DECONVOLUTION IN REAL TIME OF NOISY SIGNALS

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ABSTRACT

This paper presents an analysis of the constrained least squares filter and a feedback structure is derived which shows the noise cancelling properties of the filter. Using an identification algorithm, it is shown how the constrained least squares filter can be replaced by a finite impulse response filter which can be implemented on-line. The limitations of this F.I.R. filter are discussed. The non-linear behaviour of the constrained least squares (C.L.S.) filter is investigated and, using the proposed feedback structure, it is indicated how a non-linear recursive filter can be identified for on-line implementation.

I INTRODUCTION

Deconvolution of signals, distorted by transmission through a linear system, is a problem that arises in many areas. However, the inversion of a large ill-conditioned matrix(1) makes the solution extremely sensitive in the presence of noisy signals, and this prevents the application of direct deconvolution. By ill-conditioning it is meant that a small perturbation in y gives rise to a large perturbation in x, where the system is given in standard notation (Fig.1) as

$$\underline{y} = H\underline{x} + \underline{e} \quad [1]$$

This phenomenon is inconsistent with the a priori knowledge that the true solution is smooth. Phillips(2) proposed a method for deconvolution where a constraint is introduced to serve as a criterion by which to judge the solution. The constraint used was a smoothness condition based on the second difference. However, the solution is determined off-line and iteratively, and thus is costly in terms of computation time and memory requirements. Furthermore, the solution is only valid for a particular signal to noise ratio. Much work has been carried out(3),(4) to improve the efficiency of this algorithm. However, the need for an on-line filter to compensate for the dynamic distortion introduced by the convolutional processes is evident. Recent work has been along these lines(5),(6).

II THE CONSTRAINED LEAST SQUARES FILTER

The distorting linear system is shown in Fig.1. The received signal is given by

$$y(k) = \sum_{i=k-m}^{k} h(k-i)\, x(i) + e(k) \quad [2]$$

where the channel memory is finite (=mT, where T is the sampling time).

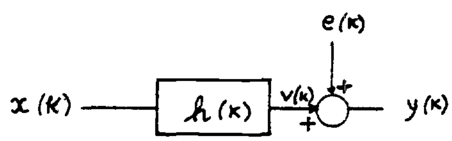

Fig: 1

The additive noise sequence e(k) is assumed to be a stationary zero-mean noise process, with variance σ_e^2. For computational efficiency, the convolution in [2] is replaced by circular convolution. The constrained least squares filter minimises the inner product

$$\langle C\underline{x}, C\underline{x}\rangle \quad [3]$$

subject to the constraint

$$\langle(\underline{y}-H_c\underline{x}),\ (\underline{y}-H_c\underline{x})\rangle = \langle\underline{e},\underline{e}\rangle \quad [4]$$

and has the well known solution

$$\underline{x} = (H_c^T H_c + \gamma C^T C)^{-1} H_c^T \underline{y} \quad [5]$$

The parameter γ is found iteratively using an estimate for $\langle\underline{e},e\rangle$ where

$$\langle\underline{e},\underline{e}\rangle \approx (N-1)\ \sigma_e^2 \quad [6]$$

C is the circulant approximation to the second difference matrix, H_C is the circulant extension of H, and N is the record length, where it has been extended to suppress the double convolution in equation [5] of the point spread function. Since the Discrete Fourier Transform diagonalises a circulant matrix(4), it is possible to write equation [5] in the discrete frequency domain as

$$\hat{x}_n = \frac{h_n^*}{|h_n|^2 + \gamma |c_n|^2} y_n \qquad n=0,\ldots,N-1 \qquad [7]$$

where * denotes the complex conjugate. Using equation [7], it is possible to obtain a feedback structure for the C.L.S. filter in the frequency domain. This is shown in Fig.2

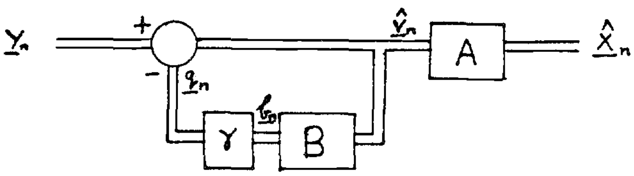

Fig: 2

where $A = \text{diag}\ \{1/h_1,\ 1/h_2, \ldots\ 1/h_N\}$ [8]

$$B = \text{diag}\ \{|\frac{c_1}{h_1}|^2,\ |\frac{c_2}{h_2}|^2 \ldots\ldots |\frac{c_N}{h_N}|^2\} \qquad [9]$$

It is possible to absorb A into the feedback structure, but this only complicates the feedback path. Furthermore, since the system is linear (for constant γ), A may reasonably be placed before the feedback structure. However, in the nonlinear case, since the convolution operator is not commutative, it is necessary to specify which structure is being considered. This will be dealt with in greater detail in section V. Using the structure shown in Fig.2, q_n can be written as

$$q_n = \frac{\gamma |c_n|^2}{|h_n|^2 + \gamma |c_n|^2} y_n \qquad [10]$$

Using equation [4], the residuals can be written as

$$\rho_n = y_n - h_n \hat{x}_n \qquad [11]$$

where $\hat{x}_n$ is found from equation [7]. Hence it is easy to show that

$$\rho_n = q_n \qquad n=0,\ldots N-1 \qquad [12]$$

This suggests that the C.L.S. filter matches the variance of high pass filtered signal q to the estimated variance of the noise.

III LINEAR IDENTIFICATION OF THE C.L.S. FILTER

For a given signal to noise ratio, the C.L.S. filter is linear, provided the complete record is used to reconstruct the transmitted signal. The finite impulse response approximation to the C.L.S. filter is obtained using the techniques of linear system identification. Consider the system shown in Fig.3.

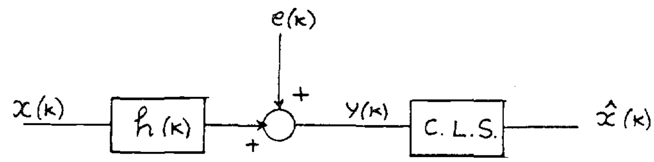

Fig: 3

Since h(K) is assumed known, it is possible to construct an experiment, consistent with Fig.3, where the noise sequence is assumed known. This condition is necessary to characterise the behaviour of the C.L.S. filter. The C.L.S. filter assumes an input to the channel which is smooth, and for this purpose, a sinewave was chosen as the signal to be transmitted, the period of which is assumed known. Hence it is possible to consider the C.L.S. filter as a system being driven by white noise with a periodic mean. Defining

$$y_1(k) = y(k) - y_p(k) \qquad [13]$$

$$\hat{x}_1(k) = \hat{x}(k) - \hat{x}_p(k) \qquad [14]$$

where the subscript p denotes the periodic mean, it is possible to use a cross correlation technique to identify the finite impulse response approximation. Since the C.L.S. filter is off-line, it can compensate for any time delay in the channel impulse response. Therefore it is necessary to use a shifted version of the sequence $y_1(k)$ in the computation of the cross-correlation function between $y_1(k)$ and $x_1(k)$.

The above procedure was carried out for a channel with point spread function [0.04, -0.15, 0.96, 0.20, -0.05] for three different signal to noise ratios, where the SNR is defined as

$$\text{SNR} = 10 \log_{10} \{\frac{\sigma_v^2}{\sigma_e^2}\} \qquad [15]$$

and for three different input frequencies. The identified F.I.R. filter was truncated to 16 points in all cases, and no windowing was used. The parameter γ was found such that it reflected a 5% total error in the satisfaction of the constraint. The following results were obtained, where the input was a sinewave in all cases.

a) For a given signal to noise ratio (14dB), the frequency of the input sinewave is related to the bandwidth of the identified inverse filter (see Table 1). When low frequencies are used in the identification, then the response degrades in the RMS sense, as the frequency content of the input signal gets higher. Here, the RMS error is defined as

$$RMS = \sum_{i=1}^{N} (x(i) - \hat{x}(i))^2/N)^{\frac{1}{2}} \qquad [16]$$

However, when identified at a high input frequency, the filter performed well at lower frequencies.

b) Table II compares the RMS error as the SNR is varied. The input frequency was held constant at $\Pi/8$. It is apparent from the results that the filter performs best when operated at the SNR at which it was identified. When identified at low SNR, the gain of the filter is too low when operated at high SNR, and vice versa.

c) The filter performs well for different types of signals. Fig.4B shows the output from the channel when the input is a damped sinewave

$$x(k) = 4.(0.98)^k \sin(\Pi/8)k \quad k=0,...127 \qquad [17]$$

and noise variance $\sigma_e^2 = 0.32$. Fig.4A shows that the reconstruction using the F.I.R. filter, identified using $W_k = \Pi/8$ at 14dB, stands comparison with that using the CLS filter.

* ω_k \ ω_k	$\pi/4$	$\pi/8$	$\pi/16$
$\pi/4$	0.594	1.444	2.03
$\pi/8$	0.417	0.434	0.746
$\pi/16$	0.41	0.394	0.486

* identified using this frequency

TABLE I

These studies reveal that the approximating linear F.I.R. system for the C.L.S. filter is sensitive to both the signal frequency and the signal to noise ratio. It is seen that for fixed SNR, the approximating F.I.R. can be usefully employed over a frequency range up to the signal frequency at which it was identified. However, the sensitivity of the approximation to SNR limits the practical usefulness of the linear F.I.R. and motivates the study of the non-linear behaviour introduced by the dependence of γ on SNR.

* SNR \ SNR	0dB	14dB	31dB
0dB	1.611	1.57	1.941
14dB	1.254	0.434	0.417
31dB	1.25	0.319	0.147

* identified using this SNR.

TABLE II

IV NONLINEAR BEHAVIOUR OF THE CLS FILTER

Consideration of equations [3] to [6] shows that as the SNR changes, then so too does the parameter γ. Hence it is of particular interest to know how γ behaves as a function of noise variance for constant signal variance. Furthermore, how does γ affect the frequency response of the CLS filter? From equation [6] it is noted that the optimum value of γ is found as a function of the residuals. Fig.5 shows how γ varies as a function of σ_e^2 for different record lengths (N). The function is obviously nonlinear and the difference between the trajectories for different N can be attributed to the actual sample noise variance. In Fig.6 the nonlinear behaviour of γ as the variance of b changes is shown.

The effect that γ has on equations [7] and [10] is shown in Figs.7 and 8. As γ increases, i.e. the SNR decreases, then for this particular channel, the inverse filter has a lower gain and lower cut-off frequency, whereas the noise filter (equation [10]) becomes more high-pass. This confirms the observation that the CLS filter can be viewed as the cascade of a form of noise cancellor with an inverse filter.

NONLINEAR IDENTIFICATION OF THE CLS FILTER

It is apparent from the previous section that the parameter γ causes the non-linear behaviour, and it was shown that it is dependent on the SNR. The parameter γ was isolated within a feedback structure in section II. It is possible to characterise γ as a polynomial function of σ_b^2, and then, approximating σ_b^2 with the instantaneous value b^2, to consider γ as a static non-linear element, determined from the polynomial expansion. Hence the problem reduces to identifying the following structure where both $\underline{A}$ and $\underline{B}$ are assumed to contain static non-linear elements in cascade with linear dynamics (Fig.10). The analysis of this class of systems was considered by George(7) and Brilliant(8). Recently, considerable work has been carried out to identify such systems(9),(10).

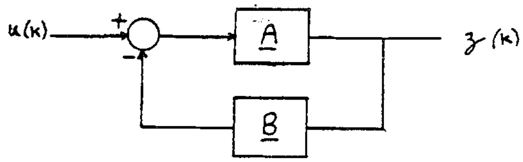

FIG: 9

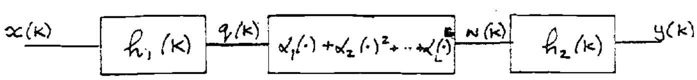

FiG : 10

The systems $\underline{A}$ and $\underline{B}$ are assumed to have Volterra series expansions, and hence it is possible to write down the first two kernels of the feedback system (Fig.9) as (using the notation of George)

$$K_1 = (I + A_1 * B_1)^{-1} * A_1 \qquad [18]$$

$$K_2 = (I + A_1 * B_1)^{-1} * \{-A_1 * B_2 \circ (K_1)^2\} + A_2 \circ (I - B_1 * K_1)^2 \qquad [19]$$

where the subscripts 1,2 indicate the first and second terms, respectively, in the expansions. To avoid truncation errors(11), a restricted class of systems $\underline{A},\underline{B}$ are considered, where $\underline{A}$ and $\underline{B}$ can be the linear, Hammerstein or Wiener models. The structure of Fig.2 falls into one such class. There is sufficient information in K_1 and K_2 to identify the linear subsystems, and hence it is necessary to isolate these kernels. Using separable processes(12) and multilevel inputs(9), it is possible to compute an estimate of K_1 and K_2 from the first and second order cross-correlation functions $\phi_{u'z'}(m), \phi_{u'u'z'}(m)$ where u' and z' indicate zero mean sequences. Having identified the linear subsystems, it is then possible to obtain a polynomial approximation to the non-linear element.

In the case of the CLS filter, the SNR is varied in a random manner for each record of length N_1. The CLS filter then operates on each record and the input and output records are then concatenated. Having computed the periodic mean, as in section III, the identification algorithm can then be applied. This is the subject of current research, and the results will be reported at a later date.

CONCLUSION

The structure of the CLS filter has been shown to comprise of an approximate noise canceller in cascade with an inverse filter. An identified F.I.R. approximation to the CLS filter has been examined, and studies reveal that subject to restrictions on the variation in SNR, the F.I.R. approximation can be usefully employed for deconvolution. However, the exposed limitations of the linear approximation motivate an examination of the SNR dependent nonlinearity which is fundamental to the CLS filter. It is indicated how a non-linear recursive approximation can be identified for on-line implementation.

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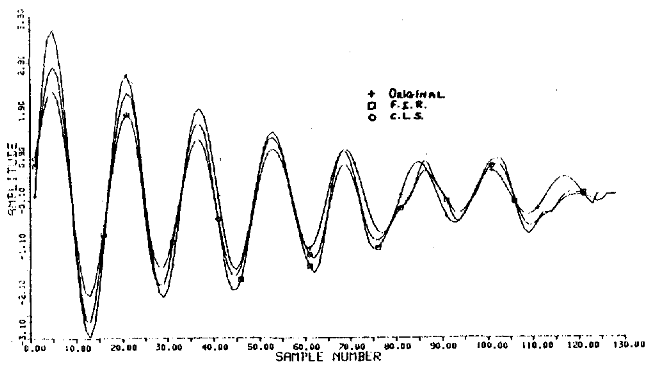

Fig: 4A

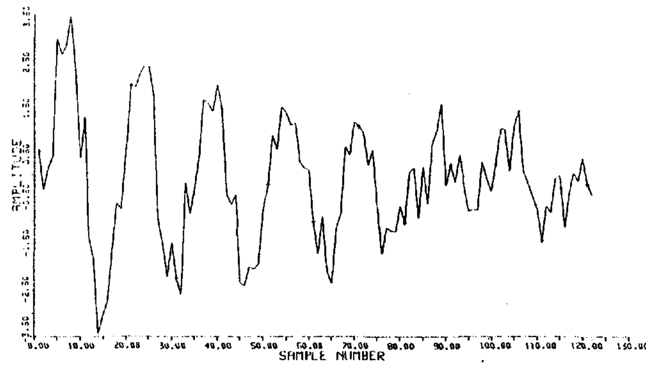

Fig: 4B

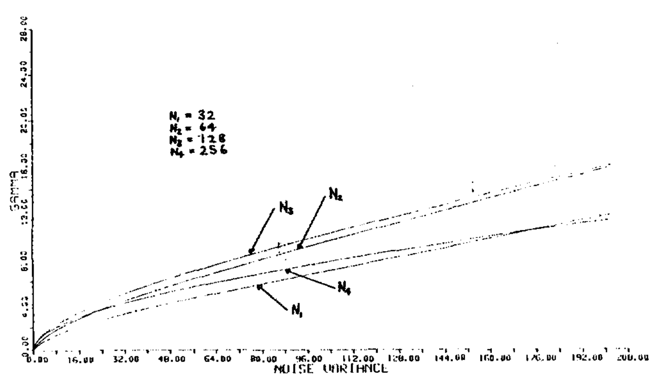

Fig: 5

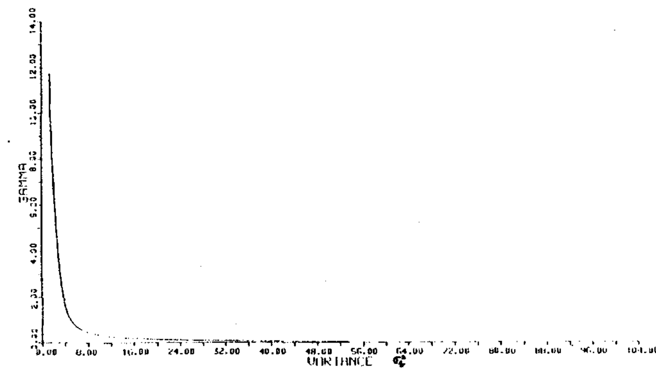

Fig: 6

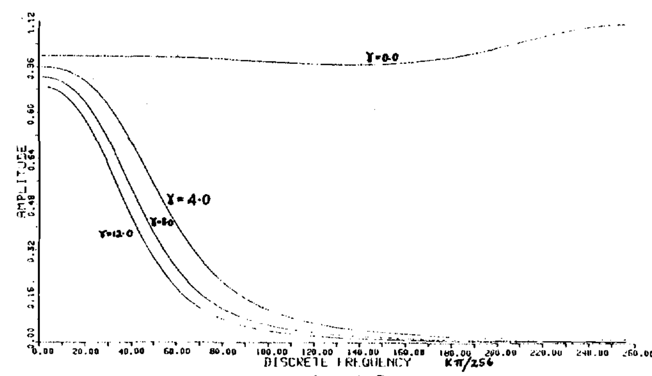

Fig: 7

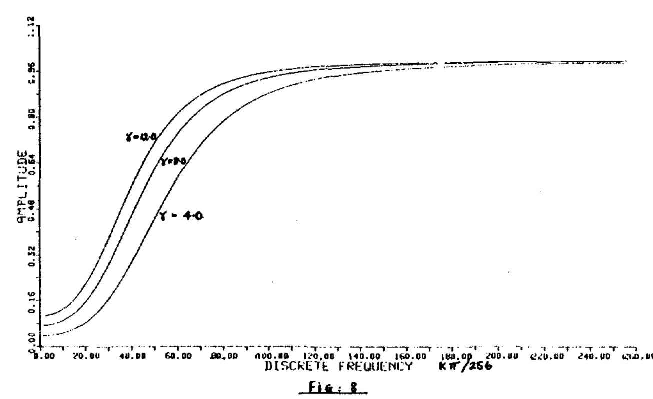

Fig: 8