grassman-valued Hilbert space functional with the required quantum numbers,

$$\mathcal{O}(x) = [\bar{\psi}^{(f_1)} \Gamma \psi^{(f_2)}](x) \quad (3.3)$$

at lattice site x where f_1 and f_2 are quark flavor identifiers. The path integral can then be written in momentum space (with 0 momentum) as

$$C(t) = \frac{1}{Z} \int \mathcal{D}[\psi, \bar{\psi}] \mathcal{D}[U] e^{-S[U, \psi, \bar{\psi}]} \sum_{\vec{y}} [\bar{\psi}_a^\alpha \Gamma_{\alpha\beta} \psi_a^\beta](\vec{y}, t) \sum_{\vec{x}} [\bar{\psi}_b^\gamma \Gamma_{\gamma\delta} \psi_b^\delta](\vec{x}, 0) \quad (3.4)$$

where U is the functional representing the gauge field, S is the QCD action and where we have assumed the same flavour for all quarks and have explicitly written out the spin ($\alpha, \beta, \dots$) and color (a, b) indices.

The fermionic part of the path integral can be performed by Gaussian integration of the Grassman functionals. We first separate the action into a pure

gauge field part and a fermionic part and add source terms θ and $\bar{\theta}$:

$$-S[U, \psi, \bar{\psi}] = \bar{\psi} M[U] \psi + \bar{\theta} \psi + \theta \bar{\psi} \quad (3.5)$$

Here, M is a matrix representation of the gauge dependent Dirac operator. To perform the Gaussian integration, we complete the square using M^{-1} and use Wick contractions of the quark fields to arrive at:

$$\begin{aligned} C(t) = & \frac{1}{Z} \int \mathcal{D}[U] e^{-S[U]} \det M[U] \times \\ & \left[\sum_{\vec{x}, \vec{y}} \Gamma_{\alpha\beta} M^{-1}(\vec{y}, t | \vec{x}, 0)_{ba}^{\gamma\beta} M^{-1}(\vec{x}, 0 | \vec{y}, t)_{ab}^{\alpha\delta} \Gamma_{\gamma\delta} - \right. \\ & \left. \sum_{\vec{y}} \Gamma_{\alpha\beta} M^{-1}(\vec{y}, t | \vec{y}, t)_{aa}^{\alpha\beta} \sum_{\vec{x}} M^{-1}(\vec{x}, 0 | \vec{x}, 0)_{bb}^{\gamma\delta} \Gamma_{\gamma\delta} \right] \end{aligned} \quad (3.6)$$

Here, $M^{-1}(\vec{y}, t | \vec{x}, 0)$ is the lattice representation of the Green's function of the Dirac operator and propagates a quark from lattice site $(\vec{x}, 0)$ to $(\vec{y}, t)$. Note that the second term in 3.6 is a consequence of the bilinears containing the same quark flavor and represents the disconnected Feynman diagram involving $\bar{\psi}\psi$ annihilation at times 0 and t . Using two different quark flavors would result in only one term, but there would be two distinct Dirac propagators and an extra $\det M$ term.

The fermionic part of the path integral is now complete, but the remaining gauge field path integral must be computed using Monte Carlo integration. This is done by computing the integrand (the bracketed terms in 3.6) for a sample of gauge field configurations that have been constructed with importance sampling. The contributions for each sample are consequently weighted with the probability measure $\frac{1}{Z} e^{-S[U]} \det M[U]$ when summed up to approximate the integral.

As the spinor and color indices fully contract in 3.6, we can rewrite the correlation function in condensed form, omitting the term associated with the disconnected diagram, as follows:

$$C(t) = \text{Tr} [\Gamma M^{-1}(t, 0) \Gamma M^{-1}(0, t)]. \quad (3.7)$$

3.2 Distillation

As we are primarily interested in measuring the ground state energy of a particle, we wish to construct optimised interpolators with a significant overlap with low

energy modes. This can be done through *smearing*, where a linear operator is first applied to the quark fields to produce spatially extended wave functions at each time slice. A particular form of smearing, known as *distillation*[16] reduces the rank, $M = N_c \times N_x \times N_y \times N_z$ (where N_c is the number of colors and N_x etc. gives the spatial extent of the lattice), of the vector space of the colored fermionic field on a particular time slice.

This can be done by choosing distillation operators $V(t)$ of rank $N \ll M$ and defining the smearing operator as an $M \times M$ matrix, $\square(t) = V(t)V^\dagger(t)$. Here, V is an $M \times N$ matrix whose columns are made up of N eigenvectors of the gauge covariant Laplacian,

$$\nabla^2(U; t)\vec{v}(U; t) = \lambda(U; t)\vec{v}(U; t), \quad (3.8)$$

and therefore $\square$ can be rewritten as

$$\square_{xy}(t) = \sum_{k=1}^N \vec{v}_x^{(k)}(t) \vec{v}_x^{(k)\dagger}(t), \quad (3.9)$$

where $\vec{v}_x^{(k)}(t)$ is the k^{th} eigenvector of ∇^2 evaluated at lattice site (x, t) .

After inserting the smearing operator, the meson interpolator becomes

$$\left[\bar{\psi}_x^\alpha \square_{xy} \Gamma_{\alpha\beta} \square_{yz} \psi_z^\beta \right] (t) = \left[\bar{\psi}_x^\alpha v_x^{(k)} v_y^{\dagger(k)} \Gamma_{\alpha\beta} v_y^{(l)} v_z^{\dagger(l)} \psi_z^\beta \right] (t) \quad (3.10)$$

where, for convenience, we have omitted color indices. After performing the Wick contractions, the first term in the correlation function can then be written as

$$[v_{z_1}^{\dagger(n)} M_{z_1 x_2}^{-1\delta\alpha} v_{x_2}^{(k)}](t, 0) [v_{y_2}^{\dagger(k)} \Gamma_{\alpha\beta} v_{y_2}^{(l)}](t) [v_{z_2}^{\dagger(l)} M_{z_2 x_1}^{-1\beta\gamma} v_{x_1}^{(m)}](0, t) [v_{y_1}^{\dagger(m)} \Gamma_{\gamma\delta} v_{y_1}^{(n)}](0). \quad (3.11)$$

We now define the *perambulator*, $\tau_{\alpha\beta}(0, t) = V^{\dagger(l)}(t) M_{\alpha\beta}^{-1}(t, 0) V(0)$, and $\Phi_{\alpha\beta}(t) = V^\dagger(t) \Gamma_{\alpha\beta}(t) V(t)$ and write the correlation function in simplified form as

$$C(t) = \text{Tr} [\Phi(t) \tau(t, 0) \Phi(0) \tau(0, t)]. \quad (3.12)$$

Note that the perambulator is independent of the meson operators and only depends on the flavour of the quark propagator M^{-1} and the distillation eigenvectors. They can therefore be computed *a priori* and utilised with a wide variety of hadron interpolators.

3.3 Tetraquark two-point correlation functions

We now approach the tetraquark two-point correlation function by generalising the meson function from the previous section. We define a charmed tetraquark interpolator, with down and anti-up quarks and a distillation operator $\square(t)$ as follows:

$$\chi(t) = \Delta^{abcd}[(\bar{c}_u \square_{uv})_a^\alpha \Gamma_{\alpha\beta}(\square_{vw} d_w)_b^\beta (\bar{u}_x \square_{xy})_c^\gamma \hat{\Gamma}_{\gamma\delta}(\square_{yz} c_z)_d^\delta](t). \quad (3.13)$$

Here, $a, b, \dots$ denote color indices, $\alpha, \beta, \dots$ denote spinor indices, $u, v, w, x, \dots$ denote lattice spatial indices and Δ and Γ are color and spinor linear operators defined in Section 2.

The connected part of the a tetraquark two-point correlation function can then be written as

$$\begin{aligned} \langle \chi(t) \chi^\dagger(0) \rangle &= \Delta^{abcd} \Delta^{d'c'b'a'} \times \\ &\quad \left\langle [(\bar{c}_{u_2} \square_{u_2 x_2})_a^\alpha \Gamma_{\alpha\beta}(\square_{x_2 w_2} d_{w_2})_b^\beta (\bar{u}_{v_2} \square_{v_2 y_2})_c^\gamma \hat{\Gamma}_{\gamma\delta}(\square_{y_2 z_2} c_{z_2})_d^\delta](t) \right\rangle \times \\ &\quad \left| [(\bar{c}_{z_1} \square_{z_1 y_1})_{d'}^{\delta'} \hat{\Gamma}_{\delta'\gamma'}(\square_{y_1 v_1} u_{v_1})_{c'}^{\gamma'} (\bar{d}_{w_1} \square_{w_1 x_1})_{b'}^{\beta'} \Gamma_{\beta'\alpha'}(\square_{x_1 u_1} c_{u_1})_{a'}^{\alpha'}](0) \right\rangle \\ &= \Delta^{abcd} \Delta^{d'c'b'a'} \Gamma_{\alpha\beta} \hat{\Gamma}_{\gamma\delta} \hat{\Gamma}_{\delta'\gamma'} \Gamma_{\beta'\alpha'} \times \\ &\quad v_{x_1}^{(i)a'} v_{x_2}^{(j)b} v_{y_1}^{(k')c'} v_{y_2}^{(l)d} \times \\ &\quad \tau_{\mathbf{c}\delta\delta'}^{(l,l')}(t, 0) \tau_{\mathbf{u}\gamma\gamma'}^{(k',k)}(0, t) \tau_{\mathbf{d}\beta\beta'}^{(j,j')}(t, 0) \tau_{\mathbf{c}\alpha\alpha'}^{(i',i)}(0, t) \times \\ &\quad v_{x_2}^{\dagger(i)a} v_{x_1}^{\dagger(j')b'} v_{y_2}^{\dagger(k)c} v_{y_1}^{\dagger(l')d'} \end{aligned} \quad (3.14)$$

where, for example, $v_{x_1}^{(i)}$ is the i^{th} eigenvector of the distillation operator at lattice site x_1 and

$$\tau_{\mathbf{c}\delta\delta'}^{(l,l')}(t, 0) \quad (3.15)$$

is the perambulator with the charm propagator.

4 Lattice QCD correlation function calculations

In this section we describe the Lattice QCD calculation of D meson and tetraquark correlation functions and effective mass plots.

4.1 Setup

The calculation was carried out using configuration data from the Hadron Spectrum Collaboration. Light and Charm quark perambulator files together with meson and tetraquark operator data were used on a $16^3 \times 128$ lattice. Up to 24 distillation space eigenvectors were used with 31 separate configurations.

The generation of the configurations is described in [2]. The gauge action used is given in [2], equation (1), and is Symanzik-improved at tree-level using tadpole-improved co-efficients, with spatial and temporal tadpole factors $u_s = 0.7336$ and $u_t = 1$, respectively. The coupling, g , is specified via $\beta = 2N_c/g^2 = 1.5$, where N_c is the number of colors. An anisotropic clover fermion action was adopted and is specified in [2], equation (2). 2 light flavors and 1 strange flavor were used and the ratio of spatial and temporal scales is given as $\xi = a_s/a_t = 3.5$. The specification of the light and strange quark masses using a range of meson interpolators corresponding to $\bar{l}l$, $s\bar{s}$ and K mesons is also described in [2].

The Charm quark fields were not treated dynamically, or quenched, during the configuration generation. The tuning of the bare Charm quark mass was determined using the ratio of the experimental masses of the η_c meson and the Ω baryon and is described in [3]. The mass of the Ω baryon was calibrated using a result of 0.2951(22) lattice units. With a mass of 1672.45(29) MeV, this yields a lattice scale of $a_t^{-1} = 5667(42)$ MeV.

The Lattice QCD computation of the meson and tetraquark two-point correlation functions is implemented in the *redstar* application, also from the Hadron Spectrum Collaboration. It consists of three components and calculates correlation functions in the following steps:

1. *redstar_gen_graph* accepts an XML file which defines the source and sink operators, calculates the allowed Feynman diagrams and determines the data requirements for the calculations.
2. *hadron_node* accepts a second XML file which defines the number of distillation space vectors and extracts the required data.

Operator	J^P	Eff. Mass	Err.	Exp. Mass
Dneg_pion_2xD0_J0__J0_A1	0^-	1905	51	$D(1870)^\pm$
Dneg_pionxD0_J0__J0_A1	0^-	1905	47	$D(1870)^\pm$
Dneg_rho_2xD0_J0__J1_T1	1^-	2023	60	$D^*(2010)^\pm$
Dneg_rho_2xD2_J0__J1_T1	1^-	2029	64	$D^*(2010)^\pm$
Dbarpos_rho_2xD1_J1__J0_A1	0^+	2483	325	$D_0^*(2400)^0$
Dbarpos_rhoxD1_J1__J0_A1	0^+	2517	449	$D_0^*(2400)^0$
Dbarpos_pionxD1_J1__J1_T1	1^+	2555	471	$D_1(2420)^0$
Dbarpos_pion_2xD1_J1__J1_T1	1^+	2578	591	$D_1(2420)^0$

Table 1: D meson operators used in two-point correlation function calculation with masses and statistical errors given in MeV. Corresponding experimentally observed particle names and masses are also given.

3. *redstar_npt* performs the correlation function calculation using the output of the first two steps.

4.2 D meson calculation

Correlation functions were calculated for a number of D meson interpolators as shown in Table 1, where the *redstar* operators and the corresponding quantum numbers are given. Here, *pion* and *rho* determine the spin content of the meson, while *D0*, *D1* and *D2* specify if a derivative operator of order 0,1, or 2 is applied to the fermion in the bilinear as part of the Γ operator. The combination of these components gives the overall angular momentum as referenced in the final *J* in the operator name. *A1* or *T1* at the end of the operator name specifies, respectively, the trivial or vector irreducible representations of the cubic point group, which is the lattice version of the (continuous) SU(2) spin group.

A linear fit was applied to the log correlation functions for time separations of greater than 9, as shown in Figure 1. For each correlation function, the Effective Mass was calculated as

$$m_{eff}(n_t + \frac{1}{2}) = \log \frac{\langle C(n_t) \rangle}{\langle C(n_t + 1) \rangle} \sim E_0 \quad (4.1)$$

with the average values shown in Table 1. The effective mass plots are shown in Figure 2, where the contributions of excited states can be observed for small values of *t*. A jackknife estimate, using the variance of leave-one-out resamples, was used to calculate the standard errors and we observe that the calculations for the p-wave states shown in the bottom 4 rows of Table 1 and the right hand

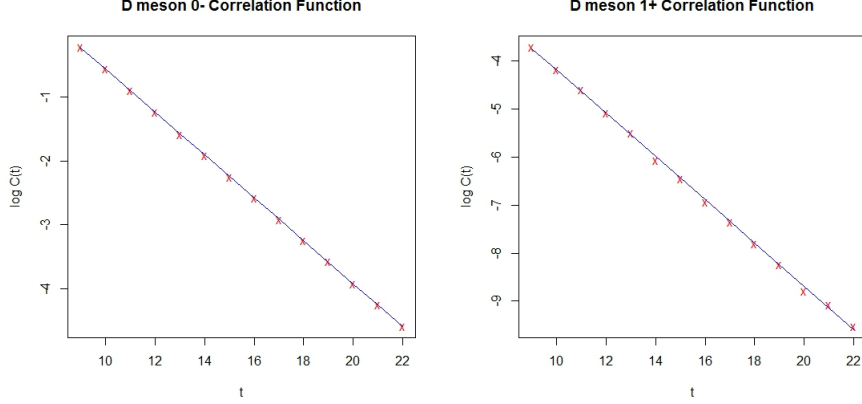


Figure 1: Correlation functions of 0^- and 1^+ D mesons.

panel of Figure 2 exhibit much greater uncertainty than the other s-wave states.

The last column of Table 1 gives the experimentally observed masses for each particle [17] and we note that in all cases, the lattice effective mass is within 1 standard error of the experimental mass.

4.3 Tetraquark calculation

The tetraquark operators utilised by *redstar* are of the following form

tetra2I2S0pM_pionxF3xC3bar_pionbarxF3barxC3__F8_2I2_S0_Gm__J0_A1pM.

The various components of the operator are separated by underscores with the following encoding:

- tetra2I2S0pM specifies double the isospin as 2 ($I_3 = 1$); strangeness or charm content, S=0; parity p for plus or +1; and charge conjugation M for minus or -1.
- The next two sections determine the quantum numbers for a diquark and anti-diquark. For example, $pion \times F3 \times C3bar$ specifies the spin, flavor and color representations of the first diquark.
- Following the diquark specification, F8 specifies the flavor SU(3) representation of the whole tetraquark, so here we have $\mathbf{3} \otimes \mathbf{3} \rightarrow \mathbf{8}$.
- Next, the isospin, strangeness and G-parity of the tetraquark are given, with Gm specifying a G-parity of -1.

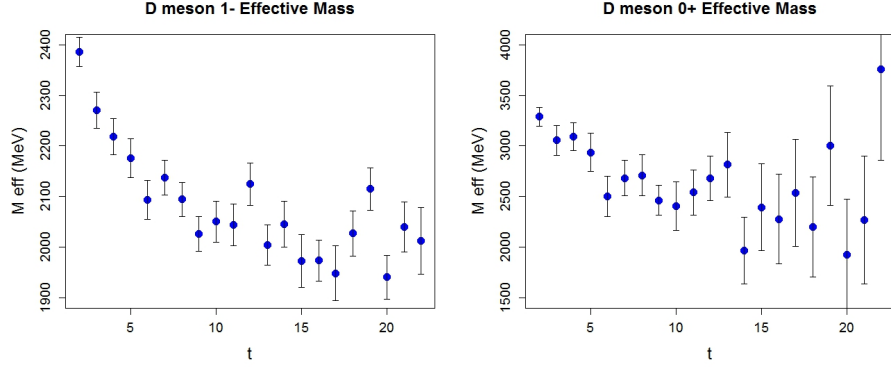


Figure 2: Effective Mass plots of 1^- and 0^+ mesons.

Operator	$I^G J^{PC}$	Eff. Mass	Std. Err.
tetra2I2S0pM pionxF3xC3bar_pionbarxF3barxC3 F8_2I2_S0_Gm__J0_A1pM	$1^- 0^{+-}$	3856	116
tetra2I2S0pP rhoxF3xC3bar_pionbarxF3barxC3 F8_2I2_S0_Gp__J1_T1pP	$1^+ 1^{++}$	3820	113
tetra2I2S0pM rhoxF3xC3bar_pionbarxF3barxC3 F8_2I2_S0_Gm__J1_T1pM	$1^- 1^{+-}$	4072	129

Table 2: Tetraquark operators used in two-point correlation function calculation with effective masses and standard errors in MeV.

- Finally, the total angular momentum projection of the tetraquark is specified, which in this case is $J = 0$.

Applying Wick contractions to the tetraquark correlation function leads to a connected diagram (as explicitly written out in 3.14) and a disconnected diagram with charmonium and pion annihilation for time separations of 0. The contribution to the effective mass of the tetraquark from the disconnected diagram is small and we omit it in order to reduce the amount of data required for the calculation. This is done by utilising an "e-quark" in the *redstar* configuration, which is a dummy quark with the same mass as a charm quark, but with a different flavour. Flavour content of the form $\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}$ ensures that each

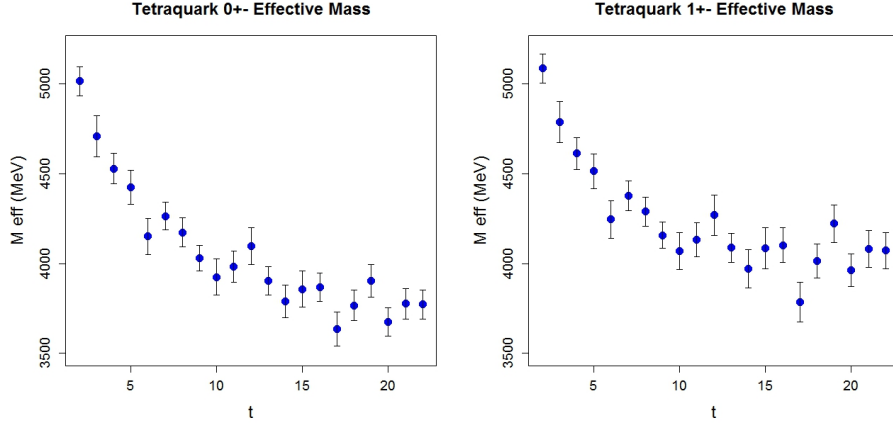


Figure 3: Effective mass plots of 1^-0^{+-} and 1^-1^{+-} tetraquark.

Num. e-vects	Eff. Mass	Std. Err.
4	3811	487
8	3772	245
16	3783	141
24	3820	113

Table 3: Effective mass and standard errors in MeV for 1^{++} tetraquarks by number of distillation space eigenvectors.

(anti-) diquark contains one (anti-) charm and one (anti-) light quark. This works well for tetraquarks with $I_3 = 1$, but for $I_3 = 0$ *redstar* interprets the $\mathbf{3}$ in the diquark as being the result of $\bar{\mathbf{3}} \otimes \bar{\mathbf{3}}$ and permits tetraquarks with quark content of the form $s\bar{l}\bar{s}\bar{l}$. The end result is that without strange propagator data, it does not appear to be possible with the current version of *redstar* to calculate correlation functions for any $I_3 = 0$ tetraquarks.

As before, we calculate the effective masses for the three tetraquarks shown in Table 2. This was again done with 24 distillation space eigenvectors on 31 configurations, with the computation time for each configuration about 50 minutes. Here we see that the effective masses for the first two tetraquarks are within 1 standard error of the $D^{*0}\bar{D}^0$ mass threshold of 3872. The effective mass plots for the 1^-0^{+-} and 1^-1^{+-} tetraquarks are shown in Figure 3.

The stability of the effective mass results for varying numbers of distillation

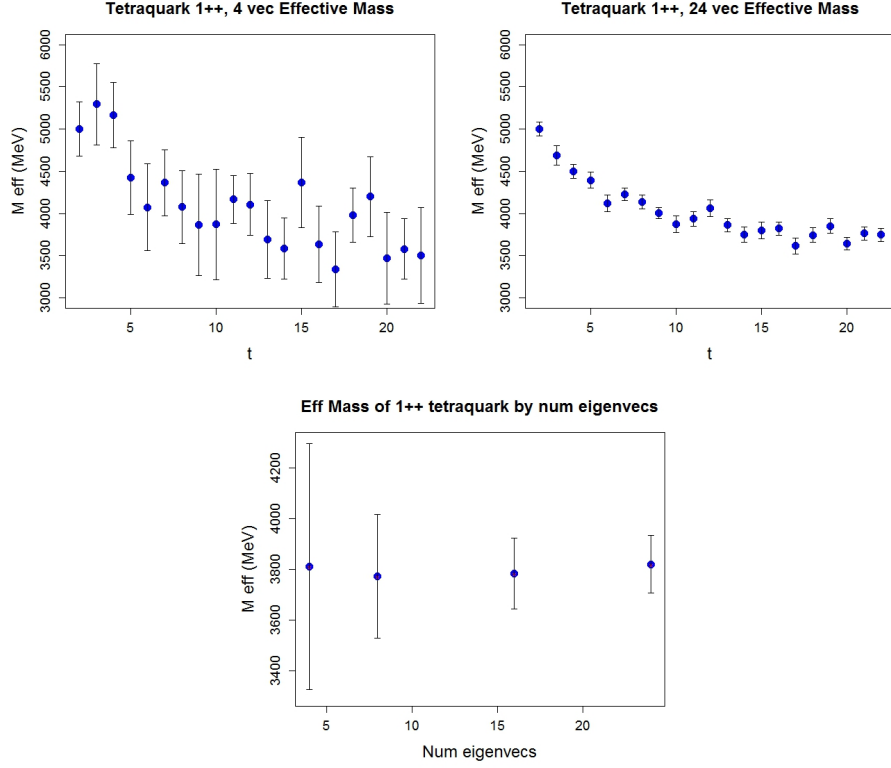


Figure 4: Effective mass plots for 1^{++} tetraquark calculated with 4 and 24 distillation space eigenvectors (upper panel) with effective mass by number of distillation space eigenvectors below.

space eigenvectors was investigated by repeating the calculation for the 1^+1^{++} tetraquark with 4, 8, 16 and 24 eigenvectors. The results are shown in Table 3 and plotted in Figure 4, where we also show the effective mass plots for this tetraquark using 4 and 24 eigenvectors. These results show that, by increasing the number of eigenvectors from 4 to 24, the standard errors decrease by more than a factor of 4. However, all of the effective masses are with 50 MeV of each other, which is less than half the smallest standard error of 113 MeV. This shows that the tetraquark effective masses are broadly stable under varying numbers of eigenvectors.

The effective masses of tetraquarks can be compared to those of the D mesons by means of a comparative effective mass plot. Whereas the effective mass is the log of the ratio of the correlation function of an operator at subsequent times,

D	$\times$	$\overline{D}$	1^-0^{+-}	1^+1^{++}	1^-1^{+-}
0^-	$\times$	0^-	-50(85)	-11(103)	-257(85)
0^-	$\times$	1^-	68(92)	107(110)	-140(79)
1^-	$\times$	1^-	185(127)	224(143)	-22(109)
0^-	$\times$	0^+	473(471)	512(486)	266(433)
1^-	$\times$	0^+	491(483)	630(498)	384(444)
0^-	$\times$	1^+	496(491)	635(499)	389(475)
1^-	$\times$	1^+	700(580)	752(508)	506(482)

Table 4: Slope of correlation function $\lim_{n_t \rightarrow \infty} \delta(n_t)$ giving energy differences in MeV between tetraquark and two D mesons (with standard errors in brackets) for various tetraquark and $D\overline{D}$ combinations.

a comparative effective mass can be defined as the ratio of correlation functions for different operators at the *same* time:

$$\delta(n_t) = \log \frac{C^T(n_t)}{C^{D_1}(n_t)C^{D_2}(n_t)} \quad (4.2)$$

where the superscript T denotes a tetraquark correlation function and D_1 and D_2 are distinct D mesons. In the ground state limit, this correlation function evolves as

$$\lim_{n_t \rightarrow \infty} \delta(n_t) \sim \log \frac{A_0^T e^{-E_0^T n_t}}{A_0^{D_1} e^{-E_0^{D_1} n_t} A_0^{D_2} e^{-E_0^{D_2} n_t}} = A - (E_0^T - (E_0^{D_1} + E_0^{D_2}))n_t. \quad (4.3)$$

Therefore, if $\lim_{n_t \rightarrow \infty} \delta(n_t)$ has a positive slope, this indicates that the tetraquark's effective mass is less than that of the two D mesons and there is potentially binding of those D mesons in the tetraquark.

The three tetraquark operators in Table 2 were compared to various $D\overline{D}$ combinations. In all cases, a linear fit was applied to the function $\delta(n_t)$ and the slope was determined. The results of this exercise are shown in Table 4, with sample comparative mass plots in Figure 5. We note the following regarding these results:

- Both the 1^-0^{+-} and 1^+1^{++} tetraquarks lie just below the threshold of two s-wave mesons, $D^*\overline{D} = 1^-0^-$, with positive binding energies within 1 standard error.

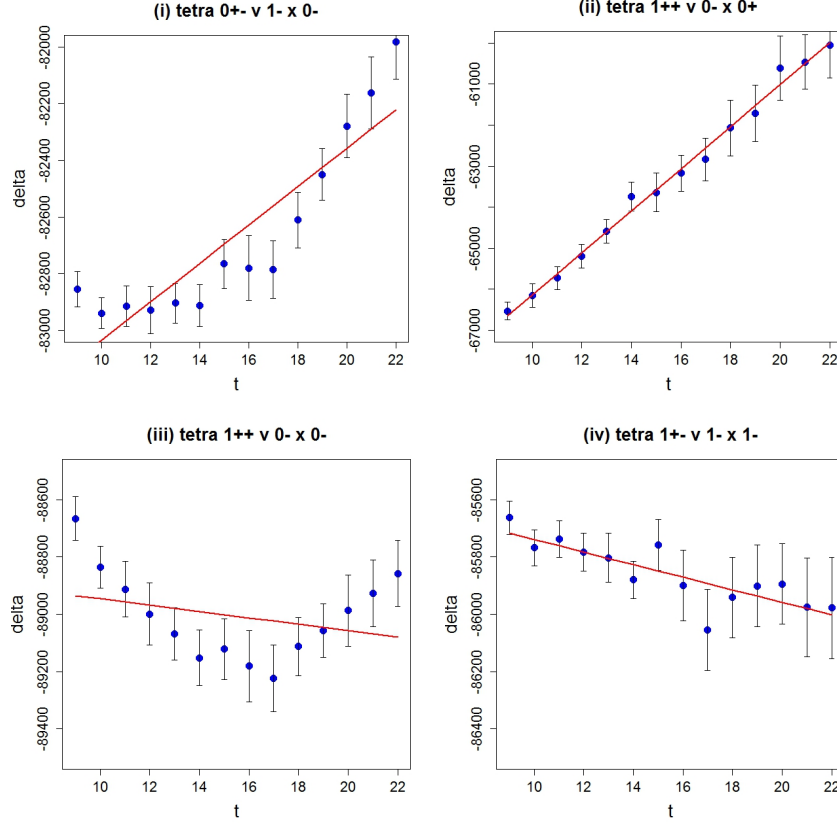


Figure 5: Comparative effective mass plots for various tetraquark and $D\bar{D}$ combinations: (i) 1^-0^{+-} and (ii) 1^+1^{++} show binding at $D^*\bar{D}$ threshold; (iii) 1^+1^{++} just above the $D\bar{D}$ threshold; and, (iv) 1^-1^{+-} shows no binding at the minimum $D^*\bar{D}^*$ threshold.

- These same tetraquarks lie just above the $D\bar{D} = 0^-0^-$ with small negative binding energies.
- All three tetraquarks show positive binding energies when compared to combinations of s- and p-wave mesons, with $D^*\bar{D} = 0^+0^-$, where the 1^-0^{+-} and 1^+1^{++} tetraquarks show binding energies in excess of 1 standard error.
- The 1^-1^{+-} tetraquark does not show positive binding energy in excess of 1 standard error, except when compared to the threshold for the s-p-wave $D^*\bar{D}^* = 1^-1^+$.

Operator	$I^G J^{PC}$	Mass Threshold	Decay
$X(3872)$	0^+1^{++}	$D^{*0}\bar{D}^0$	$D^{*0}\bar{D}^0, \pi^+\pi^-J/\psi\dots$
$Z(3900)$	1^+1^{+-}	$D^{*+}\bar{D}^0$	$D^{*+}\bar{D}^0, \pi J/\psi$
$Z(4020)$	$1^+?^{? -}$	$D^*\bar{D}^*$	$D^*\bar{D}^*, \pi h_c$
$Z_1(4050)$	$1^-?^{? +}$	$D^*\bar{D}^*$	$\pi^\pm\chi_{c1}$
$Z_c(4055)$	1^+0^{--}	$D^*\bar{D}^*$	$\pi^\pm\psi(2S)$

Table 5: Tetraquark candidate particles with observed masses close to those measured by lattice QCD.[6]

4.4 Comparison to tetraquark candidates

The experimentally derived properties of X and Z particles in the mass region of the three tetraquarks operators in Table 2 are given in Table 5.

The $X(3872)$ and $Z(3900)$ particles have masses close to the lattice calculated effective masses of the 1^-0^{+-} and 1^+1^{++} operators, 3856 and 3820 respectively. Furthermore, as shown in the previous section, these tetraquarks show binding at the $D^*\bar{D}$ threshold and it is known that $X(3872)$ and $Z(3900)$ decay to those mesons. However, either the spin or isospin of the $X(3872)$ does not match when compared to each of the tetraquark operators, while the $Z(3900)$ differs from the 1^+1^{++} operator in its charge conjugation.

Similarly, the $Z(4020)$, $Z_1(4050)$ and $Z_c(4055)$ all have masses close to the calculated effective mass of the 1^-0^{+-} operator, which shows binding at the $D^*\bar{D}^*$ mass threshold. While the quantum numbers of all of these particles are not fully known, in each case, either the G-parity or the charge conjugation do not match those of the tetraquark operators.

5 Conclusions

In this investigation, the effective masses of three tetraquark operators have been estimated using lattice QCD. We have shown that the calculation is stable under varying numbers of distillation space operators.

We have also compared the results to the effective masses of a variety of D and D^* mesons. In this comparison, it has been illustrated that the 1^-0^{+-} and 1^+1^{++} operators have positive binding energies at the $D^*\bar{D}$ threshold while the 1^-1^{+-} operator has positive binding energy at the $D^*\bar{D}^*$ threshold. However, these binding energies are close to the statistical standard errors of the calculation.

The lattice QCD tetraquark operators have also been compared to the experimentally derived properties of a number of candidate tetraquark X and Z particles. While the masses of the lattice operators compare reasonably well to the experimental evidence, the quantum numbers of the operators do not completely match those to the observed particles.

The results of this analysis, while investigative and preliminary, show encouraging signs of producing results that match experimental observation. Future work should extend the calculation to include more than 31 configurations in order to reduce statistical errors. Also, the number of lattice operators should be extended to include a large basis of interpolating operators, which would facilitate the use of variational techniques in the extraction of ground and excited state effective masses.

alpha	beta	gamma	delta	delta'	gamma'	beta'	alpha'	prod
0	3	0	2	0	2	0	3	-1
0	3	0	2	0	2	1	2	-1
0	3	0	2	0	2	2	1	1
0	3	0	2	0	2	3	0	1
0	3	0	2	1	3	0	3	-1
0	3	0	2	1	3	1	2	-1
0	3	0	2	1	3	2	1	1
0	3	0	2	1	3	3	0	1
0	3	0	2	2	0	0	3	-1
0	3	0	2	2	0	1	2	-1
0	3	0	2	2	0	2	1	1
0	3	0	2	2	0	3	0	1
0	3	0	2	3	1	0	3	-1
0	3	0	2	3	1	1	2	-1
0	3	0	2	3	1	2	1	1
0	3	0	2	3	1	3	0	1
0	3	1	3	0	2	0	3	-1
0	3	1	3	0	2	1	2	-1
0	3	1	3	0	2	2	1	1
0	3	1	3	0	2	3	0	1
0	3	1	3	1	3	0	3	-1
0	3	1	3	1	3	1	2	-1
0	3	1	3	1	3	2	1	1
0	3	1	3	1	3	3	0	1

Table 6: Results of first 24 spinor index product combinations for $\Gamma_{\alpha\beta}\hat{\Gamma}_{\gamma\delta}\hat{\Gamma}_{\delta'\gamma'}\Gamma_{\beta'\alpha'}$.

A Appendices

A.1 Sample spinor product results

Here we present some sample results for the spinor product in a tetraquark creation and annihilation operators, $\Gamma_{\alpha\beta}\hat{\Gamma}_{\gamma\delta}\hat{\Gamma}_{\delta'\gamma'}\Gamma_{\beta'\alpha'}$, as in correlation function 3.14, with $\Gamma = \gamma^1$ and $\hat{\Gamma} = \gamma^0$. The table was produced with the R function *getOpProd*, as shown in Appendix A.2. For brevity, only the first 24 of 256 rows are shown.

A.2 Sample R code

```
#####  
# Contains matrix definations and general transformation functions  
#####  
# GLOBAL Basis Declaration  
# MINKOWSKI or EUCLIDEAN  
METRIC <- "EUCLIDEAN"  
# CHIRAL or DIRAC  
BASIS <- "CHIRAL"  
print(paste("Metric:", METRIC))  
print(paste("Basis:", BASIS))  
  
# standard matrices  
O2 <- matrix(as.complex(rep(0,4)),2,2)  
I2 <- matrix(as.complex(c(1,0,0,1)),2,2)  
sigma1 <- matrix(as.complex(c(0,1,1,0)),2,2)  
sigma2 <- matrix(as.complex(c(0,1i,-1i,0)),2,2)  
sigma3 <- matrix(as.complex(c(1,0,0,-1)),2,2)  
I <- rbind(cbind(I2, O2),cbind(O2, I2))  
O <- matrix(as.complex(rep(0,16)),4,4)  
  
# gamma matrices, upper indices  
gamma <- array(dim=c(4,4,4))  
gammag5 <- gamma  
sigma <- array(rep(0,4^4),dim=c(4,4,4,4))  
# lower indices  
gamma_ <- gamma  
g5gamma_ <- gamma  
sigma_ <- sigma  
# rotation generators for 4d vector representation  
J <- array(rep(0,48),dim=c(4,4,3))  
  
mkwIdx <- 0  
if(METRIC == "EUCLIDEAN"){  
  gamma[, ,1] <- 1i*rbind(cbind(O2, sigma1), cbind(-sigma1, O2))  
  gamma[, ,2] <- 1i*rbind(cbind(O2, sigma2), cbind(-sigma2, O2))  
  gamma[, ,3] <- 1i*rbind(cbind(O2, sigma3), cbind(-sigma3, O2))  
  
  if(BASIS == "CHIRAL"){  
    gamma[, ,4] <- rbind(cbind(O2, I2), cbind(I2, O2))  
    gamma[, ,2] <- -gamma[, ,2]  
  }  
  else{gamma[, ,4] <- rbind(cbind(I2, O2), cbind(O2, -I2))  
  }  
  C <- gamma[, ,2] %*% gamma[, ,4]
```

```

g <- I
P <- -I
P[4,4] <- 1
Pidx <- 4
J[2,3,1] <- -1i; J[3,2,1] <- 1i; J[1,3,2] <- 1i;
J[3,1,2] <- -1i; J[1,2,3] <- -1i; J[2,1,3] <- 1i

} else if (METRIC == "MINKOWSKI"){

  if (BASIS == "CHIRAL"){
    gamma[, , 1] <- rbind(cbind(O2, I2), cbind(I2, O2))
  } else {
    gamma[, , 1] <- rbind(cbind(I2, O2), cbind(O2, -I2))
  }
  gamma[, , 2] <- rbind(cbind(O2, sigma1), cbind(-sigma1, O2))
  gamma[, , 3] <- rbind(cbind(O2, sigma2), cbind(-sigma2, O2))
  gamma[, , 4] <- rbind(cbind(O2, sigma3), cbind(-sigma3, O2))

  C <- 1i*gamma[, , 1] %*% gamma[, , 3]
  g <- rbind(cbind(sigma3, O2), cbind(O2, -I2))
  P <- g
  Pidx <- 1
  mkwIdx <- 1
  J[3,4,1] <- -1i; J[4,3,1] <- 1i; J[2,4,2] <- 1i;
  J[4,2,2] <- -1i; J[2,3,3] <- -1i; J[3,2,3] <- 1i
}

gamma5 <- 1i* gamma[, , 1] %*% gamma[, , 2] %*% gamma[, , 3] %*% gamma[, , 4]
gammag5[, , 1] <- gamma[, , 1] %*% gamma5
gammag5[, , 2] <- gamma[, , 2] %*% gamma5
gammag5[, , 3] <- gamma[, , 3] %*% gamma5
gammag5[, , 4] <- gamma[, , 4] %*% gamma5

for (nu in 2:4){
  for (mu in 1:(nu-1)){
    sigma[, , mu, nu] <- (gamma[, , mu] %*% gamma[, , nu] - gamma[, , nu] %*% gamma[, , mu])*1i/2
  }
}

for (mu in 1:4){
  gamma_[, , mu] <- gamma[, , mu]*g[mu, mu]
}

g5gamma_[, , 1] <- gamma5 %*% gamma_[, , 1]
g5gamma_[, , 2] <- gamma5 %*% gamma_[, , 2]
g5gamma_[, , 3] <- gamma5 %*% gamma_[, , 3]

```

```

g5gamma_[, , 4] <- gamma5 %*% gamma_[, , 4]

for (nu in 2:4){
  for (mu in 1:(nu-1)){
    sigma_[, , mu, nu] <- g[mu, mu]*g[nu, nu]*sigma[, , mu, nu]
  }
}

# rotation generators for spinor representation
S <- array(dim=c(4,4,3))
S[, , 3] <- sigma[, , 1+mkwIdx, 2+mkwIdx]/2;
S[, , 2] <- -sigma[, , 1+mkwIdx, 3+mkwIdx]/2;
S[, , 1] <- sigma[, , 2+mkwIdx, 3+mkwIdx]/2
if(METRIC == "EUCLIDEAN"){
  S <- -S
}

# Levi Civita symbol in 3d - function should be used
epsilon <- array(rep(0,27),c(3,3,3))
epsilon[1,2,3] <- 1; epsilon[2,1,3] <- -1; epsilon[1,3,2] <- -1;
epsilon[3,1,2] <- 1; epsilon[3,2,1] <- -1; epsilon[2,3,1] <- 1

Epsilon <- function(i,j,k){
  sum(epsilon[i,j,k])
}

# utility function: for two integers from 1,2,3 gives the remain one
kIndex <- function(i,j){
  if(i != j){
    if(i == 3 || j == 3){
      if(i == 2 || j == 2){
        return(1)
      } else{
        return(2)
      }
    } else{
      return(3)
    }
  }
  return(0)
}

# ensures gamma matrices obey Clifford algebra
print("Clifford_Algebra_Test")
for (mu in 1:4){
  for (nu in 1:4){

```

```

        if(prod((gamma[, , mu] %*% gamma[, , nu] + gamma[, , nu] %*% gamma[, , mu]) == 2 * g[mu, nu] * I) == 0)
            print(paste(mu, nu))
    }
}
print("Clifford_Algebra_Test_Done")

# tests the Lie Algebra for spinor rotations of gamma matrices
print("Rotation_Generator_Test")
for(i in 1:3){
    for(j in 1:3){
        if(prod((S[, , i] %*% gamma[, , j+mkwIdx] - gamma[, , j+mkwIdx] %*% S[, , i]) ==
            1i * Epsilon(i, j) * gamma[, , kIndex(i, j)+mkwIdx]) == 0){
            print(paste(i, j))
        }
    }
}
print("Rotation_Generator_Test_Done")

# generates 4d rotation matrix for given set of generators (spinor or vector) and angles.
# Computes the matrix via Taylor expansion to given order.
rotation4 <- function(gens, theta=runif(dim(gens)[3], -pi, pi), order=20){
    generators <- O
    for(k in 1:dim(gens)[3]){
        generators <- generators + 1i*gens[, , k]*theta[k]
    }

    rot <- I
    genPower <- I
    for(k in 1:order){
        genPower <- genPower %*% generators
        rot <- rot + genPower/factorial(k)
    }
    rot
}

# transforms arbitrary matrix M with spinor rep of Lorentz group
spinorTxfm <- function(LambdaS, M){
    MASS::ginv(LambdaS) %*% M %*% LambdaS
}

# transforms arbitrary array M from a vector of arrays using vector rep of Lorentz group
# Can also perform inverse transformations or transformation on a table specifying
# an operator product.
vectorTxfm <- function(LambdaV, mu=2, mArray=gamma, inv=F, prodTable=F){
    txfm <- LambdaV

```

```

if(inv){
  txfm <- MASS::ginv(LambdaV)
}

gNew <- O
if(prodTable){
  gNew <- mArray[,mu]
  gNew[,5] <- 0
}

for(a in 1:dim(gNew)[1]){
  if(prodTable){
    for(nu in 1:4){
      gNew[a,5] <- gNew[a,5] + txfm[mu,nu] * mArray[a,5,nu]
    }
  } else{
    for(b in 1:dim(gNew)[2]){
      for(nu in 1:4){
        gNew[a,b] <- gNew[a,b] + txfm[mu,nu] * mArray[a,b,nu]
      }
    }
  }
}
gNew <- round(gNew,4)
if(prodTable){
  gNew <- gNew[gNew[,5] != 0, ]
}
gNew
}

#####
# Implements product of spinor operators.
# cycles through all spinor indices in tetraquark product and outputs the indices
# and product values where the product is non-zero.
#####
getOpProd <- function(Gamma1, Gamma2, toConsole=T, filename="output.txt"){

  # holds the output data
  out <- data.frame()
  prod_t2 <- data.frame()
  prod_t1 <- data.frame()
  # cycle through all possible indices
  for(a in 0:3){
    for(b in 0:3){
      for(c in 0:3){
        for(d in 0:3){

```

```

      if (Gamma1[a+1,b+1]*Gamma2[c+1,d+1] != 0){
        prod_t2 <- rbind(prod_t2,
          data.frame(a,b,c,d,Gamma1[a+1,b+1]*Gamma2[c+1,d+1]))
      }
    }
  }
}
for(dp in 0:3){
  for(cp in 0:3){
    for(bp in 0:3){
      for(ap in 0:3){
        if (Gamma2[dp+1,cp+1]*Gamma1[bp+1,ap+1] != 0){
          prod_t1 <- rbind(prod_t1,
            data.frame(dp,cp,bp,ap,Gamma2[dp+1,cp+1]*Gamma1[bp+1,ap+1]))
        }
      }
    }
  }
}

for(i in 1:dim(prod_t2)[1]){
  for(j in 1:dim(prod_t1)[1]){
    if(prod_t2[i,5]*prod_t1[j,5] != 0){
      out <- rbind(out, data.frame(prod_t2[i,1:4],prod_t1[j,1:4],prod_t2[i,5]*prod_t1[j,5]))
    }
  }
}

names(out) <- c("a","b","c","d","d'","c'","b'","a',"prod")
names(prod_t2) <- c("a","b","c","d","prod")
names(prod_t1) <- c("d'","c'","b'","a',"prod")
if(toConsole){
  return(list(out,cbind(prod_t2,prod_t1)))
}else{
  write.table(out, paste(filepath,filename,".txt",sep=""), row.names=F)
}
}

#####
# Implements Rotation and Parity Symmetry tests
#####
# get spinor and vector reps of random rotation
# set.seed(1);
theta=runif(dim(S)[3],-pi,pi)
LambdaS <- rotation4(S, theta)

```

```

LambdaV <- rotation4(J, theta)

# rotate spinor indices of gamma matrices
gammaRot <- gamma
gammaRot[,1] <- spinorTxfm(LambdaS, gamma[,1])
gammaRot[,2] <- spinorTxfm(LambdaS, gamma[,2])
gammaRot[,3] <- spinorTxfm(LambdaS, gamma[,3])
gammaRot[,4] <- spinorTxfm(LambdaS, gamma[,4])

# perform inverse vector txfm and check results are gamma matrices
print("gamma_matrices_rotation_test")
print(vectorTxfm(LambdaV, 1, gammaRot, T, F) == gamma[,1])
print(vectorTxfm(LambdaV, 2, gammaRot, T, F) == gamma[,2])
print(vectorTxfm(LambdaV, 3, gammaRot, T, F) == gamma[,3])
print(vectorTxfm(LambdaV, 4, gammaRot, T, F) == gamma[,4])
print("gamma_matrices_rotation_test_done")

# get gamma^mu gamma^0 products
g0Idx <- 1
if(METRIC == "EUCLIDEAN"){
  g0Idx <- 4
}
prod1 <- getOpProd(gamma[,1], gamma[, g0Idx])
prod <- array(dim=c(dim(prod1)[2], 1:5), 4)
prod[,1] <- as.matrix(prod1[[2]][1:5])
prod[,2] <- as.matrix(getOpProd(gamma[,2], gamma[, g0Idx])[[2]][1:5])
prod[,3] <- as.matrix(getOpProd(gamma[,3], gamma[, g0Idx])[[2]][1:5])
prod[,4] <- as.matrix(getOpProd(gamma[,4], gamma[, g0Idx])[[2]][1:5])

# get new products with rotated matrices
prod1Rot <- getOpProd(gammaRot[,1], gammaRot[, g0Idx])
prodRot <- array(dim=c(dim(prod1Rot)[2], 1:5), 4)
prodRot[,1] <- as.matrix(prod1Rot[[2]][1:5])
prodRot[,2] <- as.matrix(getOpProd(gammaRot[,2], gammaRot[, g0Idx])[[2]][1:5])
prodRot[,3] <- as.matrix(getOpProd(gammaRot[,3], gammaRot[, g0Idx])[[2]][1:5])
prodRot[,4] <- as.matrix(getOpProd(gammaRot[,4], gammaRot[, g0Idx])[[2]][1:5])

# perform inverse vector txfm on rotated products and test
# results are same as unrotated products
print("product_rotation_test")
print(vectorTxfm(LambdaV, 1, prodRot, T, T) == prod[,1])
print(vectorTxfm(LambdaV, 2, prodRot, T, T) == prod[,2])
print(vectorTxfm(LambdaV, 3, prodRot, T, T) == prod[,3])
print(vectorTxfm(LambdaV, 4, prodRot, T, T) == prod[,4])
print("product_rotation_test_done")

```

```

# Parity transform of gamma matrices
gammaPar <- gamma
for (mu in 1:4){
  gammaPar[,mu] <- gamma[,Pidx] %*% gamma[,mu] %*% gamma[,Pidx]
}

print("gamma_product_parity_test")
for (mu in 1:4){
  if(prod(as.matrix(getOpProd(gammaPar[,mu], gammaPar[,g0Idx]))[[2]][,5])
      == P[mu,mu]*prod[,5,mu]) == 0){
    print(paste("gamma",mu))
  }
}
print("gamma_product_parity_test_done")

```

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