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Stochastic responses analysis of vibro-impact system via direct probability integral method

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ABSTRACT: Vibro-impact system, as an important type of non-smooth system in mechanical engineering, has received much research interest in the past few decades. The behaviors of vibro-impact system are complex and unusual, because it behaves as continuous dynamical system between two successive impacts and develops a discrete behavior when reaching a constraint. Note that vibro-impact system is inevitably subjected to random perturbations, and thus it is essential to develop powerful approach for stochastic dynamic analysis of vibro-impact system. This paper proposes a unified and efficient direct probability integral method (DPIM) to calculate the probability density functions of the unilateral vibro-impact system under random excitations, and explores its complicated dynamical behavior. The DPIM is a versatile approach with high accuracy and efficiency to facilitate stochastic response analysis for general systems, including linear and nonlinear systems, time independent and time dependent systems, and the systems with stochastic parameters under various random excitations. Firstly, based on the principle of probability conservation and Dirac delta function, the probability density integral equation (PDIE) for the unilateral vibro-impact system is derived. Secondly, the PDIE is solved by using the point selection technique based on generalized F discrepancy and the smoothing of Dirac delta function. Compared with Monte Carlo simulation, some numerical examples illustrate the merits of the proposed approach for the stochastic response of unilateral vibro-impact system under Gaussian white noise excitation. The influence of some parameters of the vibro-impact system on the stochastic dynamic responses is also scrutinized.

1. INTRODUCTION

The stochastic dynamic response computation of structural systems under random excitation is an important issue in reliability analysis. An overlarge response may be a great threat to the safety of life and property. Therefore, many efforts were made on the evaluation of the stochastic responses of these systems.

In particular, vibro-impact systems have attracted much attention because the systems can simulate many relevant problems in the field of engineering. Vibro-impact systems widely exist in the field of mechanical and structural engineering, such as structural interaction in piping systems (Ivanov, 1994), nonlinear soil–structure interaction (Jing and Young, 1990), heat exchanger tubes under aerodynamic excitation, vibratory pile drivers, gear-pair systems with backlash, and ship's motion when colliding against fenders (Ibrahim, 2009).

Due to the discontinuous impact interaction, the dynamical behavior of vibro-impact system is highly nonlinear and very complex. In order to suppress adverse effects and take advantages of vibro-impact dynamics, it is of great significance to reveal the fundamental characteristics of vibro-impact dynamical systems.

The behavior of vibro-impact system is unusual, because it behaves as continuous dynamical system between two successive impacts and develops a discrete behavior when reaching a constraint. Assuming that the impact is completed instantaneously, the energy dissipation is measured by the restitution coefficient. Therefore, the stochastic dynamic behavior of vibro-impact systems under random excitation needs to be studied deeply. In this field, the stochastic responses of vibro-impact systems are usually evaluated by statistical moments or probability density functions (PDFs).

On this basis, a large number of researches have been carried on. The non-smooth coordinate transformation techniques (Zhuravlev, 1976; Ivanov, 1994) were introduced, so that the traditional techniques such as stochastic averaging method (Xu et al. 2016), the path integration (Dimentberg and Iourtchenko 2004), the exponential polynomial closure (EPC) method (Zhu, 2015) and the iterative method of weighted residue (Chen et al. 2019) were advanced for investigating vibro-impact systems. In addition, Monte Carlo simulation is one of the most commonly methods for calculating the dynamic responses of stochastic vibro-impact systems, whereas, its computational efficiency is low and it present random convergence. The method of generalized cell mapping was used by Wang et al. (2018) to obtain the stochastic response of impulsive systems. However, the investigation on the PDF solution of vibro-impact systems mostly relies on stochastic averaging method (Zhu and Lin 1991). The stochastic averaging method is suitable for the dynamic system with small damping under weak wide-band excitation. These limitations motivate this study on the PDF solution of vibro-impact systems under random excitation with new powerful technique.

In this paper, a unified and efficient direct probability integral method (DPIM) (Chen and Yang 2019, 2021) is developed to calculate the PDF of the unilateral vibro-impact system under random excitation. DPIM can directly solve the equations of motion to obtain the accurate dynamic response solutions, thus avoiding non-smooth coordinate transformation. This study considers a one-sided constraint and the system collides with the constraint by instantaneous repetitive impacts. Firstly, the probability density integral equation (PDIE) of concerned response for multi-degree-of-freedom system is constructed in terms of the principle of probability conservation. Then, the response PDFs are acquired by DPIM. Lastly, compared with Monte Carlo simulation the high efficiency of the proposed approach is validated. Additionally, the effects of different noise intensities on transient responses of vibro-impact systems under Gaussian white noise are further investigated.

2. DIRECT PROBABILITY INTEGRAL METHOD

In this section, the fundamental concept of DPIM is introduced briefly.

2.1. The principle of probability conservation

There are conservation laws such as conservation of energy, conservation of mass, etc. Actually, the probability carried by random event is also conserved without introducing other stochastic factors, which is the statement of the principle of probability conservation. For a dynamic system, the principle of probability conservation can be expressed by (Chen and Yang, 2019)

$$\int_{\Omega_{\mathbf{Y}}} p_{\mathbf{Y}}(\mathbf{y},t)\mathrm{d}\mathbf{y} = \int_{\Omega_{\mathbf{\Theta}}} p_{\mathbf{\Theta}}(\mathbf{\theta})\mathrm{d}\mathbf{\theta} \tag{1}$$

where $\mathbf{\Theta}$ is a stochastic input vector collecting all the stochastic factors introduced into the system at the initial instant; $\mathbf{Y}$ represents the stochastic output vector; $\mathbf{\Omega_Y}$ indicates the total sample space of output random vector $\mathbf{Y}$; $\mathbf{\Omega_\Theta}$ denotes the input sample space; $p_{\mathbf{Y}}(\mathbf{y},t)$ means PDF of $\mathbf{y}$ at any time t; $p_{\mathbf{\Theta}}(\mathbf{\theta})$ denotes PDF of input random vector $\mathbf{\Theta}$. Specifically, Eq. (1) indicates that the total probability measure of output equals to that of input and is invariant in the evolution of multi-degree-of-freedom dynamic system.

2.2. PROBABILITY DENSITY INTEGRAL EQUATION

The relation between the output response vector $\mathbf{Y}$ and input vector $\mathbf{\Theta}$ can be written as the deterministic mapping $\mathcal{G}: \mathbf{\Theta} \to \mathbf{Y}$, i.e.

$$\mathcal{G}: \mathbf{Y}(t) = g(\mathbf{\Theta},t) \tag{2}$$

With the aid of the Dirac function, the propagation of randomness from the input space to the output space is described. Chen and Yang (2019) derived the PDIE of stochastic dynamic system, thus the uncertainty propagation from the input vector $\mathbf{\Theta}$ to the output vector $\mathbf{Y}$ was explicitly described and the uncertainty propagation was obtained.

PDIE is derived through the deterministic mapping $\mathcal{G}$ and Dirac delta function, The PDIE at time t can be expressed as

$$p_{\mathbf{Y}}(\mathbf{y},t)=\int_{\Omega_{\mathbf{\Theta}}}\delta[\mathbf{y}-g(\mathbf{\theta},t)]p_{\mathbf{\Theta}}(\mathbf{\theta})\,\mathrm{d}\mathbf{\theta} \tag{3}$$

Accordingly, the PDIE is essentially equivalent to the probability density differential equation, and the former can deduce the latter. Moreover, it should be pointed out that the PDIE for stochastic dynamic system holds true at all instant, which is the reflection of the principle of probability conversation. In addition, once the certain physical evolution paths are determined, the response PDF at any time instant will be obtained without having to consider the evolution process before that moment, since the probabilistic information has been implied in the process of physical evolution.

2.3. *Basic principle of direct probability integral method*

Generally, analytical solutions of the PDIEs are difficult to obtain due to the complexity of mapping $\mathcal{G}$.Furthermore, the non-smooth Dirac delta function limits the accuracy of numerical integral of Eq. (3). To this end, Chen and Yang (Chen and Yang 2019) proposed the direct probability integral method by introducing the following two key techniques: (1) partition of input probability space; (2) smoothing of Dirac delta function. The first technique is to generate representative points and assigned probabilities, which can be implemented via the points selection method based on generalized F discrepancy (GF-discrepancy). Moreover, the Voronoi cell based partition technique is utilized to partition the input probability space into a set of non-overlapping representative region.

The second technique aims at improving the integral accuracy by smoothing the Dirac delta function in Eq. (3). Many functions, e.g., Gaussian function, Sine function, Poisson function, etc., can be used to achieve this purpose. Without loss of generality, the Gaussian function is adopted in this study. Therefore, the PDF of the mapping $\mathcal{G}$ for stochastic dynamic systems is separately calculated as follows:

$$p_Y(y,t)=\sum_{q=1}^{N}\left\{\hat{\delta}(y,t;\sigma)P_q\right\}=\sum_{q=1}^{N}\left\{\frac{1}{\sqrt{2\pi\sigma}}e^{-[y-g(\mathbf{\theta}_q,t)]^2/2\sigma^2}P_q\right\} \tag{4}$$

where N is the total number of representative points; $\mathbf{\theta}_q$ represents the qth representative point in the input probability space; σ stands for the smoothing parameter of Dirac delta function, which can be determined by an adaptive formula based on kernel density estimation (Chen and Yang 2021); P_q denotes the assigned probability of qth representative point; $g(\mathbf{\theta}_q,t)$ are the representative outputs of the mapping $\mathcal{G}$ for stochastic dynamic systems.

As deduced from the above descriptions, the uncertainty propagation analysis is decoupled as the computation of probability density integral equation and structural physical equation by using DPIM, which provides a powerful tool to perform stochastic response analyses for dynamic systems under random excitation.

3. NUMERICAL EXAMPLE

In the following, the highly accurate transient solutions and their PDFs of nonlinear vibro-impact systems under Gaussian white noise are obtained. The vibro-impact systems are commonly used to model various structural and mechanical systems and therefore are of practical significance. The numerical results by means of Monte Carlo simulation for the unilateral vibro-impact system are shown for the purpose of comparison with the computational results obtained by DPIM.

3.1. *System description*

Consider a general nonlinear single-degree-of-freedom inelastic vibro-impact system under random excitations with a rigid unilateral barrier at the equilibrium position. The corresponding configuration of this system is illustrated in Figure 1.

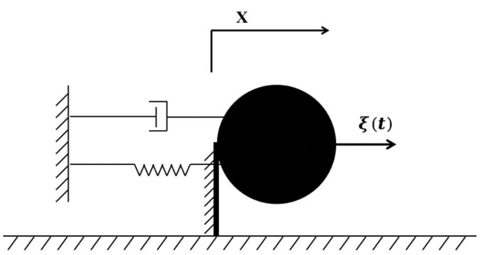


Figure 1: Stochastic vibro-impact system with unilateral barrier.

The motion equation of this system is:

$$\ddot{x} + g(x) + f(x,\dot{x}) = \sum_{k=1}^{m} h_k(x,\dot{x})\xi_k(t) \qquad (5\text{-a})$$

$$\dot{x}_+ = -r\dot{x}_-, x = 0 \qquad (5\text{-b})$$

where $f(x,\dot{x})$ is a nonlinear function of displacement and velocity, which represents linear and nonlinear damping forces; $g(x)$ is a linear or nonlinear function of the displacement, denoting the restoring force of the system; $h_k(x,\dot{x})$ denotes the amplitude of random perturbation, where $k = 1, 2, \ldots$ m; $\xi(t)$ are Gaussian white noises with zero mean, whose correlation functions are $E[\xi_k(t)\xi_l(t+\tau)] = 2D_{kl}\delta(\tau)$. For DPIM, the random function-based spectral representation method (Liu et al. 2016) is employed to generate representative samplings of Gaussian white noise excitation. Eq. (5-a) governs the system's motion between impacts. Eq. (5-b) describes the impact law, meaning that the velocity of oscillator has an instantaneous jump from $\dot{x}_-$ to $\dot{x}_+$ when it impacts with the rigid barrier at $x = 0$, where $\dot{x}_-$ and $\dot{x}_+$ are rebound and impact velocities of the oscillate; r is the restitution coefficient describing impact energy loss.

By introducing $\dot{x} = y$, the first-order differential equation can be derived:

$$\dot{x} = y$$

$$\dot{y} = \sum_{k=1}^{m} h_k(x,y)\xi_k(t) - g(x) - f(x,y) \qquad (6)$$

DPIM can solve the motion equation directly by applying the fourth-order Runge-Kutta, which can ensure the computational accuracy. Due to the unique property of the vibro-impact system, special process of impact is required when performing the numerical solution. If the $x < 0$ is satisfied, the impact is considered to occur at this moment and the system velocity variation is obtained from Eq. (5-b).

3.2. Application

Consider the following nonlinear vibro-impact oscillator under Gaussian white noise external excitation, the equation of motion is described by,

$$\ddot{x} + \alpha\dot{x} + \gamma\dot{x}^3 + x = \xi(t), x > 0$$

$$\dot{x}_+ = -r\dot{x}_-, x = 0 \qquad (7)$$

where α and γ are linear and nonlinear damping coefficients, respectively.

In this example, the numerical results of the transient response of the vibro-impact system under Gaussian white noise are obtained by DPIM with 1000 representative points, which means DPIM only needs 1000 times of dynamic computation of vibro-impact system). The marginal probability densities of displacement x and velocity $\dot{x}$ are given in Figure 2(a) and 2(b), respectively, with the following parameter values: $\alpha = 0.03$, $r = 0.98$, $D = 0.03$. The solid line denotes the numerical result obtained by DPIM while the dots are the results obtained from Monte Carlo simulation with 10^5 sampling. From Figure 2(a) and 2(b), it is obvious that the resluts of Monte Carlo simulation give powerful validation with DPIM and DPIM has high efficiency and accuracy. Furthermore, Figure 3(a) and 3(b) demonstrate the joint PDFs for the displacement x and velocity $\dot{x}$. They also agree well with each other.

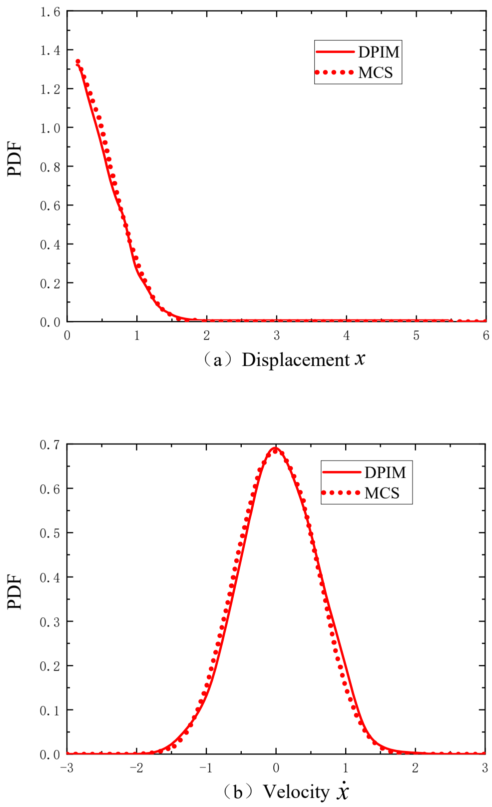
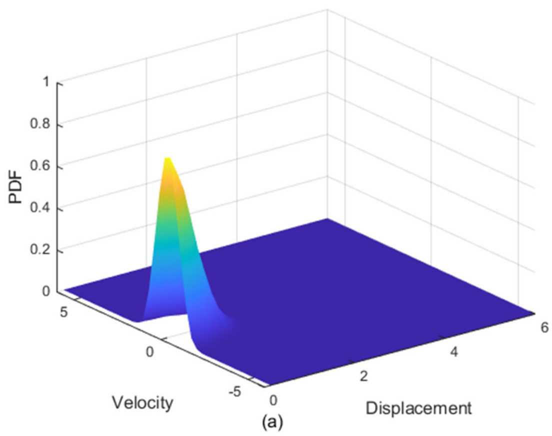


Figure 2: The probability density function curves of displacement and velocity of system (7) under D = 0.3. (a) PDFs of displacement x; (b) PDFs of velocity $\dot{x}$.



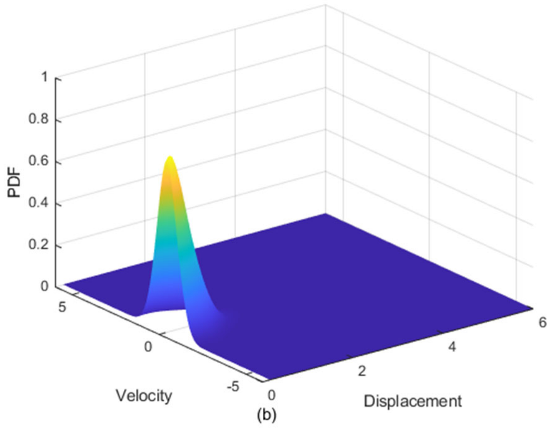


Figure 3: The joint probability density function curves of displacement and velocity of system (7) under D = 0.3. (a) the Monte Carlo simulation result; (b) the DPIM result.

In order to illustrate the influence of the intensity of Gaussian white noise excitation on the responses of the above vibro-impact system, Figure 4 show stationary PDFs under different values of the external Gaussian noise intensity when other system parameters are fixed as: $\alpha = 0.03$, $\gamma = 0.03$, $r = 0.98$. It is observed from those figures, all the results obtained by DPIM are full of high precision by comparison to Monte Carlo simulation. It can be seen in Figure 4(a) and 4(b) that the PDFs of displacement x and velocity $\dot{x}$ present lower peaks and become more flat with the increase of noise intensity. It means larger noise intensity can lead to smaller probability that the system will stay close to the equilibrium state. Besides, it is clear that the PDFs of the velocity response are symmetry and smooth.

As observed in Table1, the CPU times taken by MCS are about 1000 times more than those of DPIM at least. It provides that DPIM obtains response's PDF with high efficiency.

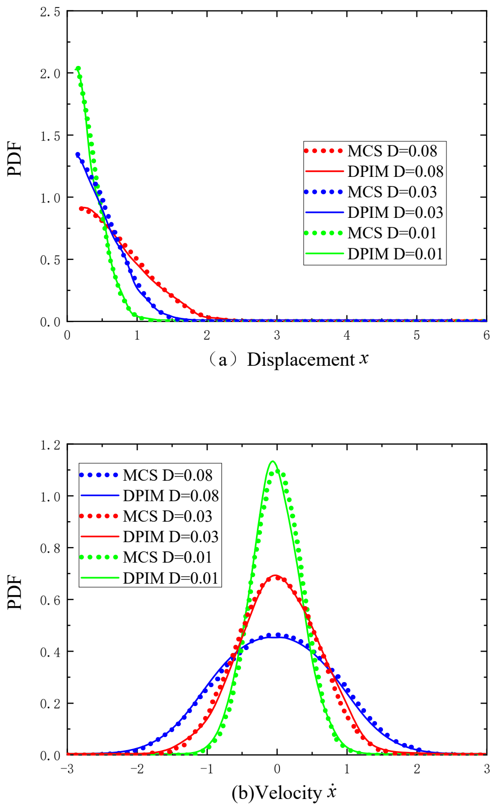


Figure 4: The PDF curves of system (7) for different values of the intensity D. (a) PDFs of displacement x; (b) PDFs of velocity $\dot{x}$.

Table 1: The CPU time of the system (7) calculated by DPIM and MCS under different values of the noise intensity D.

Methods	Noise intensity D	CPU time (s)
DPIM	0.01	24.51
	0.03	31.92
	0.08	26.82
MCS	0.01	1.36×10^4
	0.03	2.03×10^3
	0.08	1.38×10^4

4. CONCLUSIONS

In this paper, an efficient DPIM is proposed and applied to calculate transient responses and their PDFs of the single-degree-of-freedom vibro-impact systems with a rigid barrier at the static equilibrium position under Gaussian white noise excitation. DPIM can directly solve the equations of motion to obtain the exact solution without applying non-smooth coordinate transformation technique. Compare with Monte Carlo simulation, some numerical results illustrate the high efficiency and accuracy of DPIM for computing the stochastic dynamic response of the vibro-impact system. Specifically, the effect of noise intensities on the stochastic responses of the vibro-impact system is examined by using DPIM and Monte Carlo simulation. As the noise intensity increases, it is found that the system will stay close to the equilibrium state with smaller probability.

Therefore, DPIM is quite promising and can be extended to more general vibro-impact systems with inelastic impact under various random excitations including non-Gaussian excitation.

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5. REFERENCES

Chen, G. H., and Yang, D. X. (2019). “Direct probability integral method for stochastic response analysis of static and dynamic structural systems.” *Computer Methods in Applied Mechanics and Engineering*, 357(1), 12612.

Chen, G. H., and Yang, D. X. (2021). “A unified analysis framework of static and dynamic structural reliabilities based on direct probability integral method.” *Mechanical Systems and Signal Processing*, 158, 107783.

Chen, L. C., Qian, J. M., Zhu, H. T., and Sun, J. Q. (2019). “The closed-form stationary probability distribution of the stochastically excited vibro-impact oscillators.” *Journal of Sound and Vibration*, 2019, 439, 260-270.

Dimentberg, M. F., and Iourtchenko, D.V. (2004). “Random vibrations with impacts: a review.” *Nonlinear Dynamics*, 36(2), 229–254.

Ibrahim, R. A. (2009). “*Vibro-impact Dynamics: Modeling, Mapping and Applications*.” Berlin, Springer.

Ivanov, A. P. (1994). “Impact oscillations: linear theory of stability and bifurcations.” *Journal of Sound and Vibration*, 178(3), 361–378.

Jing, H. S., and Young, M. (1990). “Random response of a single-degree-of-freedom vibro-impact system with clearance.” *Earthquake Engineering and Structural Dynamics*, 19(6), 789–798.

Liu, Z. J., Liu, W., and Peng, Y. (2016). “Random function based spectral representation of stationary and non-stationary stochastic processes.” *Probabilistic Engineering Mechanics*, 45, 115–126.

Wang, L., Ma, S., Sun, C., Jia, W., and Xu, W. (2018). “The stochastic response of a class of impact systems calculated by a new strategy based on generalized cell mapping method.” *Journal of Applied Mechanics*, 85(5), 054502.

Xu, W., Li, C., Yue, X. L., and Rong, H. (2016). “Stochastic responses of a vibro-impact system with additive and multiplicative colored noise excitations.” *International Journal of Dynamics and Control*, 4(4), 393–399.

Zhu, H.T. (2015). “Stochastic response of a vibro-impact Duffing system under external Poisson impulses.” *Nonlinear Dynamics*, 82(1), 1001–1013.

Zhu, W. Q., and Lin, Y. K. (1991). “Stochastic averaging of energy envelope.” *Journal of Engineering Mechanics*, 117(8), 1890-1905.

Zhuravlev, V. F. (1976). “A method for analyzing vibration-impact systems by means of special functions.” *Mechanics of Solids*, 11(2), 23‑27.