REMARKS ON THE RIGIDITY OF CR-MANIFOLDS

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Abstract. We propose a procedure to construct new smooth CR-manifolds whose local stability groups, equipped with their natural topologies, are subgroups of certain (finite-dimensional) Lie groups but not Lie groups themselves.

1. Introduction

Given a germ (M,p) of a real submanifold of $\mathbb{C}^n$, its basic invariant is the *local stability group* $\mathsf{Aut}(M,p)$, i.e. the group of all germs at p of local biholomorphic maps of $\mathbb{C}^n$ fixing p and preserving the germ (M,p). By the work of several authors [CM74, BER97, Z97, ELZ03, LM05] it is known that this group is a (finite-dimensional) Lie group (in the natural inductive limit topology) for germs of *real-analytic* submanifolds satisfying certain nondegeneracy conditions, e.g. those having nondegenerate Levi form. On the other hand, in the absence of the nondegeneracy conditions, the group $\mathsf{Aut}(M,p)$ can possibly be infinite-dimensional (in the sense that it contains Lie groups of arbitrarily large dimension). (E.g. the local stability group of $(\mathbb{R},0)$ in $\mathbb{C}$ consists of all convergent power series with real coefficients.) Furthermore, recent results in [BRWZ04] show that a similar principle also holds for *global CR-automorphisms*, both real-analytic and smooth.

One purpose of this paper is to show that, in contrast with the behaviour mentioned above, a similar alternative does not anymore hold for the *local stability group* of a *smooth* real submanifold. In particular, we show that, for any $n \geq 2$, there exists a germ (M,p) of a smooth strongly pseudoconvex hypersurface in $\mathbb{C}^n$ with $\mathsf{Aut}(M,p)$ being (topologically) isomorphic to a countable dense subgroup of the circle $S^1 \subset \mathbb{C}$ and hence not being a Lie group. In fact, $\mathsf{Aut}(M,p)$ can be arranged to be isomorphic to any increasing countable union of finite subgroups of S^1, for instance, to the subgroup

$$\{e^{2\pi i \frac{l}{2^m}} : l, m \in \mathbb{N}\} \subset S^1. \tag{1.1}$$

Furthermore, our construction yields similar properties also for the (generally larger) *local CR stability group* $\mathsf{Aut}_{\mathrm{CR}}(M,p)$, consisting of all germs at a point p of smooth CR-automorphisms of M fixing p. Recall that a germ of a smooth transformation $\varphi\colon (M,p) \to (M,p)$ is a *CR-automorphism* if it preserves the subbundle T^cM and and the restriction

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of its differential $d\varphi|_{T^cM}$ is $\mathbb{C}$-linear, where

$$T^cM := TM \cap iTM.$$

Another purpose of this paper is to provide a general construction of new smooth generic submanifolds with certain prescribed local CR stability groups (recall that a real submanifold $M \subset \mathbb{C}^n$ is *generic* if $T_qM + iT_qM = T_q\mathbb{C}^n$ for all $q \in M$). More precisely, we show the following:

Theorem 1.1. *Let (M,p) be a germ of a smooth generic submanifold in $\mathbb{C}^n$ of positive codimension and of finite type and assume that it is invariant under an increasing countable union G of finite subgroups of $\mathsf{Aut}(\mathbb{C}^n,p)$. Then there exists a G-invariant germ of another smooth generic submanifold $(\widetilde{M},p)$ of the same dimension as (M,p), which is tangent to (M,p) of infinite order and has the following properties:*

(i) $\mathsf{Aut}(\widetilde{M},p) = G$;
(ii) $\mathsf{Aut}_{\mathrm{CR}}(\widetilde{M},p) = \{g|_M : g \in G\}$.

We use here the notion of finite type due to Kohn [K72] and Bloom-Graham [BG77]): a germ (M,p) is of *finite type*, if all germs at p of smooth real vector fields on M tangent to T^cM span together with their iterated commutators the full tangent space T_pM.

We now illustrate Theorem 1.1 by an example, where it can be applied.

Example 1.2. Consider a real hypersurface $M \subset \mathbb{C}^{n+1}$, $n \geq 1$, given in coordinates $(z_1,\dots,z_n,w) \in \mathbb{C}^{n+1}$ by

$$\mathsf{Im}\, w = \varphi(|z_1|^2, z_2,\dots,z_n,\bar z_2,\dots,\bar z_n,\mathsf{Re}\, w),$$

where φ is any smooth function such that $0 \in M$ and $(M,0)$ is of finite type. Then $(M,0)$ is clearly invariant under the rotation group consisting of all transformations $(z_1,z_2,\dots,z_n,w) \mapsto (e^{2\pi i\theta}z_1,z_2,\dots,z_n,w)$ for all real θ. Now we can take the subgroup G consisting of all these transformations corresponding to $\theta = l/2^m$ with l,m being positive integers. Then G clearly satisfies the assumptions of Theorem 1.1. We then conclude that there exists a new real submanifold $\widetilde{M} \subset \mathbb{C}^{n+1}$ such that both $\mathsf{Aut}(\widetilde{M},p)$ and $\mathsf{Aut}_{\mathrm{CR}}(\widetilde{M},p)$ are (topologically) isomorphic to G, which is a topological subgroup of S^1 but is not itself a Lie group. Similar examples can be obtained for other S^1-actions or $S^1 \times S^1$-actions or actions by more general compact groups leaving (M,p) invariant, where M can also be of any codimension.

As a remarkable consequence of Theorem 1.1 and the mentioned results [BER97, Z97], our construction provides germs of smooth generic submanifolds (even of strongly pseudoconvex hypersurfaces) that are not CR-equivalent to any germ of any real-analytic CR-manifold. Recall that (M,p) is called *finitely nondegenerate* if

$$\mathsf{span}_{\mathbb{C}}\left\{L_1\dots L_k\rho^j_Z(p) : k \geq 0,\, 1 \leq j \leq d\right\} = \mathbb{C}^n, \qquad (1.2)$$

where $\rho = (\rho^1, \ldots, \rho^d)$ is a defining function of M near p with $\partial\rho^1 \wedge \cdots \wedge \partial\rho^d \neq 0$, ρ^j_Z denotes the complex gradient of ρ^j in $\mathbb{C}^n$ and the span is taken over all collections of germs at p of smooth $(0,1)$ vector fields $L_1 \ldots L_k$ on M. We have:

Corollary 1.3. *For any germ (M,p) of a smooth generic submanifold in $\mathbb{C}^n$, which is finitely nondegenerate and of finite type, and which is invariant under the group $S^1 := \{z \in \mathbb{C} : |z| = 1\}$ acting on $\mathbb{C}^n$ by multiplication, there exists a germ of another smooth generic submanifold $(\widetilde{M},p)$, tangent to (M,p) of infinite order, which is not CR-equivalent to any germ of a real-analytic generic submanifold of $\mathbb{C}^n$.*

Indeed, since S^1 has many dense subgroups G satisfying the assumptions of Theorem 1.1 (e.g. the subgroup in (1.1)), Theorem 1.1 yields a germ $(\widetilde{M},p)$ whose local CR stability group is isomorphic to G and hence is not topologically isomorphic to any Lie group (and not locally compact). On the other hand, the local CR stability group of any real-analytic generic submanifold of $\mathbb{C}^n$, which is CR-equivalent to $(\widetilde{M},p)$ (and hence is also finitely nondegenerate and of finite type), is known to be always a Lie group (see [BER97] for hypersurfaces and [Z97] for higher codimension). Since the local CR stability group is a CR-invariant, $(\widetilde{M},p)$ cannot be CR-equivalent to any germ of a real-analytic generic submanifold of $\mathbb{C}^n$.

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2. Jet spaces and jet groups

Here we recall the jet terminology and introduce the notation that will be used throughout the paper. Recall that, given two complex manifolds X and X' and an integer $k \geq 0$, a *k-jet of a holomorphic map* is an equivalence class of holomorphic maps from open neighborhoods of x in X into X' with fixed partial derivatives at x up to order k. Denote by $J^k_x(X,X')$ the set of all such k-jets. The union $J^k(X,X') := \bigcup_{x\in X} J^k_x(X,X')$ carries a natural fiber bundle structure over X. For a holomorphic map f from a neighborhood of x in X into X', denote by $j^k_x f \in J^k_x(X,X')$ the corresponding k-jet. In local coordinates, $j^k_x f$ can be represented by the coordinates of the reference point x and all partial derivatives of f at x up to order k. If X and X' are smooth algebraic varieties, $J^k_x(X,X')$ and $J^k(X,X')$ are also of this type. We also denote by $J^k_{x,x'}(X,X')$ the space of all k-jets sending x into x'. The subset $G^k_x(X) \subset J^k_{x,x}(X,X)$ of all *invertible* k-jets forms an algebraic group with respect to composition. Completely analogously *k-jets of smooth maps* between smooth real manifolds M and M' are defined, for which we shall use the same notation $J^k(M,M')$. The possible confusion will be eliminated by the convention that we write $X_{\mathbb{R}}$ whenever we consider a complex manifold X as a real manifold. Thus, if X and X' are complex manifolds, $J^k(X,X')$ is the space of all k-jets of holomorphic maps and $J^k(X_{\mathbb{R}},X'_{\mathbb{R}})$ is the space of all k-jets of smooth maps.

Furthermore, we shall need k-jets of real submanifolds of fixed dimension of a smooth real manifold M. Let $\mathcal{C}_x^{k,m}(M)$ be the set of all germs at x of real C^k-smooth m-dimensional submanifolds of M through x. We say that two germs $V, V' \in \mathcal{C}_x^{k,m}(M)$ are *k-equivalent*, if, in a local coordinate neighborhood of x of the form $U_1 \times U_2$, V and V' can be given as graphs of smooth maps $\varphi, \varphi' \colon U_1 \to U_2$ such that $j_{x_1}^k \varphi = j_{x_1}^k \varphi'$, where $x = (x_1, x_2) \in U_1 \times U_2$. Denote by $J_x^{k,m}(M)$ the set of all k-equivalence classes of $\mathcal{C}_x^{k,m}(M)$ and by $J^{k,m}(M)$ the union $\bigcup_{x \in M} J_x^{k,m}(M)$ with the natural fiber bundle structure over M. Furthermore, for any real C^k-smooth m-dimensional submanifold $V \subset M$ through x, denote by $j_x^k(V) \in J_x^{k,m}(M)$ the corresponding k-jet. The space $J_x^{k,m}(M)$ carries a natural real (nonsingular) algebraic variety structure.

We now introduce the notions of equivalence and rigidity that will be crucial in the sequel.

Definition 2.1. (1) Two k-jets of real submanifolds of the same dimension $\Lambda_j \in J_{p_j}^{k,m}(\mathbb{C}_{\mathbb{R}}^n)$, $j = 1, 2$, are called *biholomorphically equivalent* if there exists a germ of a biholomorphic map $(\mathbb{C}^n, p_1) \to (\mathbb{C}^n, p_2)$ sending Λ_1 to Λ_2.
(2) A C^k-smooth generic submanifold $M \subset \mathbb{C}^n$ is called *totally rigid of order k*, if for any $p_1 \neq p_2 \in M$, the jets $j_{p_1}^k(M)$ and $j_{p_2}^k(M)$ are not biholomorphically equivalent in the sense of (1).
(3) A k-jet $\Lambda \in J_p^{k,m}(\mathbb{C}_{\mathbb{R}}^n)$ is called *totally rigid* if any C^k-smooth submanifold passing through p and having Λ as its k-jet at p, contains a neighborhood of p that is totally rigid of order $k-1$ in the sense of (2).

Example 2.2. Any 0-jets of real submanifolds at p are obviously biholomorphically equivalent and 1-jets are equivalent if and only if their CR-dimensions at p are the same. Two 2-jets of generic submanifolds are equivalent if and only their Levi forms at p are linearly equivalent (e.g. of the same rank and signature in the hypersurface case). Furthermore it follows from the Chern-Moser theory [CM74] that two k-jets of Levi-nondegenerate hypersurfaces of the same signature are always biholomorphically equivalent for $k \leq 5$ in case $n = 2$ and for $k \leq 3$ in case $n > 2$, but may not be equivalent in general for k larger.

It is crucial for our method to consider the total rigidity of order $k-1$ in (3) (rather then e.g. of order k) for the representing submanifolds M with $j_p^k(M) = \Lambda$. This allows us to achieve the total rigidity of M of order $k-1$ near p by ensuring that the first order derivatives of $j_q^{k-1}(M)$ at p as function of $q \in M$ have suitable transversality property with respect to the submanifolds (orbits) of biholomorphically equivalent $(k-1)$-jets (see the proof of Proposition 2.3 below for more details). Thus we need Λ to be of higher order than $k-1$ to include the extra derivatives. More precisely, the existence of totally rigid jets is guaranteed by the following statement.

Proposition 2.3. *For fixed integers $n < m < 2n$, a point $p \in \mathbb{C}^n$ and sufficiently large k (depending on n and m but not on p), the set of all totally rigid k-jets in $J_p^{k,m}(\mathbb{C}_{\mathbb{R}}^n)$ contains an open dense subset.*

In fact we show that the number k in Proposition 2.3 can be chosen such that the following inequality holds:

$$(2n-m)\left(\binom{k+m-1}{m}-1\right)-2n\left(\binom{k+n-1}{n}-1\right)\geq m. \tag{2.1}$$

The proof will be based on the following lemmas (of which the first is standard and provided with a short proof for the reader's convenience). We write $\|\cdot\|_{C^l}$ for the standard C^l norm.

Lemma 2.4. *Let $\varphi\colon \mathbb{R}^n\to\mathbb{R}^m$ be a smooth map and V be an open neighborhood in $\mathbb{R}^n$ of a point $a\in\mathbb{R}^n$. Then for any $\varepsilon>0$ and any integers $0\leq k\leq l$, there exists $\delta>0$ such that, if $\Lambda\in J^k_a(\mathbb{R}^n,\mathbb{R}^m)$ is a k-jet with $\|\Lambda-j^k_a\varphi\|<\delta$, then there exists another smooth map $\widetilde{\varphi}\colon\mathbb{R}^n\to\mathbb{R}^m$ such that $j^k_a\widetilde{\varphi}=\Lambda$, $\widetilde{\varphi}|_{\mathbb{R}^n\setminus V}=\varphi|_{\mathbb{R}^n\setminus V}$ and $\|\widetilde{\varphi}-\varphi\|_{C^l}<\varepsilon$.*

Proof. Without loss of generality, V is bounded. We shall look for a map $\widetilde{\varphi}\colon\mathbb{R}^n\to\mathbb{R}^m$ of the form

$$\widetilde{\varphi}(x):=\varphi(x)+\chi(x)\cdot(\psi(x)-\varphi(x)), \tag{2.2}$$

where $\psi\colon\mathbb{R}^n\to\mathbb{R}^m$ is a smooth map with $j^k_a\psi=\Lambda$ and $\chi\colon\mathbb{R}^n\to\mathbb{R}$ is a fixed smooth function which is 1 in a neighborhood of a and 0 outside V. Then $j^k_a\widetilde{\varphi}=\Lambda$ and $\widetilde{\varphi}|_{\mathbb{R}^n\setminus V}=\varphi|_{\mathbb{R}^n\setminus V}$. Furthermore, there exists $C>0$ depending on χ but not on ψ such that

$$\|\widetilde{\varphi}-\varphi\|_{C^l}<C\|\psi-\varphi\|_{C^l}.$$

If δ is sufficiently small and Λ satisfies our assumption, we can always choose ψ with $\|\psi-\varphi\|_{C^l}<\varepsilon/C$ on the closure $\overline{V}$. Then the map $\widetilde{\varphi}$ given by (2.2) satisfies the required properties. □

Lemma 2.5. *Let $k,l,s\geq 0$ and $0\leq r\leq m$ be any integers, $U\subset J^l(\mathbb{R}^m,\mathbb{R}^s)$ be an open set and $F\colon U\to\mathbb{R}^r$ be a smooth map of constant rank r. Then, for any nonempty open set $B\subset\mathbb{R}^m$ and a smooth map $f\colon B\to\mathbb{R}^s$ satisfying $j^l_xf\in U$ for all $x\in B$, there exists another nonempty open subset $\widehat{B}\subset B$ and another smooth map $\widehat{f}\colon\widehat{B}\to\mathbb{R}^s$ such that the following holds:*

1. *the C^k norm of $\widehat{f}-(f|_{\widehat{B}})$ can be chosen arbitrarily small;*
2. *$j^l_x\widehat{f}\in U$ for all $x\in\widehat{B}$;*
3. *the map $x\in\widehat{B}\mapsto F(j^l_x\widehat{f})\in\mathbb{R}^r$ is also of constant rank r.*

Proof. Without loss of generality, the integer k (used only in (1)) is $\geq l$. We prove the lemma by induction on r. For $r=0$ the statement is trivial. Suppose it holds for any $r<r_0\leq m$ and we are given a map $F\colon U\to\mathbb{R}^{r_0}$ as in the lemma. Consider the standard splitting $\mathbb{R}^{r_0}=\mathbb{R}^{r_0-1}\times\mathbb{R}$ and the corresponding components $F_1\colon U\to\mathbb{R}^{r_0-1}$ and $F_2\colon U\to\mathbb{R}$ of F (so that $F=(F_1,F_2)$). Then the induction assumption for F_1 yields a map $\widehat{f}\colon\widehat{B}\to\mathbb{R}^s$ such that $\|\widehat{f}-(f|_{\widehat{B}})\|_{C^k}$ is arbitrarily small, $j^l_x\widehat{f}\in U$ for all $x\in\widehat{B}$ and the map $\Phi_1(x):=F_1(j^l_x\widehat{f})\in\mathbb{R}^{r_0-1}$ is of constant rank $r_0-1<m$ for $x\in\widehat{B}$. By shrinking $\widehat{B}$ if

necessary, we may assume it is connected and of the form $\widehat{B} = \widehat{B}_1 \times \widehat{B}_2 \subset \mathbb{R}^{m-r_0+1} \times \mathbb{R}^{r_0-1}$ and such that, for some $c \in \mathbb{R}^{r_0-1}$, the level set $C := \{x \in \widehat{B} : \Phi_1(x) = c\}$ is a graph of a smooth map $\varphi\colon \widehat{B}_1 \to \widehat{B}_2$.

We now consider two points $x_1 \neq x_2 \in C$ and look for a small perturbation $\widetilde{f}$ of $\widehat{f}$ and $\widetilde{x}_2$ of x_2 such that x_1 and $\widetilde{x}_2$ still belong to the same level set $\widetilde{C}$ of $\widetilde{\Phi}_1(x) := F_1(j^l_x \widetilde{f})$ but $F_2(j^l_{x_1}\widetilde{f}) \neq F_2(j^l_{\widetilde{x}_2}\widetilde{f})$. More precisely, suppose that $F(j^l_{x_1}\widehat{f}) = F(j^l_{x_2}\widehat{f})$ (otherwise no perturbation is needed). By the assumption of the lemma, the map $F = (F_1, F_2)$ has constant rank r_0 in U. Since $r_0 \leq m \leq \dim U$, we can find a point $\widetilde{x}_2 \in \widehat{B}$ and a jet $\Lambda \in U \cap J^l_{\widetilde{x}_2}(\mathbb{R}^m, \mathbb{R}^s)$ arbitrarily close to $j^l_{x_2}\widehat{f}$ such that $F_1(\Lambda) = F_1(j^l_{x_2}\widehat{f})$ but $F_2(\Lambda) \neq F_2(j^l_{x_2}\widehat{f})$. Since Λ can be chosen arbitrarily close to $j^l_{x_2}\widehat{f}$, it can be represented by a smooth map $\widetilde{f}$ that is arbitrarily close to $\widehat{f}$ in the C^k norm and differs from it on a neighborhood of x_2 with compact support in $\widehat{B} \setminus \{x_1\}$ in view of Lemma 2.4. By choosing the norm $\|\widetilde{f} - \widehat{f}\|_{C^k}$ sufficiently small, we shall preserve the properties that $j^l_x\widetilde{f} \in U$ for all $x \in \widehat{B}$, the rank of $\widetilde{\Phi}_1(x) := F_1(j^l_x\widetilde{f})$ is still $r_0 - 1$ and the level set $\widetilde{C} := \{x \in \widehat{B} : \widetilde{\Phi}_1(x) = c\}$ is still a graph of a smooth map $\widetilde{\varphi}\colon \widehat{B}_1 \to \widehat{B}_2$. Hence $\widetilde{C}$ is a connected manifold containing two points $x_1, \widetilde{x}_2 \in \widetilde{C}$ such that $F_2(j^l_{x_1}\widetilde{f}) \neq F_2(j^l_{\widetilde{x}_2}\widetilde{f})$. The latter fact implies that the function $\widetilde{\Phi}_2(x) := F_2(j^l_x\widetilde{f})$ is not constant on $\widetilde{C}$ and therefore its differential is somewhere nonzero. Putting this property together with the rank property of $\widetilde{\Phi}_1$, we conclude that the rank of $\widetilde{\Phi}(x) := F(j^l_x\widetilde{f})$ is r_0 at some point $x_0 \in \widehat{B}$. The required conclusion is obtained by replacing $\widehat{f}$ with $\widetilde{f}$ and $\widehat{B}$ with a sufficiently small open neighborhood of x_0. □

Proof of Proposition 2.3. We may assume $p = 0$. Consider the natural action of the group $G^{k-1}_0(\mathbb{C}^n)$ (consisting of all $(k-1)$-jets at 0 of local biholomorphic maps of $\mathbb{C}^n$) on the space $J^{k-1,m}_0(\mathbb{C}^n_{\mathbb{R}})$ (consisting of all $(k-1)$-jets at 0 of real m-dimensional submanifolds of $\mathbb{C}^n_{\mathbb{R}}$ passing through 0. The dimensions of the jet spaces can be computed directly:

$$\dim_{\mathbb{R}} G^{k-1}_0(\mathbb{C}^n) = 2n\left(\binom{k+n-1}{n} - 1\right), \quad \dim_{\mathbb{R}} J^{k-1,m}_0(\mathbb{C}^n_{\mathbb{R}}) = (2n-m)\left(\binom{k+m-1}{m} - 1\right). \tag{2.3}$$

Hence the inequality (2.1) is equivalent to

$$\dim_{\mathbb{R}} J^{k-1,m}_0(\mathbb{C}^n_{\mathbb{R}}) - \dim_{\mathbb{R}} G^{k-1}_0(\mathbb{C}^n) \geq m. \tag{2.4}$$

In particular, for any sufficiently large k, all orbits of $G^{k-1}_0(\mathbb{C}^n)$ in $J^{k-1,m}_0(\mathbb{C}^n_{\mathbb{R}})$ have their (real) codimension at least m. In the rest of the proof we shall assume that (2.4) is satisfied.

It is easy to see that $G^{k-1}_0(\mathbb{C}^n)$ is an algebraic group acting rationally on $J^{k-1,m}_0(\mathbb{C}^n_{\mathbb{R}})$ by calculating the group operation and the action in local coordinates. Hence the orbits of

$G_0^{k-1}(\mathbb{C}^n)$ form, on an open dense subset $\Omega \subset J_0^{k-1,m}(\mathbb{C}^n_{\mathbb{R}})$, a foliation into real submanifolds of a fixed constant codimension $\geq m$. Consider any k-jet $\Lambda_0 \in J_0^{k,m}(\mathbb{C}^n_{\mathbb{R}})$ represented by the graph of a C^∞-smooth map $\varphi_0 \colon \mathbb{R}^m \to \mathbb{R}^{2n-m}$ with $\varphi_0(0) = 0$, where we choose a suitable identification of $\mathbb{C}^n_{\mathbb{R}}$ with $\mathbb{R}^m \times \mathbb{R}^{2n-m}$ (after a possible permutation of the real coordinates). Thus $\Lambda_0 = j_0^k((\mathsf{id} \times \varphi_0)(\mathbb{R}^m))$. By the density of Ω, we can find another C^∞-smooth map $\varphi \colon \mathbb{R}^m \to \mathbb{R}^{2n-m}$ with $\varphi(0) = 0$ and $\Theta := j_0^k\varphi$ arbitrarily close to $j_0^k\varphi_0$ such that the $(k-1)$-jet $\Lambda \in J_0^{k-1,m}(\mathbb{C}^n_{\mathbb{R}})$ at 0 of the graph of φ is contained in Ω. (By this choice, also Λ is arbitrarily close to Λ_0.) We can now find an open neighborhood U of $j_0^{k-1}\varphi$ in $J^{k-1}(\mathbb{R}^m, \mathbb{R}^{2n-m})$ and a smooth map $F \colon U \to \mathbb{R}^m$ of constant rank m and constant on the orbits (recall that m does not exceed the orbit codimension), such that two $(k-1)$-jets $\Lambda_j \in J^{k-1,m}_{(x_j,x'_j)}(\mathbb{C}^n_{\mathbb{R}})$, $j = 1, 2$, near Λ_0 (where $(x_j, x'_j) \in \mathbb{R}^m \times \mathbb{R}^{2n-m}$), which are represented by graphs of some smooth maps $\varphi_1, \varphi_2 \colon \mathbb{R}^m \to \mathbb{R}^{2n-m}$, are biholomorphically inequivalent (in the sense of Definition 2.1) whenever $F(j^{k-1}_{x_1}\varphi_1) \neq F(j^{k-1}_{x_2}\varphi_2)$. Here F can be obtained by taking the first m coordinates in any real coordinate system $(x, y) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$, for which the orbits are given by $x = \text{const}$. Then Lemma 2.5 can be applied to U, F, $f := \varphi$ and an arbitrarily small neighborhood of $B \subset \mathbb{R}^m$ of 0. Let $\widehat{B}$ and $\widehat{f} \colon \widehat{B} \to \mathbb{R}^{2n-m}$ be given by the lemma.

We claim that, for any $x_0 \in \widehat{B}$, the k-jet $\Lambda(x_0) \in J^{k,m}(\mathbb{C}^n_{\mathbb{R}})$ of the graph of $\widehat{f}$ at $(x_0, \widehat{f}(x_0))$ is totally rigid in the sense of Definition 2.1 (3). Indeed, fix any $x_0 \in \widehat{B}$ and consider any C^k-smooth real m-dimensional submanifold $V \subset \mathbb{C}^n_{\mathbb{R}}$ passing through $(x_0, \widehat{f}(x_0))$ with $j^k_{(x_0,\widehat{f}(x_0))}(V) = \Lambda(x_0)$. By shrinking V, if necessary, we may assume that V is a graph of a smooth map $g \colon B(x_0) \to \mathbb{R}^{2n-m}$, where $B(x_0)$ is a suitable open neighborhood of x_0. Then $j^k_{x_0}g = j^k_{x_0}\widehat{f}$ and therefore, the ranks of the maps $x \mapsto F(j^{k-1}_x g)$ and $x \mapsto F(j^{k-1}_x \widehat{f})$ coincide at $x = x_0$. (Here is the step, where we use the different integers k and $k-1$ for x_0 and points nearby respectively.) By property (3) in Lemma 2.5, the rank of the second map is m and hence, so is the rank of the first map. But the latter fact implies that $F(j^{k-1}_{x_1}g) \neq F(j^{k-1}_{x_2}g)$ for any $x_1 \neq x_2$ sufficiently close to x_0. In view of the choice of F, it follows that the jets $j^{k-1}_{(x_1,g(x_1))}(V)$ and $j^{k-1}_{(x_2,g(x_2))}(V)$ are biholomorphically inequivalent for any such $x_1 \neq x_2$, which is precisely what is needed to show that $\Lambda(x_0)$ is totally rigid. It remains to observe, that any translation of $\Lambda(x_0)$ is also totally rigid, hence we can find totally rigid k-jets also in $J_0^{k,m}(\mathbb{C}^n_{\mathbb{R}})$ arbitrarily close to the original jet Λ_0. The proof is complete. □

3. Realization of certain groups as CR stability groups

In the sequel we shall use the same letter for a germ and its representative unless there will be a danger of confusion. We begin with a standard lemma, whose proof is given here for the reader's convenience.

Lemma 3.1. *Let $(G_k)_{k\geq 1}$ be an increasing sequence of finite groups of germs of local biholomorphic maps of $\mathbb{C}^n$ in a neighborhood of a point p fixing that point. Let M be a smooth real submanifold of $\mathbb{C}^n$ passing through p. Let $(D_k)_{k\geq 1}$ be a sequence of domains containing p and such that, for each k, the germs from G_k can be represented by biholomorphic self-maps of D_k. Then there exist a sequence of points $p_k \in M$, $k \geq 1$, converging to p and a sequence of mutually disjoint open neighborhoods V_k of p_k in $\mathbb{C}^n$ such that $\overline{V}_k \subset D_k$ and, if $g(\overline{V}_k) \cap \overline{V}_l \neq \emptyset$ for some k, l and $g \in G_k$, then necessarily $k = l$ and $g \equiv \mathsf{id}$.*

Note that the existence of domains D_k easily follows from the finiteness of each G_k. Indeed, if $\widetilde{D}_k$ is any domain where all germs from G_k biholomorphically extend, it suffices to take $D_k := \bigcap_{g\in G_k} g(\widetilde{D}_k)$.

Proof. We shall construct $p_k \in M$ and $V_k \subset \mathbb{C}^n$ inductively. Let $k = 1$. Since G_1 is finite, the set of points $x \in D_1$, such that there are two elements $g_1 \neq g_2 \in G_1$ with $g_1(x) = g_2(x)$, is a complement of a proper analytic subset. Hence we can choose $p_1 \in M$ and a neighborhood $V_1 \subset\subset D_1$ of p_1 in $\mathbb{C}^n$ with $p \notin \overline{V}_1$ such that $g(\overline{V}_1) \cap \overline{V}_1 \neq \emptyset$ for $g \in G_1$ implies $g \equiv \mathsf{id}$.

Now suppose that p_l and V_l with $p \notin \overline{V}_l$ have been chosen for all $l < k$. Since G_k is finite, we can choose a neighborhood U of p in D_k such that $g(U) \cap \overline{V}_l = \emptyset$ for all $l < k$ and $g \in G_k$. Using the same argument as before we can choose p_k arbitrarily close to p and V_k with $p \notin \overline{V}_k$ such that $p_k \in V_k \subset \overline{V}_k \subset U$ and, for any $g \not\equiv \mathsf{id} \in G_k$, $g(\overline{V}_k) \cap \overline{V}_k = \emptyset$. Since $\overline{V}_k \subset U$ and $g(U) \cap \overline{V}_l = \emptyset$ for all $l < k$ and $g \in G_k$, it follows that $g(\overline{V}_k) \cap \overline{V}_l \neq \emptyset$ can hold for some $l \leq k$ and $g \in G_k (\supset G_l)$ if and only if $k = l$ and $g \equiv \mathsf{id}$. It is easy to see that the sequences (p_k) and (V_k) so constructed satisfy the required properties. □

Proof of Theorem 1.1. Since the statement is local, we fix an identification $\mathbb{C}^n_{\mathbb{R}} \cong \mathbb{R}^m \times \mathbb{R}^{2n-m}$ near p such that M is represented by the graph of a smooth map $\varphi\colon \mathbb{R}^m \to \mathbb{R}^{2n-m}$ with $\|\varphi\|_{C^1}$ sufficiently small. We shall write $B_r(a)$ for the open ball with center a and radius r with respect to the product metric of the Euclidean metrics of $\mathbb{R}^m$ and $\mathbb{R}^{2n-m}$. With this choice of the metric on $\mathbb{R}^m \times \mathbb{R}^{2n-m}$ and $\|\varphi\|_{C^1}$ sufficiently small, we have the property that, for any $a \in M$, the intersection $B_r(a) \cap M$ coincides with the graph of φ over the projection of $B_r(a)$ to $\mathbb{R}^m$ (which is an Euclidean ball in $\mathbb{R}^m$). In the course of the proof we shall consider small perturbations of M obtained as graphs of small perturbations of φ. We shall always assume that the C^1 norms of these perturbations are still small, so that the mentioned relation between ball intersections with their graphs and graphs over balls still holds.

By the assumption, $G = \bigcup_k G_k$, where G_k, $k \geq 1$, is an increasing sequence of finite groups of local biholomorphic maps of $\mathbb{C}^n$ in a neighborhood of p, fixing p and preserving the germ (M, p). As indicated above, we can choose a decreasing sequence of open neighborhoods D_k of p in the unit ball $B_1(p)$ in $\mathbb{C}^n$ centered at p, such that, for each k, all germs in G_k can be represented by biholomorphic self-maps of a neighborhood of $\overline{D}_k$. In the sequel we shall identify the germs from G_k with their biholomorphic representatives.

Let $p_k \in D_k \cap M$ and V_k be given by Lemma 3.1. Moreover, we can make the above choice of p_k, D_k, V_k inductively such that, in addition,

$$\max\Big(\sup_{z\in D_{k+1},g\in G_{k+1}} \|g'(z)\|, \sup_{z\in D_k,g\in G_k} \|g'(z)\|\Big)\frac{\sup_{z\in V_{k+1}}|z-p|}{\inf_{z\in V_k}|z-p|} \to 0, \quad k\to\infty, \tag{3.1}$$

where g' is the Jacobian matrix of g.

For an l-jet $\Lambda \in J^{l,m}_x(\mathbb{C}^n_{\mathbb{R}})$, we denote by $\mathsf{Orb}(\Lambda) \subset J^{l,m}(\mathbb{C}^n_{\mathbb{R}})$ the set of all l-jets $\widetilde{\Lambda} \in J^{l,m}_y(\mathbb{C}^n_{\mathbb{R}})$ for all $y \in \mathbb{C}^n_{\mathbb{R}}$ that are biholomorphically equivalent to Λ (in the sense of Definition 2.1). Then it follows from (2.3) that, for l sufficiently large, the subset $\bigcup_{x\in M\cap D_1}\mathsf{Orb}(j^l_x M) \subset J^{l,m}(\mathbb{C}^n_{\mathbb{R}})$ has Lebesgue measure zero and therefore its complement is dense in $J^{l,m}(\mathbb{C}^n_{\mathbb{R}})$. The same argument obviously applies to any other real submanifold of $\mathbb{C}^n$ of the same dimension as M. We shall use this property to choose jets that are not in the certain unions of orbits. We shall consider l sufficiently large so that this choice is always possible.

We next consider a sequence ε_k, $0<\varepsilon_k<1$, converging to 0 and such that

$$\varepsilon_k\Big(\sup_{x\in D_k,g\in G_k}\|g'(x)\|\Big) \to 0, \quad k\to\infty. \tag{3.2}$$

Then, as a consequence of Lemma 2.4 and Proposition 2.3, we can find a sequence of neighborhoods U_k of p_k in V_k with $\overline{U}_k \subset V_k$ and a sequence of graphs N_k of smooth maps $\varphi_k\colon \mathbb{R}^m \to \mathbb{R}^{2n-m}$ with

$$\|\varphi_k - \varphi\|_{C^k} < \varepsilon_k, \quad N_k\setminus\overline{U}_k = M\setminus\overline{U}_k, \tag{3.3}$$

and such that the following holds. There exist points $q_k \in N_k$ and real numbers $\delta_k > 0$ such that, for each k, the $2\delta_k$-neighborhood of q_k in N_k, $B_{2\delta_k}(q_k)\cap N_k$, is contained in U_k, totally rigid (in the sense of Definition 2.1) and

$$j^l_{q_k}N_k \notin \bigcup_{x\in M\cap D_1}\mathsf{Orb}(j^l_x M). \tag{3.4}$$

As the next step, we define $X_k \subset N_k\cap U_k$ to be the subset of all points $y\neq q_k$ for which there exists a CR-diffeomorphism from $B_{\varepsilon_k\delta_k}(y)\cap N_k$ into $B_{\delta_k}(q_k)\cap N_k$, sending y into q_k. Since the CR-manifold $B_{2\delta_k}(q_k)\cap N_k$ is totally rigid by our construction, it is clear that for any $y_1\neq y_2\in X_k$, the neighborhood $B_{\varepsilon_k\delta_k}(y_1)\cap N_k$ cannot contain y_2. Hence the neighborhoods $B_{\frac{\varepsilon_k\delta_k}{2}}(y)\cap N_k$, $y\in X_k$, do not intersect and so X_k must be a finite set. It is also clear that $X_k\cap B_{2\delta_k}(q_k)\cap N_k=\emptyset$ again by the total rigidity of $B_{2\delta_k}(q_k)\cap N_k$. Furthermore we must have $X_k\subset N_k\cap U_k$ in view of (3.3) and (3.4).

We next choose a sequence η_k, $0<\eta_k<\varepsilon_k\delta_k$ and apply again Lemma 2.4 to obtain a sequence of graphs M_k of smooth maps $\psi_k\colon\mathbb{R}^m\to\mathbb{R}^{2n-m}$ with

$$\|\psi_k-\varphi_k\|_{C^k}<\varepsilon_k \tag{3.5}$$

and finite subsets $\widetilde{X}_k \subset M_k$ with

$$j_y^l M_k \notin \bigcup_{x\in N_k\cap D_1} \mathsf{Orb}(j_x^l N_k) \quad \forall y \in \widetilde{X}_k, \tag{3.6}$$

and such that, for $W_k := \bigcup_{y\in\widetilde{X}_k} B_{\eta_k}(y)$,

$$X_k \cap M_k = \emptyset, \quad M_k \setminus \overline{W_k} = N_k \setminus \overline{W_k}, \quad \overline{W_k} \cap B_{\delta_k}(q_k) = \emptyset. \tag{3.7}$$

We may in addition assume that η_k is sufficiently small so that $\overline{W}_k \subset V_k$.

Finally, we define the new generic submanifold $\widetilde{M} \subset \mathbb{C}^n$ by replacing $g(M \cap V_k)$ with $g(M_k \cap V_k)$ for every sufficiently large k and every $g \in G_k$, i.e.

$$\widetilde{M} := \Big(M \setminus \bigcup_{k\geq k_0, g\in G_k} g(M\cap V_k)\Big) \bigcup \Big(\bigcup_{k\geq k_0, g\in G_k} g(M_k \cap V_k)\Big), \tag{3.8}$$

where k_0 is sufficiently large. (Note that all neighborhoods $g(V_k)$, $g \in G_k$, $k \geq k_0$, are disjoint together with their closures since they are given by Lemma 3.1.) Then $\widetilde{M}$ is a smooth submanifold through p and, if ε_k have been chosen sufficiently rapidly converging to 0, $\widetilde{M}$ is tangent to M of infinite order at p in view of (3.3) and (3.5). Consequently $\widetilde{M}$ is also of finite type at p. Furthermore, the germ $(\widetilde{M}, p)$ is clearly invariant under the action of G, i.e. the group $\mathsf{Aut}_{\mathrm{CR}}(\widetilde{M}, p)$ of germs at p of all local CR-automorphisms of M fixing p contains G.

We now claim that $\mathsf{Aut}_{\mathrm{CR}}(\widetilde{M}, p) = G$. Indeed, fix any $f \in \mathsf{Aut}_{\mathrm{CR}}(\widetilde{M}, p)$ and its representative defined in some neighborhood of p in $\widetilde{M}$, denoted by the same letter. Then for k sufficiently large, f is defined in $D_k \cap \widetilde{M}$ with $f(D_k \cap \widetilde{M}) \subset D_1$. By the construction, each $q_k \in N_k \cap U_k$ is not contained in W_k, hence it is in $M_k \cap V_k$ and therefore in $\widetilde{M}$ so that we can evaluate $f(q_k)$. Then (3.4) implies that $f(q_k) \notin M$ and hence $f(q_k) \in g(U_s) \subset g(V_s)$ for some s and some $g \in G_s$. Thus we have the estimates

$$(\sup_{D_s} \|(g^{-1})'\|)^{-1} \frac{\inf_{z\in V_s} |z-p|}{\sup_{z\in V_k} |z-p|} \leq \frac{|f(q_k)-p|}{|q_k - p|} \leq \sup_{D_s} \|g'\| \frac{\sup_{z\in V_s}|z-p|}{\inf_{z\in V_k}|z-p|}. \tag{3.9}$$

On the other hand, since f is a local diffeomorphism of $\widetilde{M}$ fixing p, there exist constants $0 < c < C$ such that

$$c \leq \frac{|f(z)-p|}{|z-p|} \leq C, \quad c \leq \|f'(z)\| \leq C \tag{3.10}$$

for $z \neq p \in \widetilde{M}$ sufficiently close to p. Then if k is sufficiently large, we must have $s = k$ in view of (3.1). Hence $f(q_k) \in g_k(U_k)$ for suitable $g_k \in G_k$. Setting $f_k := g_k^{-1} \circ f \in \mathsf{Aut}_{\mathrm{CR}}(M, p)$, we have $f_k(q_k) \in U_k \cap \widetilde{M}$. In view of (3.8), this means $f_k(q_k) \in U_k \cap M_k$.

We now claim that, for k sufficiently large, we must have $B_{\varepsilon_k\delta_k}(f_k(q_k)) \cap \widetilde{M} \subset N_k$. Indeed, otherwise, in view of (3.7), we would have $W_k \cap B_{\varepsilon_k\delta_k}(f_k(q_k)) \neq \emptyset$ and, since

$\eta_k < \varepsilon_k\delta_k$, it would imply $\widetilde{X}_k \cap B_{2\varepsilon_k\delta_k}(f_k(q_k)) \neq \emptyset$. However, in view of (3.2) and (3.10), this would mean that, for k sufficiently large and some point $y \in \widetilde{X}_k$, the inclusion $f_k^{-1}(y) \in B_{\delta_k}(q_k) \cap \widetilde{M}$ would hold. By our construction, $B_{\delta_k}(q_k) \cap \widetilde{M} \subset N_k$ and hence we would have a contradiction with (3.6). Hence we have $B_{\varepsilon_k\delta_k}(f_k(q_k)) \cap \widetilde{M} \subset N_k$ as claimed. Thus $B_{\varepsilon_k\delta_k}(f_k(q_k)) \cap \widetilde{M} = B_{\varepsilon_k\delta_k}(f_k(q_k)) \cap N_k$. Again, using (3.2) and (3.10), we conclude that, for k sufficiently large, f_k^{-1} sends $B_{\varepsilon_k\delta_k}(f_k(q_k)) \cap N_k$ into $B_{\delta_k}(q_k) \cap N_k$. By our construction of the set $X_k \subset N_k$, the latter conclusion means either $f_k(q_k) \in X_k$ or $f_k(q_k) = q_k$. The first case is impossible in view of the first condition in (3.7). Hence we have $f_k(q_k) = q_k$ and, since a neighborhood of q_k in M_k is totally rigid, this means $f_k \equiv \mathsf{id}$ in a neighborhood of q_k. Since $(\widetilde{M}, p)$ is of finite type, we have $f_k \equiv \mathsf{id}$ as germs at p, and hence $f \equiv g_k \in G$ implying the desired conclusion. □

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